Vector Calculus

Vector Calculus or Vector Analysis deals with the differentiation and integration of both of vectors ~~described ing~~ describing particular bodies, such as the velocity of a particle, and of vector fields, in which a vector is defined as a function of the coordinates throughout some volume.

- **Vector:**
  A vector is a ~~physical~~ quantity having both magnitude and direction.
  Eg:- displacement, velocity, etc.

- **Unit Vector:**
  A unit vector is a vector having unit magnitude.
  If $\vec{A}$ is a vector with magnitude $|\vec{A}| \neq 0$, then $\dfrac{\vec{A}}{|\vec{A}|}$ is a unit vector having the same direction as $\vec{A}$.

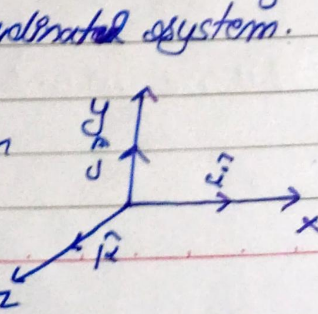
  i.e. $\hat{a} = \dfrac{\vec{A}}{|\vec{A}|}$

  $\cancel{\vec{A} = \hat{a}|\vec{A}| \quad \hat{a}|\cdot|}$

  $\vec{A} = |\vec{A}|\,\hat{a}$

- **Rectangular Unit Vectors ($\hat{i}, \hat{j}, \hat{k}$):**
  These are an important set of unit vectors having the directions of the positive x, y and z axes of a three dimensional rectangular coordinate system.

$\therefore$ Any vector $\vec{A}$ in 3 dimensions can be written as



$$\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$$

where A_1, A_2 & A_3 are called the rectangular components or simply components of $\vec{A}$ in the x, y and z directions respectively.

The magnitude of $\vec{A}$ is

$$|\vec{A}| = \sqrt{A_1^2 + A_2^2 + A_3^2}$$

* Notes :

1. $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$; $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$
2. $\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = 0$; $\hat{i}\times\hat{j} = \hat{k}$; $\hat{j}\times\hat{k} = \hat{i}$; $\hat{k}\times\hat{i} = \hat{j}$
3. If $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$
   $\vec{B} = B_1\hat{i} + B_2\hat{j} + B_3\hat{k}$

Then,

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

• Vector Field:

If to each point (x, y, z) of a region R in space there corresponds a vector $\vec{V}(x, y, z)$, then $\vec{V}$ is called a vector function of position or vector point function and we say that a vector field $\vec{V}$ has been defined in R.

eg. If the velocity at any point (x, y, z) within a moving fluid is known ~~at~~ at a certain time, then a vector field is defined.

- **Vector Differential Operator:**

The operator $\vec{\nabla}$ is known as vector differential operator and is given by

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

The operator $\vec{\nabla}$ is also known as nabla (del or nabla).

* $\phi \rightarrow$ Scalar function
$\vec{V} \rightarrow$ Vector function

- **Gradient:**

Let $\phi(x,y,z)$ be defined and differentiable at each point (x,y,z) in a certain region of space, i.e., ϕ defines a differentiable scalar field. Then the gradient of ϕ is defined by

$$\cancel{\vec{\nabla}\phi \text{ or } \mathrm{grad}\,\phi = \left(\frac{\partial}{\partial x}\hat{i}\right) -}$$

$$\vec{\nabla}\phi \text{ or } \mathrm{grad}\,\phi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\phi$$

$$= \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

Thus $\vec{\nabla}\phi$ defines a vector field.

Q. If $\phi(x,y,z) = 3x^2y - y^3z^2$, find $\vec{\nabla}\phi$ at the point $1,-2,-1$

Ans.
$$\vec{\nabla}\phi = 6xy\hat{i} + (3x^2 - 3y^2z^2)\hat{j} - 2z\hat{k}$$

$$= (-2)6\hat{i} + (3 - 3(-2)^2(-1)^2)\hat{j} + 2\hat{k}$$

$$\cancel{= 6\hat{i} - 9\hat{j} + 2\hat{k}}$$

$$= -12\hat{i} - 9\hat{j} - 16\hat{k}$$

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Q. find grad ϕ at $\phi = \ln|\vec{r}|$, $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Ans. $\vec{\nabla}\phi = \vec{\nabla} \ln\sqrt{x^2+y^2+z^2}$

$= \frac{1}{2} \vec{\nabla} \ln(x^2+y^2+z^2)$

$= \frac{1}{2} \left[\frac{1}{x^2+y^2+z^2} \left(2x\hat{i} + 2y\hat{j} + 2z\hat{k}\right)\right.$

$= \frac{x+y+z}{x^2+y^2+z^2} = \frac{\vec{r}}{|\vec{r}|^2}$

Q. Find gradient of $e^{(x^2+y^2+z^2)}$ at $(1,1,1)$.

Ans. grad $\phi = e^{x^2+y^2+z^2}(2x\hat{i} + 2y\hat{j} + 2z\hat{k})$

$= e^3(2\hat{i} + 2\hat{j} + 2\hat{k})$

$= 2e^3(\hat{i} + \hat{j} + \hat{k})$

• Properties of Gradient:

1. $\vec{\nabla}\phi$ is a vector normal to the surface $\phi(x,y,z) = c$, where c is ~~any~~ a constant

⇒ Proof:

The total differential $d\phi$ for a function $\phi(x,y,z)$ is given by

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$$= \left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right)\cdot(dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= \vec{\nabla}\phi \cdot d\vec{r} \quad \text{— (1)}$$

But

$\phi(x,y,z) = c$

On differentiating ϕ, we get

$d\phi = 0$

$\Rightarrow \vec{\nabla}\phi \cdot d\vec{r} = 0$ [Using eqn (1)]

$\Rightarrow \vec{\nabla}\phi$ and $d\vec{r}$ are perpendicular to each other
But $d\vec{r}$ is in the direction of tangent to the given surface.
Hence, grad ϕ is a vector normal to the surface
$\phi(x, y, z) = c$.

2. The component of $\vec{\nabla}\phi$ in the direction of a vector $\vec{A}$ is equal to $\vec{\nabla}\phi \cdot \hat{a}$ and is called the Directional Derivative of ϕ in the direction of $\vec{A}$.

i.e. Directional derivative $= \vec{\nabla}\phi \cdot \hat{a}$

$$= \frac{\vec{\nabla}\phi \cdot \vec{A}}{|\vec{A}|}$$

Physically, this is the rate of change of ϕ at (x, y, z) in the direction of $\vec{A}$.

* Note: Directional derivative is "Maximum" along the direction $\vec{\nabla}\phi$.

i.e. Maximum directional derivative

$$= \frac{\vec{\nabla}\phi \cdot \vec{\nabla}\phi}{|\vec{\nabla}\phi|}$$

$$= \frac{|\vec{\nabla}\phi|^2}{|\vec{\nabla}\phi|}$$

$$= |\vec{\nabla}\phi|$$

Q Find a unit vector normal to the surface
$x^2 + 3y^2 + 2z^2 = 6$ at $P(2, 0, 1)$.

Ans
$2x\hat{i} + 6y\hat{j} + 4z\hat{k}$.
$4\hat{i} + 4\hat{k} = 0$
$\vec{a} = \hat{i} + \hat{k} = 0$
So $\hat{a} = \frac{1}{\sqrt{2}}(\hat{i} + \hat{k})$

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Q Find the directional derivative of $\phi = xy + yz + zx$ at the point $(1, 2, 0)$ in the direction of the vector $\hat{i} + 2\hat{j} + 2\hat{k}$.

Ans $\vec{\nabla}\phi = (y+z)\hat{i} + (x+z)\hat{j} + (y+x)\hat{k}$

$= (2+0)\hat{i} + (1+0)\hat{j} + (1+2)\hat{k}$

$= 2\hat{i} + \hat{j} + 3\hat{k}$

So $\dfrac{(2i + j + 3k)(\hat{i} + 2\hat{j} + 2\hat{k})}{3}$

$= \dfrac{2 + 2 + 6}{3}$

$= \dfrac{10}{3}$

Q Evaluate the directional derivative of the function $\phi = x^2 - y^2 + 2z^2$ at the point $P(1, 2, 3)$ in the direction of the line PQ where Q has coordinates $(5, 0, 4)$.

Ans $2x - 2y + 4z$

$2 \cdot 1 - 2 \cdot 2 + 4 \cdot 3$

$2\hat{i} - 4\hat{j} + \cancel{16\hat{k}}\ 12\hat{k}$

$(5-1)\hat{i} + (0-2)\hat{j} + (4-3)\hat{k}$

$4\hat{i} - 2\hat{j} + \hat{k}$

So ~~8 − 12~~ $\dfrac{8 + 8 + 12}{16 + 4 + 1}$

$= \dfrac{28}{\sqrt{21}}$

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Q a In which direction from the point (2, 1, -1) is the directional derivative of $\phi = x^2 y z^3$ maximum.
b What is the magnitude of this maximum.

Ans
$$\vec{\nabla}\phi = 2xyz^3\,\hat{i} + x^2z^3\hat{j} + 3x^2yz^2\,\hat{k}$$
$$= 2\cdot 2\cdot 1\cdot (-1)^3\,\hat{i} + 2^2(-1)^3\hat{j} + 3(2)^2\,1\,(-1)^2\,\hat{k}$$
$$= -4\hat{i} - 4\hat{j} + 12\hat{k}$$

$$|\vec{\nabla}\phi| = \sqrt{4^2+4^2+12^2}$$
$$= 4\sqrt{1+1+3^2}$$
$$= 4\sqrt{11}$$

• Divergence:
Let $\vec{V}(x,y,z) = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$ be defined and differentiable at each point (x, y, z) on a certain region of space i.e., $\vec{V}$ defines a differentiable vector field. Then the divergence of $\vec{V}$ is defined by
$$\vec{\nabla}\cdot\vec{V} \text{ or } \operatorname{div}\vec{V} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(V_1\hat{i} + V_2\hat{j} + V_3\hat{k}\right)$$
$$= \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$$

* $\vec{\nabla}\cdot\vec{V} \neq \vec{V}\cdot\vec{\nabla}$

Q If $\vec{A} = x^2z\,\hat{i} - 2y^3z^2\hat{j} + xy^2z\,\hat{k}$, find div.A at the point (1, -1, 1).

Ans
$$\operatorname{div}A = \left(2xz - 6y^2z^2 + xy^2\right)$$
$$= 2 - 6 + 1$$
$$= -3$$

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12 x

Q. find the value of $\nabla^2\left(\frac{1}{|\vec{r}|}\right) = 0$

Ans $|\vec{r}| = \sqrt{x^2+y^2+z^2}$

~~$\vec{\nabla}\left|\frac{1}{\vec{r}}\right| = \frac{1}{2\sqrt{x^2+y^2+}}$~~

$$\vec{\nabla}\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) = \frac{-1}{2}\left[\frac{1}{(x^2+y^2+z^2)^{3/2}}\left[2x\hat{i}+2y\hat{j}+2z\hat{k}\right]\right]$$

$$= \frac{-1}{(x^2+y^2+z^2)^{3/2}}\left[x\hat{i}+y\hat{j}+z\hat{k}\right]$$

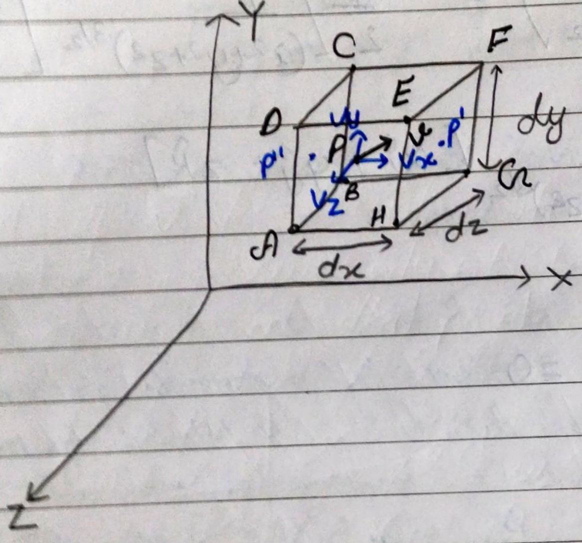
$$= \frac{-\vec{r}}{|\vec{r}|^3}$$

$$\nabla^2\left(\frac{1}{|\vec{r}|}\right) = 0$$

Physical Interpretation of Divergence:

Consider a fluid, say, water flowing through a small rectangular parallelopiped of dimensions dx, dy and dz parallel to x, y and z axes respectively

Let $\vec{V} = V_x \hat{i} + V_y \hat{j} + V_z \hat{k}$ be the velocity of the fluid at the center P of the parallelopiped.



$PP' = \frac{dx}{2}$

$PP'' = -\frac{dx}{2}$

Let $\frac{\partial V_x}{\partial x}$ = Rate of change of V_x along X-axis.

Then, the magnitude of X-component of $\vec{V}$ at face $EFGH = V_x + \frac{\partial V_x}{\partial x}\frac{dx}{2}$

Similarly the magnitude of x component of $\vec{V}$ at face $ABCD = V_x - \frac{\partial V_x}{\partial x}\frac{dx}{2}$

Now, the outward flux at face EFGH

$$= \left(V_x + \frac{\partial V_x}{\partial x}\frac{dx}{2}\right)dy\,dz$$

Now, the inward flux at face ABCD

$$= \left(V_x - \frac{\partial V_x}{\partial x}\frac{dx}{2}\right)dy\,dz$$

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So net outward flux along x-direction will be

$$= \left[\left(V_x + \frac{\partial V_x}{\partial x}\frac{dx}{2} \right) dy\,dz - \left(V_x - \frac{\partial V_x}{\partial x}\frac{dx}{2} \right) dy\,dz \right]$$

$$= \frac{\partial V_x}{\partial x}\, dx\,dy\,dz$$

Similarly, net outward flux along Y-direction

$$= \frac{\partial V_y}{\partial y}\, dx\,dy\,dz$$

and, net outward flux along z-direction

$$= \frac{\partial V_z}{\partial z}\, dx\,dy\,dz$$

~~So~~. Now, net outward flux through the parallelopiped ABCDEFGH $= \left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} \right) dx\,dy\,dz$

$\therefore$ Net outward flux per unit volume through the parallelopiped $= \dfrac{\left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} \right) dx\,dy\,dz}{dx\,dy\,dz}$

$$= \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \right) \cdot \left(V_x\hat{i} + V_y\hat{j} + V_z\hat{k} \right)$$

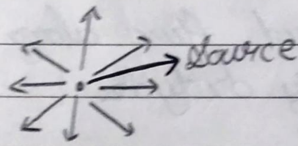
$$= \vec{\nabla} \cdot \vec{V}$$

i.e. $\vec{\nabla} \cdot \vec{V}$ = Net outward flux per unit volume

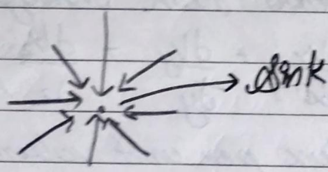
$\therefore$ Divergence of a vector field signifies the quantitative measure of how much a vector field diverges or spreads out.

* Note:
$\vec{\nabla}\cdot\vec{V}$ at a point can be +ve, -ve or 0.

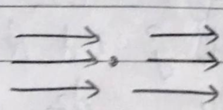
Case I: If $\vec{\nabla}\cdot\vec{V}$ = +ve, then it shows that there is some source of fluid at that point or fluid is expanding from the point.



Case II: If $\vec{\nabla}\cdot\vec{V}$ = -ve, then it shows that there is some sink of fluid at at that point or the fluid is contracting at that point.



Case III: If $\vec{\nabla}\cdot\vec{V} = 0$, it shows that neither source nor sink exists at that point. In this case, fluid entered is equal to the fluid leaving out at a given point.



Any vector $\vec{V}$ for which $\vec{\nabla}\cdot\vec{V} = 0$ is said to be solenoidal.

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Q Determine the constant 'a' so that the vector $\vec{V} = (x+3y)\hat{i} + (y-2z)\hat{j} + (x+az)\hat{k}$ is solenoidal.

Ans
$$\frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(x+az) = 0$$
$$1 + 1 + a = 0$$
$$a = -2$$

- Curl:

If vector $\vec{V}(x,y,z) = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$ defines a differentiable vector field, then the curl or rotation of $\vec{V}$ is defined by

$$\vec{\nabla} \times \vec{V} \text{ or curl } \vec{V} \text{ or rot } \vec{V}$$
$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \times \left(V_1\hat{i} + V_2\hat{j} + V_3\hat{k}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$= \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z}\right)\hat{i} - \left(\frac{\partial V_3}{\partial x} - \frac{\partial V_1}{\partial z}\right)\hat{j} + \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}\right)\hat{k}$$

Q If $\vec{A} = xz^3\hat{i} - 2x^2yz\hat{j} + 2yz^4\hat{k}$, find curl A at the point $(1,-1,1)$.

Ans $(2z^4 + 2x^2y)\hat{i} - (0 - 3z^2x)\hat{j} + (-4xyz - 0)\hat{k}$

$(2 + 2(1)(-1))\hat{i} + 3\hat{j} + 4\hat{k}$

$= \cancel{4\hat{i}}\ 3\hat{j} + 4\hat{k}$

Q $\vec{A} = x^2y\,\hat{i} - 2xz\,\hat{j} + 2yz\,\hat{k}$, find curl curl $\vec{A}$

ans $(2z - (-2x))\hat{i} - (0-0)\hat{j} + (-2z - x^2)\hat{k}$

$(2z + 2x)\hat{i} - (2z + x^2)\hat{k}$

$(0-0)\hat{i} - (-2x - 2)\hat{j} + (0-0)$

$= (2x+2)\hat{j}$

Q Given $\phi = 2x^3y^2z^4$, find $\vec{\nabla}\cdot\vec{\nabla}\phi$

ans $\vec{\nabla}\phi = 6x^2y^2z^4 + 4x^3yz^4 + 8x^3y^2z^3$

$\vec{\nabla}\cdot\vec{\nabla}\phi = 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2$

- Physical Interpretation of Curl:

We know that

$\vec{v} = \vec{\omega}\times\vec{r}$; $\vec{\omega} = \omega_1\hat{i} + \omega_2\hat{j} + \omega_3\hat{k}$

$\vec{r} = r_1\hat{i} + r_2\hat{j} + r_3\hat{k}$

where $\vec{\omega}$ is the angular velocity, $\vec{v}$ is the linear velocity and $\vec{r}$ is the position vector of a point on the rotating body.

Now, $\vec{\nabla}\times\vec{v} = \vec{\nabla}\times(\vec{\omega}\times\vec{r})$

$= \vec{\nabla}\times\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \omega_1 & \omega_2 & \omega_3\\ r_1 & r_2 & r_3\end{vmatrix}$

~~$= \vec{\nabla}\times[(\omega_2 r_3 - \omega_3 r_2)\hat{i} - (\omega_1 r_3 - \omega_3 r_1)\hat{j} + (\omega_1 r_2 - \omega_2 r_1)\hat{k}]$~~

~~$\vec{\nabla}$~~

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$$= \vec{\nabla} \times [(\omega_2 z - \omega_3 y)\hat{i} - (\omega_1 z - \omega_3 x)\hat{j} + (\omega_1 y - \omega_2 x)\hat{k}]$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (\omega_2 z - \omega_3 y) & -(\omega_1 z - \omega_3 x) & (\omega_1 y - \omega_2 x) \end{vmatrix}$$

$$= \hat{i}(\omega_1 + \omega_1) - \hat{j}(-\omega_2 - \omega_2) + \hat{k}(+\omega_3 + \omega_3)$$

$$= 2(\omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k})$$

$$= 2\vec{\omega}$$

$$\vec{\nabla} \times \vec{v} = 2\vec{\omega}$$

This shows that the curl of a vector field is connected with rotational property of the vector field and justifies the name rotation used for curl.
Hence, curl of a vector quantity signifies how much the vector quantity curls or twists.

* A vector $\vec{v}$ is called IRROTATIONAL if $\vec{\nabla} \times \vec{v} = 0$

- Formula ~~regarding~~ involving $\vec{\nabla}$:
If $\vec{A}$ and $\vec{B}$ define differentiable vector fields, and ϕ defines a differentiable scalar field of position (x, y, z), then.

i. $\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{A}\cdot(\vec{\nabla}\times\vec{B})$

ii. $\vec{\nabla}\times(\vec{A}\times\vec{B}) = (\vec{B}\cdot\vec{\nabla})\vec{A} - \vec{B}(\vec{\nabla}\cdot\vec{A}) - (\vec{A}\cdot\vec{\nabla})\vec{B} + \vec{A}(\vec{\nabla}\cdot\vec{B})$

iii. $\vec{\nabla}(\vec{A}\cdot\vec{B}) = (\vec{B}\cdot\vec{\nabla})\vec{A} + (\vec{A}\cdot\vec{\nabla})\vec{B} + \vec{B}\times(\vec{\nabla}\times\vec{A})$
$+ \vec{A}\times(\vec{\nabla}\times\vec{B})$

iv. $\vec{\nabla}\cdot(\vec{\nabla}\phi) = \nabla^2\phi = \dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} + \dfrac{\partial^2\phi}{\partial z^2}$

where $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$ is called the Laplacian operator.

v. $\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}$

vi. $\vec{\nabla}\times(\vec{\nabla}\phi) = 0$

vii. $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0$

- Line Integrals:

In line or path integrals, some quantity related to the field is integrated between two given points in space, A and B, along a prescribed curve C that joins them.

In general, we may encounter line integrals of the forms

$$\int_C \phi\cdot d\vec{r}, \quad \int_C \vec{F}\cdot d\vec{r}, \quad \int_C \vec{F}\times d\vec{r},$$

where ϕ is a scalar field and $\vec{F}$ is a vector field. In physical application, line integrals of the second type are the most common.

* Note:

i. If $\vec{F}$ represents the force acting on a particle along the arc AB, then $\int_A^B \vec{F} \cdot d\vec{r}$ gives the total work done.

ii. If $\vec{V}$ represents the velocity of a liquid, then $\oint \vec{V} \cdot d\vec{r}$ is called the Circulation of $\vec{V}$ around the curve C.
($\oint$ means that the path of integration is a closed curve).

• Properties of Line Integral:

1. Reversing the path of integration changes the sign of the integral.
If the path along which the integrals are evaluated has A and B as its end-points, then

$$\int_A^B \vec{F} \cdot d\vec{r} = -\int_B^A \vec{F} \cdot d\vec{r}$$

This implies that if the path C is a loop then integrating around the loop in the opposite direction changes the sign of the integral.

ii. If the path of integration is subdivided into smaller segments, then the sum of the seperate line integrals along each segment is equal to the line integral along the whole path.
So, if P is any point on the path of integration that lies between the paths end-points A and B, then $\int_A^B \vec{F} \cdot d\vec{r} = \int_A^P \vec{F} \cdot d\vec{r} + \int_P^B \vec{F} \cdot d\vec{r}$

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Q. If $\vec{F} = (3x^2 + 6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}$, evaluate $\int_C \vec{F} \cdot d\vec{r}$ from $(0,0,0)$ to $(1,1,1)$ along the path such that $x = t$, $y = t^2$, $z = t^3$.

Ans

$$\int \vec{F} \cdot d\vec{r} = \int (3x^2 + 6y)\,dx - 14yz\,dy + 20xz^2\,dz$$

$$= (3t^2 + 6t^2)\,dt - 14t^5 \cdot 2t\,dt + 20t^7\,3t^2\,dt$$

$$= \int_0^1 9t^2\,dt - \int_0^1 28t^6\,dt + \int_0^1 60t^9\,dt$$

$$= \frac{9}{3}[1] - \frac{28}{7}[1] + \frac{60}{10}[1]$$

$$= 3 - 4 + 6$$

$$= 5$$

Q. If a force $\vec{F} = 2x^2y\hat{i} + 3xy\hat{j}$ displaces a particle on the xy-plane from $(0,0)$ to $(1,4)$ along a curve $y = 4x^2$. Find the work done.

Ans

$$2x^2y\,dx + 3xy\,dy$$

$$= 2x^2 \cdot 4x^2\,dx + 3x \cdot 4x^2 \cdot 8x\,dx$$

$$= 8x^4\,dx + 96x^4\,dx$$

$$= \frac{8}{5}[1] + \frac{96}{5}$$

$$= 1.6 + 19.2$$

$$= 21.8$$

- **Surface Integral:**

Let $\vec{F}$ be a vector field and S be the given surface.

Surface integral of a vector field $\vec{F}$ over the surface S is defined as the integral of the components of $\vec{F}$ along the normal to the surface.

$\therefore$ Surface integral of $\vec{F}$ over S

$$= \iint_S \vec{F}\cdot d\vec{S}$$

$$= \iint_S \vec{F}\cdot \hat{n}\, dS \; ; \; d\vec{S} = \hat{n}\, dS$$

where $\hat{n}$ is the unit vector normal to an element dS. (outward to surface)

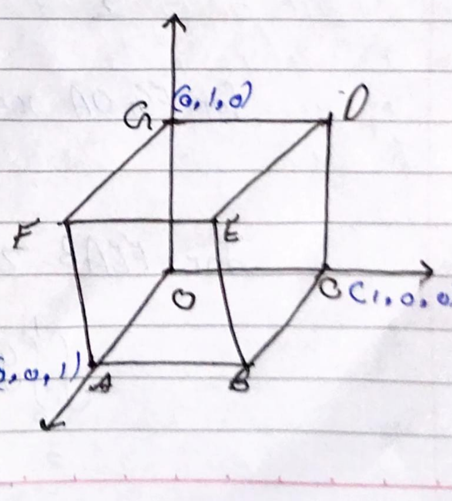
As analoguous of the line integrals, we may also encounter surface integrals of the forms,

$$\iint_S \phi\, dS \;,\; \iint \phi\, d\vec{S} \;,\; \iint_S \vec{F}\times d\vec{S}$$

Q Find the value of $\iint_S \vec{F}\cdot\hat{n}\, dS$, where $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ and S is the surface of the cube bounded by the planes, $x=0, x=1; y=0, y=1; z=0, z=1;$

Ans OABC $\to$ XZ plane $\to y = 0$

GFED $\to$ XZ plane $\to y = 1$



$$S = \iint_{OABC} \vec{F}\cdot\hat{n}\, dS + \iint_{GFED} \vec{F}\cdot\hat{n}\, dS$$

$$+ \iint_{OGFA} \vec{F}\cdot\hat{n}\, dS + \iint_{CDEB} \vec{F}\cdot\hat{n}\, dS$$

$$+ \iint_{GOCD} \vec{F}\cdot\hat{n}\, dS + \iint_{FABE} \vec{F}\cdot\hat{n}\, dS$$

Now $\iint_{OABC} \vec{F}\cdot\hat{n}\,ds = \iint (4xz\hat{i} - y^2\hat{j} + yz\hat{k}) \cdot -\hat{j}\,dx\,dz$

$= \int_0^1\int_0^1 y^2\,dx\,dz$

for OABC $y = 0$

So $= 0$

for GFED $= \int_0^1\int_0^1 -\,dx\,dz$

$= \int_0^1 1\cdot dz$

$= -1$

for BECD $x = \phi$

$\int_0^1\int_0^1 4xz\,dy\,dz$

$= \int 4x\left[\frac{z^2}{2}\right] dy$

$2\int_0^1 2x\,dy$

$= 2$

for FGOA $x = 0$ so $\iint_{FGOA} \vec{F}\cdot d\vec{s} = 0$

for FEAB $z = 1$, $\hat{n} = \hat{k}$

$\iint_0^1 yz\,dy\,dx$

$= \frac{1}{2}$

for C_1OCD $z=0$ so $\iint_{C_1OCD} \vec{F}\cdot d\vec{s} = 0$

So $\iint_S \vec{F}\cdot\hat{n}\,ds = -1 + 2 + \frac{1}{2}$

$= \frac{3}{2} = 1.5$

- Volume ~~integration~~ Integral:

Let $\vec{F}$ be a vector field and v be the volume enclosed by a closed surface

Then, volume integral $= \iiint_v \vec{F}\,dv$

Volume or space integrals are generally simpler than line or surface integrals since the element of volume dv is a scalar quantity.

We may also encounter volume integral of the form $\iiint_v \phi\,dv$.

Q) If $\vec{F} = 2z\hat{i} - x\hat{j} + y\hat{k}$, evaluate $\iiint \vec{F}\,dv$ where V is the region bounded by the surfaces $x=0, x=2$; $y=0, y=4$; $z=x^2$, $z=2$.

ans

$$\int_0^2\int_0^4\int_{x^2}^2 (2z\hat{i} - x\hat{j} + y\hat{k})\,dx\,dy\,dz$$

$$= \iint \left[z^2\hat{i} - xz\hat{j} + yz\hat{k}\right]_{x^2}^2$$

$$= \iint (4-x^4)\hat{i} - x(2-x^2)\hat{j} + y(2-x^2)\hat{k}$$

$$= \iint_0^4 (4-x^4)\hat{i} - \hat{j}(2x-x^3) + (2y-x^2y)\hat{k}\, dx\, dy$$

$$= \int (4-x^4)\cdot 4\hat{i} - 4\hat{j}(2x-x^3) + (2-x^2)\frac{y^2}{2}\, dx$$

$$= \int_0^2 4(4-x^4)\hat{i} - 4\hat{j}(2x-x^3) + 8(2-x^2)\, dx$$

$$= 4\int_0^2 \left[\left(4x - \frac{x^5}{5}\right)\hat{i} - \hat{j}\left(x^2 - \frac{x^4}{4}\right) + 2\left(2x - \frac{x^3}{3}\right)\hat{k}\right]$$

$$= 4\left[\left(4\cdot 2 - \frac{32}{5}\right)\hat{i} - \hat{j}[4-4] + 2\left[4 - \frac{8}{3}\right]\hat{k}\right.$$

$$= 4\left[\cancel{8}\,\frac{8}{5}\hat{i} + 2\left[\frac{4}{3}\right]\hat{k}\right]$$

$$= \frac{32}{5}\hat{i} + \frac{32}{3}\hat{k}$$

- Gauss's Divergence Theorem:

(Relation b/w surface integral & volume integral)

~~Staterrel~~

Statement: Gauss's Divergence theorem states that the surface integral of the normal components of a vector field $\vec{F}$ taken around a closed surface S is equal to the volume integral of the divergence of $\vec{F}$ taken over the volume V enclosed by the surface S.

Mathematically,

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \cancel{\text{div}}(\vec{\nabla}\cdot\vec{F})\,dv$$

or

$$\iint \vec{F}\cdot d\vec{s} = \iiint (\vec{\nabla}\cdot\vec{F})\,dv$$

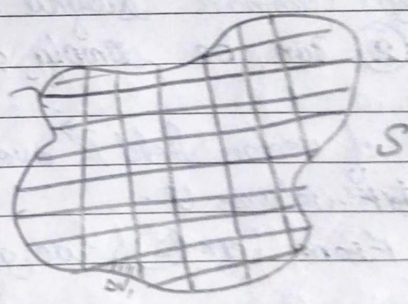
Proof:

Consider a closed surface s which enclosed volume v in a vector field $\vec{F}$.

Let us divide the given ~~vector~~ volume V into a large number, say N of infinitesimally small volume elements each having volumes $\Delta V_1, \Delta V_2, \ldots \Delta V_i, \ldots, \Delta V_n$ enclosed by the surfaces $\Delta s_1, \Delta s_2, \ldots \Delta s_i, \ldots \Delta s_N$ respectively.

Now, the flux of vector field $\vec{F}$ through volume element '1' is given by

$$\iint_{\Delta s_1} \vec{F}\cdot d\vec{s} = (\vec{\nabla}\cdot\vec{F})\,\Delta V_1$$



$(\vec{\nabla}\cdot\vec{V})$ volume = Flux

Similarly, the flux of vector field $\vec{F}$ through volume element 2

$$\iint_{\Delta s_2} \vec{F}\cdot d\vec{s} = (\vec{\nabla}\cdot\vec{F})\,\Delta V_2$$

$$\vdots \qquad \vdots$$

$$\iint_{\Delta s_N} \vec{F}\cdot d\vec{s} = (\vec{\nabla}\cdot\vec{F})\,\Delta V_n$$

Adding above equations,

$$\iint_{\Delta S_1} \vec{F}\cdot d\vec{s} + \iint_{\Delta S_2} \vec{F}\cdot d\vec{s} + \ldots\ldots + \iint_{\Delta S_N} \vec{F}\cdot d\vec{s} = (\vec{\nabla}\cdot\vec{F})(\Delta V_1 + \Delta V_2 + \cdots + \Delta V_n)$$

$$\sum_{i=1}^{N} \iint_{\Delta S_i} \vec{F}\cdot d\vec{s} = \sum_{i=1}^{N} (\vec{\nabla}\cdot\vec{F})\,\Delta V_i \quad - (i)$$

when $N \to \infty$, $\Delta V_i \to 0$

As a result, summation on RHS of eqn. (1) changes into volume ~~integration~~ integral.

$$\therefore \sum_{i=1}^{N} \iint_{\Delta S_i} \vec{F}\cdot d\vec{s} = \iiint_{V} (\vec{\nabla}\cdot\vec{F})\, dV \quad - (2)$$

As the outward of two neighbouring volume elements through common surface are equal and opposite, LHS of eqn. (2) can be simply written as $\iint_S \vec{F}\cdot d\vec{s}$ which is flux of vector field $\vec{F}$ over surface S enclosing the ~~vector~~ volume V.

$\therefore$ From eqn. (2), we get

$$\iint_S \vec{F}\cdot d\vec{s} = \iiint_V (\vec{\nabla}\cdot\vec{F})\, dV$$

Q Use Gauss Divergence theorem to evaluate $\iint \vec{F}\cdot\hat{n}\, ds$ where $\vec{F} = 4xz\,\hat{i} - y^2\hat{j} + yz\,\hat{k}$ and S is the surface of the cube bounded by

$x=0, x=1;\quad y=0, y=1;\quad z=0, z=1.$

Ans $\vec{\nabla}\cdot\vec{F} = 4z - 2y + y$

$= 4z - y$

$$\int_0^1\int_0^1\int_0^1 (\vec{\nabla}\cdot\vec{F})\,dV\; dx\,dy\,dz = \iiint (4z-y)\,dx\,dy\,dz$$

$$= \iint [2z^2 - yz]_0^1\, dx\,dy$$

$$= \iint (2-y)\,dx\,dy$$

$$= \int \left[2y - \frac{y^2}{2}\right]_0^1 dx$$

$$= \int_0^1 \left(2 - \frac{1}{2}\right) dx$$

$$= \frac{3}{2}\int_0^1 dx$$

$$= \frac{3}{2} = 1.5$$

Q. Use Gauss's Divergence theorem to evaluate $\iint_S \vec{A}\cdot d\vec{s}$ where $\vec{A} = x^3\hat{i} + y^3\hat{j} + z^3\hat{k}$ & S is surface of the sphere $x^2+y^2+z^2 = a^2$ ('a' is the radius of the sphere).

Ans $\vec{\nabla}\cdot\vec{F} = 3(x^2+y^2+z^2)$

$$\cancel{\iiint \vec{\nabla}\cdot\vec{F}\,dx\,dy\,dz = 3\times 8\int_0^a\int_0^a\int_0^a x^2+y^2+z^2}$$

$$24 \iint \left[x^2 z + y^2 z + \frac{z^3}{3} \right]_0^a$$

$$= 24 \iint (x^2+y^2)a + a^3$$

$$= 24a \iint x^2+y^2+\frac{a^2}{3}$$

$$= 24a \iint (x^2+a^2)a + \frac{a^3}{3}$$

$$= 24a^2 \int x^2 + \frac{a^2}{3} + \frac{a^2}{3}$$

$$= 24a^2 \quad \frac{x^3}{3} + \left(\frac{2a^2}{3}\right) a$$

$$= 24a^2 \quad \frac{a^3}{3} + \frac{2a^3}{3}$$

$$= 24a^2 * \frac{a^3}{3}$$

$$= 8a^5$$

$$= 3 \iiint_V (x^2+y^2+z^2)\, dV$$

$$= 3 \iiint_V a^2\, dV$$

$$= 3a^2 \iiint_V dV$$

$$= 3a^2 \times \frac{4}{3}\pi a^3$$

$$= 4\pi a^5$$

- **Stokes Theorem:**

(Relation b/w line integral and surface integral)

Statement:

Stokes theorem states that the surface integral of the components of curl $\vec{F}$ along the normal to the surface S, taken over the surface S bounded by curve C is equal to the line integral of the vector field $\vec{F}$ taken along the closed curve C.

Mathematically,

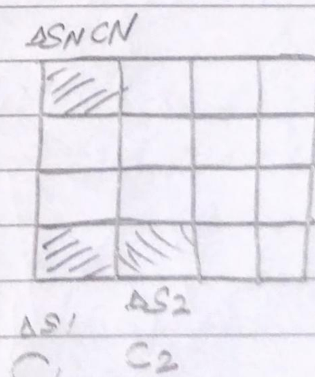
$$\iint_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}\,dS = \oint_C \vec{F}\cdot d\vec{r}$$

or $$\iint_S \vec{\nabla}\times\vec{F}\cdot d\vec{S} = \oint_C \vec{F}\cdot d\vec{r}$$

where $\hat{n}$ is the unit vector normal to the surface element dS.

Proof:

Consider an open surface S bounded by closed path C, placed in a vector field $\vec{F}$.
Let the surface S be divided into a large number, say N, of infinitesimally small surface elements $\Delta S_1, \Delta S_2, \Delta S_3 \ldots\ldots \Delta S_N$ having curve boundaries $C_1, C_2, \ldots\ldots C_n$ respectively.



We know that

$\vec{\nabla}\times\vec{F}$ = Line integral of $\vec{F}$ along closed curve per unit area

$\therefore$ For one surface element ΔS_i bounded by curve C_i, we can write

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$$(\vec{\nabla}\times\vec{F})\cdot\Delta\vec{S_i} = \oint_{C_i} \vec{F}\cdot d\vec{r}$$

This equation is valid for each surface element. Adding such equations for all the surface elements, we get

$$\sum_{i=1}^{N} (\vec{\nabla}\times\vec{F})\cdot\Delta\vec{S_i} = \sum_{i=1}^{N}\oint_{C_i} \vec{F}\cdot d\vec{r} \quad \text{— (1)}$$

When $N\to\infty$, $\Delta S_i \to 0$

As a result, summation on LHS of eqn. (1) converts into surface integral.

Thus we can write

$$\iint_S (\vec{\nabla}\times\vec{F})\cdot d\vec{S} = \sum_{i=1}^{N}\oint_{C_i} \vec{F}\cdot d\vec{r} \quad -(2)$$

In fact, the line integrals along the common sides b/w two adjacent surface elements will be traversed in opposite directions and hence cancel each other.

Thus, the RHS of eqn. (2) represent sum of line integrals only along the sides which are on the boundary of the curve C.

$$\therefore \iint_S \vec{\nabla}\times\vec{F}\cdot d\vec{S} = \oint_C \vec{F}\cdot d\vec{r}$$

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Q Use Stokes's theorem to evaluate $\oint_C (yz\,dx + xz\,dy + xy\,dz)$ where C is the curve $x^2+y^2=1,\ z=2$.

Ans $\vec{F} = yz\hat{i} + xz\hat{j} + xy\hat{k}$

$$\cancel{\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (yz\hat{i} + xz\hat{j} + xy\hat{k}) \cdot (dx\ldots)\ \hat{k}\cdot ds}$$

$$\cancel{\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} xy\,dx\,dy}$$

By Stokes's theorem

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix} = \hat{i}(x-x) - \hat{j}(y-y) + \hat{k}(z-z) = 0$$

So $\iint \vec{\nabla}\times\vec{F}\cdot\hat{n}\,ds = 0$

$\therefore \oint_C yz\,dx + xz\,dy + xy\,dz = 0$

Q Using Stokes's theorem, evaluate $\oint_C [(2x-y)\,dx - yz^2\,dy - y^2z\,dz]$ where C is circle $x^2+y^2=1$, corr. to surface of sphere of unit radius.

Ans $\vec{F} = (2x-y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix} = \hat{i}(-2yz+2yz) - \hat{j}(0-0) + \hat{k}(0+1) = \hat{k}$$

$\hat{n} = \hat{k}$

$$\iint \hat{n}\cdot\hat{n}\, ds$$

$$= \iint dx\,dy$$

$$= \pi r^2 \quad ; r = 1$$

$$= \pi$$

$ds = dx\,dy$, Cartesian (x, y)

$\Downarrow$ Polar

$r\,dr\,d\theta$

$$\int_0^1 \int_0^{2\pi} r\,dr\,d\theta$$

$$= \int_0^1 2\pi r\,dr$$

$$= \pi \left[r^2\right]_0^1$$

$$= \pi$$