Linear Algebra Primer

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Linear Algebra-Pauli Matrices

We will be using these matrices quite frequently.

$$\sigma_0 \equiv I \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \sigma_1 \equiv \sigma_x \equiv X \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\sigma_2 \equiv \sigma_y \equiv Y \equiv \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma_3 \equiv \sigma_z \equiv Z \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Linear Algebra-Operator Functions

- A function f from the set of complex numbers to the set of complex numbers can be generalised to normal operators.
- After a spectral decomposition, we just apply the function to its diagonal values.
- Note that this could not be done for any general matrix because the functions wouldnt be functions.
- Let A be a matrix which has the spectral decomposition

$$A = \sum_i \lambda_i |i\rangle \langle i|$$

Then $f(A)$ will be defined as

$$f(A) = \sum_i f(\lambda_i) |i\rangle \langle i|$$

Linear Algebra- Decompositions

The Spectral Decomposition

Any normal operator M on a vector space V is diagonal with respect to some orthonormal basis for V. Conversely, any diagionalizable operator is normal.

Polar Decomposition

Let A be a linear operator on a vector space V. Then there exists unitary U and positive operators J, K such that

$$A = UJ = KU$$

where, $J = \sqrt{A^\dagger A}, K = \sqrt{AA^\dagger}$.Further, if A is invertible then U is unique.

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Singular Value Decomposition

Let A be a square matrix.Then there exist unitaries U, V and a diagonal matrix D with non-negative entries such that

$$A = UDV$$

Linear Algebra- Commutator and Anti-Commutator

- The commutator between two operators is defined as $[A, B] = AB - BA$
- The anticommutator between two operators is defined as $A, B = AB + BA$

Simultaneous Diagonalization

Let A, B be hermitian operators on a vector space V. , $[A, B] = 0 \Leftrightarrow$ there exists an orthonormal basis for V which can diagonalize both A and B

Linear Algebra-Tensor Products

- The tensor product is a way of combining two vector spaces to form larger vector spaces
- Suppose V, W are vector spaces of dimension m, n respectively. Then $V \otimes W$ is a vector space of dimension mn.
- Tensor products are distributive and linear over the input.
- Suppose A is a m by n matrix and B is a p by q matrix, then the tensor product can be represented in Kronecker product form

$$A \otimes B \equiv \left.\overbrace{\begin{bmatrix} A_{11}B & A_{12}B & \dots & A_{1n}B \\ A_{21}B & A_{22}B & \dots & A_{2n}B \\ \vdots & \vdots & \vdots & \vdots \\ A_{m1}B & A_{m2}B & \dots & A_{mn}B \end{bmatrix}}^{nq}\right\} mp\,.$$

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