- **Stokes Theorem:**

(Relation b/w line integral and surface integral)

Statement:

Stokes theorem states that the surface integral of the components of curl $\vec{F}$ along the normal to the surface S, taken over the surface S bounded by curve C is equal to the line integral of the vector field $\vec{F}$ taken along the closed curve C.

Mathematically,

$$\iint_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}\, dS = \oint_C \vec{F}\cdot d\vec{r}$$

or $$\iint_S \vec{\nabla}\times\vec{F}\cdot d\vec{S} = \oint_C \vec{F}\cdot d\vec{r}$$

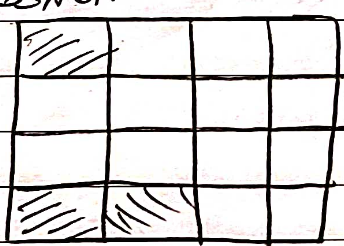
where $\hat{n}$ is the unit vector normal to the surface element dS.

Proof:

Consider an open surface S bounded by closed path C, placed in a vector field $\vec{F}$.

Let the surface S be divided into a large number, say N, of infinitesimally small surface elements $\Delta S_1, \Delta S_2, \Delta S_3 \ldots\ldots \Delta S_N$ having curve boundaries $C_1, C_2, \ldots C_n$ respectively.

$\Delta S_N C_N$



ΔS_1 C_1 ΔS_2 C_2

We know that

$\vec{\nabla}\times\vec{F}$ = Line integral of $\vec{F}$ along closed curve per unit area

$\therefore$ For one surface element ΔS_i bounded by curve C_i, we can write

$$(\vec{\nabla}\times\vec{F})\cdot\Delta\vec{S_i} = \oint_{C_i} \vec{F}\cdot d\vec{r}$$

∴ This equation is valid for each surface element. Adding such equations for all the surface elements we get

$$\sum_{i=1}^{N} (\vec{\nabla}\times\vec{F})\cdot\Delta\vec{S_i} = \sum_{i=1}^{N}\oint_{C_i} \vec{F}\cdot d\vec{r} \quad \text{— (1)}$$

When $N\to\infty$, $\Delta S_i \to 0$

As a result, summation on LHS of eqn.(1) converts into surface integral.

Thus we can write

$$\iint_S (\vec{\nabla}\times\vec{F})\cdot d\vec{S} = \sum_{i=1}^{N}\oint_{C_i} \vec{F}\cdot d\vec{r} \quad -(2)$$

In fact, the line integrals along the common sides b/w two adjacent surface elements will be traversed in opposite directions and hence cancel each other.

Thus, the RHS of eqn.(2) represent sum of line integrals only along the sides which are on the boundary of the curve C.

$$\therefore \iint_S \vec{\nabla}\times\vec{F}\cdot d\vec{S} = \oint_C \vec{F}\cdot d\vec{r}$$

Q Use Stoke's theorem to evaluate $\oint_C (yz\,dx + xz\,dy + xy\,dz)$ where C is the curve $x^2+y^2=1$, $z=2$.

Ans $\vec{F} = yz\,\hat{i} + xz\,\hat{j} + xy\,\hat{k}$

$$\cancel{\int_{-1}^{1}\int_{-1-x^2}^{1-x^2} (yz\hat{i}+xz\hat{j}+xy\hat{k}) \cdot \hat{k}\cdot ds}$$

$$\cancel{\int_{-1}^{1}\int_{-1-x^2}^{1-x^2} xy\,dx\,dy}$$

By Stoke's theorem

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix} = \hat{i}(x-x) - \hat{j}(y-y) + \hat{k}(z-z) = 0$$

So $\iint \vec{\nabla}\times\vec{F}\cdot\hat{n}\,ds = 0$

$\therefore \oint yz\,dx + xz\,dy + xy\,dz = 0$

Q Using Stoke's theorem, evaluate $\oint_C (2x-y)\,dx - yz^2\,dy - y^2z\,dz$] where C is circle $x^2+y^2=1$, corr. to surface of sphere of unit radius.

Ans $\vec{F} = (2x-y)\hat{i} - yz^2\hat{j} - y^2z\,\hat{k}$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix} = \hat{i}(-2yz+2yz) - \hat{j}(0-0) + \hat{k}(0+1) = \hat{k}$$

$\hat{n} = \hat{k}$

$$\iint R\cdot R\, ds$$

$$= \iint dx\,dy$$

$$= \pi r^2 \quad ; r = 1.$$

$$= \pi$$

$ds = dx\,dy$, Cartesian (x, y)
$\Downarrow$ Polar
$r\,dr\,d\theta$

$$\int_0^1\int_0^{2\pi} r\,dr\,d\theta$$

$$= \int_0^1 2\pi r\,dr$$

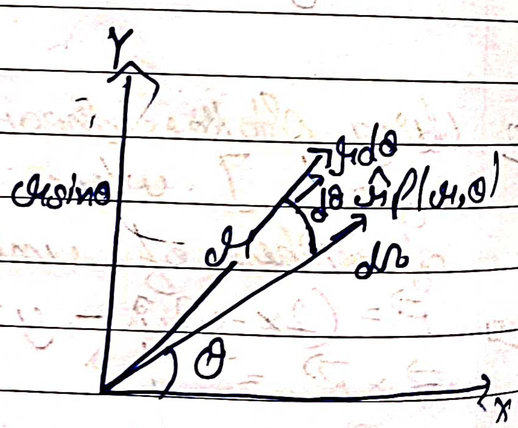
$$= \pi\left[r^2\right]_0^1$$

$$= \pi$$

- Different coordinate systems:

1. Polar Coordinate System:

$x = r\cos\theta$, $y = r\sin\theta$
(Cartesian coordinates in terms of polar coordinates)



$$r = \sqrt{x^2+y^2}, \quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

(polar coordinates in terms of cartesian coordinates)

* The unit vectors $\hat{r}$ and $\hat{\theta}$ point in the direction of increase of the corresponding coordinates.

The infinitesimal length element in $\hat{r}$ direction is
$$dl_r = dr$$

Similarly, the infinitesimal length element in $\hat{\theta}$ direction is $dl_\theta = r\,d\theta$

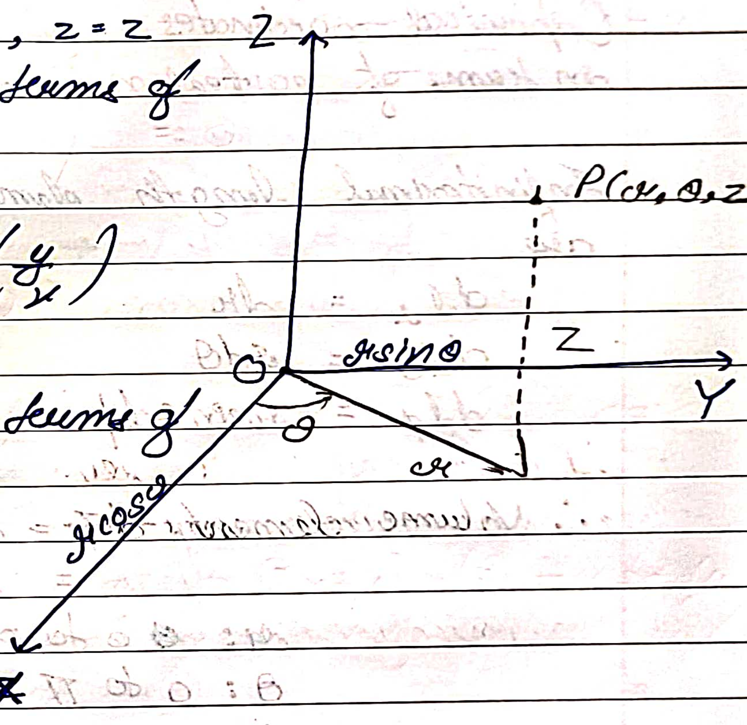
$\therefore$ Surface element, $ds = dl_r \cdot dl_\theta$
$$= r\,dr\,d\theta$$

$$r : 0 \text{ to } \infty$$
$$\theta : 0 \text{ to } 2\pi$$

2. Cylindrical Coordinate System:

$x = r\cos\theta$, $y = r\sin\theta$, $z = z$

(Cartesian coordinates in terms of cylindrical coordinates)



$r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$z = z$

(Cylindrical coordinates in terms of cartesian coordinates)

Infinitesimal length elements in $\hat{r}$, $\hat{\theta}$ and $\hat{z}$ directions are
$$dl_r = dr$$
$$dl_\theta = r\,d\theta$$
$$dl_z = dz$$

$\therefore$ Volume element, dV or $d\tau = dl_r\,dl_\theta\,dl_z$
$$= r\,dr\,d\theta\,dz$$

$$r : 0 \text{ to } \infty$$
$$\theta : 0 \text{ to } 2\pi$$
$$z : -\infty \text{ to } +\infty$$

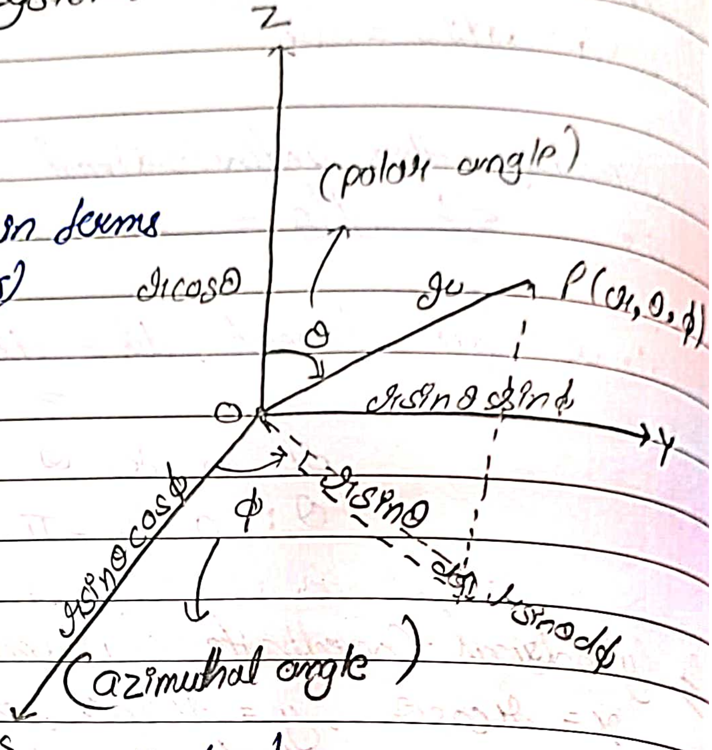
3: Spherical Coordinate System:

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

(Cartesian coordinates in terms of spherical coordinates)



$r = \sqrt{x^2+y^2+z^2}$

$\theta = \tan^{-1}\dfrac{\sqrt{x^2+y^2}}{z}$

$\phi = \tan^{-1}\left(\dfrac{y}{x}\right)$

(Spherical coordinates in terms of cartesian coordinates)

Infinitesimal length elements in $\hat{r}$, $\hat{\theta}$, and $\hat{\phi}$ directions are

$dl_r = dr$

$dl_\theta = r\,d\theta$

$dl_\phi = r\sin\theta\,d\phi$

$\therefore$ Volume element, $d\tau = dl_r\,dl_\theta\,dl_\phi$

$= r^2\sin\theta\,dr\,d\theta\,d\phi$

$r: 0$ to ∞

$\theta: 0$ to π

$\phi: 0$ to 2π

- **Gradient:**

Cartesian: (x, y, z)

$\hat{x}, \hat{y}, \hat{z}$

If F is a scalar field then

$$\vec{\nabla} F = \frac{\partial F}{\partial x}\hat{x} + \frac{\partial F}{\partial y}\hat{y} + \frac{\partial F}{\partial z}\hat{z}$$

Cylindrical:

(r, θ, z)

$\hat{r}, \hat{\theta}, \hat{z}$

$$\vec{\nabla} F = \frac{\partial F}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial F}{\partial \theta}\hat{\theta} + \frac{\partial F}{\partial z}\hat{z}$$

Spherical:

(r, θ, ϕ)

$\hat{r}, \hat{\theta}, \hat{\phi}$

$$\vec{\nabla} F = \frac{\partial F}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial F}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial F}{\partial \phi}\hat{\phi}$$

⇒ In spherical coordinates,

$$\nabla F = \frac{\partial F}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial F}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial F}{\partial \phi}\hat{\phi}$$

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

Then $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$

$= (r\sin\theta\cos\phi)\hat{x} + (r\sin\theta\sin\phi)\hat{y} + (r\cos\theta)\hat{z}$

Now $\hat{r} = \dfrac{\frac{\partial \vec{r}}{\partial r}}{\left|\frac{\partial \vec{r}}{\partial r}\right|}$

$$\hat{\theta} = \frac{\frac{\partial \vec{r}}{\partial \theta}}{\left|\frac{\partial \vec{r}}{\partial \theta}\right|}$$

$$\phi = \frac{\frac{\partial \vec{r}}{\partial \phi}}{\left|\frac{\partial \vec{r}}{\partial \phi}\right|}$$

$$\hat{r} = \frac{\frac{\partial \vec{r}}{\partial r}}{\left|\frac{\partial \vec{r}}{\partial r}\right|} = \frac{\sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}}{\sqrt{\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta}}$$

$\cancel{\sin\theta}$

$$\hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z} \quad -(1)$$

$$\hat{\theta} = \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z} \quad -(2)$$

$$\hat{\phi} = \frac{-\sin\theta\sin\phi\,\hat{x} + \sin\theta\cos\phi\,\hat{y}}{\sin\theta}$$

$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y} \quad -(3)$$

Multiply eqn. (1) by $\sin\theta$ and (2) by $\cos\theta$.

$$\sin\theta\,\hat{r} = \sin^2\theta\cos\phi\,\hat{x} + \sin^2\theta\sin\phi\,\hat{y} + \sin\theta\cos\theta\,\hat{z} \quad -(4)$$

~~$\sin\theta\,\hat{\theta} = \sin\theta\cos\theta\cos\phi\,\hat{x} + \sin\theta\cos\theta$~~

$$\cos\theta\,\hat{\theta} = \cos^2\theta\cos\phi\,\hat{x} + \cos^2\theta\sin\phi\,\hat{y} - \cos\theta\sin\theta\,\hat{z} \quad (5)$$

Adding eqn. (4) & (5),

$$\sin\theta\,\hat{r} + \cos\theta\,\hat{\theta} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y} \quad -(6)$$

Multiply eqn. (6) by $\cos\phi$ and eqn. (3) by $\sin\phi$.

$$\cos\phi\sin\theta\,\hat{r} + \cos\theta\cos\phi\,\hat{\theta} = \cos^2\phi\,\hat{x} + \sin\phi\cos\phi\,\hat{y} \quad -(7)$$

$$\sin\phi\,\hat{\phi} = -\sin^2\phi\,\hat{x} + \cos\phi\sin\phi\,\hat{y} \quad -(8)$$

Subtracting eqn. (8) from (7)

$$\sin\theta\cos\phi\,\hat{r} + \cos\theta\cos\phi\,\hat{\theta} - \sin\phi\,\hat{\phi} = \hat{x}$$

$$\therefore \hat{x} = \sin\theta\cos\phi\,\hat{r} + \cos\theta\cos\phi\,\hat{\theta} - \sin\phi\,\hat{\phi}$$

$$\hat{\phi} = -\sin\phi\sin\theta\cos\phi\,\hat{r} + \sin\phi\cos\theta\cos\phi\,\hat{\theta} - \sin^2\phi\,\hat{\phi} + \cos\phi\,\hat{y}$$

$$\hat{\phi}(1+\sin^2\theta) + \sin\phi\cos\phi\sin\theta\,\hat{r} - \sin\phi\cos\phi\cos\theta = \cos\phi\,\hat{y}$$

$$\hat{\phi}\frac{(1+\sin^2\theta)}{\cos\phi} + \sin\phi\sin\theta\,\hat{r} - \sin\phi\cos\theta$$

Multiply eqn. (6) by $\sin\phi$ and (3) by $\cos\phi$,

$$\sin\theta\sin\phi\,\hat{r} + \cos\theta\sin\phi\,\hat{\theta} = \sin\phi\cos\phi\,\hat{x} + \sin^2\phi\,\hat{y} \quad -(9)$$

$$\cos\phi\,\hat{\phi} = -\sin\phi\cos\phi\,\hat{x} + \cos^2\phi\,\hat{y} \quad -(10)$$

Adding eqn. 9 & 10

$\sin\theta \sin\phi \hat{r} + \cos\theta \sin\phi \hat{\theta} + \cos\phi \hat{\phi} = \hat{y}$

$\therefore \hat{y} = \sin\theta \sin\phi \hat{r} + \cos\theta \sin\phi \hat{\theta} + \cos\phi \hat{\phi}$

Again, multiply eqn. (1) by $\cos\theta$ and eqn. (2) by $\sin\theta$,

$$\cos\theta \hat{r} = \sin\theta\cos\theta\cos\phi \hat{x} + \sin\theta\cos\theta\sin\phi \hat{y} + \cos^2\theta \hat{z} \quad -(11)$$

$$\sin\theta \hat{\theta} = \sin\theta\cos\theta\cos\phi \hat{x} + \sin\theta\cos\theta\sin\phi \hat{y} - \sin^2\theta \hat{z} \quad -(12)$$

Subtracting eqn. (12) from (11)

$\cos\theta \hat{r} - \sin\theta \hat{\theta} = \hat{z}$

$\therefore \hat{z} = \cos\theta \hat{r} - \sin\theta \hat{\theta}$

Now $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial F}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial F}{\partial \phi}\frac{\partial \phi}{\partial x}$

(Chain rule)

$r = \sqrt{x^2+y^2+z^2}$

$\theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right)$

$\phi = \tan^{-1}\frac{y}{x}$

$$\frac{\partial r}{\partial x} = \frac{1}{2\sqrt{x^2+y^2+z^2}} \times 2x = \frac{x}{\sqrt{x^2+y^2+z^2}}$$

$$= \frac{r\sin\theta\cos\phi}{r} = \sin\theta\cos\phi$$

$$\frac{\partial \theta}{\partial x} = \frac{1}{1+\frac{x^2+y^2}{z^2}} \times \frac{2x}{z^2} \; \frac{1}{2\sqrt{\frac{x^2+y^2}{z}}} \times \frac{2x}{z}$$

$$= \frac{z^2}{x^2+y^2+z^2} \times \frac{2x}{z^2} \; \frac{x}{\sqrt{x^2+y^2}}$$

$$= \frac{\cancel{2}x\, z^2}{x^2+y^2+z^2}$$

$$= \frac{r\sin\theta\cos\phi \cdot r^2\cos^2\theta}{r^2}$$

$$=$$

$$\frac{\partial \theta}{\partial x} = \frac{xz}{(x^2+y^2+z^2)(\sqrt{x^2+y^2})} = \frac{r\sin\theta\cos\phi \cdot r\cos\theta}{r^2 \times r\sin\theta} = \frac{\cos\theta\cos\phi}{r}$$

$$\frac{\partial \phi}{\partial x} = \frac{x^2}{x^2+y^2} \; \frac{\partial}{\partial x}\left(\frac{y}{x}\right)$$

$$= \frac{x^2}{x^2+y^2} \times -\frac{y}{x^2}$$

$$= \frac{-y}{x^2+y^2}$$

$$= -\frac{r\sin\theta\sin\phi}{r^2\sin^2\theta\cos^2\phi + r^2\sin^2\theta\sin^2\phi}$$

$$= \frac{-r\sin\theta\sin\phi}{r^2\sin^2\theta}$$

$$= \frac{-1\sin\phi}{r\sin\theta}$$

$$\therefore \frac{\partial F}{\partial x} = \sin\theta\cos\phi\frac{\partial F}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial F}{\partial \theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial F}{\partial \phi}$$

Similarly

$$\frac{\partial F}{\partial y} = \sin\theta \sin\phi \frac{\partial F}{\partial r} + \frac{\cos\theta \, \cancel{\sin\theta} \sin\phi}{r} \frac{\partial F}{\partial \theta} + \frac{\cos\phi}{r\sin\theta} \frac{\partial F}{\partial \phi}$$

$$\frac{\partial F}{\partial z} = \cos\theta \frac{\partial F}{\partial r} - \frac{\sin\theta}{r} \frac{\partial F}{\partial \theta}$$

$$\because \vec{\nabla} F = \frac{\partial F}{\partial x} \hat{x} + \frac{\partial F}{\partial y} \hat{y} + \frac{\partial F}{\partial z} \hat{z}$$

$$= \frac{\partial F}{\partial x} (\sin\theta \cos\phi \, \hat{r} + \cos\theta \cos\phi \, \hat{\theta} - \sin\phi \, \hat{\phi})$$
$$+ \frac{\partial F}{\partial y} (\sin\theta \sin\phi \, \hat{r} + \cos\theta \sin\phi \, \hat{\theta} + \cos\phi \, \hat{\phi})$$
$$+ \frac{\partial F}{\partial z} (\cos\theta \, \hat{r} - \sin\theta \, \hat{r})$$

$$\vec{\nabla} F = \frac{\partial F}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial F}{\partial \theta} \hat{\theta} + \frac{1}{r\sin\theta} \frac{\partial F}{\partial \phi} \hat{\phi}$$

- Divergence :

1. Cartesian :

Let $\vec{F} = F_x \hat{x} + F_y \hat{y} + F_z \hat{z}$ be a vector field,

$$\vec{\nabla} \cdot \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

2. Cylindrical : (r, θ, z)

$$\vec{F} = F_r \hat{r} + F_\theta \hat{\theta} + F_z \hat{z}$$

$$\vec{\nabla} \cdot \vec{F} = \cancel{\partial} \frac{1}{r} \frac{\partial}{\partial r} (r F_r) + \frac{1}{r} \frac{\partial F_\theta}{\partial \theta} + \frac{\partial F_z}{\partial z}$$

3. Spherical :

$$\vec{F} = F_r \hat{r} + F_\theta \hat{\theta} + F_\phi \hat{\phi}$$

$$\vec{\nabla}\cdot\vec{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 F_r) + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta F_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi}$$

- Curl:

1. Cartesian:

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

$$= \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{x} - \left(\frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z}\right)\hat{y} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{z}$$

2. Cylindrical:

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \frac{1}{r}\hat{r} & \hat{\theta} & \frac{1}{r}\hat{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ F_r & rF_\theta & F_z \end{vmatrix}$$

$$= \left(\frac{1}{r}\frac{\partial F_z}{\partial\theta} - \frac{\partial F_\theta}{\partial z}\right)\hat{r} - \left(\frac{\partial F_z}{\partial r} - \frac{\partial F_r}{\partial z}\right)\hat{\theta} + \left(\frac{1}{r}\frac{\partial}{\partial r}(rF_\theta) - \frac{1}{r}\frac{\partial F_r}{\partial\theta}\right)\hat{z}$$

3. Spherical:

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \frac{1}{r^2\sin\theta}\hat{r} & \frac{1}{r\sin\theta}\hat{\theta} & \frac{1}{r}\hat{\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ F_r & rF_\theta & r\sin\theta F_\phi \end{vmatrix}$$

$$= \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(\sin\theta F_\phi) - \frac{\partial F_\theta}{\partial\phi}\right)\hat{r} - \frac{1}{r}\left(\frac{\partial}{\partial r}(rF_\phi) - \frac{1}{\sin\theta}\frac{\partial F_r}{\partial\phi}\right)\hat{\theta} + \frac{1}{r}\left(\frac{\partial}{\partial r}(rF_\theta) - \frac{\partial F_r}{\partial\theta}\right)\hat{\phi}$$

- **Dirac Delta Function:**

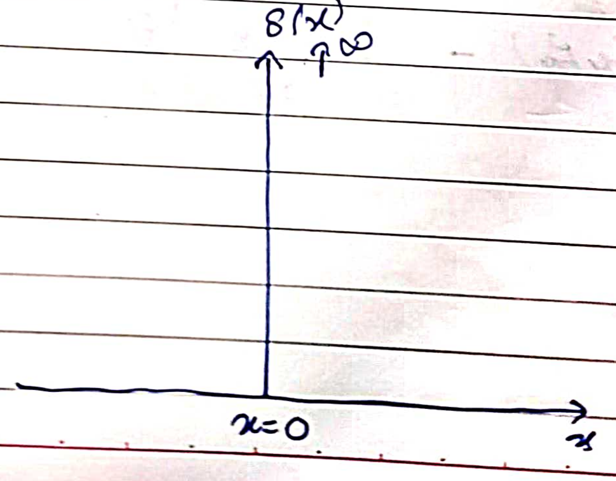
Dirac delta function $\delta(x)$ is the function which is zero everywhere except at $x=0$ where the function tends to infinity.

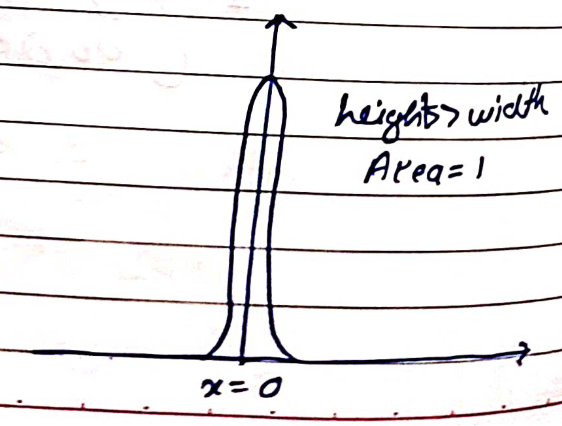
Thus,

$$\delta(x) = \begin{cases} 0, & \text{if } x \neq 0 \\ \infty, & \text{if } x = 0 \end{cases}$$

Dirac delta function can be pictured as an infinitely high, infintesimally narrow spike with area under the curve equal to unity.

i.e. $$\int_{-\infty}^{+\infty} \delta(x)\,dx = 1$$





Properties of Dirac delta function:

i) $\delta(-x) = \delta(x)$

i.e., Dirac delta function is an even function.

ii) $x\,\delta(x) = 0$

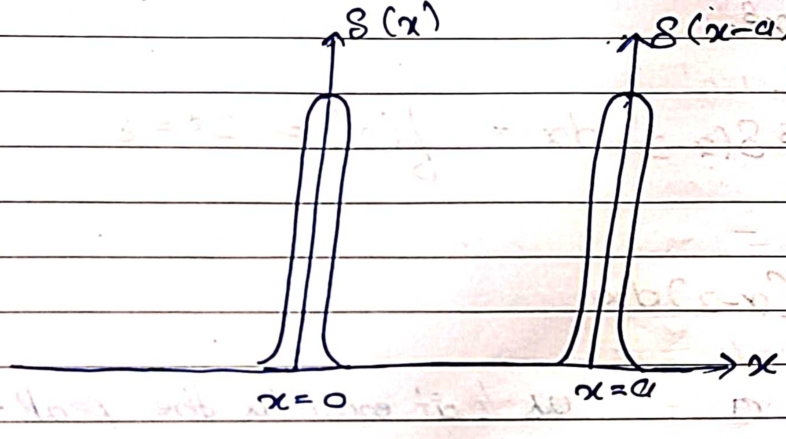
iii) $x \frac{d}{dx}\delta(x) = -\delta(x)$

iv) $\frac{d}{dx}\delta(-x) = -\frac{d}{dx}(\delta(x))$

v) $\delta(kx) = \frac{1}{|k|}\delta(x)$, k is a constant

vi) Shifting property:

According to this property, the Dirac delta function defined at $x = 0$ can be shifted to some other point, $x = a$.



Thus,

$$\delta(x-a) = \begin{cases} 0, & \text{if } x \neq a \\ \infty, & \text{if } x = a \end{cases}$$

with $\int_{-\infty}^{+\infty} \delta(x-a)\,dx = 1$

Here, $\delta(x-a)$ is called the Translated Dirac delta function.

vii) $\int_{-\infty}^{+\infty} f(x)\,\delta(x-a)\,dx = f(a)$

In 3D,

$\delta^3(\vec{r}) = \delta(x)\,\delta(y)\,\delta(z)$

where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is the position vector extending from the origin to the point (x, y, z). The three-dimensional Dirac Delta function is zero everywhere except at $(0, 0, 0)$, where it blows up and its volume integral is 1.

i.e. $\int_{\text{all space}} \delta^3(\vec{r})\, d\tau = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \delta(x)\,\delta(y)\,\delta(z)\, dx\, dy\, dz =$

Q Evaluate the integral $\int_0^3 x^3 \delta(x-2)\, dx$

Ans $f(x) = x^3$

$a = 2$

$\int_0^3 x^3 \delta(x-2)\, dx = f(2) = 2^3 = 8$

Q $\int_0^1 x^3 \delta(x-2)\, dx$

$= 0$ as it excludes the peak.

Q Evaluate the following integrals:

i $\int_2^6 (3x^2 - 2x - 1)\,\delta(x-3)\, dx \rightarrow 20$

ii $\int_0^3 x^3 \delta(x+1)\, dx \rightarrow 0$

iii $\int_{-2}^{2} (2x+3)\,\delta(3x)\, dx \rightarrow 1$

iv $\int_0^2 (x^3 + 3x + 2)\,\delta(1-x)\, dx \rightarrow 6$

Camlin Page
Date / /

v. $\int_{-1}^{1} 9x^2 \, \delta(3x+1)\, dx \rightarrow 1/3$

i. $f(a) = f(3) = 3 \cdot 9 - 2 \cdot 3 - 1$
$= 20$

ii. 0 as $x = -1$ has peak & its excluded

iii. $\delta(Kx) = \frac{1}{|K|}\delta(x)$
$= \frac{1}{3}\delta(x)$

$\int_{-2}^{2} \frac{(2x+3)}{3}\delta(x)$ so $x = 0$ $\frac{3}{3} = 1$

iv. $\int_{0}^{2} (x^3+3x+2)\,\delta(1-x)\,dx$

$x = 1$

$1 + 3 + 2 = 6$

v. $\frac{1}{3}\left(9 \times \frac{1}{9}\right) = \frac{1}{3}$ as $9x^2\,\delta\left(3\left(x + 1/3\right)\right)$

Electrostatics

- **Gauss's Law:**

Gauss's law states that the total electric flux through a closed surface is $1/\varepsilon_0$ times the net charge enclosed by the surface.

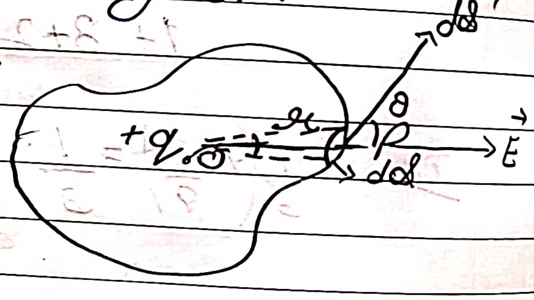
Mathematically,

$$\phi_E = \frac{1}{\varepsilon_0} q_{enclosed}$$

Proof:

Consider a charge $+q$ ~~at~~ situated at a point O within a closed surface S of arbitrary shape. Let P be a point on the surface at a distance r from O.

Now, take a small area dS around P. The normal to the surface dS is represented by the vector $\vec{dS}$ which makes an angle θ with the direction of electric field $\vec{E}$ along OP.



The magnitude of electric field intensity at point P is given by

$$E = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \quad -(1)$$

Electric flux passing through small area dS enclosing the point P is

$$d\phi_E = \vec{E} \cdot \vec{dS}$$
$$= E\,dS\cos\theta \quad -(2)$$

Then, the total electric flux passing through the closed surface is

$$\phi_E = \oint_S d\phi_E$$

$$= \oint_S E\, dS \cos\theta \quad [\text{using eqn. } ②]$$

$$= \oint_S \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2}\, dS \cos\theta \quad [\text{using eqn. } ①]$$

$$= \frac{q}{4\pi\varepsilon_0} \oint_S \frac{dS \cos\theta}{r^2} \quad \left[\frac{\text{Area}}{\text{radius}^2} = \Omega\right]$$

$$= \frac{q}{4\pi\varepsilon_0} \oint_S d\Omega$$

where $d\Omega = \frac{dS\cos\theta}{r^2}$ is the solid angle subtended by area dS at point O. By $\oint_S d\Omega$ = solid angle due to the entire closed surface S at an internal point.

$$\therefore \phi_E = \frac{q}{4\pi\varepsilon_0} \times 4\pi$$

$$\phi_E = \frac{q}{\varepsilon_0}$$

Camlin Date / Page

- Gauss's Law in Integral and Differential form:

By Gauss's law.

Total electric flux, $\phi_E = \frac{1}{\epsilon_0} q_{enclosed}$

$\therefore$ Gauss's law in integral form is given by

$$\oint_S \vec{E} \cdot d\vec{s} = \frac{1}{\epsilon_0} q_{enclosed} \quad -(1)$$

where q_{enc} is the total charge enclosed by the closed surface S.

Let v be the volume bounded by the closed surface. Then, q_{enc} can be written in terms of volume charge density ρ as

$$q_{enc} = \int_v \rho \, dv \quad -(2)$$

Using eqn. (2) in (1), we get

$$\oint_S \vec{E} \cdot d\vec{s} = \frac{1}{\epsilon_0} \int_v \rho \, dv \quad -(3)$$

According to Gauss's Divergence theorem,

$$\oint_S \vec{E} \cdot d\vec{s} = \int_v (\vec{\nabla} \cdot \vec{E}) \, dv$$

$\therefore$ Eqn. (3) can be written as

$$\int_v (\vec{\nabla} \cdot \vec{E}) \, dv = \int_v \left(\frac{\rho}{\epsilon_0} \right) dv$$

Since this holds for any volume, the integrands must be equal.

i.e., $\boxed{\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}}$ - (4)

Eqn. (4) is known as Gauss's law in Differential form.

Thus, Gauss's law in differential form states that the divergence of $\vec{E}$ at any point in space is equal to $\frac{1}{\varepsilon_0}$ times the total volume charge density at that point.

- Applications of Gauss Law:

i Electric field due to uniformly charged SOLID SPHERE.

Consider a solid sphere of radius R and centre O with uniform charge q.

Then, volume charge density,

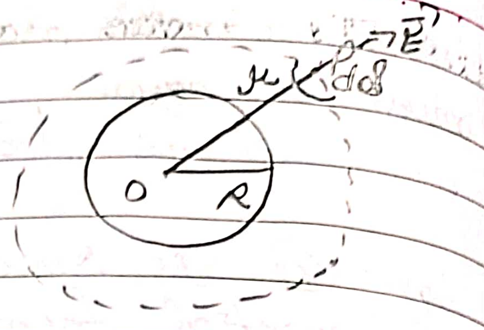
$$\rho = \frac{q}{\frac{4}{3}\pi R^3}$$

$$\Rightarrow q = \rho \cdot \frac{4}{3}\pi R^3$$

⇒ Case I: Outside the sphere

Consider a point P outside the solid sphere at a distance 'r' from the centre O of the solid sphere i.e. $r > R$

A spherical surface called Gaussian surface is drawn at radius r passing through the point P.



Let ds be the small area element on the Gaussian surface around the point P.
By integral form of Gauss's law,

$$\oint_S \vec{E} \cdot \vec{ds} = \frac{1}{\varepsilon_0} \times q$$

$$\oint_S E\,ds \cos 0^\circ = \frac{1}{\varepsilon_0} \rho \times \frac{4}{3} \pi R^3$$

$$E \times 4\pi r^2 = \frac{1}{\varepsilon_0} \times \rho \times \frac{4}{3} \pi R^3$$

$$E = \frac{\rho R^3}{3\varepsilon_0 r^2}$$

i.e. $E_{outside} \propto \frac{1}{r^2}$

⇒ Case II: On the surface of solid sphere
Consider a point P on the surface of the solid sphere i.e. $r = R$.

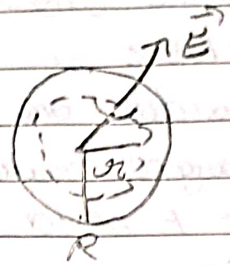
$$\therefore E = \frac{\rho R^3}{3\varepsilon_0 R^2}$$

$$E_{surface} = \frac{\rho R}{3\varepsilon_0}$$

⇒ Case III : Inside the Solid Sphere

$r < R$

By Gauss's law,



$$\oint_S \vec{E} \cdot d\vec{s} = \frac{1}{\varepsilon_0} \times q'$$

$$\oint_S E\, ds \cos 0 = \frac{1}{\varepsilon_0} \times \rho \times \frac{4}{3} \pi r^3$$

$$E \times 4\pi r^2 = \frac{\rho}{\varepsilon_0} \times \frac{4\pi r^3}{3}$$

$$E = \frac{\rho r}{3\varepsilon_0}$$

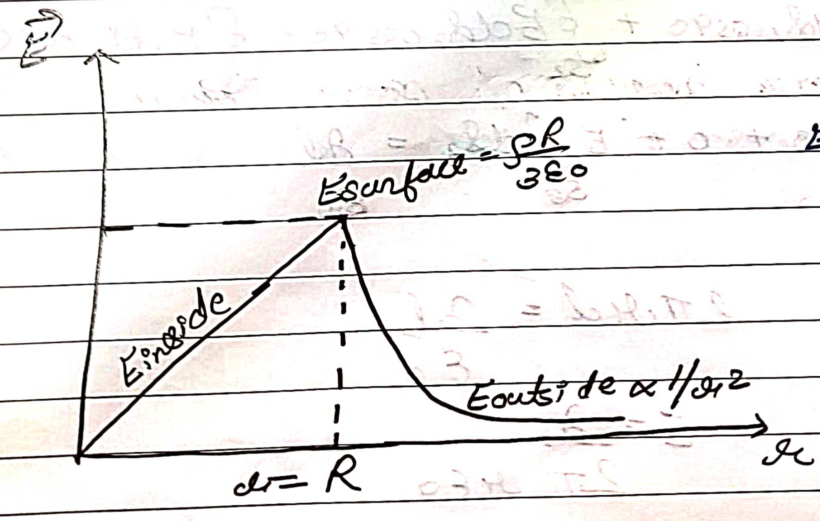
$E_{inside} \propto r$

⇒ Case IV : At the center of solid sphere.

$r = 0$

$$\therefore \quad E = \frac{\rho}{3\varepsilon_0} \times 0$$

$$\Rightarrow E_{centre} = 0$$



$E_{centre} = 0$

$E_{inside} \propto r$

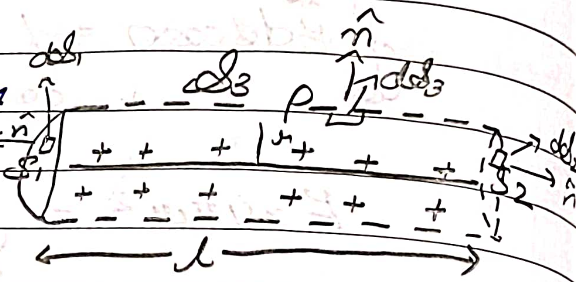
$E_{surface} = \frac{\rho R}{3\varepsilon_0}$

$E_{outside} \propto \frac{1}{r^2}$

iii) Electric field due to uniformly charged STRAIGHT WIRE:

Consider an infinitely long straight uniformly positively charged wire having line charge density λ. Let P be a point at normal distance 'r' from the wire.

Then, a cylindrical gaussian surface of radius 'r' and length 'l' co-axial with the wire is drawn. This Gaussian surface can be divided into three surfaces S_1, S_2, S_3 with respective area elements dS_1, dS_2 and dS_3.



By Gauss's law,

$$\oint_S \vec{E}\cdot d\vec{S} = \frac{1}{\varepsilon_0} q$$

$$\oint_{S_1} \vec{E}\cdot d\vec{S_1} + \oint_{S_2} \vec{E}\cdot d\vec{S_2} + \oint_{S_3} \vec{E}\cdot d\vec{S_3} = \frac{1}{\varepsilon_0} \times \lambda l$$

$$\oint_{S_1} E\, dS_1 \cos 90 + \oint_{S_2} E\, dS_2 \cos 90 + \oint_{S_3} E\, dS_3 \cos 0 = \frac{\lambda l}{\varepsilon_0}$$

$$0 + 0 + E \oint_{S_3} dS_3 = \frac{\lambda l}{\varepsilon_0}$$

$$2\pi r l = \frac{\lambda l}{\varepsilon_0}$$

$$E = \frac{\lambda}{2\pi r \varepsilon_0}$$

Camlin Page
Date / /

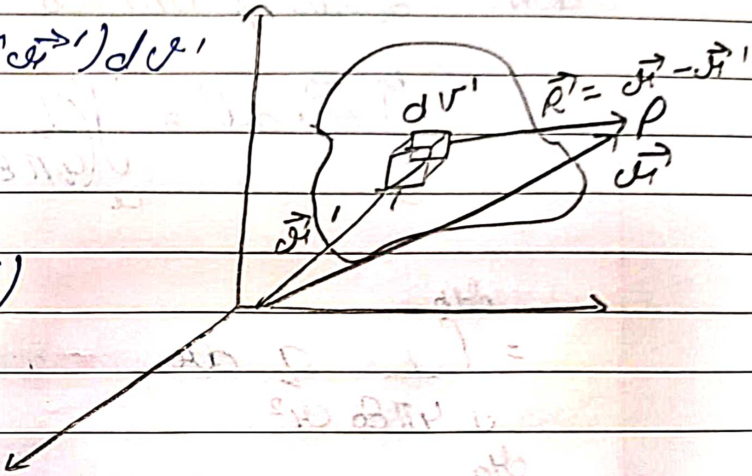
· Divergence of $\vec{E}$:

We know that the electric field for a volume charge is

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_{\text{all space}} \frac{\hat{R}}{R^2}\rho(\vec{r}')\,dv'$$

where $\vec{R} = \vec{r} - \vec{r}'$

Now $$\vec{\nabla}\cdot\vec{E} = \frac{1}{4\pi\varepsilon_0}\int_{\text{allspace}} \vec{\nabla}\cdot\frac{\hat{R}}{R^2}\rho(\vec{r}')\,dv'$$



$$\vec{\nabla}\cdot\left(\frac{\hat{R}}{R^2}\right) = 4\pi\delta^3(\vec{R})$$

$$\therefore \vec{\nabla}\cdot\vec{E} = \frac{1}{4\pi\varepsilon_0}\int_{\text{allspace}} 4\pi\delta^3(\vec{R})\rho(\vec{r}')\,dv'$$

$$= \frac{1}{\varepsilon_0}\int_{\text{allspace}} \delta^3(\vec{r}-\vec{r}')\rho(\vec{r}')\,dv'$$

$$= \frac{1}{\varepsilon_0}\rho(\vec{r})$$

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho(\vec{r})}{\varepsilon_0}$$

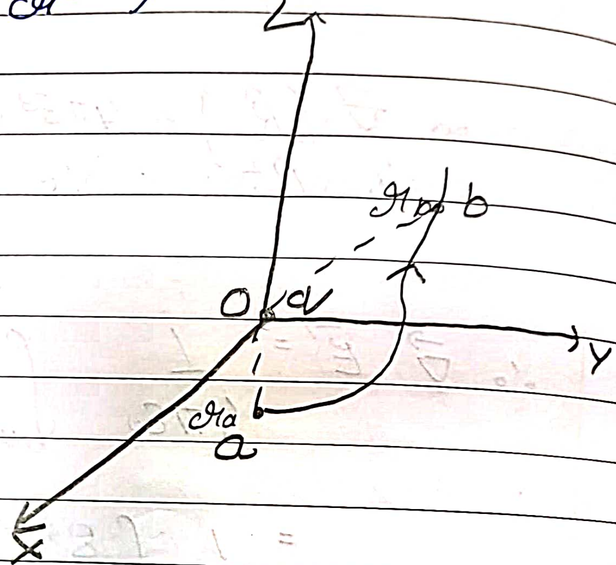
- **Curl of $\vec{E}$:**

Electric flux for a point charge at origin is

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\,\hat{r} \quad \text{— (1)}$$

Now, line integral of $\vec{E}$ from some point 'a' to some other point 'b' is given by

$$\int_a^b \vec{E}\cdot d\vec{l} = \int_a^b \left(\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\,\hat{r}\right)\cdot\left(dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\dots\right.$$



$$= \int_{r_a}^{r_b} \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\,dr$$

$$= \frac{q}{4\pi\varepsilon_0}\int_{r_a}^{r_b} \frac{1}{r^2}\,dr$$

$$= \frac{q}{4\pi\varepsilon_0}\left[-\frac{1}{r}\right]_{r_a}^{r_b}$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r_a} - \frac{1}{r_b}\right]$$

where r_a is the distance from the origin to the point 'a' and r_b is the distance to the point 'b'.

Then, the integral around the closed path is

$$\oint_C \vec{E}\cdot d\vec{l} = \int_a^b \vec{E}\cdot d\vec{l} + \int_b^a \vec{E}\cdot d\vec{l}$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r_a} - \frac{1}{r_b}\right] + \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r_b} - \frac{1}{r_a}\right]$$

$$= 0$$

i.e. $\oint_C \vec{E}\cdot d\vec{l} = 0$ — (2)

According to stoke's theorem,

$$\oint_C \vec{E}\cdot d\vec{l} = \iint_S (\vec{\nabla}\times\vec{E})\cdot d\vec{s}$$

∴ Eqn. (2) can be written as

$$\iint_S (\vec{\nabla}\times\vec{E})\cdot d\vec{s} = 0$$

$$\Rightarrow \boxed{\vec{\nabla}\times\vec{E} = 0}$$

Electrostatic Potential:
We know that
$\vec{\nabla}\times\vec{E} = 0$
⇒ Line integral of $\vec{E}$ around a closed path is zero.
⇒ Line integral of $\vec{E}$ from point 'a' to point 'b' is the same for all paths.
i.e., the line integral of $\vec{E}$ is independent of path.

∴ We can define a function called Electric potential as

$$V(r) = -\int_{Ref}^{r} \vec{E}\cdot d\vec{l}\ ;$$ where Ref is the reference point.

Then, the potential difference b/w two points 'a' and 'b' is

$$V(b) - V(a) = \left(-\int_{Ref}^{b} \vec{E}\cdot d\vec{l}\right) - \left(-\int_{Ref}^{a} \vec{E}\cdot d\vec{l}\right)$$

$$= -\int_{Ref}^{b} \vec{E}\cdot d\vec{l} - \int_{a}^{Ref} \vec{E}\cdot d\vec{l}$$

$$= -\int_{a}^{Ref} \vec{E}\cdot d\vec{l} - \int_{Ref}^{b} \vec{E}\cdot d\vec{l}$$

$$= -\int_{a}^{b} \vec{E}\cdot d\vec{l} \quad - (1)$$

But, the fundamental theorem for gradient states that $\int_{a}^{b} \vec{\nabla} V\cdot d\vec{l} = V(b) - V(a)$

$$\Rightarrow \int_{a}^{b} \vec{\nabla} V\cdot d\vec{l} = -\int_{a}^{b} \vec{E}\cdot d\vec{l} \quad [\text{using eqn. (1)}]$$

Since this is true for any points 'a' and 'b', the integrands must be equal.

$$\therefore \vec{E} = -\vec{\nabla} V$$

Camlin Page
Date / /

i.e. Electric field is the negative of the gradient of scalar potential.

Q. Find the potential inside and outside a spherical shell of radius R, which carries a uniform surface charge +q. Set the reference point at infinity.

Ans
$$\vec{E}_{outside} = \frac{\rho R^3}{3\varepsilon_0 r^2}\hat{r}$$

$$\int \vec{E}\cdot d\vec{l} = -\frac{\rho R^3}{3\varepsilon_0 r^2}dr$$

$$= -\frac{\rho R^3}{3\varepsilon_0 r}$$

$$= +\frac{q}{\frac{4}{3}\pi R^3} \times \frac{R^3}{3\varepsilon_0 r}$$

$$= +\frac{q}{4\pi\varepsilon_0 r}$$

$$E_{inside} = \frac{\rho r}{3\varepsilon_0} = \frac{q}{\frac{4}{3}\pi r^3} \times \frac{r}{3\varepsilon_0}$$

$$= \frac{q}{4\pi\varepsilon_0 r^2}dr$$

$$= \frac{q}{4\pi\varepsilon_0 r}$$

$$= \frac{q}{\frac{4}{3}\pi R^3} \times \frac{r}{3\varepsilon_0}$$

$$= \frac{q\, r}{4\pi\varepsilon_0 R^3}$$

$$V = -\frac{q\, r^2}{8\pi\varepsilon_0 R^3}$$

For a point P' inside the spherical shell,

$$V_{inside} = -\int_{\infty}^{R} \vec{E}\cdot d\vec{l} - \int_{R}^{r} \vec{E}\cdot d\vec{l}$$

$$= \frac{-q}{4\pi\varepsilon_0}\int_{\infty}^{r} \frac{1}{r^2}\,dr - 0 \quad [\vec{E}_{inside} = 0]$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q}{R}$$

⇒ Notes:

In general, the potential of a point charge q is

$$V(r) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$

For a continuous charge distribution, the potential of line, surface and volume charges are

$$\frac{1}{4\pi\varepsilon_0}\int_L \frac{\lambda\, dl}{r}, \quad \frac{1}{4\pi\varepsilon_0}\iint_S \frac{\sigma\, ds}{r} \text{ and}$$

$$\frac{1}{4\pi\varepsilon_0}\iiint_V \frac{\rho\, dv}{r}, \text{ respectively.}$$

Poisson's equation and Laplace's equation:

We know that

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$$

But

$$\vec{E} = -\vec{\nabla}V$$

$$\therefore \quad \vec\nabla \cdot (-\vec\nabla V) = \frac{\rho}{\varepsilon_0}$$

$$\Rightarrow \quad -\vec\nabla \cdot \vec\nabla V = \frac{\rho}{\varepsilon_0}$$

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0}$$

This is known as Poisson's equation.

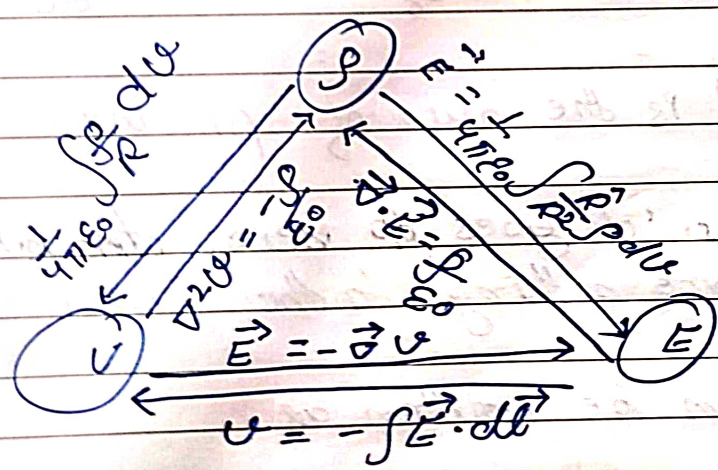
~~If~~ In regions where there is no charge, so that $\rho = 0$

$$\nabla^2 V = 0$$

This is called Laplace's equation.

⇒ Notes:

The three fundamental quantities of electrostatics are ρ, $\vec E$ and V; and the interrelation between them is given below:

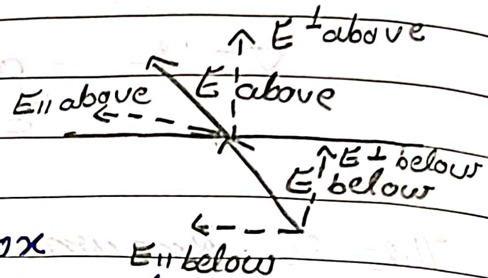


- **Boundary Conditions:**

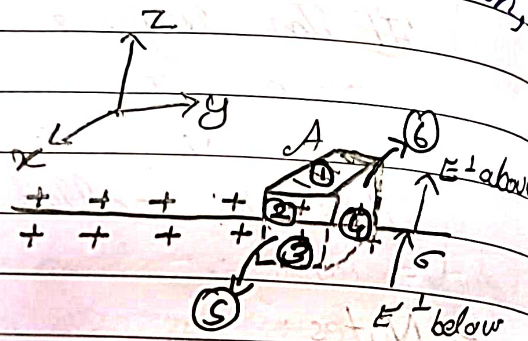
1. **Boundary condition of Electric field:**

Equation to find out how Electric field will change when it crosses any surface having surface charge density σ.

2. **Boundary Condition of Perpendicular component of electric field:**



A wafer-thin Gaussian pillbox extending just barely over the edge in each direction, is drawn.



Gauss's Law

$$\oint_S \vec{E}\cdot d\vec{s} = \frac{1}{\varepsilon_0} q_{enc}$$

5 → front face

6 → behind face

$$\Rightarrow \int_1 \vec{E}\cdot d\vec{s} + \int_2 \vec{E}\cdot d\vec{s} + \int_3 \vec{E}\cdot d\vec{s} + \int_4 \vec{E}\cdot d\vec{s} + \int_5 \vec{E}\cdot d\vec{s} + \int_6 \vec{E}\cdot d\vec{s} = \frac{1}{\varepsilon_0} \times \sigma A; \quad -(1)$$

where A is the area of pill box.

As the thickness 'd' goes to zero, the sides of the pillbox contribute nothing to the flux.

$\therefore$ Eqn. (1) can be written as

$$\int_1 \vec{E}\cdot d\vec{s} + \int_2 \vec{E}\cdot d\vec{s} = \frac{\sigma A}{\varepsilon_0}$$

Camlin Page
Date / /

$$= \int_1 (E^{\perp}_{above}\,\hat{z}) \cdot \hat{z}\, ds + \int_2 (E^{\perp}_{below}\,\hat{z}) \cdot (-\hat{z})\, ds = \frac{\sigma A}{\varepsilon_0}$$

$$= E^{\perp}_{above} \int_1 ds - E^{\perp}_{below} \int_2 ds = \frac{\sigma A}{\varepsilon_0}$$

$$= E^{\perp}_{above} A - E^{\perp}_{below} A = \frac{\sigma A}{\varepsilon_0}$$

$$= E^{\perp}_{above} - E^{\perp}_{below} = \frac{\sigma}{\varepsilon_0}$$

$\therefore$ The normal / perpendicular component of $\vec{E}$ is DISCONTINUOUS by an amount $\frac{\sigma}{\varepsilon_0}$ at any boundary.

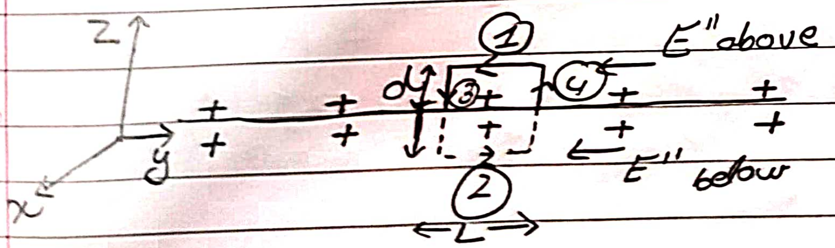
If there is no surface charge, for instance at the surface of a uniformly charged solid sphere, $\sigma = 0$

Then,

$$E^{\perp}_{above} - E^{\perp}_{below} = 0$$

i.e. The normal component of $\vec{E}$ is continuous where there is no surface charge.

ii. Boundary Condition for Parallel component of electric field:



A thin rectangular loop is drawn.
We know that the line integral of $\vec{E}$ around a closed path is zero.

i.e. $\oint \vec{E} \cdot d\vec{l} = 0$

$$\Rightarrow \int_1 \vec{E} \cdot d\vec{l} + \int_2 \vec{E} \cdot d\vec{l} + \int_3 \vec{E} \cdot d\vec{l} + \int_4 \vec{E} \cdot d\vec{l} = 0 \quad -(1)$$

As the distance b/w 2 points, $d \to 0$ the integration along the paths 3 and 4 can be neglected.
Then, eqn. 1 can be written as

$$\int_1 \vec{E} \cdot d\vec{l} + \int_2 \vec{E} \cdot d\vec{l} = 0$$

$$\Rightarrow \int_1 E''_{above} (-\hat{y}) \cdot (-\hat{y})\, dl + \int_2 E''_{below} (-\hat{y})(\hat{y})\, dl = 0$$

$$\Rightarrow \cancel{E''_{above}}\ E''_{above} \int_1 dl - E''_{below} \int_2 dl = 0$$

$$\Rightarrow E''_{above} L - E''_{below} L = 0$$

$$\Rightarrow E''_{above} - E''_{below} = 0$$

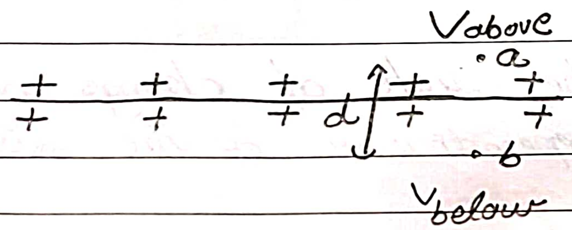
Thus, the parallel or tangential component of $\vec{E}$ is always CONTINUOUS.

The boundary condition of $\vec{E}$ can, therefore be combined into a single formula :

$$\boxed{\vec{E}_{above} - \vec{E}_{below} = \frac{\sigma}{\varepsilon_0}\hat{n}}$$

where $\hat{n}$ is a unit vector perpendicular to the surface, pointing from "below" to "above".

2. Boundary condition for Electric potential:



By the definition of potential difference,

$$V_{above} - V_{below} = -\int_{b}^{a} \vec{E} \cdot d\vec{l}$$

As the path length 'd' shrinks to zero, the integral becomes negligible.

$$\therefore \boxed{V_{above} - V_{below} = 0}.$$

Thus, potential is CONTNUOUS across any boundary.
However, the gradient of V inherits the discontinuity in $\vec{E}$.
we know that

$$\vec{E}_{above} - \vec{E}_{below} = \frac{\sigma}{\varepsilon_0}\hat{n}$$

Since $\vec{E} = -\vec{\nabla} V$, the above equation can be written as

$$-\vec{\nabla} V_{above} - (-\vec{\nabla} V_{below}) = \frac{\sigma}{\varepsilon_0} \hat{n}$$

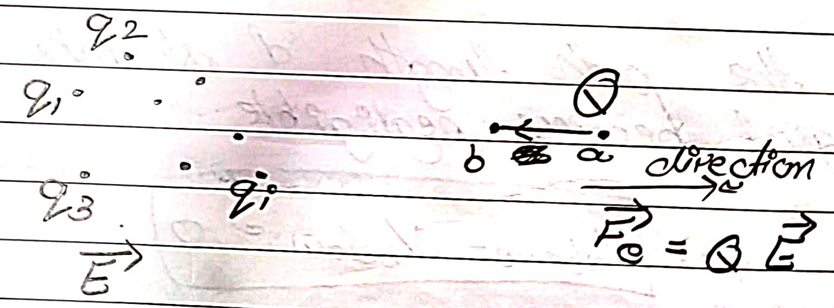
$$\vec{\nabla} V_{above} - \vec{\nabla} V_{below} = -\frac{\sigma}{\varepsilon_0} \hat{n}$$

$$\boxed{\frac{\partial V_{above}}{\partial n} - \frac{\partial V_{below}}{\partial n} = -\frac{\sigma}{\varepsilon_0}}$$

where $\frac{\partial V}{\partial n} = \vec{\nabla} \cdot V \cdot \hat{n}$, denotes the normal derivative of V i.e., the rate of change of V in the direction perpendicular to the surface.

- **Work and Energy:**

Consider a collection of source charges $q_1, q_2, q_3, \ldots, q_i$ having electric field $\vec{E}$. And we want to find work done in moving a test charge Q from point 'a' to point 'b', as shown in the following figure.



$$\vec{F}_{ext} = -\vec{F_e}$$

Since the test charge is placed in electric field $\vec{E}$, force on 'Q' due to collection of the source charges is

$$\vec{F_e} = Q\vec{E}$$