Introduction to Quantum Mechanics

Siddhant Midha and Aaryan Gupta

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Density Operators

- What if the system in consideration interacts with the environment?
- Can describe their combined behaviour by a $|\psi\rangle$.
- May not be able to describe the behaviour of the system with a pure state.
- Density operators.
- Encoding some classical uncertainity within the already present quantum uncertainity.

Density Operators

- If a system exists in the states $|\psi_i\rangle$ with probabilities p_i, then its density operator ρ is defined as

$$\rho = \sum_i p_i \left|\psi_i\right\rangle \left\langle\psi_i\right|$$

- The density operator is positive and has unit trace. The converse holds.
- Unitary evolution

$$\rho = \sum_i p_i \left|\psi_i\right\rangle \left\langle\psi_i\right| \to \sum_i p_i U \left|\psi_i\right\rangle \left\langle\psi_i\right| U^* = U\rho U^*$$

- Measurements

$$p(m) = Tr(M_m^* M_m \rho)$$

$$\rho_m = \frac{M_m \rho M_m^*}{Tr(M_m^* M_m \rho)}$$

Some more points

- We can even have a mixture of density operators $\{\rho_i, p_i\}$, such that $\rho = \sum_i p_i \rho_i$ describes the state of the system. Can be viewed as $\sum_{ij} q_{ij} \left|\psi_{ij}\right\rangle \left\langle\psi_{ij}\right|$. A simple example is the action of $\mathcal{M} = \{M_m\}$ on ρ without knowing the outcome.
- The postulates can be reformed in the density operator language.
- A ρ is pure iff $tr(\rho^2) = 1$.
- Set of density ops is convex, and the pure states occupy the boundary of this set. Intuitive example: two qubits.

Formation of density operators

Freedom in generation of the density operator.

Theorem 2.6 of QCQI

The vectors $\left|\tilde{\phi}_i\right\rangle$ and $\left|\tilde{\psi}_j\right\rangle$ generate the same density matrix (i.e., $\sum_i \left|\tilde{\phi}_i\right\rangle \left\langle\tilde{\phi}_i\right| = \sum_j \left|\tilde{\psi}_j\right\rangle \left\langle\tilde{\psi}_j\right|$) iff

$$\left|\tilde{\phi}_i\right\rangle = \sum_j u_{ij} \left|\tilde{\psi}_j\right\rangle$$

where $U = [u_{ij}]$ is a unitary matrices.

Note In the above theorem, we absorbed the probabilities into the vectors as $\left|\tilde{\psi}\right\rangle \equiv \sqrt{p}\,|\psi\rangle$.

Analyzing subsystems

If we have systems A and B, described by the density operator ρ_{AB}, we define the density operator for the subsystem A as

$$\rho_A \equiv Tr_B(\rho_{AB})$$

with the partial trace operation Tr_B being defined as

$$Tr_B(|a_1\rangle\langle a_2| \otimes |b_1\rangle\langle b_2|) \equiv |a_1\rangle\langle a_2|\, tr(|b_1\rangle\langle b_2|)$$

and for general states ρ_{AB} the definition extends by superposition. Or,

$$Tr_B(\rho_{AB}) := \sum_j (I_A \otimes \langle j|_B)\rho_{AB}(I_A \otimes |j\rangle_B)$$

A nice example

Consider

$$|\psi\rangle_{AB} = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

Thus, $\rho_{AB} = |\psi\rangle_{AB} \langle\psi|_{AB}$. Now, we can compute $\rho_A = Tr_B(\rho_{AB})$.
(compute)

(comment on $S(\cdot)$)

Why the partial trace?

- The unique operation that preserves observable statistics.
- Let $M = \sum_m mP_m$ be an observable on the system A
- Let M' be the corresponding observable on the system AB. Thus, any state $|m\rangle_A |\psi\rangle_B$ of AB should have $p(m) = 1$.
- Thus, the corresponding projector for measuring m on AB is $P_m \otimes I_B$ and thus

$$M' = \sum_m mP_m \otimes I_B = M \otimes I_B$$

Why the partial trace?

- Now, we want a map $f(\cdot)$ such that $f(\rho_{AB})$ gives us the density operator describing the state of A.
- We need

$$tr(Mf(\rho_{AB})) = tr((M \otimes I_B)\rho_{AB}) \text{ –}(\#)$$

- Let M_i be an orthonormal basis of the space of hermitian operators w.r.t. the inner product $(A, B) \equiv Tr(AB)$. Expand $f(\rho_{AB})$ as

$$f(\rho_{AB}) = \sum_i M_i tr(M_i f(\rho_{AB})) = \sum_i M_i tr((M \otimes I_B)\rho_{AB})$$

- Thus such an f is unique.
- Now see that $f = tr_B$ satisfies $(\#)$.

Schmidt Decomposition

This is a very useful theorem in quantum information.

Theorem 2.7 of QCQI

Let $|\psi\rangle \in \mathcal{H}_{AB}$. Then there exist orthonormal states $|i_A\rangle \in \mathcal{H}_A$ and orthonormal states $|i_B\rangle \in \mathcal{H}_B$ such that

$$|\psi\rangle = \sum_i \lambda_i |i_A\rangle \otimes |i_B\rangle$$

where $\lambda_i \geq 0$ and $\sum_i \lambda_i^2 = 1$.

Purification

- Suppose we are given the state ρ_A for the system A.
- With the help of the previous theorem, we come up with an auxilliary system R, such that the combined state of these two systems is a pure state.
- Purification.
- Suppose

$$\rho_A = \sum_i p_i \left| i_A \right\rangle \left\langle i_A \right|$$

- Introduce R with $\mathcal{H}_R = \mathcal{H}_A$ and a orthonormal basis $\left| i_R \right\rangle$.
- Define

$$\left| \psi_{AR} \right\rangle \equiv \sum_i \sqrt{p_i} \left| i_A \right\rangle \left| i_R \right\rangle$$

- See that

$$\rho_A = Tr_R(\left| \psi_{AR} \right\rangle \left\langle \psi_{AR} \right|)$$