Divergence of a vector function

The divergence of a vector function $\vec{F}$ is denoted by div F and is defined as

Let $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ then

$$\text{div}\,\vec{F} = \vec{\nabla}\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(\hat{i}F_1 + \hat{j}F_2 + \hat{k}F_3)$$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Note: If $\text{div}\,\vec{v} = 0$ then v is called a Solenoidal vector function.

Ex. Find the value of n for which the vector $r^n\vec{r}$ is solenoidal, where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Sol: Divergence of $\vec{F}$

$$\vec{F} = \vec{\nabla}\vec{F} = \vec{\nabla}(r^n\vec{r})$$

$$= \vec{\nabla}(x^2+y^2+z^2)^{n/2}(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\Big[(x^2+y^2+z^2)^{n/2}x\hat{i} + (x^2+y^2+z^2)^{n/2}y\hat{j} + (x^2+y^2+z^2)^{n/2}z\hat{k}\Big].$$

$$= \frac{n}{\cancel{2}}(x^2+y^2+z^2)^{\frac{n}{2}-1}\cancel{2}x^2 + (x^2+y^2+z^2)^{n/2} +$$

$$\frac{n}{\cancel{2}}(x^2+y^2+z^2)^{\frac{n}{2}-1}\cancel{2}y^2 + (x^2+y^2+z^2)^{n/2} +$$

$$\frac{n}{\cancel{2}}(x^2+y^2+z^2)^{\frac{n}{2}-1}\cancel{2}z^2 + (x^2+y^2+z^2)^{n/2}$$

$$= n(x^2+y^2+z^2)^{\frac{n}{2}-1}(x^2+y^2+z^2) + 3(x^2+y^2+z^2)^{n/2}.$$

$$= n(x^2+y^2+z^2)^{\frac{n}{2}\cancel{-1}\cancel{+1}} + 3(x^2+y^2+z^2)^{n/2}.$$

$$= n(x^2+y^2+z^2)^{\frac{n}{2}} + 3(x^2+y^2+z^2)^{n/2}.$$

$$= (x^2+y^2+z^2)^{\frac{n}{2}}(n+3)$$

If $r^n\vec{r}$ is a solenoidal then

$$(n+3)(x^2+y^2+z^2)^{n/2} = 0$$

or $n+3=0$ or $n=-3$ //

Ex. Show that $\nabla\left[\frac{\vec{a}\cdot\vec{r}}{r^n}\right] = \frac{\vec{a}}{r^n} - \frac{n(\vec{a}\cdot\vec{r})\vec{r}}{r^{n+2}}$

Sol: we have, $\frac{\vec{a}\cdot\vec{r}}{r^n} = \frac{(a_1\hat{i}+a_2\hat{j}+a_3\hat{k})\cdot(x\hat{i}+y\hat{j}+z\hat{k})}{r^n}$

$$= \frac{a_1x+a_2y+a_3z}{r^n}$$

Let $\phi = \frac{\vec{a}\cdot\vec{r}}{r^n} = \frac{a_1x+a_2y+a_3z}{r^n}$

$$\therefore \frac{\partial\phi}{\partial x} = \frac{r^n\cdot a_1 - (a_1x+a_2y+a_3z)\,n\,r^{n-1}\,\partial r/\partial x}{r^{2n}}$$

But $r^2 = x^2+y^2+z^2 \Rightarrow 2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$

$$\therefore \frac{\partial\phi}{\partial x} = \frac{a_1r^n - (a_1x+a_2y+a_3z)\,n\,r^{n-2}\cdot x}{r^{2n}}$$

$$= \frac{a_1}{r^n} - \frac{n(a_1x+a_2y+a_3z)x}{r^{n+2}}$$

$$\therefore \nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$$= \frac{1}{r^n}(a_1\hat{i}+a_2\hat{j}+a_3\hat{k}) - \frac{n}{r^{n+2}}\left[(a_1x+a_2y+a_3z)(x\hat{i}+y\hat{j}+z\hat{k})\right]$$

$$= \frac{\vec{a}}{r^n} - \frac{n}{r^{n+2}}(\vec{a}\cdot\vec{r})\,\vec{r}$$

//

The Curl : If $V(x,y,z)$ is a differentiable vector field then the curl or rotation of V. written $\nabla \times V$ curl V is

$$(\nabla \times V) = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right) \times (V_1 i + V_2 j + V_3 k).$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_2 & V_3 \end{vmatrix} i - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ V_1 & V_3 \end{vmatrix} j + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ V_1 & V_2 \end{vmatrix} k$$

$$= \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) i + \left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x}\right) j + \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right) k.$$

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The Curl.

Ex. $A = xz^3 i - 2x^2yz j + 2yz^4 k$. Find $\nabla \times A$ (or curl A) at the point $(1,-1,1)$

$$\nabla \times A = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^3 & -2x^2yz & 2yz^4 \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(2yz^4) - \frac{\partial}{\partial z}(-2x^2yz)\right] i + \left[\frac{\partial}{\partial z} xz^3 - \frac{\partial}{\partial x} 2yz^4\right] j$$

$$+ \left[\frac{\partial}{\partial x}(-2x^2yz) - \frac{\partial}{\partial y}(xz^3)\right] k$$

$$= (2z^4 + 2x^2y) i + (3xz^2) j \; 6xyz + (-4xyz) k$$

at $(1,-1,1)$

$$= (2 + (-2)) i + 3.1.1^2 j + (-4(1)(-1)1) k = 3j + 4k.$$

Ex. $A = x^2 y\,i - 2xz\,j + 2yz\,k$. Find Curl Curl A.

Curl. Curl $A = \nabla \times (\nabla \times A)$

$$\nabla \times A = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y & -2xz & 2yz \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(2yz) - \frac{\partial}{\partial z}(-2xz)\right] i - j\left[\frac{\partial}{\partial z}(x^2 y) - \frac{\partial}{\partial x} 2yz\right]$$
$$+ \left[\frac{\partial}{\partial x}(-2xz) + \frac{\partial}{\partial y}(x^2 y)\right] k$$

$$= (2\cancel{y}z + 2x) i + j(0 - 0) + k(-2z - x^2)$$

$$\nabla \times A = (2\cancel{y}z + 2x) i - k(2z + x^2)$$

$$\nabla \times (\nabla \times A) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2\cancel{y}z + 2x & 0 & -2z + x^2 \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(-2z + x^2) - 0\right] i + \left[\frac{\partial}{\partial z}(2\cancel{y}z + 2x) - \frac{\partial}{\partial x}(-2z - x^2)\right] j$$
$$+ \left[0 - \frac{\partial}{\partial y}(2\cancel{y}z + 2x)\right] k$$

$$= 0i + [0 \cancel{-} (-2x)]\cancel{i} + \cancel{[-2]} j$$

$$= 0i + [2 - (-2x)] j + 0.k$$

$$= (2 + 2x) j$$

Ex If $V = \omega \times r$ prove $\omega = \frac{1}{2} \text{curl } V$. where ω is a constant vector.

$\text{Curl } V = \nabla \times V = \nabla \times (\omega \times r)$

$$= \nabla \times \begin{vmatrix} i & j & k \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}$$

$$= \nabla \times [(\omega_2 z - \omega_3 y) i + (\omega_3 x - \omega_1 z) j + (\omega_1 y - \omega_2 x) k$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \omega_2 z - \omega_3 y & \omega_3 x - \omega_1 z & \omega_1 y - \omega_2 x \end{vmatrix}$$

$$= (\omega_1 i - (-\omega_1) i) + (\omega_2 - (-\omega_2)) j + (\omega_3 - (-\omega_3)) k$$

$$= 2\omega_1 i + 2\omega_2 j + 2\omega_3 k = 2\omega.$$

$\Rightarrow \text{curl } V = 2\omega.$

$\Rightarrow \boxed{\omega = \frac{1}{2} \text{curl } V.}$

Vector Integration:

Let $R(u) = R_1(u)i + R_2(u)j + R_3(u)k$ be a vector depending on a single scalar variable u. then

$$\int R(u)\,du = i\int R_1(u)\,du + j\int R_2(u)\,du + k\int R_3(u)\,du$$

is called an indefinite integral of $R(u)$.

If there exist a vector $S(u)$ such that $R(u) = \frac{d}{du} S(u)$

then $\int R(u)\,du = \int \frac{d}{du}(S(u))\,du = S(u) + C.$

Line Integral:

Let $r(u) = x(u)i + y(u)j + z(u)k$, where $r(u)$ is the position vector of $(x\,y\,z)$ define a curve C joining points P_1 and P_2, where $u = u_1$ and $u = u_2$

Let $A(x\,y\,z) = A_1 i + A_2 j + A_3 k$. Then the integral of the tangential component of A along C from P_1 to P_2 written as,

$$\int_{P_1}^{P_2} A \cdot dr = \int_C A \cdot dr = \int_C A_1\,dx + A_2\,dy + A_3\,dz.$$ of

C is a closed curve then

$$\oint A \cdot dr = \oint A_1\,dx + A_2\,dy + A_3\,dz$$

In general, any integral which is to be evaluated along a curve is called a line integral.

Theorem: If $A = \nabla\phi$ everywhere in a region R of space, defined by $a_1 \le x \le a_2$, $b_1 \le y \le b_2$, $c_1 \le z \le c_2$ where $\phi(x,y,z)$ is a single valued and has continuous derivatives in R. then

1. $\int_{P_1}^{P_2} A \cdot dr$ is independent of the path C in R joining P_1 and P_2.

2. $\oint_C A \cdot dr = 0$ around any closed curve C in R.

In such a case A is called a conservative vector field and ϕ is its scalar potential.

A vector field is conservative iff $\nabla \times A = 0$ or $A = \nabla\phi$. In such case

$$A \cdot dr = A_1\,dx + A_2\,dy + A_3\,dz = d\phi$$

* Surface Integrals:

A unit normal 'n' to any positive side of S is called a positive or outward unit normal. and dS a surface area whose magnitude is dS and direction n. Then $dS = n\,dS$. Then the integral

$$\iint_S A \cdot dS = \iint_S A \cdot n\,dS$$ is an example of a surface integral called the flux of A over S. Other S.I are $\iint_S \phi\,dS$, $\iint_S \phi\,n\,dS$, $\iint_S A \times dS$

Volume integral ;

consider a closed surface in space enclosing a volume V. Then

$$\iiint_V A\,dv \quad \text{and} \quad \iiint_V \phi\,dv$$ are example of volume integral

Q1. If $R(u) = (u-u^2)i + 2u^3 j - 3k$. (a) $\int R(u)\,du$ (b) $\int_1^2 R(u)\,du$.

(a) $\int R(u)\,du = \int [(u-u^2)i + 2u^3 j - 3k]\,du$

$= i\int (u-u^2)\,du + j\int 2u^3\,du - \cancel{k}\int 3k\,du$

$= i\left(\frac{u^2}{2} - \frac{u^3}{3} + c_1\right) + j\left(\frac{2u^4}{4} + c_2\right) + (-3ku) + c_3$

$= \left(\frac{u^2}{2} - \frac{u^3}{3}\right)i + \frac{u^4}{2}j - 3uk + c_1 i + c_2 j + c_3 k.$

$= \left(\frac{u^2}{2} - \frac{u^3}{3}\right)i + \frac{u^4}{2}j - 3uk + C$ where C is the constant vector $c_1 i + c_2 j + c_3 k$.

(b) $\int_1^2 R(u)\,du = \left[\frac{u^2}{2} - \frac{u^3}{3}\,i + \frac{u^4}{2}j - 3uk + c\right]_1^2$

$= \left(\frac{4}{2} - \frac{8}{3}\right)i + \frac{16}{2}j - 3.2k + c] - [(\frac{1}{2} - \frac{1}{3})i + \frac{1}{2}j - 3.1k + c].$

$= \cancel{-\frac{5}{6}i}\ \cancel{+8}\left[\left(\frac{-4}{6}\right)i + 8j - 6k + c\right] - \left[\frac{1}{6}i + \frac{1}{2}j - 3k + c\right]$

$= -\frac{5}{6}i + \frac{15}{2}j - 3k$ //

Line Integral.

Ex. If $A = (3x^2 + 6y)i - 14yz j + 20xz^2 K$, evaluate $\int_C A \cdot dr$ from $(0,0,0)$ to $(1,1,1)$ along the following path C.

(a) $x = t,\ y = t^2,\ z = t^3$

(b) Straight lines from $(0,0,0)$ to $(1,0,0)$ then $(1,1,0)$ and then to $(1,1,1)$.

(c) The straight line joining $(0,0,0)$ and $(1,1,1)$

Sol: $\int_C A \cdot dr = \int [(3x^2 + 6y)i - 14yz j + 20xz^2 K] \cdot [dx i + dy j + dz k]$

$= \int (3x^2 + 6y)\, dx + \int 14yz\, dy + \int 20xz^2\, dz$

$= \cancel{\frac{3x^3}{3}} + \cancel{6xy}$

(a) at $x = t,\ y = t^2,\ z = t^3$ points $(0,0,0)$ and $(1,1,1)$ Correspon $t=0$ and $t=1$.

$\int A \cdot dr = \int_0^1 (3t^2 + 6t^2)\, dt - \int 14 t^2 t^3\, dt^2 + \int 20\, t (t^3)^2\, dt^3$

$= \int_0^1 9t^2\, dt - 28t^6\, dt + 60 t^9\, dt$

$= \left[\frac{9t^3}{3} - \frac{28t^7}{7} + \frac{60\, t^{10}}{10}\right]_0^1$

$= \frac{9}{3} - 4 + 6 = \frac{9 - 12 + 18}{3} = \frac{15}{3} = 5.$

Another method - Along C, $A = 9t^2 i - 14t^5 + 20t^7 K$ and

$r = xi + yj + zK = ti + t^2 j + t^3 k$ and $dr = (i + 2tj + 3t^2 K)\, dt$

then $\int A \cdot dr = \int_0^1 (9t^2 i - 14t^5 j + 20t^7 K) \cdot (i + 2tj + 3t^2 K)\, dt$

$= \int_0^1 (9t^2 - 28t^6 + 60t^9)\, dt = 5$

$\left[\frac{9t^3}{3} - \frac{28}{7} t^7 + \frac{60\, t^{10}}{10}\right]_0^1 = \frac{9}{3} - \frac{28}{7} + \frac{60}{10} = 3 - 4 + 6 = 5$

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Ex// Find the total work done in moving a particle in a force field given by $F = 3xyi - 5zj + 10xk$ along the curve $x = t^2+1$, $y = 2t^2$, $z = t^3$ from $t=1$, to $t=2$.

Total work done $= \int_C F \cdot dr$

$$= \int (3xyi - 5zj + 10xk)(dxi + dyj + dzk).$$

$$= \int 3xy\,dy - 5z\,dy + 10x\,dz.$$

$$= \int_1^2 3(t^2+1)(2t^2)\,d(t^2+1) - 5t^3\,d(2t^2) + 10(t^2+1)\,d(t^3)$$

$$= \int_1^2 (3t^2+3)\,2t^2 \cdot 2t\,dt - 5t^3 \cdot 2 \cdot 2t\,dt + (10t^2+10) \cdot 3t^2\,dt$$

$$= \int_1^2 (3t^2+3)\,4t^3 - 20t^4\,dt + 30t^4 + 30t^2\,dt$$

$$= \int_1^2 12t^5 + 12t^3 - 20t^4 + 30t^4 + 30t^2.$$

$$= \int_1^2 (12t^5 + 10t^4 + 12t^3 + 30t^2)\,dt$$

$$= \left[12\frac{t^6}{6} + 10\frac{t^5}{5} + 12\frac{t^4}{4} + 30\frac{t^3}{3}\right]_1^2$$

$$= \frac{\cancel{12}}{\cancel{6}} + \frac{\cancel{10}}{\cancel{5}} + \frac{12}{4} + \frac{30}{3}$$

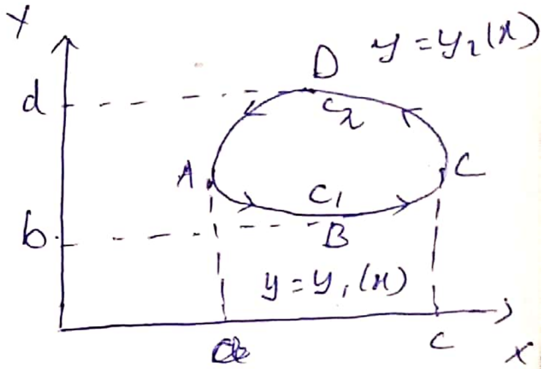
$$= 303$$

Green Theoram:

Statement: If $\phi(x,y)$; $\psi(x,y)$, $\frac{\partial \phi}{\partial y}$, $\frac{\partial \psi}{\partial x}$ be continuous function over a region R bounded by a simple closed curve C in xy-plane then

$$\oint_C \phi\, dx + \psi\, dy = \iint_R \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx\, dy.$$

proof: Let the curve C be divided into two curves C_1 (ABC) and C_2 (CDA)



Let the equation of the curve c_1 (ABC) be $y = y_1(x)$ and equation of the curve c_2 (CDA) be $y = y_2(x)$

then

$$\iint_R \frac{\partial \phi}{\partial y}\, dx\, dy = \int_a^c \left[\int_{y=y_1(x)}^{y=y_2(x)} \frac{\partial \phi}{\partial y}\, \partial y\right] dx$$

$$= \int_a^c \left[\phi(x,y)\right]_{y=y_1(x)}^{y=y_2(x)} dx = \int_a^c \phi(x, y_2)\, dx - \int_a^c \phi(x, y_1)\, dx$$

$$= -\int_c^a \phi(x, y_2)\, dx - \int_a^c \phi(x, y_1)\, dx.$$

$$= -\left[\int_{C_2} \phi(x,y)\, dx + \int_{C_1} \phi(x,y)\, dx\right] = -\oint \phi(x,y)\, dx.$$

Similarity Thus $\oint_C \phi\, dx = -\iint_R \frac{\partial \phi}{\partial y}\, dx\, dy$. —(1).

Similarly it can be shown that $\oint \psi \, dy = \iint \frac{\partial \psi}{\partial x} \, dx\, dy$ — (2).

Adding (1) and (2).

$$\oint \phi \, dx + \psi \, dy = \iint_R \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y} \right) dx\, dy.$$

Note: Green theorem in vector form

$$\oint \bar{F} \cdot d\bar{r} = \iint_R (\nabla \times \bar{F}) \cdot \hat{k} \, dR$$

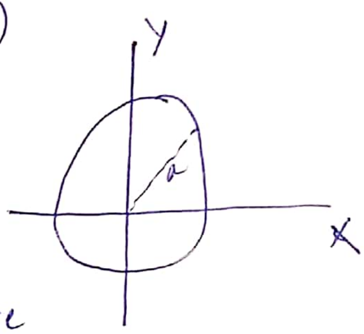
where $\bar{F} = \phi i + \psi j$. $\bar{r} = x\hat{i} + y\hat{j}$,

$\hat{k}$ is a unit vector along z-axis and $dR = dx\,dy$.

Ex. A vector field $\bar{F}$ given by $\bar{F} = \sin y\,\hat{i} + x(1+\cos y)\,\hat{j}$. Evaluate the line integral $\int_C \bar{F}\cdot d\bar{r}$ where C is the circular path given by $x^2 + y^2 = a^2$.

Sol: $\bar{F} = \sin y\,\hat{i} + x(1+\cos y)\,\hat{j}$

$$\int_C \bar{F}\cdot d\bar{r} = \int_C \left(\sin y\,\hat{i} + x(1+\cos y)\,\hat{j}\right)\cdot(\hat{i}\,dx + \hat{j}\,dy)$$



$$= \int_C \sin y\,dx + x(1+\cos y)\,dy.$$

On applying green theorem, we have

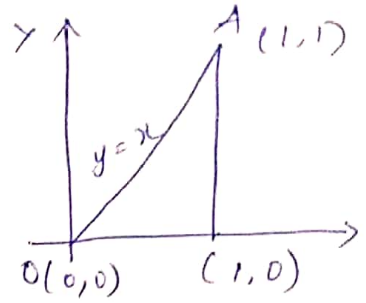
$$\oint \phi\,dx + \psi\,dy = \iint \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx\,dy$$

$$= \iint \left((1+\cos y) - \cos y\right) dx\,dy.$$

where S is the circular plane surface of radius a.

$$= \iint dx\,dy = \text{Area of circle} = \pi a^2.$$

Q. Using Green's Theorem, evaluate $\int_c (x^2y\,dx + x^2dy)$, where c is the boundary described counter clockwise of the triangle with vertices $(0,0), (1,0)\,(1,1)$.



Sol: By Green's theorem we have

$$\oint \phi\,dx + \psi\,dy = \iint \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx\,dy.$$

$$\frac{\partial \psi}{\partial x} = \frac{\partial}{\partial x}(x^2) = 2x.$$

$$\frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y}(x^2y) = x^2$$

$$\therefore \int_C x^2y\,dx + x^2dy = \iint_R (2x - x^2)\,dx\,dy.$$

$$= \int_0^1 (2x - x^2)\,dx \int_0^x dy.$$

$$= \cancel{\left[\frac{2x^2}{2} - \frac{x^3}{3}\right.} \int_0^1 (2x - x^2)\,dx\,[y]_0^x$$

$$= \int_0^1 (2x - x^2)\,dx \cdot x = \int_0^1 (2x^2 - x^3)\,dx$$

$$= \left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^1$$

$$= \frac{2}{3} - \frac{1}{4} = \frac{8-3}{12} = \frac{5}{12}$$

E1. Using Green's theorem to evaluate

$\int_C (x^2+xy)\,dx + (x^2+y^2)\,dy$. where C is the square

formed by the line $y=\pm 1$, $x=\pm 1$

Sol: $\int (x^2+xy)\,dx + (x^2+y^2)\,dy = \iint \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx\,dy.$

$\frac{\partial \psi}{\partial x} = \frac{\partial}{\partial x}(x^2+y^2) = 2x.$ $\frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y}(x^2+xy) = x$

$\therefore \int_{-1}^{1}\int_{-1}^{1} (2x-x)\,dx\,dy = \int_{-1}^{1} x\,dx \int_{-1}^{1} dy$

$= \int_{-1}^{1} x\,dx\,[y]_{-1}^{1}$

$= \int_{-1}^{1} x\,dx\,[1+1]. = \int_{-1}^{1} 2x\,dx.$

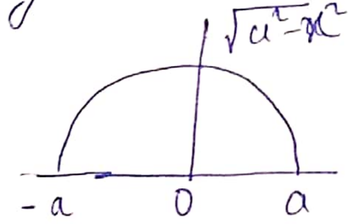
$= \cancel{2}\left[\frac{x^2}{\cancel{2}}\right]_{-1}^{1}$

$= [1-1] = 0$

Ex. Apply Green's theorem to apply $\int_C [(2x^2-y^2)\,dx + (x^2+y^2)\,dy]$, where C is the boundary of the area enclosed by the x-axis and the upper half of circle $x^2+y^2=a^2$.

Sol: By Green theorem $\iint_R \left(\frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y}\right) dx\,dy$.

$$= \int_{-a}^{a}\int_{0}^{\sqrt{a^2-x^2}} \left[\frac{\partial}{\partial x}(x^2+y^2) - \frac{\partial}{\partial y}(2x^2-y^2)\right] dx\,dy$$



$$= \int_{-a}^{a}\int_{0}^{\sqrt{a^2-x^2}} (2x+2y)\,dx\,dy.$$

$$= 2\int_{-a}^{a} dx \int_{0}^{\sqrt{a^2-x^2}} (x+y)\,dy. = 2\int_{-a}^{a} dx \left[xy + \frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}}$$

$$= 2\int_{-a}^{a} dx \left[x\sqrt{a^2-x^2} + \frac{a^2-x^2}{2}\right]$$

$$= 2\int_{-a}^{a} x\sqrt{a^2-x^2}\,dx + \int_{-a}^{a}(a^2-x^2)\,dx.$$

$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$, f is even

$= 0$, f is odd.

~ 0 ~

$$= 2\int_0^a (a^2-x^2)\,dx$$

$$= 2\left(a^2x - \frac{x^3}{3}\right)_0^a = 2\left(3a^2a - \frac{a^3}{3}\right)$$

$$= 2\left(\frac{3a^3-a^3}{3}\right) = \frac{4a^3}{3}$$

8 Surface Integrals:

The divergence Theorem: Stoke's theorem and

9 Green theorem:

The divergence theorem of Gauss: It states that if V is the volume bounded by a closed surface S and A is a vector function of position with continuous derivatives then

$$\iiint_V \nabla . A \, dV = \iint_S A . n \, dS = \oint_S A . dS.$$

STOKES Theorem: States that if S is open, two sided surface bounded by a closed, non-intersecting curve C, then if A has continuous derivatives.

$$\oint_C A . dr = \iint_S (\nabla \times A) . n \, dS = \iint_S (\nabla \times A) . dS$$

Green Theorem: If R is a closed region of the xy plane bounded by a simple closed curve C and if M and N are continuous function of x and y having continuous derivatives in R, then

$$\oint M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx \, dy$$

STOKE's Theorem

$$\oint \bar{F}.d\bar{r} = \iint_S \text{curl } \bar{F} \; \hat{n} \, ds. = \iint_S \text{curl } \bar{F} \, d\bar{S}$$

When $\bar{F} = F_x \hat{i} + F_y \hat{j} + F_z \hat{k}$ $\quad d\bar{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

Ex. using Stroke theorem evaluate $\int_C (2x-y)dx - yz^2dy - y^2z\,dz$.

where C is the circle $x^2+y^2=1$. Corresponding to the Surface of sphere.

Sol: $\int (2x-y)\,dx - yz^2dy - y^2z\,dz$.

$= \int ((2x-y)\hat{i} - yz^2\hat{j} - y^2z\hat{k})\;(\hat{i}dx + \hat{j}dy + \hat{k}dz)$.

By Stroke theorem $\iint \text{curl } \bar{F}.\hat{n}\,ds = \int \bar{F}.d\bar{r}$ —①.

$$\text{curl } \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (2x-y) & -yz^2 & -y^2z \end{vmatrix}$$

$$= \hat{i}\,|-2yz + 2yz| - \hat{j}\,|0-0| + \hat{k}\,|0-(-1)| = \hat{k}.$$

Putting the value of curl $\vec{F}$ in ①.

$$ds = \frac{dx\,dy}{\hat{n}\,\hat{k}}$$

$$= \iint \hat{k}\,\hat{n}\,\frac{dx\,dy}{\hat{n}\,\hat{k}}$$

$$= \iint dx\,dy = \pi$$

Evaluate: $\int_C \bar{F}\cdot d\bar{r}$, where $F(x,y,z) = -y^2\hat{i} + x\hat{j} + z^2\hat{k}$ and C is the curve of intersection of the plane $y+z=2$ and cylinder $x^2+y^2=1$.

Sol: $\int \bar{F}\cdot d\bar{r} = \iint_S \text{curl } \bar{F}\cdot\hat{n}\,ds$ — (1).

$F = -y^2\hat{i} + x\hat{j} + z^2\hat{k}$

$$\text{curl } F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{vmatrix}$$

$= \hat{i}\,|0-0| - \hat{j}\,|0-0| + \hat{k}\,|1+2y|$

$= (1+2y)\hat{k}$

$\nabla F = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(y+z-2)$ ~~$(1+2y)\hat{k}$~~.

$= (\hat{j} + \hat{k})$

Unit normal $\hat{n} = \dfrac{\hat{j}+\hat{k}}{\sqrt{1^2+1^2}} = \dfrac{\hat{j}+\hat{k}}{\sqrt{2}}$.

and $ds = \dfrac{dx\,dy}{\hat{n}\cdot\hat{k}}$.

putting the values in eq (1).

$$= \iint (1+2y)\hat{k}\cdot\frac{\hat{j}+\hat{k}}{\sqrt{2}}\cdot\frac{dx\,dy}{\left(\frac{\hat{j}+\hat{k}}{\sqrt{2}}\right)\hat{k}}$$

$$= \iint (1+2y)\,dx\,dy.$$

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$$= \int_0^{2\pi} \int_0^1 (1 + 2r\sin\theta)\, r \, d\theta \, dr.$$

$$= \int_0^{2\pi} d\theta \int_0^1 r + 2r^2 \sin\theta \, dr.$$

$$= \int_0^{2\pi} d\theta \left[\frac{r^2}{2} + \frac{2r^3}{3} \sin\theta \right]_0^1$$

$$= \int_0^{2\pi} d\theta \left[\frac{1}{2} + \frac{2}{3} \sin\theta \right] = \int_0^{2\pi} \left[\frac{1}{2} + \frac{2}{3} \sin\theta \right] d\theta$$

$$= \int_0^{2\pi} \frac{1}{2} d\theta + \int_0^{2\pi} \frac{2}{3} \sin\theta \, d\theta.$$

$$= \frac{1}{2} [\theta]_0^{2\pi} + \frac{2}{3} [-\cos\theta]_0^{2\pi}. \qquad / \cos 2\pi = 1.$$

$$= \frac{\cancel{2}\pi}{\cancel{2}} + \frac{2}{3} [-\cos 2\pi - (-\cos 0)].$$

$$= \pi + [-1 - (-1)] = \pi$$

Ex. use Stokes Theorem to evaluate $\int \bar{v} \cdot d\bar{r}$ where $\vec{v} = y^2 \hat{i} + xy \hat{j} + xz \hat{k}$ and C is the curve of the hemisphere $x^2 + y^2 + z^2 = 9$, $z > 0$

Sol: $\int \bar{v} \cdot d\bar{l} = \iint \text{curl}\, \bar{v} \cdot \hat{n} \, ds$

$$\text{curl}\, \bar{v} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & xy & xz \end{vmatrix} = i(0 - 0) - j(z - 0) + k(y - 2y)$$

$$= -z\hat{j} - y\hat{k}$$

$$\nabla\phi = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(x^2+y^2+z^2-9)$$

$$\nabla\phi = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\hat{i}+2y\hat{j}+2z\hat{k}}{\sqrt{4x^2+4y^2+4z^2}} = \frac{2(x\hat{i}+y\hat{j}+z\hat{k})}{\sqrt{4\times 9}} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{3}$$

$$ds = \frac{dx\,dy}{\hat{n}\cdot\hat{k}} \Rightarrow ds\,\hat{n}\cdot\hat{k} = dx\,dy.$$

$$\Rightarrow \frac{x\hat{i}+y\hat{j}+z\hat{k}}{3}\cdot\hat{k}\,ds = dx\,dy \Rightarrow \frac{z}{3}ds = dx\,dy.$$

$$\Rightarrow ds = \frac{3\,dx\,dy}{z}$$

$$\text{curl}\,\bar{V}\cdot\hat{n} = (-z\hat{j}-y\hat{k})\left(\frac{x\hat{i}+y\hat{j}+z\hat{k}}{3}\right)$$

$$= \frac{-zy-zk}{3} = \frac{-2zy}{3}$$

$$\therefore \iint(\nabla\times\bar{V})\cdot\hat{n}\,ds = \iint\frac{-2zy}{3}\cdot\frac{3}{z}\,dx\,dy.$$

$$= \iint -2y\,dx\,dy.$$

$$= -\int_0^{2\pi}\int_0^3 2r\sin\theta\, r\,d\theta\,dr. \Rightarrow -\int_0^{2\pi}\sin\theta\,d\theta\int_0^3 2r^2\,dr.$$

$$= -\left[-\cos\theta\right]_0^{2\pi}\cdot 2\left[\frac{r^3}{3}\right]_0^3$$

$$= -\left[-\cos 2\pi + \cos 0\right]\cdot 2\cdot 9 = -2[-1+1]\cdot 9 = 0.$$

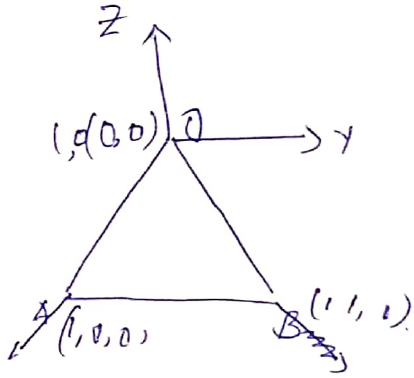
Example: Evaluate $\oint \bar{F} \cdot d\bar{r}$ by Stroke Theorem, where $\bar{F} = y^2\hat{i} + x^2\hat{j} - (x+z)\hat{k}$ and C is the boundary of triangle with vertices $(0,0,0)$; $(1,0,0)$ and $(1,1,0)$

Sol: We have $\text{curl } \bar{F} = \nabla \times \vec{F}$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & -(x+z) \end{vmatrix} = \hat{i}\,|0-0| - \hat{j}\,|-1-0| + \hat{k}\,|2x-2y)$$

$$= \hat{j} + 2(x-y)\hat{k}$$

We observe that the z coordinate of each of the triangle is zero. Therefore the triangle lies in the xy-plane.



$\therefore \hat{n} = \hat{k}$

$\therefore \text{curl } \vec{F} \cdot \hat{n} = (\hat{j} + 2(x-y)\hat{k})\,\hat{k}$

$$= 2(x-y).$$

In the figure only xy plane is considered. Therefore equation of the line OB is $y = x$.

By Stroke theorem $\oint \bar{F} \cdot d\bar{r} = \iint \text{curl } \bar{F}\ \hat{n}\ ds.$

$$= \int_0^1 \int_0^x 2(x-y)\, dx\, dy.$$

$$= 2\int_0^1 \left[xy - \frac{y^2}{2}\right]_0^x dx.$$

$$= 2\int_0^1 \left[x^2 - \frac{x^2}{2}\right] dx = 2\int_0^1 \frac{x^2}{2}\, dx = \int_0^1 x^2 dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$ (7)

Ex. using Stroke theorem evaluate $\int_C [(x+2y)\,dx + (x-z)\,dy + (y-z)\,dz]$ where c is the boundary of the triangle with vertices $(2,0,0)$, $(0,3,0)$ and $(0,0,6)$ oriented in the anti-clockwise direction.

Sol: $\int (x+2y)\,dx + (x-z)\,dy + (y-z)\,dz$

$$= \int (x+2y)\,i + (x-z)\,\hat{j} + (y-z)\,\hat{k})\,(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz).$$

$$\oint \bar{F}\cdot d\bar{r} = \iint \text{curl } \bar{F}\ \hat{n}\ ds$$

$$\text{curl } \bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x+2y) & (x-z) & (y-z) \end{vmatrix} = i\,|1-(-1)| - \hat{j}\cdot|0-0| + k\,|1-2| = 2\hat{i} - \hat{k}$$

$$\nabla\phi = \left|i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right| \left[\frac{x}{2} + \frac{y}{3} + \frac{z}{6}\right]$$

$$= \frac{\hat{i}}{2} + \frac{\hat{j}}{3} + \frac{\hat{k}}{6} = \frac{1}{6}(3\hat{i} + 2\hat{j} + \hat{k}).$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{\cancel{\frac{1}{6}}(3\hat{i} + 2\hat{j} + \hat{k})}{\cancel{\frac{1}{6}}\sqrt{9+4+1}} = \frac{1}{\sqrt{14}}(3\hat{i} + 2\hat{j} + \hat{k}).$$

$$\therefore \text{curl } \bar{F}\cdot\hat{n} = (2\hat{i} - \hat{k})\cdot\frac{1}{\sqrt{14}}(3\hat{i} + 2\hat{j} + \hat{k}).$$

$$= \frac{6}{\sqrt{14}} + \frac{1}{\sqrt{14}} = \frac{5}{\sqrt{14}}$$

$\int \bar{F} \cdot d\bar{r} = \iint \text{curl} \bar{F} \, \hat{n} \, ds.$

$= \iint \frac{5}{\sqrt{14}} ds.$ $\qquad \Big| \; ds = \frac{dx\,dy}{\hat{n} \cdot \hat{k}}.$

$= \frac{5}{\sqrt{14}} \iint \frac{dx\,dy}{\left[\frac{1}{\sqrt{14}}(3i + 2j + k)\right] \cdot \hat{k}}$

$= \frac{5}{\sqrt{14}} \iint_{R} \frac{dx\,dy}{\frac{1}{\sqrt{14}} \cdot 1}.$

$= 5 \iint dx\,dy. =$

$= 5 \times$ area of the triangle OAD $= \frac{5 \times (2 \times 3)}{2} = 15.$