Quantum Operations and Noise

Siddhant Midha and Aaryan Gupta

Last updated on July 2, 2022

Some classical intuition

- A register X with the alphabet Σ may be in a probabilistic state p.
- Some noise causes the state to change to q. We can use the law of total probability

$$q(a) = \sum_{b \in \Sigma} \mathbb{P}(X' = a | X = b)\mathbb{P}(X = b) \ \forall a \in \Sigma$$

- Thus we have

$$q = Ep$$

for some matrix E *such that*

1. Positivity is held: all entries of E are non negative.
2. Completeness is fulfilled $\sum_{a \in \Sigma} \mathbb{P}(X' = a | X = b) = 1$ (columns sum to one).

Quantum Operations

- We know that systems may not be represented by a pure quantum states.
- Introduced density operator formalism.
- Now how does a system in the state ρ evolve?
- It might not be a closed system.
- Close the system by including the environment. Thus the net state is $\rho \otimes \rho_{env}$.
- The postulate now follows, so we have the new state $\mathcal{E}(\rho)$ of the system as

$$\mathcal{E}(\rho) = Tr_{env}(U(\rho \otimes \rho_{env})U^*)$$

- Issue! Why must the combined state be a product one?
- Observation Input and output spaces needn't be the same.

Don't want the environment

- Goal: describe general quantum operations on open systems without accounting for the environment.
- Question arises: What kind of a map must $\mathcal{E}$ be so that it represents a valid quantum operation?
- Axioms;
  1. $Tr[\mathcal{E}(\rho)] \in [0,1] \forall \rho$ ($Tr[\mathcal{E}(\rho)]$ is the probability that ρ undergoes the transformation $\mathcal{E}$).
  2. Convex linearity
  $$\mathcal{E}(\sum_i p_i \rho_i) = \sum_i p_i \mathcal{E}(\rho_i)$$
  for all density matrices ρ_i and probabilities p_i s.t. $\sum_i p_i = 1$
  3. $\mathcal{E}$ is completely positive. Not only does $\mathcal{E}$ preserve positivity, $(I \otimes \mathcal{E})$ also preserves positivity for I being the identity on an aribtrarily dimensional system's hilbert space.

Complete Positivity vs. Positivity

Consider the transpose operation on a single qubit. By definition, this map transforms

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longrightarrow \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

This map obviously preserves positivity of a single qubit.
However, consider a two qubit system in $|\beta_{00}\rangle$ with the transpose operation being applied to the first of the two qubits. The density operator of the system after the dynamics has been applied is

$$\frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This is not a positive denisty operator. Therefore, the trace preserves positivity of operators when applied to principal system but not when applied to systems which are subsystems of the principal system.

Example

For a single qubit state ρ, a measurement in the computational basis can be described by the operations $\mathcal{E}_0(\rho) \equiv |0\rangle\langle 0|\rho|0\rangle\langle 0|$ and $\mathcal{E}_1(\rho) \equiv |1\rangle\langle 1|\rho|1\rangle\langle 1|$ with probabilities given as $tr[\mathcal{E}_0(\rho)]$ and $tr[\mathcal{E}_1(\rho)]$. The final state is

$$\frac{\mathcal{E}_i(\rho)}{Tr([\mathcal{E}_i(\rho)])} \text{for some } i \in \{0, 1\}$$

That is, if no measurement is happening, the map $\mathcal{E}$ would be a *completely positive trace preserving (CPTP)* map.

The operator sum representation

Theorem 8.1 of QCQI

The map $\mathcal{E}$ satisfies the axioms for a valid quantum operation iff there exists a set of operators $\{E_i\}$ such that

$$\mathcal{E}(\rho) = \sum_i E_i \rho E_i^*$$

for all valid density matrices ρ and $0 \preccurlyeq \sum_i E_i^* E_i \preccurlyeq I$

Note: $A \preccurlyeq B$ if $B - A$ is PSD. So if we are just dealing with CPTP maps, then these E_i satisfy $\sum_i E_i^* E_i = I$, and are called the *kraus operators.*

S+E $\rightarrow$ Kraus

- We have, the system coupled to an environment under unitary evolution as
$$U(\rho \otimes |e_0\rangle \langle e_0|)U^*$$
- Trace out the environment,
$$\sum_k (\mathbb{1} \otimes \langle e_k|)U(\rho \otimes |e_0\rangle \langle e_0|)U^*(\mathbb{1} \otimes |e_k\rangle)$$
- See that
$$(\rho \otimes |e_0\rangle \langle e_0|) = (\mathbb{1} \otimes |e_0\rangle)\rho(\mathbb{1} \otimes \langle e_0|)$$
- Define
$$E_k \equiv (\mathbb{1} \otimes \langle e_k|)U(\mathbb{1} \otimes |e_0\rangle)$$
and see the equivalence.

More physically ...

With the notation in the previous slide, define

$$\rho_k \equiv \frac{E_k \rho E_k^*}{tr(E_k \rho E_k^*)}$$

Thus, we can consider the act of applying $\mathcal{E}$ as applying U to $\rho \otimes |e_0\rangle \langle e_0|$ and then measuring the environment in the $|e_k\rangle$ basis.
That is equivalent to replacing ρ randomly by ρ_k with the probability $p(k) = tr(E_k \rho E_k^*)$, thus resulting in

$$\mathcal{E}(\rho) = \sum_k p(k)\rho_k = \sum E_k \rho E_k^*$$

as expected.

(More General) S+E $\rightarrow$ Kraus

Let P_m be a projective measurement on the S+E. Define

$$\mathcal{E}_m(\rho) = tr_E(P_m U(\rho \otimes \sigma) U^* P_m)$$

where $\sigma = \sum_i q_j \left|j\right\rangle \left\langle j\right|$. And let $\left|e_k\right\rangle$ be a basis for $\mathcal{H}^E$.Thus,

$$\mathcal{E}_m(\rho) = \sum_{jk} q_j(\mathbb{1} \otimes \left\langle e_k\right|) P_m U(\rho \otimes \left|j\right\rangle \left\langle j\right|) U^* P_m(\mathbb{1} \otimes \left|e_k\right\rangle)$$

Define

$$E^m_{jk} = \sqrt{q_j}(\mathbb{1} \otimes \left\langle e_k\right|) P_m U(\mathbb{1} \otimes \left|j\right\rangle)$$

and we get

$$\mathcal{E}_m(\rho) = \sum_{jk} E^m_{jk} \rho E^{m*}_{jk}$$

It is easy to see that $\sum_{jk} E^{m*}_{jk} E^m_{jk} \preccurlyeq \mathbb{1}$, not $= \mathbb{1}$. The state evolves to $\mathcal{E}_m(\rho)$ with probability $tr(\mathcal{E}_m(\rho))$.

Kraus $\rightarrow$ S+E

- We are given the set $\{\mathcal{E}_m\}$. We shall construct a S+E (+measurement) model.
- Let E^m_k be the kraus rep. for the operation $\mathcal{E}_m$.
- Introduce env E with an orthonormal basis $|m, k\rangle$ with indices in 1-1 correspondence.
- Let $|e_k\rangle$ be a basis for $\mathcal{H}^E$ and define

$$U|\psi\rangle \otimes |e_0\rangle \equiv \sum_{mk} E_{mk}|\psi\rangle|m, k\rangle$$

  As done earlier, we know this can be extended to a unitary on the composite system.
- Define

$$P_m \equiv \sum_{m} |m, k\rangle\langle m, k|$$

Kraus $\to$ S+E

Let $\rho = \sum p_i \left|i\right\rangle \left\langle i\right|$ be a state of our system.

Consider

$$\begin{aligned} U(\rho \left|e_0\right\rangle \left\langle e_0\right|)U^* &= \sum_j p_i U(\left|j\right\rangle \left\langle j\right| \otimes \left|e_0\right\rangle \left\langle e_0\right|)U^* \\ &= \sum_{mjk} p_j (E_{mk} \left|j\right\rangle \left\langle j\right| E_{mk}^* \otimes \left|m,k\right\rangle \left\langle m,k\right|) \end{aligned}$$

Now it is easy (and a bit annoying to write) to see that measuring P_m will result in $\mathcal{E}_m(\rho)$ with probability $tr(\mathcal{E}_m(\rho))$.

Concluding Points

- Kraus operators are not unique. Unitary equivalence exists as in the case of ensembles.
- Physical motivation for kraus ops: Unitary evolution on joint state, and then measurement[1] of the environment in some basis.
- Non trace preserving maps are those which have unitary evolution of the sys+env followed by projective measurement of the two. Thus trace of $\mathcal{E}_m(\rho)$ represents the probability that E_m took place out of all possible m.[2].
- Given an opsum representation, we can cook up an environment s.t. unitary evolution (plus possibly projective measurement) followed by tracing out environment describes the map.
- For a d dimensional system, a general CPTP map can be represented by atmost d^2 kraus operators.[3]

[1] *without knowing the outcome*

[2] that is, a single non trace preserving map $\mathcal{E}$ does NOT describe the dynamics fully, you need the set $\{\mathcal{E}_m\}$

[3] a consequence of the unitary freedom theorem

Depolarising Channels

- A depolarising channel is a type of quantum noise where we take a single qubit, and with a probability p, it is completely depolarised, i.e. replaced with $\frac{I}{2}$. With probability $1-p$, it is left untouched.

$$\mathcal{E}(\rho) = \frac{pI}{2} + (1-p)\rho$$

- Notice that

$$\frac{I}{2} = \frac{\rho + X\rho X + Y\rho Y + Z\rho Z}{4}$$

Substituting for $\frac{I}{2}$, we get

$$\mathcal{E}(\rho) = \left(1 - \frac{3p}{4}\right)\rho + \frac{p}{4}(X\rho X + Y\rho Y + Z\rho Z)$$

$$\mathcal{E}(\rho) = (1-p)\rho + \frac{p}{3}(X\rho X + Y\rho Y + Z\rho Z)$$

- The operator set for the second representation can be written as $\{\sqrt{1-p}I, \sqrt{\frac{p}{3}}X, \sqrt{\frac{p}{3}}Y, \sqrt{\frac{p}{3}}Z\}$

Geometric picture of Single qubit operations

- A general state can be written as

$$\rho = \frac{I + \vec{r} \cdot \vec{\sigma}}{2} = \frac{1}{2} \begin{bmatrix} 1 + r_z & r_x - ir_y \\ r_x + ir_y & 1 - r_z \end{bmatrix}$$

- It turns out that an arbitrary trace-preserving quantum operation is equivalent to the map $\vec{r} \xrightarrow{\mathcal{E}} \vec{r'} = M\vec{r} + \vec{c}$ where M is a real matrix and $\vec{c}$ is a constant vector. This is an affine map, mapping vectors from the Bloch sphere to itself.
- Suppose that the E_i's generating the operator sum representation for $\mathcal{E}$ can be written as $E_i = \alpha_i I + \sum_{k=1}^{3} a_{ik}\sigma_k$. We then use the completeness relation for E_i's to calculate expressions for M and $\vec{c}$.
- We can unitarily decompose M as OS where O is a real orthogonal matrix with $|O| = 1$ and S is a real symmetric matrix. Viewed this way, thsi operation is basically a deformation of the Bloch sphere along principal axes determined by S, followed by a proper rotation by O and a displacement due to $\vec{c}$.

Bit Flip

The bit-flip channel changes the state of a qubit from $|0\rangle$ to $|1\rangle$(and vice-versa) with a probability p. It is easy to see that it has operation elements

$$E_0 = \sqrt{p}I = \sqrt{p}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } E_1 = \sqrt{1-p}X = \sqrt{1-p}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

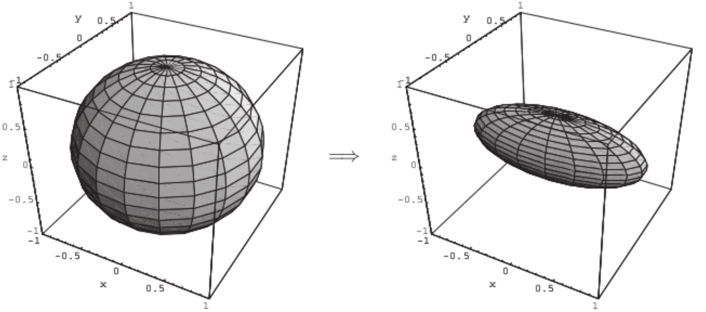

Figure 8.8. The effect of the bit flip channel on the Bloch sphere, for $p = 0.3$. The sphere on the left represents the set of all pure states, and the deformed sphere on the right represents the states after going through the channel. Note that the states on the $\hat{x}$ axis are left alone, while the $\hat{y}$-$\hat{z}$ plane is uniformly contracted by a factor of $1 - 2p$.

Phase Flip

The phase-flip has operation elements

$$E_0 = \sqrt{p} I = \sqrt{p} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } E_1 = \sqrt{1-p} Z = \sqrt{1-p} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

The case $p = \frac{1}{2}$ corresponds to the map $(r_x, r_y, r_z) \to (0, 0, r_z)$. (Discuss)

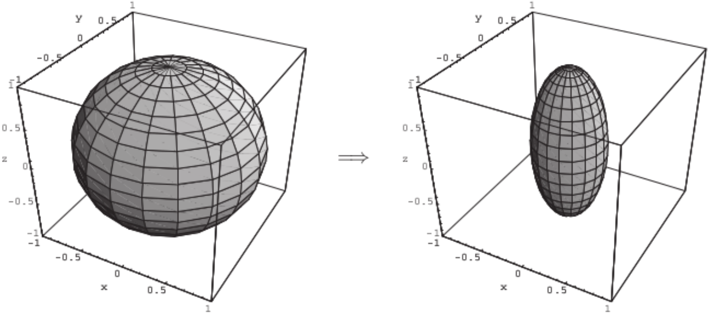

Figure 8.9. The effect of the phase flip channel on the Bloch sphere, for $p = 0.3$. Note that the states on the $\hat{z}$ axis are left alone, while the $\hat{x} - \hat{y}$ plane is uniformly contracted by a factor of $1 - 2p$.

Bit-Phase Flip

The bit-phase flip has operation elements

$$E_0 = \sqrt{p} I = \sqrt{p} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } E_1 = \sqrt{1-p} Y = \sqrt{1-p} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

This is a combination of both bit and phase flips as $Y = iXZ$.

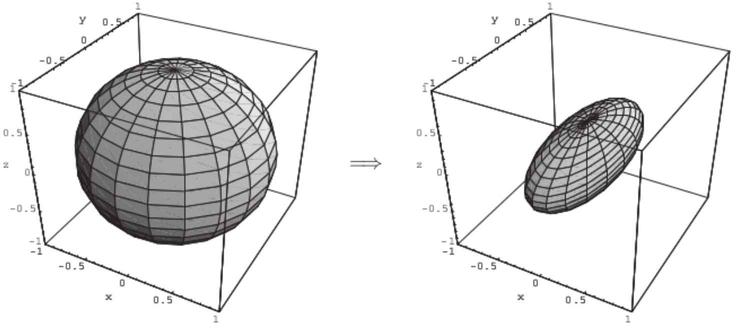

Figure 8.10. The effect of the bit–phase flip channel on the Bloch sphere, for $p = 0.3$. Note that the states on the $\hat{y}$ axis are left alone, while the $\hat{x}$-$\hat{z}$ plane is uniformly contracted by a factor of $1 - 2p$.