Limits

Q. Evaluate $\lim_{\substack{x\to 0\\ y\to 0}} (x^3+y^3)$

Sol^n

$$\lim_{\substack{x\to 0\\ y\to 0}} (x^3+y^3) = \lim_{y\to 0} (0^3+y^3) = 0+0 = 0.$$

limit exist

$\lim_{\substack{x\to x_0\\ y\to y_0}} f = \lim_{\substack{y\to y_0\\ x\to x_0}}$

$$\lim_{\substack{x\to 0\\ y\to 0}} (x^3+y^3) = \lim_{x\to 0} (x^3+0^3) = 0+0 = 0$$

Putting $y = mx$:-

$$\lim_{\substack{x\to 0\\ y\to 0}} (x^3+m^3x^3) = \lim_{x\to 0} x^3(1+m^3) = 0$$

$y = mx^2$

$$\lim_{\substack{x\to 0\\ y\to 0}} (x^3+m^3x^6) = \lim_{x\to 0} x^3(1+m^3x^2) = 0$$

$\Rightarrow$ limit exists and limit value is 0.

Q. $\lim_{\substack{x\to 0\\ y\to 0}} \frac{y^2-x^2}{y^2+x^2}$, $x\neq 0, y\neq 0$.

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{y^2-x^2}{y^2+x^2} = \lim_{y\to 0} \frac{\cancel{y^2}}{\cancel{y^2}} = 1$$

$\neq$

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{y^2-x^2}{y^2+x^2} = \lim_{x\to 0} \frac{-\cancel{x^2}}{\cancel{x^2}} = -1$$

$\therefore$ limit does not exist.

Q. $\lim_{\substack{x \to 1 \\ y \to 2}} \frac{x^2+2y}{x+y^2}$

$\Rightarrow \lim_{\substack{x \to 1 \\ y \to 2}} \frac{x^2+2y}{x+y^2} = \lim_{y \to 2} \frac{1+2y}{1+y^2} = \frac{1+4}{1+4} = \underline{\underline{1}}$

$\Rightarrow \lim_{\substack{x \to 1 \\ y \to 2}} \frac{x^2+2y}{x+y^2} = \lim_{x \to 1} \frac{x^2+4}{x+4} = \frac{1+4}{1+4} = 1.$

$\therefore$ limit exists with value $\underline{\underline{1}}$.

Q. $\lim_{\substack{x \to \infty \\ y \to 3}} \frac{2x-3}{x^3+4y^3}$

$\lim_{\substack{x \to \infty \\ y \to 3}} \frac{2x-3}{x^3+4y^3} = \lim_{x \to \infty} \frac{2x-3}{x^3+108} = \lim_{x \to \infty} \frac{\cancel{x}(2+3/x)}{x^2 \cancel{x^3}(1+108/x^3)} = \underline{\underline{0}}$

$\lim_{\substack{x \to \infty \\ y \to 3}} \frac{2x-3}{x^3+4y^3} = \lim_{y \to 3} \left(\lim_{x \to \infty} \frac{x(2-3/x)}{x^3(1+4y^3/x^3)} \right) = \lim_{y \to 3} (0) = \underline{\underline{0}}$

Q. $\lim_{\substack{x \to 1 \\ y \to 2}} \frac{2x^2+y^2}{2xy}$

$\lim_{\substack{x \to 1 \\ y \to 2}} \frac{2x^2+y^2}{2xy} = \lim_{y \to 2} \frac{2+y^2}{2y} = \frac{6}{4} = 3/2$

$\lim_{\substack{x \to 1 \\ y \to 2}} \frac{2x^2+y^2}{2xy} = \lim_{x \to 1} \frac{2x^2+4}{4x} = \frac{6}{4} = 3/2$

Q. $\lim_{\substack{x \to 2 \\ y \to 3}} \frac{x^3+y^2}{x^2-y} = \lim_{y \to 3} \frac{8+y^2}{4-y} = \frac{8+9}{4-3} = \frac{17}{1} = \underline{\underline{17}}$

$\lim_{\substack{x \to 2 \\ y \to 3}} \frac{x^3+y^2}{x^2-y} = \lim_{x \to 2} \frac{x^3+9}{x^2-3} = \frac{8+9}{4-3} = \underline{\underline{17}}$

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Q) $\lim_{\substack{x\to\infty\\ y\to 3}} \frac{2xy-3}{x^3+4y^3} = \lim_{x\to\infty} \frac{6x-3}{x^3+108} = \lim_{x\to\infty} \frac{x(6-3/x)}{x^3(1+108/x^3)} = 0$

$\lim_{\substack{x\to\infty\\ y\to 3}} \frac{2xy-3}{x^3+4y^3} = \lim_{y\to 3}\left(\lim_{x\to\infty}\left(\frac{x(2y-3/x)}{x^3(1+4y^3/x^3)}\right)\right) = \lim_{y\to 3}(0) = 0$

Q. $\lim_{\substack{x\to 0\\ y\to 0}} \frac{xy}{y-x^2} = \lim_{y\to 0} \frac{0(y)}{y-0^2} = 0$

$\lim_{\substack{x\to 0\\ y\to 0}} \frac{xy}{y-x^2} = \lim_{x\to 0} \frac{x(0)}{0-x^2} = 0$

put $y = mx$

$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x(mx)}{mx - x^2} = \lim_{\substack{x\to 0\\ y\to 0}} \frac{mx^2}{mx-x^2} = \frac{0}{0} \Rightarrow$ limit dne

Q ~~$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x-y}{x^2+y^2} = \lim_{y\to 0} \frac{0-y}{0+y^2} =$~~

Q $\lim_{\substack{x\to 1\\ y\to 1}} \frac{xy-2x}{xy-2y} = \lim_{y\to 1} \frac{y-2}{y-2} = 1$

$\lim_{\substack{x\to 1\\ y\to 1}} \frac{xy-2x}{xy-2y} = \lim_{x\to 1} \frac{x-2x}{x-2} = \frac{1-2}{1-2} = 1$

limit exists with value 1

Q7) $\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^3+2y^3}{x^2+4y^2} = \lim_{y\to 0} \frac{2y^3 y}{4y^2} = 0$

$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^3+2y^3}{x^2+4y^2} = \lim_{x\to 0} \frac{x^3}{x^2} = 0$

$y = mx \Rightarrow \lim_{\substack{x\to 0\\ y\to 0}} \frac{x^3+2m^3x^3}{x^2+4m^2x^2} = \lim_{\substack{x\to 0\\ y\to 0}} \frac{x\,x^2(2m^3+1)}{x^2(1+4m^2)} = 0$

$y = mx^2 \Rightarrow \lim_{x\to 0} \frac{x^3+2m^3x^6}{x^2+4m^2x^4} = \lim_{x\to 0} \frac{x^3(1+2m^3x^3)}{x^2(1+4m^2x^2)} = 0$ | limit exists with value 0

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(8) $\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2y^3}{x^2+y^2} = \lim_{y\to 0} 0 = 0$

$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2y^3}{x^2+y^2} = \lim_{x\to 0} 0 = 0$

$y = mx$;- $\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2(m^3x^3)}{x^2+m^2x^2} = \lim_{x\to 0} \frac{m^3x^3}{1+m^2} = 0$

$y = mx^2$;- $\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2(m^3x^6)}{x^2+m^2x^4} = \lim_{x\to 0} \frac{m^3x^6}{1+m^2x^2} = 0$

limit exist with value 0

(9) $\lim_{\substack{x\to 0\\ y\to 0}} \frac{xy+2}{x^2+y^2} = \lim_{y\to 0} \frac{(0+2)}{0+y^2} = \infty$

$\lim_{\substack{x\to 0\\ y\to 0}} \frac{xy+2}{x^2+y^2} = \lim_{x\to 0} \left(\frac{0+2}{x^2+0}\right) = \infty$

$y = mx$;- $\lim_{\substack{x\to 0\\ y\to 0}} \frac{mx^2+2}{x^2+m^2x^2} = \frac{0+2}{0+0} = \infty$

$y = mx^2$;- $\lim_{\substack{x\to 0\\ y\to 0}} \frac{mx^3+2}{x^2+m^2x^4} = \frac{2}{0} = \infty$

$\sqrt{p - x^2}$

Continuity of a function:- a func^n $f(x,y)$ is cont: at pt (x_0,y_0).

Rule:- (a) limit exists i.e., $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists.

(b) func^n should be well defined at (x_0,y_0)

(c) ~~f(x) = lim~~ $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$.

A function is continuous if it is continous at every point of its domain.

Q Show that $f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$ is continuous at every pt. ~~of~~ except the origin.

(a) limit exists at (0,0)?

$$\lim_{\substack{x\to0\\y\to0}} \frac{2xy}{x^2+y^2} = \lim_{y\to0} 0 = 0.$$

$$\lim_{\substack{x\to0\\y\to0}} \frac{2xy}{x^2+y^2} = \lim_{x\to0} 0 = 0$$

put, $y = mx$:- $\lim_{\substack{x\to0\\y\to0}} \frac{2x(mx)}{x^2+m^2x^2} = \lim_{\substack{x\to0\\y\to0}} \frac{2m}{1+m^2} = \frac{2m}{1+m^2}$ //

$\therefore$ limit does not exists.//

Discont.

Q Show that the function $f(x,y) = \dfrac{2x^2y}{x^4+y^2}$ has no limits as (x,y) approaches to $(0,0)$.

$$\lim_{\substack{x\to0\\y\to0}} f(x,y) = \lim_{\substack{x\to0\\y\to0}} \frac{2x^2y}{x^4+y^2} = \lim_{y\to0} 0 = 0.$$

$$\lim_{\substack{x\to0\\y\to0}} f(x,y) = \lim_{\substack{x\to0\\y\to0}} \frac{2x^2y}{x^4+y^2} = \lim_{x\to0} 0 = 0.$$

Putting $y = mx$

$$\lim_{\substack{x\to0\\y\to0}} \frac{2mx^3}{x^4+m^2x^2} = \lim_{x\to0} \frac{2mx}{x^2+m^2} = 0.$$

Putting $y = mx^2$

$$\lim_{\substack{x\to0\\y\to0}} \frac{2x^4m}{x^4+m^2x^4} = \lim_{x\to0} \frac{2m}{1+m^2} = \frac{2m}{1+m^2}$$

$\therefore$ limit does not exists at $(x,y) = (0,0)$

Q. $\lim_{(x,y)\to(0,0)} \frac{3x^2-y^2+5}{x^2+y^2+2}$

$\lim_{\substack{x\to 0\\ y\to 0}} \frac{3x^2+y^2+5}{x^2+y^2+2} = \lim_{y\to 0} \frac{y^2+5}{y^2+2} = \frac{5}{2}$

$\lim_{\substack{x\to 0\\ y\to 0}} \frac{3x^2+y^2+5}{x^2+y^2+2} = \lim_{x\to 0} \frac{3x^2+5}{x^2+2} = \frac{5}{2}.$

At what points (x,y) in the plane are the functions cont?

Q. $f(x,y) = \sin(x+y)$

Domain $= R$ $\therefore$ it is cont. at every point.

Q. $f(x,y) = \frac{x+y}{x-y}$ Domain $= R-(x=y)$

Q. $g(x,y) = \frac{x^2+y^2}{x^2-3x+2}$

$\frac{x^2+y^2}{x^2-x-2x+2} = \frac{x^2+y^2}{(x-1)(x-2)}$

Domain $= R-(x=1, x=2)$

Q. Discuss the continuity of $f(x,y) = \begin{cases} \frac{x}{\sqrt{x^2+y^2}}, & x\neq 0, y\neq 0 \\ 2, & x=0, y=0 \end{cases}$ at origin.

(a) func^n is defined at $(0,0)$ as $f(x,y)=2$ at $(0,0)$

(b) limit exists?

$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x}{\sqrt{x^2+y^2}} = \lim_{y\to 0} 0 = 0.$

$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x}{\sqrt{x^2+y^2}} = \lim_{x\to 0} \frac{x}{\sqrt{x^2}}$

contd

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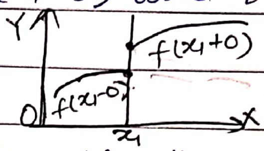
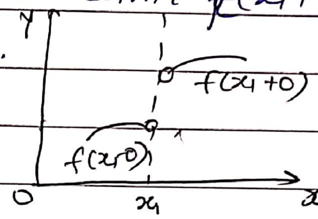
Types of discontinuity :-

① **First Kind :-** $f(x)$ is said to have discontinuity of first kind at point $x = x_1$ if right limit $f(x_1+0)$ and left limit $f(x_1-0)$ exists but not equal.

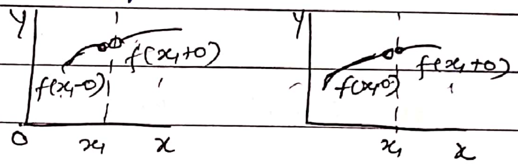


② **Second Kind :-** $f(x)$ is said to have discontinuity of second kind at $x = x_1$ if ~~neither~~ neither right limit $f(x_1+0)$ exists nor left limit $f(x_1-0)$ exists.



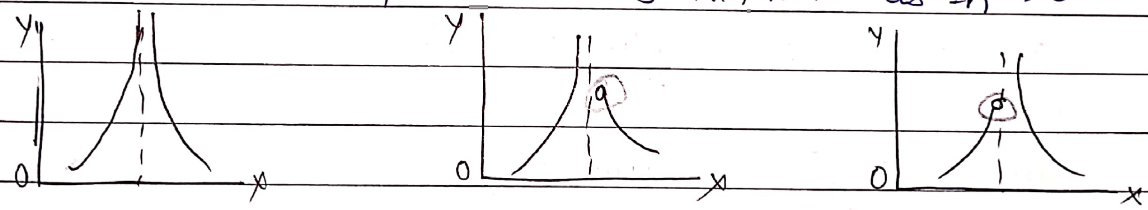
③ **Third Kind :-** (Mixed discontinuity)

One of the Right limit $f(x_1+0)$ or left limit $f(x_1-0)$ exists.

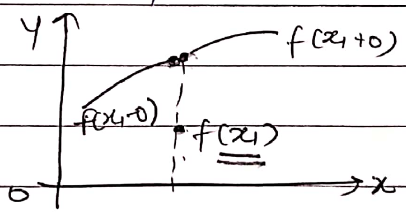


④ **Fourth Kind (Indefinite discontinuity) :-** Either one or both limits right limit $f(x_1+0)$ & left limit $f(x_1-0)$ are infinit.

If both limits does not exist & if $f(x_1 \pm h)$ oscillates b/w limits one of which is infinite as $\pm h \to 0$.



⑤ **Fifth Kind :-** (Removable Discontinuity) :- if Right limit is equal to left limit is not equal to $f(x_1)$.



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Q. Show that the given funcⁿ is discont. at all the points (2, −2).

$$f(x,y) = \begin{cases} \dfrac{x^2+xy+x+y}{x+y}, & (x,y) \neq (2,-2) \\ 4, & (x,y) = (2,-2) \end{cases}$$

Q. Test for continuity :-

① $$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2}, & \text{when } x \neq 0, y \neq 0 \\ 0, & \text{when } x = 0, y = 0. \end{cases}$$, at origin.

$$\lim_{\substack{x \to 0 \\ y \to 0}} \frac{xy(x^2-y^2)}{x^2+y^2} = \lim_{y \to 0} 0 = 0.$$

$$\lim_{\substack{x \to 0 \\ y \to 0}} \frac{xy(x^2-y^2)}{x^2+y^2} = \lim_{x \to 0} 0 = 0.$$

$y = mx$

$$\lim_{\substack{x \to 0 \\ y \to 0}} \frac{mx^2(x^2-m^2x^2)}{x^2+m^2x^2} = \lim_{x \to 0} \frac{mx^2(1-m^2)}{1+m^2} = 0$$

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Q.4 $f(x,y) = \begin{cases} x^3+y^3, & x\neq 0, y\neq 0 \\ 0, & x=0, y=0 \end{cases}$, at origin.

Q.5 $f(x,y) = \begin{cases} \frac{x^2+2y}{x+y^2}, & \text{When } x=1, x=2 \\ 1, & \end{cases}$ at pt $(1,2)$

Q.6

Q.4 (4)

$$\lim_{\substack{x\to 0\\ y\to 0}} x^3+y^3 = \lim_{y\to 0} y^3 = \underline{\underline{0}}$$

$$\lim_{\substack{x\to 0\\ y\to 0}} x^3+y^3 = \lim_{x\to 0} x^3 = 0.$$

$y = mx$

$$\lim_{\substack{x\to 0\\ y\to 0}} x^3+m^3x^3 = \lim_{x\to 0} x^3(1+m^3) = \underline{\underline{0}}$$

$y = mx^2$

$$\lim_{\substack{x\to 0\\ y\to 0}} x^3+m^3x^6 = \lim_{x\to 0} x^3(1+m^3x^3) = \underline{\underline{0}}$$

~~f(0,0) =~~ continuous.

5)

$$\lim_{\substack{x\to 1\\ y\to 2}} \frac{x^2+2y}{x+y^2} = \lim_{y\to 2} \frac{1+2y}{1+y^2} = \frac{5}{5} = 1$$

$$\lim_{\substack{x\to 1\\ y\to 2}} \frac{x^2+2y}{x+y^2} = \lim_{x\to 1} \frac{x+4}{x+4} = \frac{5}{5} = 1$$

Limit exists,

$f(1,2) = 1$

$\therefore$ It is continuous.

6) $f(x,y) = \begin{cases} 2x^2+4 & , (x,y) \neq (1,2) \\ 0 & (x,y) = (1,2) \end{cases}$

$\lim_{\substack{x\to 1 \\ y\to 2}} 2x^2+4 = \lim_{y\to 2} 2+4 = 6$ ✓

$\lim_{\substack{x\to 1 \\ y\to 2}} 2x^2+4 = \lim_{x\to 1} 2x^2+4 = 6$ ✓

limit exists at (1, 2), with value 6

$f(1,2) = 0 \neq 6$ ✓

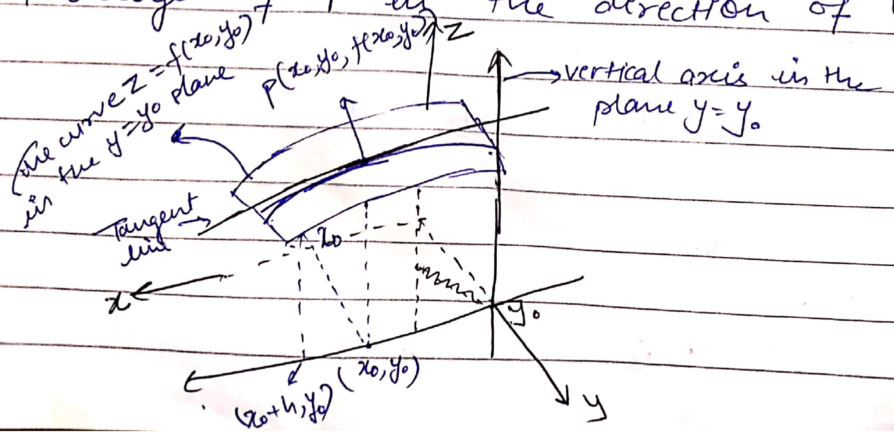
discontinuous

Partial Derivatives :-

The partial derivative of $f(x,y)$ with respect to x at the point (x_0,y_0) is $\left.\frac{\partial f}{\partial x}\right|_{x_0,y_0} = \left.\frac{d}{dx} f(x,y_0)\right|_{x=x_0} = \lim_{h\to 0} \frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}$

(keeping y constant) provided the limit exists.

The slope of the curve $z=f(x,y_0)$ at the point $P(x_0,y_0,f(x_0,y_0))$ in the plane $y=y_0$ is the value of partial derivative of f with respect to x at (x_0,y_0). The tangent line to the curve at P is the line in the plane $y=y_0$ that passes through P with this slope. The partial derivative $\partial f/\partial x$ at (x_0,y_0) gives the rate of change of f with respect to x when y is held fixed at y_0. This is the rate of change of f in the direction of i at (x_0,y_0).



The partial derivative of $f(x,y)$ w.r.t. 'y' at the point (x_0, y_0) is ~~the~~ $\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = \frac{d f(x_0,y)}{dy} = \lim_{h\to 0} \frac{f(x_0, y_0+h) - f(x_0,y_0)}{h}$

provided the limit exists.

Q $f(x,y) = x^2+3xy+y-1$, find $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$ at $(4,-5)$.

$$\frac{\partial f}{\partial x} = 2x+3y+0-0 = 2x+3y$$

$$\left.\frac{\partial f}{\partial x}\right|_{(x,y)=(4,-5)} = 2(+4)+3(-5) = -7$$

$$\frac{\partial f}{\partial y} = 0+3x+1-0$$

$$\left.\frac{\partial f}{\partial y}\right|_{(4,-5)} = 3(4)+1 = 13$$

Q $f(x,y) = y\sin(xy)$, find $\frac{\partial f}{\partial y}$.

$$\frac{\partial f}{\partial y} = 1\times\sin(xy) + y\cos(xy)\cdot x = \sin(xy)+\cos(xy)\cdot xy.$$

$$\frac{\partial f}{\partial x} = y\cos(xy)\cdot y = y^2\cos(xy).$$

Q find f_x, if $f(x,y) = \frac{2y}{y+\cos x}$
$\partial f/\partial x$

$$f_x = \frac{\partial f}{\partial x} = \frac{2y(-1)\,(0+(-\sin x))}{(y+\cos x)^2} = \frac{2y\sin x}{(y+\cos x)^2}$$

$$f_y = \frac{\partial f}{\partial y} = \frac{2}{y+\cos x} + \frac{2y\times(-1)\,(1+0)}{(y+\cos x)^2} = \frac{2}{y+\cos x} - \frac{2y}{(y+\cos x)^2}$$

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Some notations:-

(1) $\frac{\partial z}{\partial x} = p$; $\frac{\partial z}{\partial y} = q$; $\frac{\partial^2 z}{\partial x^2} = r$, $\frac{\partial^2 z}{\partial x \partial y} = s$; $\frac{\partial^2 z}{\partial y^2} = t$

Q. If $z(x+y) = x^2+y^2$, show that $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1-\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$ (dep. / indep.)

$\therefore z = \frac{x^2+y^2}{x+y}$

$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$ $\quad \frac{\partial z}{\partial x} = \frac{2x(x+y) - (x^2+y^2)(1)}{(x+y)^2} = \frac{x^2-y^2+2xy}{(x+y)^2}$

$\frac{\partial z}{\partial y} = \frac{2y(x+y) - (x^2+y^2)(1)}{(x+y)^2} = \frac{y^2-x^2+2xy}{(x+y)^2}$

$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = \left(\frac{x^2-y^2+2\cancel{xy} - y^2+x^2-2\cancel{xy}}{(x+y)^2}\right)^2 = \left(\frac{2x^2-2y^2}{(x+y)^2}\right)^2$

$= 4\left(\frac{(x-y)}{(x+y)}\right)^2$

$4\left(1-\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right) = 4\left(1-\left(\frac{\cancel{x^2}-\cancel{y^2}+2xy + \cancel{y^2}-\cancel{x^2}+2xy}{(x+y)^2}\right)\right)$

$= 4\left(1 - \frac{4xy}{(x+y)^2}\right) = 4\left(\frac{(x+y)^2-4xy}{(x+y)^2}\right)$

$= 4\left(\frac{(x-y)^2}{(x+y)^2}\right)$

$\therefore \left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1-\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$

Hence

Q. $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$, find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$

$x\left[\frac{1}{\sqrt{1-\frac{x^2}{y^2}}} \times \frac{1}{y} + \frac{1}{1+\frac{y^2}{x^2}} \times y\left(\frac{-1}{x^2}\right)\right] + y\left[\frac{1}{\sqrt{1-\frac{x^2}{y^2}}} \times x\left(\frac{-1}{y^2}\right) + \frac{1}{1+\frac{y^2}{x^2}} \times \frac{1}{x}\right]$

$\Rightarrow \cancel{\frac{x}{y} \times \frac{1}{\sqrt{1-\frac{x^2}{y^2}}}} - \cancel{\frac{y}{x} \times \frac{1}{1+\frac{y^2}{x^2}}} - \cancel{\frac{x}{y}\frac{1}{\sqrt{1-\frac{x^2}{y^2}}}} + \cancel{\frac{y}{x}\cdot\frac{1}{1+\frac{y^2}{x^2}}}$

$= 0$

Partial derivative of Higher orders :-

Let, $z = f(x,y)$ then $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ being the functions of x and y can further be differentiated partially with respect to x and y.

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} \text{ or } \frac{\partial^2 f}{\partial x^2} \text{ or } f_{xx} \quad (r)$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} \text{ or } \frac{\partial^2 f}{\partial y \partial x} \text{ or } f_{yx} \quad (s)$$

$$\parallel$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} \text{ or } \frac{\partial^2 f}{\partial x \partial y} \text{ or } f_{xy}. \quad (s)$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} \text{ or } \frac{\partial^2 f}{\partial y^2} \text{ or } f_{yy}. \quad (t)$$

* $$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}. \quad \#$$

Q. $y = f(x+at) + g(x-at)$ satisfies, $\frac{\partial^2 y}{\partial t^2} = a^2\left(\frac{\partial^2 y}{\partial x^2}\right)$

$$\frac{\partial y}{\partial t} = f'(x+at).a + g'(x-at)(-a)$$

$$\frac{\partial^2 y}{\partial t^2} = f''(x+at).a^2 + g''(x-at)(+a^2) \quad \text{—— (i)}$$

& $$\frac{\partial y}{\partial x} = f'(x+at).(1) + g'(x-at)(1)$$

$$\frac{\partial^2 y}{\partial x^2} = f''(x+at) + g''(x-at). \quad \text{—— (ii)}$$

from (i) & (ii)

$$\frac{\partial^2 y}{\partial t^2} = a^2\left(\frac{\partial^2 y}{\partial x^2}\right)$$

Proved

Q. If $z = \tan(y + ax) + (y - ax)^{3/2}$, show that $\frac{\partial^2 z}{\partial x^2} = a^2 \frac{\partial^2 z}{\partial y^2}$

Q. $u = e^{xyz}$, find the value of $\frac{\partial^3 u}{\partial x \partial y \partial z}$

$$\frac{\partial u}{\partial z} = e^{xyz}(xy)$$

$$\frac{\partial^2 u}{\partial y \partial z} = x\left[e^{xyz} + y \cdot e^{xyz} \cdot xz\right]$$

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = \left(e^{xyz} + xyz \cdot e^{xyz}\right) + x\left[yz \cdot e^{xyz} + yz\left(e^{xyz} + xyz e^{xyz}\right.\right.$$

Q. $U = \log(x^3 + y^3 + z^3 - 3xyz)$, show that $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 U = -\frac{9}{(x+y+z)^2}$

$$\frac{\partial u}{\partial x} = \frac{3x^2 - 3yz}{x^3 + y^3 + z^3 - 3xyz}$$

$$\frac{\partial u}{\partial y} = \frac{3y^2 - 3xz}{x^3 + y^3 + z^3 - 3xyz}$$

$$\frac{\partial u}{\partial z} = \frac{3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz}$$

$$\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right) = \left(\frac{3(x^2 + y^2 + z^2) - 3(xy + yz + zx)}{x^3 + y^3 + z^3 - 3xyz}\right)$$

$$= \frac{3\cancel{(x^2 + y^2 + z^2 - xy - yz - zx)}}{(x+y+z)\cancel{(x^2 + y^2 + z^2 - xy - yz - zx)}}$$

$$= \frac{3}{(x+y+z)} \#$$

Now, $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right) = \frac{-3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} - \frac{3}{(x+y}$$

$$= \frac{-9}{(x+y+z)^2}$$ Ans.

Q. $\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1$, show that,

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 2\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}\right)$$

Q. $z^3 - 3yz - 3x = 0$, show that $z\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y}$ and

$$z\left(\frac{\partial^2 z}{\partial x \partial y} + \left(\frac{\partial z}{\partial x}\right)^2\right) = \frac{\partial^2 z}{\partial y^2}.$$

$z^3 = 3yz + 3x.$

partial diff. w.r.t. x.

$$3z^2\frac{\partial z}{\partial x} = 3y\frac{\partial z}{\partial x} + 3$$

$$\frac{\partial z}{\partial x}\left(3z^2 - 3y\right) = 3$$

$$\frac{\partial z}{\partial x} = \frac{1}{z^2-y}.$$

Partial diff w.r.t. y :-

$$\cancel{3}z^2\frac{\partial z}{\partial y} = \cancel{3}\left(z + y\frac{\partial z}{\partial y}\right) + 0.$$

$$(z^2-y)\frac{\partial z}{\partial y} = z$$

$$\frac{\partial z}{\partial y} = z \times \left(\frac{1}{z^2-y}\right)$$

$$\left[\frac{\partial z}{\partial y} = z\frac{\partial z}{\partial x}\right]$$

Homogenous Function :-

A function $f(x,y)$ is said to be homogenous function in which the power of each term is same.

A function $f(x,y)$ is a homogeneous func. of order n if the degree of each term in x & y is equal to n,

$$a_0 x^n y^0 + a_1 x^{n-1} y^1 + a_2 x^{n-2} y^2 + \cdots + a_n x^0 y^n. \quad \text{---} (1)$$

The polynomial func. (1) can be written as :-

$$x^n \left[a_0 + a_1 \frac{y}{x} + a_2 \left(\frac{y}{x}\right)^2 + \cdots + a_{n-1} \left(\frac{y}{x}\right)^{n-1} + a_n \left(\frac{y}{x}\right)^n \right]$$

$$= x^n \phi\left(\frac{y}{x}\right) \quad \text{---} (2)$$

ex (i) The func. $x^3 \left[1 + \frac{y}{x} + 3\left(\frac{y}{x}\right)^2 + 5\left(\frac{y}{x}\right)^3 \right]$ is a homogeneous func. of order 3.

(ii) $$\frac{\sqrt{x} + \sqrt{y}}{x^2 + y^2} = \frac{\sqrt{x}\left[1 + (y/x)^{1/2}\right]}{x^2\left[1 + \left(\frac{y}{x}\right)^2\right]} = x^{-3/2} \left[\frac{1 + (y/x)^{1/2}}{1 + (y/x)^2}\right]$$

$\therefore$ It is a homogeneous func of order $-3/2$.

Euler's Theorem on Homogeneous equation:-

"If z is a homogeneous function of x, y of order n then, $x.\frac{\partial z}{\partial x} + y.\frac{\partial z}{\partial y} = nz$."

If f is a homogeneous func. of x, y, z of order n then, $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = nf$.

Remark:- If f is a Homogeneous function in x, y of order n, then $x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2} = n(n-1)f$.

Proof:-

If f is a homogeneous func^n in x, y of order n, then by Euler's theorem:-

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf \quad \text{—— (1)}$$

diff. eq^n (1) partially w.r.t. 'x', keeping 'y' as constant.

then, $$x\frac{\partial^2 f}{\partial x^2} + \frac{\partial f}{\partial x} + y\frac{\partial^2 f}{\partial x \partial y} = n\frac{\partial f}{\partial x}$$

multiplying by 'x'

$$x^2\frac{\partial^2 f}{\partial x^2} + x\frac{\partial f}{\partial x} + xy\frac{\partial^2 f}{\partial x \partial y} = nx\frac{\partial f}{\partial x} \quad \text{—— (2)}$$

diff^n (1) w.r.t. 'y' keeping 'x' as constant.

$$x\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial f}{\partial y} + y\frac{\partial^2 f}{\partial y^2} = n\frac{\partial f}{\partial y}$$

multiply by 'y'

$$xy\frac{\partial^2 f}{\partial x \partial y} + y\frac{\partial f}{\partial y} + y^2\frac{\partial^2 f}{\partial y^2} = ny\frac{\partial f}{\partial y} \quad \text{—— (3)}$$

eq^n (2) + eq^n (3)

$$x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2} + x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = n\left(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\right)$$

$$x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2} = n(nf) - nf \quad (\text{from } 1)$$

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$$\Rightarrow \boxed{x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2} = n(n-1)f}$$

Q. Verify Euler's theorem for func^n, $U = (x^{1/2}+y^{1/2})(x^n+y^n)$

$$U = x^{1/2}\left[1+\left(\frac{y}{x}\right)^{1/2}\right]x^n\left[1+\left(\frac{y}{x}\right)^n\right] = x^{\frac{1}{2}+n}\left[1+\sqrt{\frac{y}{x}}\right]\left[1+\left(\frac{y}{x}\right)^n\right]$$

$\hookrightarrow$ order $= \frac{1}{2}+n$

$$x\frac{\partial U}{\partial x} + y\frac{\partial U}{\partial y} = \underset{\text{order}}{n}\,U$$

$$x\frac{\partial U}{\partial x} + y\frac{\partial U}{\partial y} = \left(\frac{1}{2}+n\right)U$$

LHS

$$x\left[\frac{1}{2}\times\frac{1}{x^{1/2}}(x^n+y^n) + (x^{1/2}+y^{1/2})(nx^{n-1})\right] + y\left[\frac{1}{2(y)^{1/2}}(x^n+y^n) + (x^{1/2}+y^{1/2})ny^{n-1}\right]$$

$= \cancel{\left(\frac{1}{2}+n\right)}$

$$\frac{1}{2}(x^{1/2})(x^n+y^n) + (x^{1/2}+y^{1/2})n(x^n) + \frac{1}{2}(y)^{1/2}(x^n+y^n) + (x^{1/2}+y^{1/2})ny^n$$

$$= \frac{1}{2}(x^n+y^n)(x^{1/2}+y^{1/2}) + n(x^{1/2}+y^{1/2})(x^n+y^n)$$

$$= \frac{1}{2}U + nU = \left(\frac{1}{2}+n\right)U$$

$= \underline{RHS}$

Hence it ~~too~~ verifies Euler's theorem.

Q. If $U = \tan^{-1}\frac{x^3+y^3}{x-y}$, Prove that $x\frac{\partial U}{\partial x} + y\frac{\partial U}{\partial y} = \sin 2U$.

$$\cancel{x\left[\frac{1}{1+\left(\frac{x^3+y^3}{x-y}\right)^2}\right]\left[3y^2(x-y) - (x^3+y^3)\right]}$$

here u is not a homogeneous func^n in x, y.

$$\Rightarrow \tan u = \frac{x^3+y^3}{x-y} = \frac{x^3(1+y^3/x^3)}{x(1-y/x)} = x^2\phi(y/x)$$

$\Rightarrow \tan u$ is a homogeneous func^n of order 2.

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Q. If $u = \sin^{-1}\left(\frac{x+2y+3z}{\sqrt{x^3+y^3+z^3}}\right)$, show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = -3\tan u.$$

Q. If $u = \sin^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$, Prove that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}$

$= \frac{-\sin u \cos 2u}{4\cos^3 u}$

↓
not a homogeneous func^n.

Now,

$$\sin u = \frac{x+y}{\sqrt{x}+\sqrt{y}} = \sqrt{x}\left(\frac{1+y/x}{1+\sqrt{y/x}}\right) = \sqrt{x}\,\phi(y/x)$$ is a

homogeneous func^n of order $1/2$.

By Euler's theorem

$$x\frac{\partial \sin u}{\partial x} + y\frac{\partial \sin u}{\partial y} = \frac{1}{2}\sin u.$$

$$x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = \frac{1}{2}\sin u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u \quad \text{--- (1)}$$

partial diff. w.r.t. 'x'

$$\frac{\partial u}{\partial x} + x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial u}{\partial x \partial y} = \frac{1}{2}\sec^2 u\frac{\partial u}{\partial x}$$

multiplying by 'x'

$$x\frac{\partial u}{\partial x} + x^2\frac{\partial^2 u}{\partial x^2} + xy\frac{\partial^2 u}{\partial x \partial y} = \frac{x}{2}\sec^2 u\frac{\partial u}{\partial x} \quad \text{--- (2)}$$

partial diff. of (1) w.r.t. 'y'

$$x\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial y} + y\frac{\partial^2 u}{\partial y^2} = \frac{1}{2}\sec^2 u\frac{\partial u}{\partial y}$$

multiplying by 'y'

$$xy\frac{\partial^2 u}{\partial x \partial y} + y\frac{\partial u}{\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{y}{2}\sec^2 u\frac{\partial u}{\partial y} \quad \text{--- (3)}$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{\sec^2 u}{2}\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} + \frac{1}{2}\tan u = \frac{\sec^2 u}{2} \times \frac{1}{2}\tan u$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{1}{4\cos^2 u} \times \frac{\sin u}{\cos u} - \frac{1}{2}\frac{\sin u}{\cos u}$$

$$= \frac{\sin u - 2\sin u \cos^2 u}{4\cos^3 u}$$

$$= \frac{\sin u(1 - 2\cos^2 u)}{4\cos^3 u}$$

$$= \frac{-\sin u \cos 2u}{4\cos^3 u}$$

Hence Proved

Total Derivative :-

If $u = f(x,y)$, where $x = \phi(t)$, $y = \psi(t)$ then we can express u as a function of 't' alone by substituting the values of x and y in $f(x,y)$. Then we can find the ordinary derivative $\frac{du}{dt}$ which is called the total derivative of u, to distinguish it from partial diff. $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}.$$

Corollary - I :-

If $u = f(x,y,z)$ and x, y & z are function of 't' then,

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}.$$

Derivation of Implicit function :-

If $f(x,y) = c$. be an implicit ~~function~~ relation b/w x and y which defined as a differentiable funcn of 'x', then.

$$0 = \frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

$$\Rightarrow \frac{df}{dx} = 0.$$

$$\Rightarrow \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\partial f/\partial y} = -\frac{f_x}{f_y}, \text{ provided } f_y \neq 0.$$

Corollary II :-

If $z = f(x, y)$, where $x = \phi(u, v)$, $y = \psi(u, v)$, then z is called a composite functions of u & v, so that,

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

similarly,
$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}.$$

Corollary III :-

If $u = f(x, y)$ where, $y = \phi(x)$, then u is a composite function of x, so that,

$$\frac{du}{dx} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dx} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx}$$

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx}.$$

Ques :- If $u = f(y-z, z-x, x-y)$, prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.

Let $X = y-z$, $Y = z-x$, $Z = x-y$

then u is a composite function of X, Y, Z.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial X}\cdot\cancelto{0}{\frac{\partial X}{\partial x}} + \frac{\partial u}{\partial Y}\cancelto{(-1)}{\frac{\partial Y}{\partial x}} + \frac{\partial u}{\partial Z}\cancelto{(+1)}{\frac{\partial Z}{\partial x}}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial Z} - \frac{\partial u}{\partial Y} \quad \text{—— (1)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial X}\cancelto{(1)}{\frac{\partial X}{\partial y}} + \frac{\partial u}{\partial Y}\cancelto{(0)}{\frac{\partial Y}{\partial y}} + \frac{\partial u}{\partial Z}\cancelto{(-1)}{\frac{\partial Z}{\partial y}}$$

$$= \frac{\partial u}{\partial X} - \frac{\partial u}{\partial Z} \quad \text{—— (2)}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial X}\cancelto{(-1)}{\frac{\partial X}{\partial z}} + \frac{\partial u}{\partial Y}\cancelto{(1)}{\frac{\partial Y}{\partial z}} + \frac{\partial u}{\partial Z}\cancelto{(0)}{\frac{\partial Z}{\partial z}} = \frac{\partial u}{\partial Y} - \frac{\partial u}{\partial X} \quad \text{—— (3).}$$

(1) + (2) + (3)

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$$

Hence Proved

Q. $W = f(x,y)$, where $x = r\cos\theta$, $y = r\sin\theta$ then show that

$$\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2.$$

$$w \to (x,y) \longrightarrow (r,\theta)$$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r} = \frac{\partial w}{\partial x}\cos\theta + \frac{\partial w}{\partial y}\sin\theta$$

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial \theta} = \frac{\partial w}{\partial x}(-r\sin\theta) + \frac{\partial w}{\partial y}(r\cos\theta)$$

$$\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = \left(\frac{\partial w}{\partial x}\right)^2\cos^2\theta + \left(\frac{\partial w}{\partial y}\right)^2\sin^2\theta + \cancel{2\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\cos\theta\sin\theta}$$

$$+ \left(\frac{\partial w}{\partial x}\right)^2\sin^2\theta + \left(\frac{\partial w}{\partial y}\right)^2\cos^2\theta - \cancel{2\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\sin\theta\cos\theta}$$

$$= \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2$$

Hence Proved

Q. (1) If $u = f(r,s,t)$ & $r = x/y$, $s = y/z$, $t = z/x$, Prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0.$$

(2) If $U = x^2 - y^2$, $V = 2xy$, $f(x,y) = \phi(u,v)$, show that –

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 4(x^2-y^2)\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right).$$

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Jacobians :-

If u & v are functions of two independent variables x and y, then the determinant $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ is called jacobians of u, v, with respect to x, y and is denoted by $J\left(\frac{u,v}{x,y}\right)$ or $\frac{\partial(u,v)}{\partial(x,y)}$.

If u, v, w are funcⁿ of x, y, z, then the jacobian of u, v, w, w.r.t. x, y, z is -

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} \text{ or } J\left(\frac{u,v,w}{x,y,z}\right) \text{ or } \frac{\partial(u,v,w)}{\partial(x,y,z)}.$$

Properties of Jacobians :-

① If u, v are functions of r, s where r, s are funcⁿ of x, y, then $\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(x,y)}$

② If J_1 is the jacobians of u, v w.r.t. x, y
J_2 " " " " x, y w.r.t. u, v, then.

$$J_1 J_2 = 1 \Rightarrow \frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = 1$$

Q. If $r = \sqrt{x^2+y^2}$; $\theta = \tan^{-1}\left(\frac{y}{x}\right)$, find $\frac{\partial(r,\theta)}{\partial(x,y)}$.

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1(\cancel{2}x)}{\cancel{2}\sqrt{x^2+y^2}} & \frac{1(\cancel{2}y)}{\cancel{2}\sqrt{x^2+y^2}} \\ \frac{1}{\left(1+\frac{y^2}{x^2}\right)} \cdot \frac{y(-1)}{x^2} & \frac{1}{\left(1+\frac{y^2}{x^2}\right)} \times \frac{1}{x} \end{vmatrix}$$

$$\Rightarrow \frac{\cancel{x}}{\sqrt{x^2+y^2}} \times \frac{1}{\cancel{x}\left(1+\frac{y^2}{x^2}\right)} + \frac{y^2}{\sqrt{x^2+y^2}} \times \frac{1}{x^2\left(1+\frac{y^2}{x^2}\right)}$$

$$\Rightarrow \frac{1}{\sqrt{x^2+y^2}} \times \frac{1}{\cancel{\left(1+\frac{y^2}{x^2}\right)}} \left[\cancel{1+\frac{y^2}{x^2}}\right]$$

$$= \frac{1}{\sqrt{x^2+y^2}} \text{ Ans.}$$

Q. If $x = r\cos\theta$, $s = r\sin\theta$, find J.

$$\frac{\partial(x,s)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial s}{\partial r} & \frac{\partial s}{\partial \theta} \end{vmatrix} = \cos\theta \times r\cos\theta - (-r\sin\theta)(\sin\theta)$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r(\sin^2\theta + \cos^2\theta)$$

$$= r \text{ Ans.}$$

Polar Coordinate:- $x, y \rightarrow r, \theta$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

Spherical Polar Coordinate:-

$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

find J.

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = (\sin\theta\cos\phi)\left[(r\cos\theta\sin\phi \times 0 - r\sin\theta \times (-r\sin\theta)\right]$$

$$- r\cos\theta\cos\phi(\sin\theta\sin\phi \times 0 - r\sin\theta\cos\phi \times$$

$$+ r\sin\theta(-\sin\phi)(\sin\theta\sin\phi \times r(-\sin\theta) - r\cos\theta \times \cos$$

$$= r^2\sin^3\theta\cos^2\phi - r\cos\theta\cos\phi\sin\theta\sin\phi(0) + r^2\sin\theta\cos^2\phi\cos^2\theta + r^2\sin^3\theta\sin^2\phi$$

$$+ r^2\sin\theta\cos^2\theta\sin^2\phi$$

$$\Rightarrow r^2\sin^3\theta + r^2\sin\theta\cos^2\theta$$

$$\Rightarrow r^2\sin\theta(\sin^2\theta + \cos^2\theta)$$

$$= r^2\sin\theta \text{ Ans.}$$

$$* \boxed{\frac{\partial(x,y)}{\partial(r,\theta)} = r \quad \text{and} \quad \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta}$$

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Cylindrical coordinate :-

In cylindrical coordinate :-

$x = r\cos\phi, \quad y = r\sin\phi, \quad z = z.$

find $\frac{\partial(x,y,z)}{\partial(r,\phi,z)}$

$$= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial z} \end{vmatrix} = \cos\phi(r\cos\phi - 0) - r(-\sin\phi)(\sin\phi \times \phi - 0)$$

$$= r\cos^2\phi + r\sin^2\phi$$

$$= \underline{\underline{r}}$$

$$\Rightarrow \frac{\partial(x,y,z)}{\partial(r,\phi,z)} = r$$

Q. If $u = x(1-y)$, $v = xy$, Prove that $JJ' = 1$.

$J = \frac{\partial(u,v)}{\partial(x,y)}, \quad J' = \frac{\partial(x,y)}{\partial(u,v)}$

$$J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = (1-y)(x) - (-x)(y) = x - xy + xy = \underline{\underline{x}}$$

$v = xy \Rightarrow x = v/y.$

$\cancel{x = u} \Rightarrow u = \frac{v}{y}(1-y)$

$uy = v - vy$

$$\boxed{y = \frac{v}{u+v}}$$

$x = \frac{v}{v/(u+v)} \Rightarrow \boxed{x = u+v}$

$$\Rightarrow J' = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = 1 \times \left[\frac{1(u+v) - v(1)}{(u+v)^2}\right] - 1 \times \frac{v(-1)}{(u+v)^2}$$

$$= \frac{u+v}{(u+v)^2} = \frac{1}{u+v} = \frac{1}{x}$$

$JJ' = 1$ Ans.

Q. ① If $u = \frac{yz}{x}$, $v = \frac{zx}{y}$, $w = \frac{xy}{z}$ show that

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = 4.$$

② If $u = \frac{x}{y-x}$, $v = \frac{y}{z-x}$, $w = \frac{z}{x-y}$ show that.

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0.$$

③ If $x = r\cos\theta$, $y = r\sin\theta$, show that $\frac{\partial(r,\theta)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(r,\theta)} = 1$

Jacobians of Implicit functions :-

If (u_1, u_2, u_3) instead of being given implicit in terms x_1, x_2, x_3 be connected with them by eqns such as $f_1(u_1, u_2, u_3, x_1, x_2, x_3) = 0$, $f_2(u_1, u_2, u_3, x_1, x_2, x_3) = 0$ and $f_3(u_1, u_2, u_3, x_1, x_2, x_3) = 0$, then,

$$\left[\frac{\partial(u_1,u_2,u_3)}{\partial(x_1,x_2,x_3)} = (-1)^3 \frac{\partial(f_1,f_2,f_3)}{\partial(x_1,x_2,x_3)} \div \frac{\partial(f_1,f_2,f_3)}{\partial(u_1,u_2,u_3)}\right].$$

Q. if $u = x y z$, $v = x^2+y^2+z^2$, $w = x+y+z$, find $\frac{\partial(x,y,z)}{\partial(u,v,w)}$.

$u - xyz = 0$, $v - x^2 - y^2 - z^2 = 0$, $w - x - y - z = 0$.

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = (-1)^3 \frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)} \div \frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)}$$

$$= (-1) \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w} \end{vmatrix} \div \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix}$$

$$= (-1)\left[1(1\times1 - 0\times0) - 0 + 0\right] \div (-yz)\left((-2y)(-1) - (-2z)(-1)\right) - \left((-2x)(-1) - (-2z)(-1)\right) + (-xy)\left((-2x)(-1) - (-2y)(-1)\right)$$

$$\Rightarrow -1 \div \left[-2y^2z + 2yz^2 + 2x^2z - 2xz^2 - 2x^2y + 2xy^2\right]$$

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Q. $u = x^2 - y^2$, $v = 2xy$, $\frac{\partial(x,y)}{\partial(u,v)}$

$f_1 \Rightarrow u - x^2 + y^2 = 0$, $f_2 = v - 2xy = 0$

$$\frac{\partial(x,y)}{\partial(u,v)} = (-1) \frac{\partial(f_1,f_2)}{\partial(u,v)} \div \frac{\partial(f_1,f_2)}{\partial(x,y)}$$

$$= (-1) \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{vmatrix} \div \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{vmatrix}.$$

$$= (-1)[1 \times 1 - 0 \times 0] \div [-2x \times (-2x) - (2y)(-2y)]$$

$$= \frac{-1}{4x^2 + 4y^2}$$

Gradient :- The rate of change of ϕ w.r.t. distance of a variable quantity (Temp, pressure etc.) in the direction of maximum change.

Mathematically, Let $\phi(x,y,z)$ be a function ~~determining~~ defining a scalar field then the vector $\frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$ is called the Gradient of the scalar field ϕ. It is denoted by $\nabla\phi$ or grad ϕ, i.e., grad $\phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}$

Properties of Gradient :-

1) If ϕ is a constant scalar point function then $\nabla\phi = 0$.

2) If ϕ_1 & ϕ_2 are two scalar point functions, then,

(a) $\nabla(\phi_1 \pm \phi_2) = \nabla\phi_1 \pm \nabla\phi_2$

(b) $\nabla(C_1\phi_1 + C_2\phi_2) = C_1\nabla\phi_1 + C_2\nabla\phi_2$

(c) $\nabla(\phi_1 \cdot \phi_2) = \phi_1(\nabla\phi_2) + \phi_2(\nabla\phi_1)$

(d) $\nabla(\phi_1/\phi_2) = \frac{\phi_2(\nabla\phi_1) - \phi_1(\nabla\phi_2)}{\phi_2^2}$, provided $\phi_2 \neq 0$.

Q. Find grad ϕ, where ϕ is given by $\phi = 3x^2y - y^3z^2$ at $(1,-2,-1)$

$$(\text{grad } \phi)_{(1,-2,-1)} = [(6xy) - 0]\hat{i} + (3x^2 - 3y^2z^2)\hat{j} + (0 - 2y^3z)\hat{k}$$

$$= -12\hat{i} - 9\hat{j} - 16\hat{k}$$

Remark :- The unit vector normal to the surface,

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} \quad \#$$

Q. Find the unit vector normal to the surface $x^3 + y^3 + 3xyz = 3$ at $(1,-2,-1)$.

$$\nabla\phi = (3x^2 + 3yz)\hat{i} + (3y^2 + 3xz)\hat{j} + 3xy\,\hat{k}$$

$$\nabla\phi|_{(1,-2,-1)} = (9\hat{i} + 9\hat{j} - 6\hat{k})$$

$$\hat{n} = \frac{9\hat{i} + 9\hat{j} + 6\hat{k}}{\sqrt{9^2 + 9^2 + 36}} = \frac{9\hat{i} + 9\hat{j} - 6\hat{k}}{\sqrt{198}} = \frac{3\hat{i} + 3\hat{j} - 2\hat{k}}{\sqrt{22}} \quad \text{Ans}$$

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Q find the unit normal vector to the surface xy^2 ... at $(3,-2,1)$.

$$\nabla\phi = (y^2+yz)\hat{i} + (2xy+xz)\hat{j} + (xy)\hat{k}$$
$$= 2\hat{i} - 9\hat{j} - 6\hat{k}$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\hat{i}-9\hat{j}-6\hat{k}}{\sqrt{4+81+36}} = \frac{2\hat{i}-9\hat{j}-6\hat{k}}{11}$$

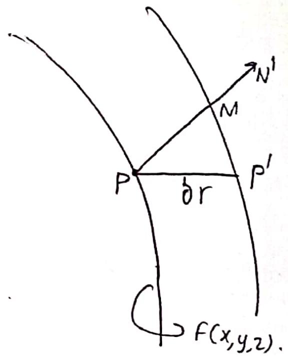
Q find the gradient of the scalar field - (i) $f(x,y) = y$ ... at $(1,2)$ (ii) $x^2y^2 + xy^2 - z^2$ at $(3,1,1)$.

Directional Derivatives :-

Defⁿ:- The rate at which any function changes at particular point in a fixed direction.

Mathematically,

If δr denotes the length of PP' and N' is a unit vector in the direction of PP', then the limiting v... of $\lim_{\delta r \to 0} \frac{\partial f}{\partial r}$ is called the directional derivative o... at P along the direction PP'.



$$\therefore \frac{\partial f}{\partial r} = N \cdot \nabla f$$
$$= N \cdot (\text{grad } f).$$

Q: find the directional derivative of $f(x,y,z) = xy^2 + 4xyz + \ldots$ the point $(1,2,3)$ in the direction of $3\hat{i} + 4\hat{j} - 5\hat{k}$.

$$\frac{\partial f}{\partial r} = N \cdot (\nabla f)$$
$$= \frac{(3\hat{i}+4\hat{j}-5\hat{k})}{\sqrt{50}} \cdot [(y^2+4yz)\hat{i} + (2xy+4xz)\hat{j} + (4xy+2z)\hat{k}]$$
$$= \frac{(3\hat{i}+4\hat{j}-5\hat{k})}{\sqrt{50}} \cdot [28\hat{i} + 16\hat{j} + 14\hat{k}]$$
$$= \frac{78}{5\sqrt{2}}$$