Entanglement and its Applications

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Entanglement

- After Bell's work, entanglement not only became a point to see where QM was different from classical mechanics, but was also looked at as a potentially valuable resource.
- Entanglement is basically when two quantum states are such that measuring one affects the measurement of the other.
- More mathematically, a composite system having the property that it cannot be written as a product of its subsystems is said to be entangled.
- We will look at two applications of quantum entanglement in the following slides after a short problem:
  1. Superdense Coding
  2. Quantum Teleportation

Entangled Bell State

Problem. Prove that the Bell state $|\psi\rangle = \frac{|00\rangle+|11\rangle}{2}$ cannot be written as a product state i.e. there are no single qubit states such that $|\psi\rangle = |a\rangle |b\rangle$.
Solution. Let $|a\rangle = \alpha |0\rangle + \beta |1\rangle$ and $|b\rangle = \gamma |0\rangle + \delta |1\rangle$. Then if $|\psi\rangle$ can be written as a product then

$$|\psi\rangle = \alpha\gamma |00\rangle + \alpha\delta |01\rangle + \beta\gamma |10\rangle + \beta\delta |11\rangle$$

We see here that the product of the coefficients of the $|00\rangle$ and $|11\rangle$ terms is $\alpha\beta\gamma\delta$ and the product of the coefficients of $|01\rangle$ and $|10\rangle$ is $\alpha\beta\gamma\delta$ so they are equal. But for the Bell state the former is $\frac{1}{2}$ and the latter is 0 leading to a contradiction.

Superdense Coding: The Problem

- There are two people Alice and Bob that are far away and wish to transmit some classical information from Alice to Bob.
- Alice has two classical bits of information which she has to send to Bob.
- She is only allowed to send a single qubit to Bob.
- Is it possible?
- The answer is YES.

Superdense Coding: The Solution

- Consider a third party manufacturing an entangled two-qubit state $|\psi\rangle = \frac{|00\rangle+|11\rangle}{\sqrt{2}}$ and send the first qubit to Alice and the second qubit to Bob.
- If the two bits of classical information she wishes to send Bob is "00", she does nothing to her qubit. If its "01", she applies the phase flip Z to her qubit. If its "10", she applies the quantum NOT gate. If its "11", she applies the gate iY to her qubit.
- After this operation, Bob receives the first qubit and now has both qubits. The four possible states based on the classical information Alice wanted to send are:

$$\begin{array}{ll} 00 : \frac{|00\rangle+|11\rangle}{\sqrt{2}} & 01 : \frac{|00\rangle-|11\rangle}{\sqrt{2}} \\ 10 : \frac{|10\rangle+|01\rangle}{\sqrt{2}} & 11 : \frac{|01\rangle-|10\rangle}{\sqrt{2}} \end{array}$$

- These states are orthogonal and can be distinguished by Bob using appropriate quantum measurement and he can figure out the two bits Alice wanted to send.

Quantum Teleportation: The Problem

- Alice and Bob met long ago and generated an EPR pair $\frac{|00\rangle+|11\rangle}{\sqrt{2}}$ of qubits.
- From the EPR pair, Alice takes the first qubit, leaves Bob with the second qubit and goes far away.
- Now she wishes to send another qubit $|\psi\rangle$ to Bob. (Note that Alice is now in possession of two qubits, one from the EPR pair and the other one is $|\psi\rangle$.
- The constraint is that she can only send classical information to Bob.
- Lets state some problems that Alice must overcome:
  1. She does not know the state $|\psi\rangle$ of the qubit in her possession.
  2. Even if she did know $|\psi\rangle$, a quantum state has an infinite amount of classical information as it takes its values in a continuous space.
  3. Somehow utilise the EPR qubit to send $|\psi\rangle$ to Bob.

Quantum Teleportation: The Solution - Outline

A rough outline of the solution is as follows:
1. Alice interacts with the qubit $|\psi\rangle$ in her possession through quantum gates.
2. Alice then performs quantum measurement on the two qubits she has and obtains one of the four results: 00, 01, 10, 11.
3. Alice sends this classical information to Bob.
4. Depending on Alice's classical information, Bob performs one of four operations on his half of the EPR pair.
Amazingly, the the state of the qubit Bob has after performing the operation is the same as the state $|\psi\rangle$ which Alice wanted to send.

Quantum Teleportation: The Solution - Circuit

Let use the quantum circuit from Nielson and Chuang to explain the detailed procedure:

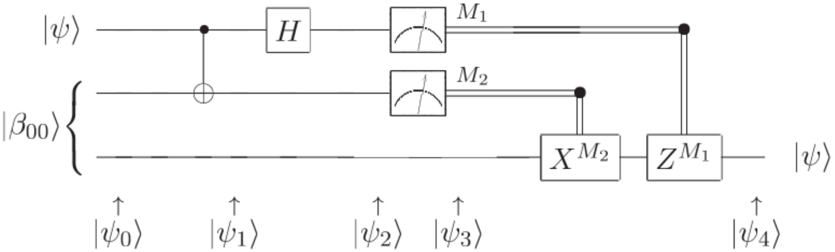

Figure 1.13. Quantum circuit for teleporting a qubit. The two top lines represent Alice's system, while the bottom line is Bob's system. The meters represent measurement, and the double lines coming out of them carry classical bits (recall that single lines denote qubits).

Quantum Teleportation: The Solution - Input and CNOT

- The state to be teleported is $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$.
- The state input in the circuit is $|\phi_0\rangle = |\psi\rangle|\beta_{00}\rangle$ where $|\beta_{00}\rangle = \frac{|00\rangle+|11\rangle}{\sqrt{2}}$ represents the state of the EPR pair.

$$|\psi_0\rangle = |\psi\rangle|\beta_{00}\rangle = \frac{1}{\sqrt{2}}[\alpha|0\rangle(|00\rangle + |11\rangle) + \beta|1\rangle(|00\rangle + |11\rangle)]$$

- The first two qubits here belong to Alice and the last one belongs to Bob, and Alice's second qubit and Bobs's qubit start out in an EPR state.
- Alice send her two qubits through a CNOT gate obtaining:

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}[\alpha|0\rangle(|00\rangle + |11\rangle) + \beta|1\rangle(|10\rangle + |01\rangle)]$$

Quantum Teleportation: The Solution - Hadamard

- Alice then send her first qubit through a Hadamard gate to obtain

$$|\psi_2\rangle = \frac{1}{2}\left[\alpha(|0\rangle + |1\rangle)(|00\rangle + |11\rangle) + \beta(|0\rangle - |1\rangle)(|10\rangle + |01\rangle)\right]$$

- Regrouping terms in this state we get:

$$|\psi_2\rangle = \frac{1}{2}[|00\rangle (\alpha |0\rangle + \beta |1\rangle) + |01\rangle (\alpha |1\rangle + \beta |0\rangle) +$$

$$|10\rangle (\alpha |0\rangle - \beta |1\rangle) + |11\rangle (\alpha |1\rangle - \beta |0\rangle)]$$

- In this new expression we have the first two qubits to be in possession of Alice and the term multiplied to it to be the state of Bob's qubit.

Quantum Teleportation: The Solution - Measurement

- Now Alice performs a quantum measurement on her two qubits.
- Notice that if the result of her measurement is 00, then Bob's state collapses to $\alpha\,|0\rangle + \beta\,|1\rangle$, which is the same as $|\psi\rangle$, the state which Alice wanted to deliver.

$$00 \implies |\psi_3(00)\rangle = (\alpha\,|0\rangle + \beta\,|1\rangle)$$

$$01 \implies |\psi_3(01)\rangle = (\alpha\,|1\rangle + \beta\,|0\rangle)$$

$$10 \implies |\psi_3(10)\rangle = (\alpha\,|0\rangle - \beta\,|1\rangle)$$

$$11 \implies |\psi_3(11)\rangle = (\alpha\,|1\rangle - \beta\,|0\rangle)$$

- For the other measurements Bob has to fix up his state using the gate $Z^{M_1}X^{M_2}$ where M_1 and M_2 are the measurement outcomes of the two qubits in Alice's possession respectively to get the state $\psi_4 = \alpha\,|0\rangle + \beta\,|1\rangle = |\psi\rangle$ in all cases as desired.

Quantum Teleportation: Afterthoughts

Some puzzles and paradoxes around quantum teleportation are resolved below:

- Doesn't teleportation allow communication faster than the speed of light?
  No. The communication of the two classical bits prevents this from happening.
- This seems to violate the no–cloning theorem.
  No it does not. At the end of the process, the information of $|\psi\rangle$ is only with Bob as the qubit Alice had had been changed by measurement.

Quantum teleportation emphasises interchangeability of different resources in quantum mechanics. One EPR pair and a pair of classical qubits is a resources equal to least one qubit.