Bessel's Equation

The differential equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y=0 \quad (1)$$ is called the

Bessel's differential equation

Consider $y = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \dots \} \quad (2)$

or $y = \sum a_r x^{m+r}$

then we have $\frac{dy}{dx} = \sum a_r (m+r)x^{m+r-1}$ and

$$\frac{d^2y}{dx^2} = \sum a_r (m+r)(m+r-1)x^{m+r-2}$$

Substituting these value in eq. (1) we have

$$x^2\sum a_r(m+r)(m+r-1)x^{m+r-2} + x\sum a_r(m+r)x^{m+r-1} + (x^2-n^2)\sum a_r x^{m+r} = 0.$$

$$\Rightarrow \sum a_r(m+r)(m+r-1)x^{m+r} + \sum a_r(m+r)x^{m+r} + \sum a_r x^{m+r+2} - n^2\sum a_r x^{m+r} = 0$$

$$\Rightarrow \sum a_r x^{m+r}[(m+r)(m+r-1)+(m+r) - n^2] + \sum a_r x^{m+r+2} = 0$$

$$\Rightarrow \sum a_r x^{m+r}[(m+r)^2-n^2] + \sum a_r x^{m+r+2} = 0$$

Then equating the coefficient of x^m to zero (i.e $r=0$)

we have

$$a_0 [(m+0)^2 - n^2] = 0$$

$$\Rightarrow m^2 = n^2 \quad \text{i.e } m = n.$$

Ily equating the coefficient of x^{m+1} to zero (i.e $r=1$)

$$a_1 [(m+1)^2 - n^2] = 0 \Rightarrow (m+1)^2 - n^2 \neq 0 \text{ for } a_1 = 0$$

Ily Equating the coefficient of x^{m+r+2} to zero,

$$a_{r+2} [(m+r+2)^2 - n^2] + a_r = 0$$

$$\Rightarrow \quad a_{r+2} = -\frac{1}{(m+r+2)^2 - n^2} \times a_r \quad - \quad (*)$$

And since $a_1 = 0 \therefore a_3 = a_5 = a_7 = \dots = 0$.

If $r=0$, From $(*)$ $a_2 = -\frac{1}{(m+2)^2 - n^2} \cdot a_0$

If $r=2$, $a_4 = -\frac{1}{(m+4)^2 - n^2} \cdot a_2$

$$= -\frac{1}{[(m+4)^2 - n^2]} \left(-\frac{1}{(m+2)^2 - n^2} \cdot a_0 \right)$$

$$a_4 = \frac{1}{[(m+4)^2 - n^2][(m+2)^2 - n^2]} \cdot a_0$$

Substituting $a_0, a_2, a_4 \ldots$ in eq. (2)

$$y = a_0 x^m - \frac{a_0}{(m+2)^2 - n^2} x^{m+2} + \frac{a_0}{[(m+4)^2 - n^2][(m+2)^2 - n^2]} x^{m+4} + \ldots\ldots$$

$$\therefore y = a_0 x^m \left[1 - \frac{1}{(m+2)^2 - n^2} x^2 + \frac{1}{[(m+4)^2 - n^2][(m+2)^2 - n^2]} x^4 + \ldots\right]$$

Since $m = n$

$$y = a_0 x^n \left[1 - \frac{1}{4(n+1)} x^2 + \frac{1}{4^2 \cdot 2!\,(n+1)(n+2)} x^4 - \ldots\right].$$

where a_0 is arbitiary constant.

For if $m = -n$.

$$y = a_0 x^{-n} \left[1 - \frac{1}{4(-n+1)} x^2 + \frac{1}{4^2 \cdot 2!\,(-n+1)(-n+2)} x^4 - \ldots\right].$$

Which is the general solution of Bessel's Equation.

Bessel's Function $[J_n(x)]$:

The Bessel's equation is $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2-n^2)y = 0$ — ①

And the solution of ① is given by

$$y = a_0 x^n \left[1 - \frac{x^2}{2.2(n+1)} + \frac{x^4}{2.4.2^2(n+1)(n+2)} - \cdots + \right.$$

$$\left. (-1)^r \frac{x^{2r}}{(2^r r!).2^r(n+1)(n+2)(n+3)\cdots(n+r)} + \cdots \right].$$

$$y = a_0 x^n \sum (-1)^r \frac{x^{2r}}{2^{2r} r!\,(n+1)(n+2)\cdots(n+r)} \quad \text{— ②}$$

where a_0 is an arbitrary constant.

If $a_0 = \frac{1}{2^n \sqrt{(n+1)}}$

The above solution is called Bessel's function denoted by $J_n(x)$.

Thus From ②.

$$y = \frac{1}{2^n\sqrt{(n+1)}} \sum_{r=0}^{\infty} \frac{(-1)^r x^{n+2r}}{2^{2r} r!\,(n+1)(n+2)\cdots(n+r)}$$

$\Rightarrow$

$$J_n(x) = \left(\frac{x}{2}\right)^n \left\{ \frac{1}{\sqrt{(n+1)}} - \frac{1}{1!\sqrt{(n+2)}}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\sqrt{(n+3)}}\left(\frac{x}{2}\right)^4 - \frac{1}{3!\sqrt{(n+4)}}\left(\frac{x}{2}\right)^6 + \dots \right.$$

$$\boxed{\cancel{\Rightarrow J_n(x) = \frac{x^n}{2^n \sqrt{n+1}} \sqrt{1+1} \; \frac{x^2}{2}}}$$

$$\Rightarrow \quad J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\sqrt{(n+r)!}} \left(\frac{x}{2}\right)^{n+2r} \quad \text{---} \; (*)$$

$\sqrt{n+1} = n!$

$\sqrt{n+r+1} = \sqrt{(n+r)}!$

If $n=0$, $J_0(x) = \sum \frac{(-1)^r}{r!^2}\left(\frac{x}{2}\right)^{2r}$

or $J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} - \frac{x^6}{2^2.4^2.6^2} + \dots$

If $n=1$, $J_1(x) = \frac{x}{2} - \frac{x^3}{2^2.4} + \frac{x^5}{2^2.4^2.6} - \dots$

Replacing n by $-n$ in $(*)$, we get

$$J_{-n}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\sqrt{-n+r+1}} \left(\frac{x}{2}\right)^{-n+2r}$$

Example: Prove that $J_{-n}(x) = (-1)^n J_n(x)$ where n is a positive integer.

Solution: We know,

$$J_{-n}(x) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{r!\,\Gamma(-n+r+1)} \left(\frac{x}{2}\right)^{-n+2r}$$

$$= \sum_{r=0}^{n-1} \frac{(-1)^r \left(\frac{x}{2}\right)^{-n+2r}}{r!\,\Gamma(-n+r+1)} + \sum_{n=k}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{-n+2r}}{r!\,\Gamma(-n+r+1)}$$

$$= 0 + \sum_{n=k}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{-n+2r}}{r!\,\Gamma(-n+r+1)}$$

on putting $r = n+k$.

$$J_{-n}(x) = \sum_{k=0}^{\infty} \frac{(-1)^{n+k} \left(\frac{x}{2}\right)^{-n+2n+2k}}{(n+k)!\,\Gamma(-n+n+k+1)}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^n (-1)^k \left(\frac{x}{2}\right)^{n+2k}}{(n+k)!\,\Gamma(k+1)}$$

$$= (-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{n+2k}}{(n+k)!\,k!}$$

$$\boxed{J_{-n}(x) = (-1)^n \cdot J_n(x)}$$

Hence proved.

Prove that $J_{1/2}(x) = \sqrt{\left(\frac{2}{\pi x}\right)} \sin x$

Sol: We know, $J_n(x) = \frac{x^n}{2^n \sqrt{(n+1)}} \left[1 - \frac{x^2}{2.2(n+1)} + \frac{x^4}{2.4.2^2(n+1)(n+2)} \ldots\right]$ — (1).

Substitute $n = \frac{1}{2}$ in (1)

$$J_{1/2}(x) = \frac{x^{1/2}}{2^{1/2}\sqrt{\left(\frac{1}{2}+1\right)}} \left[1 - \frac{x^2}{2.2\left(\frac{1}{2}+1\right)} + \frac{x^4}{2.4.2^2(1/2+1)(1/2+2)} \ldots\ldots\right]$$

$$= \frac{\sqrt{x}}{\sqrt{2}\sqrt{3/2}} \left[1 - \frac{x^2}{2.3!} + \frac{x^4}{2.4.2^2.3/2.5/2} \ldots\ldots\right].$$

$$= \frac{\sqrt{x}}{\sqrt{2}.\sqrt{3/2}} \left[1 - \frac{x^2}{2.3!} + \frac{x^4}{2.3.4.5} \ldots\ldots\right].$$

$$= \frac{\sqrt{x}}{\sqrt{2}.\frac{1}{2}\sqrt{\frac{1}{2}}} . \frac{1}{x}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\right].$$

$$= \frac{1}{\sqrt{2x}.\frac{1}{2}\sqrt{\pi}} \sin x = \sqrt{\left(\frac{2}{\pi x}\right)} \sin x \quad //$$

where $\sin x = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right)$

Similarly $J_{-1/2}(x) = \sqrt{\left(\frac{2}{\pi x}\right)} \cos x$ by putting $n = \frac{-1}{2}$ in (1)

(Try yourself).

Recurrence Formula.

Prove that $x J_n' = n J_n - x J_{n+1}$

Sol: We know $J_n = \sum_{r=0}^{\infty} \frac{(-1)^r}{r! \sqrt{(n+r+1)}} \left(\frac{x}{2}\right)^{n+2r}$

Differentiating w.r.t. 'x'.

$$J_n' = \sum \frac{(-1)^r}{r! \sqrt{(n+r+1)}} \cdot (n+2r)\left(\frac{x}{2}\right)^{n+2r-1} \cdot \frac{1}{2}.$$

$$\Rightarrow x J_n' = n \sum \frac{(-1)^r}{r! \sqrt{(n+r+1)}} \left(\frac{x}{2}\right)^{n+2r} + x \sum_{r=0}^{\infty} \frac{(-1)^r \cdot 2r}{2 \cdot r! \sqrt{(n+r+1)}} \left(\frac{x}{2}\right)^{n+2r-1}$$

$$= n J_n + x \sum_{r=0}^{\infty} \frac{(-1)^r}{(r-1)! \sqrt{(n+r+1)}} \left(\frac{x}{2}\right)^{n+2r-1} \quad \Big| \; r! = r(r-1)!$$

$$= n J_n + x \sum_{0}^{\infty} \frac{(-1)^{s+1}}{s! \sqrt{n+s+2}} \left(\frac{x}{2}\right)^{n+2s+1} \quad [\text{Put } r-1=s].$$

$$= \cancel{n J_n - x J_{n+1}}$$

$$= n J_n - x \sum_{0}^{\infty} \frac{(-1)^s}{s! \sqrt{(n+1)+s+1}} \left(\frac{x}{2}\right)^{(n+1)+2s}.$$

$$x J_n' = n J_n - x J_{n+1}.$$

//

lly $x J_n' = -n J_n + x J_{n-1}$ (Try yourself).

Show that $2J_n' = J_{n-1} - J_{n+1}$

Sol: We know,

$$xJ_n' = nJ_n - xJ_{n+1} \quad — (1)$$

$$xJ_n' = -nJ_n + xJ_{n-1} \quad — (2).$$

Adding (1) and (2),

$$2xJ_n' = -xJ_{n+1} + xJ_{n-1}$$

$$2J_n' = J_{n-1} - J_{n+1}$$

Show that $2nJ_n = x(J_{n-1} + J_{n+1})$.

(Subtract (2) from (1)).

Show that $\dfrac{d}{dx}(x^{-n} \cdot J_n) = -x^{-n} J_{n+1}$

Multiply eq (1) by x^{-n-1} we obtain

$$x^{-n} J_n' = n x^{-n-1} J_n - x^{-n} J_{n+1}$$

i.e $\quad x^{-n} J_n' - n x^{-n-1} J_n = -x^{-n} J_{n+1}$

or $\quad \dfrac{d}{dx}(x^{-n} J_n) = -x^{-n} J_{n+1}$ //

Show that $\dfrac{d}{dx}(x^n J_n) = x^n J_{n-1}$ (Try yourself).

Try yourself:

Show that $4J_n''(x) = J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)$

Prove that $\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$.

* Orthogonality of Bessel Function:

If α and β are the roots of $J_n(x)=0$ then

$$\int_0^1 x J_n(\alpha x) \cdot J_n(\beta x)\, dx = 0$$

Ex. prove that $\int_0^1 x [J_n(\alpha x)]^2 dx = \frac{1}{2}[J_{n+1}(\alpha)]^2$

Sol: We know that

$$(\beta^2-\alpha^2)\int_0^1 x J_n(\alpha x)\cdot J_n(\beta x)\, dx = \alpha J_n'(\alpha)\cdot J_n(\beta) - \beta J_n'(\beta)\cdot J_n(\alpha).$$

when $\beta=\alpha$, $J_n(\alpha)=0$.

Let β be a neighbouring of α, which tends to α.

then

$$\lim_{\beta\to\alpha}\int_0^1 x J_n(\alpha x).J_n(\beta x)\, dx = \lim_{\beta\to\alpha}\frac{0+\alpha J_n'(\alpha)\cdot J_n(\beta)}{\beta^2-\alpha^2}$$

As limit is of the form $\frac{0}{0}$, by applying L'Hospital rule we have

$$\int_0^1 x J_n^2(\alpha x)\,dx = \lim_{\beta \to \alpha} \frac{\alpha J_n'(\alpha) \cdot J_n'(\beta)}{2\beta}$$

$$= \frac{1}{2}\left[J_n'(\alpha)\right]^2$$

* Generating function for $J_n(x)$.

Prove that $J_n(x)$ is the coefficient of z^n in the expansion of $e^{\frac{x}{2}\left(z - \frac{1}{z}\right)}$

Sol:- We know that $e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots$

$$e^{\frac{xz}{2}} = 1 + \left(\frac{xz}{2}\right) + \frac{1}{2!}\left(\frac{xz}{2}\right)^2 + \frac{1}{3!}\left(\frac{xz}{2}\right)^3 + \ldots \quad -(1).$$

$$e^{-\frac{x}{2z}} = 1 - \left(\frac{x}{2z}\right) + \frac{1}{2!}\left(\frac{x}{2z}\right)^2 - \frac{1}{3!}\left(\frac{x}{2z}\right)^3 + \ldots \quad (2).$$

On multiplying (1) and (2) we get

$$e^{\frac{x}{2}\left(z - \frac{1}{z}\right)} = \left[1 + \left(\frac{xz}{2}\right) + \frac{1}{2!}\left(\frac{xz}{2}\right)^2 + \frac{1}{3!}\left(\frac{xz}{2}\right)^3 + \ldots\right] \times$$

$$\left[1 - \frac{x}{2z} + \frac{1}{2!}\left(\frac{x}{2z}\right)^2 - \frac{1}{3!}\left(\frac{x}{2z}\right)^3 + \ldots\right] - (3).$$

From (3). the coefficient of z^n is

$$= \frac{1}{n!}\left(\frac{x}{2}\right)^n - \frac{1}{(n+1)!}\left(\frac{x}{2}\right)^{n+2} + \frac{1}{2!}\left(\frac{x}{2}\right)^{n+4} \ldots\ldots$$

$$= J_n(x).$$

Similarly coefficient of z^{-n} in ③ is $= -J_n(x)$

$\therefore$ $e^{\frac{x}{2}\left(z-\frac{1}{z}\right)} = J_0 + zJ_1 + z^2 J_2 + \dots + z^{-1}J_{-1} + z^{-2}J_{-2} + \dots$

$$= \sum_{-\infty}^{\infty} z^n J_n(x)$$

Hence $e^{\frac{x}{2}\left(z-\frac{1}{z}\right)}$ is known as the generating function of Bessel functions

\# Trigonometric expansion involving Bessel Function.

we know that

$$e^{\frac{x}{2}\left(z-\frac{1}{z}\right)} = J_0 + zJ_1 + z^2J_2 + z^3J_3 + \dots + z^{-1}J_{-1} + z^{-2}J_{-2} + \dots \quad ①$$

putting $z = e^{i\theta}$ in ① we get

$$e^{\frac{x}{2}\left(e^{i\theta} - e^{-i\theta}\right)} = J_0 + J_1 e^{i\theta} + J_2 e^{2i\theta} + \dots J_{-1} e^{-i\theta} + J_{-2} e^{-2i\theta} + \dots$$

since $\dfrac{e^{i\theta} - e^{-i\theta}}{2i} = \sin\theta$

$$\therefore e^{ix\sin\theta} = J_0 + J_1 e^{i\theta} + J_2 e^{2i\theta} + J_3 e^{3i\theta} + \dots - J_1 e^{-i\theta} + J_2 e^{-2i\theta} - \dots$$

since $J_{-n} = (-1)^n J_n$.

Now by D'moivre theorem $e^{i\theta} = \cos\theta + i\sin\theta$

$\therefore \cos(x\sin\theta) + i\sin(x\sin\theta) = J_0 + J_1\left(e^{i\theta} - e^{-i\theta}\right) + J_2\left(e^{2i\theta} + e^{-2i\theta}\right) \dots$

Prove that $J_0^2 + 2J_1^2 + 2J_2^2 + \dots = 1$

Sol: We know

$$\cos(x\sin\theta) = J_0 + 2J_2\cos 2\theta + 2J_4\cos 4\theta + \dots \quad -(1)$$

$$\sin(x\sin\theta) = 2J_1\sin\theta + 2J_3\sin 3\theta + 2J_5\sin 5\theta + \dots \quad -(2).$$

Now squaring (1) and integrating w.r.t. 'θ' between the limits 0 and π, we get

$$J_0^2\pi + 2J_2^2\pi + 2J_4^2\pi \dots = \int_0^\pi \cos^2(x\sin\theta)\,d\theta \quad -(3).$$

Since $\int_0^\pi 2\sin^2 n\theta\, d\theta = \pi$, $\int_0^\pi 2\cos^2 n\theta\, d\theta = \pi$

Also squaring (2) and integrating w.r.t. 'θ' between the limits 0 to π., we get

$$2J_1^2\pi + 2J_3^2\pi + 2J_5^2\pi + \dots = \int_0^\pi \sin^2(x\sin\theta)\,d\theta \quad -(4).$$

Adding (3) and (4).

$$\pi(J_0^2 + 2J_1^2 + 2J_2^2 + 2J_3^2 + \dots) = \int_0^\pi \cos^2(x\sin\theta) + \sin^2(x\sin\theta)\,d\theta$$

$$\pi(J_0^2 + 2J_1^2 + 2J_2^2 + 2J_3^2 + \dots) = \int_0^\pi d\theta.$$

$$= [\theta]_0^\pi$$

$$\cancel{\pi}(J_0^2 + 2J_1^2 + 2J_2^2 + \dots) = \cancel{\pi}$$

Hence, $J_0^2 + 2J_1^2 + 2J_2^2 + \dots = 1$ //