Vector Calculus

vector calculus or vector analysis deals with the differentiation and integration of both of vectors describing particular bodies, such as velocity of a particle and of vector fields, in which a vector is defined as a function of the coordinates throughout some volume. (one, two or three dimensional.)

Vector - A vector is a quantity having both magnitude and direction such as force, acceleration and displacement.

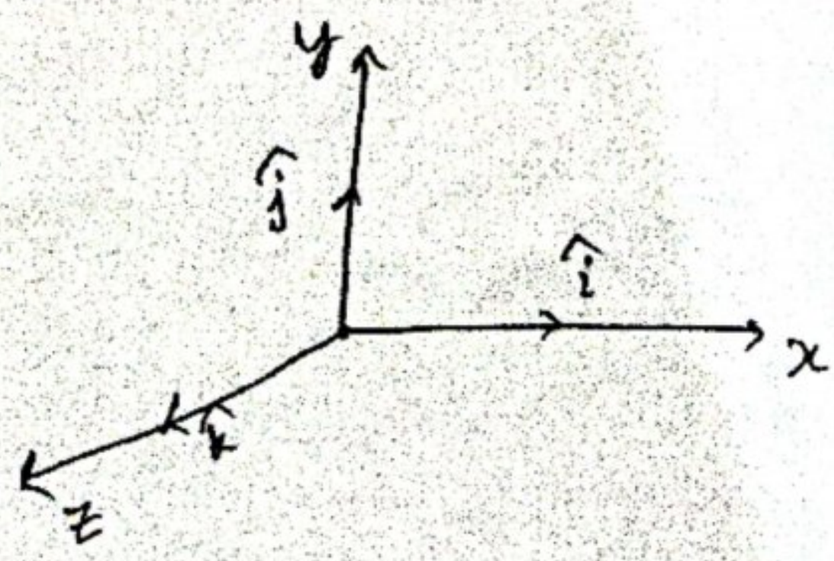
unit vector - A unit vector is a vector having unit magnitude, if $\vec{A}$ is a vector with magnitude $A \neq 0$, then $\frac{\vec{A}}{A}$ is a unit vector having the same direction as $\vec{A}$.

i.e. $$\hat{a} = \frac{\vec{A}}{A}$$

$$\Rightarrow \vec{A} = A\hat{a}$$

Rectangular unit vectors $(\hat{i}, \hat{j}, \hat{k})$ -

These are an important set of unit vectors having the dir^{n}s of positive x, y, z axis of a three dimensional rectangular coordinate system.



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∴ Any vector $\vec{A}$ in 3 dimension can be written as-

$$\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$$

where A_1, A_2 and A_3 are called the rectangular components or simply components of $\vec{A}$ in the x, y and z directions respectively.

The magnitude of $\vec{A}$ is

$$A = |\vec{A}|$$

$$= \sqrt{A_1^2 + A_2^2 + A_3^2}$$

1) $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$

$\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$

2) $\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = 0$

$\hat{i}\times\hat{j} = \hat{k}$, $\hat{j}\times\hat{k} = \hat{i}$, $\hat{k}\times\hat{i} = \hat{j}$

3) If $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$

and $\vec{B} = B_1\hat{i} + B_2\hat{j} + B_3\hat{k}$

then,

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

$$= \hat{i}\begin{vmatrix} A_2 & A_3 \\ B_2 & B_3 \end{vmatrix} - \hat{j}\begin{vmatrix} A_1 & A_3 \\ B_1 & B_3 \end{vmatrix} + \hat{k}\begin{vmatrix} A_1 & A_2 \\ B_1 & B_2 \end{vmatrix}$$

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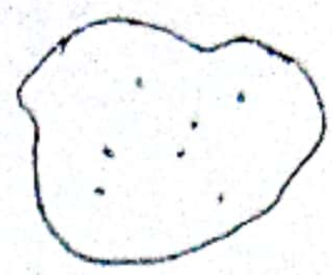
Vector field -

If to each point (x, y, z) of a region R in space there corresponds a vector $\vec{V}(x, y, z)$, then $\vec{V}$ is called a vector function of position or vector point function and we say that a vector field $\vec{V}$ has been defined in R.



eg. If the velocity at any point (x, y, z) within a moving fluid is known at a certain time, then a vector field is defined.

Vector differential operator :

$\partial \rightarrow$ partial
$d \rightarrow$ total

(nabla)
del operator

$$\vec{\nabla} \equiv \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

The operator $\vec{\nabla}$ is also known as **nabla.**
It operates both in scalar and vector field.

Gradient :

Let $\phi(x, y, z)$ be defined and differentiable at each point (x, y, z) in a certain region of space **i.e.**, ϕ defines a differentiable scalar field. Then the gradiant of ϕ, is defined by -

$$\vec{\nabla}\phi \text{ or } \operatorname{grad}\phi = \overbrace{\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)}^{\text{vector}} \overbrace{\phi}^{\text{scalar}}$$

$$= \left(\frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}\right)$$

$\vec{\nabla}\phi$ thus defines a vector field.

The component of $\vec{\nabla}\phi$ in the direction of a unit vector $\hat{a}$ is given by -

$\vec{\nabla}\phi \cdot \hat{a}$ and is called the directional derivative of ϕ in the direction $\hat{a}$.

$\vec{\nabla}\phi \cdot \hat{a}$ physically, this is the rate of change of $\phi(x, y, z)$ in the direction $\hat{a}$.

Ques. If $\phi(x,y,z) = 3x^2y - y^3z^2$, find $\vec{\nabla}\phi$ at the point $(1,-2,-1)$

$$\vec{\nabla}\phi = \left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right)$$

$$\frac{\partial\phi}{\partial x} = 6xy$$

$$\frac{\partial\phi}{\partial y} = 3x^2 - 3y^2z^2$$

$$\frac{\partial\phi}{\partial z} = -y^3 \times 2z$$

$$\vec{\nabla}\phi = 6xy\,\hat{i} + (3x^2 - 3y^2z^2)\hat{j} + (-2y^3z)\hat{k}$$

$$= 6\times(1)(-2)\hat{i} + (3\times1 - 3\times(-2)^2(-1)^2)\hat{j} + (-2\times(-2)^3\times(-1))\hat{k}$$

$$= -12\hat{i} + (3-12)\hat{j} + (-16)\hat{k}$$

$$= -12\hat{i} - 9\hat{j} - 16\hat{k}$$

Ques - Find $\vec{\nabla}\phi$ if

a) $\phi = \ln|\vec{r}|$

b) $\phi = \frac{1}{r}$

where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

a) $$\vec{\nabla}\phi = \left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right)$$

$$|\vec{r}| = \sqrt{x^2+y^2+z^2}$$

$$\phi = \ln(x^2+y^2+z^2)^{1/2} = \frac{1}{2}\ln(x^2+y^2+z^2)$$

$$\frac{\partial \phi}{\partial x} = \frac{1 \times 2x}{2(x^2+y^2+z^2)}$$

$$\frac{\partial \phi}{\partial y} = \frac{1}{2}\frac{2y}{(x^2+y^2+z^2)}$$

$$\frac{\partial \phi}{\partial z} = \frac{1}{2}\frac{2z}{(x^2+y^2+z^2)}$$

$$\vec{\nabla}\phi = \left(\frac{x\hat{i}}{(x^2+y^2+z^2)} + \frac{y\hat{j}}{(x^2+y^2+z^2)} + \frac{z\hat{k}}{(x^2+y^2+z^2)}\right)$$

$$= \frac{(x\hat{i}+y\hat{j}+z\hat{k})}{(x^2+y^2+z^2)}$$

$$\vec{\nabla}\phi = \frac{\vec{r}}{r^2}$$

b) $\vec{\nabla}\phi = \left(\frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}\right)$ $\quad -\frac{\vec{r}}{r^3}$

$$\phi = \frac{1}{\sqrt{x^2+y^2+z^2}}$$

$$r = \sqrt{x^2+y^2+z^2}$$

$$\frac{\partial \phi}{\partial x} = \frac{-1 \times 2x}{2(x^2+y^2+z^2)^{3/2}} = \frac{-x}{(\sqrt{x^2+y^2+z^2})^3}$$

$$= \frac{-x}{r^3}$$

$$\frac{\partial \phi}{\partial y} = \frac{-1}{2} \times \frac{2y}{(x^2+y^2+z^2)^{3/2}} = \frac{-y}{(\sqrt{x^2+y^2+z^2})^3} = \frac{-y}{r^3}$$

$$\frac{\partial \phi}{\partial z} = \frac{-1}{2} \times \frac{2z}{(x^2+y^2+z^2)^{3/2}} = \frac{-z}{(\sqrt{x^2+y^2+z^2})^3} = \frac{-z}{r^3}$$

$$\vec{\nabla}\phi = \left(\frac{-x}{r^3}\hat{i} + \frac{(-y)}{r^3}\hat{j} + \frac{(-z)}{r^3}\hat{k}\right)$$

$$\vec{\nabla}\phi = -\frac{(x\hat{i}+y\hat{j}+z\hat{k})}{r^3}$$

$$\vec{\nabla}\phi = -\frac{\vec{r}}{r^3}$$

Qu Show that $\vec{\nabla} r^n = n r^{n-2} \vec{r}$.

$$\phi = r^n$$
$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$
$$r = |\vec{r}|$$
$$= \sqrt{x^2+y^2+z^2}$$

$$\phi = r^n$$
$$\phi = \left(\sqrt{x^2+y^2+z^2}\right)^n$$
$$\phi = (x^2+y^2+z^2)^{n/2}$$

$\frac{n}{2} - 1$

$\frac{n-2}{2}$

$$\frac{\partial \phi}{\partial x} = \frac{n}{\cancel{2}}(x^2+y^2+z^2)^{\frac{n-2}{2}} \times \cancel{2}x$$
$$= n\left(\sqrt{x^2+y^2+z^2}\right)^{n-2} \times x = n r^{n-2} x$$

$$\frac{\partial \phi}{\partial y} = \frac{n}{\cancel{2}}(x^2+y^2+z^2)^{\frac{n-2}{2}} \times \cancel{2}y$$
$$= n\left(\sqrt{x^2+y^2+z^2}\right)^{n-2} \times y = n r^{n-2} y$$

$$\frac{\partial \phi}{\partial z} = \frac{n}{\cancel{2}}(x^2+y^2+z^2)^{\frac{n-2}{2}} \times \cancel{2}z$$
$$= n\left(\sqrt{x^2+y^2+z^2}\right)^{n-2} \times z = n r^{n-2} z$$

$$\vec{\nabla}\phi = \left(\frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}\right)$$

$$\vec{\nabla}\phi = \left(n r^{n-2} x\hat{i} + n r^{n-2} y\hat{j} + n r^{n-2} z\hat{k}\right)$$

$$\vec{\nabla}\phi = n r^{n-2}(x\hat{i} + y\hat{j} + z\hat{k})$$

$\hat{i}\cdot\hat{i} = 1$

$$\boxed{\vec{\nabla}\phi = n r^{n-2}\vec{r}}$$

$$\boxed{\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}}$$

$\nabla^2 \Rightarrow$ Laplacian operator.

Divergence:

Let $\vec{V}(x,y,z) = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$ be defined and differentiable at each point (x,y,z) in a certain region of space, i.e., $\vec{V}$ defines a differentiable vector field. Then the divergence of $\vec{V}$ is defined by

$$\vec{\nabla}\cdot\vec{V} \text{ or } \operatorname{div}\vec{V} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot(V_1\hat{i} + V_2\hat{j} + V_3\hat{k})$$

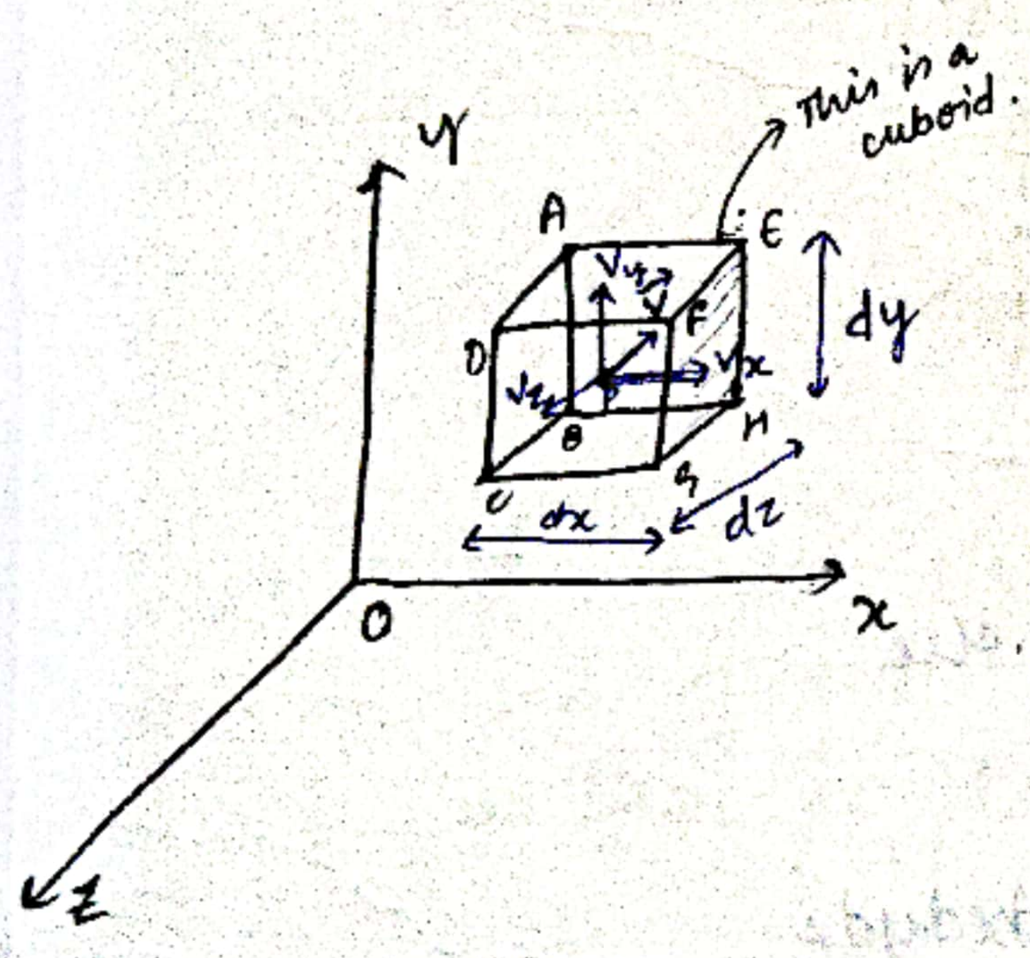
$$\boxed{\vec{\nabla}\cdot\vec{V} \text{ or } \operatorname{div}\vec{V} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}}$$

$\vec{A}\cdot\vec{B} = AB\cos\theta$
$\vec{B}\cdot\vec{A} = BA\cos\theta$
$\boxed{\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}}$
commutative property.

$\because \vec{\nabla}\cdot\vec{V} \neq \vec{V}\cdot\vec{\nabla}$

so, this is not commutative.

Physical Interpretation of Divergence:



Consider a fluid, say water flowing through a small rectangular parallelopiped of dimensions dx, dy, dz, parallel to x, y, z axis respectively.

Let $\vec{V} = V_x\hat{i} + V_y\hat{j} + V_z\hat{k}$ be the velocity of fluid at the center P of the parallelopiped.

Let $\frac{\partial V_x}{\partial x}$ = rate of change of V_x along X-axis

Then, the magnitude of X-component of V at face EFGH

$$= V_x + \frac{\partial V_x}{\partial x} \times \frac{dx}{2}$$

Now, the outward flux at face EFGH

$$= \left(V_x + \frac{\partial V_x}{\partial x} \times \frac{dx}{2}\right) \times dy\,dz$$

Similarly, the inward flux at face ABCD

$$= \left(V_x - \frac{\partial V_x}{\partial x} \times \frac{dx}{2}\right) \times dy\,dz$$

Net outward flux in x-direction

$$= \left[\left(V_x + \frac{\partial V_x}{\partial x} \times \frac{dx}{2}\right) - \left(V_x - \frac{\partial V_x}{\partial x} \times \frac{dx}{2}\right)\right] dy\,dz$$

$$= \frac{\partial V_x}{\partial x}\, dx\,dy\,dz$$

Similarly, netoutward flux in y-direction

$$= \frac{\partial V_y}{\partial y}\, dx\,dy\,dz$$

And, net outward flux in z-direction

$$= \frac{\partial V_z}{\partial z}\, dx\,dy\,dz$$

∴ Net outward flux through the parallelopiped ABCDEFGH

$$= \left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right) dx\,dy\,dz$$

Then, net outward flux per unit volume through parallelopiped.

$$= \frac{\left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right) dx\,dy\,dz}{dx\,dy\,dz}$$

$$= \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \right) \cdot \left(V_1\hat{i} + V_2\hat{j} + V_3\hat{k} \right)$$

$$= \vec{\nabla} \cdot \vec{V}$$

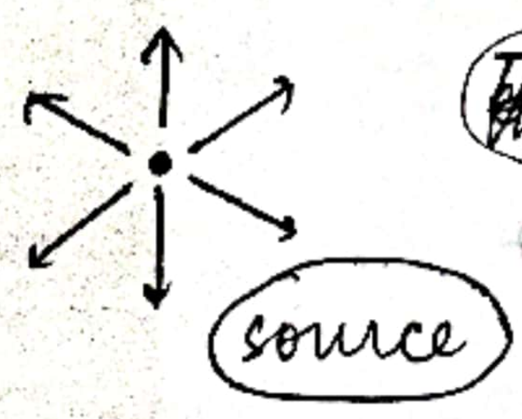
i.e., $\vec{\nabla} \cdot \vec{V}$ = Net outward flux per unit volume

Therefore, divergence of a vector field signifies the quantitative measure of how much a vector field, diverges or spreadups.

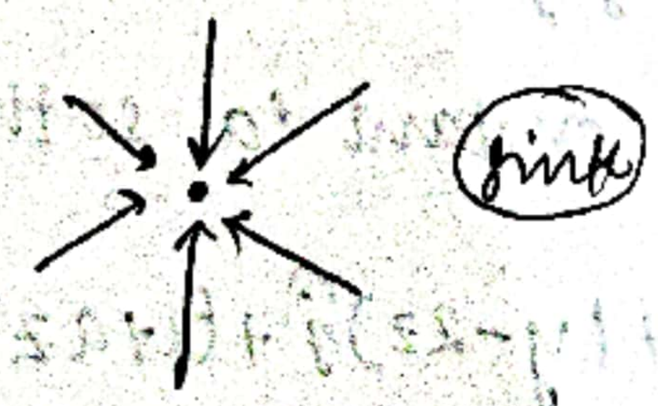
<u>Note</u>:

$\vec{\nabla} \cdot \vec{V}$ at a point can be +ve, -ve or zero.

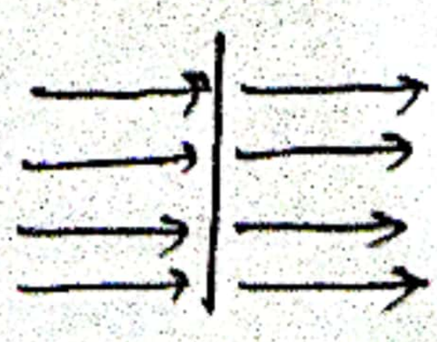
1) If $\vec{\nabla} \cdot \vec{V}$ = +ve, then it shows that there is some source of fluid at that point or fluid is expanding from the point.



2) If $\vec{\nabla} \cdot \vec{V}$ = -ve, then it shows that there is some sink of fluid at that point or fluid is contracting at that point.



3) If $\vec{\nabla} \cdot \vec{V} = 0$, then it shows that neither source nor sink exists at that point. In this case, fluid entered is equal to the fluid leaving out at a given point.



For an incompressible fluid, $\vec{\nabla}\cdot\vec{V}=0$ and $\vec{V}$ is called <u>solenoidal vector</u>.

Ques1. If $\vec{A} = x^2z\,\hat{i} - 2y^3z^2\,\hat{j} + xy^2z\,\hat{k}$, find $\vec{\nabla}\cdot\vec{A}$ at the point $(1,-1,1)$

$$\vec{\nabla}\cdot\vec{A} = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$$

$$\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$$

$$\frac{\partial A_1}{\partial x} = 2xz = 2\times(1)\times(1) = 2$$

$$\frac{\partial A_2}{\partial y} = -6y^2z^2 = -6\times(-1)^2\times(1)^2 = -6$$

$$\frac{\partial A_3}{\partial z} = xy^2 = 1\times(-1)^2 = 1$$

$$\vec{\nabla}\cdot\vec{A} = 2-6+1 = \underline{\underline{-3}}$$

Ques2. Given $\phi = 2x^3y^2z^4$

a) Find $\vec{\nabla}\cdot\vec{\nabla}\phi$

b) Show that $\vec{\nabla}\cdot\vec{\nabla}\phi = \nabla^2\phi$

Ques3. Prove that $\nabla^2\left(\frac{1}{r}\right)=0$, $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$

Ques4. Determine that constant 'a' so that the vector

$$\vec{V} = (x+3y)\hat{i} + (y-2z)\hat{j} + (x+az)\hat{k}$$

is solenoidal.

(2) $\phi = 2x^3y^2z^4$

$$\vec{\nabla}\phi = \left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right)$$

$$\frac{\partial\phi}{\partial x} = 6x^2y^2z^4$$

$$\frac{\partial\phi}{\partial y} = 4x^3yz^4$$

$$\frac{\partial\phi}{\partial z} = 8x^3y^2z^3$$

$$\vec{\nabla}\phi = \left(6x^2y^2z^4\hat{i} + 4x^3yz^4\hat{j} + 8x^3y^2z^3\hat{k}\right)$$

$$\vec{\nabla}.\vec{\nabla}\phi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right).\left(6x^2y^2z^4\hat{i} + 4x^3yz^4\hat{j} + 8x^3y^2z^3\hat{k}\right)$$

$$\vec{\nabla}.\vec{\nabla}\phi = \frac{\partial}{\partial x}(6x^2y^2z^4) + \frac{\partial}{\partial y}(4x^3yz^4) + \frac{\partial}{\partial z}(8x^3y^2z^3)$$

$$\vec{\nabla}.\vec{\nabla}\phi = 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2$$

$$\nabla^2\phi = \left(\frac{\partial^2}{\partial x^2}\phi + \frac{\partial^2}{\partial y^2}\phi + \frac{\partial^2}{\partial z^2}\phi\right)$$

$$\frac{\partial^2\phi}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\phi\right) = \frac{\partial}{\partial x}(6x^2y^2z^4) = 12xy^2z^4$$

$$\frac{\partial^2\phi}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\phi\right) = \frac{\partial}{\partial y}\left(4x^3yz^4\right) = 4x^3z^4$$

$$\frac{\partial^2\phi}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial}{\partial z}\phi\right) = \frac{\partial}{\partial z}(8x^3y^2z^3) = 24x^3y^2z^2$$

$$\nabla^2\phi = 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2$$

$$\vec{\nabla}.\vec{\nabla}\phi = \nabla^2\phi$$

proved.

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(3) $\nabla^2\left(\frac{1}{r}\right) = 0$ $\qquad \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$$

$$\nabla^2\left(\frac{1}{r}\right) = \frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{1}{r}\right) + \frac{\partial^2}{\partial z^2}\left(\frac{1}{r}\right)$$

$$\frac{\partial^2}{\partial x^2}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\frac{1}{\sqrt{x^2+y^2+z^2}}\right)$$

$$= \frac{\partial}{\partial x}\left(-\frac{x}{(\sqrt{x^2+y^2+z^2})^3}\right)$$

$$= \frac{-1\times(\sqrt{x^2+y^2+z^2})^3 + x\times \frac{3}{2}(\sqrt{x^2+y^2+z^2})\times 2x}{(x^2+y^2+z^2)^3}$$

$$= \frac{\sqrt{x^2+y^2+z^2}\,(-x^2-y^2-z^2+3x^2)}{(x^2+y^2+z^2)^3}$$

$$\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right) = \frac{\sqrt{x^2+y^2+z^2}\,(2x^2-y^2-z^2)}{(x^2+y^2+z^2)^3}$$

Similarly,

$$\frac{\partial^2}{\partial y^2}\left(\frac{1}{r}\right) = \frac{\sqrt{x^2+y^2+z^2}\,(2y^2-x^2-z^2)}{(x^2+y^2+z^2)^3}$$

$$\frac{\partial^2}{\partial z^2}\left(\frac{1}{r}\right) = \frac{\sqrt{x^2+y^2+z^2}\,(2z^2-x^2-y^2)}{(x^2+y^2+z^2)^3}$$

$$\nabla^2\left(\frac{1}{r}\right) = \frac{\sqrt{x^2+y^2+z^2}\,(2x^2-y^2-z^2+2y^2-x^2-z^2+2z^2-x^2-y^2)}{(x^2+y^2+z^2)^3}$$

$$\boxed{\nabla^2\left(\frac{1}{r}\right) = 0}$$

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(4) $\vec{V} = (x+3y)\hat{i} + (y-2z)\hat{j} + (x+az)\hat{k}$

$= V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$

$$\vec{\nabla}\cdot\vec{V} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$$

for solenoidal vector -

$\vec{\nabla}\cdot\vec{V} = 0$

$$\frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z} = 0$$

$$\frac{\partial V_1}{\partial x} = \frac{\partial}{\partial x}(x+3y) = 1$$

$$\frac{\partial V_2}{\partial y} = \frac{\partial}{\partial y}(y-2z) = 1$$

$$\frac{\partial V_3}{\partial z} = \frac{\partial}{\partial z}(x+az) = a$$

$1+1+a=0$

$\boxed{a=-2}$

Curl -

If $\vec{V}(x,y,z)$ is a differentiable vector field, then the curl or rotation of $\vec{V}$ is defined by

$\vec{\nabla}\times\vec{V}$ or curl $\vec{V}$ or rot $\vec{V}$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \times \left(V_1\hat{i} + V_2\hat{j} + V_3\hat{k}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix} = \hat{i}\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_2 & V_3 \end{vmatrix} - \hat{j}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ V_1 & V_3 \end{vmatrix} + \hat{k}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ V_1 & V_2 \end{vmatrix}$$

$$\boxed{\vec{\nabla}\times\vec{V} = \hat{i}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right) - \hat{j}\left(\frac{\partial V_3}{\partial x} - \frac{\partial V_1}{\partial z}\right) + \hat{k}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)}$$

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Physical interpretation of Curl :

we know that

$$\vec{V} = \vec{\omega} \times \vec{r}$$

where $\vec{V} = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$, is the linear velocity;

$\vec{\omega} = \omega_1\hat{i} + \omega_2\hat{j} + \omega_3\hat{k}$, is the angular velocity;

and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, is the position vector of a point on the rotating body.

Now,

$$\vec{\nabla} \times \vec{V} = \vec{\nabla} \times (\vec{\omega} \times \vec{r})$$

$$= \vec{\nabla} \times \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}$$

$$= \vec{\nabla} \times \left[(z\omega_2 - y\omega_3)\hat{i} - (z\omega_1 - x\omega_3)\hat{j} + (y\omega_1 - x\omega_2)\hat{k} \right]$$

$$\neq \cancel{\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (z\omega_2 - y\omega_3) & & \end{vmatrix}}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (z\omega_2 - y\omega_3) & -(z\omega_1 - x\omega_3) & (y\omega_1 - x\omega_2) \end{vmatrix}$$

= solve it properly for exam.

$$= 2\omega_1\hat{i} + 2\omega_2\hat{j} + 2\omega_3\hat{k}$$

$$= 2(\omega_1\hat{i} + \omega_2\hat{j} + \omega_3\hat{k})$$

$$= 2\vec{\omega}$$

$$\Rightarrow \vec{\nabla} \times \vec{V} = 2\vec{\omega}$$

→ This shows that curl of a vector field is connected with rotational property of the vector field and justifies the name rotation used for curl.

Therefore, curl of a vector quantity signifies how much the vector quantity curls or twist.

** A vector $\vec{V}$ is called irrotational if curl $\vec{V} = 0$

$$\boxed{\vec{\nabla} \times \vec{V} = 0}$$

Ques 1. If $\vec{A} = xz^3\hat{i} - 2x^2yz\hat{j} + 2yz^4\hat{k}$, find $\vec{\nabla} \times \vec{A}$ at the point $(1,-1,1)$.

$$\vec{\nabla} \times \vec{A} = \hat{i}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) - \hat{j}\left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z}\right) + \hat{k}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)$$

$$\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$$

$$\frac{\partial A_1}{\partial y} = 0 \qquad \frac{\partial A_2}{\partial x} = -4xyz \qquad \frac{\partial A_3}{\partial x} = 0$$

$$\frac{\partial A_1}{\partial z} = 3xz^2 \qquad \frac{\partial A_2}{\partial z} = -2x^2y \qquad \frac{\partial A_3}{\partial y} = 2z^4$$

$$= \hat{i}(2z^4 + 2x^2y) - \hat{j}(0 - 3xz^2) + \hat{k}(-4xyz - 0)$$

$$= \hat{i}(2-2) - \hat{j}(-3) + \hat{k}(-4(1)(-1)(1) - 0)$$

$$\vec{\nabla} \times \vec{V} = \underline{3\hat{j} + 4\hat{k}}$$

Ques 2. If $\vec{A} = x^2y\hat{i} - 2xz\hat{j} + 2yz\hat{k}$, find $\vec{\nabla} \times \vec{\nabla} \times \vec{A}$

$(2x+2)\hat{j}$

$$\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$$

$$\vec{\nabla} \times \vec{A} = \hat{i}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) - \hat{j}\left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z}\right) + \hat{k}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)$$

$$\frac{\partial A_3}{\partial y} = 2z \qquad \frac{\partial A_2}{\partial z} = -2x \qquad \frac{\partial A_1}{\partial y} = x^2$$

$$\frac{\partial A_3}{\partial x} = 0 \qquad \frac{\partial A_2}{\partial x} = -2z \qquad \frac{\partial A_1}{\partial z} = 0$$

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$$\vec{R} = \vec{\nabla}\times\vec{A} = \hat{i}(2z+2x) - \hat{j}(0-0) + \hat{k}(-2z-x^2)$$
$$= \hat{i}(2z+2x) - \hat{k}(2z+x^2)$$
$$= R_1\hat{i} - R_3\hat{k}$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}\times\vec{R}$$
$$= \hat{i}\left(\frac{\partial R_3}{\partial y} - \frac{\partial R_2}{\partial z}\right) - \hat{j}\left(\frac{\partial R_3}{\partial x} - \frac{\partial R_1}{\partial z}\right) + \hat{k}\left(\frac{\partial R_2}{\partial x} - \frac{\partial R_1}{\partial y}\right)$$

$$\frac{\partial R_3}{\partial y} = 0 \qquad \frac{\partial R_1}{\partial y} = 0$$

$$\frac{\partial R_3}{\partial x} = -2x \qquad \frac{\partial R_1}{\partial z} = 2$$

$$= \hat{i}(0-0) - \hat{j}(-2x-2) + \hat{k}(0-0)$$

$$\boxed{\vec{\nabla}\times\vec{\nabla}\times\vec{A} = (2x+2)\hat{j}}$$

Formula involving $\vec{\nabla}$

If $\vec{A}$ and $\vec{B}$ are differentiable vector fields, and ϕ is a differentiable scalar field of position (x, y, z) then -

1) $\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{A}(\vec{\nabla}\times\vec{B})$

2) $\vec{\nabla}\times(\vec{A}\times\vec{B}) = (\vec{B}\cdot\vec{\nabla})\vec{A} - \vec{B}(\vec{\nabla}\cdot\vec{A}) - (\vec{A}\cdot\vec{\nabla})\vec{B} + \vec{A}(\vec{\nabla}\cdot\vec{B})$

3) $\vec{\nabla}(\vec{A}\cdot\vec{B}) = (\vec{B}\cdot\vec{\nabla})\vec{A} + (\vec{A}\cdot\vec{\nabla})\vec{B} + \vec{B}\times(\vec{\nabla}\times\vec{A}) + \vec{A}\times(\vec{\nabla}\times\vec{B})$

4) $\vec{\nabla}\cdot(\vec{\nabla}\phi) = \nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$

where $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is called the <u>Laplacian operator</u>.

(5) $\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}$

(6) $\vec{\nabla}\times(\vec{\nabla}\phi) = 0$

(7) $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0$

$\vec{\nabla}\phi$ = Gradient
$\vec{\nabla}\cdot\vec{\nabla}$ = divergence
$\vec{\nabla}\times\vec{\nabla}$ = curl

Line Integral

In line or path integrals, some quantity related to the field is integrated between two given points in space, A and B, along a prescribed curve C that joins them.

In general, we may encounter line integrals of the forms.

line element vector quantity

$$\int_C \phi\, d\vec{r}\ ,\ \int_C \vec{F}\cdot d\vec{r}\ ,\ \int_C \vec{F}\times d\vec{r}$$

$d\vec{r}$ = line element

where ϕ is a scalar field and $\vec{F}$ is a vector field. In physical applications, line integrals of the second type are the most common.

* If $\vec{F}$ represents the force acting on a particle along arc AB, then $\int_A^B \vec{F}\cdot d\vec{r}$ gives the total work done.

* If $\vec{F}$ represents the velocity of a liquid, then $\oint_C \vec{F}\cdot d\vec{r}$ is called the circulation of $\vec{F}$ round the curve. → any closed area

($\oint$ means that the path of integration is a closed curve)

Properties of Line Integration -

(1) Reversing the path of Integration changes the sign of the Integral. If the path C along which the integrals are evaluated has A and B as its end-points, then

$$\int_A^B \vec{F}\cdot d\vec{r} = -\int_B^A \vec{F}\cdot d\vec{r}$$

This implies that if the path C is a loop then integrating around the loop in the opposite direction changes the sign of the integral.

(2) If the path of integration is subdivided into small segments then the sum of the separate line integrals along each segment is equal to the line integral along the whole path. So, if P is any point on the path of integration that lies between the path's end-points A and B, then.

$$\int_A^B \vec{F}\cdot d\vec{r} = \int_A^P \vec{F}\cdot d\vec{r} + \int_P^B \vec{F}\cdot d\vec{r}$$

Ques - If a force $\vec{F} = 2x^2y\,\hat{i} + 3xy\,\hat{j}$ displaces a particle in the XY-plane from (0,0) to (2,4) along a curve $y = 4x^2$. Find the work done.

$\vec{r} = x\hat{i} + y\hat{j}$
$d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$

$y = 4x^2$
$dy = 8x\,dx$

$$W.d. = \int_C \vec{F}\cdot d\vec{r}$$

$$= \int (2x^2y\,\hat{i} + 3xy\,\hat{j}).(dx\,\hat{i} + dy\,\hat{j})$$

$$= \int (2x^2y\,dx + 3xy\,dy)$$

$$= \int 2x^2 \times 4x^2\,dx + \int 3x \times 4x^2 \times 8x\,dx$$

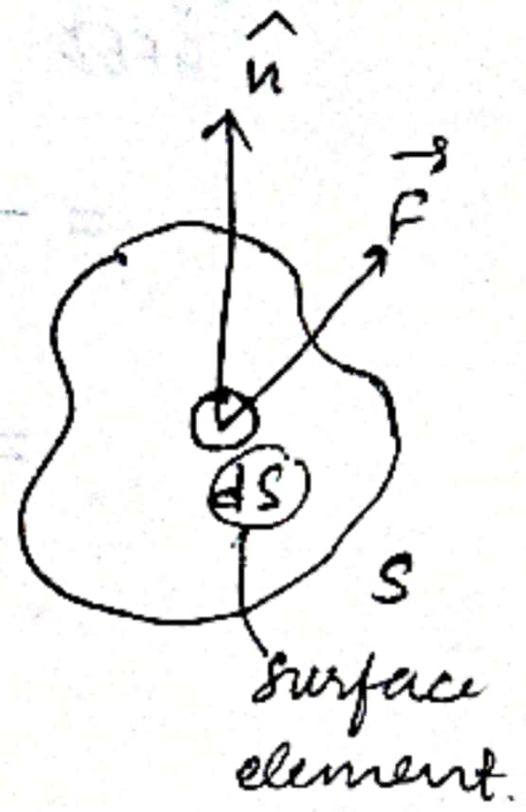
$$= 8\int x^4 dx + 96\int x^4 dx$$

$$= 104\int_0^1 x^4 dx$$

work done $= 104 \times \frac{x^5}{5}\Big|_0^1 = \frac{104}{5}$ Ans

Surface Integral -

Let $\vec{F}$ be a vector field and S be the given surface. Surface integral of a vector field $\vec{F}$ over the surface S is defined as the integral of the components of $\vec{F}$ along the normal to the surface.



$\therefore$ surface integral of $\vec{F}$ over S

$$= \iint_S (\vec{F}\cdot\hat{n})\, dS \quad \text{OR} \iint_S \vec{F}\cdot d\vec{S}$$

vector quantity

where $\hat{n}$ is the unit vector normal to an element of dS.

$d\vec{S}$ = surface element

As analogues of the line integrals, we may also encounter surface integrals of the forms,

$$\iint_S \phi\, dS, \iint_S \phi\, d\vec{S}, \iint_S \vec{F}\cdot d\vec{S}, \iint_S \vec{F}\times d\vec{S}$$

$d\vec{S} = \hat{n}\cdot dS$

Ques- Find the value of $\iint_S \vec{F}\cdot\hat{n}\, dS$, where $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ and S is the surface of the cube bounded by the planes,

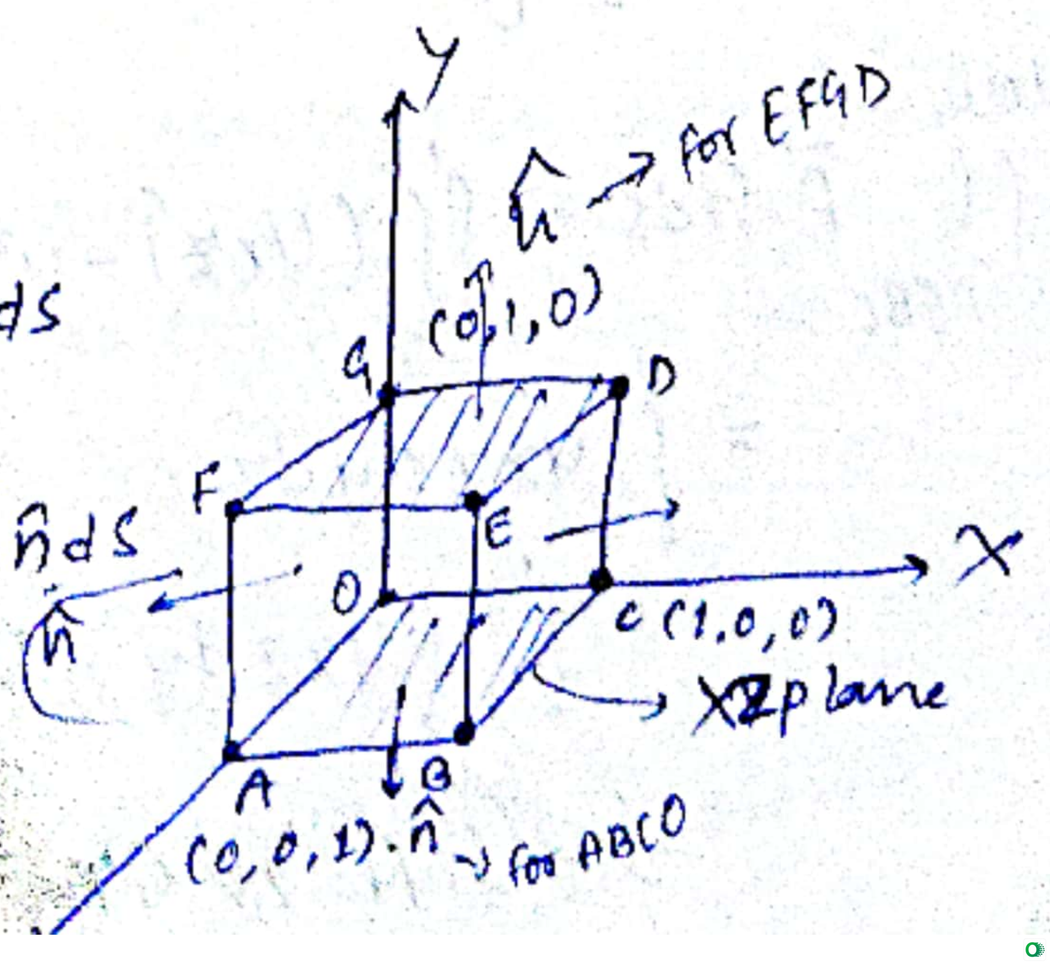
$x=0, x=1, y=0, y=1, z=1, z=0$

Sol-

$$\iint_S \vec{F}\cdot\hat{n}\, dS =$$

$$\iint_{OABC} \vec{F}\cdot\hat{n}\, dS + \iint_{GFED} \vec{F}\cdot\hat{n}\, dS + \iint_{OAFG} \vec{F}\cdot\hat{n}\, dS$$

$$+ \iint_{DEBC} \vec{F}\cdot\hat{n}\, dS + \iint_{OCDG} \vec{F}\cdot\hat{n}\, dS + \iint_{ABEF} \vec{F}\cdot\hat{n}\, dS$$



Now,

on OABC,

$\hat{n} = -\hat{j}$

$ds = dx\,dz$

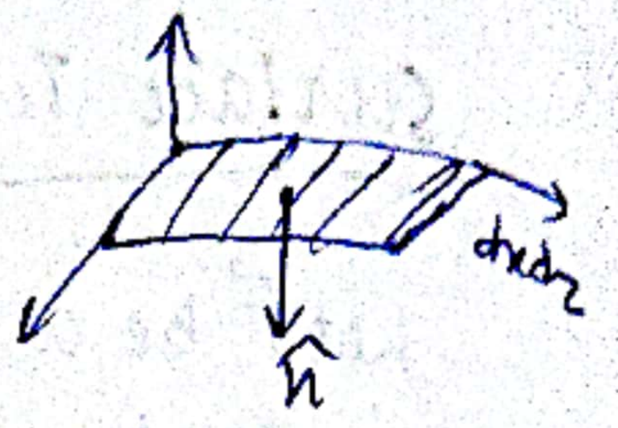
$y = 0$

$$\iint_{OABC} \vec{F}\cdot\hat{n}\,ds$$

$$= \int_0^1\int_0^1 (4xz\hat{i} - y^2\hat{j} + yz\hat{k})\cdot(-\hat{j})\,dx\,dz$$

$$= \int_0^1\int_0^1 y^2\,dx\,dz$$

$$= 0$$



Now,

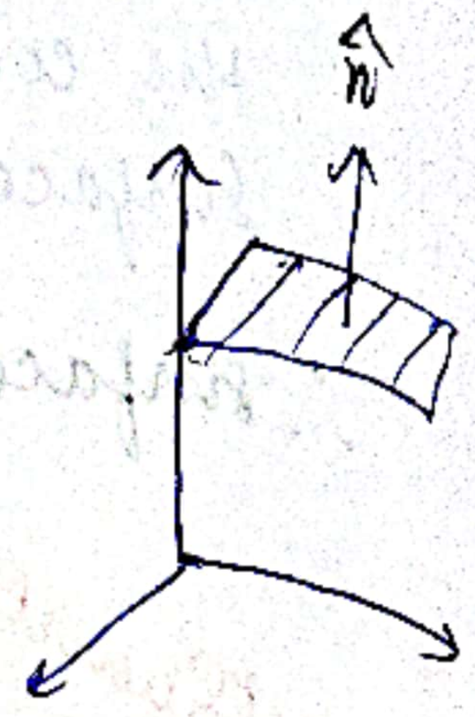
$$\iint_{GFED} \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 (4xz\hat{i} - y^2\hat{j} - yz\hat{k})\cdot\hat{j}\,dx\,dz$$

on GFED,

$\hat{n} = \hat{j}$

$ds = dx\,dz$

$y = 1$



$$= -\int_0^1\int_0^1 y^2\,dx\,dz$$

$$= -\int_0^1\int_0^1 dx\,dz$$

$$= -\int_0^1 dx\int_0^1 dz$$

$$= -\int_0^1 dx\,[z]_0^1$$

$$= -\int_0^1 dx \times 1 = -[x]_0^1 = -1$$

Now,

$$\iint_{OAFG} \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 (4xz\hat{i} - y^2\hat{j} + yz\hat{k})\cdot(-\hat{i})\,dy\,dz$$

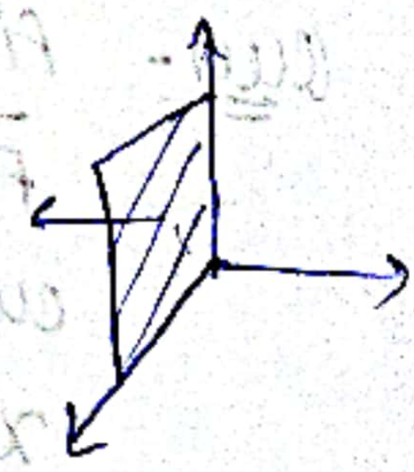
on OAFG,

$\hat{n} = -\hat{i}$

$ds = dy\,dz$

$x = 0$

$z = 1$



$$= -\int_0^1\int_0^1 4xz\,dz$$

$$= 0$$

Now,

$$\iint_{DEBC} \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 (4xz\hat{i} - y^2\hat{j} + yz\hat{k})\cdot(\hat{i})\,dy\,dz$$

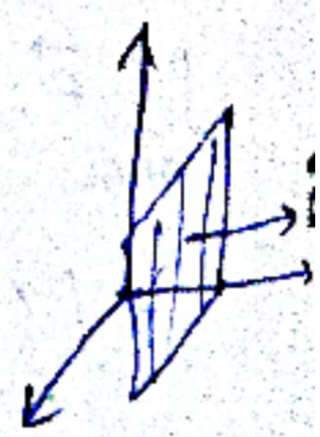
on DEBC,

$\hat{n} = \hat{i}$

$ds = dy\,dz$

$x = 1$

$z = 1$



$$= \int_0^1\int_0^1 4xz\,dy\,dz$$

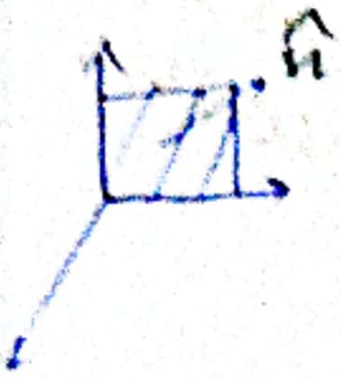
$$= \int_0^1 \left(\int_0^1 4xz\,dy\right) dz$$

$$= \int_0^1 [4xzy]_0^1\,dz$$

$$= \int_0^1 4xz\,dz$$
$$= 4x\frac{z^2}{2} = 2xz^2$$
$$= 2\times1\times1^2$$
$$= 2$$

Now,

$$\iint_{OCDG} \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 (4xz\,\hat{i} - y^2\hat{j} + yz\,\hat{k})\cdot(-\hat{k})\,dx\,dy$$



$$= -\int_0^1\int_0^1 yz\,dx\,dy$$
$$= 0$$

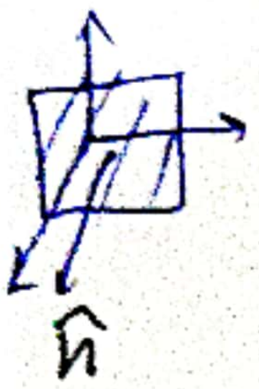
on OCDG,
$\hat{n} = -\hat{k}$
$dS = dx\,dy$
$y = 1,\ z = 0$

Now,

$$\iint_{ABEF} \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 (4xz\,\hat{i} - y^2\hat{j} + yz\,\hat{k})(\hat{k})\,dx\,dy$$

$$= \int_0^1\int_0^1 yz\,dx\,dy$$

on ABEF,
$\hat{n} = \hat{k}$
$dS = dx\,dy$
$y = 1,\ z = 1$



$$= \int_0^1\left(\int_0^1 yz\,dx\right)dy$$
$$= \int_0^1 (yzx)\,dy$$
$$= 2\times\frac{y^2}{2} = 1\times1\times\frac{1}{2} = 1/2$$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = 0 - 1 + 0 + 2 + 0 + 1/2$$
$$= 3/2$$

Volume Integral

Volume or space Integrals are generally simpler than line or surface integrals since the element of volume dV is a scalar quantity. We may encounter volume integrals of the forms

$$\iiint_V \phi\,dV,\quad \iiint_V \vec{F}\,dV$$

scalar quantity

$dV = dx\,dy\,dz$

dV = volume element

where V is the volume enclosed by a closed surface in space.

Ques - If $\vec{F} = 2z\hat{i} - x\hat{j} + y\hat{k}$, evaluate $\iiint_V \vec{F}\,dV$ where V is the region bounded by the surfaces

$$x=0,\ x=2,\ y=0,\ y=4,\ z=x^2,\ z=2$$

Ans. $\left(\frac{32}{5}\hat{i} + \frac{32}{3}\hat{k}\right)$

notation in integration is wrong so see ques② of Gauss's divergence theorem.

$$\iiint_V \vec{F}\,dV$$

$$= \iiint_V (2z\hat{i} - x\hat{j} + y\hat{k})\,dx\,dy\,dz$$

$$x=0,\ x=2,\ y=0,\ y=4,\ z=x^2,\ z=2$$

$$= \iiint (2z\hat{i} - x\hat{j} + y\hat{k})\,dz\,dy\,dx$$

$$= \iint\int_{x^2}^{2} (2z\,dz\,\hat{i} - x\,dz\,\hat{j} + y\,dz\,\hat{k})\,dy\,dx$$

$$= \iint_0^4 \left[\frac{2z^2}{2}\hat{i} - xz\hat{j} + yz\hat{k}\right]_{x^2}^{2} dy\,dx$$

$$= \iint_0^4 \left(\left[4\hat{i} - 2x\hat{j} + 2y\hat{k}\right] - \left[x^4\hat{i} - x^3\hat{j} + x^2y\hat{k}\right]\right) dy\,dx$$

$$= \iint_0^4 (4\hat{i} - 2x\hat{j} + 2y\hat{k})\,dy\,dx - \iint_0^4 (x^4\hat{i} - x^3\hat{j} + x^2y\hat{k})\,dy\,dx$$

$$= \int_0^2\int_0^4 (4\,dy\,\hat{i} - 2x\,dy\,\hat{j} + 2y\,dy\,\hat{k})\,dx - \int_0^2\int_0^4 (x^4\,dy\,\hat{i} - x^3\,dy\,\hat{j} + x^2y\,dy\,\hat{k})\,dx$$

$$= \int_0^2 \left(\left[4y\,\hat{i} - 2xy\,\hat{j} + y^2\hat{k}\right]_0^4\right) dx - \int_0^2 \left(\left[x^4y\,\hat{i} - x^3y\,\hat{j} + \frac{x^2y^2}{2}\hat{k}\right]_0^4\right) dx$$

$$\Rightarrow \int_0^2 (16\hat{i} - 8x\hat{j} + 16\hat{k})\,dx - \int_0^2 (4x^4\hat{i} - 4x^3\hat{j} + 8x^2\hat{k})\,dx$$

$$= \left[16x\hat{i} - 4x^2\hat{j} + 16x\hat{k}\right]_0^2 - \left[4\times\frac{x^5}{5}\hat{i} - x^4\hat{j} + \frac{8x^3}{3}\hat{k}\right]_0^2$$

$$\Rightarrow \left[32\hat{i} - 16\hat{j} + 32\hat{k}\right] - \left[\frac{32\times 4}{5}\hat{i} - 16\hat{j} + \frac{64}{3}\hat{k}\right]$$

$$= \frac{32\times 5 - 32\times 4}{5}\hat{i} - \cancel{16\hat{j}} + \cancel{16\hat{j}} + 32\hat{k} - \frac{64}{3}\hat{k}$$

$$= \frac{32}{5}\hat{i} + \frac{32}{3}\hat{k}$$

$$\boxed{\iiint_V \vec{F}\, dV = \frac{32}{5}\hat{i} + \frac{32}{3}\hat{k}}$$

Gauss's divergence theorem

(Relation between surface integral and volume integral)

Gauss's divergence theorem states that the surface integral of the normal component of a vector field $\vec{F}$ taken around a closed surface S is equal to the volume integral of the divergence of $\vec{F}$ taken over the volume V enclosed by the surface S.

Mathematically,

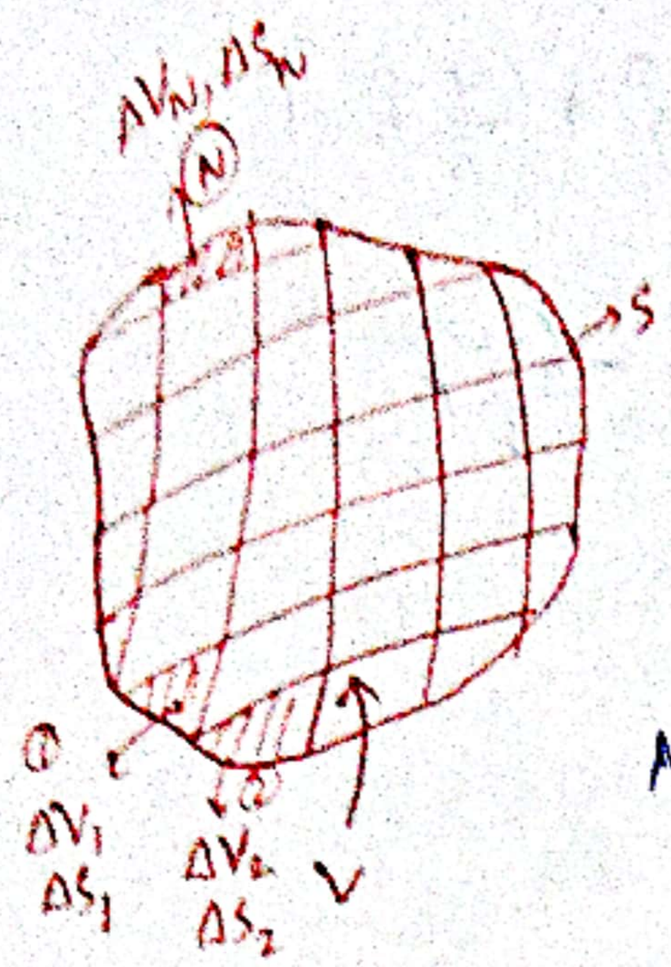
$$\iint_S \vec{F}\cdot\hat{n}\, dS = \iiint_V (\vec{\nabla}\cdot\vec{F})\, dV$$

or

$$\boxed{\iint_S \vec{F}\cdot d\vec{S} = \iiint_V (\vec{\nabla}\cdot\vec{F})\, dV}$$

Proof- consider a closed surface S which encloses volume V in a vector field $\vec{F}$.

Let us divide the given volume V into a large number, say N, of infinitesimally small volume elements each having volume $\Delta V_1, \Delta V_2, \ldots \Delta V_N$ ~~ed~~ enclosed by the surfaces $\Delta S_1, \Delta S_2, \ldots, \Delta S_N$ respectively.



$$\vec{\nabla}\cdot\vec{F} = \frac{\text{Flux of } \vec{F}}{\text{Volume}}$$

$$\text{Flux of } \vec{F} = (\vec{\nabla}\cdot\vec{F}) \times \text{Volume}$$

Now, the flux of vector field $\vec{F}$ through volume element '1' is given by

$$\underbrace{\iint_{\Delta S_1} \vec{F}\cdot d\vec{S}}_{\text{Flux}} = (\vec{\nabla}\cdot\vec{F})\Delta V_1$$

similarly, flux of vector field $\vec{F}$ through volume element '2' is -

$$\iint_{\Delta S_2} \vec{F}\cdot d\vec{S} = (\vec{\nabla}\cdot\vec{F})\Delta V_2$$

$$\vdots \qquad \vdots$$

$$\iint_{\Delta S_N} \vec{F}\cdot d\vec{S} = (\vec{\nabla}\cdot\vec{F})\Delta V_N$$

Adding the above equations,

$$\iint_{\Delta S_1} \vec{F}\cdot d\vec{S} + \iint_{\Delta S_2} \vec{F}\cdot d\vec{S} + \cdots + \iint_{\Delta S_N} \vec{F}\cdot d\vec{S} = (\vec{\nabla}\cdot\vec{F})(\Delta V_1 + \Delta V_2 + \cdots + \Delta V_N)$$

$$\Rightarrow \sum_{i=1}^{N} \iint_{\Delta S_i} \vec{F}\cdot d\vec{S} = \sum_{i=1}^{N} (\vec{\nabla}\cdot\vec{F})\Delta V_i \quad \text{—(1)}$$

when $N \to \infty$, $\Delta V_i \to 0$

As a result, summation on RHS of eqn (1) can be replaced by volume integral.

$$\therefore \sum_{i=1}^{N} \iint_{\Delta S_i} \vec{F}\cdot d\vec{S} = \iiint_V \vec{\nabla}\cdot\vec{F}\, dV \quad \text{—(2)}$$

As the outward flux of two neighbouring volume elements through common surface Area equal and opposite, LHS of

equation (2) can be simply written as
$\iint_S \vec{F}\cdot d\vec{s}$ which is flux of vector field $\vec{F}$ over surface S enclosing the volume V.

$\therefore$ From eq^n (2), we get

$$\boxed{\iint_S \vec{F}\cdot d\vec{s} = \iiint_V \vec{\nabla}\cdot\vec{F}\,dV}$$

Ques. Use Gauss's divergence theorem to evaluate $\iint_S \vec{A}\cdot d\vec{s}$, where $\vec{A} = x^3\hat{i} + y^3\hat{j} + z^3\hat{k}$ and S is the surface of the sphere $x^2+y^2+z^2=a^2$ (where a is the radius of the sphere.)

By Gauss's divergence theorem,

$$\iint_S \vec{A}\cdot d\vec{s} = \iiint_V \vec{\nabla}\cdot\vec{A}\,dV$$

$$\vec{\nabla}\cdot\vec{A} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(x^3\hat{i} + y^3\hat{j} + z^3\hat{k}\right)$$

$$\vec{\nabla}\cdot\vec{A} = \frac{\partial}{\partial x}x^3 + \frac{\partial}{\partial y}y^3 + \frac{\partial}{\partial y}z^3$$

$$\vec{\nabla}\cdot\vec{A} = 3x^2 + 3y^2 + 3z^2$$

$$\iint_S \vec{A}\cdot d\vec{s} = \iiint_V (3x^2+3y^2+3z^2)\,dV$$

$x^2+y^2+z^2=a^2$

$$\iint_S \vec{A}\cdot d\vec{s} = 3\iiint_V (x^2+y^2+z^2)\,dV$$

$$\iint_S \vec{A}\cdot d\vec{s} = 3\iiint_V a^2\,dV$$

volume of the sphere. $= \frac{4}{3}\pi a^3$

$$\iint_S \vec{A}\cdot d\vec{s} = 3a^2\iiint_V dV$$

$$\iint_S \vec{A}\cdot d\vec{s} = \cancel{3}a^2 \times \frac{4}{\cancel{3}}\pi a^3$$

$$\iint_S \vec{A}\cdot d\vec{s} = 4\pi a^5$$

Que 2- Use Gauss's divergence theorem to evaluate the surface integral,

$\iint_S \vec{F}\cdot\hat{n}\,ds$, where $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$

and S is the surface of the cube

bounded by -

$x=0,\ x=1,\ y=0,\ y=1,\ z=0,\ z=1$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_S \vec{F}\cdot d\vec{S}$$

By Gauss's divergence theorem,

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_V (\vec{\nabla}\cdot\vec{F})\,dV$$

$$\vec{\nabla}\cdot\vec{F} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(4xz\hat{i} - y^2\hat{j} + yz\hat{k}\right)$$

$$= \frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz)$$

$$= 4z - 2y + y$$

$$\vec{\nabla}\cdot\vec{F} = 4z - y$$

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_V (4z-y)\,dV$$

$dV = dx\,dy\,dz$

$$= \int_{x=0}^{x=1}\int_{y=0}^{y=1}\int_{z=0}^{z=1} (4z-y)\,dx\,dy\,dz$$

$$= \int_0^1 dx \int_0^1 dy \left[4x\frac{z^2}{2} - yz\right]_0^1$$

$$= \int_0^1 dx \int_0^1 dy\,(2-y)$$

$$= \int_0^1 dx \left[2y - \frac{y^2}{2}\right]_0^1$$

$$= \int_0^1 dx \times 3/2 \quad = \frac{3}{2}[x]_0^1 = \frac{3}{2} \times 1 = 3/2$$

Stoke's Theorem -

(Relation between line integral and surface integral)

Stoke's theorem states that surface integral of the component of curl $\vec{F}$ along the normal to the surface S, taken over the surface S bounded by curve C is equal to the line Integral of the vector field $\vec{F}$ taken along the closed curve C.

Mathematically,

$$\iint_S \vec{\nabla} \times \vec{F} \cdot \hat{n}\, ds = \oint_C \vec{F} \cdot d\vec{r}$$

or

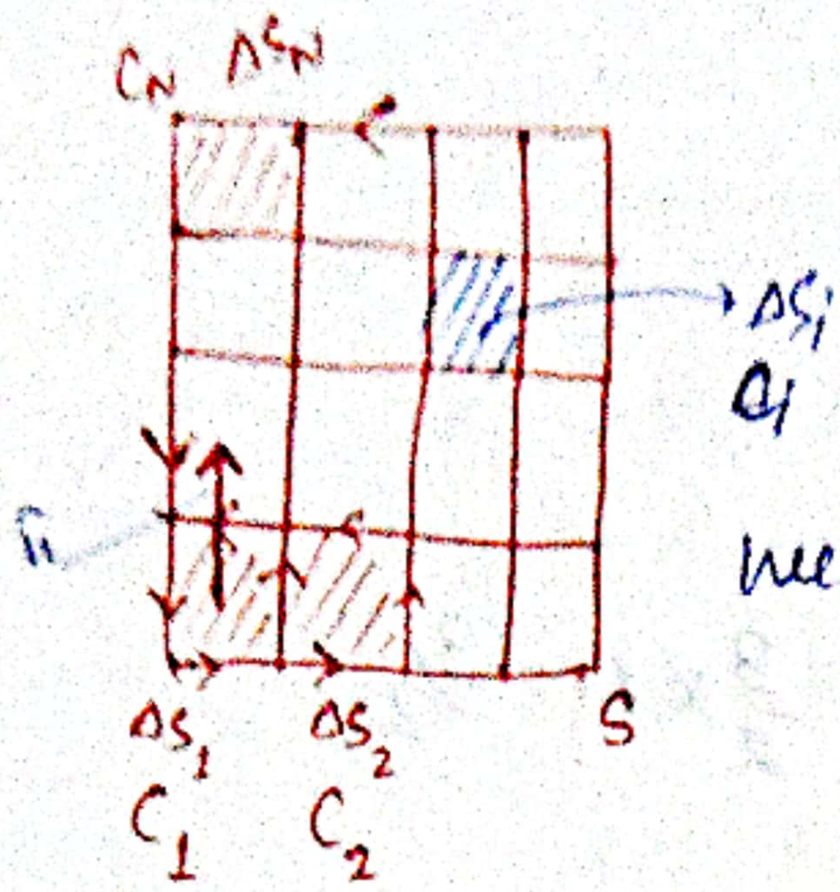
$$\boxed{\iint_S \vec{\nabla} \times \vec{F} \cdot d\vec{s} = \oint_C \vec{F} \cdot d\vec{r}}$$

Proof- Consider an open surface 'S' bounded by closed path C, placed in a vector field $\vec{F}$.

Let the surface 'S' be divided into a large number say N, of infinitesimally small surface elements $\Delta S_1, \Delta S_2, \ldots, \Delta S_N$ having curve boundaries $C_1, C_2, C_3 \ldots C_N$ respectively.

anticlock wise arrow sign → vector dirn

clock wise arrow sign → down (normal vector dirn)



We know that,

$\vec{\nabla}\times\vec{F}$ = Line integral of $\vec{F}$ per unit area.

∴ For one surface element ΔS_i bounded by curve C_i, we can write,

$$(\vec{\nabla}\times\vec{F}).\Delta\vec{S_i} = \oint_{C_i} \vec{F}.d\vec{r}$$

This equation is valid for each surface element. Adding such equations for all the surface elements, we get -

$$\sum_{i=1}^{N} (\vec{\nabla}\times\vec{F}).\Delta\vec{S_i} = \sum_{i=1}^{N} \oint_{C_i} \vec{F}.d\vec{r} \quad \text{—— (1)}$$

when $N \to \infty$, $\Delta S_i \to 0$

As a result, summation on LHS of eq^n (1) can be replaced by surface integral.

Thus, we can write

$$\iint_S (\vec{\nabla}\times\vec{F}).d\vec{S} = \sum_{i=1}^{N} \oint_{C_i} \vec{F}.d\vec{r} \quad \text{—— (2)}$$

In fact, the line integrals along the common side between two adjacent elements will be traversed in opposite directions and hence cancel each other.

Thus, RHS of eq^n (2) will represent sum of line integrals only along the sides which are on the boundary of the curve C.

$$\boxed{\therefore \iint_S (\vec{\nabla}\times\vec{F}).d\vec{s} = \oint_C \vec{F}.d\vec{r}}$$

Notes -

1) For a circle, $x^2+y^2=r^2$

polar coordinates are $x = r\cos\theta$, $y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

2) For a cylinder,

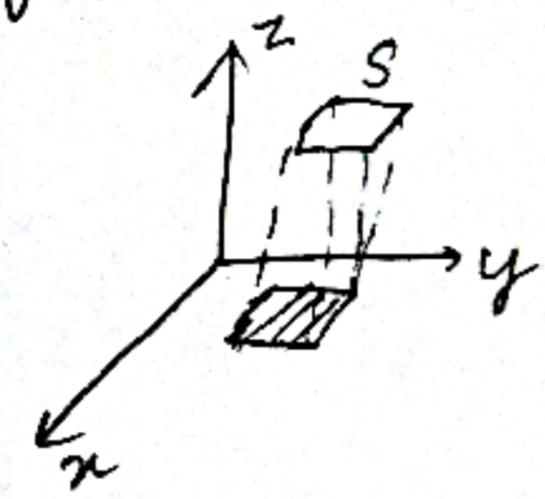
$$x^2+y^2=r^2, \quad z=a$$

cylindrical form is $x = r\cos\theta$, $y = r\sin\theta$, $z = z$

3) If the projection of surface S is in xy plane,

then $\hat{n} = \hat{k}$

Thus, $ds = dx\,dy$



4) Let S is the surface of shape ϕ,

then normal vector

$$\hat{n} = \frac{\text{grad}\,\phi}{|\text{grad}\,\phi|} \qquad \boxed{\text{grad}\,\phi = \vec{\nabla}.\phi}$$

Ques - use stoke's theorem to evaluate

$$\oint_C (yz\,dx + xz\,dy + xy\,dz),$$

where C is the curve $x^2+y^2=1$, $z=2$

By stoke's theorem -

$$\iint_S (\vec{\nabla}\times\vec{F}).d\vec{s} = \oint_C \vec{F}.d\vec{r}$$

$x^2+y^2 = 1^2$

$x = 1\times\cos\theta = \cos\theta$ $\qquad z = 2$

$y = 1\times\sin\theta = \sin\theta$

$$\oint_c (yz\,dx + xz\,dy + xy\,dz)$$

$$= \oint_c (yz\hat{i} + xz\hat{j} + xy\hat{k}) \cdot (dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

$$= \oint_c \vec{F} \cdot d\vec{r} \quad , \text{ where } \vec{F} = (yz\hat{i} + xz\hat{j} + xy\hat{k})$$

By Stoke's theorem -

$$\oint_c \vec{F} \cdot d\vec{r} = \iint_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{S}$$

$$= \iint_S (\vec{\nabla} \times \vec{F}) \cdot \hat{n}\, ds$$

Now,

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}(xz)\right) - \hat{j}\left(\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial z}(yz)\right)$$

$$+ \hat{k}\left(\frac{\partial}{\partial x}(xz) - \frac{\partial}{\partial y}(yz)\right)$$

$$= 0$$

$$\therefore \oint_c (yz\,dx + xz\,dy + xy\,dz) = \oint_c \vec{F} \cdot d\vec{r}$$

$$= \iint_S \vec{\nabla} \times \vec{F} \cdot \hat{n}\, ds$$

$$= 0$$

<u>Ques</u> - Using Stoke's theorem, evaluate

$$\oint_c [(2x - y)\,dx - yz^2\,dy - y^2 z\,dz]$$

where C is the circle $x^2 + y^2 = 1$, corresponding to the surface of sphere of unit radius.

$$\Rightarrow \oint_C \left[(2x-y)dx - yz^2dy - y^2z\,dz\right]$$

$$= \oint_C \left[(2x-y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}\right]\cdot(dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= \oint_C \vec{F}\cdot d\vec{r}, \text{ where } \vec{F} = (2x-y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}$$

By Stoke's Theorem -

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\vec{\nabla}\times\vec{F})\cdot d\vec{S}$$

$dx\,dy = r\,dr\,d\theta$

$$= \iint_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}\,dS$$

Now,

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial}{\partial y}(-y^2z) - \frac{\partial}{\partial z}(-yz^2)\right) - \hat{j}\left(\frac{\partial}{\partial x}(-y^2z) - \frac{\partial}{\partial z}(2x-y)\right) + \hat{k}\left(\frac{\partial}{\partial x}(-yz^2) - \frac{\partial}{\partial y}(2x-y)\right)$$

$$= \hat{i}\left(-2yz + 2yz\right) - \hat{j}(0-0) + \hat{k}(0-(-1))$$

$$\vec{\nabla}\times\vec{F} = \hat{k}$$

$$\oint_C \left[(2x-y)dx - yz^2dy - y^2z\,dz\right] = \oint_C \vec{F}\cdot d\vec{r}$$

$$= \iint_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}\,dS$$

$$= \iint_S \hat{k}\cdot\hat{k}\,dx\,dy$$

$$= \iint_S dx\,dy$$

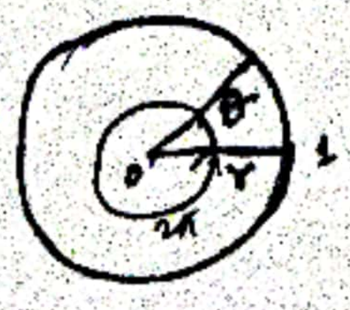
$$= \int_{r=0}^{r=1}\int_{\theta=0}^{\theta=2\pi} r\,dr\,d\theta$$

* In x-y plane

$ds = dx\,dy$

$\hat{n} = \hat{k}$ $\quad \hat{k}\cdot\hat{k} = 1$

* $dx\,dy = r\,dr\,d\theta$



$$= \int_0^1 r\,dr \int_0^{2\pi} d\theta$$

$$= \int_0^1 r\,dr\,[\theta]_0^{2\pi}$$

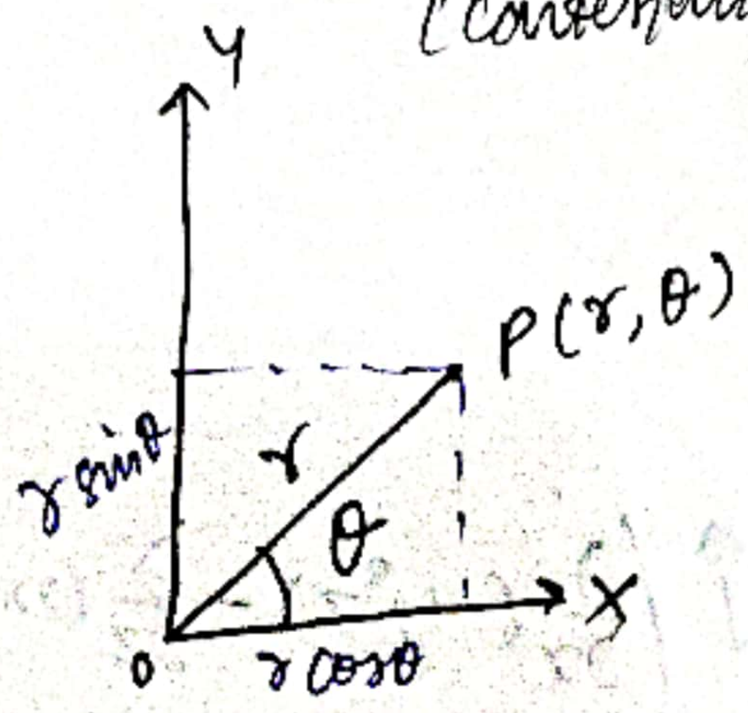
$$= \int_0^1 r\,dr \times 2\pi$$

$$= 2\pi \int_0^1 r\,dr = 2\pi \left[\frac{r^2}{2}\right]_0^1$$

$$= 2\pi \times \frac{1}{2} = \underline{\underline{\pi}}$$

Polar coordinates:

(cartesian)



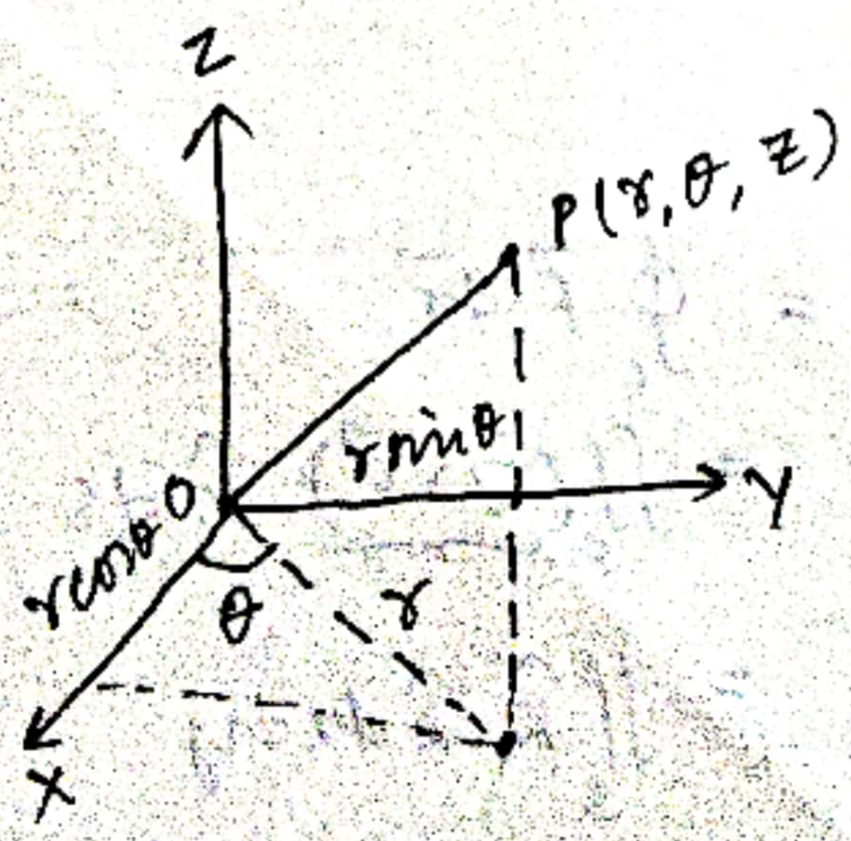
$x = r\cos\theta$
$y = r\sin\theta$ (cartesian coordinates in terms of polar coordinates)

$x^2 + y^2 = r^2(\cos^2\theta + \sin^2\theta)$

$\Rightarrow x^2 + y^2 = r^2 \Rightarrow r^2 = x^2 + y^2$ (polar coordinates in terms of cartesian coordinates)

$\tan\theta = \frac{y}{x}$

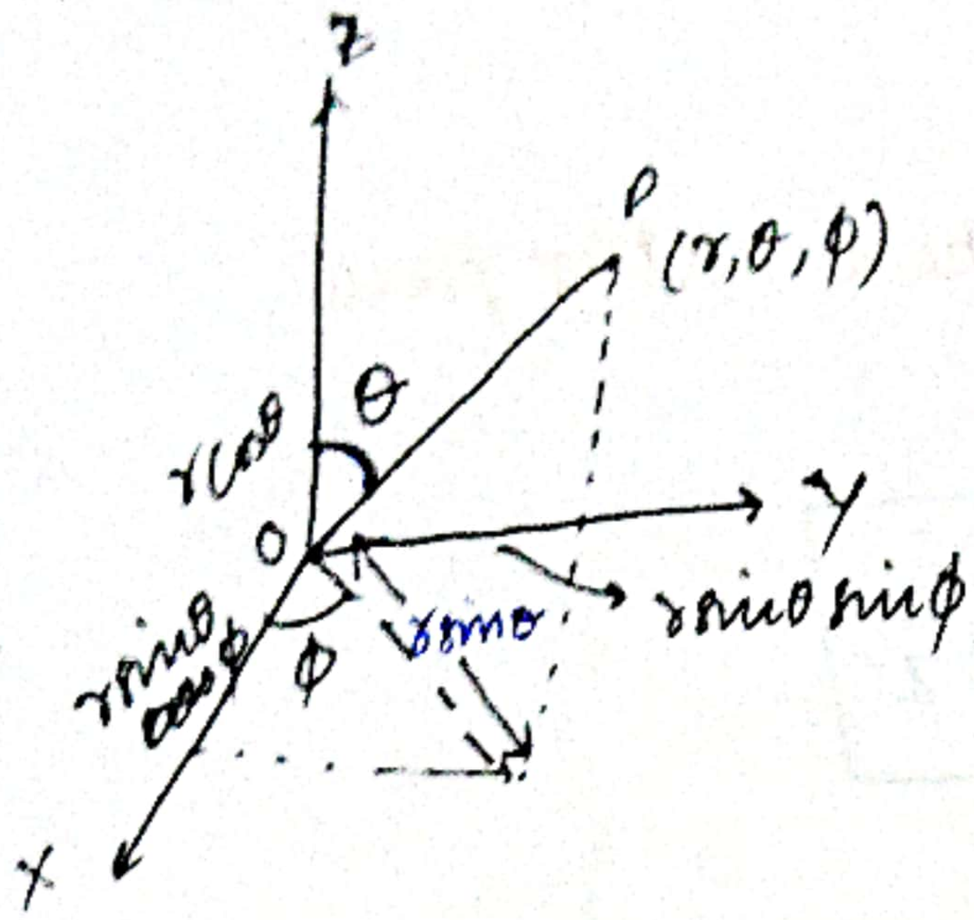
Cylindrical coordinates:



$x = r\cos\theta$
$y = r\sin\theta$ (cartesian coordinates in terms of cylindrical coordinates)
$z = z$

$r^2 = x^2 + y^2$
$\tan\theta = \frac{y}{x}$ (cylindrical coordinates in terms of cartesian coordinates)
$z = z$

Spherical coordinates :



$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

(cartesian coordinates in terms of spherical coordinates)

$r^2 = x^2+y^2+z^2$

$\tan\phi = \frac{y}{x}$

$\tan\theta = \frac{\sqrt{x^2+y^2}}{z}$

(spherical coordinates in terms of cartesian coordinates)

Gradient :-

cartesian : (x, y, z)
$\hat{x}, \hat{y}, \hat{z}$

If F is a scalar field,

$$\boxed{\vec{\nabla}F = \frac{\partial F}{\partial x}\hat{x} + \frac{\partial F}{\partial y}\hat{y} + \frac{\partial F}{\partial z}\hat{z}}$$

cylindrical: (r, θ, z)
$\hat{r}, \hat{\theta}, \hat{z}$

$$\boxed{\vec{\nabla}F = \frac{\partial F}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial F}{\partial \theta}\hat{\theta} + \frac{\partial F}{\partial z}\hat{z}}$$

$\theta \to r$

$\phi \to r\sin\theta$

spherical : (r, θ, ϕ)
$\hat{r}, \hat{\theta}, \hat{\phi}$

$$\boxed{\vec{\nabla}F = \frac{\partial F}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial F}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial F}{\partial \phi}\hat{\phi}}$$

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Divergence :

Cartesian : (x, y, z)

Let $\vec{F} = F_x \hat{x} + F_y \hat{y} + F_z \hat{k}$ be a vector field.

Then,

$$\boxed{\vec{\nabla}\cdot\vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}}$$

cylindrical: (r, θ, z)

$$\vec{F} = F_r \hat{r} + F_\theta \hat{\theta} + F_z \hat{z}$$

$$\boxed{\vec{\nabla}\cdot\vec{F} = \frac{1}{r}\frac{\partial(rF_r)}{\partial r} + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z}}$$

r is constant for θ & z.

$\theta \to r$

$\phi \to r\sin\theta$

L.H.S.

$\Rightarrow r^2 \sin\theta$

$$\vec{\nabla}\cdot\vec{F} = \frac{1}{r}\frac{\partial(rF_r)}{\partial r} + \frac{1}{r}\times\frac{1}{r}\frac{\partial(rF_\theta)}{\partial \theta} + \frac{1}{r}\frac{\partial(rF_z)}{\partial z}$$

$$= \frac{1}{r}\frac{\partial(rF_r)}{\partial r} + \frac{1}{r}\times\frac{r}{r}\frac{\partial F_\theta}{\partial \theta} + \frac{r}{r}\frac{\partial F_z}{\partial z}$$

$$= \frac{1}{r}\frac{\partial(rF_r)}{\partial r} + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z}$$

Spherical : (r, θ, ϕ)

$$\vec{F} = F_r \hat{r} + F_\theta \hat{\theta} + F_\phi \hat{\phi}$$

$$\boxed{\vec{\nabla}\cdot\vec{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 F_r) + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta F_\theta)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial \phi}}$$

1st step:-

$$\vec{\nabla}\cdot\vec{F} = \frac{\partial F_r}{\partial r} + \frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_\phi}{\partial \phi}$$

2nd step:-

$$\vec{\nabla}\cdot\vec{F} = \frac{\partial F_r}{\partial r} + \frac{1}{r}\frac{\partial F_\theta}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial \phi}$$

3rd step:-

$$\vec{\nabla}\cdot\vec{F} = \frac{1}{r^2\sin\theta}\frac{\partial(r^2\sin\theta F_r)}{\partial r} + \frac{1}{r}\cdot\frac{1}{r^2\sin\theta}\frac{\partial(r^2\sin\theta F_\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\times\frac{1}{r\sin\theta}\frac{\partial(r^2\sin\theta F_\phi)}{\partial \phi}$$

$$= \frac{1}{r^2 \cancel{\sin\theta}} \cancel{\sin\theta} \frac{\partial(r^2 F_r)}{\partial r} + \frac{1}{r} \times \frac{1}{r^2 \sin\theta} \times r^2 \frac{\partial(\sin\theta F_\theta)}{\partial \theta}$$

$$+ \frac{r^2 \cancel{\sin\theta}}{r \sin\theta \times r^2 \cancel{\sin\theta}} \frac{\partial(F_\phi)}{\partial \phi}$$

$$= \frac{1}{r^2} \frac{\partial(r^2 F_r)}{} + \frac{1}{r\sin\theta} \frac{\partial(\sin\theta F_\theta)}{\partial \theta} + \frac{1}{r\sin\theta} \frac{\partial F_\phi}{\partial \phi}$$

Curl :-

cartesian : (x, y, z)

$$\boxed{\vec{\nabla} \times \vec{F} = \hat{x}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) - \hat{y}\left(\frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z}\right) + \hat{z}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)}$$

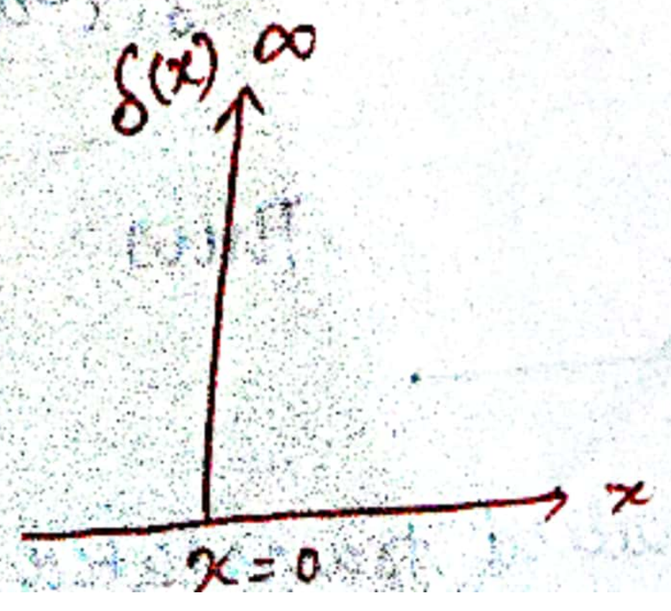
cylindrical : (r, θ, z)

$$\boxed{\vec{\nabla} \times \vec{F} = \hat{r}\left(\frac{1}{r}\frac{\partial F_z}{\partial \theta} - \frac{\partial F_\theta}{\partial z}\right) - \hat{\theta}\left(\frac{\partial F_z}{\partial r} - \frac{\partial F_r}{\partial z}\right) + \hat{z}\left(\frac{1}{r}\frac{\partial (r F_\theta)}{\partial r} - \frac{1}{r}\frac{\partial F_r}{\partial \theta}\right)}$$

spherical : (r, θ, ϕ)

$$\boxed{\begin{aligned}\vec{\nabla} \times \vec{F} = \frac{1}{r\sin\theta} \hat{r}\left(\frac{\partial}{\partial \theta}(\sin\theta F_\phi) - \frac{\partial F_\theta}{\partial \phi}\right) - \frac{1}{r}\hat{\theta}\left(\frac{\partial}{\partial r}(r F_\phi) - \frac{1}{\sin\theta}\frac{\partial F_r}{\partial \phi}\right) \\ + \frac{1}{r}\hat{\phi}\left(\frac{\partial}{\partial r}(r F_\theta) - \frac{\partial F_r}{\partial \theta}\right)\end{aligned}}$$

Dirac delta function -

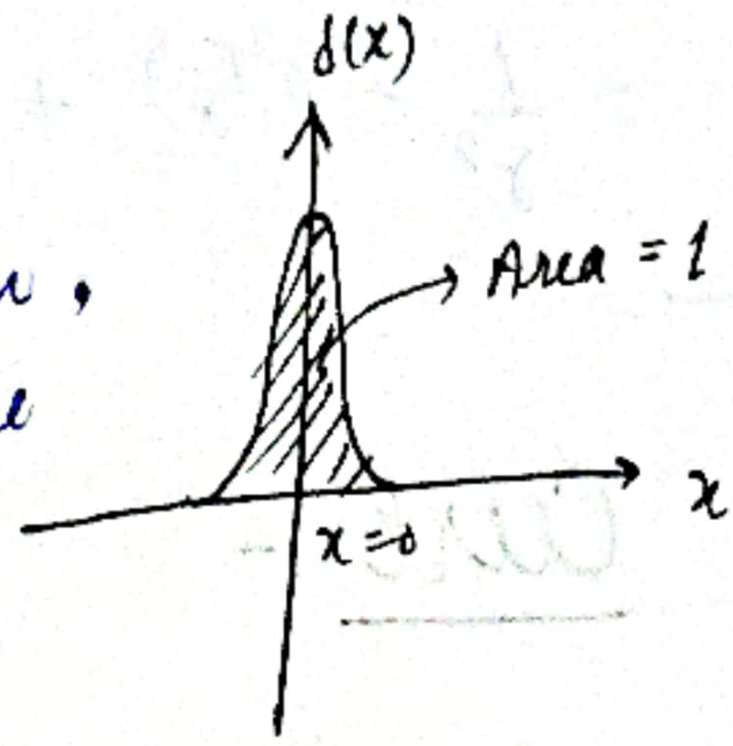


Dirac Delta function $\delta(x)$ is the function which is zero everywhere except at $x=0$ where the function tends to infinity.

Thus,

$$\delta(x) = \begin{cases} 0 & , x \neq 0 \\ \infty & , x = 0 \end{cases}$$

Dirac Delta function can be pictured as infinitely high, infinitesimally narrow spike with area under the curve equal to unity.



i.e.

$$\int_{-\infty}^{\infty} \delta(x)dx = 1$$

Properties of Dirac Delta function -

(1) Dirac Delta function is an even function.

i.e., $\delta(x) = \delta(-x)$

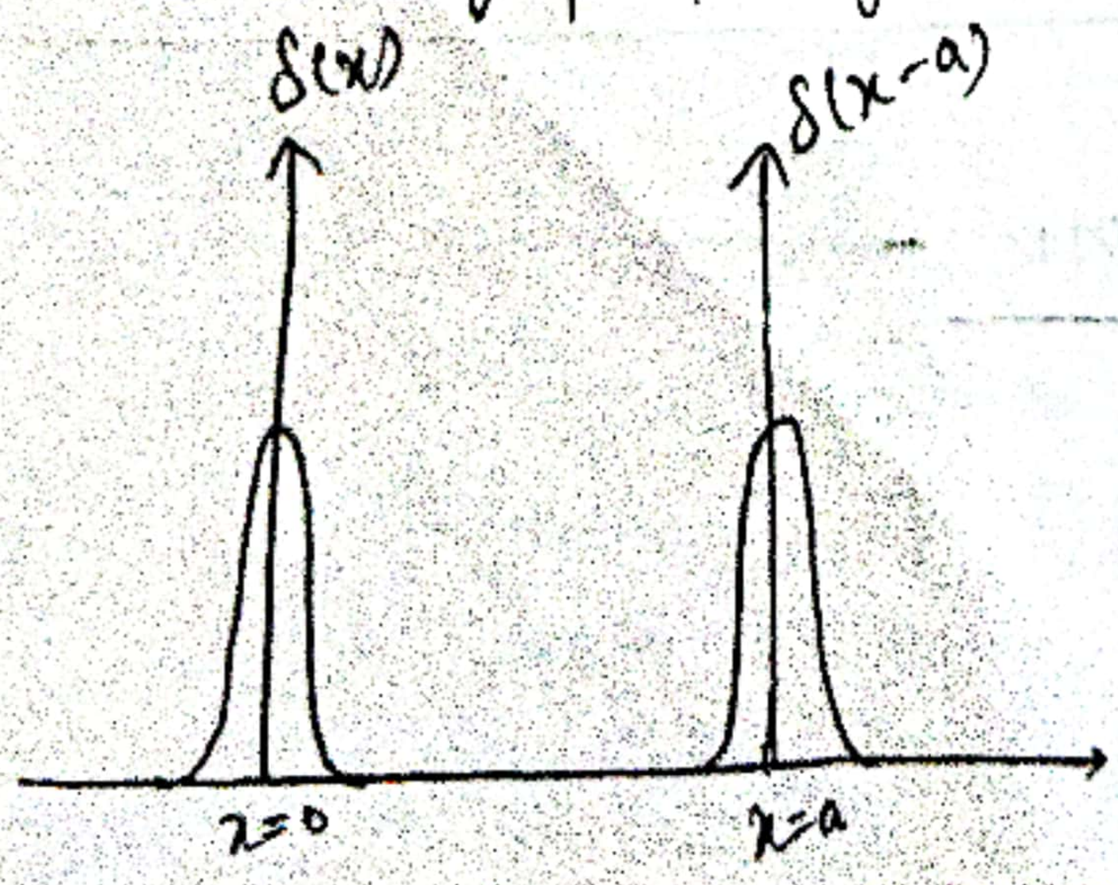
(2) $x\delta(x) = 0$

(3) $x \frac{d}{dx}[\delta(x)] = -\delta(x)$

(4) $\frac{d}{dx}[\delta(-x)] = -\frac{d}{dx}[\delta(x)]$

(5) $\delta(kx) = \frac{1}{|k|}\delta(x)$

(6) Shifting property :



According to this property, the Dirac delta function defined at $x=0$ can be shifted at $x=a$.

$$\delta(x-a) = \begin{cases} 0 & , x \neq a \\ \infty & , x = a \end{cases}$$

Thus $\int_{-\infty}^{\infty} \delta(x-a)dx = 1$

$\delta(x-a)$ is called translated Dirac delta function.

Electrostatics

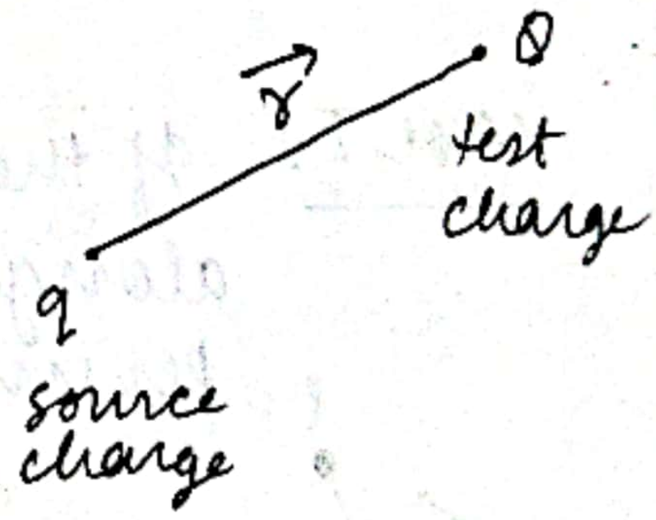
Electrostatics -

Electrostatics is a branch of physics that studies the phenominon and properties of electric charges at rest.

Electrostatics phenominon arise from the forces that electric charges exert on each other and are described by coulomb's law.

Coulomb's law -

According to coulomb's law, the force on a test charge Q due to a single point charge q which is at rest a distance 'r' away, is given by -



$$\boxed{\vec{F} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{qQ}{r^2}\hat{r}}$$

where ε_0 $(= 8.85 \times 10^{-12} C^2/Nm^2)$ is the permittivity of free space.

→ If we have several point charges $q_1, q_2, q_3 \cdots q_n$ at distances $r_1, r_2, \cdots r_n$ from Q, then total force on Q is given by -

$$\vec{F} = \vec{F_1} + \vec{F_2} + \cdots$$

$$\vec{F} = \frac{Q}{4\pi\varepsilon_0}\left(\frac{q_1}{r_1^2}\hat{r}_1 + \frac{q_2}{r_2^2}\hat{r}_2 + \cdots\right)$$

∴ Electric field of the source charges is –

$$\vec{E} = \frac{\vec{F}}{q}$$

$$\boxed{\vec{E} = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \frac{q_i}{r_i^2} \hat{r}_i}$$, for discrete point charges q_i

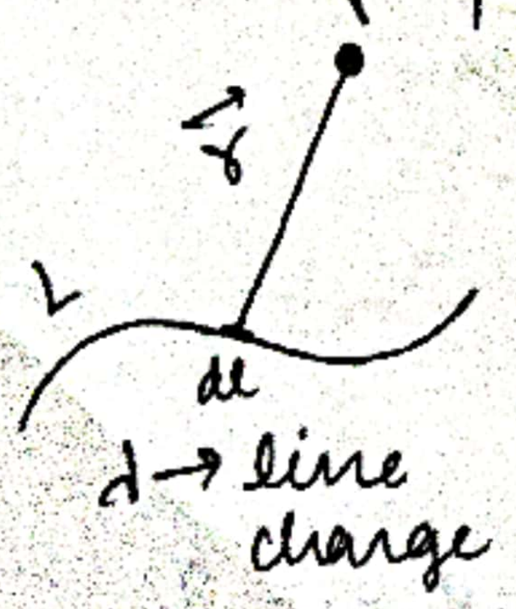
If charges q_i will be continuous then integration.

If the charge is distributed continuously over the region,

$$\boxed{\vec{E} = \frac{1}{4\pi\epsilon_0} \int \frac{1}{r^2} \hat{r}\, dq}$$

1) line, λ = line charge
2) surface, σ = surface charge
3) volume, ρ = volume charge

<u>case I</u>: If the charge is spread out along a line, with charge per unit length (λ), then



$$dq = \lambda\, dl$$

where dl is an element of length along the line.

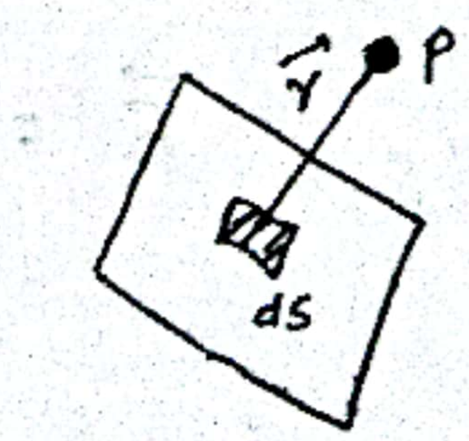
∴ Electric field for a line charge is –

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \int_L \frac{\lambda}{r^2} \hat{r}\, dl$$

<u>case II</u>: If the charge is smeared out over a surface, with charge per unit area (σ), then

$$dq = \sigma\, dS$$

where dS is an element of area on the surface.



$\sigma \rightarrow$ surface charge

∴ Electric field for a surface charge is -

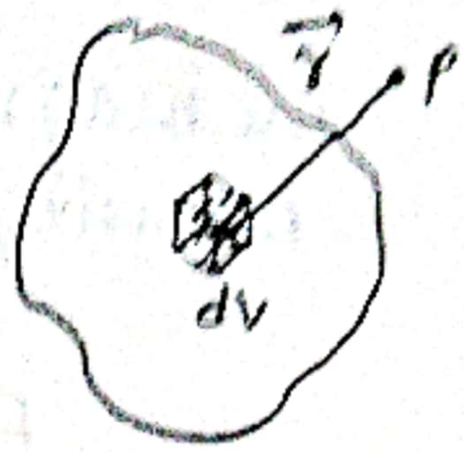
$$\vec{E} = \frac{1}{4\pi\epsilon_0}\iint_S \frac{\sigma}{r^2}\,\hat{r}\,ds$$

<u>case III</u> : If the charge fills a volume, with charge per unit volume (ρ), then

$$dq = \rho\, dV$$

where dV is an element of volume

∴ Electric field for a volume charge is -



$$\vec{E} = \frac{1}{4\pi\epsilon_0}\iiint_V \frac{\rho}{r^2}\,\hat{r}\,dV$$

Electric flux -

Electrix flux through a surface S is a measure of the number of field lines passing through S and is given by -

$$\boxed{\phi_E = \int_S \vec{E}\cdot d\vec{S}}$$

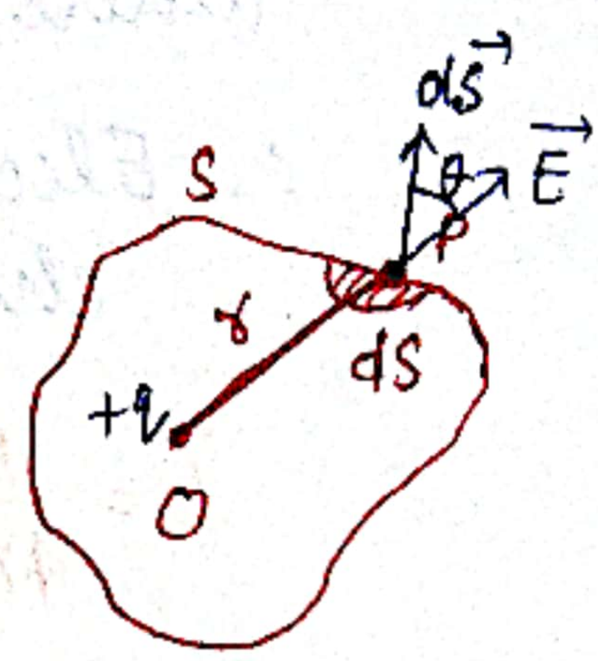
<u>Gauss's Law</u> - Gauss's law states that the total electric flux through a closed surface is $\frac{1}{\epsilon_0}$ times the net charge enclosed by the surface.

Mathematically,

$$\boxed{\phi_E = \frac{1}{\epsilon_0} q_{enclosed}}$$

Proof -

consider a charge $+q$ situated at a point O within a closed surface S of arbitrary shape. Let P be a point on the surface at a distance r from O. Now take a small area dS around P. The normal to the surface dS is represented by the vector $d\vec{S}$ which makes an angle θ with the direction of electric field $\vec{E}$ along OP.



The magnitude of electric field intensity at point P is given by -

$$E = \frac{1}{4\pi\epsilon_0} \cdot \frac{q}{r^2} \quad \text{—(1)}$$

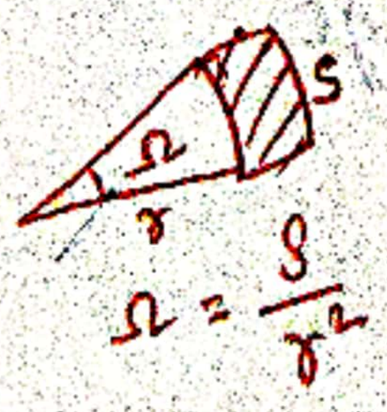
Electric flux passing through area dS enclosing point P is -

$$d\phi_E = \vec{E} \cdot d\vec{S}$$
$$= E\,dS\cos\theta \quad \text{—(2)}$$

Then, total electric flux passing through the closed surface S is given by -

$$\phi_E = \oint_S d\phi_E = \oint_S E\,dS\cos\theta \quad [\text{using eq}^n (2)]$$

$$= \oint_S \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} dS\cos\theta \quad [\text{using eq}^n (1)]$$



$$= \frac{q}{4\pi\epsilon_0} \oint_S \underbrace{\frac{dS\cos\theta}{r^2}}_{d\Omega}$$

$$= \frac{q}{4\pi\epsilon_0} \oint_S d\Omega$$ ← solid angle

where $d\Omega = \frac{dS\cos\theta}{r^2}$ is the solid angle subtended by area dS at point O

But $\oint_S d\Omega$ = solid angle due to the entire closed surface S at an internal point O.

$$= 4\pi$$

$$\therefore \phi_E = \frac{q}{4\pi\epsilon_0} \times 4\pi$$

$$\Rightarrow \boxed{\phi_E = \frac{q}{\epsilon_0}}$$

Gauss's Law in integral and differential form

By Gauss's law,

Total electric flux, $\phi_E = \frac{1}{\epsilon_0} q_{enclosed}$

where $q_{enclosed}$ is the total charge enclosed by the closed surface S.

$\therefore$ Gauss's law in integral form is given by -

$$\boxed{\oint_S \vec{E}\cdot d\vec{s} = \frac{1}{\epsilon_0} q_{enclosed}} \quad \text{—(1)}$$

Let V be the volume bounded by the closed surface S.

Then $q_{enclosed}$ can be written in terms of volume charge density ρ as

$$\boxed{\begin{aligned} \rho &= \frac{dq}{dV} \\ dq &= \rho\, dV \\ q &= \int dq \end{aligned}}$$

$$q_{enc.} = \int_V \rho\, dV \quad \text{—(2)}$$

using (2) in (1), we get

$$\oint_S \vec{E}\cdot d\vec{s} = \frac{1}{\epsilon_0}\int_V \rho\, dV \quad \text{—(3)}$$

According to Gauss's divergence theorem -

$$\oint_S \vec{E}\cdot d\vec{s} = \int_V (\vec{\nabla}\cdot\vec{E})\, dV$$

$\therefore$ Eq^n (3) can be written as-

$$\int_V (\vec{\nabla}\cdot\vec{E})\, dV = \int_V \left(\frac{\rho}{\epsilon_0}\right) dV$$

since this holds for any volume, the integrands must be equal.

ie,

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} \quad \text{— (4)}$$

Eq^n (4) is known as Gauss's law in differential form.

Thus, Gauss's Law in differential form states that the divergence of $\vec{E}$ at any point in space is equal to $\frac{1}{\epsilon_0}$ times total charge density at that point.

Applications of Gauss's Law :-

1) Electric field due to uniformly charged solid sphere -

consider a solid sphere of radius R and center O with uniformly charge q.

Then, volume charge density, $\rho = \dfrac{q}{\frac{4}{3}\pi R^3}$

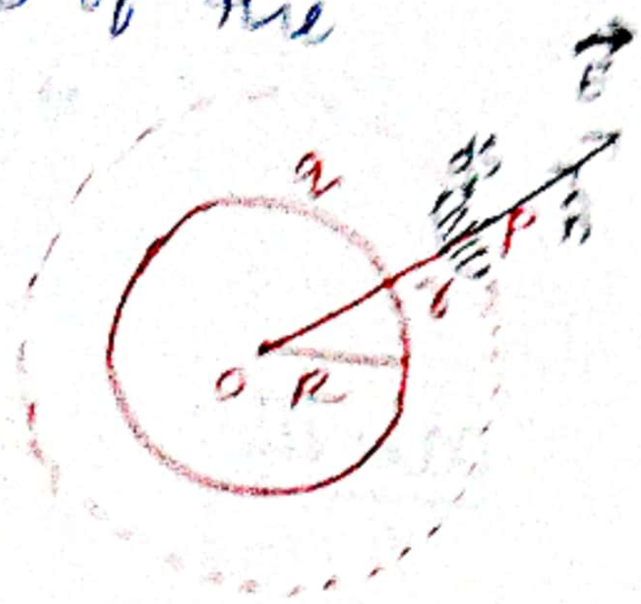
$$q = \rho \times \frac{4}{3}\pi R^3$$

Case I :- Outside the solid sphere

(AE-BAS)

consider a point P outside the solid sphere at a distance 'r' from centre O of the solid sphere.

ie., $r > R$



A spherical surface called Gaussian surface is drawn at radius r (r>R) passing through the point P.

Let ds be the small area element on the Gaussian surface at P.

By Gauss's law,

$$\oint_s \vec{E}\cdot d\vec{s} = \frac{1}{\epsilon_0} q$$

$$\Rightarrow \oint_s E\, ds \cos 0^\circ = \frac{1}{\epsilon_0} \times \rho \times \frac{4}{3}\pi R^3$$

$$\Rightarrow E \oint_s ds = \frac{1}{\epsilon_0} \times \rho \times \frac{4}{3}\pi R^3$$

$$\Rightarrow E \times 4\pi r^2 = \frac{1}{\epsilon_0} \times \rho \times \frac{4}{3}\pi R^3$$

$$\Rightarrow \boxed{E_{outside} = \frac{\rho R^3}{3\epsilon_0 r^2}} \longrightarrow \text{outside}$$

ie,

$$\boxed{E_{outside} \propto \frac{1}{r^2}}$$

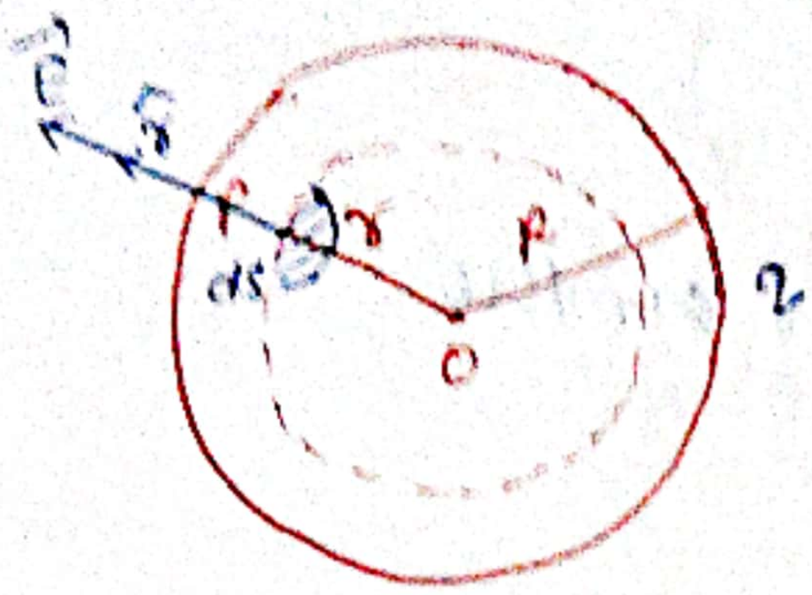
Case II :- On the surface of solid sphere

consider a point P on the surface of the solid sphere ie., $r = R$

$$\therefore E = \frac{\rho R^3}{3\epsilon_0 R^2}$$

$$\Rightarrow \boxed{E = \frac{\rho R}{3\epsilon_0}}$$ — on the surface

Case III :- Inside the solid sphere -



consider a point P inside the solid sphere at a distance r from centre O of the solid sphere i.e., $r < R$.

A Gaussian surface of radius r is drawn through P and ds is the small area element at P.

By Gauss's law,

$$\oint_S \vec{E}\cdot d\vec{S} = \frac{1}{\epsilon_0}(\rho \times \text{Volume of Gaussian surface})$$

$$\Rightarrow \oint_S E\,ds\cos 0^\circ = \frac{1}{\epsilon_0} \times \frac{q}{\frac{4}{3}\pi R^3} \times \frac{4}{3}\pi r^3$$

$$\Rightarrow E\oint_S ds = \frac{qr^3}{\epsilon_0 R^3}$$

$$\Rightarrow E \times 4\pi r^2 = \frac{qr^3}{\epsilon_0 R^3}$$

$$\Rightarrow \boxed{E_{inside} = \frac{qr}{4\pi\epsilon_0 R^3}}$$

i.e., $\boxed{E_{inside} \propto r}$

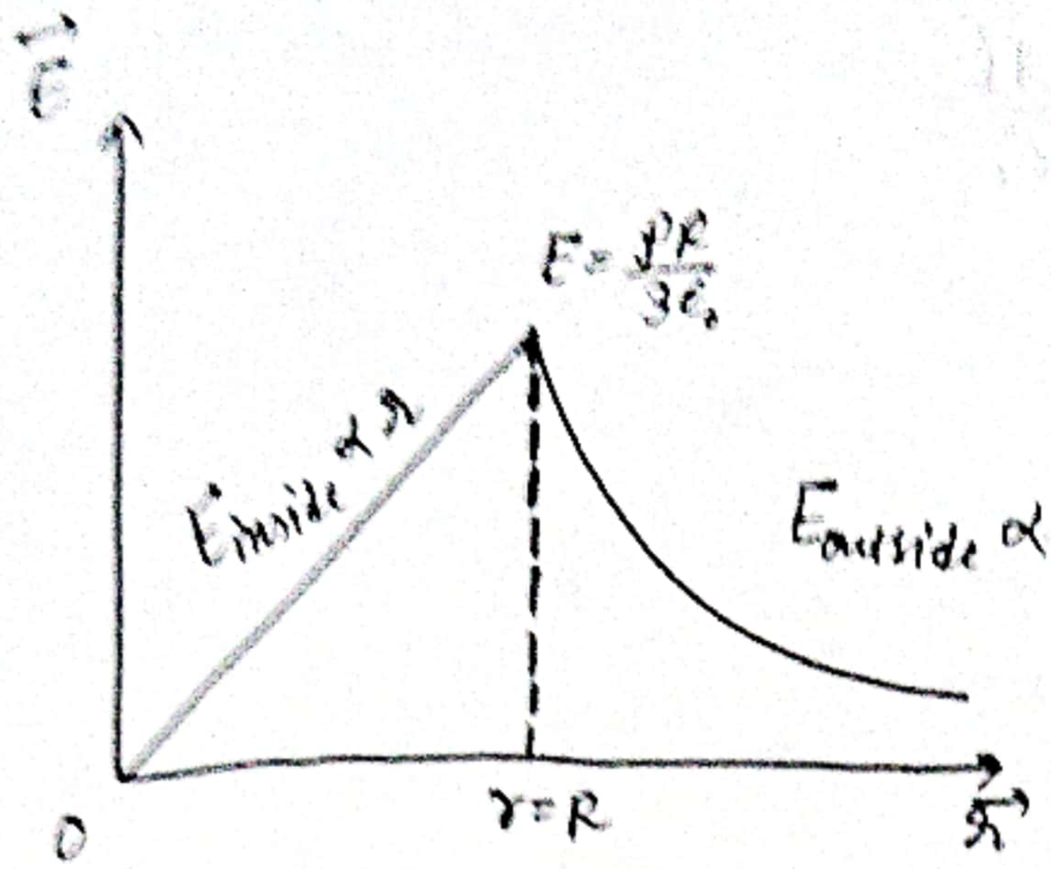
Case IV :- At the centre of solid sphere -

$r = 0$

$$\therefore E_{center} = \frac{q}{4\pi\epsilon_0 R^3} \times 0 = 0$$

$$\boxed{\therefore E_{center} = 0}$$

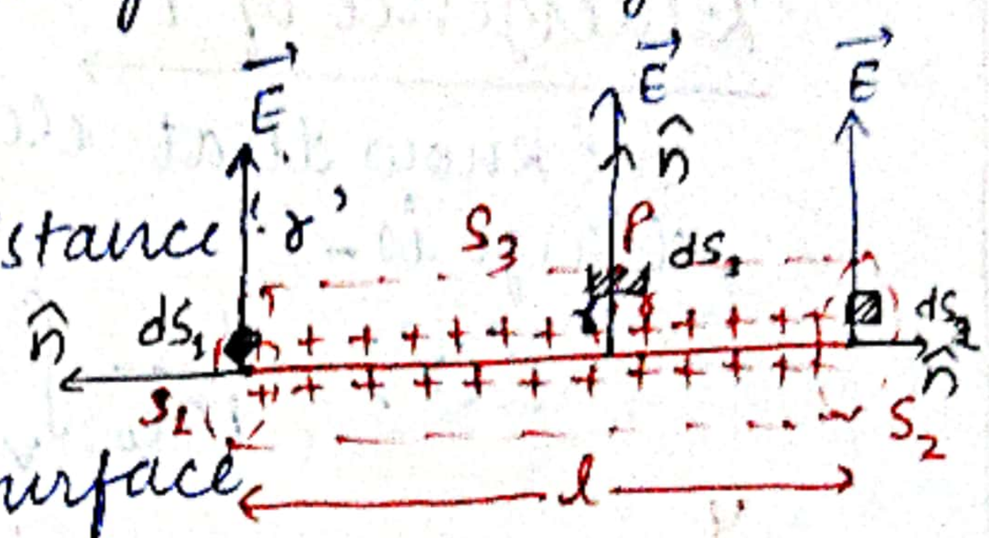
On center. $E = 0$
inside. $E \propto r$
on the surface. $E = \frac{\rho R}{3\epsilon_0}$
outside. $E \propto \frac{1}{r^2}$



2) Electric field due to uniformly charged straight wire –

consider an infinitely long straight uniformly positive charged wire having line charge density λ.

Let P be a point at normal distance 'r' from the wire



Then, a cylindical Gaussian surface of radius 'r' and length 'l' co-axial with the wire is drawn. This Gaussian surface can be divided into 3 surfaces S_1, S_2, S_3 with respective area elements ds_1, ds_2, ds_3 are shown in the figure.

charge enclosed by the Gaussian surface is given by –

$$q = \lambda l$$

$$\boxed{\lambda = \frac{q}{\text{length}}}$$

By Gauss's law –

$$\oint_S \vec{E} \cdot \vec{ds} = \frac{1}{\epsilon_0} q$$

$$\Rightarrow \int_{S_1} \vec{E}\cdot d\vec{S_1} + \int_{S_2} \vec{E}\cdot d\vec{S_2} + \int_{S_3} \vec{E}\cdot d\vec{S_3} = \frac{1}{\epsilon_0} \times \lambda l$$

$$\Rightarrow \int_{S_1} E\,dS_1 \cos 90^\circ + \int_{S_2} E\,dS_2 \cos 90^\circ + \int_{S_3} E\,dS_3 \cos 0^\circ = \frac{1}{\epsilon_0} \times \lambda l$$

$$\Rightarrow 0 + 0 + E\int_{S_3} dS_3 = \frac{\lambda l}{\epsilon_0}$$

$$\Rightarrow \quad E\int_{S_3} dS_3 = \frac{\lambda l}{\epsilon_0}$$

$$E \times 2\pi r l = \frac{\lambda l}{\epsilon_0}$$

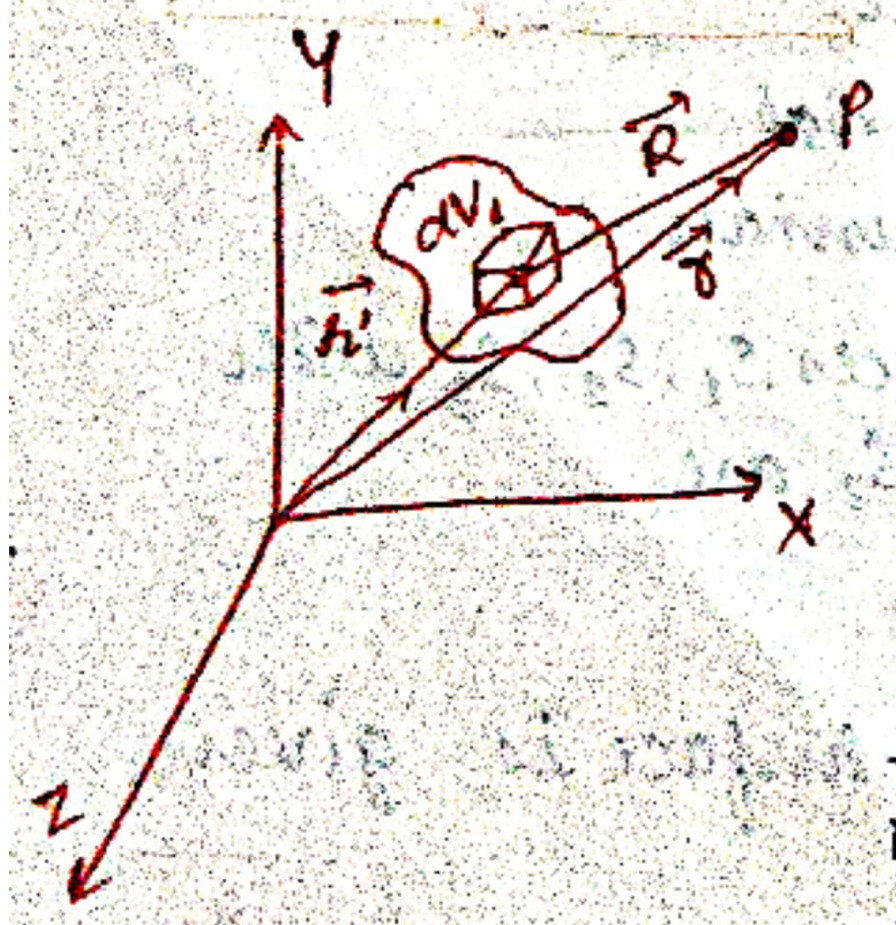
$$\boxed{E = \frac{\lambda}{2\pi\epsilon_0 r}}$$

i.e. $\boxed{E \propto \frac{1}{r}}$

Divergence of $\vec{E}$

we know that electric field for a volume charge is –

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\hat{R}}{R^2} \rho(\vec{r}')\,dV'$$



since $\rho = 0$ in the exterior region of the volume occupied by the charge. the integration can be extended to all space.

Thus, we can write –

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{\text{all space}} \frac{\hat{R}}{R^2} \rho(\vec{r}')\,dV'$$

where $\vec{R} = \vec{r} - \vec{r}'$

Now, $\vec{\nabla}\cdot\vec{E} = \frac{1}{4\pi\epsilon_0} \int_{\text{all space}} \vec{\nabla}\cdot\left(\frac{\hat{R}}{R^2}\right) \rho(\vec{r}')\,dV'$

But $\vec{\nabla}\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = 4\pi\,\delta^3(\vec{\mathscr{r}})$

$$\therefore\ \vec{\nabla}\cdot\vec{E} = \frac{1}{4\pi\epsilon_0}\int_{\text{all space}} 4\pi\,\delta^3(\vec{\mathscr{r}}-\vec{\mathscr{r}}')\,\rho(\vec{\mathscr{r}}')\,dV'$$

$$= \frac{1}{\epsilon_0}\int_{\text{all space}} \delta^3(\vec{\mathscr{r}}-\vec{\mathscr{r}}')\,\rho(\vec{\mathscr{r}}')\,dV'$$

$\boxed{\int_{-\infty}^{\infty} f(x)\,\delta(x-a)\,dx = f(a)}$

$\Rightarrow$ $\boxed{\vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon_0}\rho(\vec{\mathscr{r}})}$, using the property of dirac-delta function

Curl of $\vec{E}$ →

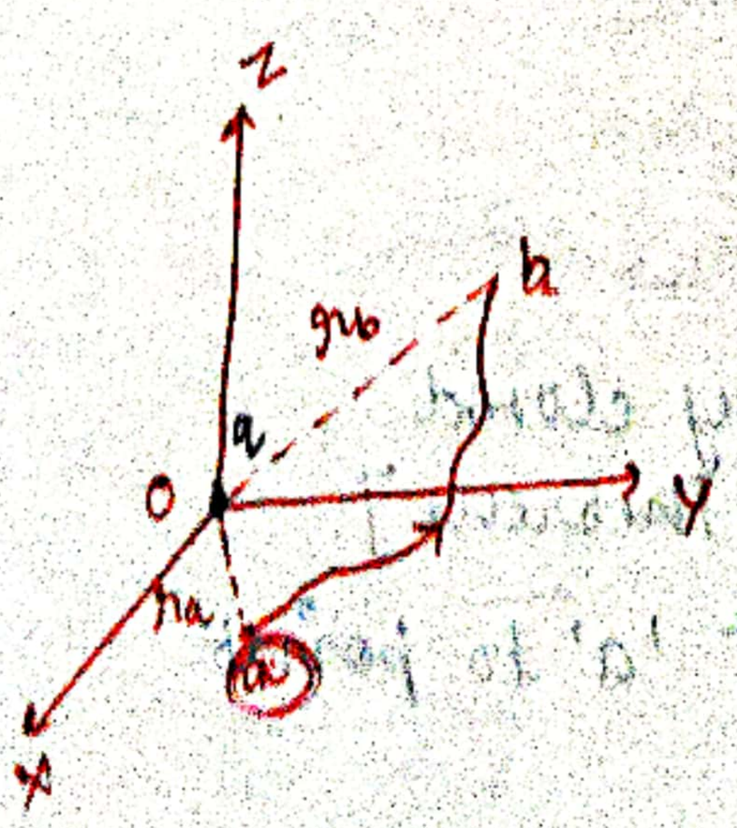
Electric field for a point charge at origin is -

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}^2}\hat{\mathscr{r}}$$

Now, line integral of $\vec{E}$ from some point 'a' to some other point 'b' is given by-

$$\int_a^b \vec{E}\cdot d\vec{l} = \int_a^b \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}^2}\,d\mathscr{r}$$

, since $d\vec{l} = d\mathscr{r}\,\hat{\mathscr{r}} + \mathscr{r}\,d\theta\,\hat{\theta} + \mathscr{r}\sin\theta\,d\phi\,\hat{\phi}$, in spherical coordinates.



$$= \frac{q}{4\pi\epsilon_0}\int_{r_a}^{r_b}\frac{1}{r^2}\,dr$$

$$= -\frac{q}{4\pi\epsilon_0}\left[\frac{1}{r}\right]_{r_a}^{r_b}$$

$$= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{r_a} - \frac{1}{r_b}\right]$$

where r_a is the distance from the origin to the point 'a' and r_b is the distance to 'b'.

Then, the integral around closed path is given by -

$$\oint \vec{E}\cdot d\vec{l} = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{r_a} - \frac{1}{r_b}\right] + \frac{q}{4\pi\epsilon_0}\left[\frac{1}{r_b} - \frac{1}{r_a}\right]$$

$$= 0$$

$$\Rightarrow \oint \vec{E}\cdot d\vec{l} = 0 \quad \text{—①}$$

By stoke's theorem -

$$\oint \vec{E}\cdot d\vec{l} = \int_s (\vec{\nabla}\times\vec{E})\cdot d\vec{s}$$

$\therefore$ Eq^n ① becomes -

$$\int_s (\vec{\nabla}\times\vec{E})\cdot d\vec{s} = 0$$

$$\Rightarrow \boxed{\vec{\nabla}\times\vec{E} = 0}$$

curl - divergence -
$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$$
$$\vec{\nabla}\times\vec{E} = 0$$

Electrostatic potential :

we know that

$$\vec{\nabla}\times\vec{E} = 0$$

$\Rightarrow$ Line integral of $\vec{E}$ around any closed loop is zero. [From Stoke's Theorem]

$\Rightarrow$ Line integral of $\vec{E}$ from point 'a' to point 'b' is the same for all paths.

i.e, the line integral of $\vec{E}$ is independent of path.

$\therefore$ we can define a function called electric potential as -

$$V(\vec{r}) = -\int_{Ref}^{r} \vec{E}\cdot d\vec{l}$$

where Ref denotes some <u>reference point</u>.

Thus, the potential difference between two points 'a' and 'b' is given by,

$$V(b) - V(a) = -\int_{Ref}^{b} \vec{E}\cdot d\vec{l} - \left(-\int_{Ref}^{a} \vec{E}\cdot d\vec{l}\right)$$

$$= \int_{Ref}^{a} \vec{E}\cdot d\vec{l} - \int_{Ref}^{b} \vec{E}\cdot d\vec{l}$$

$$= -\int_{a}^{Ref} \vec{E}\cdot d\vec{l} - \int_{Ref}^{b} \vec{E}\cdot d\vec{l}$$

$$V(b) - V(a) = -\int_{a}^{b} \vec{E}\cdot d\vec{l} \quad \text{—} (1)$$

But, the fundamental theorem for gradients states that -

$$\int_{a}^{b} (\vec{\nabla} V)\cdot d\vec{l} = V(b) - V(a) \quad \text{—} (2)$$

comparing eq^n (1) and (2), we get -

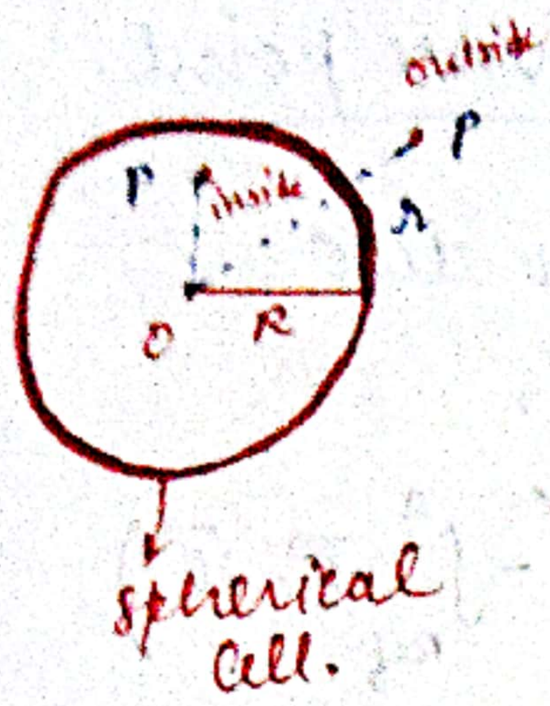
$$-\int_{a}^{b} \vec{E}\cdot d\vec{l} = \int_{a}^{b} (\vec{\nabla} V)\cdot d\vec{l}$$

Since this is true for any points 'a' and 'b', the integrals must be equal.

$$\therefore \boxed{\vec{E} = -\vec{\nabla} V}$$

i.e., Electric field is the gradient of a scalar potential.

Ques - Find the potential inside and outside a spherical shell of radius R, which carries a uniform surface charge. Set the reference point at infinity.



By Gauss's law,
the electric field outside the spherical shell is -

$$E = \frac{1}{4\pi\epsilon_0} \cdot \frac{q}{r^2}$$

where q is the total charge on the sphere.

and E is zero inside the spherical shell.

∴ For a point P outside the sphere ($r > R$), potential is given by -

$$V_{outside} = -\int_{\infty}^{r} \vec{E} \cdot d\vec{l}$$

$$= -\int_{\infty}^{r} \frac{1}{4\pi\epsilon_0} \cdot \frac{q}{r^2} dr$$

$$= \frac{-q}{4\pi\epsilon_0} \int_{\infty}^{r} \frac{1}{r^2} dr$$

$$= \frac{q}{4\pi\epsilon_0} \cdot \left[\frac{1}{r}\right]_{\infty}^{r} = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{r} - \cancel{\frac{1}{\infty}}^{\,0}\right]$$

$$V_{outside} = \frac{q}{4\pi\epsilon_0 r}$$

$$\boxed{V_{outside} = \frac{1}{4\pi\epsilon_0} \cdot \frac{q}{r}}$$

Again, for a point P inside the sphere,

$$V_{inside} = -\int_{\infty}^{R} \vec{E} \cdot d\vec{l} - \int_{R}^{r} \vec{E} \cdot d\vec{l}$$

$$V_{inside} = -\int_{\infty}^{R} \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\,dr - 0$$

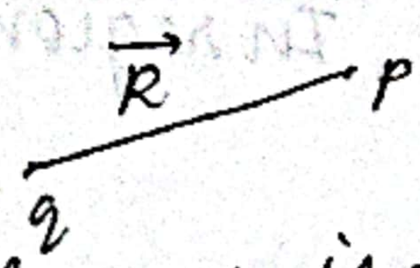
$$= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{r}\right]_{\infty}^{R}$$

$$\boxed{V_{inside} = \frac{1}{4\pi\epsilon_0}\cdot\frac{q}{R}}$$

→ inside spherical shell V_{inside} have constant value because Radius (R) is fixed.

In general, the potential of a point charge q is

$$V = \frac{1}{4\pi\epsilon_0}\frac{q}{R}$$



Then, the potential of a collection of charges is -

$$\boxed{V = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{R_i}}$$

→ for discrete charge distribution

For a continuous charge distribution, the potentials of line, surface and volume charges are -

$$\boxed{\frac{1}{4\pi\epsilon_0}\int_L \frac{\lambda\,dl}{R}}$$ line

$$\boxed{\frac{1}{4\pi\epsilon_0}\int_S \frac{\sigma\,ds}{R}}$$ surface

$$\boxed{\frac{1}{4\pi\epsilon_0}\int_V \frac{\rho\,dV}{R}}$$ volume.

Poisson's Equation and Laplace's equation

We know that,

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} \quad \text{— (1)}$$

But, electric field can be written as the gradient of a scalar potential.

i.e,

$$\vec{E} = -\vec{\nabla}V \quad \text{— (2)}$$

using eq^n ② in ①, we get -

$$\vec{\nabla}\cdot(-\vec{\nabla}V) = \frac{\rho}{\varepsilon_0}$$

$$\Rightarrow -\vec{\nabla}\cdot\vec{\nabla}V = \frac{\rho}{\varepsilon_0}$$

$$\Rightarrow -\nabla^2 V = \frac{\rho}{\varepsilon_0}$$

$$\Rightarrow \boxed{\nabla^2 V = -\frac{\rho}{\varepsilon_0}}$$

This is known as Poisson's equation.

In regions, where there is no volume charge, so that $\rho = 0$,

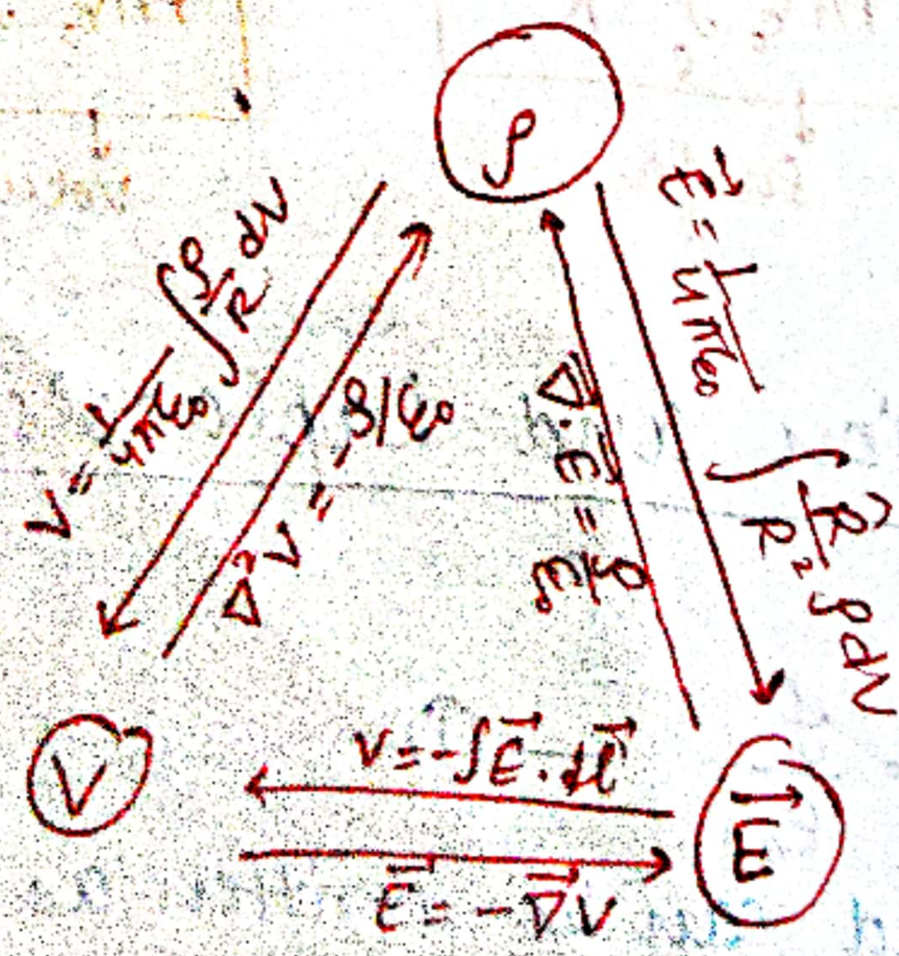
$$\boxed{\nabla^2 V = 0}$$

This is known as Laplace's equation.

Notes :-

The three fundamental quantities of electrostatics are ρ, $\vec{E}$ and V

The interrelation between them is given below :-



$\rho \rightarrow$ volume charge density.

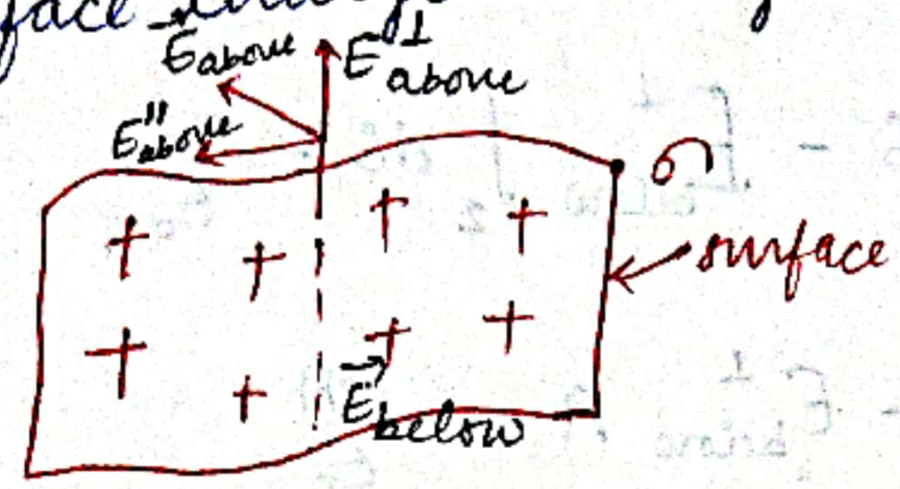
$\vec{E} \rightarrow$ Electric field

$V \rightarrow$ Electric potential

Boundary Conditions

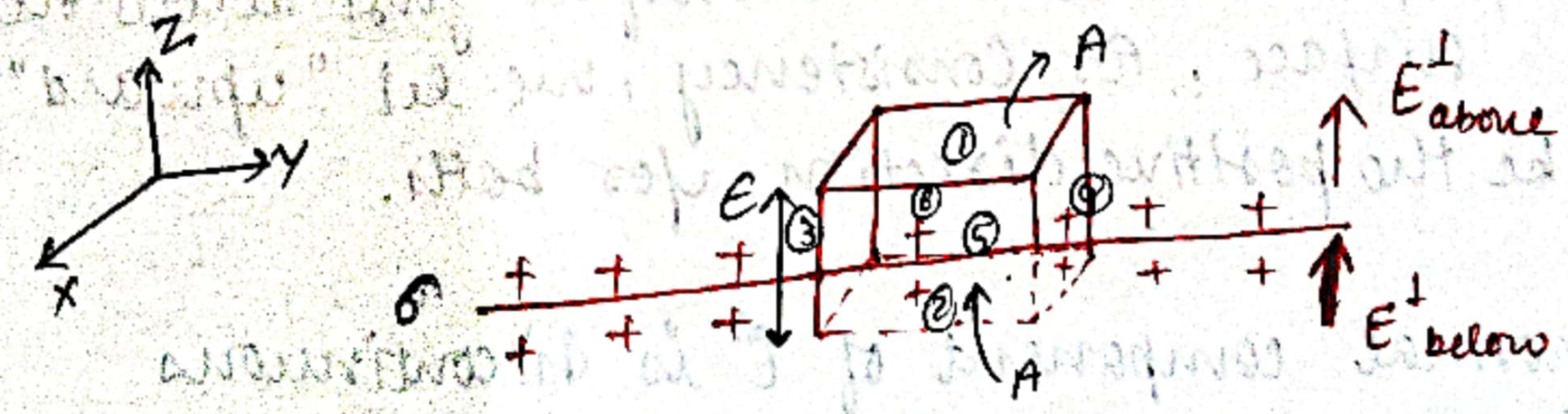
1) Boundary conditions of electric field.

Equations to find out how electric field will change when we cross any surface having surface charge density σ.



1) Boundary condition for perpendicular component of electric field -

A wafer - thin Gaussian pill box extending just barely over the edge ~~of~~ in each direction is drawn.



Gauss's law states that -

$\sigma = \frac{q}{A}$

$$\oint_S \vec{E}\cdot d\vec{s} = \frac{1}{\epsilon_0} q_{enc.}$$

$$\Rightarrow \oint_S \vec{E}\cdot\hat{n}\,ds = \frac{1}{\epsilon_0}\sigma A$$, where A is the area of pill box lid.

$$\Rightarrow \int_1 \vec{E}\cdot\hat{n}\,ds + \int_2 \vec{E}\cdot\hat{n}\,ds + \int_3 \vec{E}\cdot\hat{n}\,ds + \int_4 \vec{E}\cdot\hat{n}\,ds + \int_5 \vec{E}\cdot\hat{n}\,ds + \int_6 \vec{E}\cdot\hat{n}\,ds = \frac{\sigma A}{\epsilon_0} \quad \text{--- (1)}$$

As the thickness ϵ goes to zero, the sides of the pillbox contribute nothing to the flux.

Thus, eq^n ① can be written

$$\int_1 \vec{E}\cdot\hat{n}\,ds + \int_2 \vec{E}\cdot\hat{n}\,ds = \frac{\sigma A}{\epsilon_0}$$

$$\int_1 E^{\perp}_{above}(\hat{z})\cdot(\hat{z})\,ds + \int_2 E^{\perp}_{below}(\hat{z})\cdot(-\hat{z})\,ds = \frac{\sigma A}{\epsilon_0}$$

$$E^{\perp}_{above}\int_1 ds - E^{\perp}_{below}\int_2 ds = \frac{\sigma A}{\epsilon_0}.$$

$$E^{\perp}_{above}A - E^{\perp}_{below}A = \frac{\sigma A}{\epsilon_0}$$

$$\boxed{E^{\perp}_{above} - E^{\perp}_{below} = \frac{\sigma}{\epsilon_0}}$$

where $E^{\perp}_{above}$ is the component of $\vec{E}$ that is perpendicular to the surface immediately above and $E^{\perp}_{below}$ is the component of $\vec{E}$ that is perpendicular to the surface just below the surface. For consistency, we let "upward" be the positive direction for both.

The normal component of $\vec{E}$ is discontinuous by an amount $\frac{\sigma}{\epsilon_0}$ at any boundary.

If there is no surface charge, for instance at the surface of a uniformly charged solid sphere.

$$\sigma = 0$$

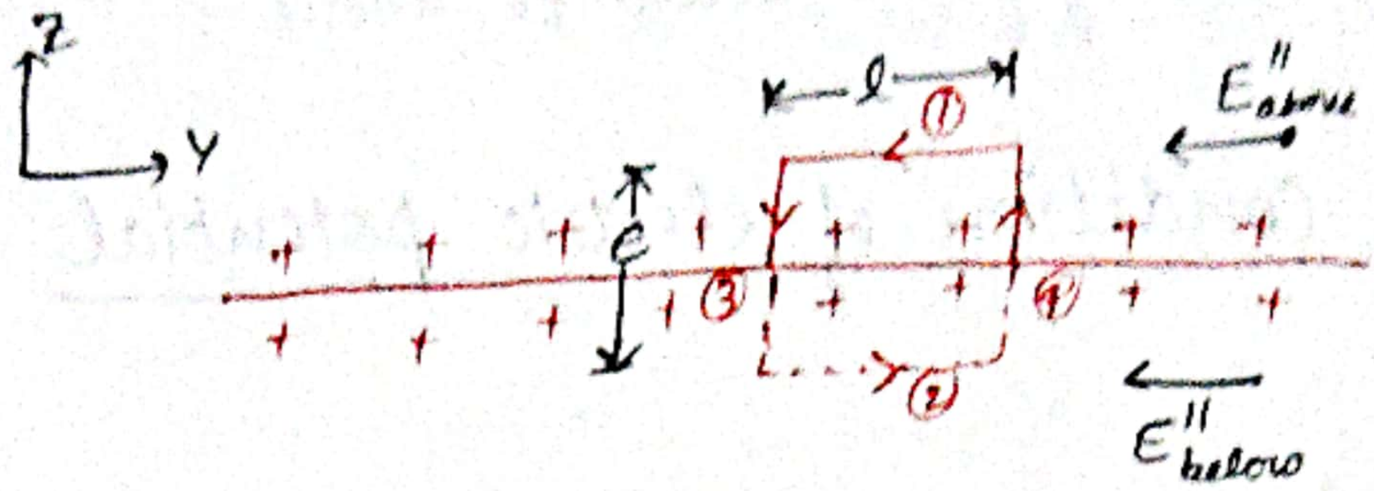
Then,

$$\boxed{E^{\perp}_{above} - E^{\perp}_{below} = 0}$$

i.e. The normal/perpendicular component of $\vec{E}$ is continuous where there is no surface charge.

(ii) Boundary condition for parallel component of electric field -

A thin rectangular loop is drawn, as shown in the figure.



we know that the line integral of $\vec{E}$ around a closed path is zero.

i.e. $\oint \vec{E}\cdot d\vec{l} = 0$

$$\Rightarrow \oint \vec{E}\cdot\hat{n}\,dl = 0$$

$$\Rightarrow \int_1 \vec{E}\cdot\hat{n}\,dl + \int_2 \vec{E}\cdot\hat{n}\,dl + \int_3 \vec{E}\cdot\hat{n}\,dl + \int_4 \vec{E}\cdot\hat{n}\,dl = 0 \quad \text{—(1)}$$

As $\epsilon \to 0$, the integration along path 3 and path 4 can be neglected.

Then, eq (1) can be written as -

$$\int_1 \vec{E}\cdot\hat{n}\,dl + \int_2 \vec{E}\cdot\hat{n}\,dl = 0$$

$$\int_1 E''_{above}(-\hat{y})\cdot(-\hat{y})\,dl + \int_2 E''_{below}(-\hat{y})\cdot\hat{y}\,dl = 0$$

$$\Rightarrow E''_{above}\int_1 dl - E''_{below}\int_2 dl = 0$$

$$\Rightarrow E''_{above}\,l - E''_{below}\,l = 0$$

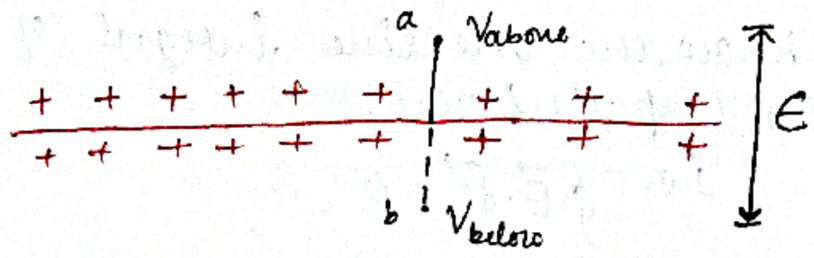
$$\Rightarrow \boxed{E''_{above} - E''_{below} = 0}$$

Thus, the parallel or tangential component of $\vec{E}$ is always continuous. The boundary conditions of $\vec{E}$ can, therefore be combined into a single formula -

$$\boxed{\vec{E}_{above} - \vec{E}_{below} = \frac{\sigma}{\epsilon_0}\hat{n}}$$

where $\hat{n}$ is a unit vector perpendicular to the surface, pointing from "below" to "above".

(2) Boundary Condition of electric potential



By the definition of ~~potential~~ potential difference,

$$V_{above} - V_{below} = -\int_b^a \vec{E}\cdot d\vec{l} \qquad \left[\because V(b) - V(a) = -\int_a^b \vec{E}\cdot d\vec{l}\right]$$

As the path length ϵ shrinks to zero, the integral becomes negligible.

$$\therefore V_{above} - V_{below} = 0$$

$$\Rightarrow V_{above} = V_{below}$$

Thus, potential is continuous across any boundary.

However, the gradient of V inherits the discontinuity in $\vec{E}$.

We know that,

$$\vec{E}_{above} - \vec{E}_{below} = \frac{\sigma}{\epsilon_0}\hat{n}$$

since,

$\vec{E} = -\vec{\nabla}V$, the above eq^n can be written as -

$$-\vec{\nabla}V_{above} - (-\vec{\nabla}V_{below}) = \frac{\sigma}{\epsilon_0}\hat{n}$$

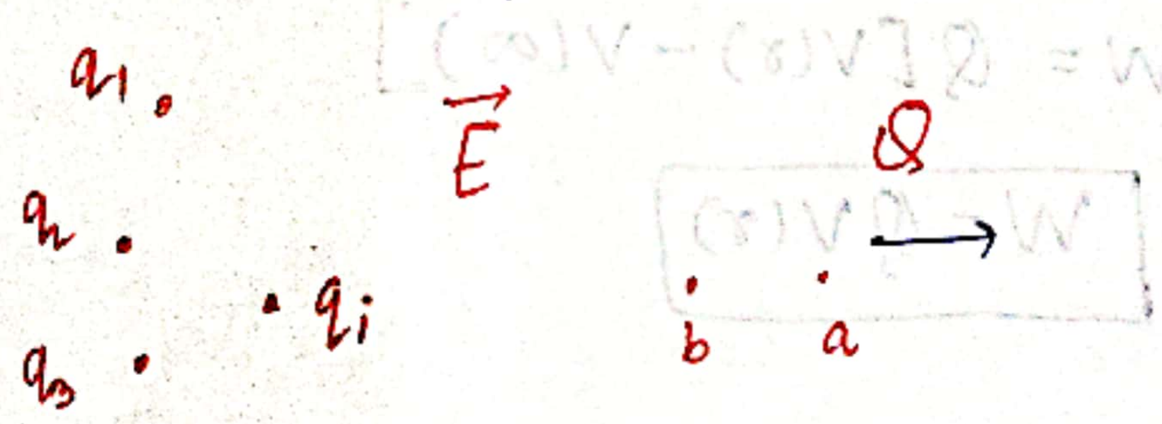
$$\Rightarrow \vec{\nabla}V_{above} - \vec{\nabla}V_{below} = -\frac{\sigma}{\epsilon_0}\hat{n}$$

$$\Rightarrow \frac{\partial V_{above}}{\partial n} - \frac{\partial V_{below}}{\partial n} = -\frac{\sigma}{\epsilon_0}$$

where $\frac{\partial V}{\partial n} = \vec{\nabla}V \cdot \hat{n}$, denotes the normal derivative of V ie, the rate of change of V in the direction perpendicular to the surface.

WORK AND ENERGY

Consider a collection of source charges $q_1, q_2, q_3, \ldots, q_i$ having electric field $\vec{E}$. And we want to find work done in moving a test charge Q from point 'a' to point 'b', as shown in the following figure.



$$\vec{F}_e = Q\vec{E}$$

$$\vec{F}_{ext} = -\vec{F}_e$$

since the test charge is placed in electric field $\vec{E}$, force on 'Q' due to collection of the source charges is -

$$\vec{F}_e = Q\vec{E}$$

Then, the minimum amount of external force to move the test charge from point 'a' to ~~point~~ point 'b' is given by -

$$\vec{F}_{ext} = -Q\vec{E}$$

∴ The work done by the external force to move the test charge from point 'a' to point 'b' is -

$$W = \int_a^b \vec{F}_{ext} \cdot d\vec{l} = -\int_a^b Q\vec{E} \cdot d\vec{l}$$

$$= -Q \int_a^b \vec{E} \cdot d\vec{l}$$

$\left[\because (V(b) - V(a)) = -\int_a^b \vec{E} \cdot d\vec{l}\right]$

$$\boxed{W = Q\,[V(b) - V(a)]}$$

(or)

$$\boxed{V(b) - V(a) = \frac{W}{Q}}$$

i.e., The potential difference b/w points 'a' and 'b' is equal to the work per unit charge required to carry a charge from point 'a' to 'b'.

If we want to bring the test charge 'Q' from a reference point at infinity to point r, the work done is given by -

$$W = Q\,[V(r) - V(\infty)]$$

$$\boxed{W = Q\,V(r)}$$

Energy of a point charge distribution

We know that the work done to bring a charge from infinity to a point 'r' is given by -

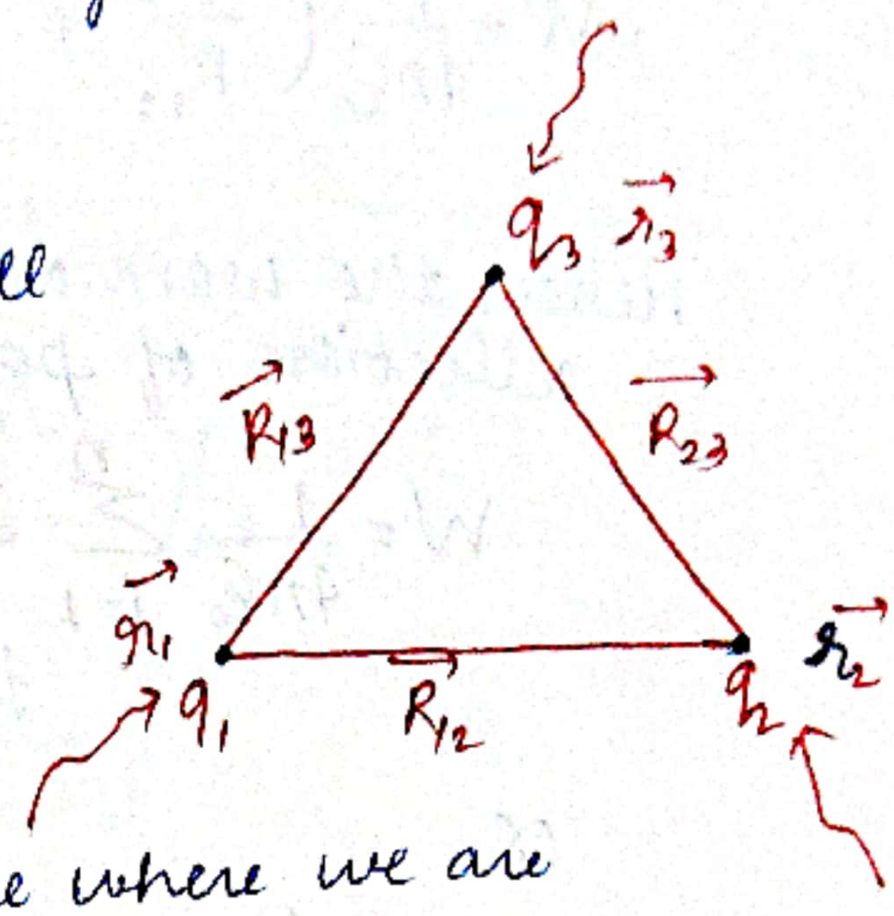
$$W = QV(r)$$

In order to find workdone to assemble an entire collection of point charges, imagine bringing in the charges, one by one, from infinity.

The first charge, q_1, takes no work, since there is no field yet to fight against.

i.e,

$$W_1 = 0$$



Now, bring in q_2. This will cost $q_2 V_1(r_2)$.

i.e,

$$W_2 = q_2 V_1(r_2)$$

where V_1 is the potential due to q_1 and r_2 is the place where we are putting q_2.

$$\Rightarrow W_2 = q_2 \times \frac{1}{4\pi\epsilon_0} \frac{q_1}{R_{12}}$$

where R_{12} is the distance between q_1 and q_2 once they are in position.

Now, bring in q_3. This requires work $q_3 V_{1,2}(r_3)$, where $V_{1,2}$ is the potential due to charges q_1 and q_2.

Thus,

$$W_3 = q_3 V_{1,2}(r_3)$$

$$= q_3 \left[\frac{1}{4\pi\epsilon_0} \frac{q_1}{R_{13}} + \frac{1}{4\pi\epsilon_0} \frac{q_2}{R_{23}} \right]$$

Similarly, the extra work to bring in q_4 will be.

$$W_4 = q_4 \left[\frac{1}{4\pi\epsilon_0} \frac{q_1}{R_{14}} + \frac{1}{4\pi\epsilon_0} \frac{q_2}{R_{24}} + \frac{1}{4\pi\epsilon_0} \frac{q_3}{R_{34}} \right]$$

$\therefore$ The total work necessary to assemble the first four charges is -

$$W = W_1 + W_2 + W_3 + W_4$$

$$W = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1 q_2}{R_{12}} + \frac{q_1 q_3}{R_{13}} + \frac{q_2 q_3}{R_{23}} + \frac{q_1 q_4}{R_{14}} + \frac{q_2 q_4}{R_{24}} + \frac{q_3 q_4}{R_{34}}\right)$$

Hence; the work done to assemble an entire collection of point charges can be written as-

$$W = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j>i}}^{n} \frac{q_i q_j}{R_{ij}}$$

or

$$W = \frac{1}{2} \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{q_i q_j}{R_{ij}}$$

$$W = \frac{1}{2} \sum_{i=1}^{n} q_i \left(\frac{1}{4\pi\epsilon_0} \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{q_j}{R_{ij}} \right)$$

$$\Rightarrow \boxed{W = \frac{1}{2} \sum_{i=1}^{n} q_i V(r_i)}$$

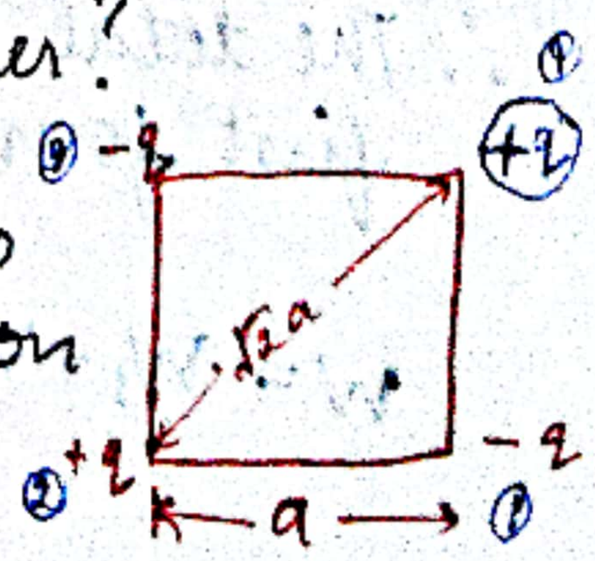
where, $V(r_i)$ is the potential at r_i (the position of q_i) due to all the other charges.

This is the required amount of work done to assemble a configuration of point charges.

This is also amount of work we would get back if we dismantled the system.

Ques- i) Three charges are situated at the corners of a square of side 'a', as shown in the figure. How much work does it take to bring in another charge, +q, from far away and place it in the fourth corner?

(ii) How much work does it take to assemble the whole configuration of four charges?



(i) $$W_4 = q_4\left[\frac{1}{4\pi\varepsilon_0}\frac{q_1}{R_{14}} + \frac{1}{4\pi\varepsilon_0}\frac{q_2}{R_{24}} + \frac{1}{4\pi\varepsilon_0}\frac{q_3}{R_{34}}\right]$$

$$W_4 = q\left[\frac{1}{4\pi\varepsilon_0}\frac{(-q)}{a} + \frac{1}{4\pi\varepsilon_0}\times\frac{q}{\sqrt{2}a} + \frac{1}{4\pi\varepsilon_0}\frac{(-q)}{a}\right]$$

$$W_4 = \frac{1}{4\pi\varepsilon_0}\times q^2\left[-\frac{1}{a} + \frac{1}{\sqrt{2}a} - \frac{1}{a}\right]$$

$$W_4 = \frac{q^2}{4\pi\varepsilon_0}\left[-\frac{2}{a} + \frac{1}{\sqrt{2}a}\right]$$

$$W_4 = \frac{q^2}{4\pi\varepsilon_0}\left[\frac{-2\sqrt{2}+1}{\sqrt{2}a}\right]$$

$$W_4 = \frac{q^2}{4\pi\varepsilon_0}\left[\frac{1-2\sqrt{2}}{\sqrt{2}a}\right]$$

$$\boxed{W_4 = \frac{1}{\sqrt{2}}\frac{q^2}{4\pi\varepsilon_0 a}\left[1-\sqrt{2}\right]}$$

Ans

(ii) Total work

$$W = W_1 + W_2 + W_3 + W_4$$

$$= \frac{1}{4\pi\varepsilon_0}\left(\frac{-q^2}{a} + \frac{+q^2}{\sqrt{2}a} + \frac{(-q^2)}{a} + \frac{(-q)^2}{a} + \frac{q^2}{\sqrt{2}a} + \frac{(-q^2)}{a}\right)$$

$$= \frac{K}{a}q^2\left(-1 + \frac{1}{\sqrt{2}} - 1 - 1 + \frac{1}{\sqrt{2}} - 1\right)$$

$$= \frac{K}{a}q^2\left(\sqrt{2} - 4\right)$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q^2}{a}\left(\sqrt{2}-4\right)$$

$$\boxed{W = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{a}\left(-4+\sqrt{2}\right)}$$

Ans

Energy of a continuous charge distribution -

we know that the work done for a ~~contin~~ point charge distribution is -

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i V(r_i)$$

Then, for a continuous charge distribution of volume charge density ρ, we can write -

$$W = \frac{1}{2}\int_{\tau} \rho V \, d\tau \quad \text{—} (1)$$

But

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$$

$$\Rightarrow \rho = \epsilon_0 (\vec{\nabla}\cdot\vec{E}) \quad \text{—} (2)$$

using eq. (2) in (1) we get -

$$W = \frac{\epsilon_0}{2}\int_{\tau} (\vec{\nabla}\cdot\vec{E}) V \, d\tau$$

$$= \frac{\epsilon_0}{2}\int_{\tau} [\vec{\nabla}\cdot(V\vec{E}) - \vec{E}\cdot\vec{\nabla}V] \, d\tau$$

$[\because \vec{\nabla}\cdot(V\vec{E}) = V(\vec{\nabla}\cdot\vec{E}) + \vec{E}\cdot\vec{\nabla}V]$

$\vec{E} = -\vec{\nabla}V$

from gauss's divergence theorem -

$$\oint_S \vec{F}\cdot d\vec{s} = \int_V (\vec{\nabla}\cdot\vec{F}) \, dV$$

$$= \frac{\epsilon_0}{2}\left[\int_{\tau} \vec{\nabla}\cdot(V\vec{E}) \, d\tau - \int_{\tau} \vec{E}\cdot(\vec{\nabla}V) \, d\tau\right]$$

$$= \frac{\epsilon_0}{2}\left[\oint_S V\vec{E}\cdot d\vec{s} - \int_{\tau} \vec{E}\cdot(-\vec{E}) \, d\tau\right]$$ [using gauss's divergence theorem and $\vec{E} = -\vec{\nabla}V$]

$$= \frac{\epsilon_0}{2}\left[\int_{\tau} E^2 \, d\tau + \oint_S V\vec{E}\cdot d\vec{s}\right]$$

If we take larger and larger volumes, the contribution from the volume integral goes up and that of the surface integral goes down in order to leave the sum intact.

$$\therefore \boxed{W = \frac{\varepsilon_0}{2}\int_{\text{all space}(\tau)} \vec{E}^2 d\tau} \Rightarrow \boxed{W = \frac{\varepsilon_0}{2}\int_{\tau} E^2 d\tau}$$

<u>Ques</u>- Find the energy of a uniformly charged spherical shell of total charge q and radius R.

$$W = \frac{\varepsilon_0}{2}\int_{\text{all space}} \vec{E}^2 d\tau.$$

$$\cancel{W = \frac{\varepsilon_0}{2}\int_0^R \frac{q}{4\pi\varepsilon_0 R^2} dR}$$

$$\cancel{W = \frac{\varepsilon_0}{2} \times \frac{q}{4\pi\varepsilon_0}\int_0^R \frac{1}{R^2} dR}$$

$$\cancel{W = \frac{q}{8\pi}\left[\frac{R^{-1}}{-1}\right]_0^R}$$

$$\cancel{W = \frac{q}{8\pi}\left[-\frac{1}{R}\right]} = \cancel{\frac{q}{8\pi R}}$$

In spherical cell-

Inside the sphere, $\vec{E} = 0$

Outside the sphere, $\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$

Then, $E^2 = \frac{1}{(4\pi\varepsilon_0)^2}\frac{q^2}{r^4}$

$\therefore$ The required energy $= \frac{\varepsilon_0}{2}\int_{\tau} E^2 d\tau$

[$\because$ inside is zero so don't req.]

$$= \frac{\varepsilon_0}{2}\frac{q^2}{(4\pi\varepsilon_0)^2}\int_{\text{outside}} \frac{1}{r^4}(r^2 \sin\theta\, dr\, d\theta\, d\phi)$$

$\phi \rightarrow 0$ to 2π

$\theta \rightarrow 0$ to π

$r \rightarrow R$ to ∞ (b/c 0 to $R = 0$)

$$= \frac{q^2}{2\varepsilon_0(4\pi)^2}\int_R^\infty \frac{dr}{r^2}\int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi$$

$$= \frac{q^2}{32\pi^2\varepsilon_0}\int_R^\infty \frac{dr}{r^2}\int_0^\pi \sin\theta\, d\theta\,[2\pi - 0]$$

$$= \frac{q^2 \times 2\pi}{32\pi^2\varepsilon_0}\int_R^\infty \frac{dr}{r^2}\int_0^\pi \sin\theta\, d\theta$$

$$= \frac{q^2}{16\pi\varepsilon_0}\int_R^\infty \frac{dr}{r^2}\,[-\cos\theta]_0^\pi$$

$$= \frac{q^2}{16\pi\varepsilon_0}\int_R^\infty \frac{dr}{r^2}\,[-\cos\pi + \cos 0]$$

$$\Rightarrow \frac{q^2}{16\pi\varepsilon_0}\int_R^\infty \frac{dr}{r^2}\,[1+1]$$

$$= \frac{2q^2}{16\pi\varepsilon_0}\int_R^\infty \frac{dr}{r^2}$$

$$= \frac{q^2}{8\pi\varepsilon_0}\left[-\frac{1}{r}\right]_R^\infty$$

$$= \frac{q^2}{8\pi\varepsilon_0}\left[-\cancel{\frac{1}{\infty}}^{\,0} + \frac{1}{R}\right]$$

$\therefore$ The required energy $= \dfrac{q^2}{8\pi\varepsilon_0 R}$

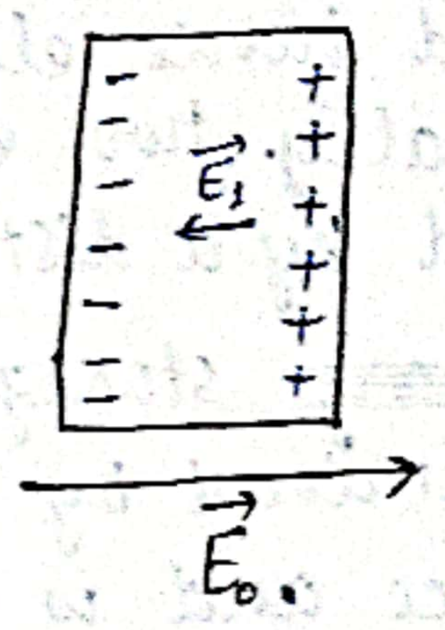
Conductors

In a metallic conductor, one or more electrons per atom are free to roam about at will through the material. And in liquid conductors such as salt water, it is ions that do the moving. In fact, a perfect conductor is a material containing an unlimited supply of completely free charges.

The basic electrostatic properties of ideal conductors are -

i) $\vec{E} = 0$ inside a conductor.

When we put a conductor into an external electric field $\vec{E}_0$, this will drive any free positive charges to the right, and negative ones to the left. When they come to the edge of the material, the charges pile up such that + on the right side and − on the left. Now these induced charges produce a field of their own, $\vec{E}_1$, which is in the opposite direction to $\vec{E}_0$. This means that the field of the induced charges tends to cancel off the original field. Charge will continue to flow until this cancellation is complete, and the resultant field inside the conductor is precisely zero. The whole process practically instantaneous.



ii) $\rho = 0$ inside a conductor.

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$$

we know that,

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$$

since $\vec{E} = 0$ inside conductor,

$$\rho = 0$$

There is still charge around, but exactly as much plus charge as minus, so the net charge density in the interior is zero.

iii) A conductor is equipotential -

If 'a' and 'b' are any two points within a given conductor,

$$V(b) - V(a) = -\int_a^b \vec{E} \cdot d\vec{l}$$

But $\vec{E} = 0$

$$\therefore V(b) - V(a) = 0$$

$$\Rightarrow \boxed{V(b) = V(a)}$$

Capacitors -

A capacitor is a device used to store electrical charge and hence electrical energy. Capacitors are generally two electrical conductors separated by a distance.

The amount of ~~charge~~ storage of charge in a capacitor is determined by a property called capacitance and is defined as the ratio of the maximum charge Q that can be ~~sent~~ stored in a capacitor to the applied voltage V across the capacitor plates.

i.e.,

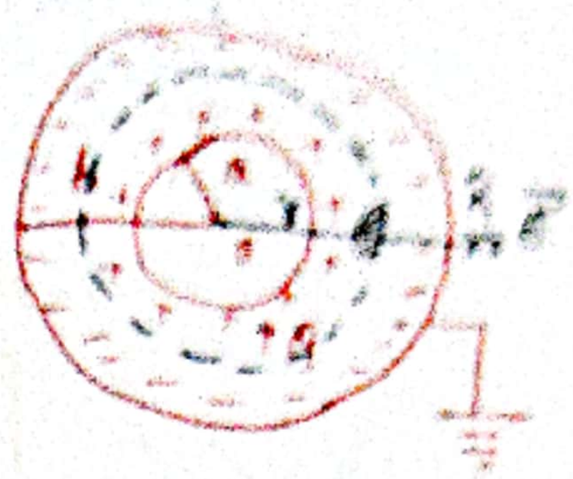
$$C = \frac{Q}{V}$$

The SI unit of capacitance is farad (F). Thus, 1F is defined as the ability of capacitor to store 1C of charge when the potential difference between its plates is only 1V.

Capacitance of spherical capacitor.

A spherical capacitor consists of two concentric spherical shells of radii 'a' and 'b'.

A Gaussian surface of radius 'r' concentric with the two spherical shells is drawn.



According to Gauss's law,

$$\oint \vec{E}\cdot d\vec{s} = \frac{1}{\epsilon_0} \times Q$$

$$\Rightarrow \oint E\,ds\cos 0^\circ = \frac{Q}{\epsilon_0}$$

$$\Rightarrow E\oint ds = \frac{Q}{\epsilon_0}$$

$$\Rightarrow E \times 4\pi r^2 = \frac{Q}{\epsilon_0}$$

$$\Rightarrow E = \frac{1}{4\pi\epsilon_0}\cdot\frac{Q}{r^2}$$

Then, the potential difference between the two spherical shells is –

$$V = -\int_b^a \vec{E}\cdot d\vec{l}$$

$$V = -\frac{Q}{4\pi\epsilon_0}\int_b^a \frac{1}{r^2}\,dr \qquad \left[\because d\vec{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\right]$$

$$V = -\frac{Q}{4\pi\epsilon_0}\left[\frac{-1}{r}\right]_b^a$$

$$V = \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{a} - \frac{1}{b}\right]$$

$$V = \frac{Q(b-a)}{4\pi\epsilon_0 ab}$$

$\therefore$ capacitance,

$$C = \frac{Q}{V} = \frac{Q}{\frac{Q(b-a)}{4\pi\epsilon_0 ab}} = 4\pi\epsilon_0\frac{ab}{(b-a)}$$

$$\boxed{C = 4\pi\epsilon_0\frac{ab}{(b-a)}}$$

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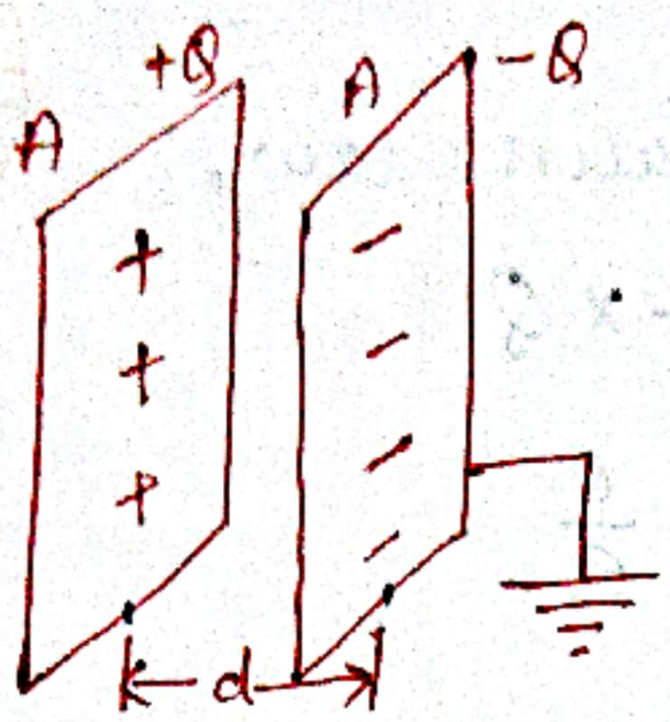
Capacitance of Parallel plate capacitor -

A parallel plate capacitor consists of two thin conducting plates, each of area 'A' held parallel to each other at a distance 'd' such that $d << A$.



The magnitude of electric field intensity at any point between two plates is given by -

$$E = \frac{\sigma}{\epsilon_0} \quad [\text{using Gauss's Law}]$$

$$\left[\sigma = \frac{Q}{A}\right] \Rightarrow E = \frac{Q}{\epsilon_0 A}$$

Then, the potential difference between the plates is -

$$V = E \times d$$

$$\Rightarrow V = \frac{Qd}{\epsilon_0 A}$$

$\therefore$ capacitance, $C = \frac{Q}{V} = \frac{Q}{\frac{Qd}{\epsilon_0 A}}$

$$\Rightarrow \boxed{C = \frac{\epsilon_0 A}{d}}$$

i.e, $C \propto A$

$C \propto \frac{1}{d}$

Energy stored in capacitor -

In order to find work done to charge the capacitor up to a final amount Q, let us assume that at some intermediate stage in the process, the charge on the positive plate is q.

Then,

Potential difference, $V = \frac{q}{C}$ $\quad [\because C = \frac{q}{V}]$

$\therefore$ The work done to transport the next piece of charge dq is -

$$dW = \left(\frac{q}{C}\right)dq \qquad W = QV$$

Hence, the total work necessary to go from q=0 to q=Q is -

$$W = \int_0^Q \left(\frac{q}{C}\right)dq = \frac{Q^2}{2C} \Rightarrow \boxed{W = \frac{Q^2}{2C}}$$

or $W = \frac{(CV)^2}{2C}$

$$\boxed{W = \frac{1}{2}CV^2}$$

where V is the final potential of the capacitor.

Ques - A parallel plate capacitor having plate area 200 cm^2 and separation between the plates is 5mm. If it is charged by a battery of 100 V, calculate the energy stored in the capacitor.

$A = 200\ cm^2 = 2\times10^2\times(10^{-4})\ m^2 = 2\times10^{-2}\ m^2$

$d = 5mm = 0.5\ cm = 0.005\ m$

$V = 100\ V$

$\epsilon_0 = 0.05\times10^{-12}\ Fm^{-1}$

$$C = \frac{\epsilon_0 A}{d} = \frac{0.05\times10^{-12}\times2\times10^{-2}}{0.005}$$

$$= \frac{0.05\times10^{-14}\times10^{3}\times2}{5}$$

$$C = 0.05\times10^{-14}\times400\ F$$

$$W = \frac{1}{2} C V^2$$

$$W = \frac{1}{2} \times 400 \times 0.85 \times 10^{-14} \times 10^{4}$$

$$W = 200 \times 0.85 \times 10^{-10}$$

$$W = 2 \times 0.85 \times 10^{-8}$$

$$W = 17.70 \times 10^{-8}$$

$$W = 1.7 \times 10^{-7} \text{ Joule}$$

$$\begin{array}{r} 0.85 \\ \times 2 \\ \hline 17.70 \end{array}$$

Laplace's Equation -

$$\nabla^2 V = 0$$

In cartesian coordinates -

$$\underbrace{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)}_{\text{Lapalacian operator}} V = 0$$

$$\Rightarrow \frac{\partial^2}{\partial x^2} V + \frac{\partial^2}{\partial y^2} V + \frac{\partial^2}{\partial z^2} V = 0 \quad \text{—} (1)$$

Let $V(x, y, z) = X(x)\, Y(y)\, Z(z)$ —(2)

using eq. (2) in (1), we get -

$$\frac{\partial^2}{\partial x^2}(XYZ) + \frac{\partial^2}{\partial y^2}(XYZ) + \frac{\partial^2}{\partial z^2}(XYZ) = 0$$

$$\Rightarrow YZ \frac{\partial^2 X}{\partial x^2} + XZ \frac{\partial^2 Y}{\partial y^2} + XY \frac{\partial^2 Z}{\partial z^2} = 0$$

→ divide by XYZ both sides -

$$\underbrace{\frac{1}{X} \frac{\partial^2 X}{\partial x^2}}_{F_1} + \underbrace{\frac{1}{Y} \frac{\partial^2 Y}{\partial y^2}}_{F_2} + \underbrace{\frac{1}{Z} \frac{\partial^2 Z}{\partial z^2}}_{F_3} = 0 \quad \text{—} (3)$$

Eq^n (3) is of the type -

$$F_1(x) + F_2(y) + F_3(z) = 0$$

This will be true only if $F_1(x)$, $F_2(y)$ and $F_3(z)$ are constants.

Let the constants are $-m^2, -n^2$ and p^2, respectively such that -

$$-m^2 - n^2 + p^2 = 0$$

$$\Rightarrow p = \sqrt{m^2+n^2} \quad \text{—} (4)$$

Now, we can write -

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -m^2 \quad \text{—} (5)$$

$$\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -n^2 \quad \text{—} (6)$$

$$\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = p^2 \quad \text{—} (7)$$

From eq^n (5) -

$$\frac{d^2X}{dx^2} + m^2 X = 0 \quad \text{—} (8)$$

The general solution of eq^n (8) is -

$$X(x) = C_1 \sin mx + C_2 \cos mx$$

Again, from eq^n (6) -

$$\frac{d^2Y}{dy^2} + n^2 Y = 0$$

$$\therefore Y(y) = C_3 \sin ny + C_4 \cos ny$$

And, from eq^n (7) -

$$\frac{d^2Z}{dz^2} - p^2 Z = 0$$

$$\therefore Z(z) = C_5 e^{pz} + C_6 e^{-pz}$$

Hence,

$$V(x,y,z) = X(x)\,Y(y)\,Z(z)$$

$$= (C_1 \sin mx + C_2 \cos mx)(C_3 \sin ny + C_4 \cos ny)(C_5 e^{pz} + C_6 e^{-pz})$$

$[\because p = \sqrt{m^2+n^2}]$

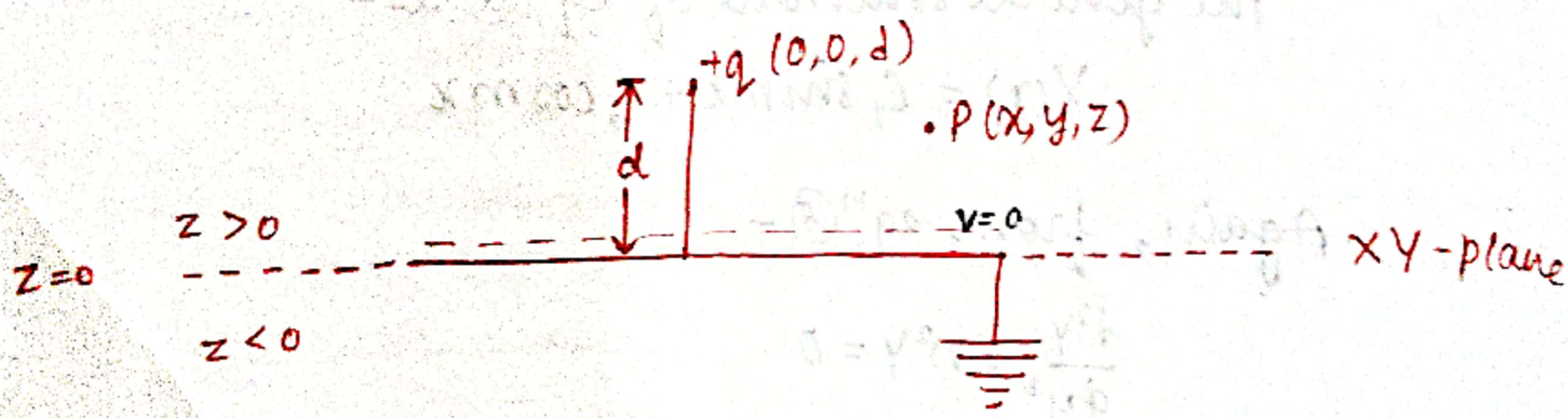
This is the solution of Laplace's equation in Cartesian coordinates.

Method of Images.

Suppose a point charge q is held a distance 'd' above an infinite grounded conducting plane.

The potential in the region above the plane is not just $\frac{1}{4\pi\varepsilon_0}\frac{q}{r}$, for q will induce a certain amount of negative charge on the nearby surface of the conductor. The total potential is due in part to q directly, and in part to this induced charge.

Such potential in which we don't know how much charge is induced or how it is distributed can be calculated by using the method of Images.



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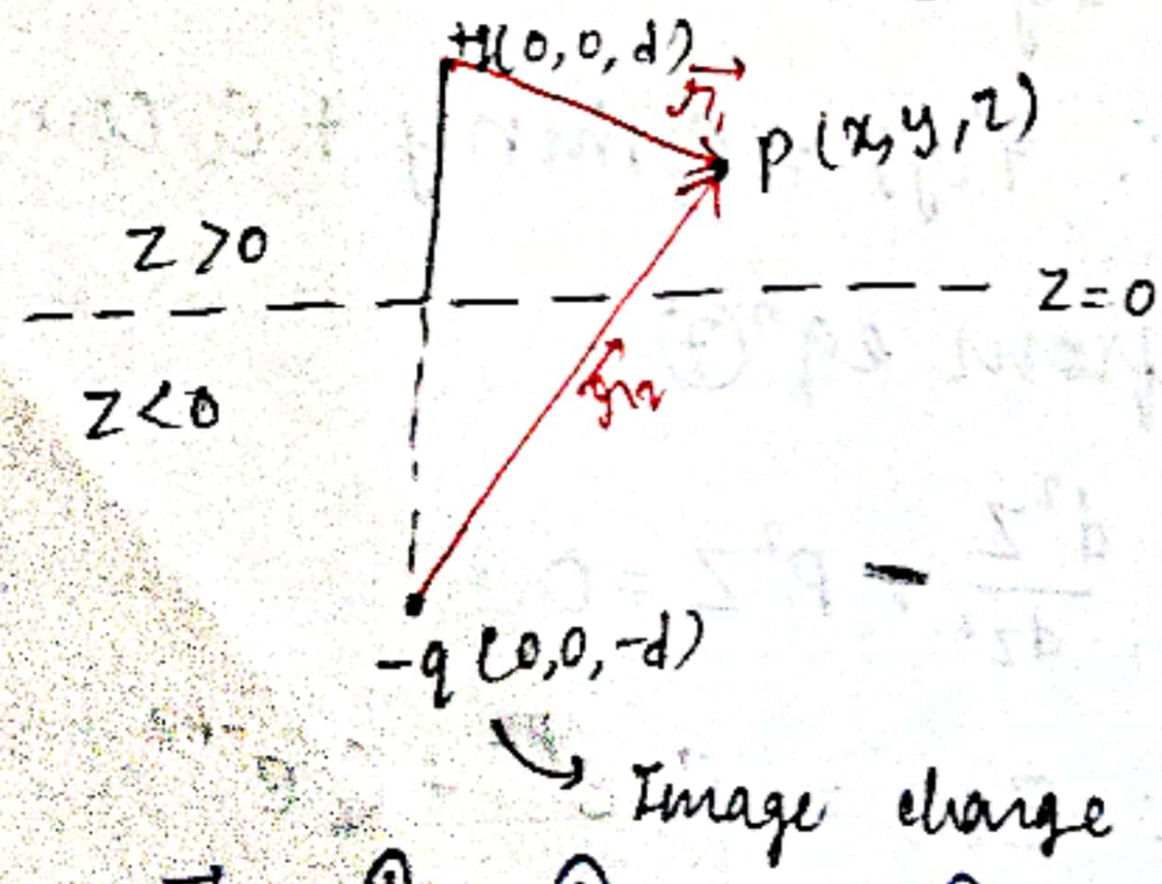


$$\vec{r}_1 = x\hat{i} + y\hat{j} + (z-d)\hat{k}$$

$$\vec{r}_2 = x\hat{i} + y\hat{j} + (z+d)\hat{k}$$

$$r_1 = \sqrt{x^2+y^2+(z-d)^2}$$

$$r_2 = \sqrt{x^2+y^2+(z+d)^2}$$

$\therefore$ The net potential at point P is given by -

$$V = V_1 + V_2$$

$$V = \frac{1}{4\pi\epsilon_0} \frac{(+q)}{r_1} + \frac{1}{4\pi\epsilon_0} \frac{(-q)}{r_2}$$

$$V = \frac{q}{4\pi\epsilon_0} \left[\frac{1}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+d)^2}} \right] \quad \text{--- (1)}$$

Since we know the potential, we can compute the surface charge density σ induced on the surface of conductor.

The boundary condition of $\vec{E}$ is -

$$\vec{E}_{above} - \vec{E}_{below} = \frac{\sigma}{\epsilon_0}\hat{n}$$

Because the field inside a conductor is zero, the field immediately outside is given by -

$$\vec{E} = \frac{\sigma}{\epsilon_0}\hat{n}$$

But $\vec{E} = -\vec{\nabla} V$

$$\therefore -\vec{\nabla} V = \frac{\sigma}{\epsilon_0}\hat{n}$$

$$\Rightarrow -\frac{\partial V}{\partial n} = \frac{\sigma}{\epsilon_0}$$; $\frac{\partial V}{\partial n} = \vec{\nabla} V \cdot \hat{n}$, the normal derivative of V at the surface.

$$\Rightarrow \sigma = -\epsilon_0 \frac{\partial V}{\partial n}$$

Here, the normal direction is the z-direction, so we can write -

$$\sigma = -\epsilon_0 \frac{\partial V}{\partial z}\Big|_{z=0}$$

$$\sigma = -\epsilon_0 \frac{\partial}{\partial z} \left[\frac{q}{4\pi\epsilon_0} \left\{ \frac{1}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+d)^2}} \right\} \right]_{z=0}$$

$$\sigma = -\epsilon_0 \times \frac{q}{4\pi\epsilon_0} \frac{\partial}{\partial z}\left[\frac{1}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+d)^2}}\right]_{z=0}$$

$$\sigma = -\frac{q}{4\pi}\left[-\frac{1}{2}\frac{2(z-d)}{\left(\sqrt{x^2+y^2+(z-d)^2}\right)^3} + \frac{1}{2}\frac{2(z+d)}{\left(\sqrt{x^2+y^2+(z+d)^2}\right)^3}\right]_{z=0}$$

$$\sigma = \frac{q}{8\pi}\left[\frac{2(z-d)}{\left(\sqrt{x^2+y^2+(z-d)^2}\right)^3} - \frac{2(z+d)}{\left(\sqrt{x^2+y^2+(z+d)^2}\right)^3}\right]_{z=0}$$

$$\sigma = \frac{q}{8\pi}\left[\frac{-2d}{\left(\sqrt{x^2+y^2+d^2}\right)^3} - \frac{2d}{\left(\sqrt{x^2+y^2+d^2}\right)^3}\right]$$

$$\sigma = \frac{-4dq}{8\pi}\frac{1}{\left(\sqrt{x^2+y^2+d^2}\right)^3}$$

$$\sigma = \frac{-q}{2\pi}\frac{d}{\left(\sqrt{x^2+y^2+d^2}\right)^3}$$

$$\Rightarrow \sigma(x,y) = \frac{-qd}{2\pi(x^2+y^2+d^2)^{3/2}}$$

Thus the induced charge is negative.

Now, the total induced charge is -

$$Q = \int \sigma\, dS$$

This integral, over the XY-plane, could be done in cartesian coordinates, with $dS = dx\,dy$.

But it is easier to use polar coordinates (r, θ) with $r^2 = x^2+y^2$ and $dS = r\,dr\,d\theta$

Then $\sigma(r) = \dfrac{-qd}{2\pi(r^2+d^2)^{3/2}}$

$$\therefore Q = \int_0^{\infty}\int_0^{2\pi} \frac{-qd}{2\pi(r^2+d^2)^{3/2}}\, r\,dr\,d\theta$$

$$Q = \int_0^\infty \frac{-qd}{2\pi(r^2+d^2)^{3/2}}\, r\,dr \int_0^{2\pi} d\theta$$

$$Q = \int_0^\infty \frac{-qd}{2\pi(r^2+d^2)^{3/2}}\, r\,dr\, [2\pi - 0]$$

$$Q = \frac{-qd}{\cancel{2\pi}} \times \cancel{2\pi} \int_0^\infty \frac{r}{(r^2+d^2)^{3/2}}\, dr$$

$$\cancel{Q = -qd\left[\frac{1\times(r^2+d^2)^{3/2} - r\times\frac{3}{2}(r^2+d^2)^{1/2}\times 2r}{(r^2+d^2)^3}\right]_0^\infty}$$

$$\cancel{Q = -qd\left[\frac{(r^2+d^2)^{3/2} - 3r^2(r^2+d^2)^{1/2}}{(r^2+d^2)^3}\right]_0^\infty}$$

$$\cancel{Q = -qd\left[\frac{(r^2+d^2)^{1/2}\left[r^2+d^2-3r^2\right]}{(r^2+d^2)^3}\right]_0^\infty}$$

$$\cancel{Q = -qd\left[\frac{d^2-2r^2}{(r^2+d^2)^{5/2}}\right]_0^\infty}$$

$$Q = -qd\int_0^\infty \frac{r\,dr}{(r^2+d^2)^{3/2}}$$

$$r^2+d^2 = u$$
$$2r\,dr = du$$
$$r\,dr = \frac{1}{2}du$$

$r \to 0$ $\quad$ $r \to \infty$

$0+d^2 = u$ $\quad$ $\infty + d^2 = u$

$u = d^2$ $\quad$ $u = \infty$

$$Q = -qd\int_{d^2}^\infty \frac{1}{2}\frac{du}{u^{3/2}}$$

$$Q = -\frac{qd}{2}\left[\frac{u^{-1/2}}{-1/2}\right]_{d^2}^\infty$$

$$Q = qd\left[\frac{1}{\sqrt{u}}\right]_{d^2}^\infty$$

$$Q = qd\left[\frac{1}{\infty} - \frac{1}{d^2}\right]$$

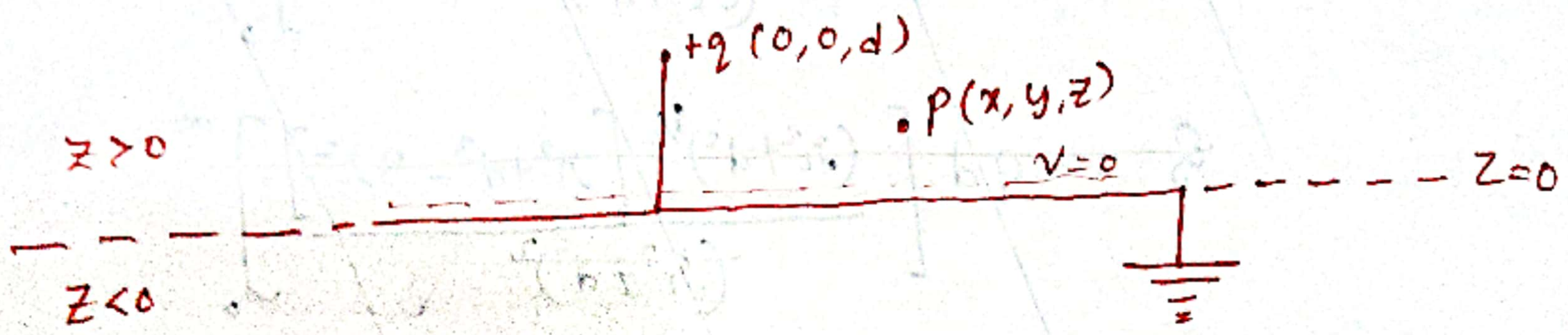
$$= qd\left[0 - \frac{1}{d}\right]$$

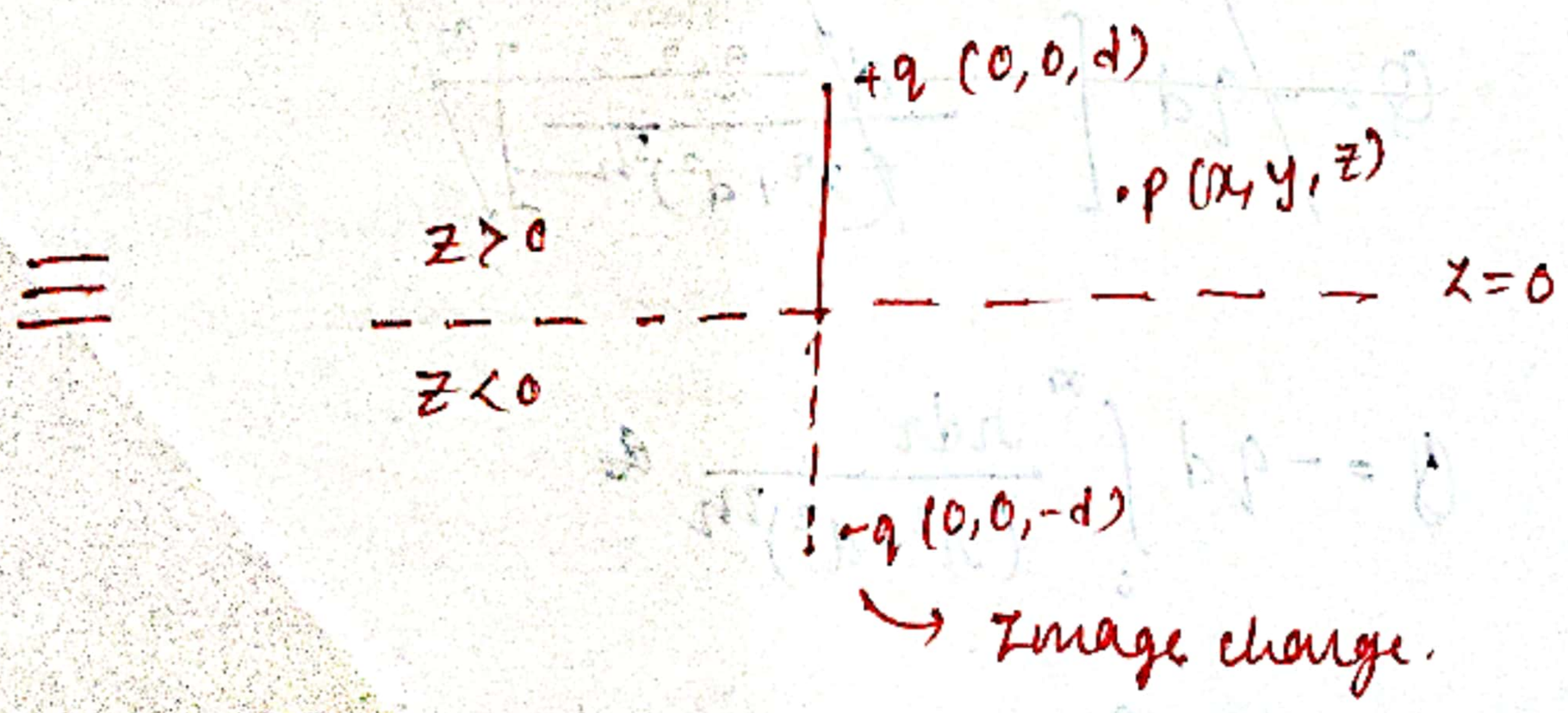
$$= qd \times -\frac{1}{d}$$

$$\boxed{Q = -q}$$

Hence, the total charge induced on the conducting plane is $-q$.

Method of Images





It is obvious that the charge $+q$ is attracted towards the plane, because of the negative induced charge. Let's calculate the force of attraction.

Since the potential in the vicinity of $+q$ is the same as in the problem with $+q$ and $-q$ but no conductor, so also is the electric field and therefore, the force.

$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{q(-q)}{(2d)^2}\hat{z}$$

$$\Rightarrow \boxed{\vec{F} = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{4d^2}\hat{z}}$$

And $\vec{E}$ at point $p = \vec{E}_1 + \vec{E}_2$

$$= \frac{1}{4\pi\epsilon_0}\frac{(+q)}{(x^2+y^2+(z-d)^2)} \cdot \frac{x\hat{i}+y\hat{j}+(z-d)\hat{k}}{\sqrt{x^2+y^2+(z-d)^2}}$$

$$+ \frac{1}{4\pi\epsilon_0}\frac{(-q)}{(x^2+y^2+(z+d)^2)} \frac{x\hat{i}+y\hat{j}+(z+d)\hat{k}}{\sqrt{x^2+y^2+(z+d)^2}}$$

$$= \frac{q}{4\pi\epsilon_0}\left[\frac{x\hat{i}+y\hat{j}+(z-d)\hat{k}}{(x^2+y^2+(z-d)^2)^{3/2}} - \frac{x\hat{i}+y\hat{j}+(z+d)\hat{k}}{(x^2+y^2+(z+d)^2)^{3/2}}\right]$$

Energy stored in $z>0$ region :
with the two point charges and no conductor,

$W = qV(r)$

$$W = q \times \frac{1}{4\pi\epsilon_0} \cdot \frac{-q}{2d}$$

But for a single charge and conducting plane, the energy is half of this.

$$\therefore \boxed{W = -\frac{1}{2}\frac{1}{4\pi\epsilon_0}\frac{q^2}{2d}}$$

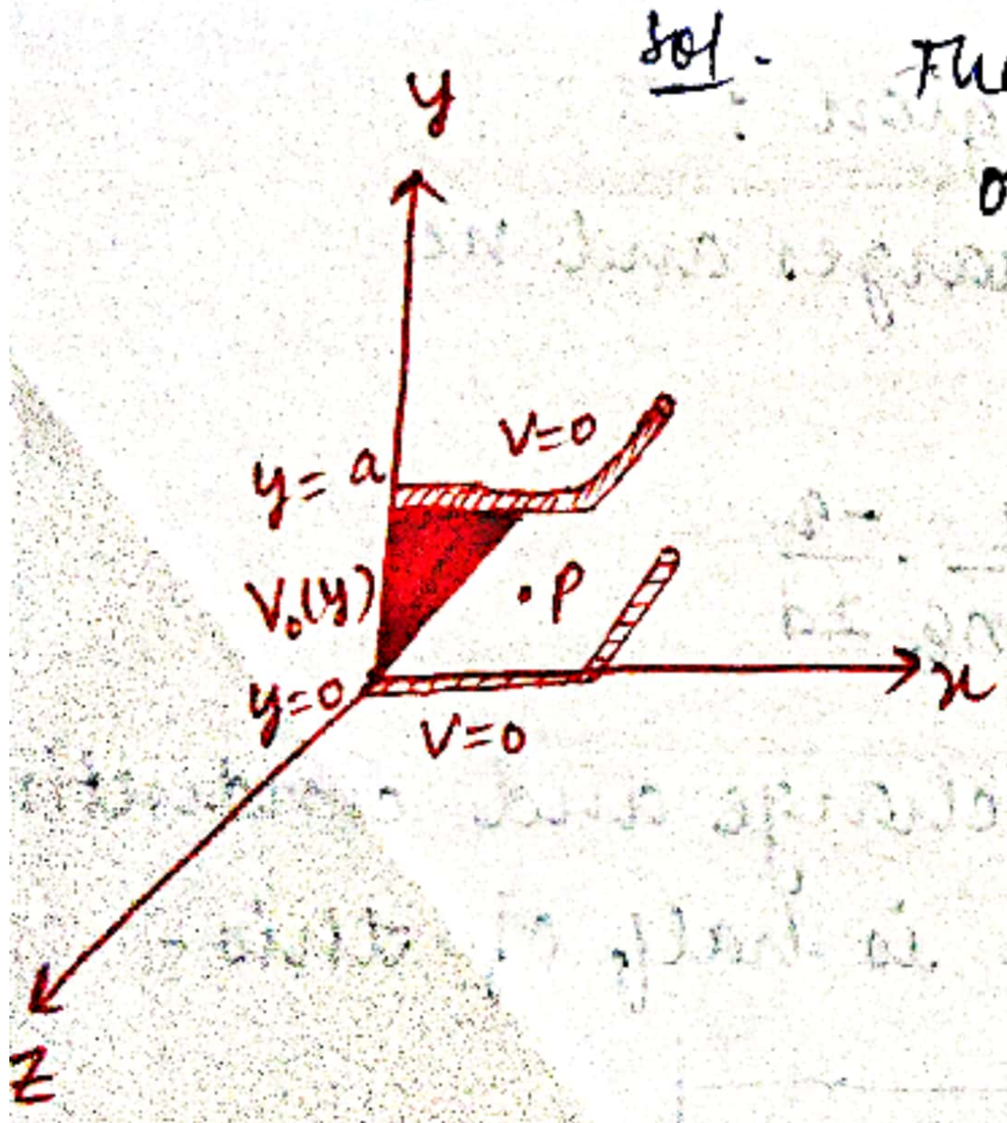
As we move charge q towards the conductor, we do work only on q. It is true that induced charge is moving in over the conductor but this cost nothing, since the whole conductor is at potential zero.

By contrast, if we simultaneously bring in

two point charges with no conductor, we do work on both of them and the total is twice as grea.

Boundary value problem in Cartesian Coordinate systems-

Ques. Two infinite grounded metal plates lie parallel to the xz-plane, one at $y=0$, the other at $y=a$. The left end, at $x=0$, is closed off with an infinite strip insulated from the two plates and maintained at a specific potential $V_0(y)$. Find the potential inside this slot.



Sol. The configuration is independent of z, so this is a two-dimensional problem.

In mathematical terms, we must solve Laplace's equation,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \quad \text{—(1)}$$

subject to the boundary conditions-

$$\left.\begin{array}{l} \text{(i) } V=0 \text{ at } y=0 \\ \text{(ii) } V=0 \text{ at } y=a \\ \text{(iii) } V=V_0(y) \text{ when } x=0 \\ \text{(iv) } V\to 0 \text{ as } x\to\infty \end{array}\right\} \text{—(2)}$$

Let $V(x,y) = X(x)\,Y(y)$ — (3)

Putting eqn (3) in (1), we obtain

$$Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} = 0$$

$$\Rightarrow \underbrace{\frac{1}{X}\frac{d^2X}{dx^2}} + \underbrace{\frac{1}{Y}\frac{d^2Y}{dy^2}} = 0$$, dividing both sides by XY — (4)

Here the first term depends only on x and the second term only on y. In other words, we have an equation of the form -

$$f(x) + g(y) = 0 \quad \text{— (5)}$$

This will be true only if f and g must both be constant.

ie, $\frac{1}{X}\frac{d^2X}{dx^2} = C_1$ and $\frac{1}{Y}\frac{d^2Y}{dy^2} = C_2$,

with $C_1 + C_2 = 0$

one of these constants is positive and the other is negative.

Thus, $\frac{1}{X}\frac{d^2X}{dx^2} = k^2 \Rightarrow \frac{d^2X}{dx^2} - k^2X = 0$

And $\frac{1}{Y}\frac{d^2Y}{dy^2} = -k^2 \Rightarrow \frac{d^2Y}{dy^2} + k^2Y = 0$

$\therefore X(x) = Ae^{kx} + Be^{-kx}$

and $Y(y) = C\sin ky + D\cos ky$

so that

$$V(x,y) = (Ae^{kx} + Be^{-kx})(C\sin ky + D\cos ky)$$

condition (iv) requires that A equal zero.

Applying the cond. D, we are left with

$$V(x,y) = e^{-kx}(C\sin ky + D\cos ky)$$

condition (i) now demands that D equal zero.

so -

$$V(x,y) = Ce^{-kx}\sin ky$$

condition (ii) yields

$$\sin ka = 0 = \sin n\pi$$

$$\Rightarrow ka = n\pi$$

$$\Rightarrow k = \frac{n\pi}{a}; n = 1, 2, 3 \ldots$$

$$\therefore V(x,y) = Ce^{-\frac{n\pi x}{a}}\sin\left(\frac{n\pi y}{a}\right)$$

A much more general solution is-

$$V(x,y) = \sum_{n=1}^{\infty} C_n e^{-\frac{n\pi x}{a}}\sin\left(\frac{n\pi y}{a}\right) \quad \text{--- (6)}$$

Applying condition (iii);

$$V(0,y) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi y}{a}\right) = V_0(y) \quad \text{--- (7)}$$

In order to determine the coefficients C_n, multiply eq^n by $\sin\left(\frac{n'\pi y}{a}\right)$ (where n' is a positive integer) and integrate from 0 to a.

$$\sum_{n=1}^{\infty} C_n \int_0^a \sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{n'\pi y}{a}\right) dy$$

$$= \int_0^a V_0(y)\sin\left(\frac{n'\pi y}{a}\right) dy$$

But $\int_0^a \sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{n'\pi y}{a}\right) dy = \begin{cases} 0, & \text{if } n' \neq n \\ a/2, & \text{if } n' = n \end{cases}$

$$\therefore C_{n'} \times \frac{a}{2} = \int_0^a V_0(y) \sin\left(\frac{n'\pi y}{a}\right) dy$$

$$\Rightarrow C_n = \frac{2}{a}\int_0^a V_0(y) \sin\left(\frac{n\pi y}{a}\right) dy \qquad n'=n$$

Suppose the strip at $x=0$ is a metal plate with constant potential V_0.

then,

$$C_n = \frac{2V_0}{a}\int_0^a \sin\left(\frac{n\pi y}{a}\right) dy$$

$$= \frac{2V_0}{n\pi}(1-\cos n\pi) \qquad \cos 2\pi = 1, \quad 1-1=0, \quad \cos\pi = -1$$

$$= \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{4V_0}{n\pi}, & \text{if } n \text{ is odd} \end{cases}$$

Hence, eq^n (6) becomes-

$$\boxed{V(x,y) = \frac{4V_0}{\pi}\sum_{n=1,3,5\ldots} \frac{1}{n} e^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right)}$$

Dielectrics -

(1) conductor
(2) semiconductor
(3) Insulator (external electric field)

Dielectrics are insulating materials in which all charges are attached to specific atoms or molecules. Thus, the charges are on a tight leash and all they can do is move a bit within the atom or molecule.

Such ~~an~~ microscopic displacements are not as dramatic as the wholesale ~~or~~ rearrangement of charge in a conductor, but there cumulative effects account for the ~~[illegible]~~ characteristic behaviour of dielectric materials.

Distinction between Dielectric and Insulator

Dielectric	Insulator
1. It can develope an internal electric field which nullifies the externally applied field.	1. It can not develope an internal electric field.
2. It can be easily polarized.	2. It can not be polarized.
3. It contains relatively weak bonds within atoms.	3. It contains strong co-valent bond.
4. All dielectrics are Insulators.	4. All Insulator are not Dielectrics.

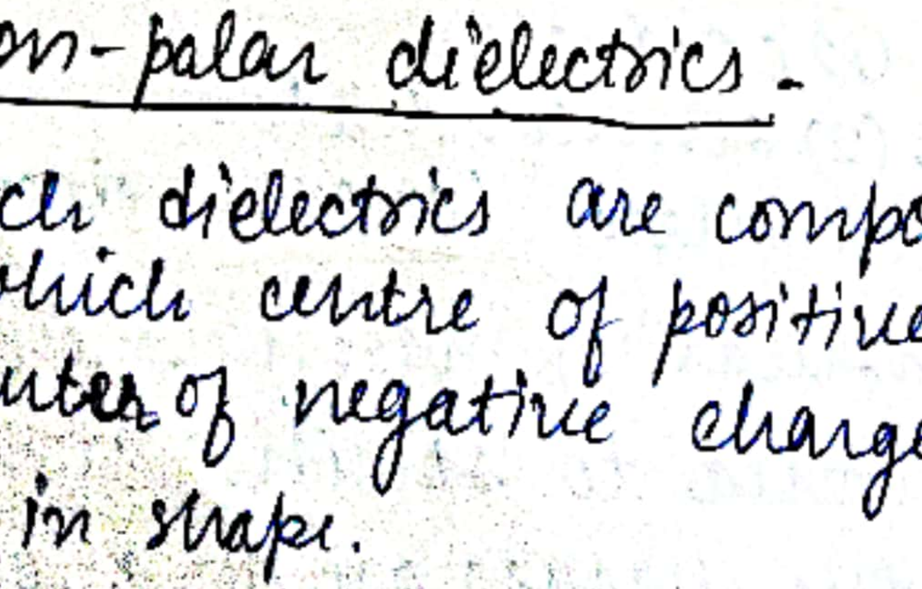
Types of Dielectrics :-

1) Non-polar dielectrics
2) Polar dielectrics

1) Non-polar dielectrics -

Such dielectrics are composed of molecules in which centre of positive charge ~~coincide~~ coincides with center of negative charge and are symmetric in shape.

eg. O_2, CO_2, CH_4 etc.



Non-polar dielectrics

In non-polar dielectrics, the dipole moment of each molecule is zero.

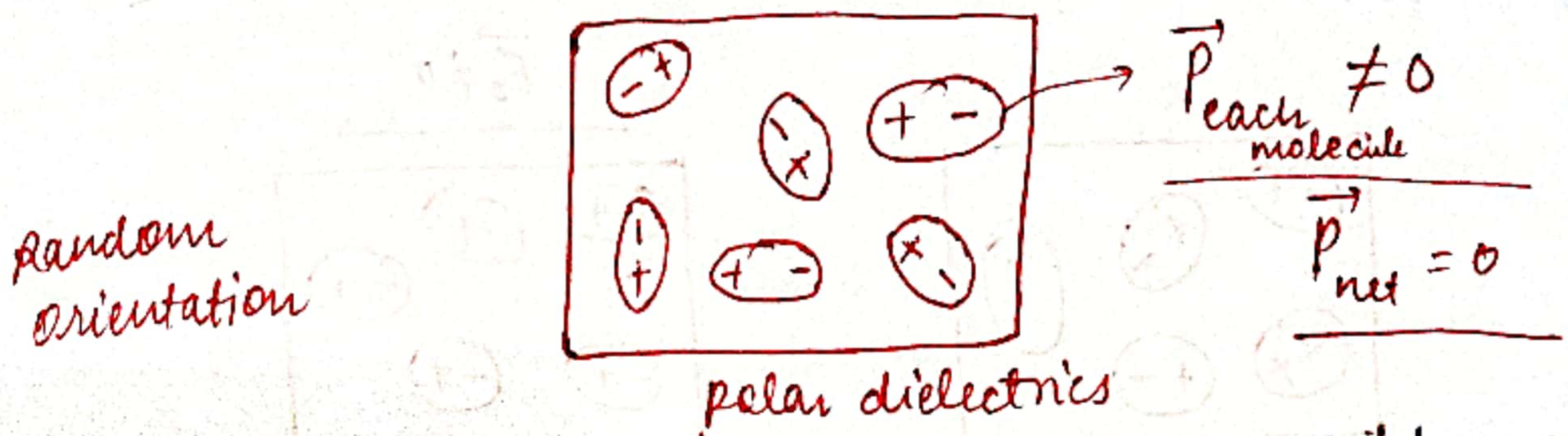
∴ Net dipole moment of non-polar dielectric,

$$\vec{p}_{net} = 0$$

2) <u>Polar dielectrics</u> -

Such dielectrics are composed of molecules in which center of +ve charge do not coincide with the center of −ve charge and are not symmetric in shape.

eg. HCl, H_2O, NH_3 etc.



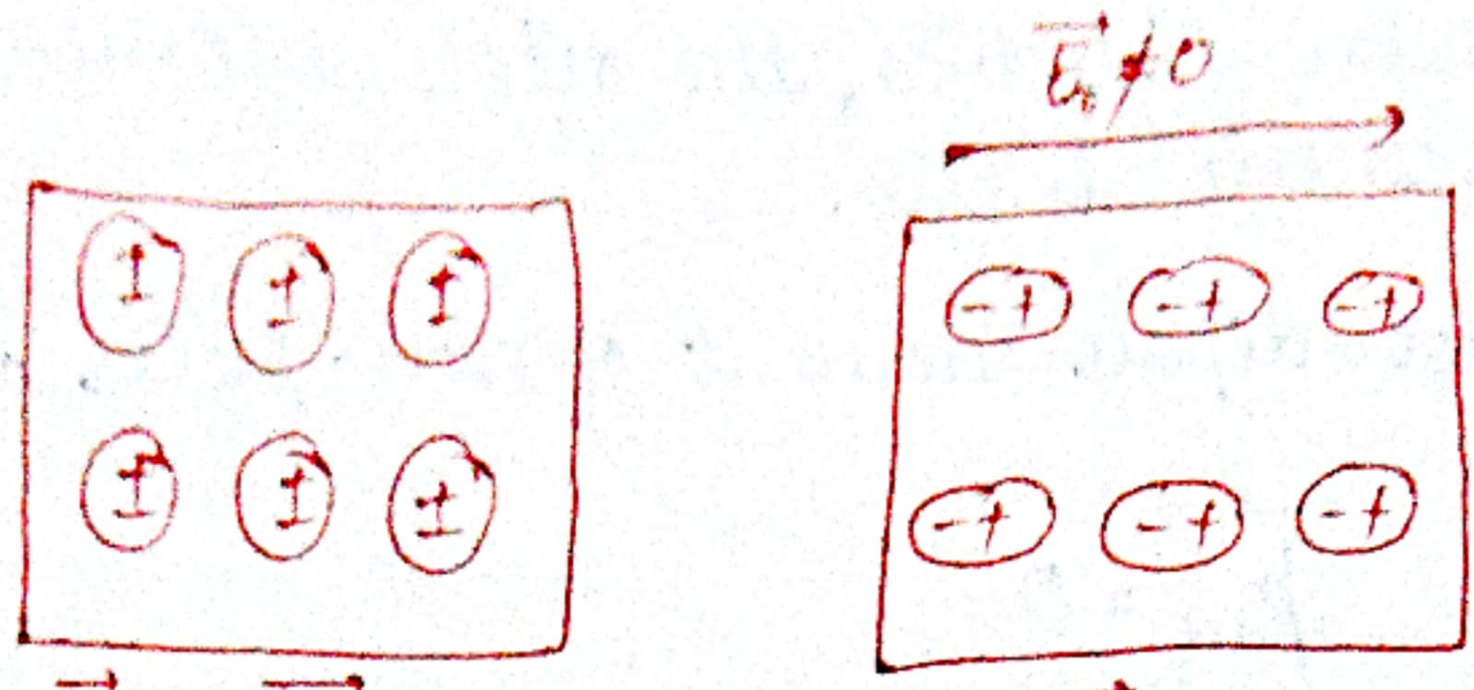
In polar dielectrics, each molecule has non-zero dipole moment. But the molecules (or dipoles) are randomly oriented due to thermal agitations thereby cancelling the dipole moments of each other.

∴ Net dipole moment of a polar dielectric,

$$\vec{p}_{net} = 0.$$

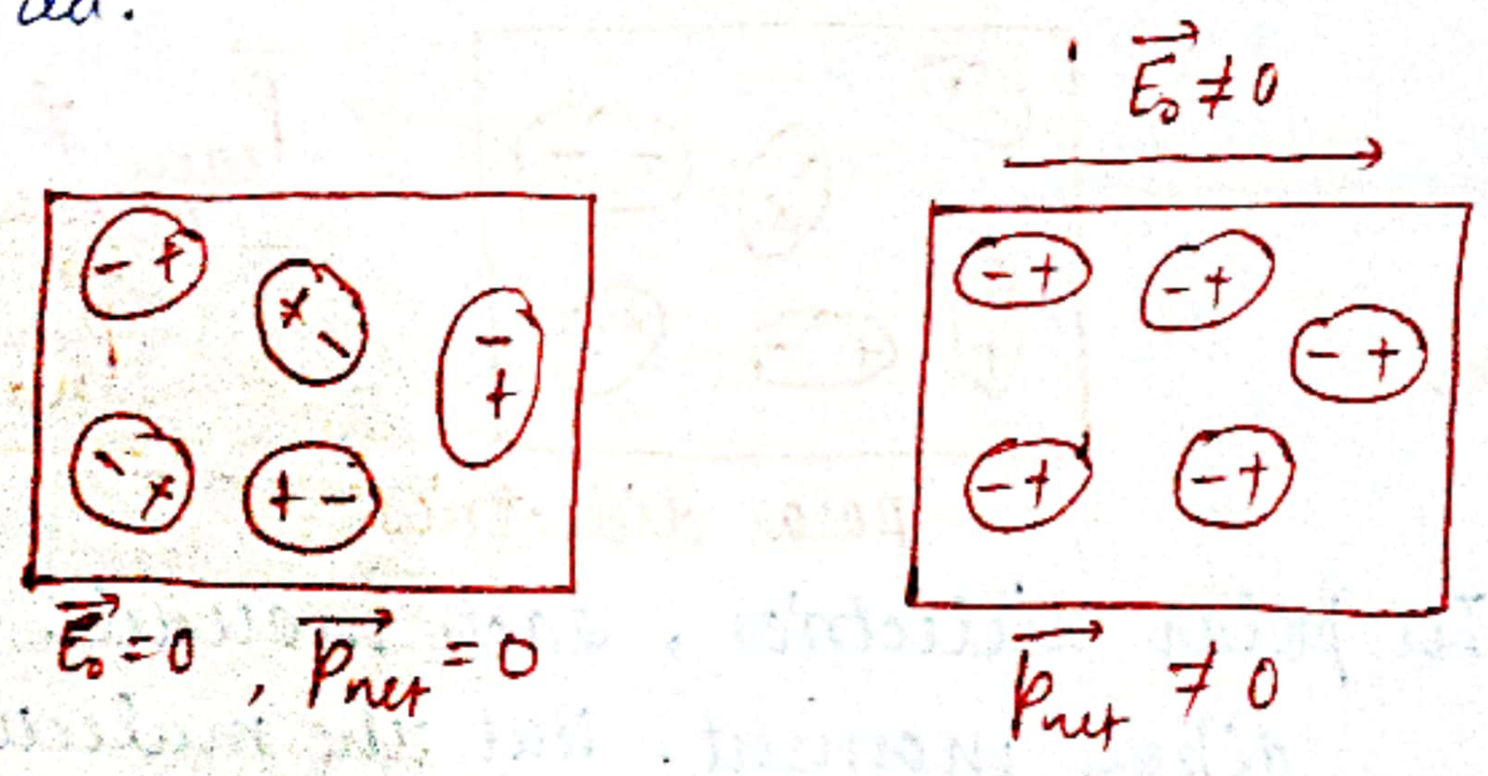
<u>Polarization</u> -

1) When a non-polar dielectric is held in an external electric field $\vec{E}_0$, then the center of +ve charge is pulled in the direction of $\vec{E}_0$ and the center of −ve charge is pulled in the direction opposite to $\vec{E}_0$. In this way, the center of +ve and −ve charges are separated. The molecules get distorted and are said to be polarized.



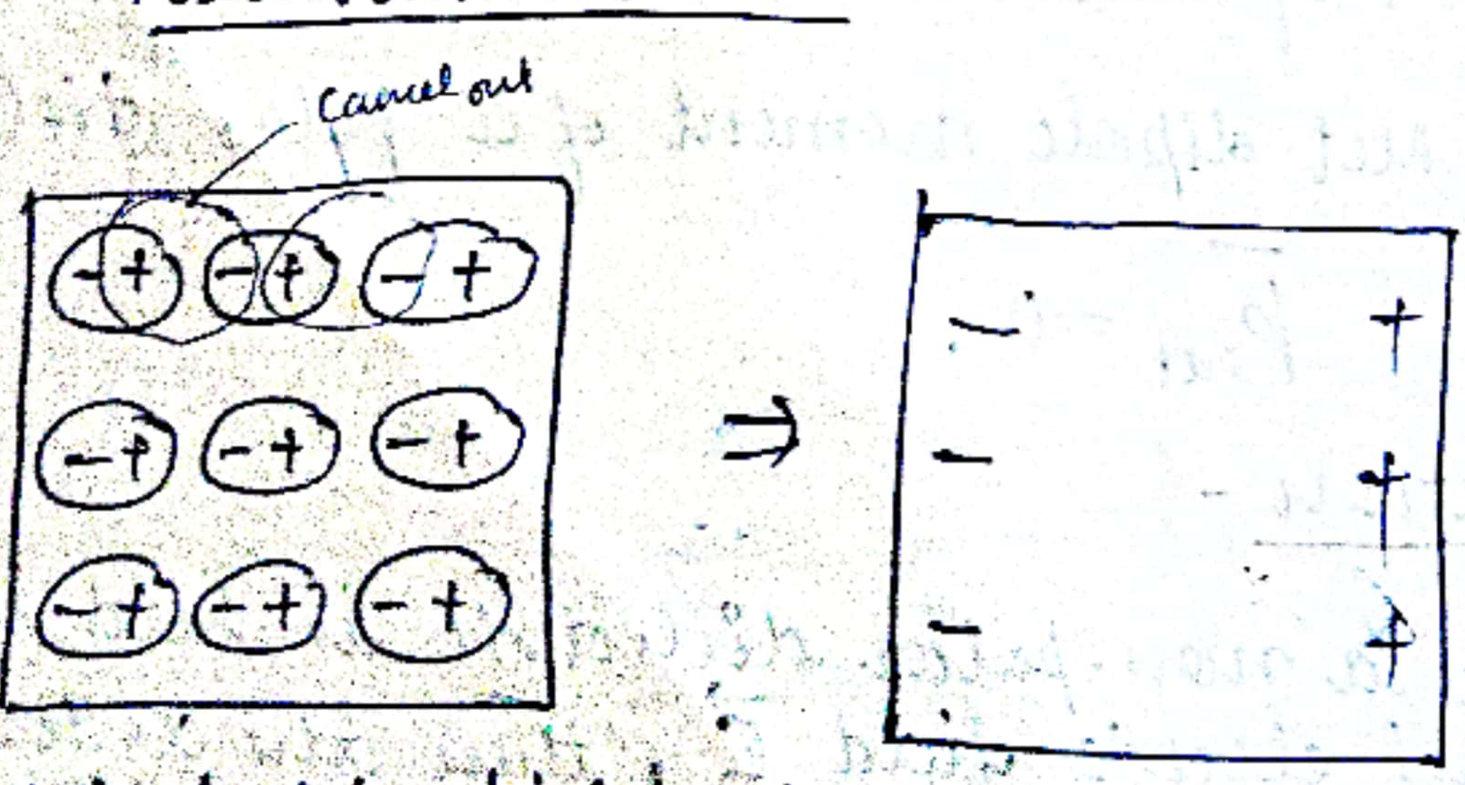
(Deformation by stretching)

2) when a polar dielectric is placed in an external electric field $\vec{E}_0$, the individual dipole moments tends to align in the direction of $\vec{E}_0$ and there will be net dipole moment in the direction of $\vec{E}_0$ ie, the dielectric is polarized.



(Deformation by Rotation)

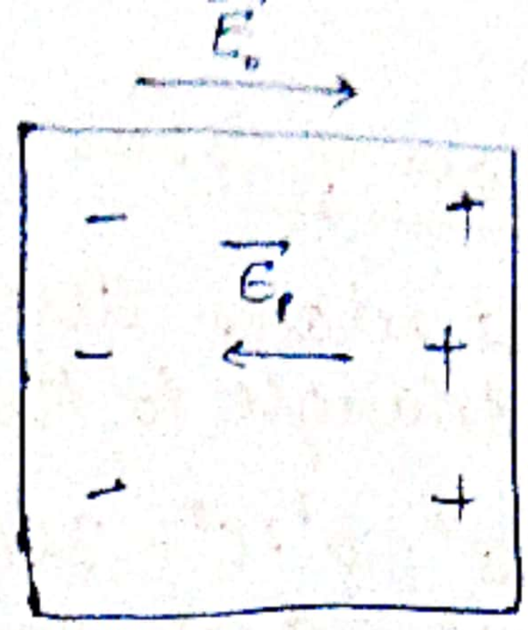
Polarized dielectric



Net electric field for a polarized dielectric,

$$\vec{E} = \vec{E}_0 - \vec{E}_p$$

Thus polarization causes a decrease in the net electric field within the dielectric.



$\vec{p} \rightarrow$ dipole moment

$\vec{P} \rightarrow$ induced dipole moment per unit volume (Polarization)

ie, $\vec{P} = \dfrac{\vec{p}_{net}}{V}$

And $\vec{P}$ is directly proportional to the effective (net) electric field $\vec{E}$ of the polarized dielectric.

ie, $\vec{P} \propto \vec{E}$

$\Rightarrow \vec{P} = \epsilon_0 \chi \vec{E}$

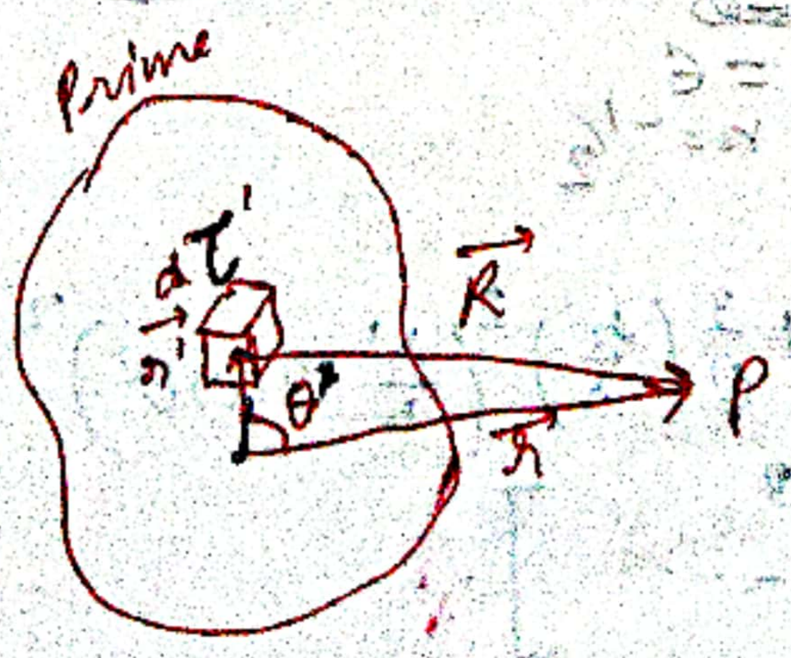
where ϵ_0 is the absolute permittivity of free space and χ is called electric susceptibility.

For vaccum,

$\chi = 0$

$\vec{P} = 0$

Multipole Expansion of Potential



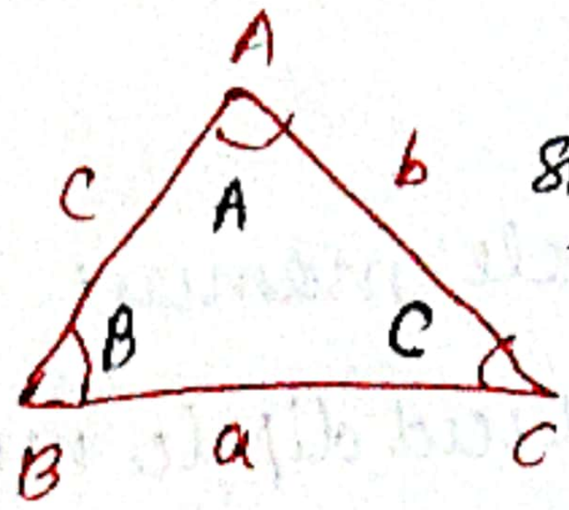
The potential at P due to an arbitrary localized charge distribution is given by -

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}')}{R} d\tau'$$

Law of Cosines

It relates the lengths of the sides of a triangle to the cosine of one of its angles.

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ca\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$



Sides of triangle

using the law of cosines,

$$R^2 = r^2 + (r')^2 - 2rr'\cos\theta'$$

$$R^2 = r^2\left[1 + \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\theta'\right]$$

$$R^2 = r^2\left[1 + \epsilon\right] \quad ; \quad \epsilon = \left[\frac{r'}{r}\right]^2 - r\left(\frac{r'}{r}\right)\cos\theta'$$

$$R = r(1+\epsilon)^{1/2}$$

$$\epsilon = \left(\frac{r'}{r}\right)\left[\frac{r'}{r} - 2\cos\theta'\right]$$

$$\frac{1}{R} = \frac{1}{r}(1+\epsilon)^{-1/2}$$

$$\frac{r'}{r} << 1$$

for points well outside the charge distribution, ϵ is much less than 1, and this invites a binomial expansion:

$$\frac{1}{R} = \frac{1}{r}\left(1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 - \frac{5}{16}\epsilon^3 + \cdots\right)$$

$$\left[\because (1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 \ldots\right]$$

$(1+\epsilon)^{-1/2}$; $x = \epsilon$, $n = -1/2$

$$\Rightarrow \frac{1}{R} = \frac{1}{r}\left[1 - \frac{1}{2}\left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\theta'\right) + \frac{3}{8}\left(\frac{r'}{r}\right)^2\left(\frac{r'}{r} - 2\cos\theta'\right)^2 - \frac{5}{16}\left(\frac{r'}{r}\right)^3\left(\frac{r'}{r} - 2\cos\theta'\right)^3 \cdots\right]$$

$$\Rightarrow \frac{1}{R} = \frac{1}{r}\left[1 + \left(\frac{r'}{r}\right)\cos\theta' + \left(\frac{r'}{r}\right)^2\left(\frac{3\cos^2\theta' - 1}{2}\right) + \left(\frac{r'}{r}\right)^3\left(\frac{5\cos^3\theta' - 3\cos\theta'}{2}\right) + \cdots\right]$$

$$\therefore V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{1}{r}\int \rho(\vec{r}')d\tau' + \frac{1}{r^2}\int r'\cos\theta'\rho(\vec{r}')d\tau' + \frac{1}{r^3}\int (r')^2\left(\frac{3}{2}\cos^2\theta' - \frac{1}{2}\right)\rho(\vec{r}')d\tau' + \cdots\right]$$

This is the required multipole expansion of V in power of $\frac{1}{r}$.

The first term is the monopole contribution (it goes like $\frac{1}{r}$); the second term is the dipole contribution (it goes like $\frac{1}{r^2}$), the third is quadrupole, the fourth is octopole contribution. and so on.

At large r, the multipole expansion is dominated by the monopole term.

Monopole $\rightarrow \frac{1}{r}$

Dipole $\rightarrow \frac{1}{r^2}$

Quadrupole $\rightarrow \frac{1}{r^3}$

Octopole $\rightarrow \frac{1}{r^4}$

i.e, $V_{monopole}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}$

where $Q = \int \rho d\tau$ is the total charge of the configuration.

If the total charge is zero, the dominant term in the potential will be the dipole.

$$V_{dipole}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int r'\cos\theta'\,\rho(\vec{r}')\,d\tau'$$

Since θ' is the angle between $\vec{r}'$ and $\vec{r}$

$$r'\cos\theta' = \vec{r}'\cdot\hat{r}$$

$$r'\cos\theta' = \hat{r}\cdot\vec{r}' \quad \rightarrow \because \text{commutative property.}$$

$$V_{dipole}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\int \hat{r}\cdot\vec{r}'\rho(\vec{r}')\,d\tau'$$

$$\therefore V_{dipole}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\hat{r}\cdot\underbrace{\int \vec{r}'\rho(\vec{r}')\,d\tau'}$$

= distance × charge
= dipole moment.

This integral, which does not depend on $\vec{r}$ at all, is called dipole moment of the distribution.

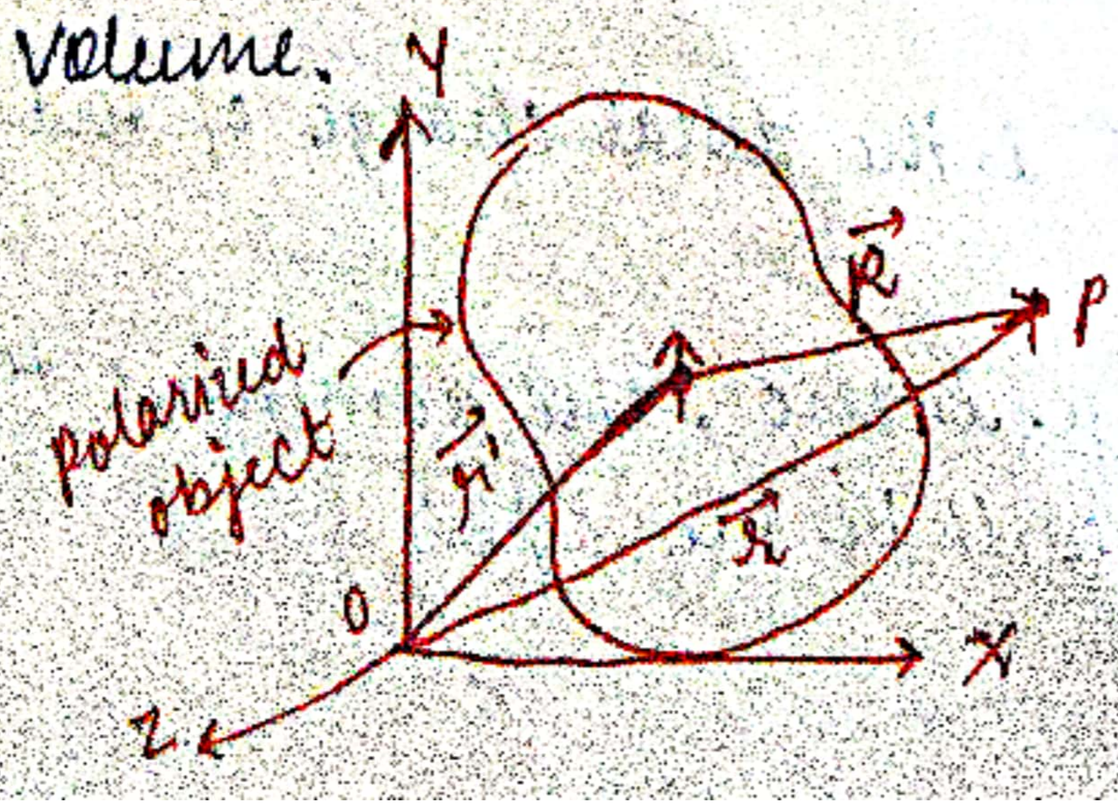
ie, $\vec{p} \equiv \int \vec{r}'\rho(\vec{r}')\,d\tau'$

Hence, the dipole contribution to the potential simplifies.

$$\boxed{V_{dipole}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{\hat{r}\cdot\vec{p}}{r^2}}$$

→ Bound charges

Consider a polarized object containing a lot of ~~dipole~~ microscopic dipoles lined up. For a volume element $d\tau'$, dipole moment is $\vec{p}$ with $\vec{P}$ as the dipole moment per unit volume.



potential at point P due to a single dipole $\vec{p}$ is -

$$dV = \frac{1}{4\pi\epsilon_0} \frac{\hat{R}\cdot\vec{p}}{R^2} \quad \text{--- (1)}$$

$\vec{p} = \frac{\vec{p}}{vol}$

But -

$\vec{p} = \vec{P}(\vec{r}')\,d\tau'$

for each volume element $d\tau'$

eq^n (1) becomes-

$$dV = \frac{1}{4\pi\epsilon_0} \frac{\hat{R}\cdot\vec{P}(\vec{r}')\,d\tau'}{R^2}$$

Then, total potential ~~pe~~ at P is -

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{\tau'} \frac{\hat{R}\cdot\vec{P}(\vec{r}')\,d\tau'}{R^2}$$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{\tau'} \frac{\vec{P}\cdot\hat{R}}{R^2}\,d\tau'$$

we know that

$$\vec{\nabla}'\left(\frac{1}{R}\right) = \frac{\hat{R}}{R^2}$$

$$\therefore V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{\tau'} \vec{P}\cdot\vec{\nabla}'\left(\frac{1}{R}\right) d\tau' \quad \text{--- (2)}$$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \left[\int_{\tau} \vec{\nabla}'\cdot\left(\frac{1}{R}\vec{P}\right) d\tau' - \int_{\tau'} \frac{1}{R}\left(\vec{\nabla}'\cdot\vec{P}\right) d\tau' \right]$$

$\because \left[\vec{\nabla}\cdot(f\vec{A}) = f\vec{\nabla}\cdot\vec{A} + \vec{A}\cdot\vec{\nabla}f \right]$

$\Rightarrow \vec{A}\cdot\vec{\nabla}f = \vec{\nabla}\cdot(f\vec{A}) - f\vec{\nabla}\cdot\vec{A}$

$\vec{A} \to \vec{P}$

$f \to \frac{1}{R}$

by gauss divergence theorem

$$\oint_S \vec{F}\cdot d\vec{s} = \int_V \vec{\nabla}\cdot\vec{F}\,dV$$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \left[\oint_S \frac{1}{R}\vec{P}\cdot d\vec{s}' - \int_{\tau'} \frac{1}{R}\left(\vec{\nabla}'\cdot\vec{P}\right) d\tau' \right]$$

$$\Rightarrow V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\oint_{S'} \frac{\vec{P}\cdot\hat{n}}{R}\, ds' + \frac{1}{4\pi\epsilon_0}\int_{\tau'} \frac{(-\vec{\nabla}'\cdot\vec{P})}{R}\, d\tau' \quad \text{--- (3)}$$

only for understand ⇒

Potential of continuous charge distribution, Line charge, surface charge, Volume charge

$$V = \frac{1}{4\pi\epsilon_0}\int_L \frac{\lambda\, dl}{R}, \quad \frac{1}{4\pi\epsilon_0}\int_S \frac{\sigma\, ds}{R}, \quad \frac{1}{4\pi\epsilon_0}\int_V \frac{\rho\, dV}{R}$$

The first term looks like the potential of a surface charge.

$\sigma_b = \vec{P}\cdot\hat{n}$, where $\hat{n}$ is the normal unit vector,

bound charge

while the second term looks like the potential of a volume charge;

$$\rho_b = -\vec{\nabla}\cdot\vec{P}$$

$\therefore$ Eq^n (3) becomes.

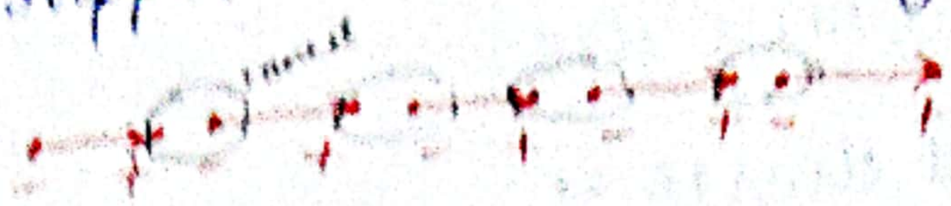
$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\oint_{S'} \frac{\sigma_b}{R}\, ds' + \frac{1}{4\pi\epsilon_0}\int_{\tau'} \frac{\rho_b}{R}\, d\tau'$$

Hence, potential of a polarized object is the same as that produced by a volume charge density $\rho_b \equiv -\vec{\nabla}\cdot\vec{P}$ plus a surface charge density $\sigma_b = \vec{P}\cdot\hat{n}$.

Physical Interpretation of Bound charges

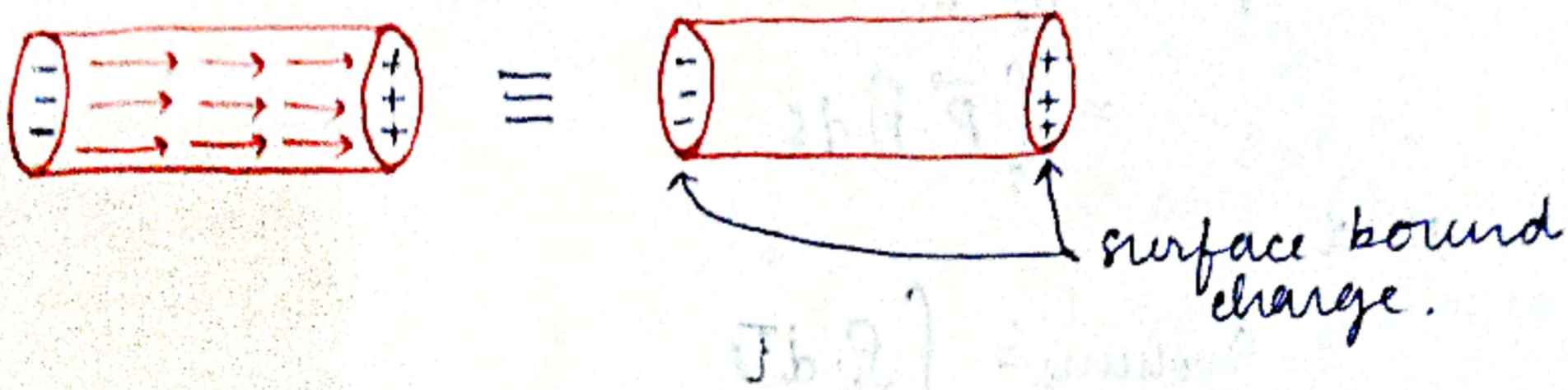
(1) <u>Surface bound charge</u> - ($\sigma_b = \vec{P}\cdot\hat{n}$)

Suppose we have a long string of dipoles.



Along the line, the head of one effectively cancels the tail of its neighbor, but at the ends there are two charges left over; plus charge at the right end and minus charge at the left.

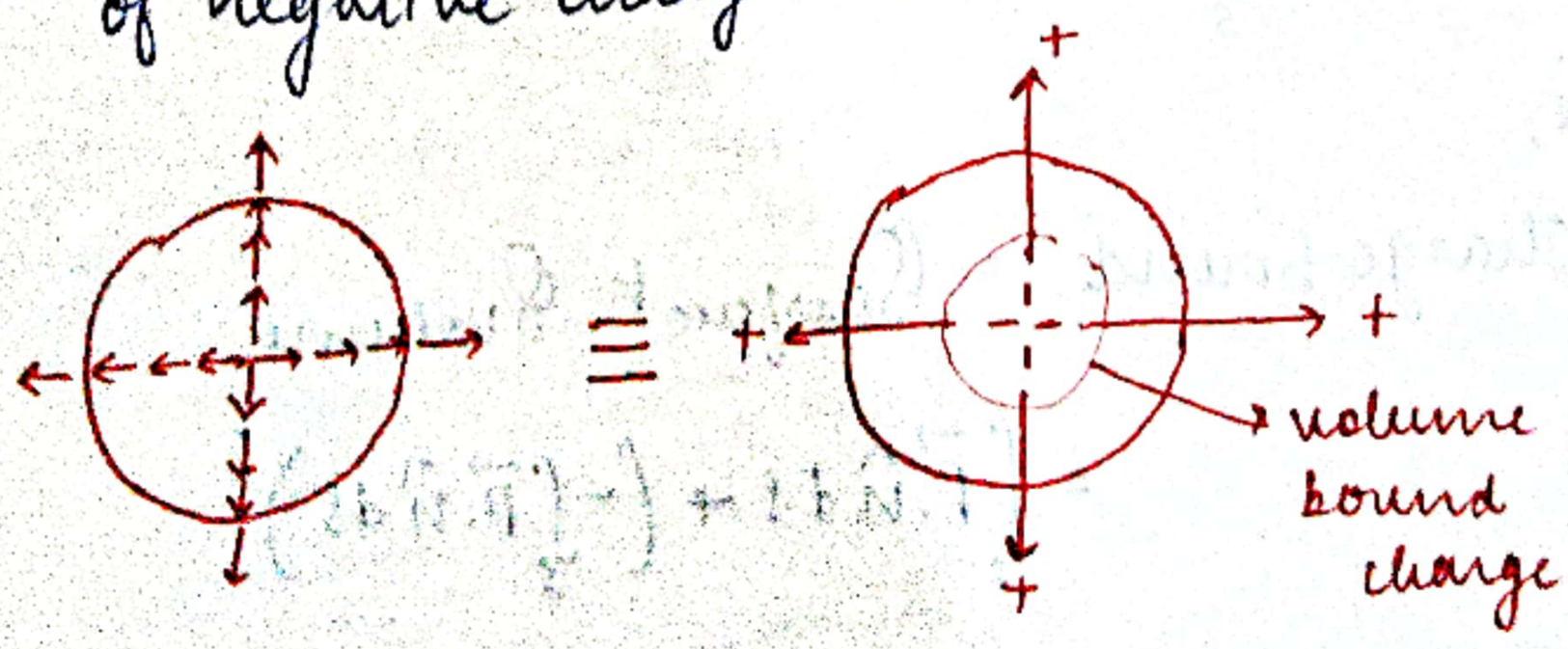
Now, consider a "tube" of dielectric.



(2) <u>Volume bound charge</u> - ($\rho_b \equiv -\vec{\nabla}\cdot\vec{P}$)

If the polarization is non-uniform, we get accumulations of bound charge within the material as well as on the surface.

In fact, a diverging $\vec{P}$ results in a pile up of negative charge.



Property of Bound charges :

* net bound charge is always zero.

$\Rightarrow$ Total surface bound charge + Total volume bound charge = zero

i.e, $\boxed{Q_{surface} + Q_{volume} = 0}$

Proof -

$\sigma = \frac{\text{charge}}{\text{Area}}$

By definition,

$$Q_{surface} = \int_S \sigma_b \, ds \qquad (\sigma_b = \vec{P}\cdot\hat{n})$$

$$= \int_S \vec{P}\cdot\hat{n} \, ds$$

And,

$$Q_{volume} = \int_\tau \rho_b \, d\tau$$

$$= -\int_\tau \vec{\nabla}\cdot\vec{P} \, d\tau$$

$$= -\int_S \vec{P}\cdot d\vec{s} \qquad \left[\text{Gauss's divergence Theorem}\right] \quad \int_S \vec{F}\cdot d\vec{s} = \int_\tau \vec{\nabla}\cdot\vec{F} \, d\tau$$

$$= -\int_S \vec{P}\cdot\hat{n} \, ds$$

Therefore,

Total charge bound $= Q_{surface} + Q_{volume}$

$$= \int_S \vec{P}\cdot\hat{n} \, ds + \left(-\int_S \vec{P}\cdot\hat{n} \, ds\right)$$

$$= \underline{\underline{0}}$$

Que. A sphere of radius R carries a polarization $\vec{P}(\vec{r}) = K\vec{r}$,
where K is a constant and $\vec{r}$ is the vector from the center.
Calculate the bound charges σ_b & ρ_b.

$$\vec{P}(\vec{r}) = K\vec{r}$$

Sol^n.

$$\sigma_b = \vec{P}\cdot\hat{n} = K\vec{r}\cdot\hat{n} \qquad \vec{r} = R\hat{n}$$

at surface $r = R$

$$= KR\,\hat{n}\cdot\hat{n}$$

$$\boxed{\sigma_b = KR.}$$

And.

$$\rho_b = -\vec{\nabla}\cdot\vec{P}$$

$$= -\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 P_r)$$

$\because$ P is independent of θ and ϕ.

$$= -\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 K r)$$

$$= -\frac{1}{r^2}K\frac{\partial}{\partial r}(r^3)$$

$$= -\frac{K}{r^2}\times 3r^2$$

$$\boxed{\rho_b = -3K}$$

$\vec{F}(r,\theta,\phi)$

$$\left[\vec{\nabla}\cdot\vec{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 F_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta F_\theta) + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi}\right]$$

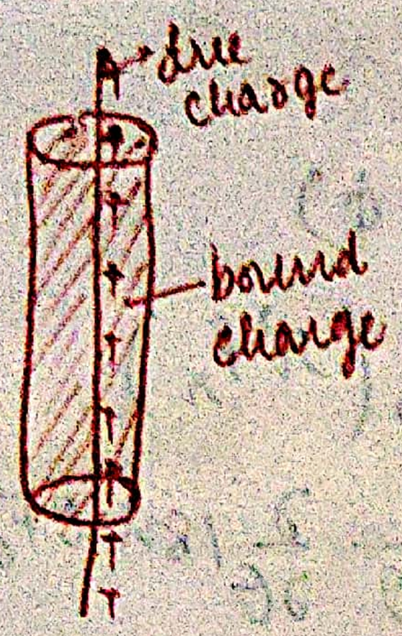
in sperical coordinates

Electric Displacement

we have found that the effect of polarization is to produce accumulations of bound charge, $\rho_b = -\vec{\nabla}\cdot\vec{P}$ within the dielectric and $\sigma_b = \vec{P}\cdot\hat{n}$ on the surface.

If free charge represents any charge that is not a result of polarization, then the total charge density can be written as -

Total charge density = Bound charge density + Free charge density.



i.e, $\underbrace{\rho}_{\text{Total charge density}} = \underbrace{\rho_b}_{\text{bound charge den.}} + \underbrace{\rho_f}_{\text{free charge den.}}$ —①

we know that the differential form of Gauss's law is -

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$$

$$\Rightarrow \epsilon_0(\vec{\nabla}\cdot\vec{E}) = \rho$$

$$\Rightarrow \epsilon_0(\vec{\nabla}\cdot\vec{E}) = \rho_b + \rho_f \quad (\text{using eq}^n ①)$$

$$\Rightarrow \epsilon_0(\vec{\nabla}\cdot\vec{E}) = -\vec{\nabla}\cdot\vec{P} + \rho_f$$

$$\Rightarrow \vec{\nabla}\cdot(\epsilon_0\vec{E} + \vec{P}) = \rho_f$$

$$\Rightarrow \vec{\nabla}\cdot\vec{D} = \rho_f \quad —②$$

where $\vec{D} \equiv \epsilon_0\vec{E} + \vec{P}$, is known as the <u>electric displacement</u>.

Eq^n ② is known as the differential form of gauss's law in presence of dielectric.

$$\vec{\nabla}\cdot\vec{D} = \rho_f \quad \text{— ②}$$

Integrating eq^n ② over a volume τ,

$$\int_\tau \vec{\nabla}\cdot\vec{D}\, d\tau = \int_\tau \rho_f\, d\tau$$

[By Gauss's Divergence Theorem.]

$$\Rightarrow \oint_S \vec{D}\cdot d\vec{S} = (Q_f)_{enclosed} \quad \text{— ③}$$

Total free charge enclosed.

Eq^n ③ is the integral form of Gauss's law in presence of dielectric.

Revise

charge

- free charge → not from polarization
- Bound charge → due to polarization.
  - surface bound charge $\sigma_b = \vec{P}\cdot\hat{n}$
  - volume bound charge $\rho_b = -\vec{\nabla}\cdot\vec{P}$

Electric displacement,

$$\boxed{\vec{D} = \epsilon_0\vec{E} + \vec{P}}$$

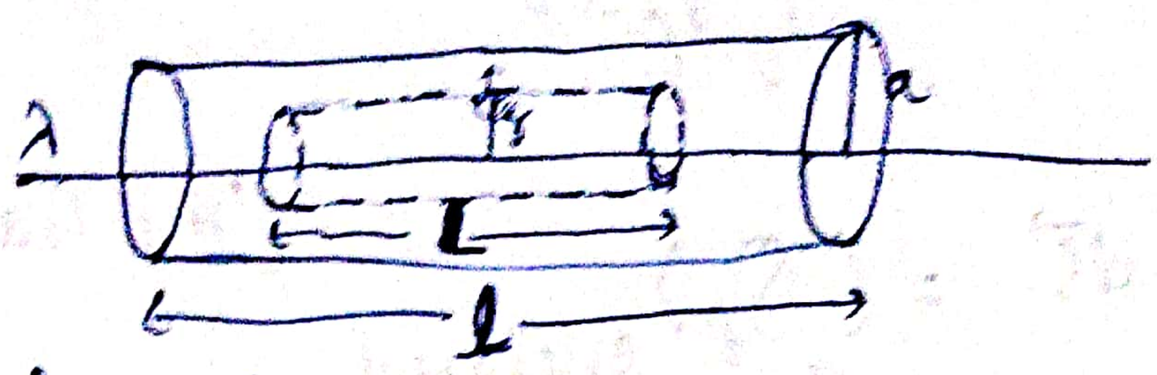
Gauss's law in presence of dielectric.

$$\boxed{\vec{\nabla}\cdot\vec{D} = \rho_f \text{ (Differential form)}}$$

$$\boxed{\oint_S \vec{D}\cdot d\vec{S} = (Q_f)_{enc}} \text{ (Integral form)}$$

Ques. A long straight wire, carrying uniform line charge λ, is surrounded by a rubber insulation out to a radius 'a'. Find the electric displacement.

Ans.



for $r<a$;

A cylindrical Gaussian surface of radius 'r' and length 'L' is drawn.

By integral form of Gauss's law in presence of dielectric,

we get

$$\oint_S \vec{D}\cdot d\vec{s} = (Q_f)_{enclosed}$$

$$\Rightarrow \oint_S D\, ds \cos 0^\circ = \int \lambda\, dl$$

$$\Rightarrow D \oint_S ds = \lambda \int dl$$

$$\Rightarrow D \times 2\pi r L = \lambda L$$

$$D = \frac{\lambda}{2\pi r}$$

$$\boxed{\vec{D} = \frac{\lambda}{2\pi r}\hat{r}}$$

Electric field for $r<a$;

$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$

$\because \vec{P}$ is not given so we can't find $\vec{E}$ for $r<a$.

for $r>a$,

$$\boxed{\vec{D} = \frac{\lambda}{2\pi r}\hat{r}}$$

[$\because \vec{D}$ depends only on free charge]

electric field.

for $r > a$, $\vec{P} = 0$

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}^{\,0}$$

$$\Rightarrow \vec{D} = \epsilon_0 \vec{E}$$

$$\boxed{\vec{E} = \frac{1}{\epsilon_0}\vec{D} = \frac{\lambda}{2\pi\epsilon_0 r}\hat{r}}$$

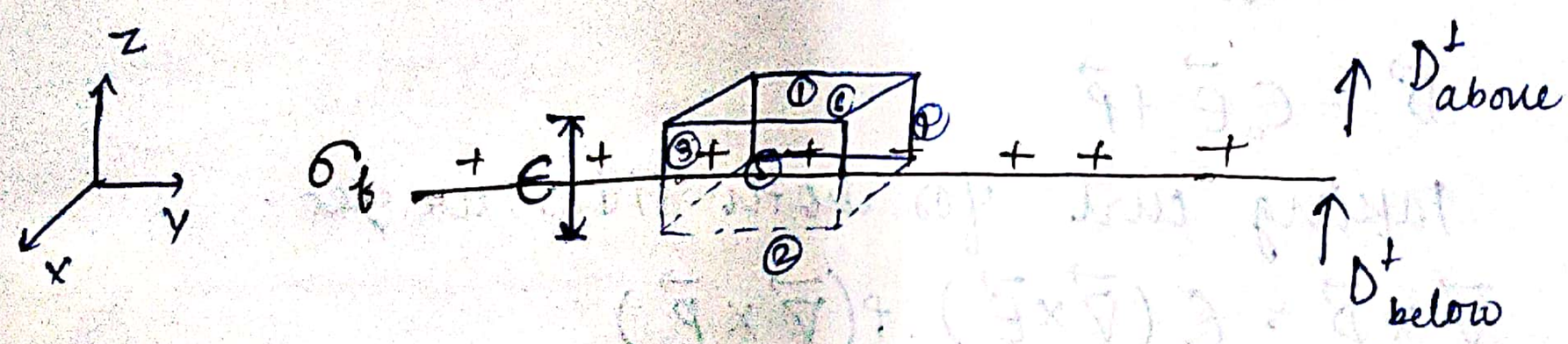
Boundary conditions in Dielectrics

$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$

$E^{\perp}_{above} - E^{\perp}_{below} = \frac{\sigma}{\epsilon_0}$

$E^{\parallel}_{above} - E^{\parallel}_{below} = 0$

1) Boundary condition for perpendicular component of $\vec{D}$.



The integral form of gauss's law in presence of dielectric reads

$$\oint_S \vec{D}\cdot d\vec{s} = (Q_f)_{enc.}$$

$$\Rightarrow \int_1 \vec{D}\cdot d\vec{s} + \int_2 \vec{D}\cdot d\vec{s} + \int_3 \vec{D}\cdot d\vec{s} + \int_4 \vec{D}\cdot d\vec{s} + \int_5 \vec{D}\cdot d\vec{s} + \int_6 \vec{D}\cdot d\vec{s} = \int_S \sigma_f \, dS \quad \text{—(1)}$$

As $\epsilon \to 0$, $ds \to 0$ for surfaces 3, 4, 5 and 6. So, eq^n (1) can be written as-

$$\int_1 \vec{D}\cdot d\vec{s} + \int_2 \vec{D}\cdot d\vec{s} = \int_S \sigma_f \, dS$$

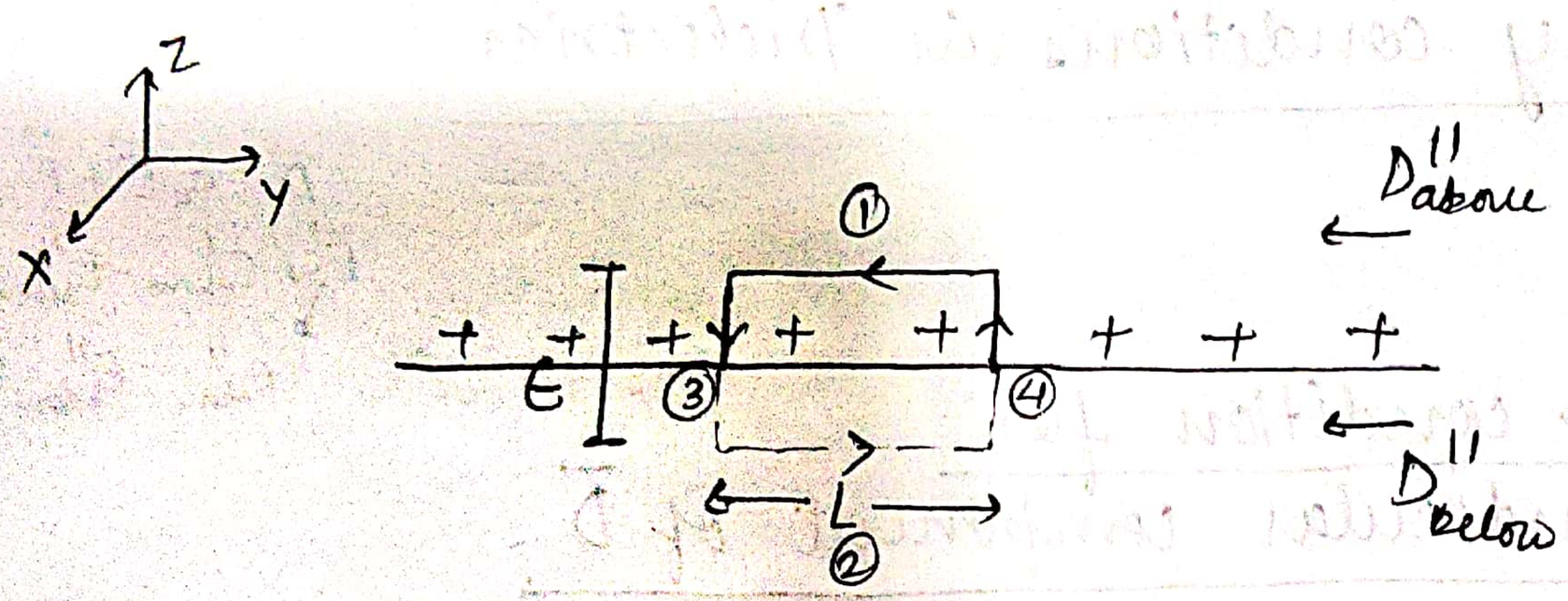
$$\Rightarrow \int_1 D^{\perp}_{above}(\hat{z})\cdot ds(\hat{z}) + \int_2 D^{\perp}_{below}(\hat{z})\cdot ds(-\hat{z}) = \int_S \sigma_f \, ds$$

$$\Rightarrow D^{\perp}_{above}\int_1 dS - D^{\perp}_{below}\int_2 dS = \sigma_f \int_S dS$$

$$\Rightarrow D^{\perp}_{above} A - D^{\perp}_{below} A = \sigma_f A$$

$$\Rightarrow \boxed{D^{\perp}_{above} - D^{\perp}_{below} = \sigma_f}$$

2) <u>Boundary condition for parallel component of $\vec{D}$.</u>



we know that -

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$

taking curl for both sides, we get.

$$\vec{\nabla}\times\vec{D} = \epsilon_0(\vec{\nabla}\times\vec{E}) + (\vec{\nabla}\times\vec{P})$$

$$\vec{\nabla}\times\vec{D} = \vec{\nabla}\times\vec{P} \qquad [\because \vec{\nabla}\times\vec{E} = 0]$$

$$\Rightarrow \int_S (\vec{\nabla}\times\vec{D})\cdot d\vec{S} = \int_S (\vec{\nabla}\times\vec{P})\cdot d\vec{S}$$

By stoke's theorem - $\left[\oint_C \vec{F}\cdot d\vec{l} = \int_S (\vec{\nabla}\times\vec{F})\cdot d\vec{s}\right]$

$$\Rightarrow \oint_C \vec{D}\cdot d\vec{l} = \oint_C \vec{P}\cdot d\vec{l}$$

$$\Rightarrow \int_1 \vec{D}\cdot d\vec{l} + \int_2 \vec{D}\cdot d\vec{l} + \int_3 \vec{D}\cdot d\vec{l} + \int_4 \vec{D}\cdot d\vec{l}$$

$$= \int_1 \vec{P}\cdot d\vec{l} + \int_2 \vec{P}\cdot d\vec{l} + \int_3 \vec{P}\cdot d\vec{l} + \int_4 \vec{P}\cdot d\vec{l}$$

As $\epsilon \to 0$, $dl \to 0$ for paths 3 and 4,

$\therefore$ Therefore, the above equation can be written as -

$$\int_1 \vec{D}\cdot d\vec{l} + \int_2 \vec{D}\cdot d\vec{l} = \int_1 \vec{P}\cdot d\vec{l} + \int_2 \vec{P}\cdot d\vec{l}$$

$$\Rightarrow \int_1 D''_{above}(-\hat{y}).dl(-\hat{y}) + \int_2 D''_{below}(-\hat{y}).dl(\hat{y})$$

$$= \int_1 P''_{above}(-\hat{y}).dl(-\hat{y}) + \int_2 P''_{below}(-\hat{y}).dl(\hat{y})$$

$$\Rightarrow D''_{above}\int_1 dl - D''_{below}\int_2 dl = P''_{above}\int_1 dl - P''_{below}\int_2 dl$$

$$\Rightarrow D''_{above}L - D''_{below}L = P''_{above}L - P''_{below}L$$

$$\boxed{\Rightarrow D''_{above} - D''_{below} = P''_{above} - P''_{below}}$$

Thus, both the perpendicular and parallel components of $\vec{D}$ are DISCONTINUOUS at the interface.

Linear Dielectrics

For linear dielectrics,

$$\vec{P} \propto \vec{E}$$

$$\boxed{\vec{P} = \epsilon_0 \chi_e \vec{E}} \quad \text{—(1)}$$

χ_e → electric susceptibility

ϵ_0 → permittivity of free space

Then,

$$\vec{D} = \epsilon_0\vec{E} + \vec{P}$$

$$\vec{D} = \epsilon_0\vec{E} + \epsilon_0\chi_e\vec{E} \quad [\text{using eq}^n \text{(1)}]$$

$$\vec{D} = \epsilon_0(1+\chi_e)\vec{E}$$

$$\Rightarrow \boxed{\vec{D} = \epsilon\vec{E}} \qquad \epsilon = \epsilon_0(1+\chi_e)$$

where $\epsilon \equiv \epsilon_0(1+\chi_e)$, is called permittivity of the material.

Energy in dielectrics -

we know that the energy stored in an electrostatic system is given by

$$W = \frac{\epsilon_0}{2}\int_{\text{all space}} E^2 d\tau$$

Then, energy of any dielectric system is

$$W = \frac{\epsilon}{2}\int_{\text{all space}} E^2 d\tau$$

$$W = \frac{\epsilon}{2}\int_{\text{all space}} \vec{E}\cdot\vec{E}\, d\tau$$

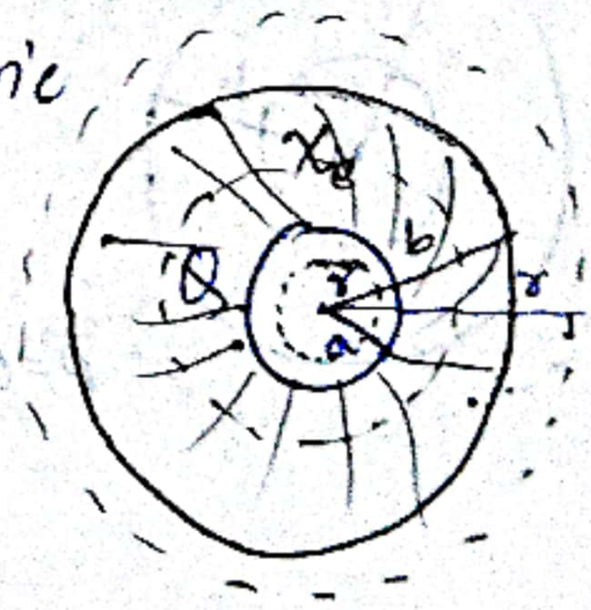
$$W = \frac{1}{2}\int_{\text{all space}} \epsilon\vec{E}\cdot\vec{E}\, d\tau$$

$$\boxed{W = \frac{1}{2}\int_{\text{all space}} \vec{D}\cdot\vec{E}\, d\tau}$$, for linear dielectrics.

Ques- A spherical conductor of radius 'a', carries a charge Q. It is surrounded by linear dielectric material of susceptability χ_e out to radius 'b'. Find the energy of this configuration.

we know that

$$W = \frac{1}{2}\int_{\text{all space}} \vec{D}\cdot\vec{E}\, d\tau$$, for linear dielectric



3 regions
→ $r<a$
$a<r<b$
$r>b$

$\vec{D} = ?$
$\vec{E} = ?$

for $r<a$, charge inclosed $= 0$

Then, $\oint_S \vec{D}\cdot d\vec{s} = (Q_f)_{inc.} = 0$

$\Rightarrow \vec{D} = 0$

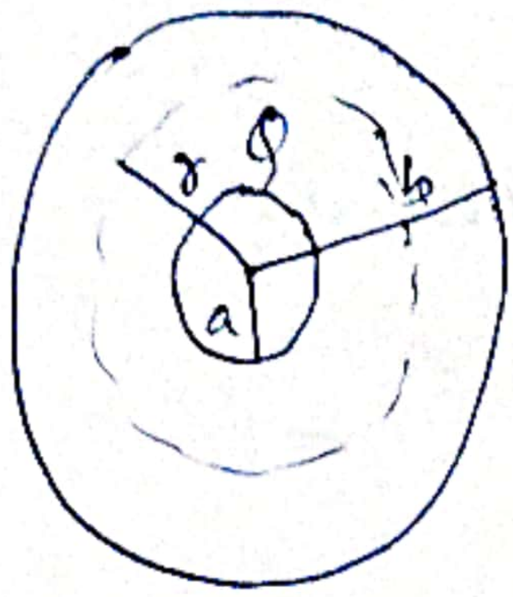
Ans

$$W = \frac{Q^2}{8\pi\epsilon_0(1+\chi_e)}\left(\frac{1}{a}+\frac{\chi_e}{b}\right)$$

Since $\vec{D} = \epsilon \vec{E}$, for linear dielectric

$$\Rightarrow \vec{E} = 0$$

For $a < r < b$,



$$\oint_s \vec{D} \cdot d\vec{s} = (Q_f)_{inc.}$$

$$\Rightarrow \oint_s D\,ds \cos 0^\circ = Q$$

$$\Rightarrow D \oint_s ds = Q$$

$$\Rightarrow D \times 4\pi r^2 = Q$$

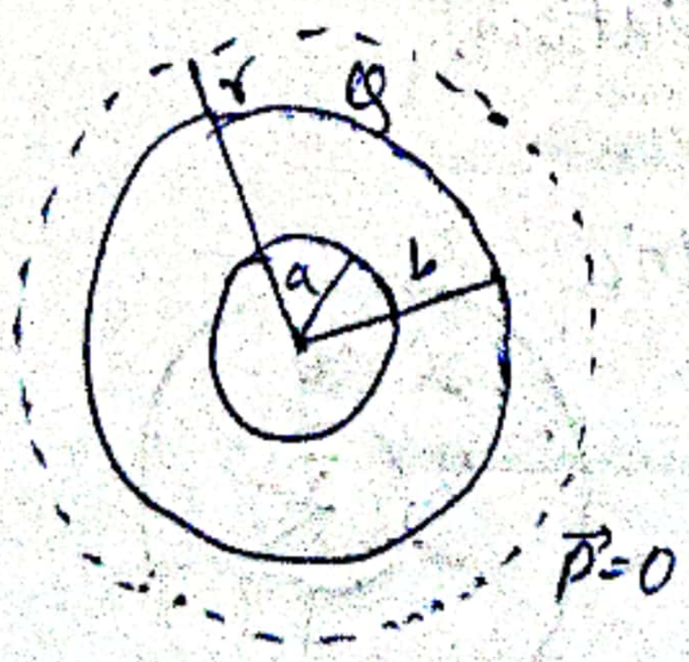
$$\Rightarrow D = \frac{Q}{4\pi r^2}$$

$$\Rightarrow \vec{D} = \frac{Q}{4\pi r^2}\hat{r}$$

Since $\vec{D} = \epsilon \vec{E}$

$$\Rightarrow \vec{E} = \frac{1}{\epsilon}\vec{D} = \frac{Q}{4\pi\epsilon r^2}\hat{r}$$

And, for $r > b$,



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$$\Rightarrow D \times 4\pi r^2 = Q$$

$$\Rightarrow D = \frac{Q}{4\pi r^2}$$

$$\Rightarrow \vec{D} = \frac{Q}{4\pi r^2}\hat{r}$$

For $r > b$,

$\vec{P} = 0$ since there is no dielectric here. $(r > b)$

Then,

$$\vec{D} = \epsilon \vec{E}$$

$$\vec{D} = \epsilon_0 \vec{E} + \cancel{\vec{P}}^{\,0}$$

$$\vec{D} = \epsilon_0 \vec{E}$$

$$\Rightarrow \vec{E} = \frac{1}{\epsilon_0} \vec{D}$$

$$\Rightarrow \vec{E} = \frac{Q}{4\pi\epsilon_0 r^2} \hat{r}$$

$\therefore$ The required energy

$$= \frac{1}{2} \int_{\text{allspace}} \vec{D} \cdot \vec{E} \, d\tau$$

$$= \frac{1}{2} \int_{r<a} \cancel{\vec{D} \cdot \vec{E}}^{\,0} \, d\tau + \frac{1}{2} \int_{a<r<b} \vec{D} \cdot \vec{E} \, d\tau + \frac{1}{2} \int_{r>b} \vec{D} \cdot \vec{E} \, d\tau$$

$d\tau = r^2 \sin\theta \, dr \, d\theta \, d\phi$

$$= \frac{1}{2} \int_a^b \int_0^{\pi} \int_0^{2\pi} \frac{Q}{4\pi r^2} \hat{r} \cdot \frac{Q}{4\pi r^2 \epsilon} \hat{r} \, r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$+ \frac{1}{2} \int_b^{\infty} \int_0^{\pi} \int_0^{2\pi} \frac{Q}{4\pi r^2} \hat{r} \cdot \frac{Q}{4\pi \epsilon_0 r^2} \hat{r} \, r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \frac{1}{2} \left(\frac{Q}{4\pi}\right)^2 \frac{1}{\epsilon} \int_a^b \int_0^{\pi} \int_0^{2\pi} \frac{1}{r^2} \sin\theta \, dr \, d\theta \, d\phi + \frac{1}{2} \left(\frac{Q}{4\pi}\right)^2 \frac{1}{\epsilon_0} \int_b^{\infty} \int_0^{\pi} \int_0^{2\pi} \frac{1}{r^2} \sin\theta \, dr \, d\theta \, d\phi$$

$$= \frac{1}{2} \left(\frac{Q}{4\pi}\right)^2 \frac{1}{\epsilon} \, 4\pi \int_a^b \frac{1}{r^2} \, dr + \frac{1}{2} \left(\frac{Q}{4\pi}\right)^2 \frac{1}{\epsilon_0} \times 4\pi \int_b^{\infty} \frac{1}{r^2} \, dr$$

$$= \frac{Q^2}{8\pi\epsilon} \left[-\frac{1}{r}\right]_a^b + \frac{Q^2}{8\pi\epsilon_0} \left[-\frac{1}{r}\right]_b^{\infty}$$

$$= \frac{Q^2}{8\pi\epsilon} \left[\frac{1}{a} - \frac{1}{b}\right] + \frac{Q^2}{8\pi\epsilon_0} \left[\frac{1}{b} - \frac{1}{\infty}\right]$$

$\because \epsilon = \epsilon_0 (1+\chi_e)$

$$= \frac{Q^2}{8\pi\epsilon_0(1+\chi_e)}\left[\frac{1}{a}-\frac{1}{b}\right] + \frac{Q^2}{8\pi\epsilon_0}\left[\frac{1}{b}\right]$$

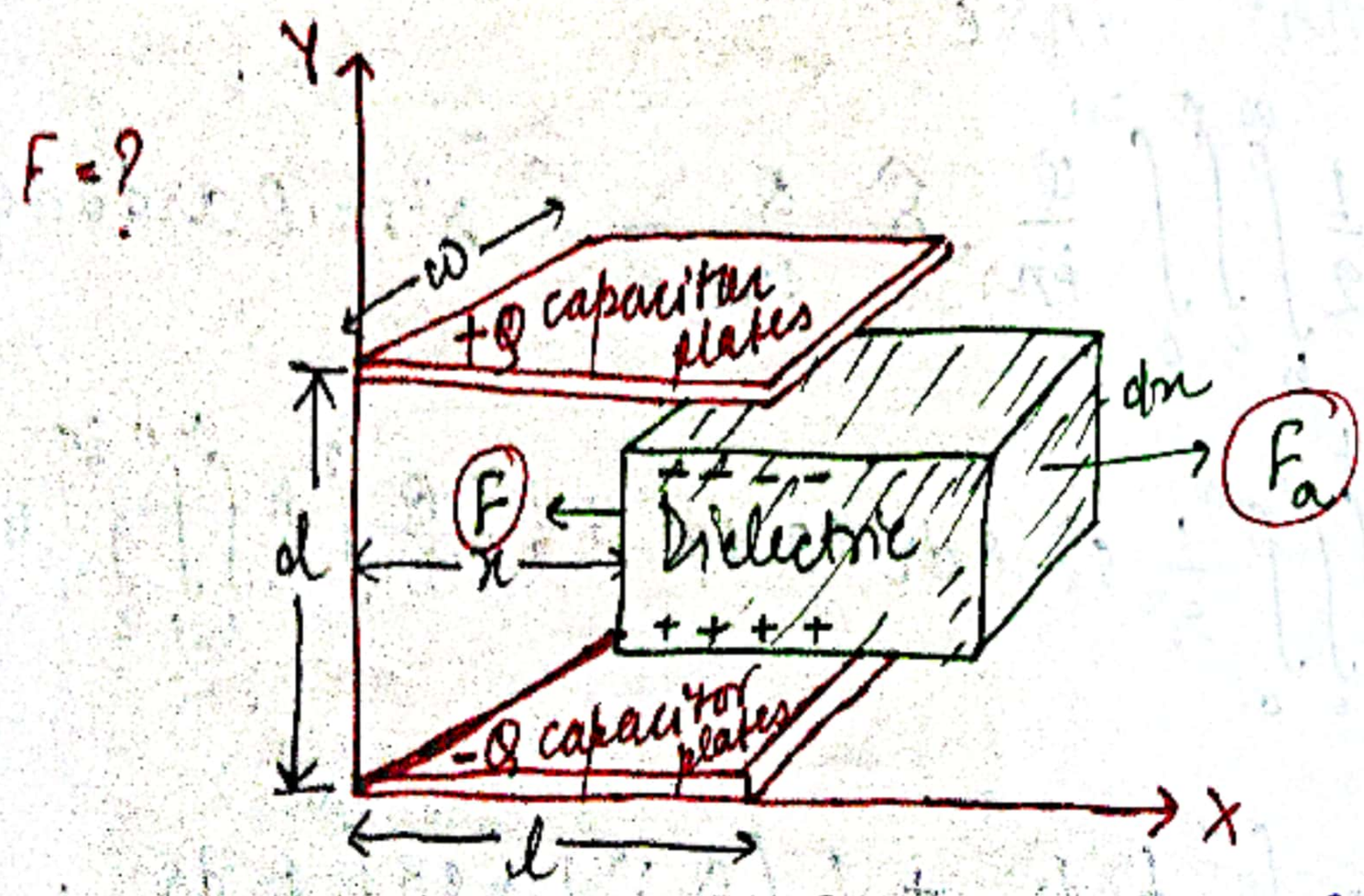
$$= \frac{Q^2}{8\pi\epsilon_0(1+\chi_e)}\left[\frac{1}{a}-\frac{1}{b}+\frac{1+\chi_e}{b}\right]$$

$$\therefore \boxed{W = \frac{Q^2}{8\pi\epsilon_0(1+\chi_e)}\left[\frac{1}{a}+\frac{\chi_e}{b}\right]}$$

Ans

FORCES on DIELECTRICS

Consider a slab of linear dielectric material, partially inserted between the plates of a parallel-plate capacitor.



~~Since~~ (non-uniform fields)

Then, there is a fringing field around the edges of the capacitor.

This non-uniform fringing field is responsible for the force on the dielectric.

Let W be the energy of the system.
If the dielectric is pulled out by an infinitesimal distance dx, the energy is changed by an amount equal to the work done.

$$dW = F_a\, dx \quad \text{—} (1)$$

where F_a is applied force to countract the electrical force of F on the dielectric, $F_a = -F$

∴ The electric force on the dielectric slab is –

$$F = -F_a$$

$$F = -\frac{dW}{dx} \quad [\text{using eq}^n (1)] \quad \text{—} (2)$$

We know that the energy stored in the capacitor is –

$$W = \frac{1}{2} CV^2$$

$$W = \frac{1}{2}\frac{Q^2}{C} \quad \text{—} (3)$$

assuming that the total charge on the plates ($Q = CV$) is held constant, as the dielectric moves.

where the capacitance C is given by –

$$C = \frac{\epsilon_0 w x}{d} + \frac{\epsilon w (l-x)}{d}$$

$C = \frac{\epsilon_0 A}{d}$

$\epsilon = \epsilon_0(1+\chi_e)$

$\epsilon_r = 1+\chi_e$

$$C = \frac{\epsilon_0 w x}{d} + \frac{\epsilon_0 (1+\chi_e) w (l-x)}{d}$$

$$C = \frac{\epsilon_0 w}{d}\left[x + (1+\chi_e)(l-x)\right]$$

$$C = \frac{\epsilon_0 w}{d}\left[x + (1+\chi_e) l - (1+\chi_e) x\right]$$

$$C = \frac{\epsilon_0 w}{d}\left[x + \epsilon_r l - x - \chi_e x\right]$$

$$C = \frac{\epsilon_0 \omega}{d}\left[\epsilon_r l - \chi_e x\right] \quad \text{—(9)}$$

$\therefore$ eq^n (2) becomes -

$$F = -\frac{dW}{dx}$$

$$F = -\frac{d}{dx}\left(\frac{1}{2}\frac{Q^2}{C}\right) \quad \text{using eq}^n \text{(9)}$$

$$F = -\frac{Q^2}{2}\frac{d}{dx}\left(\frac{1}{C}\right)$$

$$F = -\frac{Q^2}{2}x - \frac{1}{C^2}\frac{dC}{dx}$$

$$F = -\frac{Q^2}{2}x - \frac{1}{C^2}\frac{dC}{dx} = \frac{Q^2}{2C^2}\frac{dC}{dx}$$

$$F = \frac{Q^2}{2C^2}\frac{d}{dx}\left[\frac{\epsilon_0 \omega}{d}(\epsilon_r l - \chi_e x)\right] \quad \leftarrow \text{using eq}^n \text{ 4}$$

$$F = \frac{-Q^2}{2C^2}\frac{\epsilon_0 \omega}{d}\chi_e$$

$$F = -\frac{\epsilon_0 \chi_e \omega}{2d} \cdot \frac{Q^2}{C^2}$$

$$\left[\begin{array}{c} Q = CV \\ V = \frac{Q}{C} \end{array}\right]$$

$$\boxed{F = -\frac{\epsilon_0 \chi_e \omega V^2}{2d}}$$

The minus sign indicates that the force is in the NEGATIVE x-direction ie, the dielectric is pulled into the capacitor.

MAGNETOSTATICS

Steady current produces magnetic field that is constant in time. The study of steady current is called <u>magnetostatics</u>.

By (stationary) steady current, we mean a continuous flow that has been going on forever, without charge and without charge piling up anywhere.

LORENTZ FORCE LAW:

According to this law, the magnetic force on a charge Q, moving with velocity $\vec{v}$ in a magnetic field $\vec{B}$, is given by -

$$\boxed{\vec{F}_{mag} = Q(\vec{v} \times \vec{B})}$$

[An axiom, justified from the experiments]

In the presence of both electric and magnetic fields, the net force on the charge Q will be -

$$\vec{F} = \vec{F}_{electric} + \vec{F}_{magnetic}$$

$$= Q\vec{E} + Q(\vec{v} \times \vec{B})$$

$$\boxed{\vec{F} = Q[\vec{E} + (\vec{v} \times \vec{B})]}$$

Property of magnetic force -

- <u>magnetic forces do No work.</u>
  <u>i.e, work done by the magnetic force is zero.</u>

<u>proof:</u> Let $\vec{F}_{mag}$ moves a charge Q by an amount $\vec{dl}$ in time dt with velocity $\vec{v}$ such that,

$\Rightarrow \vec{v} = \frac{d\vec{l}}{dt}$

$\Rightarrow d\vec{l} = \vec{v}\,dt$

Then, the workdone by the magnetic force is given by -

$$dW_{mag} = \vec{F}_{mag} \cdot d\vec{l}$$

$$= q(\vec{v} \times \vec{B}) \cdot d\vec{l}$$

$$= q(\vec{v} \times \vec{B}) \cdot \vec{v}\,dt$$

[$\because (\vec{v} \times \vec{B})$ is perpendicular to $\vec{v}$, so $(\vec{v} \times \vec{B}) \cdot \vec{v} = 0$]

$$dW_{mag} = 0$$

Total workdone

$$\therefore W_{mag} = 0$$

Hence, workdone by the magnetic force is zero.

* Magnetic force in terms of Different current densities -

Case I:

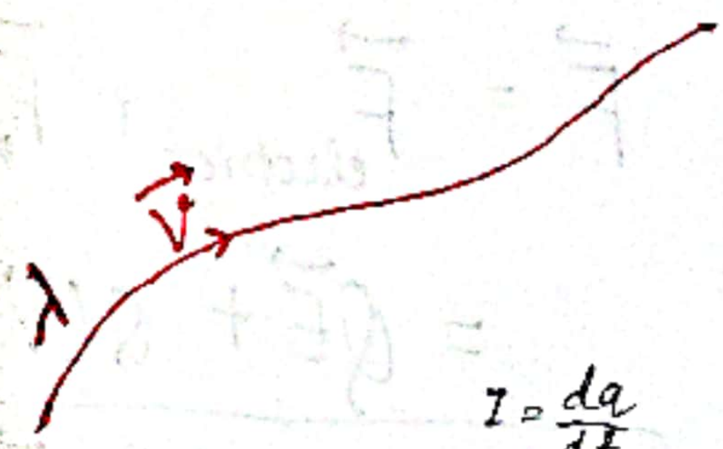
$\Rightarrow$ A line charge λ traveling down a wire at speed v constitutes a current -

$$\boxed{\vec{I} = \lambda \vec{v}}$$

$I = \frac{dq}{dt}$ $\quad dq = \lambda dl$

$\vec{I} = \lambda \frac{d\vec{l}}{dt}$

$\vec{I} = \lambda \vec{v}$



$I = \frac{dq}{dt}$

$dq = \lambda dl$

Then, the magnetic force on the line current is given by -

$$\vec{F}_{mag} = \int (\vec{v} \times \vec{B})\,dq$$

$$= \int (\vec{v} \times \vec{B})\,\lambda\,dl$$

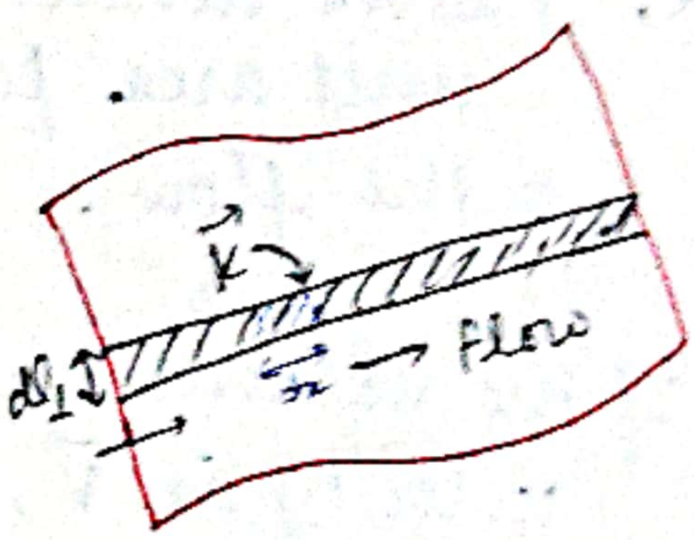
$$\boxed{\vec{F}_{mag} = \int (\vec{I} \times \vec{B})\,dl}$$

Case II

⇒ When charge flows over a surface, we describe it by the surface current density $\vec{K}$.

Consider a "ribbon" of infinitesimal width $dl_\perp$, running parallel to the flow. If the current in this ribbon is $d\vec{I}$, the surface current density is defined as -

$$\vec{K} = \frac{d\vec{I}}{dl_\perp}$$



i.e, $\vec{K}$ is the current per unit width perpendicular to the flow.

If the surface charge density is σ and its velocity is $\vec{v}$, then -

$$\boxed{\vec{K} = \sigma\vec{v}}$$

$$\vec{K} = \frac{d\vec{I}}{dl_\perp} = \frac{dq}{dt\,dl_\perp}$$

$$= \sigma \times \frac{dx\,dl_\perp}{dt\,dl_\perp}$$

$$= \sigma\vec{v}$$

∴ The magnetic force on the surface current is given by -

$$\vec{F}_{mag} = \int (\vec{v} \times \vec{B})\,dq$$

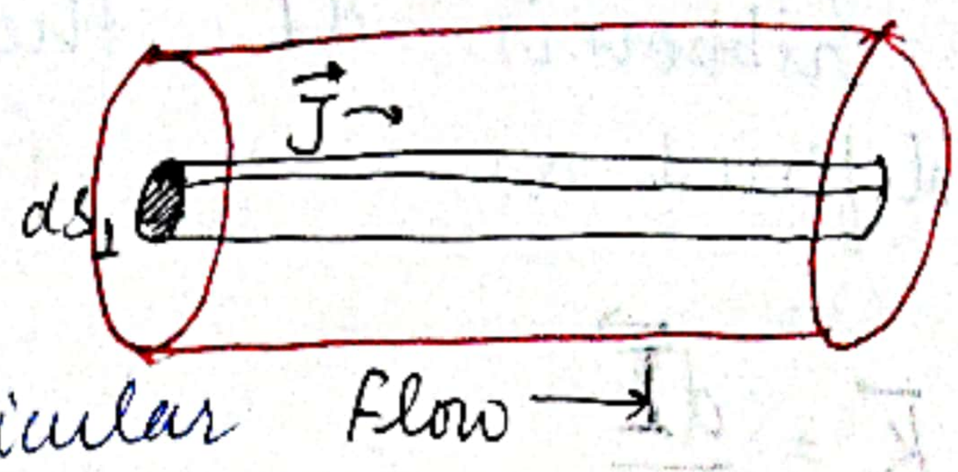
$$= \int (\vec{v} \times \vec{B})\,\sigma\,ds$$

$$\boxed{\vec{F}_{mag} = \int (\vec{K} \times \vec{B})\,ds}$$

Case III :- ⇒ When the flow of charge is distributed throughout a three-dimensional region, we describe it by the volume current density $\vec{J}$.

consider a "tube" of infinitesimal cross section $dS_\perp$, ~~running~~ running parallel to the flow. If the current in this tube is $d\vec{I}$, the volume current density is defined as -

$$\vec{J} \equiv \frac{d\vec{I}}{dS_\perp}$$



ie, $\vec{J}$ is the current per unit area perpendicular to the flow.

If the volume charge density is ρ and velocity is $\vec{v}$, then -

$$\vec{J} = \frac{d\vec{I}}{dS_\perp} = \frac{d\vec{q}}{dt\, dS_\perp} \qquad \rho = \frac{\text{charge}}{\text{Vol.}}$$

$$= \frac{\rho\, dS_\perp\, d\vec{l}}{dt\, dS_\perp}$$

$$= \rho\vec{v}$$

$$\boxed{\vec{J} = \rho\vec{v}}$$

∴ The magnetic force on the volume current is given by -

$$\vec{F}_{mag} = \int (\vec{v} \times \vec{B})\, dq$$

$$= \int (\vec{v} \times \vec{B})\rho\, d\tau$$

$$\boxed{\vec{F}_{mag} = \int (\vec{J} \times \vec{B})\, d\tau}$$

Continuity Equation:-

By definition,

volume current density, $\vec{J} \equiv \frac{d\vec{I}}{dS_\perp}$

Then, the current crossing a surface S can be written as -

$$I = \int_S \vec{J}\, dS_\perp = \int_S \vec{J} \cdot d\vec{S} \qquad d\vec{S} = \hat{n}\, dS$$

The total charge per unit time leaving a volume τ enclosed by the surface S is

$$\oint_S \vec{J}\cdot d\vec{s} = \int_\tau (\vec{\nabla}\cdot\vec{J})\,d\tau \qquad \left[\text{By Gauss's Divergence theorem}\right]$$

Since charge is conserved, the amount of charge coming out from the surface must be equal to the decrease of charge inside the volume τ.

ie,

$$\int_\tau (\vec{\nabla}\cdot\vec{J})\,d\tau = -\frac{\partial}{\partial t}\int_\tau \rho\,d\tau$$

$$\int_\tau (\vec{\nabla}\cdot\vec{J})\,d\tau = -\int_\tau \frac{\partial\rho}{\partial t}\,d\tau$$

[The minus sign reflects the fact that an outward flow decreases the charge left in volume τ.]

Since this applies to any volume, we can write -

$$\vec{\nabla}\cdot\vec{J} = -\frac{\partial\rho}{\partial t}$$

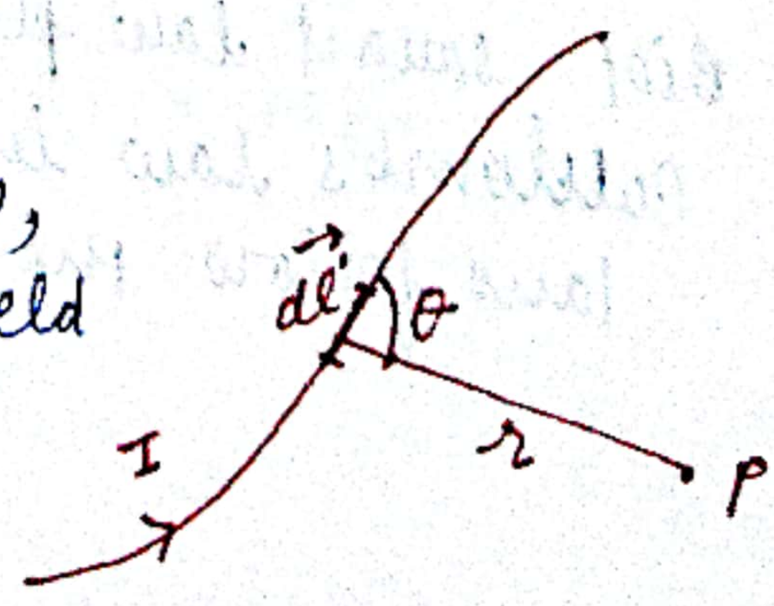
$$\Rightarrow \boxed{\vec{\nabla}\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0}$$

This is the mathematical statement of local charge conservation and is called the continuity equation.

BIOT - SAVART LAW :-

This law provides the magnetic field of a steady line current.

According to Biot-Savart Law, the magnitude of magnetic field at point P due to the small current element dl' depends



upon the following factors:

i) $\propto I$ $\qquad$ dl', I → same dir^n

ii) $\propto dl'$

iii) $\propto \frac{1}{r^2}$

and iv) $\propto \sin\theta$

Thus magnetic field at point P due to the current carrying wire is given by

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{I}\times\hat{r}}{r^2}\, dl'$$

where $\frac{\mu_0}{4\pi}$ is the constant of proportionality ($\mu_0 = 4\pi\times10^{-7}$ N/A^2, is called PERMEABILITY of free space)

Since, the direction of $\vec{I}$ and $d\vec{l}'$ are same, we can also write,

$$\boxed{\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi}\int \frac{\vec{dl'}\times\hat{r}}{r^2}}$$

For surface and volume currents, the Biot-Savart Law becomes

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{K}\times\hat{r}}{r^2}\, ds'$$

and $\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}\times\hat{r}}{r^2}\, d\tau'$, respectively

Biot-Savart law plays a role analogous to Coulomb's law in electrostatics. Both the laws follow the $\frac{1}{r^2}$ dependence.

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i) $\propto I$

ii) $\propto dl'$

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Biot-Savart law plays a role analogous to Coulomb's law in electrostatics. Both the laws follow the $\frac{1}{r^2}$ dependence.

Electrostatics — Magnetostatics

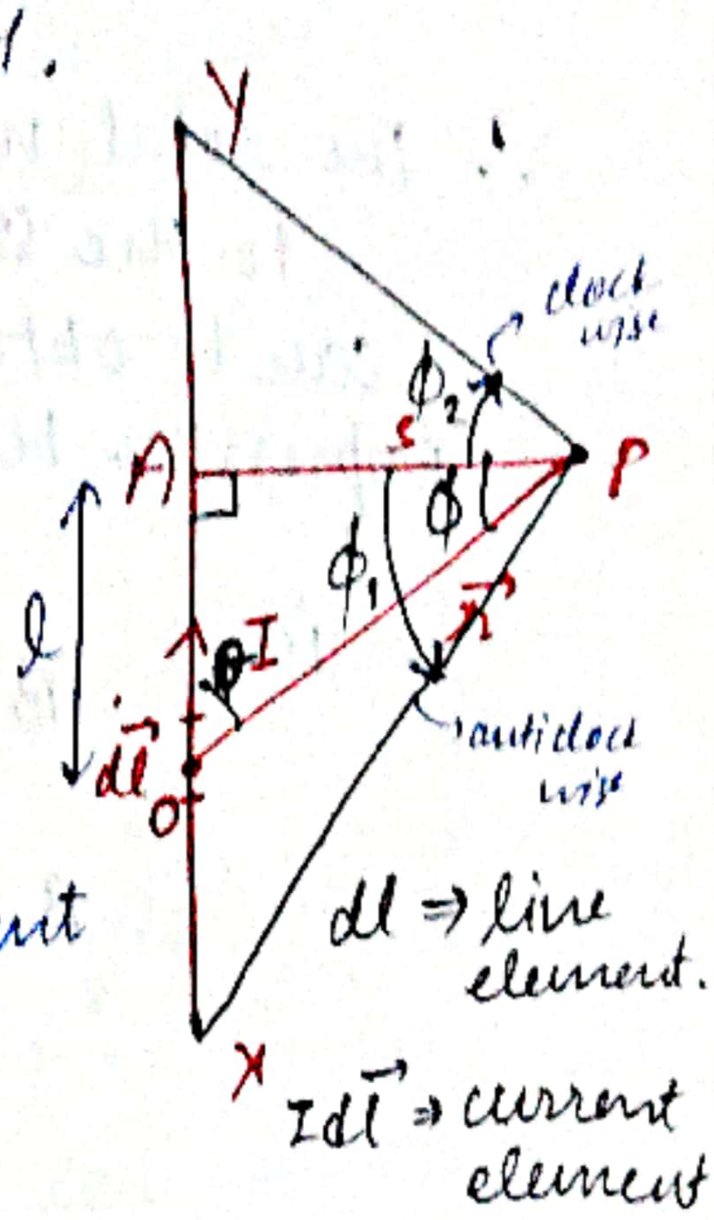
* Coulomb's law ← → Biot-Savart's Law
* Gauss's law ← → Ampere's Law

Application of Biot-Savart LAW :

* **Magnetic field due to a straight wire carrying current** –

Consider a straight wire conductor XY carrying current I in the dirn X to Y. Let 'P' be a point at a perpendicular distance 's' from the straight wire conductor.

ie, $PA = s$



Consider a small current element $Id\vec{l}$ of the straight wire conductor at 'O'. Let $\vec{r}$ be the position vector of point P w.r.t. small current element $Id\vec{l}$ and θ be the angle between $Id\vec{l}$ and $\vec{r}$.

According to Biot-Savart Law, the magnitude of magnetic field at point P due to small current element $Id\vec{l}$ is given by,

$$dB = \frac{\mu_0}{4\pi} \frac{Idl\sin\theta}{r^2} \quad \text{—(1)}$$

Now, in right ΔPAO,

$$\theta + \phi + 90^\circ = 180^\circ$$

$$\Rightarrow \theta = 90^\circ - \phi$$

$$\Rightarrow \sin\theta = \sin(90^\circ - \phi) = \cos\phi$$

$$r = \frac{s}{\cos\phi}$$

And $l = s\tan\phi$

$\Rightarrow dl = s\sec^2\phi\, d\phi$

Putting the values of $\sin\theta$, r and dl in eqn ①, we get.

$$dB = \frac{\mu_0}{4\pi} \frac{I(s\sec^2\phi\, d\phi)\cos\phi}{\left(\frac{s}{\cos\phi}\right)^2}$$

$$dB = \frac{\mu_0}{4\pi} \frac{I}{s} \cos\phi\, d\phi$$

∴ The total magnetic field at point P due to the straight wire carrying current can be obtained by integrating the above expression between the limits $-\phi_1$ to $+\phi_2$.

ie, $B = \int dB$

$$B = \frac{\mu_0 I}{4\pi s} \int_{-\phi_1}^{\phi_2} \cos\phi\, d\phi$$

$$B = \frac{\mu_0 I}{4\pi s} [\sin\phi]_{-\phi_1}^{\phi_2}$$

$(\because \sin(-\theta) = \sin\theta)$ $$B = \frac{\mu_0 I}{4\pi s} [\sin\phi_2 - \sin(-\phi_1)]$$

$$\boxed{B = \frac{\mu_0 I}{4\pi s} [\sin\phi_2 + \sin\phi_1]}$$

In the case of an infinite wire and the point P lies near the center of the wire,

$$\phi_1 = \phi_2 = 90^\circ$$

Then,

$$B = \frac{\mu_0 I}{4\pi s} [\sin 90^\circ + \sin 90^\circ]$$

$$\Rightarrow B = \frac{\mu_0 I}{4\pi s} [1+1]$$

$$\Rightarrow \boxed{B = \frac{\mu_0 I}{2\pi s}}$$

AMPERE'S LAW

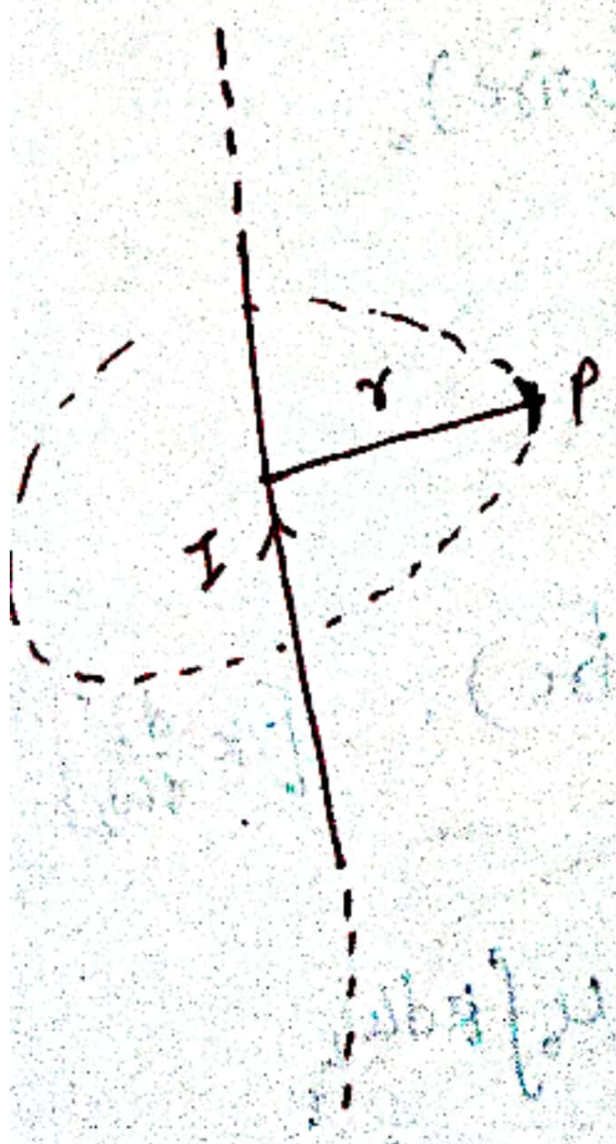
Ampere's Law states that "the line integral of magnetic field around a closed loop is equal to μ_0 times current passing through the loop.

ie,

$$\boxed{\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc.}}$$

Proof:-

Let I be the current flowing through an infinitely long straight conductor.



According to Biot-savart's law, the magnetude of magnetic field at any point P at a perpendicular distance 'r' from the conductor is given by.

$$B = \frac{\mu_0 I}{2\pi r} \quad \text{—— (1)}$$

Now, consider an Amperian loop (a closed circular curve) of radius 'r' with center at the wire and passing through the point P.

Then,

$$\oint \vec{B} \cdot d\vec{l} = \oint B\, dl \cos 0^\circ$$

$$= B \oint dl$$

$$= B \times 2\pi r$$

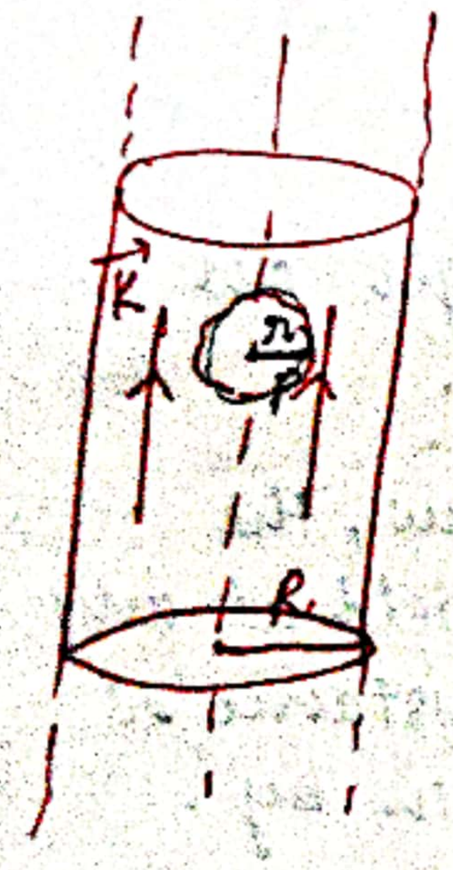
$$= \frac{\mu_0 I}{2\pi r} \times 2\pi r \quad [\text{using eq}^n \text{ (1)}]$$

$$\Rightarrow \boxed{\oint \vec{B} \cdot d\vec{l} = \mu_0 I}$$

Application of Ampere's Law :-

→ **Magnetic field due to Hollow infinite current carrying wire -**

Consider a hollow infinitely long current carrying wire of radius 'R' and surface current density $\vec{K}$.



For a point P inside the wire, Amperian loop of radius 'r' is drawn;

From Ampere's law,

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}.$$

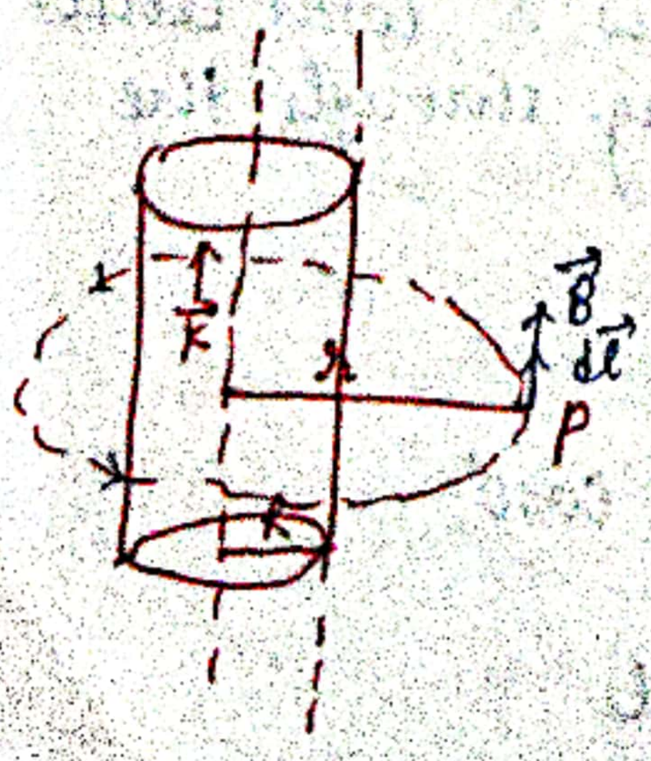
For $r < R$ (inside the wire),

$$I_{enc.} = 0$$

$$\therefore \oint \vec{B} \cdot d\vec{l} = 0$$

$$\Rightarrow \boxed{B = 0}$$

For $r > R$ (outside the wire) $\left[\because K = \frac{dI}{dl_\perp}\right]$



$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}.$$

$$\Rightarrow \oint B\,dl \cos 0^\circ = \mu_0 \int K\,dl_\perp$$

$$\Rightarrow B \oint dl = \mu_0 K \int dl_\perp$$

$$\Rightarrow B \times 2\pi r = \mu_0 K \times 2\pi R$$

$$\boxed{B = \frac{\mu_0 K R}{r}}$$

i.e,

$$B \propto \frac{1}{r}$$

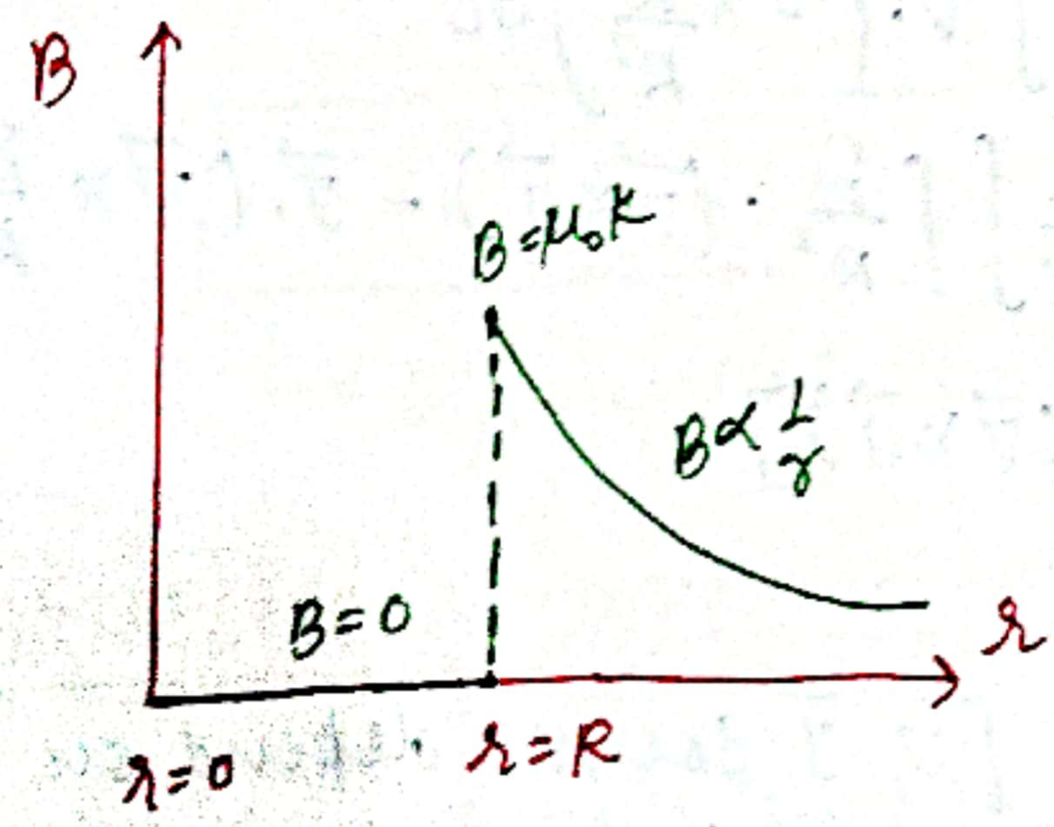
And, for $r = R$ (on the surface of the wire)

$$B = \frac{\mu_0 k R}{R}$$

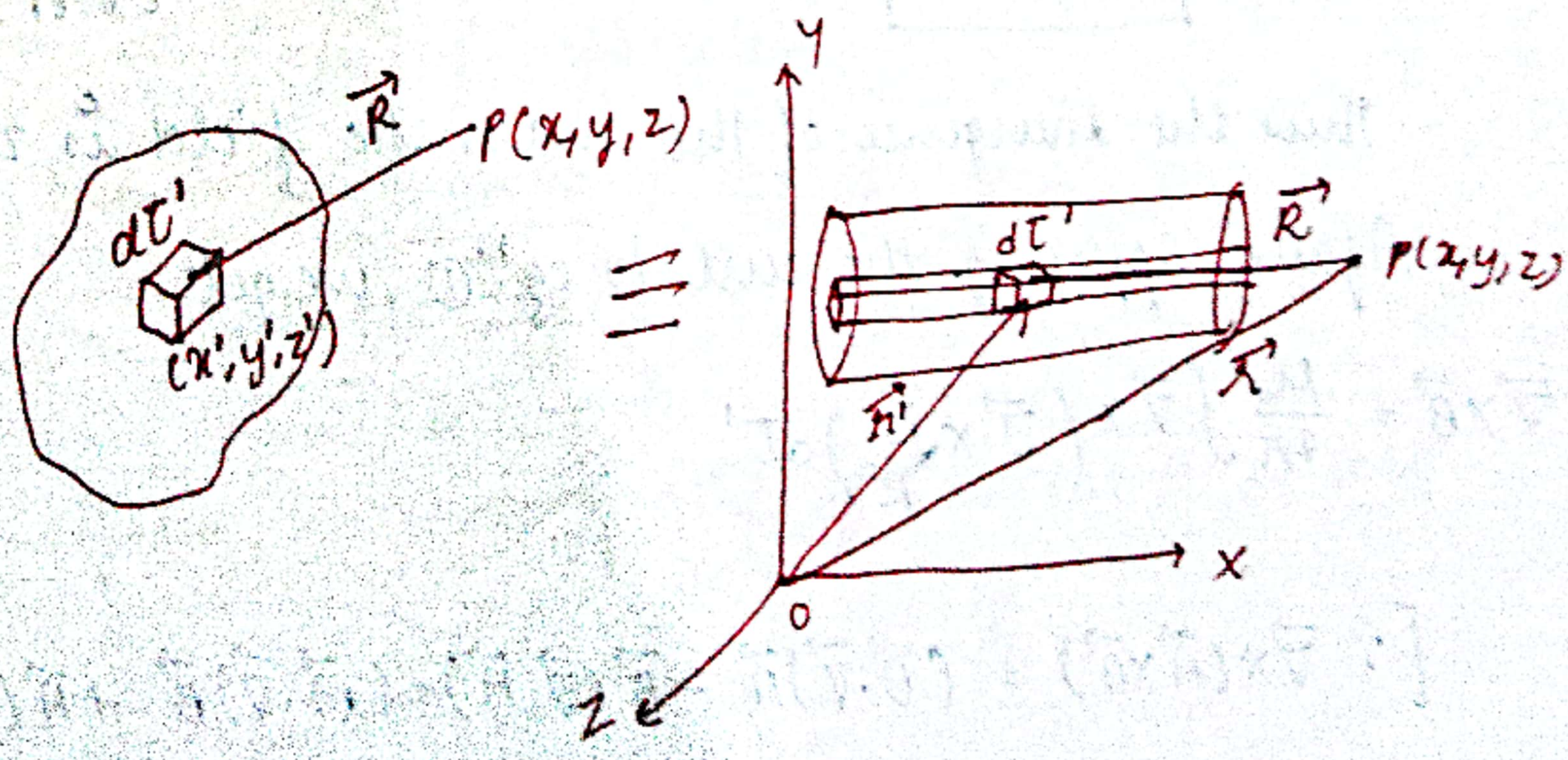
$$\Rightarrow \boxed{B = \mu_0 k}$$

inside : $B = 0$
On the wire : $B = \mu_0 k$
outside : $B \propto \frac{1}{r}$

→ <u>variation of magnetic field with distance from axis of the wire:</u>



DIVERGENCE AND CURL OF MAGNETOSTATIC FIELDS



We know that the Biot - Savart law for volume current reads -

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}')\times\hat{R}}{R^2}\,d\tau' \quad \text{--- (1)}$$

Here,

$\vec{B}$ is a function of (x, y, z)
$\vec{J}$ is a function of (x', y', z')

$$\vec{R} = (x-x')\hat{x} + (y-y')\hat{y} + (z-z')\hat{z}$$

and $d\tau' = dx'dy'dz'$

Applying the divergence to eq^n (1), we get

$$\vec{\nabla}\cdot\vec{B} = \frac{\mu_0}{4\pi}\int \vec{\nabla}\cdot\left(\vec{J}\times\frac{\hat{R}}{R^2}\right)d\tau'$$

$$= \frac{\mu_0}{4\pi}\int\left[\frac{\hat{R}}{R^2}\cdot(\vec{\nabla}\times\vec{J}) - \vec{J}\cdot\left(\vec{\nabla}\times\frac{\hat{R}}{R^2}\right)\right]d\tau'$$

$[\because \vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{A}\cdot(\vec{\nabla}\times\vec{B})]$

But ~~each~~

$\vec{\nabla}\times\vec{J} = 0$ [$\because$ $\vec{J}$ doesn't depend on the unprimed variables (x, y, z)]

and $\vec{\nabla}\times\left(\frac{\hat{R}}{R^2}\right) = 0$

$\therefore$ $\boxed{\vec{\nabla}\cdot\vec{B} = 0}$ $\Rightarrow$ magnetic monopole does not exist.

Thus the divergence of the magnetic field is zero

Again, applying the curl to eq^n (1), we get

$$\vec{\nabla}\times\vec{B} = \frac{\mu_0}{4\pi}\int \vec{\nabla}\times\left(\vec{J}\times\frac{\hat{R}}{R^2}\right)d\tau'$$

$[\because \vec{\nabla}\times(\vec{A}\times\vec{B}) = (\vec{B}\cdot\vec{\nabla})\vec{A} - \vec{B}(\vec{\nabla}\cdot\vec{A}) - (\vec{A}\cdot\vec{\nabla})\vec{B} + \vec{A}(\vec{\nabla}\cdot\vec{B})]$

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$$= \frac{\mu_0}{4\pi}\int\left[\cancel{\left(\frac{\hat{R}}{R^2}\cdot\vec{\nabla}\right)\vec{J}}^{\,0} - \cancel{\frac{\hat{R}}{R^2}(\vec{\nabla}\cdot\vec{J})}^{\,0} - \cancel{(\vec{J}\cdot\vec{\nabla})\frac{\hat{R}}{R^2}}^{\,0} + \vec{J}\left(\vec{\nabla}\cdot\frac{\hat{R}}{R^2}\right)\right]d\tau'$$

$$\Rightarrow \vec{\nabla}\times\vec{B} = \frac{\mu_0}{4\pi}\int \vec{J}\left(\vec{\nabla}\cdot\frac{\hat{R}}{R^2}\right)d\tau'$$

But $\vec{\nabla}\cdot\left(\frac{\hat{R}}{R^2}\right) = 4\pi\delta^3(\vec{R})$

Then,

curl

$$\therefore \vec{\nabla}\times\vec{B} = \frac{\mu_0}{4\pi}\int \vec{J}\,4\pi\delta^3(\vec{R})\,d\tau'$$

$$= \frac{\mu_0}{4\pi}\times 4\pi\int \vec{J}(\vec{r}')\,\delta^3(\vec{r}-\vec{r}')\,d\tau'$$

By dirac-delta function.

Alternative way.
$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$.
By stokes's theorem

$\Rightarrow$

$$\int_S (\vec{\nabla}\times\vec{B})\cdot d\vec{S} = \mu_0\int_S \vec{J}\cdot d\vec{S}$$

$$\int_S (\vec{\nabla}\times\vec{B})\cdot d\vec{S} = \int_S (\mu_0\vec{J})\,d\vec{S}$$

$$\Rightarrow \vec{\nabla}\times\vec{B} = \mu_0\vec{J}$$

$$\boxed{\vec{\nabla}\times\vec{B} = \mu_0\vec{J}(\vec{r})}$$

↳ Differential form of Ampere's law

MAXWELL'S EQUATIONS:

$$\left.\begin{aligned} &\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} \quad \text{(Gauss's law)} \\ &\vec{\nabla}\times\vec{E} = 0 \end{aligned}\right\} \text{Maxwell's eq}^n \text{ for electrostatics}$$

$$\left.\begin{aligned} &\vec{\nabla}\cdot\vec{B} = 0 \\ &\vec{\nabla}\times\vec{B} = \mu_0\vec{J} \quad \text{(Ampere's law)} \end{aligned}\right\} \text{Maxwell's eq}^n \text{ for magnetostatics}$$

MAGNETIC VECTOR POTENTIAL:

we know that

$$\vec{\nabla}\cdot\vec{B} = 0 \quad \text{—— (1)}$$

Again, divergence of curl of a vector $\vec{A}$ is zero.

i.e, $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0$ —— (2)

Comparing eq^n ① and ②, we get -

$$\vec{B} = \vec{\nabla} \times \vec{A} \quad \text{——— ③}$$

Here, $\vec{A}$ is known as MAGNETIC VECTOR POTENTIAL.

Now, By differential form of Ampere's Law,

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

$$\Rightarrow \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \mu_0 \vec{J} \quad [\text{using eq}^n \text{ ③}]$$

$$\Rightarrow \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J} \quad \text{——— ④}$$

If we choose $\vec{A}$ is such a way that $\vec{A}$ is of solenoidal nature i.e,

$\vec{\nabla} \cdot \vec{A} = 0$, then eq^n ④ can be written as -

$$-\nabla^2 \vec{A} = \mu_0 \vec{J}$$

$$\Rightarrow \boxed{\nabla^2 \vec{A} = -\mu_0 \vec{J}} \quad \text{——— ⑤}$$

eq^n ⑤ is known as POISSON'S Equation in magnetostatics.

Then,

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}\, d\tau}{r}, \text{ volume current.}$$

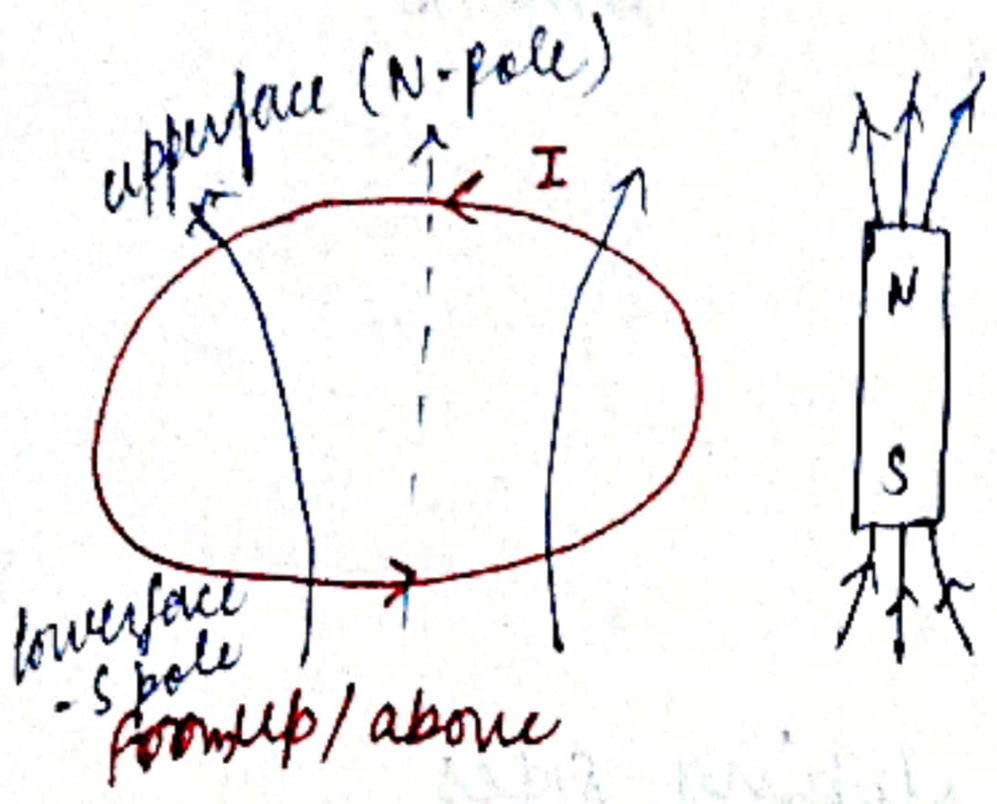
Similarly, for line and surface current,

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{I}\, dl}{r}, \text{ line current}$$

and $$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{K}\, ds}{r}, \text{ surface current}$$

FORCE AND TORQUE ON A MAGNETIC DIPOLE

* A current carrying loop behaves as a magnetic dipole.

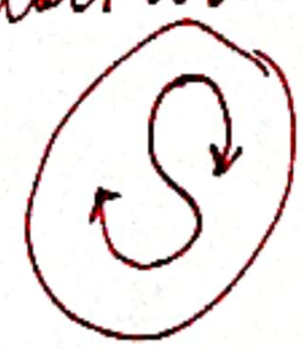


Anticlockwise - N pole

Right hand rule

from below/bottom

clockwise - S pole



The magnetic dipole moment (m) of the current loop is directly proportional to

i) current (I) in the loop

ii) area (A) of the loop

i.e, $m \propto IA$

$m = KIA$

$\Rightarrow \boxed{m = IA}$ [∵ In SI unit, K=1]

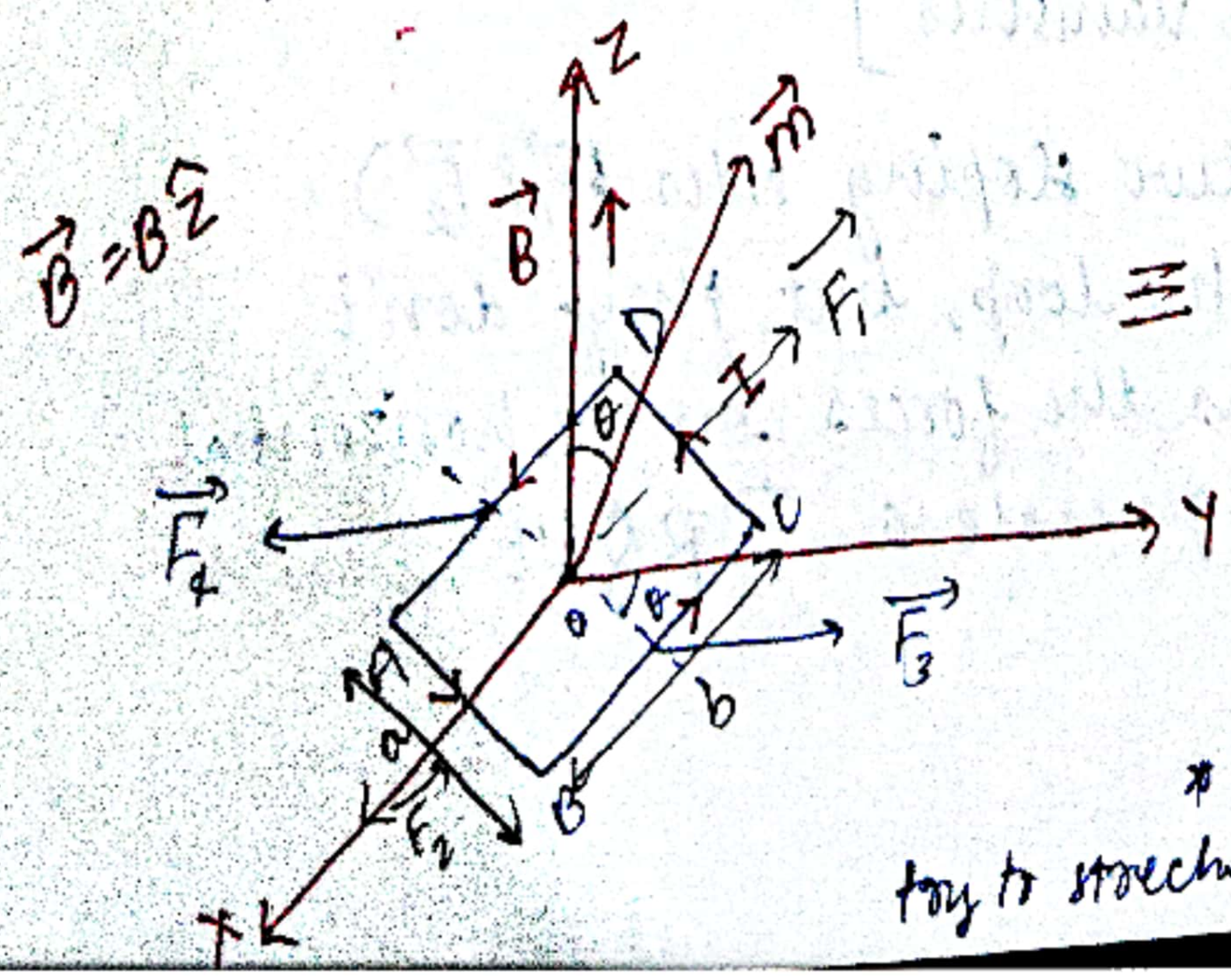
or $\boxed{\vec{m} = I\vec{A}}$

for N turns,

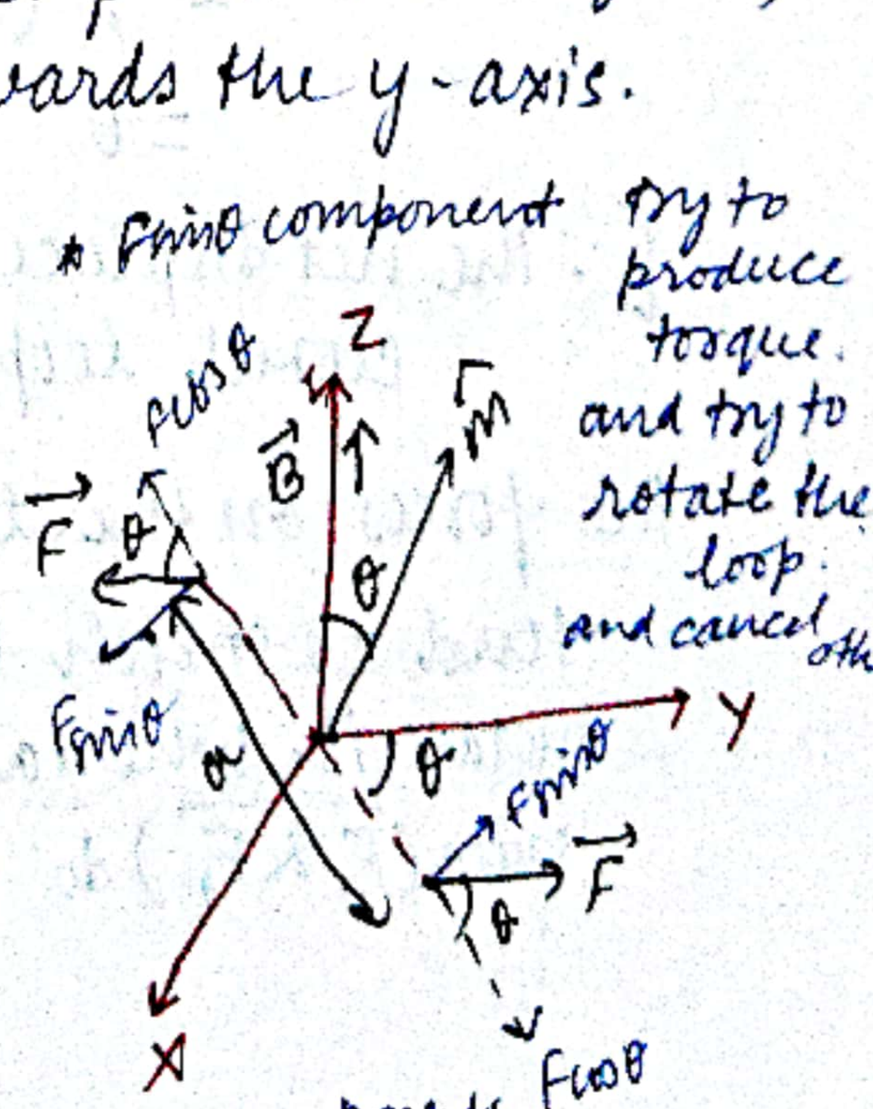
$\boxed{\vec{m} = NI\vec{A}}$

consider a rectangular current loop in a uniform magnetic field $\vec{B}$. Center the loop at the origin, and tilt it at an angle θ towards the y-axis.

Let $\vec{B}$ points in the z-direction.



≡



* F sin θ component try to produce torque and try to rotate the loop and cancel other.

* F cos θ components F cos θ try to stretch the loop and cancel other

Left hand rule
Forefinger → $\vec{B}$
Middle finger - I
Thumb → Force ~~segment~~

segment CD : $\vec{F_1}(-x)$
segment AB : $\vec{F_2}(+x)$
segment BC : $\vec{F_3}(+y)$
segment AD : $\vec{F_4}(-y)$

same length 'a'
same current 'I'
same $\vec{B}$

$|\vec{F_1}| = |\vec{F_2}|$, $|\vec{F_3}| = |\vec{F_4}|$

$$\boxed{\vec{F}_{net} = 0}$$

Thus, the forces on the two sloping sides ($\vec{F_1}$ & $\vec{F_2}$) cancel. The force on the horizontal sides ($\vec{F_3}$ & $\vec{F_4}$) are likewise equal and opposite. Hence, the net force on the rectangular current carrying loop is zero.

<u>Alternative way:-</u>

The net force on a current carrying loop is given by -

$$\vec{F}_{net} = I\oint(d\vec{l} \times \vec{B})$$
$$= I\left(\oint d\vec{l}\right) \times \vec{B}$$
$$= 0$$

[∵ The net displacement $\oint d\vec{l}$ around a closed loop vanishes]

The forces on the two sloping sides ($\vec{F_1}$ & $\vec{F_2}$) tend to stretch the loop, but they don't rotate it. Whereas the forces on the horizontal sides ($\vec{F_3}$ & $\vec{F_4}$) do generate a TORQUE.

By definition,

Torque = Perpendicular distance × Force

$$N = aF\sin\theta$$

$$\Rightarrow \vec{N} = aF\sin\theta\,\hat{x}$$

Now, force on each of these segments is

$$\vec{F} = I\int(d\vec{l}\times\vec{B})$$ (segment BC)

$$\vec{F} = I\int(dl(-\hat{x})\times B\hat{z})$$

$$\vec{F} = IB\int dl\,\hat{y}$$

$$\vec{F} = IBb\,\hat{y}$$

$$\Rightarrow |\vec{F}| = \underline{\underline{IBb}}$$

$$\therefore \vec{N} = aIBb\sin\theta\,\hat{x}$$

$$\vec{N} = \underbrace{Iab}\,B\sin\theta\,\hat{x}$$

$m = I \times$ area

$$\vec{N} = mB\sin\theta\,\hat{x}$$

(where $m = Iab$ is the magnetic dipole moment of the loop)

$$\Rightarrow \boxed{\vec{N} = \vec{m}\times\vec{B}}$$

Torque on the magnetic dipole in the presence of external magnetic field always tries to align the dipole in the direction of the magnetic field.

MAGNETIC MATERIALS :-

Magnetic materials are those materials that can be magnetized. ie, alignment of magnetic dipole in the presence of magnetic field.

On the basis of magnetization, magnetic materials can be classified into -

i) Diamagnet
ii) Paramagnet
iii) Ferromagnet

Diamagnet	Paramagnet	Ferromagnet
i) weakly repelled by a magnet.	i) weakly attracted by a magnet	i) strongly attracted by a magnet.
ii) weakly magnetized in opposite dirⁿ of magnetic field. $\chi_m = -ve$, small	ii) weakly magnetized in the same dirⁿ of magnetic field. $\chi_m = +ve$, small	ii) strongly magnetized in the same dirⁿ of magnetic field. $\chi_m = +ve$, large
iii) They lose their magnetism, on removal of external magnetic field.	iii) They lose their magnetism on removal of external magnetic field.	iii) They do not lose their magnetism, on removal of external magnetic field.
eg. copper, gold etc	eg. Al, sodium etc	eg. iron, nickel etc

MAGNETIZATION ($\vec{M}$).

In the presence of a magnetic field, a matter becomes magnetized i.e, upon microscopic examination, the matter will be found to contain many tiny dipoles, with a net alignment along some dirⁿ.

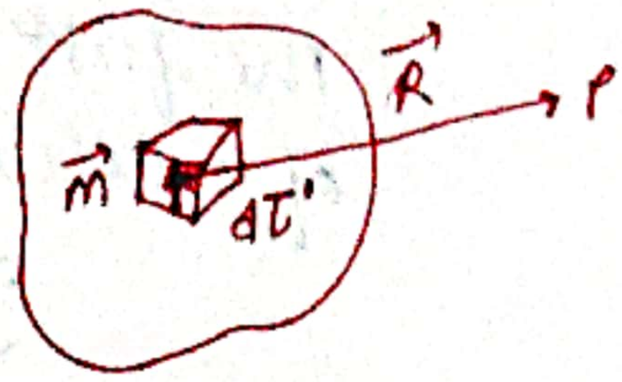
The state of magnetic polarization is described by a vector quantity $\vec{M}$ called MAGNETIZATION and is defined as the magnetic dipole moment per unit volume.

$$\therefore \vec{M} = \frac{\vec{m}}{V}$$

BOUND CURRENTS

Suppose we have a piece of magnetized material for which the magnetic dipole moment per unit volume $\vec{M}$, is given -

The vector potential of a single dipole $\vec{m}$ is given by -



$$d\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{R}}{R^2}$$

In the magnetized object, each volume element $d\tau'$ carries a dipole moment $\vec{m} = \vec{M} d\tau'$, so that the total vector potential is -

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{m}(\vec{r}') \times \hat{R}}{R^2} d\tau'$$

But $\vec{\nabla}'\left(\frac{1}{R}\right) = \frac{\hat{R}}{R^2}$

$$\therefore \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \left[\vec{m}(\vec{r}') \times \vec{\nabla}'\left(\frac{1}{R}\right)\right] d\tau'$$

$$\left[\begin{array}{l} \vec{\nabla} \times (f\vec{A}) = f(\vec{\nabla} \times \vec{A}) - \vec{A} \times \vec{\nabla} f \\ \vec{A} \times \vec{\nabla} f = f(\vec{\nabla} \times \vec{A}) - \vec{\nabla} \times (f\vec{A}) \end{array}\right]$$

$$\Rightarrow \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \left[\frac{1}{R}\left[\vec{\nabla}' \times \vec{m}(\vec{r}')\right] - \vec{\nabla}' \times \left[\frac{1}{R}\vec{m}(\vec{r}')\right]\right] d\tau'$$

$$\Rightarrow \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{1}{R}\left[\vec{\nabla}' \times \vec{m}(\vec{r}')\right] d\tau' - \frac{\mu_0}{4\pi} \int \vec{\nabla}' \times \left[\frac{\vec{m}(\vec{r}')}{R}\right] d\tau'$$

$$\left[\because \int_\tau (\vec{\nabla} \times \vec{F}) d\tau = -\oint_S \vec{F} \times d\vec{s}\right]$$

- corollary of Gauss Divergence Theorem

$$\Rightarrow \frac{\mu_0}{4\pi}\int \frac{\vec{\nabla}' \times \vec{M}(\vec{r}')}{R}\, d\tau' + \frac{\mu_0}{4\pi}\oint_S \left[\frac{\vec{M}(\vec{r}')}{R} \times d\vec{s}'\right]$$

$$\Rightarrow \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\overbrace{\vec{\nabla}' \times \vec{M}(\vec{r}')}^{\vec{J}_b}}{R}\, d\tau' + \frac{\mu_0}{4\pi}\oint_S \frac{\overbrace{\vec{M}(\vec{r}') \times \hat{n}}^{\vec{K}_b}}{R}\, ds'$$

The first term looks like the vector potential of a volume current,

$$\vec{J}_b = \vec{\nabla} \times \vec{M}$$

while the second term looks like the vector potential of a surface current,

$$\vec{K}_b = \vec{M} \times \hat{n}$$, where $\hat{n}$ is the normal unit vector.

$$\therefore \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int_{\tau'} \frac{\vec{J}_b(\vec{r}')}{R}\, d\tau' + \frac{\mu_0}{4\pi}\oint_{S'} \frac{\vec{K}_b(\vec{r}')}{R}\, ds'$$

Thus, the vector potential of a magnetized object is the same as would be produced by a volume current $\vec{J}_b = \vec{\nabla} \times \vec{M}$ throughout the material plus a surface current $\vec{K}_b = \vec{M} \times \hat{n}$ on the boundary.

AMPERE'S LAW IN MAGNETIZED MATERIALS:

We found that the effect of magnetization is to establish bound ~~cu~~ currents $\vec{J}_b = \vec{\nabla} \times \vec{M}$ within the material and $\vec{K}_b = \vec{M} \times \hat{n}$ on the surface. The field due to magnetization of the medium is just the field produced by these bound currents.

If free current represents the current other than bound current, then the total current density can be written as -

$$\vec{J} = \vec{J}_b + \vec{J}_f \quad (1)$$

, Here free current results from actual transport of charge, bound current is due to magnetization and therefore results.

Ampere's law reads,

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}.$$

$\Downarrow$ Stoke's theorem

$$\Rightarrow \int_S (\vec{\nabla} \times \vec{B}) \cdot d\vec{S} = \mu_0 \int_S \vec{J} \cdot d\vec{S}.$$

$$\Rightarrow \int_S (\vec{\nabla} \times \vec{B}) \cdot d\vec{S} = \int_S (\mu_0 \vec{J}) \cdot d\vec{S}$$

$$\Rightarrow \vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

$$\Rightarrow \frac{1}{\mu_0} (\vec{\nabla} \times \vec{B}) = \vec{J}$$

$$\Rightarrow \frac{1}{\mu_0} (\vec{\nabla} \times \vec{B}) = \vec{J}_b + \vec{J}_f \quad [\text{using eq}^n (1)]$$

$$\Rightarrow \frac{1}{\mu_0} (\vec{\nabla} \times \vec{B}) = \vec{\nabla} \times \vec{m} + \vec{J}_f \quad [\because \vec{J}_b = \vec{\nabla} \times \vec{m}]$$

$$\Rightarrow \vec{\nabla} \times \left(\frac{1}{\mu_0} \vec{B} - \vec{m} \right) = \vec{J}_f$$

$$\vec{\nabla} \times \vec{H} = \vec{J}_f \quad \text{--- (2)}$$

where $\vec{H} = \frac{1}{\mu_0} \vec{B} - \vec{M}$, is called Auxiliary field.

Eq^n (2) is the differential form of Ampere's law in magnetized material.

Taking surface integral of eq^n (2), we get -

$$\int_S (\vec{\nabla} \times \vec{H}) \cdot d\vec{S} = \int_S \vec{J}_f \cdot d\vec{S}$$

$\Downarrow$ stoke's theorem

$$\Rightarrow \oint \vec{H} \cdot d\vec{l} = (I_f)_{enc.} \quad \text{— (3)}$$

where $(I_f)_{enc.}$ is the total free current passing through the Amperian loop.

Eq^n (3) is the integral form of Ampere's Law in magnetized material.

MAGNETIC SUSCEPTIBILITY AND PERMEABILITY

For linear medium

$$\vec{m} \propto \vec{H}$$

$$\Rightarrow \boxed{\vec{m} = \chi_m \vec{H}}$$

where the constant of proportionality χ_m is called the MAGNETIC SUSCEPTIBILITY.

$\chi_m > 0$, for paramagnet, ferromagnet

$\chi_m < 0$ for diamagnet

By definition,

$$\vec{H} = \frac{1}{\mu_0} \vec{B} - \vec{M}$$

$$\Rightarrow \frac{1}{\mu_0} \vec{B} = \vec{H} + \vec{m}$$

$$\vec{B} = \mu_0 (\vec{H} + \vec{m})$$

$$\vec{B} = \mu_0 (\vec{H} + \chi_m \vec{H}) \text{, for linear medium}$$

$$\Rightarrow \vec{B} = \mu_0(1+\chi_m)\vec{H}$$

$$\Rightarrow \boxed{\vec{B} = \mu\vec{H}}$$

where $\mu \equiv \mu_0(1+\chi_m)$, is called the PERMEABILITY of the material.

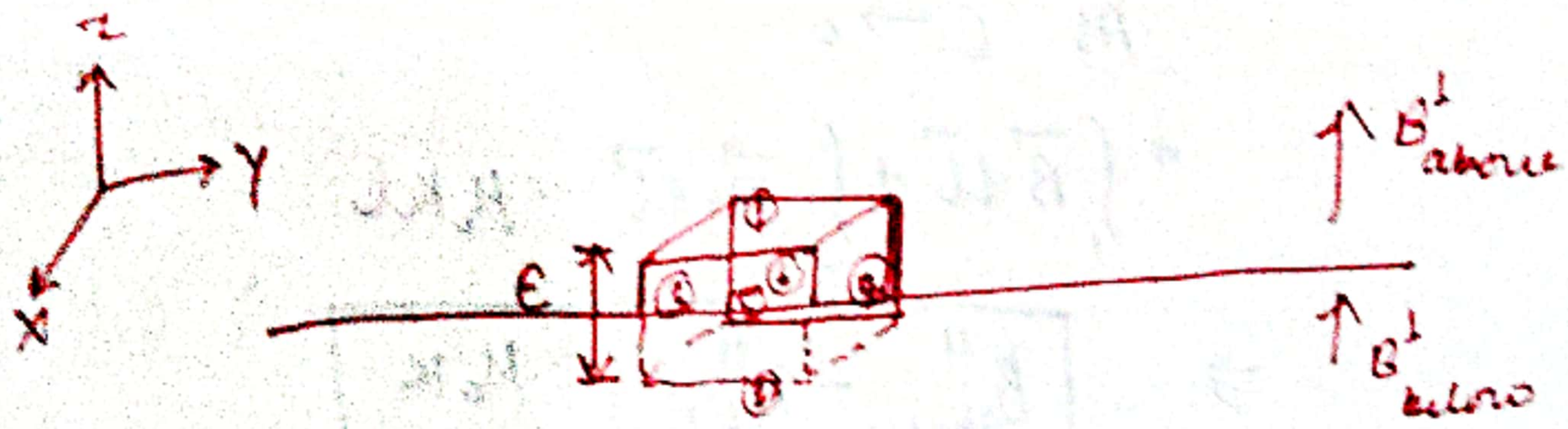
In vaccum, where there is no matter to magnetize, the susceptibility χ_m vanishes and the permeability is μ_0. That's why μ_0 is called the permeability of free space.

BOUNDARY CONDITIONS :-

$\vec{E}, \vec{B}, \vec{V}, \vec{A}$

① Boundary condition of $\vec{B}$

i) ~~Perpendicular~~ Boundary condition for Perpendicular component of $\vec{B}$ –



we know that –

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\Rightarrow \int_{\tau}(\vec{\nabla}\cdot\vec{B})\,d\tau = 0$$

$$\Rightarrow \oint_S \vec{B}\cdot d\vec{S} = 0$$

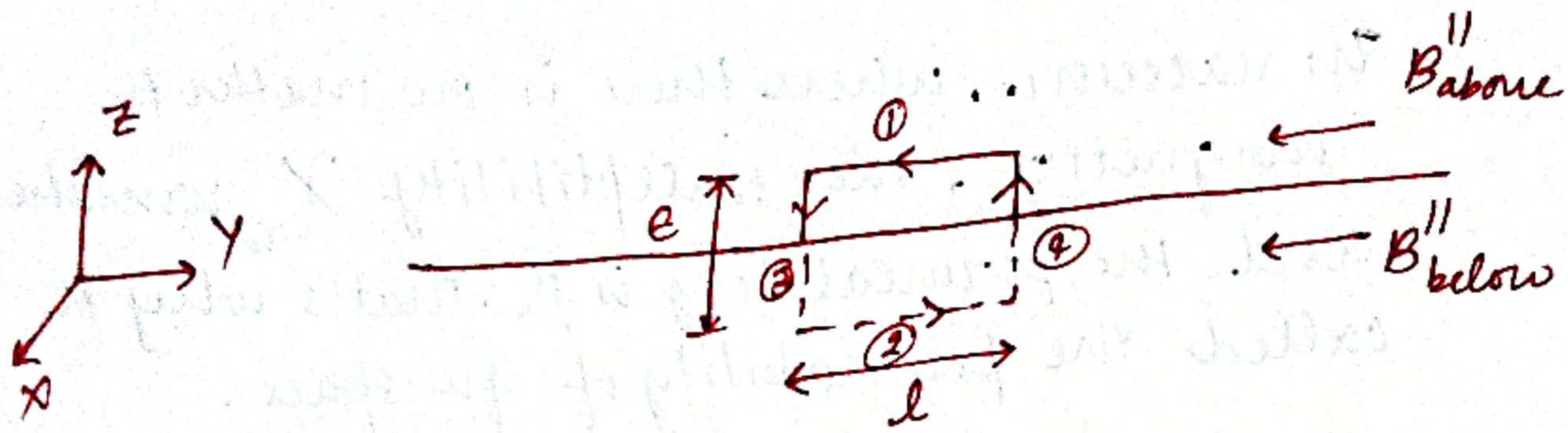
$$\Rightarrow \int_1 \vec{B}\cdot d\vec{S} + \int_2 \vec{B}\cdot d\vec{S} + \cancel{\int_3 \vec{B}\cdot d\vec{S}}^{\,0} + \cancel{\int_4 \vec{B}\cdot d\vec{S}}^{\,0} + \cancel{\int_5 \vec{B}\cdot d\vec{S}}^{\,0} + \cancel{\int_6 \vec{B}\cdot d\vec{S}}^{\,0} = 0$$

As $\epsilon \longrightarrow 0$

$$\int_1 \vec{B}\cdot d\vec{S} + \int_2 \vec{B}\cdot d\vec{S} = 0$$

$$\Rightarrow \boxed{\left[B^{\perp}_{above} - B^{\perp}_{below} \right] = 0}$$

ii) <u>parallel component</u> -



$K = \frac{dI}{dl_1}$

Ampere's law reads -

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc} = \mu_0 \int K \, dl_1$$

$$\Rightarrow \oint \vec{B} \cdot d\vec{l} = \mu_0 K \int dl_1 = \mu_0 K l$$

$$\Rightarrow \int_1 \vec{B} \cdot d\vec{l} + \int_2 \vec{B} \cdot d\vec{l} + \cancel{\int_3 \vec{B} \cdot d\vec{l}}^{\,0} + \cancel{\int_4 \vec{B} \cdot d\vec{l}}^{\,0} = \mu_0 K l$$

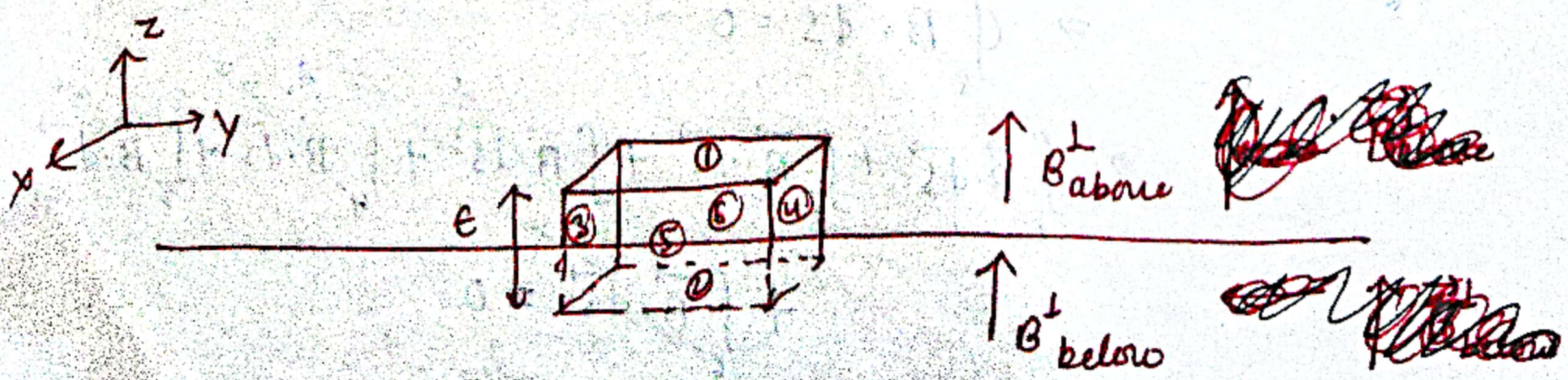
As $\epsilon \to 0$

$$\Rightarrow \int_1 \vec{B} \cdot d\vec{l} + \int_2 \vec{B} \cdot d\vec{l} = \mu_0 K l$$

$$\Rightarrow \boxed{B^{\parallel}_{above} - B^{\parallel}_{below} = \mu_0 K}$$

② <u>Boundary condition of $\vec{H}$</u>

i) <u>perpendicular component</u> -



we know that,

$$\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$$

$$\Rightarrow \vec{\nabla}\cdot\vec{H} = \frac{1}{\mu_0}(\vec{\nabla}\cdot\vec{B}) - \vec{\nabla}\cdot\vec{M}$$

$$\Rightarrow \vec{\nabla}\cdot\vec{H} = -\vec{\nabla}\cdot\vec{M} \qquad [\because \vec{\nabla}\cdot\vec{B} = 0]$$

$$\Rightarrow \int_\tau (\vec{\nabla}\cdot\vec{H})\,d\tau = -\int_\tau (\vec{\nabla}\cdot\vec{M})\,d\tau$$

$$\Rightarrow \oint_S \vec{H}\cdot d\vec{s} = -\oint_S \vec{M}\cdot d\vec{s}$$

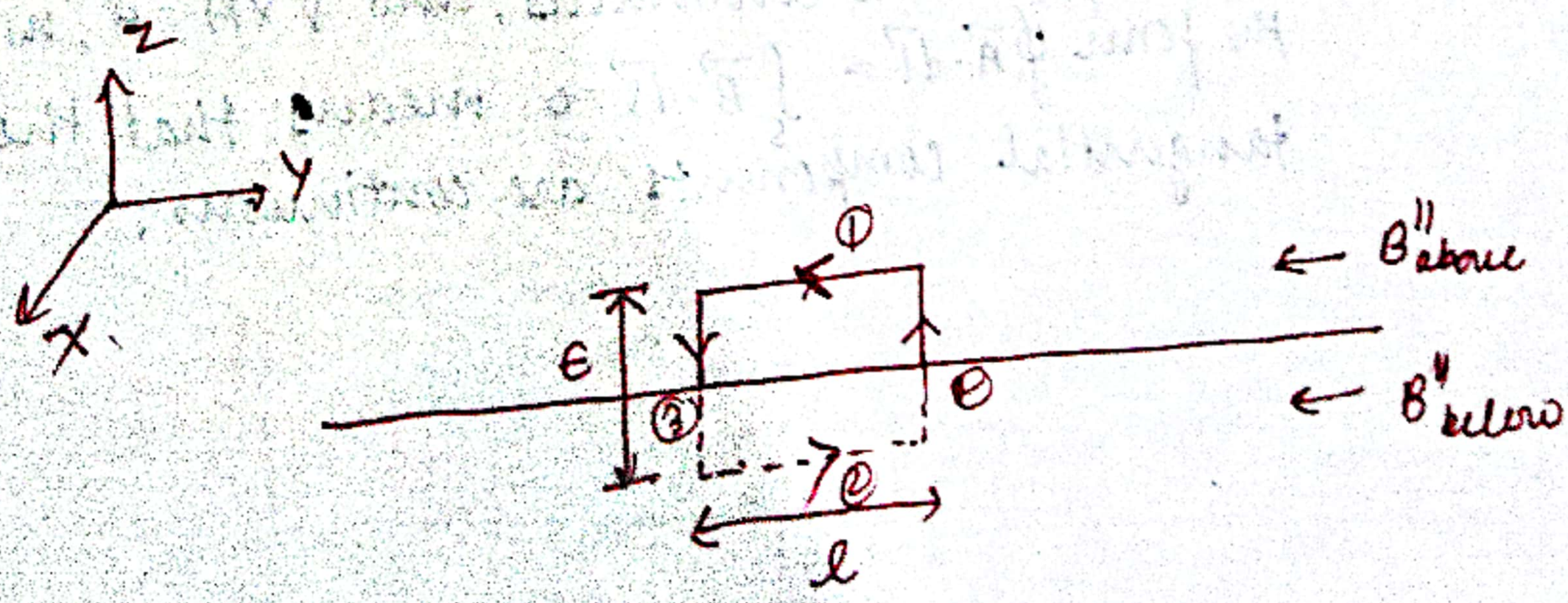
$$\Rightarrow \int_1 \vec{H}\cdot d\vec{s} + \int_2 \vec{H}\cdot d\vec{s} + \cancel{\int_3 \vec{H}\cdot d\vec{s}}^{\,0} + \cancel{\int_4 \vec{H}\cdot d\vec{s}}^{\,0} + \cancel{\int_5 \vec{H}\cdot d\vec{s}}^{\,0} + \cancel{\int_6 \vec{H}\cdot d\vec{s}}^{\,0}$$

$$= -\left[\int_1 \vec{M}\cdot d\vec{s} + \int_2 \vec{M}\cdot d\vec{s} + \cancel{\int_3 \vec{M}\cdot d\vec{s}}^{\,0} + \cancel{\int_4 \vec{M}\cdot d\vec{s}}^{\,0} + \cancel{\int_5 \vec{M}\cdot d\vec{s}}^{\,0} + \cancel{\int_6 \vec{M}\cdot d\vec{s}}^{\,0}\right]$$

As $\epsilon \to 0$

$$\boxed{H^{\perp}_{above} - H^{\perp}_{below} = -\left(M^{\perp}_{above} - M^{\perp}_{below}\right)}$$

ii) <u>Parallel component</u>



The integral form of Ampere's law in magnetized material reads

$$\oint \vec{H}\cdot d\vec{l} = (I_f)_{enc}$$

$$\Rightarrow \oint \vec{H}\cdot d\vec{l} = \int K_f \, dw_\perp = K_f \int dw_\perp = K_f \, l$$

$$\Rightarrow \int_1 \vec{H}\cdot d\vec{l} + \int_2 \vec{H}\cdot d\vec{l} + \cancel{\int_3 \vec{H}\cdot d\vec{l}} + \cancel{\int_4 \vec{H}\cdot d\vec{l}} = K_f \, l$$

As $\epsilon \to 0$.

$$\Rightarrow \int_1 \vec{H}\cdot d\vec{l} + \int_2 \vec{H}\cdot d\vec{l} = K_f \, l$$

$$\Rightarrow \boxed{H^{\parallel}_{above} - H^{\parallel}_{below} = K_f}$$

③ <u>Boundary condition of $\vec{A}$</u> —

Like the scalar potential in ~~the~~ electrostatics, the vector potential is CONTINUOUS across any boundary.

$$A^{\perp}_{above} - A^{\perp}_{below} = 0$$

$$A^{\parallel}_{above} - A^{\parallel}_{below} = 0$$

for $\vec{\nabla}\cdot\vec{A} = 0$ guarantees that the normal component is continuous, and $\vec{\nabla}\times\vec{A} = \vec{B}$, in the form $\oint \vec{A}\cdot d\vec{l} = \int_S \vec{B}\cdot d\vec{S} = 0$ means that the tangential components are continuous.

Electrostatics

$\vec{E}$:

$$E^{\perp}_{above} - E^{\perp}_{below} = \frac{\sigma}{\epsilon_0}$$

$$E^{\parallel}_{above} - E^{\parallel}_{below} = 0$$

$\vec{D}$:

$$D^{\perp}_{above} - D^{\perp}_{below} = \sigma_f$$

$$D^{\parallel}_{above} - D^{\parallel}_{below} = P^{\parallel}_{above} - P^{\parallel}_{below}$$

V:

$$V_{above} - V_{below} = 0$$

Magnetostatics

$\vec{B}$:

$$B^{\perp}_{above} - B^{\perp}_{below} = 0$$

$$B^{\parallel}_{above} - B^{\parallel}_{below} = \mu_0 K$$

$\vec{H}$:

$$H^{\perp}_{above} - H^{\perp}_{below} = -(M^{\perp}_{above} - M^{\perp}_{below})$$

$$H^{\parallel}_{above} - H^{\parallel}_{below} = K_f$$

$\vec{A}$:

$$A^{\perp}_{above} - A^{\perp}_{below} = 0$$

$$A^{\parallel}_{above} - A^{\parallel}_{below} = 0$$

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