Quantum Operations and Noise

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Some classical intuition

- A register X with the alphabet Σ may be in a probabilistic state p.
- Some noise causes the state to change to q. We can use the law of total probability

$$q(a) = \sum_{b \in \Sigma} \mathbb{P}(X' = a | X = b) \mathbb{P}(X = b) \ \forall a \in \Sigma$$

- Thus we have

$$q = Ep$$

  for some matrix E *such that*

  1. Positivity is held: all entries of E are non negative.
  2. Completeness is fulfilled $\sum_{a \in \Sigma} \mathbb{P}(X' = a | X = b) = 1$ (columns sum to one).

Quantum Operations

- We know that systems may not be represented by a pure quantum states.
- Introduced density operator formalism.
- Now how does a system in the state ρ evolve?
- It might not be a closed system.
- Close the system by including the environment. Thus the net state is $\rho \otimes \rho_{env}$.
- The postulate now follows, so we have the new state $\mathcal{E}(\rho)$ of the system as

$$\mathcal{E}(\rho) = Tr_{env}(U(\rho \otimes \rho_{env})U^*)$$

- Issue! Why must the combined state be a product one?
- Observation Input and output spaces needn't be the same.

Don't want the environment

- Goal: describe general quantum operations on open systems without accounting for the environment.
- Question arises: What kind of a map must $\mathcal{E}$ be so that it represents a valid quantum operation?
- Axioms;
  1. $Tr[\mathcal{E}(\rho)] \in [0,1] \forall \rho$ ($Tr[\mathcal{E}(\rho)]$ is the probability that ρ undergoes the transformation $\mathcal{E}$).
  2. Convex linearity
  $$\mathcal{E}(\sum_i p_i \rho_i) = \sum_i p_i \mathcal{E}(\rho_i)$$
  for all density matrices ρ_i and probabilities p_i s.t. $\sum_i p_i = 1$
  3. $\mathcal{E}$ is completely positive. Not only does $\mathcal{E}$ preserve positivity, $(I \otimes \mathcal{E})$ also preserves positivity for I being the identity on an aribtrarily dimensional system's hilbert space.

Example

For a single qubit state ρ, a measurement in the computational basis can be described by the operations
$\mathcal{E}_0(\rho) \equiv |0\rangle\langle 0|\rho|0\rangle\langle 0|$ and $\mathcal{E}_1(\rho) \equiv |1\rangle\langle 1|\rho|1\rangle\langle 1|$ with probabilities given as $tr[\mathcal{E}_0(\rho)]$ and $tr[\mathcal{E}_1(\rho)]$. The final state is

$$\frac{\mathcal{E}_i(\rho)}{Tr([\mathcal{E}_i(\rho)])} \text{for some } i \in \{0, 1\}$$

That is, if no measurement is happening, the map $\mathcal{E}$ would be a *completely positive trace preserving (CPTP)* map.

The operator sum representation

Theorem 8.1 of QCQI

The map $\mathcal{E}$ satisfies the axioms for a valid quantum operation iff there exists a set of operators $\{E_i\}$ such that

$$\mathcal{E}(\rho) = \sum_i E_i \rho E_i^*$$

for all valid density matrices ρ and $0 \preccurlyeq \sum_i E_i^* E_i \preccurlyeq I$

Note: $A \preccurlyeq B$ if $B - A$ is PSD. So if we are just dealing with CPTP maps, then these E_i satisfy $\sum_i E_i^* E_i = I$, and are called the *kraus operators.*

Concluding Points

- Kraus operators are not unique. Unitary equivalence exists as in the case of ensembles.
- Physical motivation for kraus ops: Unitary evolution on joint state, and then measurement[1] of the environment in some basis.
- Non trace preserving maps are those which have unitary evolution of the sys+env followed by projective measurement of the two. Thus trace of $\mathcal{E}_m(\rho)$ represents the probability that E_m took place out of all possible m.[2].
- Given an opsum representation, we can cook up an environment s.t. unitary evolution (plus possibly projective measurement) followed by tracing out environment describes the map.
- For a d dimensional system, a general CPTP map can be represented by atmost d^2 kraus operators.[3]

[1] *without knowing the outcome*

[2] that is, a single non trace preserving map $\mathcal{E}$ does NOT describe the dynamics fully, you need the set $\{\mathcal{E}_m\}$

[3] not in QCQI, elsewhere.