$C_s(w)$

Notes on Counting with Dual Graphs

INTRO

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Tree Decompositions

$G = (V(G), E(G)) \rightarrow$ graph
$T = (V(T), E(T)) \rightarrow$ tree

$\chi \rightarrow$ labelling of $\underbrace{\text{vertices of } T}_{\text{nodes}}$ by sets of vertices of G

$(T, \chi) \rightarrow$ tree decomposition of G iff.

1. For every $v \in V(G)$ $\exists$ a node $t \in V(T)$ s.t. $v \in \chi(t)$
2. For every $(v, w) \in E(G)$ $\exists$ $t \in V(T)$ s.t. $v, w \in \chi(t)$
3. For any three nodes $t_1, t_2, t_3 \in V(T)$, if t_2 lies on the path from t_1 to t_3, then $\chi(t_1) \cap \chi(t_3) \subseteq \chi(t_2)$

$$\text{width}_{(T,\chi)} = \max_{t \in V(T)} |\chi(t)| - 1$$

$$\text{treewidth } tw(G) = \min_{(T, \chi) \text{ for } G} \text{width}_{(T,\chi)}$$

Any tree decomposition ~~(T, χ)~~ can be converted to a "nice TD" (T, χ, r) such that (r is root node).

1. Every node of T has at most two children
2. If a node of T, t has two children t_1 and t_2, then $\chi(t) = \chi(t_1) = \chi(t_2)$
3. If $t \in V(T)$ has exactly one child t' then
   a) $|\chi(t)| = |\chi(t')| + 1$ and $\chi(t') \subset \chi(t) \Rightarrow$ introduce node t
   b) $|\chi(t)| = |\chi(t')| - 1$ and $\chi(t) \subset \chi(t') \Rightarrow$ forget node t

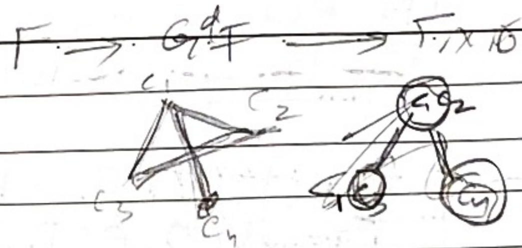
$\underset{\text{is nice}}{(T, \chi, r)} \Longrightarrow (T_t, \chi_{|V(T_t)}, t)$ is nice

$G^d(F)$ **Dual Graph** $\Rightarrow$ of CNF formula F

vertex set F (clauses) edges: C, C' are joined by an edge if $var(C) \cap var(C') = \emptyset$

AZONE

Dual tw in model Counting



~~Defn~~ What to Count??

Let (T, χ, r) be a nice tree decomp of dual graph $G^d(F)$ of CNF F.

Let t be a node of T and for each subset $A \subseteq \chi(t)$ we define $N(t, A)$ as the set of truth assignments $\tau : var(V_t) \to \{0,1\}$ for which-

1. Every clause in A is falsified by τ
2. Every clause in $V_t \setminus \chi(t)$ is satisfied by τ

→ irrespective of A.

$\chi(t) \setminus A$ clauses??

$n(t, A) = |N(t, A)|$

At each node there is a table M_t. Columns of M_t contain Boolean value encoding membership of C in A for clauses $C \in \chi(t)$ and $n(t, A)$.

Join Node

$$n(t, A) = \frac{n(t_1, A) \cdot n(t_2, A)}{2^{|var(\chi(t)) \setminus var(A)|}}$$

$A \subseteq \chi(t)$

↳ set of clauses

→ these are false under assignment

$\tau : var(V_t) = var(V_{t_1/t_2}) \to \{0,1\}$

$\boxed{x(t) = V_{t_1} \cap V_{t_2}}$ ⇒ from connectedness property

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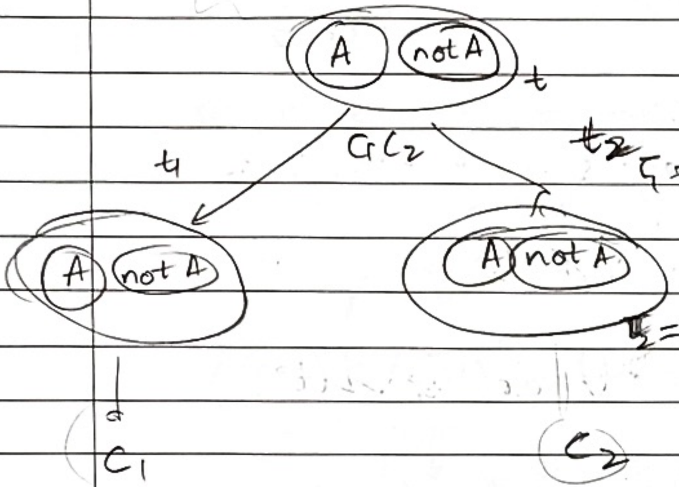
Proof for join node -

$f: \tau \mapsto (\tau|_{var(V_{t_1})}, \tau|_{var(V_{t_2})})$ is a bijection from $N(t,A)$ to

injectivity + surjectivity

N(t) → $M = \{ (\tau_1, \tau_2) \mid \tau_1 \in N(t_1, A), \tau_2 \in N(t_2, A), \tau_1(x) = \tau_2(x)$ for all $x \in var(V_{t_1}) \cap var(V_{t_2}) \}$

by τ
A → falsified, $C_1 C_2$ → sat.



$\tau_1 = \tau|_{var(V_{t_1})}$ A → falsified
C_1 → ~~falsified~~ sat

$\tau_2 = \tau|_{var(V_{t_2})}$ A → falsified
C_2 → sat.

~~Now C_1 & C_2 may have common variables~~

conn. property → C_1 and C_2 have all different variables

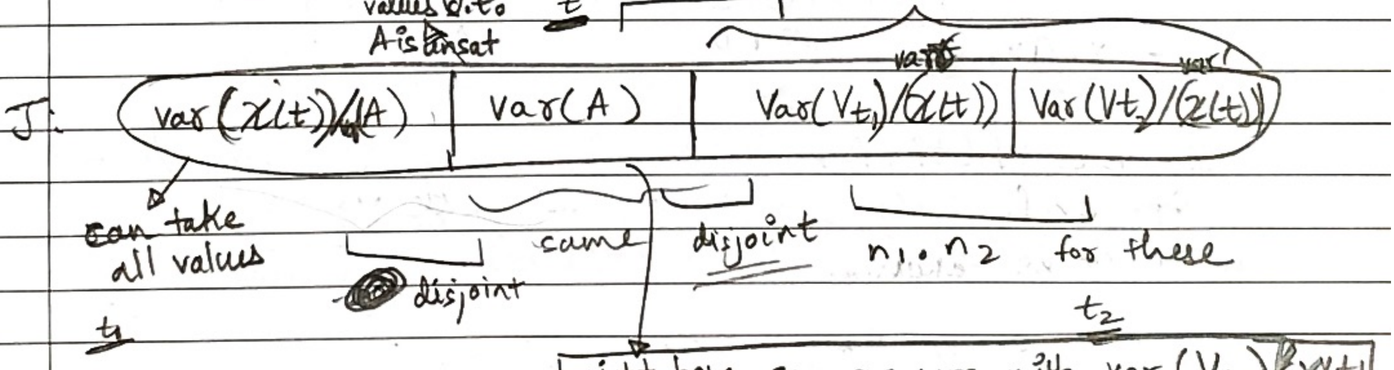
no. of assignments $\tau: Var(V_t) \to \{0,1\}$

$var(x(t)/A)$ ⇒ can take any value / disjoint

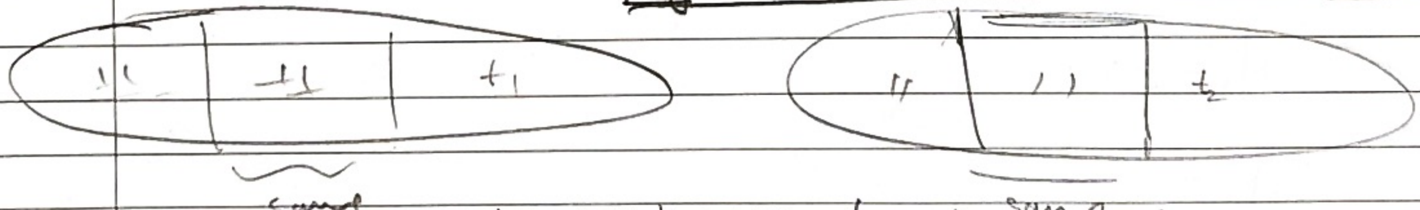
variable string

~~not disjoint~~ $Var(V_t / x(t))$

values s.t. A is ~~un~~sat



$\boxed{\text{might have common vars with } var(V_t) / x(t)}$



We want every clause in A → same false
every clause in $V_t / x(t)$ → true

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~~not disjoint~~

because of connectedness

τ	$\overbrace{\text{① } var(\chi(t))/var(A) \quad\quad \text{falsify ② } var(A)}^{var(\chi(t))}$		$\overbrace{\text{③ } var(V_{t_1})/var(\chi(t)) \quad\quad \text{④ } var(V_{t_2})/var(\chi(t))}^{var(V_t)/var(\chi(t))}$	
	① $var(\chi(t))/var(A)$	② $var(A)$	③ $var(V_{t_1})/var(\chi(t))$	④ $var(V_{t_2})/var(\chi(t))$
	can be anything $2^{\lvert var(\chi(t))\setminus var(A)\rvert}$ values	sole purpose is to falsify each clause in A	vary	vary

give rise to product terms $m_1 \cdot m_2$

τ_1: ①+②+③ $\Rightarrow$ make C_1 true

τ_2: ①+②+④ $\Rightarrow$ make C_2 true

~~$X_i = (V_{t_1} \setminus \chi(t_1)) \cup (V_{t_2} \setminus \chi(t_2))$~~

$$X_i = var(\chi(t_i)) \setminus (\underbrace{var(A)}_{\text{falsified clauses in } \chi(t_i)} \cup \underbrace{var(V_{t_i} \setminus \chi(t_i))}_{\text{true clauses in } V_{t_i}})$$

$X = X_1 \cap X_2$

why??

reason

$X = var(\chi(t)) \setminus (var(A) \cup var(V_t \setminus \chi(t)))$ b/c of connectedness

$var(\chi(t_1)) \cup var(\chi(t_2)) \setminus (var(A) \cup var(V_{t_1} \setminus \chi(t)) \cup var(V_{t_1} \setminus \chi(t)))$

$=$ ~~var~~

$A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$

$= X_1 \cap X_2$

If $\tau \in N(t_i, A)$ then for every $\tau' : var(V_{t_i}) \to \{0,1\}$ and $\tau'(x) = \tau(x)$ for all $x \in Var(V_{t_i}) \setminus X_i$ it holds that $\tau' \in N(t_i, A)$ (Obviously)

If $(\tau_1, \tau_2) \in M$ then $(\tau_1', \tau_2') \in M$

$M_1 = \{\tau|_{var(V_{t_1}) \setminus X} \mid \tau \in N(t_1, A)\}$ $\quad M_2 = \cdots$

$M' = \{(\tau_1, \tau_2) \mid \tau_1 \in M_1, \tau_2 \in M_2, \tau_1(x) = \tau_2(x) \ \forall x \in \underbrace{(var(V_{t_1}) \cap var(V_{t_2}))}_{var(\chi(t))}\}$

means??

$$(var(\chi(t_1)) \setminus var(A) \setminus X_1) \cap (var(\chi(t_2)) \setminus var(A) \setminus X_2) = \emptyset$$

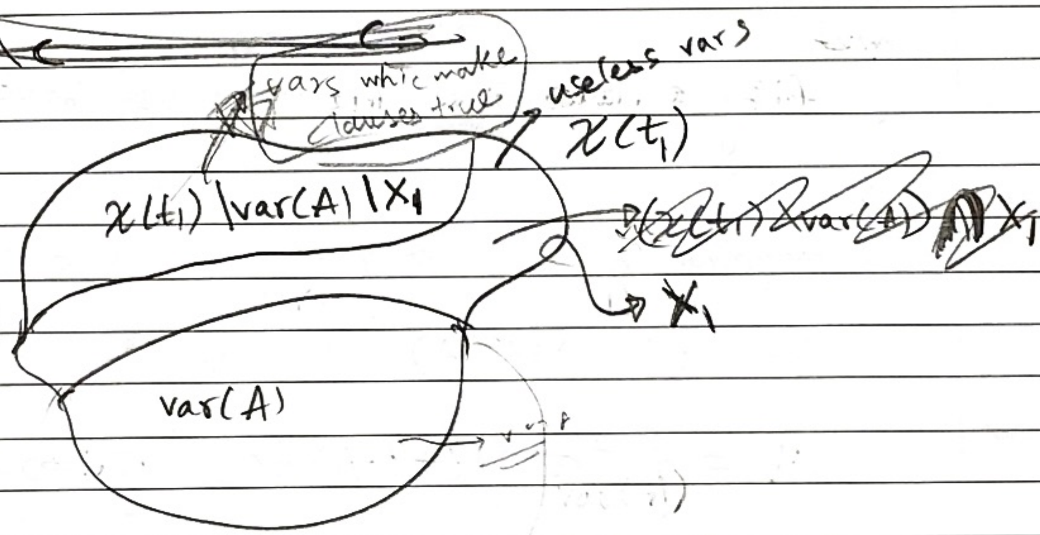
$$X_1 = var(\chi(t_1)) \setminus (var(A) \cup var(V_{t_1} \setminus \chi(t_1)))$$

A B D

t_1 t_2 $= var(\chi(t_1)) \cap var(A)$

$\cap \; var(\chi(t_1)) \setminus var(V_{t_1} \setminus \chi(t_1))$

~~A \ B \ C~~



exactly one $\tau_1' \in E_1$ s.t. $\tau_1'(x) = \tau_2(x)$

$\forall x \in (var(\chi(t_2)) \setminus var(A)) \setminus X_2$

$= X_1 \setminus X$

$$X_1 \cup X_2 = var(\chi(t)) \setminus var(A)$$

Introduce Node

t' child t and $\chi(t) = \chi(t') \cup \{C\}$ for clause C

new clause C.

$$n(t,A) = \begin{cases} 0 & \text{if } A \text{ is unfalsifiable} \\ n(t',A) \cdot 2^{|var(C) \setminus var(\chi(t'))|} & \text{if } C \notin A \\ n(t', A \setminus \{C\}) \cdot 2^{|var(C) \cap (var(\chi(t')) \setminus var(A \setminus \{C\}))|} & \text{if } C \in A \end{cases}$$

Proof Outline:

Obvious for $C \notin A$.

If $c \in A$

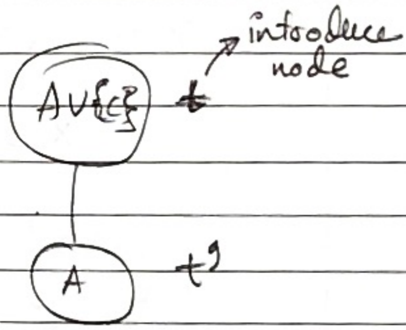
We have to $\boxed{\text{falsify } c.}$

new vars ⇒ ~~var(C) \ var(X(t))~~

$var(C) \setminus var(V_{t'})$ ✓

✓

old vars must

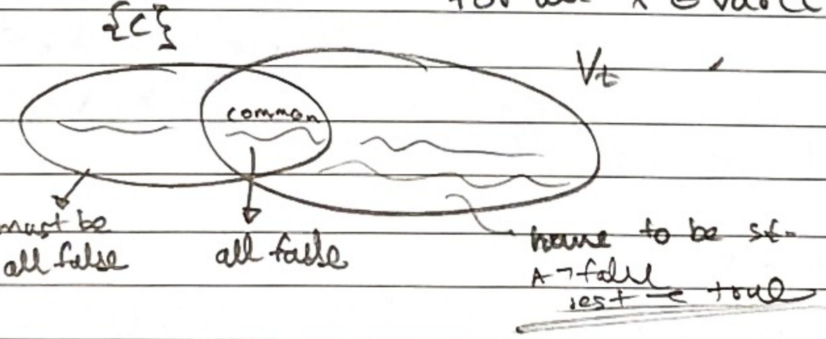


falsify a clause ⇒ all literals must be false

mapping $g: \tau \longrightarrow \tau|_{V_{t'}}$ is a bijection

from set $N(t, A)$ to M_2

$$M_2 = \{ \tau \in N(t', A \setminus \{C\}) \mid \tau(x) = 0 \text{ if } x \in C \text{ and } \tau(x) = 1 \text{ if } \neg x \in C \text{ for all } x \in var(C) \cap var(V_{t'}) \}$$



initially useless variables

$$var(C) \cap (var(X(t')) \setminus var(A \setminus \{C\}))$$

what does this represent?

that were also in C so now have to be false.

$n(t', A)$

1 - - - - - 1

2

$A\{C\}$ t'

$g : \tau \longrightarrow \tau|_{var(V_{t'})}$

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$\tau \in N(t, A)$ ~~eg~~ and $g(\tau) = \tau'$
since $C \in A$,
$\tau(x) = 0$ if $x \in X$ and $\tau(x) = 1$ if $\neg x \in C$

injectivity $\Rightarrow$ trivial by def. of M_2

surjectivity $\Rightarrow$

$\tau' \in M_2$
$\tau : Var(V_t) \rightarrow \{0, 1\}$
s.t. $\tau|_{Var(V_{t'})} = \tau'$, $\tau(x) = 0$ if $x \in C$
$\tau(x) = 1$ if $\neg x \in C$ $\forall x \in var(C) \setminus Var(V_{t'})$

$V_t \setminus \chi(t) = V_{t'} \setminus \chi(t')$

then $\tau \in N(t, A)$
So bijection proved.
Now compute $|M_2| \Longrightarrow$ prove equality of two sets

<u>Forget Node</u> $\chi(t) = \chi(t') \setminus \{c\}$

$n(t, A) = n(t', A) - n(t', A \cup \{c\})$

$\boxed{V_t = V_{t'}}$ (should be satisfied for τ in node t.

<u>Leaf Node</u>

$$n(t, A) = \begin{cases} 0 & \text{if } A \text{ is unfalsifiable} \\ 2^{|var(\chi(t) \setminus var(A)|} & \text{otherwise} \end{cases}$$

$$\boxed{\#(\models) = \sum_{i=0}^{|\chi(r)|} \left((-1)^i \sum_{A \subseteq \chi(r), |A| = i} n(r, A) \right)}$$