Introduction to Quantum Mechanics

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Projective Measurements

Given a hermitian observable M, let its spectral decomposition be

$$M = \sum_m m P_m$$

where P_m is the projector onto the eigenspace with eigenvalue m. Upon measuring this observible with the state $|\psi\rangle$, the state collapses into a projection onto one of the eigenspaces, with probability

$$p(m) = \langle\psi|P_m|\psi\rangle$$

with the collapsed state being

$$|\psi|m\rangle = \frac{P_m |\psi\rangle}{\sqrt{p(m)}}$$

It is easy to see that

$$\mathbb{E}[M] = \langle\psi|M|\psi\rangle$$

The notion of 'measuring in a basis' $|i\rangle$ can be viewed as a projective measurement with $P_i = |i\rangle\langle i|$.

Differences between General Measurements and Projective Measurements

Both general measurements and projective measurements satisfy the completeness relation i.e. $\sum_m M_m^* M_m = I$.

In addition to general measurements, projective measurements also satisfy the additional constraints-

1. P_m are hermitian.
2. $P_m P_{m'} = \delta(m, m^{'})\ M_m$

These two conditions just mean that P_m are orthogonal projectors.

Projective Measurement $\rightarrow$ General Measurements!

- The projective measurements rule together with the postulate on unitary time evolution is sufficient to derive the postulate on general measurements using the composite systems postulate.
- Suppose we have a quantum system with state space Q with measurement operators M_m. Now we introduce an *ancilla system* with the state space M with an orthonormal basis $|m\rangle$.
- Let $|0\rangle$ be a fixed state in M. Define operator U on the products $|\psi\rangle |0\rangle$ with $|\psi\rangle$ from state space Q and $|0\rangle$ as the fixed state in M as

$$U |\psi\rangle |0\rangle = \sum_m M_m |\psi\rangle |m\rangle$$

- $\langle\phi| \langle 0| U^* U |\psi\rangle |0\rangle = \sum_{m,m'} \langle\phi| M_m^* M_m |\psi\rangle \left\langle m \middle| m' \right\rangle = \sum_m \langle\phi| M_m^* M_m |\psi\rangle$ (Using orthonormality of m) $= \langle\phi|\psi\rangle$ (Using completeness of M_m)

Projective Measurement $\rightarrow$ General Measurements!

- We saw above that the operator U preserves inner products between states of the form $|\psi\rangle|0\rangle$. As $|0\rangle$ was an arbitrary state in M, we can extend the definition of U to a unitary operator on space $Q \otimes M$ generalised from $Q \otimes |0\rangle$.
- We now perform projective measurements on the two systems describes by projectors $P_m = I_Q \otimes |m\rangle\langle m|$.
- This set of projective measurements give us the probability and the final state of the system in accordance to the measurement postulate.

Projective Measurement $\rightarrow$ General Measurements!

- The probabilities are calculated as follows

$$p(m) = \langle\phi| \langle 0| U^* P_m U |\phi\rangle |0\rangle = \langle\phi| M_m^* M_m |\phi\rangle$$

- The joint state of the system QM, given that result m occurs is

$$\frac{P_m U |\psi\rangle |0\rangle}{\sqrt{\langle\psi| U^* P_m U |\psi\rangle}} = \frac{M_m |\psi\rangle |m\rangle}{\sqrt{\langle\psi| M_m^* M_m |\psi\rangle}}$$

- As the state of the system M after measurement is given by $|m\rangle$, it follows that the final state of the system Q after measurement is given by $\frac{M_m|\psi\rangle}{\sqrt{\langle\psi|M_m^* M_m|\psi\rangle}}$

Conclusion – Thus unitary dynamics and projective measurements alongwith the ability to introduce ancilla systems enables us to implement general measurements, as described in postulate three.

POVMs

- Convenient formalism of measurement.
- Useful when we care about only the measurement statistics and not the output states.
- Formally, any set of operators E_m form a POVM if
  1. Each E_m is positive.
  2. $\sum_m E_m = I$
- We have the probabilities

$$p(m) = \langle\psi|E_m|\psi\rangle$$

- Can go from a measurement set M_m to a POVM E_m by defining $E_m \equiv M_m^* M_m$

Some Points

- A physical intuition: Projective measurements are repeatable, many real world measurements may not be.
- General measurement formalism provides more control over the measurement scheme which is useful in QCQI (example?)
- POVMs are a mathematical convenience, a restriction to the general measurement formalism in which we don't care about the post measurement state, only the measurement statistics.

Distinguishing Quantum States

- Let us consider a game between two players Alice and Bob where Alice chooses a state $|\psi_i\rangle$ for $i = 1...n$ and Bobs has to guess which state Alice gave him.
- If the states are orthonormal, Bob can do quantum measurement to win the game in the following way-
  1. Define measurement operators $M_i = |\psi_i\rangle \langle\psi_i|$ for each i and define M_0 as the positive square root of the positive operator $I - \sum_{i \neq 0} |\psi_i\rangle \langle\psi_i|$
  2. The measurement operators satisfy the completeness relation and if state i occurs then $p_i = \langle\psi_i| M_i |\psi_i\rangle = 1$ and Bob wins the game.
- If the states are not orthonormal, no quantum measurement is capable of distinguishing them.
- This is intuitively explained as follows: A state $|\psi_2\rangle$ can be decomposed to a component parallel to another state $|\psi_1\rangle$ and a component perpendicular to it. Suppose Bob guesses state $|\psi_1\rangle$ when he observes outcome j. But since $|\psi_2\rangle$ has a non-zero component parallel to $|\psi_1\rangle$, there is a probability that j is observes when $|\psi_2\rangle$ is prepared. Hence, Bob might guess wrong.

Proof

- Here we try to formally prove the fact that if the states are not orthonormal, no quantum measurement is capable of distinguishing them.
- If Bob observes outcome j, he guesses the state i the system is in using the rule $i = f(j)$.
- We proceed by a proof by contradiction. Suppose such a measurement is possible. Then if the state $|\psi_1\rangle$ is prepared, the probability of measuring outcome j such that $f(j) = 1$ is 1.
- Define $E_i = \sum_{j:f(j)=i} M_j^* M_j$. Then we have $\langle\psi_1| E_1 |\psi_1\rangle = 1$ and similarly $\langle\psi_2| E_2 |\psi_2\rangle = 1$ as the observations. Since $\sum_i E_i = I$, it follows that $\sum_i \langle\psi_1| E_i |\psi_1\rangle = 1$. These imply that $\sqrt{E_2}\,|\psi_1\rangle = 0$.
- Decompose $|\psi_2\rangle = \alpha\,|\psi_1\rangle + \beta\,|\psi\rangle$ where $|\psi\rangle$ is orthogonal to $|\psi_1\rangle$, $|\alpha|^2 + |\beta|^2 = 1$, and $|\beta| < 1$. Then $\sqrt{E_2}\,|\psi_2\rangle = \beta\sqrt{E_2}\,|\psi\rangle$ which implies $\langle\psi_2| E_2 |\psi_2\rangle = |\beta|^2 \langle\psi| E_2 |\psi\rangle \le |\beta|^2 < 1$. This implies contradiction.

A use of POVMs

- Consider the states $|\psi_1\rangle = |0\rangle$ and $|\psi_2\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$. Distinguishing between these states is impossible. However, it is possible to distinguish the states by measurement for some of the time such that if an identification is made, it is always correct.
- Let the POVM $= \{E_1, E_2, E_3\}$ contain three operators

$$E_1 = \frac{\sqrt{2}}{\sqrt{2}+1} |1\rangle \langle 1|$$

$$E_2 = \frac{\sqrt{2}}{\sqrt{2}+1} \frac{(|0\rangle - |1\rangle)(\langle 0| - \langle 1|)}{2}$$

$$E_3 = I - E_1 - E_2$$

- Notice that $\langle \psi_1| E_1 |\psi_1\rangle = 0$ and $\langle \psi_2| E_2 |\psi_2\rangle = 0$. So after the POVM measurement, if we get E_1 as a result, then we can conclude that $|\psi_1\rangle$ is not the state, and $|\psi_2\rangle$ must have been the state. The same logic can be applied to the pair E_2 and $|\psi_2\rangle$. However, if we observe E_3 we cannot comment anything on the state.