Bessel's Integral

* Show that $J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x \sin\theta)\, d\theta$

Sol: We know that

$$\cos(x\sin\theta) = J_0 + 2J_2\cos 2\theta + 2J_4\cos 4\theta + \dots \quad -(1)$$

$$\sin(x\sin\theta) = 2J_1\sin\theta + 2J_3\sin 3\theta + 2J_5\sin 5\theta + \dots \quad -(2)$$

Integrating (1) between the limits 0 and π, we have

$$\int_0^{\pi}\cos(x\sin\theta) = \int_0^{\pi}(J_0 + 2J_2\cos 2\theta + 2J_4\cos 4\theta + \dots)\, d\theta$$

$$= J_0\int_0^{\pi} d\theta + 2J_2\int_0^{\pi}\cos 2\theta\, d\theta + 2J_4\int_0^{\pi}\cos 4\theta\, d\theta + \dots$$

$$= J_0[\theta]_0^{\pi} + 0 + 0$$

$$= J_0\pi \Rightarrow J_0 = \frac{1}{\pi}\int_0^{\pi}\cos(x\sin\theta)\, d\theta$$

* Show that $J_n(x) = \frac{1}{\pi}\int_0^{\pi}\cos(n\theta - x\sin\theta)\, d\theta$

Multiplying (1) by $\cos n\theta$ and integrate between the limits 0 to π

$$\int_0^{\pi}\cos(x\sin\theta)\cos n\theta\, d\theta = \int_0^{\pi} J_0\cos n\theta\, d\theta + 2J_2\cos 2\theta\cos n\theta\, d\theta + \dots$$

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$$= 2J_0 \int_0^{\pi} \cos n\theta \, d\theta + 2J_2 \int_0^{\pi} \cos 2\theta \cos n\theta \, d\theta + \ldots$$

$= 0$ if 'n' is odd — (3)

$= \pi J_n$ is 'n' is even — (4).

Again multiply (2) by $\sin n\theta$ and integrate between 0 to π

$$\int_0^{\pi} \sin(x \sin\theta) \sin n\theta \, d\theta = \int_0^{\pi} 2J_1 \sin\theta \sin n\theta + 2J_3 \sin 3\theta \sin n\theta + \ldots$$

$= 0$ if n is ~~odd~~ even — (5)

$= \pi J_n$ if n is odd — (6)

Adding (3) and (6)

$$\int_0^{\pi} \left[\cos(x \sin\theta) \cos n\theta + \sin(x \sin\theta) \sin n\theta\right] d\theta = \pi J_n$$

$$\int_0^{\pi} \cos(n\theta - x \sin\theta) \, d\theta = \pi J_n \,,$$

$$J_n = \frac{1}{\pi} \int_0^{\pi} \cos(n\theta - x \sin\theta) \, d\theta.$$

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Lecture Notes Mathematics-II
IIIT-Manipur

LEGENDRE'S EQUATION

The differential equation $$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0 \qquad ...(1)$$

is known as Legendre's equation. The above equation can also be written as

$$\frac{d}{dx}\left\{(1-x^2)\frac{dy}{dx}\right\}+n(n+1)y = 0 \qquad n \ \varepsilon \ I$$

This equation can be integrated in series of ascending or descending powers of x. *i.e.*, series in ascending or descending powers of x can be found which satisfy the equation (1).

Let the series in descending powers of x be

$$y = x^m(a_0+a_1x^{-1}+a_2x^{-2}+...) \qquad ...(2)$$

or $$y = \sum_{r=0}^{\infty} a_r x^{m-r}$$

so that $$\frac{dy}{dx} = \sum_{r=0}^{\infty} a_r(m-r)^{m-r-1}$$

and $$\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} a_r(m-r)(m-r-1)x^{m-r-2}$$

Substituting these in (1), we have

$$(1-x^2)\sum_{r=0}^{\infty} a_r(m-r)(m-r-1)x^{m-r-2}-2x\sum_{r=0}^{\infty} a_r(m-r)x^{m-r-1}+n(n+1)\sum_{0}^{\infty} a_r x^{m-r}=0$$

or $$\sum_{r=0}^{\infty} a_r(m-r)(m-r-1)x^{m-r-2}+\{n(n+1)-2(m-r)-(m-r)(m-r-1)\}x^{m-r}a_r=0$$

or $$\sum_{r=0}^{\infty}[(m-r)(m-r-1)x^{m-r-2}+\{n(n+1)-(m-r)(m-r+1)\}x^{m-r}]a_r \equiv 0 \qquad ...(3)$$

The equation (3) is an identity and therefore coefficients of various powers of x must vanish. Now equating to zero the coefficients of x^m from the above we have ($r = 0$)

$$a_0\{n(n+1)-m(m+1)\} = 0$$

But $a_0 \neq 0$, as it is the coefficient of the very first term in the series.

Hence $n(n+1) - m(m+1) = 0$ ....(4)

i.e., $n^2 + n - m^2 - m = 0$ or $(n^2 - m^2) + (n - m) = 0$

or $(n - m)(n + m + 1) = 0$

which gives $m = n$ or $m = -n - 1$ ...(5)

This is important as it determines the index.

Next, equating to zero the coefficient of x^{m-1} by putting $r = 1$,

$$a_1 [n(n+1) - (m-1)m] = 0$$

or $a_1 [(m+n)(m-n-1) = 0$

which gives $a_1 = 0$ ...(6)

Since $(m+n)(m-n-1) \neq 0.$ by (5)

Again to find a relation in successive coefficients a_r, etc., equating the coefficient of x^{m-r-2} to zero, we get

$$(m-r)(m-r-1)a_r + [n(n+1) - (m-r-2)(m-r-1)]a_{r+2} = 0$$

Now $n(n+1) - (m-r-2)(m-r-1) = n^2 + n - (m-r-1-1)(m-r-1)$

$$= -[(m-r-1)^2 - (m-r-1) - n^2 - n]$$

$$= -[(m-r-1+n)(m-r-1-n) - (m-r-1+n)]$$

$$= -[(m-r-1+n)(m-r-1-n-1)]$$

$$= (m-r+n-1)(m-r+n-2)$$

or $(m-r)(m-r-1)a_r - (m-r+n-1)(m-r-n-2)a_{r+2} = 0$

or $$a_{r+2} = \frac{(m-r)(m-r-1)}{(m-r+n-1)(m-r-n-2)} a_r \qquad ...(7)$$

Now since $a_1 = a_3 = a_5 = a_7 = \ldots = 0$

For the two values given by (5) there arises following two cases.

Case I: When $\boldsymbol{m = n}$

$$a_{r+2} = -\frac{(n-r)(n-r-1)}{(2n-r-1)(r+2)} a_r \qquad \text{from (7)}$$

so that, $$a_2 = -\frac{n(n-1)}{(2n-1)2} a_0,$$

$$a_4 = -\frac{(n-2)(n-3)}{(2n-3)\times 4} a_2 = \frac{n(n-1)(n-2)(n-3)}{(2n-1)(2n-3)2.4} a_0$$

and so on and $a_1 = a_3 = a_5 = \ldots = 0$

Hence the series (2) becomes

$$y = a_0\left[x^n - \frac{n(n-1)}{(2n-1)\cdot 2} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{(2n-1)(2n-3)2.4} \cdot x^{n-4} - \ldots\right] \qquad ...(8)$$

which is a solution of (1)

Case II: When $\boldsymbol{m = -(n+1)}$, we have

$$a_{r+2} = \frac{(n+r+1)(n+r+2)}{(r+2)(2n+r+3)} a_r \qquad \text{from (7)}$$

so that
$$a_2 = \frac{(n+1)(n+2)}{2(2n+3)} a_0;$$
$$a_4 = \frac{(n+1)(n+2)(n+3)(n+4)}{2.4(2n+3)(2n+5)} a_0 \qquad \text{and so on.}$$

Hence the series (2) in this case becomes
$$y = a_0\left[x^{-n-1} + \frac{(n+1)(n+2)}{2\cdot(2n+3)} x^{-n-3} + \frac{(n+1)(n+2)(n+3)(n+4)}{2.4(2n+3)(2n+5)} \cdot x^{-n-5} + \dots\right] \qquad ...(9)$$

This gives another solution of (1) in a series of descending powers of x.

Note. If we want to integrate the Legendre's equation in a series of ascending powers of x, we may proceed by taking
$$y = a_0 x^k + a_1 x^{k+1} + a_2 x^{k+2} + \dots = \sum_0^{\infty} a_r x^{k+r}$$

But integration in descending powers of x is more important than that in ascending powers of x.

LEGENDRE'S POLYNOMIAL $P_n(x)$.

Definition:

The Legendre's Equation is
$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)\,y = 0 \qquad ...(1)$$

The solution of the above equation in the series of descending powers of x is
$$y = a_0\left[x^n - \frac{n(n-1)}{(2n-1)2} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{(2n-1)(2n-3)2.4} x^{n-4} \dots\right]$$
where a_0 is an arbitrary constant.

Now if n is a positive integer and $a_0 = \dfrac{1.3.5 \dots (2n-1)}{n!}$ the above solution is $P_n(x)$, so that
$$P_n(x) = \frac{1.3.5 \dots (2n-1)}{n!}\left[x^n - \frac{n(n-1)}{(2n-1)\cdot 2} x^{n-2} + \dots\right]$$

Note 1. This is a terminating series.

When n is even, it contains $\frac{1}{2}n + 1$ terms, the last term being
$$(-1)^{\frac{1}{2}n} \frac{n(n-1)(n-2)\dots 1}{(2n-1)(2n-3)\dots(n+1)\,2.4.6\dots n}$$

And when n is odd it contains $\frac{1}{2}(n+1)$ terms and the last term in this case is
$$(-1)^{\frac{1}{2}(n-1)} \frac{n(n-1)\dots 3.2}{(2n-1)(2n-3)\dots(n+2)\,2.4\dots(n-1)} x$$

$P_n(x)$ is called the *Legendre's functions of the first kind.*

Note. $P_n(x)$ is that solution of Legendre's equation (1) which is equal to unity when $x = 1$.

LEGENDRE'S FUNCTION OF THE SECOND KIND *i.e.* $Q_n(x)$.

Another solution of Legendre's equation

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y = 0$$

when n is a positive integer

$$y = a_0\left[x^{-n-1} + \frac{(n+1)(n+2)}{2\cdot(2n+3)}x^{-n-3}+\dots\right]$$

If we take $$a_0 = \frac{n!}{1.3.5\dots(2n+1)}$$

the above solution is called $Q_n(x)$, so that

$$\boldsymbol{Q_n(x) = \frac{n!}{1.3.5.\dots(2n+1)}\left[x^{-n-1}+\frac{(n+1)(n+2)}{2\cdot(2n+3)}x^{-n-3}+\dots\right]}$$

The series for $Q_n(x)$ is a non-terminating series.

GENERAL SOLUTION OF LEGENDRE'S EQUATION

Since $P_n(x)$ and $Q_n(x)$ are two independent solutions of Legendre's equation, therefore the most general solution of Legendre's equation is

$$y = AP_n(x)+BQ_n(x)$$

where A and B are two arbitrary constants.

RODRIGUE'S FORMULA

$$P_n(x) = \frac{1}{2^n\cdot n!}\frac{d^n}{dx^n}(x^2-1)^n$$ (*A.M.I.ET.E., Winter 2001*)

Proof. Let $$v = (x^2-1)^n \qquad ...(1)$$

Then $$\frac{dv}{dx} = n(x^2-1)^{n-1}(2x)$$

Multiplying both sides by (x^2-1), we get

$$(x^2-1)\frac{dv}{dx} = 2n(x^2-1)^n x.$$

or $$(x^2-1)\frac{dv}{dx} = 2nvx \qquad ...(2)$$

Now differentiating (2), $(n+1)$ times by Leibnitz's theorem, we have

$$(x^2-1)\frac{d^{n+2}v}{dx^{n+2}}+{}^{(n+1)}C_1(2x)\frac{d^{n+1}v}{dx^{n+1}}+{}^{(n+1)}C_2(2)\frac{d^nv}{dx^n}=2n\left[x\frac{d^{n+1}v}{dx^{n+1}}+{}^{(n+1)}C_1(1)\frac{d^nv}{dx^n}\right]$$

or $$(x^2-1)\frac{d^{n+2}v}{dx^{n+2}}+2x[{}^{n+1}C_1-n]\frac{d^{n+1}v}{dx^{n+1}}+2[{}^{n+1}C_2-n.{}^{(n+1)}C_1]\frac{d^nv}{dx^n} = 0$$

or $$(x^2-1)\frac{d^{n+2}v}{dx^{n+2}}+2x\frac{d^{n+1}v}{dx^{n+1}}-n(n+1)\frac{d^nv}{dx^n} = 0 \qquad ...(3)$$

If we put $\frac{d^nv}{dx^n} = y$, (3) becomes

$$(x^2-1)\frac{d^2y}{dx^2}+2x\frac{dy}{dx}-n(n+1)y = 0$$

or $$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y = 0$$

This shows that $y = \dfrac{d^n v}{dx^n}$ is a solution of Legendre's equation.

$$\therefore \qquad C\frac{d^n v}{dx^n} = P_n(x) \qquad ...(4)$$

where C is a constant.

But $$v = (x^2-1)^n = (x+1)^n (x-1)^n$$

so that $$\frac{d^n v}{dx^n} = (x+1)^n \frac{d^n}{dx^n}(x-1)^n + {}^nC_1 \cdot n\,(x+1)^{n-1} \cdot \frac{d^{n-1}}{dx^{n-1}}(x-1)^n + \ldots + (x-1)^n \frac{d^n}{dx^n}(x+1)^n = 0$$

when $x = 1$, $$\frac{d^n v}{dx^n} = 2^n \cdot n\,!$$

All the other terms disappear as $(x - 1)$ is a factor in every term except first.

Therefore when $x = 1$, (4) gives

$$C \cdot 2^n \cdot n\,! = P_n(1) = 1 \qquad P_n(1) = 1$$

$$C = \frac{1}{2^n \cdot n\,!}. \qquad ...(5)$$

Substituting the value of C from (1) in (5) we have

$$P_n(x) = \frac{1}{2^n \cdot n\,!}\frac{d^n v}{dx^n}$$

$$P_n(x) = \frac{1}{2^n \lfloor n}\frac{d^n}{dx^n}(x^2-1)^n$$

Example 1. *Let $P_n(x)$ be the Legendre polynomial of degree n. Show that for any function, $f(x)$, for which the nth derivative is continuous,*

$$\int_{-1}^{1} f(x)\, P_n(x)\, dx = \frac{(-1)^n}{2^n\, n\,!}\int_{-1}^{1} (x^2-1)^n f^n(x)\, dx.$$

Solution. $$\int_{-1}^{1} f(x)\, P_n(x)\, dx = \int_{-1}^{+1} f(x) \cdot \frac{1}{2^n \lfloor n}\frac{d^n}{dx^n}(x^2-1)^n\, dx$$

$$\left[P_n(x) = \frac{1}{2^n \lfloor n}\frac{d^n}{dx^n}(x^2-1)^n \right]$$

$$= \frac{1}{2^n \lfloor n}\int_{-1}^{+1} f(x) \cdot \frac{d^n}{dx^n}(x^2-1)^n\, dx$$

Integrating by parts, we get

$$= \frac{1}{2^n \lfloor n}\left[f(x) \cdot \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n - \int f'(x) \cdot \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\, dx \right]_{-1}^{+1}$$

$$= \frac{1}{2^n \lfloor n}\left[0 - \int_{-1}^{+1} f'(x) \cdot \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\, dx \right]$$

$$= \frac{(-1)}{2^n \lfloor n}\int_{-1}^{+1} f'(x) \cdot \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\, dx$$

Again integrating by parts, we have

$$= \frac{(-1)}{2^n \underline{|n}} \left[f'(x) \frac{d^{\,n-2}}{dx^{n-2}} (x^2-1)^n - \int f''(x) \cdot \frac{d^{\,n-2}}{dx^{n-2}} (x^2-1)^n \, dx \right]_{-1}^{+1}$$

$$= \frac{(-1)^2}{2^n \underline{|n}} \int_{-1}^{+1} f''(x) \frac{d^{\,n-2}}{dx^{n-2}} \cdot (x^2-1)^n \, dx$$

Integrating $(n-2)$ times, by parts, we get

$$= \frac{(-1)^n}{2^n \underline{|n}} \int_{-1}^{+1} f^n(x) \, (x^2-1)^n \, dx$$ **Proved.**

LEGENDRE POLYNOMIALS

$$P_n(x) = \frac{1}{2^n \cdot n!} \cdot \frac{d^n}{dx^n} (x^2-1)^n$$ (Rodrigue's formula)

If $n = 0$, $\quad P_0(x) = \dfrac{1}{2^o \cdot 0!} = 1$

If $n = 1$, $\quad P_1(x) = \dfrac{1}{2^1 \cdot 1!} \dfrac{d}{dx} (x^2-1) = \dfrac{1}{2}(2x) = x$

If $n = 2$, $\quad P_2(x) = \dfrac{1}{2^2 \cdot 2!} \dfrac{d^2}{dx^2} (x^2-1)^2 = \dfrac{1}{8} \dfrac{d}{dx} [2(x^2-1)(2x)]$

$$= \frac{1}{2} [(x^2-1) \cdot 1 + 2x \cdot x] = \frac{1}{2}(3x^2-1)$$

similarly $\quad P_3(x) = \dfrac{1}{2}(5x^3 - 3x)$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$

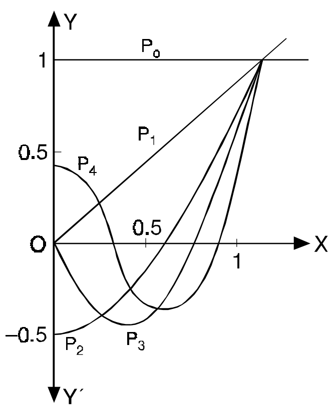
$$P_6(x) = \frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5)$$

....

$$P_n(x) = \sum_{r=0}^{N} \frac{(-1)^r (2n-2r)!}{2^n \cdot r!\,(n-r)!\,(n-2r)!} x^{n-2r}$$

where $\quad N = \dfrac{n}{2}$ if n is even.

$$N = \frac{1}{2}(n-1) \text{ if } n \text{ is odd.}$$



Note. We can evaluate $P_n(x)$ by expanding $(x^2-1)^n$ by Binomial theorem.

$$(x^2-1)^n = \sum_{r=0}^{r=n} {}^nC_r (x^2)^{n-r} (-1)^r = \sum_{r=0}^{r=n} (-1)^r \frac{n!}{r!\,(n-r)!} x^{2n-2r}$$

$$P_n(x) = \frac{1}{2^n \cdot n!} \frac{d^n}{dx^n} (x^2-1)^n = \frac{1}{2^n \cdot n!} \sum_{r=0}^{r=n} (-1)^r \frac{n!}{r!\,(n-r)!} \frac{d^n}{dx^n} (x^{2n-2r})$$

$$= \sum_{r=0}^{N} \frac{(-1)^r (2n-2r)!}{2^n \cdot r!(n-r)!(n-2r)!} x^{n-2r}$$

Either x^0 or x^1 is in the last term.

$$\therefore \qquad n - 2r = 0 \quad \text{or} \quad r = \frac{n}{2} \qquad (n \text{ is even})$$

$$\text{or} \qquad n - 2r = 1 \quad \text{or} \quad r = \frac{1}{2}(n-1) \qquad (n \text{ is odd})$$

Example 2. *Express $f(x) = 4x^3 + 6x^2 + 7x + 2$ in terms of Legendre Polynomials.*

Solution. Let

$$4x^3 + 6x^2 + 7x + 2 \equiv a\,P_3(x) + b\,P_2(x) + c\,P_1(x) + d\,P_0(x) \qquad \text{...(1)}$$

$$\equiv a\left(\frac{5x^3}{2} - \frac{3x}{2}\right) + b\left(\frac{3x^2}{2} - \frac{1}{2}\right) + c(x) + d(1)$$

$$\equiv \frac{5ax^3}{2} - \frac{3ax}{2} + \frac{3bx^2}{2} - \frac{b}{2} + cx + d$$

$$\equiv \frac{5ax^3}{2} + \frac{3bx^2}{2} + \left(\frac{-3a}{2} + c\right)x - \frac{b}{2} + d.$$

Equating the coefficients of like powers of x, we have

$$4 = \frac{5a}{2}, \quad \text{or} \quad a = \frac{8}{5}$$

$$6 = \frac{3b}{2} \quad \text{or} \quad b = 4$$

$$7 = \frac{-3a}{2} + c \quad \text{or} \quad 7 = \frac{-3}{2}\left(\frac{8}{5}\right) + c \quad \text{or} \quad c = \frac{47}{5}$$

$$2 = \frac{-b}{2} + d \quad \text{or} \quad 2 = \frac{-4}{2} + d \quad \text{or} \quad d = 4$$

Putting the values of a, b, c, d in (1), we get

$$4x^3 + 6x^2 + 7x + 2 = \frac{8}{5} P_3(x) + 4\,P_2(x) + \frac{47}{5} P_1(x) + 4\,P_0(x) \qquad \textbf{Ans.}$$

A GENERATING FUNCTION OF LEGENDRE'S POLYNOMIAL

Prove that $P_n(x)$ is the coefficient of z^n in the expansion of $(1 - 2xz + z^2)^{-1/2}$ in ascendig powers of z.

Proof. $(1 - 2xz + z^2)^{-1/2} = [1 - z(2x - z)]^{-1/2}$

$$= 1 + \frac{1}{2} z(2x - z) + \frac{-\frac{1}{2}\left(-\frac{3}{2}\right)}{2!} z^2 (2x - z)^2 + \ldots$$

$$+ \frac{-\frac{1}{2}\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{1}{2} - n + 1\right)}{n!} (-z)^n (2x - z)^n + \ldots \qquad \text{...(1)}$$

Now coefficient of z^n in

$$\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{1}{2} - n + 1\right)}{n!} (-z)^n (2x - z)^n$$

$$= \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{1}{2}-n+1\right)}{n\,!}(-1)^n(2x)^n$$

$$= \frac{1.3.5\ldots(2n-1)}{2^n \cdot n\,!}(2)^n \cdot x^n = \frac{1.3.5\ldots(2n-1)}{n\,!}x^n$$

Coefficient of z^n in

$$\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{1}{2}-n+2\right)}{(n-1)\,!}(-z)^{n-1}(2x-z)^{n-1}$$

$$= \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{1}{2}-n+2\right)}{(n-1)\,!}(-1)^{n-1}[-(n-1)(2x)^{n-2}]$$

$$= \frac{1.3.5\ldots(2n-3)}{2^{n-1}\cdot(n-1)\,!}(2)^{n-2}(n-1)\,x^{n-1} = \frac{1.3.5\ldots(2n-3)}{2\cdot(n-1)\,!}(n-1)\,x^{n-2}$$

$$= \frac{1.3.5\ldots(2n-3)}{2\cdot(n-1)\,!}\times\frac{(2n-1)}{(2n-1)}(n-1)\,x^{n-2} = \frac{1.3.5\ldots(2n-3)(2n-1)}{n\,!}\times\frac{n(n-1)}{2(2n-1)}x^{n-2}$$

Coefficient of z^n in

$$\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{1}{2}-n+3\right)}{(n-2)\,!}z^{n-2}(2x-z)^{(n-2)}$$

$$= \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{1}{2}-n+3\right)}{(n-2)\,!}\times(-1)^{n-2}\times\frac{(n-2)(n-3)}{2\,!}(2x)^{n-4}$$

$$= \frac{1.3.5.\ldots(2n-5)}{2^{n-2}(n-2)\,!}\times\frac{(n-2)(n-3)}{2\,!}(2x)^{n-4}$$

$$= \frac{1.3.5\ldots(2n-5)(2n-3)(2n-1)}{4(n-2)\,!}\times\frac{(n-2)(n-3)}{2(2n-3)(2n-1)}x^{n-4}$$

$$= \frac{1.3.5\ldots(2n-1)}{4n(n-1)(n-2)\,!}\times\frac{n(n-1)(n-2)(n-3)}{2(2n-3)(2n-1)}x^{n-4}$$

$$= \frac{1.3.5\ldots(2n-1)}{n\,!}\times\frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}x^{n-4}$$

and so on.

Thus coefficient of z^n in the expansion of (1)

$$= \frac{1.3.5\ldots(2n-1)}{n\,!}\left[x^n - \frac{n(n-1)}{2(2n-1)}\cdot x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4.(2n-1)(2n-3)}x^{n-4} - \ldots\right]$$

$$= P_n(x)$$

Thus coefficients of z, z^2, z^3 ... etc. in (1) are $P_1(x), P_2(x), P_3(x)\ldots$

Hence

$$(1-2xz+z^2)^{-1/2} = P_0(x) + zP_1(x) + z^2P_2(x) + z^3P_3(x) + \ldots + z^nP_n(x) + \ldots$$

i.e., $$(1-2xz+z^2)^{-1/2} = \sum_{n=0}^{n=\infty} P_n(x)\cdot z^n.$$ **Proved.**

Example 3. *Prove that* $P_n(1) = 1$.

Solution. We know that

$$(1-2xz+z^2)^{-1/2} = 1 + zP_1(x) + z^2P_2(x) + z^3P_3(x) + \ldots + z^nP_n(x) + \ldots$$

Substituting 1 for x in the above equation, we get

$$(1-2z+z^2)^{-1/2} = 1 + zP_1(1) + z^2P_2(1) + z^3P_3(1) + \ldots + z^nP_n(1) + \ldots$$

$$[(1-z)^2]^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(1) \quad \text{or} \quad (1-z)^{-1} = \Sigma z^n P_n(1)$$

or $$\Sigma z^n P_n(1) = (1-z)^{-1} = 1 + z + z^2 + z^3 + \ldots + z^n + \ldots$$

Equating the coefficients of z^n on both sides we get

$$P_n(1) = 1$$ **Proved.**

Example 4. *Show that*

(*i*) $P_{2n}(0) = (-1)^n \dfrac{1.3.5\ldots(2n-1)}{2.4.6\ldots 2n}$ (*ii*) $P_{2n+1}(0) = 0$.

Solution. We know that

$$\Sigma z^{2n} P_{2n}(x) = (1-2xz+z^2)^{-1/2}$$

$$\Sigma z^{2n} P_{2n}(0) = (1+z^2)^{-1/2}$$

$$= 1 + \left(-\frac{1}{2}\right)z^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(z^2)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}(z^2)^3$$
$$+ \ldots + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{1}{2}-n+1\right)}{n!}(z^2)^n + \ldots$$

Equating the coefficient of z^{2n} both sides we get

$$P_{2n}(0) = \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{1}{2}-n+1\right)}{n!}$$

$$= (-1)^n \frac{1.3.5\ldots(2n-1)}{2^n\cdot n!}$$

$$= (-1)^n \frac{1.3.5\ldots(2n-1)}{2.4.6.8\ldots 2n}$$ **Proved.**

Coefficient of $z^{2n+1} = P_{2n+1}(0) = 0$ **Proved.**

8 ORTHOGONALITY OF LEGENDRE POLYNOMIALS

$$\int_{-1}^{+1} P_m(x)\cdot P_n(x)\,dx = 0 \qquad n \neq m$$

Proof. $P_n(x)$ is a solution of

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y = 0 \quad ...(1)$$

$P_m(x)$ is the solution of

$$(1-x^2)\frac{d^2z}{dx^2}-2x\frac{dz}{dx}+m(m+1)z = 0 \quad ...(2)$$

Multiplying (1) by z and (2) by y and subtracting, we get

$$(1-x^2)\left[z\frac{d^2y}{dx^2}-y\frac{d^2z}{dx^2}\right]-2x\left[z\frac{dy}{dx}-y\frac{dz}{dx}\right]+[n(n+1)-m(m+1)]\,yz = 0$$

$$(1-x^{2)}\left[\left\{z\frac{d^2y}{dx^2}+\frac{dz}{dx}\times\frac{dy}{dz}\right\}-\left\{\frac{dy}{dx}\frac{dz}{dx}+y\frac{d^2z}{dx^2}\right\}-2x\left(z\frac{dy}{dx}-y\frac{dz}{dx}\right)+(n-m)(n+m+i)\,yz=0\right.$$

or $$\frac{d}{dx}\left[(1-x^2)\left(z\frac{dy}{dx}-y\frac{dz}{dx}\right)\right]+(n-m)(n+m+1)\,yz = 0$$

Now integrating from -1 to 1, we get

$$\left[(1-x^2)\left(z\frac{dy}{dx}-y\frac{dz}{dx}\right)\right]_{-1}^{+1}+(n-m)(n+m+1)\int_{-1}^{+1} y.z\,dx = 0.$$

or $$0+(n-m)(n+m+1)\int_{-1}^{+1} y\cdot z\,dx = 0$$

or $$\int_{-1}^{+1} P_n(x)\cdot P_m(x)\,dx = 0 \qquad \text{if } n \neq m$$ **Proved**

Example 4. *Prove that*

$$\int_{-1}^{+1} [P_n(x)]^2\,dx = \frac{2}{2n+1}$$ *(U.P. III Semester, Summer, 2004 2002)*

Solution. We know that $(1-2xz+z^2)^{-1/2} = \Sigma z^n P_n(x)$

Squaring both sides we get

$$(1-2xz+z^2)^{-1} = \Sigma z^{2n} P_n^2(x)+2\Sigma z^{m+n} P_m(x)\cdot P_n(x)$$

Integrating both sides between -1 and $+1$, we have

$$\int_{-1}^{+1} \Sigma z^{2n}\cdot P_n^2(x)\,dx+\int_{-1}^{+1} 2\Sigma z^{m+n}\cdot P_m(x)\cdot P_n(x)\,dx=\int_{-1}^{+1}(1-2xz+z^2)^{-1}\,dx$$

$$\int_{-1}^{+1}\Sigma z^{2n} P_n^2(x)\,dx+0 = \int_{-1}^{+1}\frac{1}{1-2xz+z^2}\,dx$$

or $$\Sigma z^{2n}\int_{-1}^{+1} P_n^2(x)\,dx = -\frac{1}{2z}\left[\log(1-2xz+z^2)\right]_{-1}^{+1}$$

$$= -\frac{1}{2z}\log\frac{1-2z+z^2}{1+2z+z^2} = -\frac{1}{2z}\log\left(\frac{1-z}{1+z}\right)^2$$

$$= \frac{1}{z}\log\frac{1+z}{1-z} = \frac{1}{z}[\log(1+z)-\log(1-z)]$$

$$= \frac{1}{z}\left[\left(z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \ldots + \frac{z^{2n+1}}{2n+1} + \ldots\right) - \left(-z - \frac{z^2}{2} - \frac{z^3}{3} - \frac{z^4}{4} - \ldots - \frac{z^{2n+1}}{2n+1} - \ldots\right)\right]$$

$$= \frac{2}{z}\left[z + \frac{z^3}{3} + \frac{z^5}{5} + \ldots + \frac{z^{2n+1}}{2n+1} + \ldots\right] = 2\left[1 + \frac{z^2}{3} + \frac{z^4}{5} + \ldots + \frac{z^{2n}}{2n+1} + \ldots\right]$$

Equating the coefficient of z^{2n} on both sides, we have

$$\int_{-1}^{+1} [P_n(x)]^2\, dx = \frac{2}{2n+1}.$$ **Proved.**

Hence $$\int_{-1}^{+1} P_3^2(x)\, dx = \frac{2}{2 \times 3 + 1} = \frac{2}{7}.$$

Example 5. *Assuming that a polynomial $f(x)$ of degree n can be written as*

$$f(x) = \sum_{0}^{\infty} C_m P_m(x),$$

show that $$C_m = \frac{2m+1}{2} \int_{-1}^{1} f(x) P_m(x)\, dx$$

Solution. $$f(x) = \sum_{0}^{\infty} C_m P_m(x)$$

$$= C_0 P_0(x) + C_1 P_1(x) + C_2 P_2(x) + C_3 P_3(x) + C_4 P_4(x) + \ldots + C_m P_m(x) + \ldots$$

Multiplying both sides by $P_m(x)$, we get

$$P_m(x) f(x) = C_0 P_0(x) P_m(x) + C_1 P_1(x) P_m(x) + C_2 P_2(x) P_m(x) + \ldots + C_m P_m^2(x) + \ldots$$

$$\int_{-1}^{+1} f(x) P_m(x)\, dx = \int_{-1}^{+1} [C_0 P_0(x) P_m(x) + C_1 P_1(x) P_m(x) + C_2 P_2(x) P_m(x) + \ldots + C_m P_m^2(x) + \ldots]\, dx$$

$$= \left[0 + 0 + \ldots + C_m \frac{2}{2m+1} + ..\right] = \frac{2C_m}{2m+1}$$

$$C_m = \frac{2m+1}{2} \int_{-1}^{+1} f(x) P_m(x)\, dx$$ **Proved.**

Example 6. *Using the Rodrigue's formula for Legendre function, prove that*

$$\int_{-1}^{+1} x^m P_n(x)\, dx = 0, \text{ where } m, n \text{ are positive integers and } m < n.$$

Solution. $$\int_{-1}^{+1} x^m P_n(x)\, dx = \int_{-1}^{+1} x^m \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n\, dx$$

$$= \frac{1}{2^n n!} \int_{-1}^{+1} x^m \frac{d^n}{dx^n} (x^2 - 1)^n\, dx$$

On integrating by parts we get

$$= \frac{1}{2^n\, n\,!}\left[\left\{ x^m \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n \right\}_{-1}^{+1} - \int_{-1}^{+1} m\, x^{m-1} \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\, dx\right]$$

$$= 0 - \frac{m}{2^n\, n\,!}\int_{-1}^{+1} x^{m-1}\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\, dx$$

$$\int_{-1}^{+1} x^m P_n(x)\, dx = -\frac{(-1)^2 m\,(m-1)}{2^n\, n\,!}\int_{-1}^{+1} x^{m-2}\frac{d^{n-2}}{dx^{n-2}}(x^2-1)^n\, dx$$

Integrating $m-2$ times, we get

$$= (-1)^m \frac{m\,(m-1)\,\ldots\,1}{2^n\, n\,!}\int_{-1}^{+1}\frac{d^{n-m}}{dx^{n-m}}(x^2-1)^n\, dx$$

$$= \frac{(-1)^m\, m\,!}{2^n\, n\,!}\int_{-1}^{+1}\frac{d^{n-m}}{dx^{n-m}}(x^2-1)^n\, dx$$

$$= \frac{(-1)^m\, m\,!}{2^n\, n\,!}\left[\frac{d^{n-m-1}}{dx^{n-m-1}}(x^2-1)^n\right]_{-1}^{+1} = 0$$ **Ans.**

RECURRENCE FORMULAE FOR $P_n(x)$

Formula 1. $nP_n = (2n-1)\, x\, P_{n-1} - (n-1)\, P_{n-2}$.

Solution. We know that $(1-2xz+z^2)^{-1/2} = \Sigma z^n P_n(x)$

Differentiating w.r.t. 'z', we get

$$-\frac{1}{2}(1-2xz+z^2)^{-3/2}(-2x+2z) = \Sigma n z^{n-1} P_n(x)$$

Multiplying both sides by $(1-2xz+z^2)$, we get

$$(1-2xz+z^2)^{-1/2}(x-z) = (1-2xz+z^2)\,\Sigma n z^{n-1} P_n(x)$$

$$(x-z)\,\Sigma z^n P_n(x) = (1-2xz+z^2)\,\Sigma n z^{n-1} P_n(x) \qquad \ldots(1)$$

Equating the coefficients of z^{n-1} from both sides, we get

$$x P_{n-1} - P_{n-2} = nP_n - 2x(n-1)P_{n-1} + (n-2)P_{n-2}$$

or $$nP_n = (2n-1)\, x\, P_{n-1} - (n-1)\, P_{n-2}.$$ **Proved.**

Formula II. $x P_n' - P'_{n-1} = nP_n$.

Solution. We know that $(1-2xz+z^2)^{-1/2} = \Sigma z^n P_n(x)$ ..(1)

Differentiating (1) with respect to z, we get

$$-\frac{1}{2}(1-2xz+z^2)^{-3/2}(-2x+2z) = \Sigma n z^{n-1} P_n(x)$$

or $$(x-z)(1-2xz+z^2)^{-3/2} = \Sigma n z^{n-1} P_n(x) \qquad \ldots(2)$$

Differentiating (1) with respect to x, we get

$$-\frac{1}{2}(1-2xz+z^2)^{-3/2}(-2z) = \Sigma z^n P'_n(x)$$

or $$z\,(1-2xz+z^2)^{-3/2} = \Sigma z^n P_n'(x) \qquad \ldots(3)$$

Dividing (2) by (3), we get

$$\frac{x-z}{z} = \frac{\Sigma\, nz^{n-1} P_n(x)}{\Sigma\, z^n P_n'(x)}$$

or $$(x-z)\,\Sigma\, z^n P_n'(x) = \Sigma\, nz^n P_n(x)$$

Equating coefficients of z^n from both sides, we get

$$x P_n'(x) - P'_{n-1}(x) = nP_n(x)$$ **Proved.**

Formula III. $P_n' - x P'_{n-1} = nP_{n-1}$

Solution. $$nP_n = (2n-1)\, x P_{n-1} - (n-1) P_{n-2}$$ Recurrence formula *I*

Differentiating the above formula w.r.t. 'x', we get

$$nP_n' = (2n-1) P_{n-1} + (2n-1)\, x P'_{n-1} - (n-1) P'_{n-2}$$

or $$n[P_n' - x P'_{n-1}] - (n-1)[x P'_{n-1} - P'_{n-2}] = (2n-1) P_{n-1}$$

or $$n[P_n' - x P'_{n-1}] - (n-1)[(n-1) P_{n-1}] = (2n-1) P_{n-1}$$

(From formula *II*)

or $$n[P_n' - x P'_{n-1}] = [(n-1)^2 + (2n-1)] P_{n-1} = n^2 P_{n-1}$$

or $$P_n' - x P'_{n-1} = nP_{n-1}.$$ **Proved.**

Formula IV. $P'_{n+1} - P'_{n-1} = (2n+1) P_n$

Solution. $$nP_n = (2n-1)\, x P_{n-1} - (n-1) P_{n-2}$$ (Formula *I*)

Replacing n by $(n+1)$,

$$(n+1) P_{n+1} = (2n+2-1)\, x P_n - nP_{n-1}$$

or $$(n+1) P_{n+1} = (2n+1)\, x P_n - nP_{n-1} \qquad ...(1)$$

Differentiating (1) w.r.t. 'x', we get

$$(n+1) P'_{n+1} = (2n+1) P_n + (2n+1)\, x P_n' - nP'_{n-1} \qquad ...(2)$$

$$x P_n' - P'_{n-1} = nP_n \qquad \text{(Recurrence formula } II) \qquad ...(3)$$

Substituting the value of $x P_n'$ from (3) into (2) we get

$$(n+1) P'_{n+1} = (2n+1) P_n + (2n+1)[nP_n + P'_{n+1}] - nP'_{n-1}$$

or $$(n+1) P'_{n+1} - (n+1) P'_{n-1} = (2n+1)(1+n) P_n$$

or $$P'_{n+1} - P'_{n-1} = (2n+1) P_n$$ **Proved.**

Formula V. $(x^2-1) P_n' = n[x P_n - P_{n-1}]$

Solution. $$P_n' - x P'_{n-1} = nP_{n-1} \qquad ...(1) \qquad \text{[Recurrence Formula } III]$$

$$x P_n' - P'_{n-1} = nP_n \qquad ...(2) \qquad \text{(Recurrence Formula } II)$$

Multiplying (2) by x and subtracting from (1), we get

$$(1-x^2) P_n' = n(P_{n-1} - x P_n).$$ **Proved.**

Formula VI. $(x^2-1) P_n' = (n+1)(P_{n+1} - x P_n)$

Solution. $$nP_n = (2n-1)\, x P_{n-1} - (n-1) P_{n-2}$$ (Recurrence formula *I*)

Replacing n by $(n+1)$, we get

$$(n+1) P_{n+1} = (2n+2-1)\, x P_n - nP_{n-1}$$

$$(n+1) P_{n+1} = (2n+1)\, x P_n - nP_{n-1}$$

which can be written as

$$(n+1)(P_{n+1} - x P_n) = n(x P_n - P_{n-1}) \qquad \text{...(1)}$$

But $\qquad (x^2 - 1) P_n' = n(x P_n - P_{n-1}). \qquad$ ...(2) $\qquad$ (Recurrence formula V)

From (1) and (2), we get

or $\qquad (x^2 - 1) P_n' = (n+1)(P_{n+1} - x P_n).$ **Proved.**

Example 7. *Prove that*

$$\int_{-1}^{+1} x^2 P_{n+1}(x) \cdot P_{n-1}(x)\, dx = \frac{2n(n+1)}{(2n-1)(2n+1)(2n+3)} \qquad \textit{(Bhopal 2000)}$$

Solution. The recurrence formula I is

$$(2n+1) x P_n = (n+1) P_{n+1} + n P_{n-1}$$

Replacing n by $(n+1)$ and $(n-1)$, we have

$$(2n+3) x P_{n+1} = (n+2) P_{n+2} + (n+1) P_n \qquad \text{...(1)}$$

$$(2n-1) x P_{n-1} = n P_n + (n-1) P_{n-2} \qquad \text{...(2)}$$

Multiplying (1) and (2) and integrating in the limits -1 to $+1$, we have

$$(2n+3)(2n-1) \int_{-1}^{+1} x^2 P_{n+1} \cdot P_{n-1}\, dx = n(n+1) \int_{-1}^{1} P_n^2\, dx + n(n+2) \int_{-1}^{+1} P_n \cdot P_{n+2}\, dx$$

$$+ (n^2 - 1) \int_{-1}^{+1} P_n P_{n-2}\, dx + (n-1)(n+2) \int_{-1}^{+1} P_{n+2} \cdot P_{n-2}\, dx$$

$$= n(n+1) \int_{-1}^{1} P_n^2\, dx + 0 + 0 + 0 \qquad \text{(Orthogonality Property)}$$

$$= n(n+1) \cdot \frac{2}{(2n+1)}$$

or $$\int_{-1}^{+1} x^2 \cdot P_{n+1} \cdot P_{n-1}\, dx = \frac{2n(n+1)}{(2n-1)(2n+1)(2n+3)}$$ **Proved.**