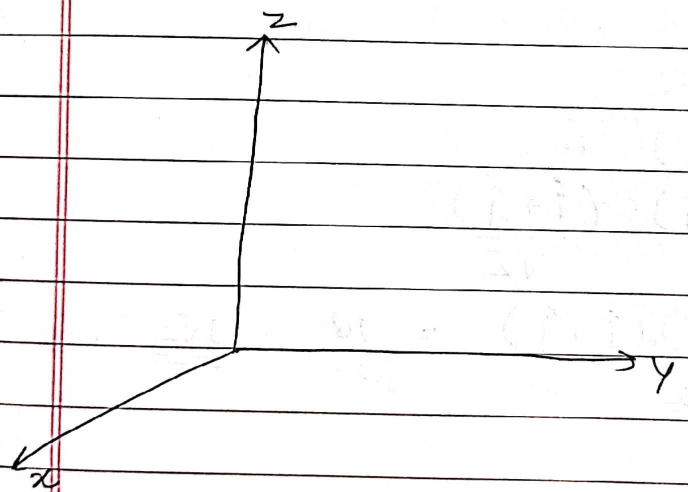
A Generalisation of Partial Differentiation :-

Suppose $u = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$ is a unit vector in the xy plane from $(x, y, 0)$ to $(x+\Delta x, y+\Delta y, 0)$. If $h = \sqrt{(\Delta x)^2 + (\Delta y)^2}$ & $h>0$, then $\cancel{v = h/u}$. $V = hu$. furthermore, let the plane $\perp^r$ to xy plane that contains these points slice the surface $z = f(x,y)$ in a curve C. We ask: What is the slope of the tangent

from the figure, we see that $\Delta x = h\cos\theta$ & $\Delta y = h\sin\theta$ so that the slope of the indicated secant line is.

$$\frac{f(x+\Delta x, y+\Delta y) - f(x,y)}{h} = \frac{f(x + h\cos\theta, y + h\sin\theta) - f(x,y)}{h}$$

—— (B)



The direc^n al der: of $z = f(x,y)$ in the direction of a unit vector

Observe that (4) is truly a generalisation of partial differentiation, since,

$\theta = 0$, implies that, $D_1 f(x,y) = \lim_{h\to 0} \frac{f(x+h, y) - f(x,y)}{h} = \frac{\partial f}{\partial x}$

$\theta = \pi/2$ implies that, $D_2 f(x,y) = \lim_{h\to 0} \frac{f(x, y+h) - f(x,y)}{h} = \frac{\partial f}{\partial y}$

Computing a directional derivative :-

If $z = f(x,y)$ is a differentiable funct of x & y, and $u = \cos\theta\hat{i} + \sin\theta\hat{j}$ then,

$$\boxed{D_u f(x,y) = \vec{\nabla} f(x,y) \cdot \hat{u}}$$

Q. consider the plane $\perp^r$ to xy plane and passes through $P(2,1)$ and $Q(3,2)$. What is the slope of the tangent line to the curve of intersection of this plane with the surface $f(x,y) = 4x^2+y^2$ at $(2,1,17)$ in the dirn of Q?

The required slope will be directional derivative along $\vec{PQ}$.

$\vec{PQ} = \hat{i} + \hat{j}$

$$\therefore \quad D_u f(x,y) = (\vec{\nabla} f(x,y)) \cdot \hat{u}$$
$$= (8x\hat{i} + 2y\hat{j}) \cdot \frac{(\hat{i}+\hat{j})}{\sqrt{2}}$$
$$= \frac{(16\hat{i} + 2\hat{j}) \cdot (\hat{i}+\hat{j})}{\sqrt{2}} = \frac{18}{\sqrt{2}} = 9\sqrt{2} \text{ ans.}$$

functions of three variables :-

for the function $f(x,y,z)$ the directional derivative is defined by,

$$D_u f(x,y,z) = \lim_{h\to 0} \frac{f(x+h\cos\alpha, y+h\cos\beta, z+h\cos\gamma) - f(x,y,z)}{h}$$

where α, β + γ are the direction angles of unit vector u measured relative to positive x, y, and z-axes.

But in the same manner as before we can show that

$$D_u f(x,y,z) = \vec{\nabla} f(x,y,z) \cdot u.$$

Q:- find the directional derivative of $F(x,y,z) = xy^2 - 4x^2y + z^2$ at $(1,-1,2)$ in the dir^n of $6\hat{i} + 2\hat{j} + 3\hat{k}$.

$$D_u f(x,y,z) = (\vec{\nabla} f(x,y,z)) \cdot \hat{u}$$

$$= \left((y^2 - 8xy)\hat{i} + (2xy - 4x^2)\hat{j} + (2z)\hat{k} \right) \cdot \hat{u}$$

$$= (9\hat{i} + 6\hat{j} + 4\hat{k}) \cdot \frac{(6\hat{i} + 2\hat{j} + 3\hat{k})}{\sqrt{36+4+9}}$$

$$= \frac{(54 - 12 + 12)}{7} = \frac{54}{7} \text{ Ans.}$$

Q find directional derivative :-

(i) $f(x,y) = 5x^3y^6$; $(-1,1)$, $\theta = \pi/6$.

$u = \sin\frac{\pi}{6}\hat{j} + \cos\frac{\pi}{6}\hat{k}$

$$D_u f(x,y) = (15x^2y^6\hat{i} + 30x^3y^5\hat{j}) \cdot (\cos\frac{\pi}{6}\hat{i} + \sin\pi/6\,\hat{j})$$

$$= (15\hat{i} - 30\hat{j}) \cdot \left(\frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}\right)$$

$$= \frac{15\sqrt{3}}{2} - 15$$

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Tangent plane :-

Let $P(x_0, y_0, z_0)$ be a point on the graph of $f(x,y,z) = c$, where ∇f is not 0. The tangent plane at P is that plane through P that is normal ~~of~~ to ∇f evaluated at P.

Equation of tangent plane :-

Let $P(x_0, y_0, z_0)$ be a point on the graph of $f(x,y,z) = c$ where $\nabla f \neq 0$. Then equation of tangent plane at P is –

$$\boxed{f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0) = 0}$$

Q: find an eq^n of the tangent plane to the graph of $x^2 - 4y^2 + z^2 = 16$ at $(2,1,4)$.

$f_x = 2x$, $f_y = -8y$, $f_z = 2z$

$f_x(2,1,4) = 4$, $f_y(2,1,4) = -8$, $f_z(2,1,4) = 8$

$\therefore$ eq^n :-

$$4(x-2) + (-8)(y-1) + 8(z-4) = 0$$

$$4x - 8y + 8z - 32 = 0$$

$$\boxed{x - 2y + 2z - 8 = 0}$$

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Normal line:- Let $P(x_0, y_0, z_0)$ be a point on the graph of $f(x,y,z) = c$, where ∇F is not 0. The line containing $P(x_0, y_0, z_0)$ that is parallel to $\nabla f(x_0, y_0, z_0)$ is called normal line to the surface at P. This line is normal to the tangent plane to the surface at P.

Q Find parametric equations for the normal line to the surface in example 4 $z = \frac{1}{2}x^2 + \frac{1}{2}y^2 + 4$ at $(1,-1,5)$.

$$\Rightarrow \quad f = \frac{1}{2}x^2 + \frac{1}{2}y^2 - z + 4.$$

$$\nabla f = x\hat{i} + y\hat{j} - \hat{k}$$

$$\nabla f(1,-1,5) = \hat{i} - \hat{j} - \hat{k}$$

Parametric equations of normal line are -

$x = 1 + t$; $y = -1 - t$; $z = 5 - t$.

Expressed as symmetric eq's the normal line to the surface $f(x,y,z) = c$ at $P(x_0, y_0, z_0)$ is given by.

$$\frac{x - x_0}{f_x(x_0, y_0, z_0)} = \frac{y - y_0}{f_y(x_0, y_0, z_0)}$$

Q-15 Find eqn of tangent plane to the surfaces -

(i) $x^2+y^2+z^2 = 9$; at $(-2,2,1)$.

$f_x(x_0,y_0,z_0)(x-x_0) + f_y(x_0,y_0,z_0)(y-y_0) + f_z(x_0,y_0,z_0)(z-z_0) = 0$

$= (2x)_{x_0=x_0}(x-x_0) + (2y)_{y=y_0}(y-y_0) + (2z)_{z=z_0}(z-z_0) = 0$

$\Rightarrow -4(x+2) + 4(y-2) + 2(z-1) = 0$

$\Rightarrow -4x + 4y + 2z - 18 = 0.$

$\boxed{2x - 2y - z + 9 = 0}$

$$\frac{x+2}{(2x)_{x=x_0}} = \frac{y-2}{(2y)_{y=y_0}} = \frac{z-1}{(2z)_{z=z_0}}$$

$$\Rightarrow \frac{x+2}{-4} = \frac{y-2}{4} = \frac{z-1}{2}.$$

Q. # $xy + yz + zx = 7$; $(1,-3,-5)$

$x^2 - y^2 - 3z^2 = 5$; $(6,2,3)$.

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Curl:-

curl of vector field $\vec{F}$, $\text{curl} \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$

Divergence:-

The divergence of vector Field $\vec{F} = P\hat{i} + Q\hat{j} + R\hat{k}$ is the ~~scalar~~. scalar,

$$\text{div} F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Q $F = (x^2y^3 - z^4)\hat{i} + 4x^5y^2\hat{j} - y^4z^6\hat{k}$ — (i) curl F
(ii) div F, (iii) div (curl f).

(i) $\text{curl } F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y^3 - z^4 & 4x^5y^2 & -4y^4z^6 \end{vmatrix}$

$= \hat{i}(-16y^3z^6) - \hat{j}(-2xy^3) + \hat{k}(20x^4y^2 - 3x^2y^2)$

(ii) $\text{div } F = 2xy^3 + 8x^5y - 6y^4z^5$

$\text{div} (\text{curl} F) = 6xy^2$ -

Q. Find out curl & divergence :.

(i) $F(x,y,z) = xz\hat{i} + yz\hat{j} + xy\hat{k}$

$\text{div} F = z + z + 0 = \underline{2z}$

$\text{curl} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & yz & xy \end{vmatrix} = (x-y)\hat{i} - \hat{j}(y-x) + (0)\hat{k}$

$= (x-y)\hat{i} + (x-y)\hat{j}$ Ans.

Q $f(x,y,z) = (x-y)^3\hat{i} + e^{-yz}\hat{j} + xye^{2y}\hat{k}$

$\text{div} f = 3(x-y)^2 + e^{-yz}(-z) + 0.$

$$\text{curl} f = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (x-y)^3 & e^{-yz} & xye^{2y} \end{vmatrix}$$

$$= [x(e^{2y} + 2ye^{2y}) - e^{-yz}(-y)]\hat{i} - (ye^{2y} - 0)\hat{j} + [3(x-y)^2(-1) - 0]\hat{k}$$

Q find div $\vec{F}$ and curl $\vec{F}$, $\vec{F} = \text{grad}(x^3+y^3+z^3-3xyz)$

Q Prove that; div $(fV) = f(\text{div} V) + (\text{grad} f)V$ where f is scalar func

Q find the constant a, b, c so that $\vec{F} = (x+2y+az)\hat{i} + (bx-3y-z)\hat{j} + (4x+cy+2z)\hat{k}$ is irrotational

Maxima and Minima of a function of two or more variable:-

Maxima:-

A function f of two variables is said to have a relative maximum at a point (x_0, y_0) if there is a disk centered at (x_0, y_0) such that $f(x_0, y_0) \geq f(x, y)$ for all points (x, y) that lie inside the disk, and f is said to have an absolute maximum at (x_0, y_0) if $f(x_0, y_0) \geq f(x, y)$ for all points (x, y) in the domain of f.

Minima:-

A func^n f to two var. is said to have relative minimum at a point (x_0, y_0) if there is a disk centered at (x_0, y_0) such that $f(x_0, y_0) \leq f(x, y)$ for all points (x, y) that lie inside the disk, and f is said to have an absolute minimum at (x_0, y_0) if $f(x_0, y_0) \leq f(x, y)$ for all points (x, y) ~~that~~ in the domain of y.

Extreme Value Theorem:-

If $f(x, y)$ is continuous on a closed and bounded set R, then f has both an absolute maximum and an absolute minimum.

Theorem:- The second Partial test:- Let f be func^n of two variables with continuous second-order partial derivative in some disk centred at a critical point (x_0, y_0) and let,

$$D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f_{xy}^2(x_0, y_0) \Rightarrow (rt - s^2).$$

(a) If $D > 0$ & $f_{xx}(x_0, y_0) > 0$, then f has a rel. min. at (x_0, y_0)

(b) If $D > 0$ & $f_{xx}(x_0, y_0) < 0$, then f has rel. max. at (x_0, y_0)

(c) If $D < 0$, f has a saddle point at (x_0, y_0)

$\Rightarrow f$ has no maximum value at (x_0, y_0)

(d) If $D = 0$, then no conclusion can be drawn.

Working Rule for Maxima/Minima :-

$z = f(x, y)$

Step I :- find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

Step II :- Solve for $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$ simultaneously. Let (a, b) & (c, d) be the solⁿ of these equations.

Step III :- find $r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 f}{\partial x^2}$ and $s = \frac{\partial^2 z}{\partial x \partial y}$ and $t = \frac{\partial^2 z}{\partial y^2}$

Step IV :- (a) If $rt - s^2 > 0$, $r < 0$ for a particular solⁿ (a, b), then $f(a, b)$ ~~is~~ has maximum value.

(b) If $rt - s^2 > 0$, $r > 0$ for a particular solⁿ (a, b) then $f(a, b)$ has minimum value.

(c) If $rt - s^2 < 0$, for a particular solⁿ (a, b), then $z = f(x, y)$ has no extreme value at $(a, b) \rightarrow$ saddle point.

(d) If $rt - s^2 = 0$, at (a, b), this case is doubtful and requires further investigation.

Q. Examine the funcⁿ $x^3 + y^3 - 3axy$ for maxima or minima.

$z = x^3 + y^3 - 3axy$.

$\frac{\partial z}{\partial x} = 3x^2 - 3ay = 0$ | $\frac{\partial z}{\partial y} = 3y^2 - 3ax = 0$

$x^2 = ay$ | $y^2 = ax$

$\left(\frac{y^2}{a}\right)^2 = ay$

$y^3 = a^3$

$\boxed{y = a} \Rightarrow (a, a)$ & $(0, 0)$.

$\Rightarrow \boxed{x = a}$

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$r = \frac{\partial^2 z}{\partial x^2} = 6x - 0 = 6x$ | $s = \frac{\partial^2 z}{\partial x \partial y} = -3a$

$t = \frac{\partial^2 z}{\partial y^2} = 6y.$

$D = rt - s^2 = 36xy - 9a^2$

at $(0,0)$

$D = -9a^2 < 0 \Rightarrow z$ has no extreme values at $(0,0)$

case: at (a,a)

$D = 36a^2 - 9a^2 = 27a^2 > 0$

and $r = 6a$

case 1: when $a = +ve$, z has a minimum value at (a, a)

and min. value of $z = a^3 + a^3 - 3a^3 = -a^3$

case 2: when $a = -ve$, $\Rightarrow r = 6a < 0$

z has a max. value at (a, a) and maximum value of $z = -a^3 + -a^3 + 3a(a)(a) = a^3$

Q. Locate all relative extrema and saddle points of $f(x,y) = 4xy - x^4 - y^4$.

$\frac{\partial f(x,y)}{\partial x} = 4y - 4x^3$ | $\frac{\partial f}{\partial y} = 4x - 4y^3$

$4y - 4x^3 = 0$ or $\boxed{x = y^3}$

$\boxed{y = x^3}$

$x = (x^3)^3 = x^9$

$x - x^9 = 0$

$x(1 - x^8) = 0.$

$x = 0, 1, -1 \Rightarrow y = 0, 1, -1$

critical pts :- $(0,0); (1,1); (-1,-1).$

$f_{xx} = -12x^2, \quad f_{yy} = -12y^2, \quad f_{xy} = 4.$

pts	$f_{xx} \cdot f_{yy} - f_{xy}^2$	
(0,0)	$0 \times 0 - 4^2 = -16 < 0$	saddle pt. $\Rightarrow (0,0)$.
(1,1)	$-12 \times -12 - 4^2 = 128 > 0, r < 0 \Rightarrow$	maxima $\Rightarrow (1,1)$
(-1,-1)	$-12 \times -12 - 4^2 = +128 > 0, r < 0 \Rightarrow$	maxima $\Rightarrow (-1,-1)$

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(8) Determinate the dimension of the rectangular box, open at the top, having a volume of $32\ ft^2$ & requiring ~~atleast~~ the least amount of material for its construction.

Let, x is length, y is breadth & z is height of the box.

$$S = xy + 2yz + 2zx$$

&

$$V = 32$$

$$xyz = 32 \Rightarrow z = \frac{32}{xy}$$

$$\therefore \quad S = xy + \frac{64}{x} + \frac{64}{y}.$$

$$\frac{\partial S}{\partial x} = y - \frac{64}{x^2} = 0 \qquad \frac{\partial S}{\partial y} = x - \frac{64}{y^2} = 0$$

$$y = \frac{64}{x^2} \Rightarrow x = \frac{64}{y^2} = \frac{64\,(x^2)^2}{(64)^2}$$

$$64x = x^4$$

$$64x - x^4 = 0 \Rightarrow x(64 - x^3) = 0$$

$$x = 0, \quad x = 4$$

dimension can't be zero

$$\therefore \boxed{x = 4} \Rightarrow y = \frac{64}{4^2} = 4$$

$$\boxed{y = 4}$$

$$\therefore \boxed{z = 2}$$ //

$$r = \frac{\partial^2 S}{\partial x^2} = -(-2)\frac{64}{x^3} = \frac{128}{x^3} = 2$$

$$t = \frac{\partial^2 S}{\partial y^2} = \frac{128}{y^3} = 2$$

$$s = \frac{\partial^2 S}{\partial x \partial y} = 1$$

$$rt - s^2 = 4 - 1 = 3 > 0$$

& $r > 0$

$\Rightarrow$ Minima

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Q. A rectangular box open at the top is to have a given capacity. find the dimension of the box requiring least material for its construction.

Ans :- $x = y = (2V)^{1/3}$

$x = y = 2z$.

Q. Examine for maxima / minima of the function.

$f(x,y) = \sin x + \sin y + \sin(x+y)$

$\frac{\partial f}{\partial x} = \cos x + \cos(x+y)$ | $\frac{\partial f}{\partial y} = \cos y + \cos(x+y)$.

$r = \frac{\partial^2 f}{\partial x^2} = -\sin x - \sin(x+y)$ | $t = \frac{\partial^2 f}{\partial y^2} = -\sin y - \sin(x+y)$

$s = \frac{\partial^2 f}{\partial x \partial y} = -\sin(x+y)$.

$\frac{\partial f}{\partial x} = 0 \Rightarrow \cos x + \cos(x+y) = 0 \Rightarrow \cos(x+y) = -\cos x$ — (1)

$\frac{\partial f}{\partial y} = \cos y + \cos(x+y) = 0 \Rightarrow \cos y - \cos x = 0$.

$\boxed{y = x}$

Now, eqn (1)

$\cos x + \cos(x+x) = 0$.

$\cos 2x = -\cos x$

$\cos 2x = \cos(\pi - x)$

$2x = \cancel{\pi} \; \pi - x \Rightarrow \boxed{x = \pi/3}, \boxed{y = \pi/3}$

$\therefore \left(\pi/3, \pi/3\right)$.

$r = -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = -\sqrt{3}$; $t = -\sqrt{3}$; $s = -\frac{\sqrt{3}}{2}$

$\therefore$ $rt - s^2 = 3 - \frac{3}{4} > 0$.

& $r < 0$

$\Rightarrow$ Maxima at $(-\sqrt{3}, -\sqrt{3})$

and the max. value is $\Rightarrow$.

In a plane ΔABC. find the max. value of ~~cos A cos B~~ $\cos A \cdot \cos B \cdot \cos C$.

$$f = \cos A \cos B \cos C$$

we know that.

$$A + B + C = 180$$

$$C = 180 - A - B.$$

$$f = \cos A \cos B \cos(180 - A - B).$$

$$\frac{\partial f}{\partial A} = \cancel{-\sin A} \cos B \left[+ \sin A \cos(180 - A - B) + \cos A \sin(180 - A - B) \right]$$

$$\frac{\partial f}{\partial B} = \cos A \left[+ \sin B \cos(180 - A - B) + \cos B \sin(180 - A - B) \right]$$

$$\frac{\partial f}{\partial A} = 0 \Rightarrow -\sin A \cos B(-\cos(A+B)) + \cos A \cos B (\sin(A+B)) = 0.$$

$$= \cos B(\sin A \cos(A+B) + \cos A \sin(A+B)) = 0$$

$$\Rightarrow \cos B \sin(2A + B) = 0.$$

case ① $\cos B = 0$.

$$\boxed{B = \frac{\pi}{2}}$$

$$\frac{\partial f}{\partial B} = 0 \Rightarrow \cos A \cdot \sin(A + 2B) = 0.$$

$$\because B = \frac{\pi}{2}$$

$$\therefore \cos A \cdot \sin(A + \pi) = 0.$$

$$-\cos A \cdot \sin A = 0$$

$$\therefore \cos A = 0 \text{ or } \sin A = 0$$

$$\therefore A = \frac{\pi}{2} \text{ or } A = 0.$$

for, $B = \frac{\pi}{2}$, $A = \frac{\pi}{2}, 0$.

case ② $\sin(2A + B) = 0$ or ~~sin~~

$$2A + B = 0$$

~~Lagrange Multiplier~~

Lagrange method of undetermined multiplier ;-

Let $f(x,y,z)$ be a func^n of x,y,z which is to be examined for minimum or maximum value.

Let the variables x,y,z be connected by the relation $\phi(x,y,z) = 0$. ——— (i)

for $f(x,y,z)$ to have a maximum or minimum value, the necessary cond^n is $\frac{\partial f}{\partial x} = 0;\ \frac{\partial f}{\partial y} = 0;\ \frac{\partial f}{\partial z} = 0$.

$$\therefore \quad \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = 0. \quad \text{——— (ii)}$$

diff. partially eq^n (i) ;

$$\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz = 0 \quad \text{——— (iii)}$$

Let, λ be the Lagrange multiplier, then

$$\text{(ii)} + \lambda \text{(iii)} \Rightarrow \left(\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x}\right) dx + \left(\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y}\right) dy + \left(\frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z}\right) dz = 0$$

$$= 0dx + 0dy + 0dz$$

$$\therefore \text{ we have, } \frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0;\ \frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0;\ \frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0 \quad \text{——— (iv)}$$

All these eq^n (iv) along with eq^n (i), gives the values of $x, y, z,$ & λ for a maximum or minimum.

Q. Find the maximum or minimum distances of the point (3, 4, 12) from the sphere $x^2+y^2+z^2=1$.

Let (x,y,z) be any point on the sphere. The distance of the point $A(3,4,12)$ from (x,y,z) is

$$\sqrt{(x-3)^2+(y-4)^2+(z-12)^2} = d$$

$$\Rightarrow d^2 = (x-3)^2+(y-4)^2+(z-12)^2.$$

Let, $f(x,y,z) = (x-3)^2+(y-4)^2+(z-12)^2$ —— (i)

& it is given, $\phi(x,y,z) = x^2+y^2+z^2-1=0$ —— (ii)

Consider a Lagrange's function $F(x,y,z) = f(x,y,z) + \lambda\phi(x,y,z)$

$\therefore F(x,y,z) = (x-3)^2+(y-4)^2+(z-12)^2+\lambda(x^2+y^2+z^2-1)$

for stationary values, $dF=0$.

$$(2(x-3)+2\lambda x)dx + (2(y-4)+2\lambda y)dy + (2(z-12)+2\lambda z)dz = 0.$$

$\therefore x-3+\lambda x = 0$, $x = \frac{3}{1+\lambda}$ —— (iii)

$\cancel{y-4+\lambda y=0}$, $y = \frac{4}{1+\lambda}$ —— (iv)

$\cancel{z-12+\lambda z=0}$, $z = \frac{12}{1+\lambda}$ —— (v)

$x \times$ (iii) $+ y\times$ (iv) $+ z\times$ (v)

$$\Rightarrow (x^2+y^2+z^2) - 3x - 4y - 12z + \lambda(x^2+y^2+z^2) = 0$$

$$3x+4y+12z = \lambda+1$$

$$\frac{3\times 3}{1+\lambda} + \frac{4\times 4}{1+\lambda} + \frac{12\times 12}{1+\lambda} = 1+\lambda$$

$$9+16+144 = (1+\lambda)^2$$

$$169 = (1+\lambda)^2$$

$$1+\lambda = \pm 13$$

$$\boxed{\lambda = +12, -14}$$

for, $\lambda = 12$; $x = \frac{3}{13}$, $y = \frac{4}{13}$, $z = \frac{12}{13}$

for $\lambda = -14$; $x = \frac{-3}{13}$, $y = \frac{-4}{13}$, $z = \frac{-12}{13}$.

at $\left(\frac{3}{13}, \frac{4}{13}, \frac{12}{13}\right)$; $d^2 = \cancel{10}\ 144$

$\boxed{d = 12}$

at $\left(\frac{-3}{13}, \frac{-4}{13}, \frac{-12}{13}\right)$; $\boxed{d = 14}$.

Multiple Integral :-

★ $$\int_c^d\int_a^b f(x,y)\,dx\,dy = \int_c^d\left[\int_a^b f(x,y)\,dx\right]dy = \int_a^b\left[\int_c^d f(x,y)\,dy\right]dx.$$

Q. $$\int_1^3\left[\int_2^4 (40-2xy)\,dy\right]dx$$

$$= \int_1^3 \left[40y - xy^2\right]_2^4 dx = \int_1^3 (80-12x)\,dx = \left[80x-6x^2\right]_1^3$$

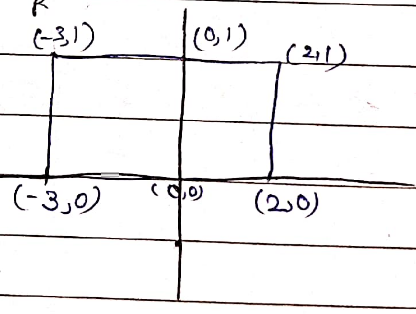
$$= 160-48 = 112 \text{ Ans.}$$

Q. Theorem (fubini's Theorem) :-

Let R be the rectangle defined by the inequaliti $a \le x \le b$, $c \le y \le d$. If $f(x,y)$ is continuous on this rectangle, then.

$$\iint_R f(x,y)\,dA = \int_c^d\int_a^b f(x,y)\,dx\,dy = \int_a^b\int_c^d f(x,y)\,dy\,dx.$$

Q. Evaluate the double integral $\iint_R y^2x\,dA$ over the rectangle $R = [(x,y) : -3 \le x \le 2,\ 0 \le y \le 1]$.



$$\iint_R y^2x\,dA = \int_{x=-3}^{2}\int_{y=0}^{1} y^2x\,dy\,dx$$

$$= \int_{-3}^{2}\left[\frac{y^3}{3}x\right]_0^1 dx = \frac{1}{3}\int_{-3}^{2} x\,dx$$

$$= \frac{1}{3}\left[x^2/2\right]_{-3}^{2} = \frac{1}{3}\left[\frac{-5}{2}\right] = \frac{-5}{6} = \frac{5}{6} \text{ sq unit.}$$

Properties

1. $\iint_R cf(x,y)\,dA = c\iint_R f(x,y)\,dA.$ (c is a constant).

2. $\iint_R [f(x,y) \pm g(x,y)]\,dA = \iint_R f(x,y)\,dA \pm \iint_R f(x,y)\,dA.$

(3) $\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$.

Here R is subdivided into two parts.

Q.1 $\int_0^2 \int_0^1 y \sin x \, dy\, dx$.

$\Rightarrow \int_0^2 \left[\frac{y^2}{2} \sin x \right]_0^1 dx = \int_0^2 \frac{\sin x}{2} dx = \left[-\frac{\cos x}{2} \right]_0^2 = -\cancel{\cos 2}$

$= -\frac{\cos 2}{2} - \left(-\frac{\cos 0}{2} \right) = \frac{-\cos 2 + 1}{2}$. ~~Ans~~.

Q.2 $\int_1^3 \int_{-1}^1 (2x - 4y)\, dy\, dx$

$= \int_1^3 \left[2xy - \frac{\cancel{4}^2 y^2}{\cancel{2}} \right]_{-1}^1 dx = \int_1^3 [4x]\, dx = \left[\frac{\cancel{4}^2 x^2}{\cancel{2}} \right]_1^3 = \underline{16}$ Ans

Q.3 $\int_3^4 \int_1^2 \frac{1}{(x+y)^2}\, dy\, dx$.

$\Rightarrow \int_3^4 \left[\frac{(x+y)^{-1}}{-1} \right]_1^2 dx =$

Q. $\iint_R 4xy^3\, dA$; $R = [(x,y) : -1 \le x \le 1,\ -2 \le y \le 2]$.

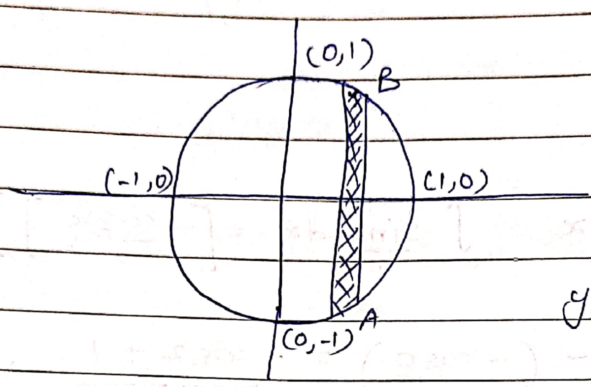
$\Rightarrow \int_{-2}^2 \int_{-1}^1 4xy^3\, dx\, dy = \int_{-2}^2 \left[2x^2 y^3 \right]_{-1}^1 dy = \int_{-2}^2 0\, dy = \underline{0}$.

Q. $\iint_R \frac{xy}{\sqrt{x^2+y^2+1}}\, dA$: $R = [(x,y) : 0 \le x \le 1 ;\ 0 \le y \le 1]$.

$= \int_{y=0}^1 \int_{x=0}^1 \frac{xy}{\sqrt{x^2+y^2+1}}\, dx\, dy$.

Q. find the area of unit circle :-

let the eq^n of circle is ; $x^2+y^2=1$.



Consider a strip AB of width dx which is parallel to y axis, then x axis varies from -1 to 1, and y varies from $-\sqrt{1-x^2}$ to $\sqrt{1-x^2}$

$$\text{Area} = \int_{-1}^{1}\left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1\, dy\right) dx$$

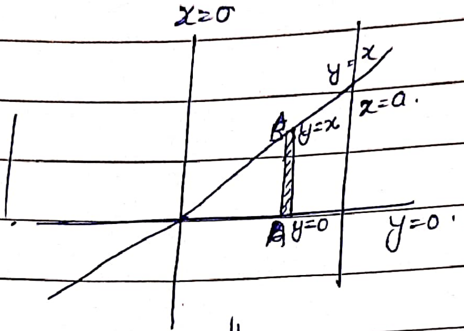
$$= \int_{-1}^{1} 2\sqrt{1-x^2}\, dx \quad = 2\int_{-1}^{1}\sqrt{1-x^2}\, dx$$

Change the order of Integration :-

Eg:- $\int_{x=0}^{a}\left(\int_{y=0}^{x} dy\right)dx$

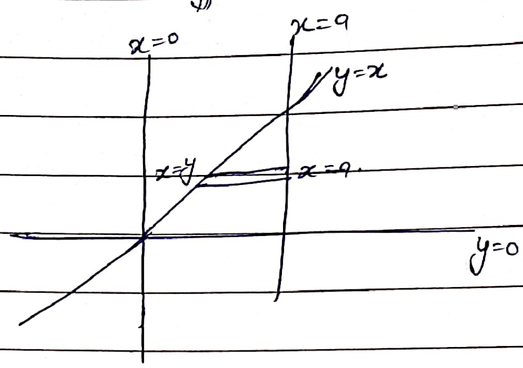
$x=0,\ x=a$

$y=0,\ y=x$



⇓

$\int_{y=0}^{a}\left(\int_{x=y}^{a} dx\right)dy$

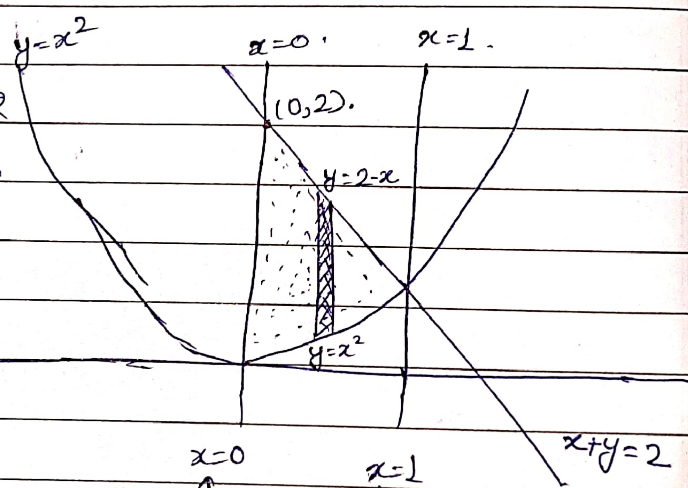


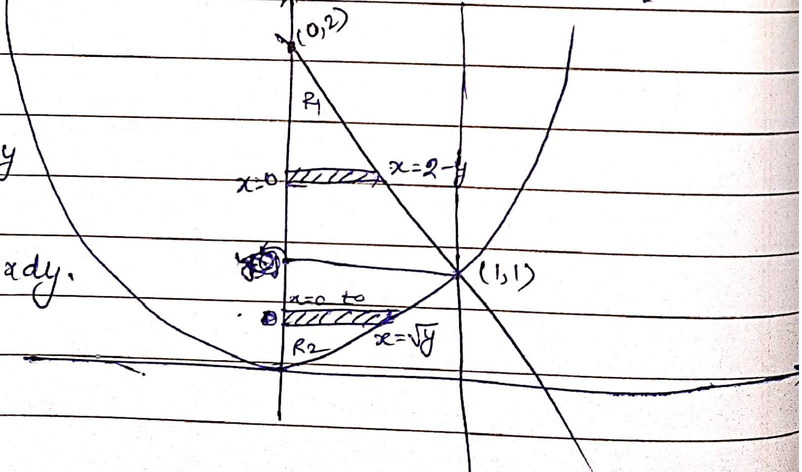
Q. Change the order of $\int_0^1\int_{x^2}^{2-x} xy\,dy\,dx$ and hence evaluate the same

Sol^n:-

Here, $y=x^2,\ y+x=2$

$x=1,\ x=0$.





$\Rightarrow \iint_{R_1} xy\,dx\,dy + \iint_{R_2} xy\,dx\,dy$

$= \int_{y=1}^{2}\int_{x=0}^{2-y} xy\,dx\,dy + \int_{0}^{1}\int_{x=0}^{\sqrt{y}} xy\,dx\,dy.$

$y \cdot \frac{(2-y)^2}{2} = y(4+y^2-4y)$

$$\Rightarrow \int_{y=1}^{2} \left[y \cdot \frac{x^2}{2} \right]_0^{2-y} dy = \int_{y=1}^{2} \frac{y(2-y)^2}{2} dy + \int_0^1 \left[\frac{x^2}{2} \cdot y \right]_0^{\sqrt{y}} dy.$$

$$= \frac{1}{2} \int_{y=1}^{2} [4y + y^3 - 4y^2] dy + \int_0^1 \frac{y^2}{2} dy.$$

$$= \frac{1}{2} \left[2y^2 + \frac{y^4}{4} - \frac{4y^3}{3} \right]_1^2 + \left[\frac{y^3}{6} \right]_0^1$$

$$= \frac{1}{2} \left[\left(8 + 4 - \frac{32}{3}\right) - \left(2 + \frac{1}{4} - \frac{4}{3}\right) \right] + \frac{1}{6}$$

$$= \frac{1}{2} \left[10 - \frac{28}{3} - \frac{1}{4} \right] + \frac{1}{6}$$

$$\Rightarrow 5 - \frac{28}{6} - \frac{1}{8} + \frac{1}{6} = 5 - \frac{1}{8} - \frac{27}{6}$$

$$= \frac{40 - 1 - 36}{8} = \frac{3}{8} \text{ Ans}$$

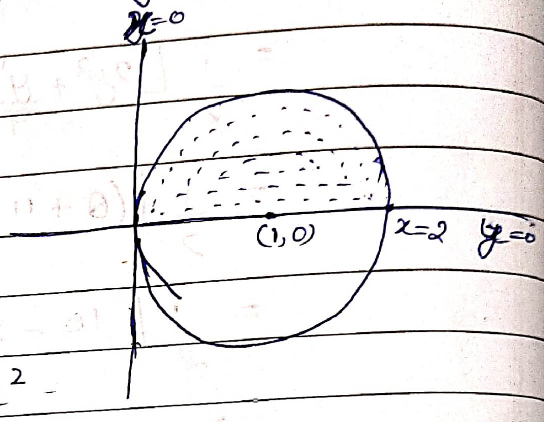
Q.1 change the order of $\int_0^3 \int_1^{\sqrt{4-y}} (x+y)\, dx\, dy$ and evaluate.

Q.2 $\int_0^1 \int_{x}^{e^x} \frac{1}{\log y} dy dx$

(3) $\int_0^1 \int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}} dx\, dy.$

Change of variables b/w cartesian and polar coordinates.

Ex: Evaluate $\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x\,dy\,dx}{\sqrt{x^2+y^2}}$ by changing into polar coordinates.



Soln

Here, $x=0,\ x=2$

$y=0,\ y=\sqrt{2x-x^2}$

$y^2+x^2-2x=0$

$(y-0)^2+(x-1)^2=1^2$

circle of radius '1' & centre $(1,0)$

In polar coordinates

$x=r\cos\theta,\quad y=r\sin\theta$.

$r^2-2r\cos\theta=0$.

$r(r-2\cos\theta)=0$.

$r=0$ or $r=2\cos\theta$

$$\int_{\theta=0}^{\pi/2}\int_{r=0}^{2\cos\theta} \frac{r\cos\theta}{\sqrt{r^2}}\,|J|\,dr\,d\theta$$

Jacobian of polar coordinate $= r$

$$\therefore \int_{\theta=0}^{\pi/2}\int_{r=0}^{2\cos\theta} \frac{r\cos\theta\,\cancel{r}\,dr\,d\theta}{\sqrt{\cancel{r^2}}} = \int_{\theta=0}^{\pi/2}\left[\frac{r^2\cos\theta}{2}\right]_0^{2\cos\theta} d\theta$$

$$= \int_{\theta=0}^{\pi/2} 2\cos^3\theta\,d\theta$$

for $\int_0^{\pi/2} \cos^{50}\theta \sin^{100}\theta \, d\theta$ type of problem, use gamma & Beta function.

$\Gamma \rightarrow$ gamma & $\beta \rightarrow$ Beta.

$5! = \Gamma 6$ #

$5! = 5\times4\times3\times2\times1 = 120.$

$\Gamma 6 = \Gamma 5+1 = 5\,\Gamma 5 = 5\,\Gamma 4+1 = 5\times4\times\Gamma 4 = 5\times4\times\Gamma 3+1$

$= 5\times4\times3\,\Gamma 3 = 5\times4\times3\times\Gamma 2+1 = 5\times4\times3\times2\,\Gamma 2$

$= 5\times4\times3\times2\,\Gamma 1+1 = 5\times4\times3\times2\times1 = \underline{\underline{120}}$

some result:-

$\Gamma 1 = 1$, $\Gamma 0 = \infty$, $\Gamma -1 = \infty$

$\Gamma n \, \Gamma 1-n = \dfrac{\pi}{\sin n\pi}$; $\Gamma \frac{1}{2} = \sqrt{\pi}$

$$\int_0^{\pi/2} \cos^m\theta \sin^n\theta \, d\theta = \frac{\Gamma \frac{m+1}{2} \; \Gamma \frac{n+1}{2}}{2\,\Gamma \frac{m+n+2}{2}} \qquad \begin{array}{l} m > -1 \\ n > -1 \end{array}$$

A simple polar region in a polar coordinate system is a region that is enclosed b/w two rays $\theta = \alpha$ & $\theta = \beta$, & two continuous polar curves, $r = r_1(\theta)$ & $r = r_2(\theta)$, where equations of the rays and polar curves satisfy the following conditions.

(i) $\alpha \le \beta$ (ii) $\beta - \alpha \le 2\pi$ (iii) $0 \le r_1(\theta) \le r_2(\theta)$

A polar rectangle is a simple polar region for which the bounding polar curves are circular arcs. for example, fig 14.3.3 shows the polar rectangle R given by: $1.5 \le r \le 2$.

Theorem:- If R is a simple polar region whose boundaries are the rays $\theta = \alpha$ and $\theta = \beta$ and the curves $r = r_1(\theta)$ and $r = r_2(\theta)$ shown in figure 14.3.8 and if ~~r(θ)~~ $f(r,\theta)$ is continuous on R, then,

$$\iint_R f(r,\theta)\, dA = \int_{\theta=\pi}^{\beta} \int_{r=r_1\theta}^{r_2\theta} f(r,\theta)\, r\, dr\, d\theta$$

Q $\iint_R \sin\theta \, dA = ?$, where R is region in first quad. that is outside circle $r=2$ and inside the cardioid $r = 2(1+\cos\theta)$.

$$\iint_R \sin\theta \, dA = \int_{\theta=0}^{\pi/2} \int_{r=2}^{2+2\cos\theta} \sin\theta \, r\, dr\, d\theta = \int_{\theta=0}^{\frac{\pi}{2}} \left[\frac{\sin\theta\, r^2}{2} \right]_2^{2(1+\cos\theta)} d\theta$$

$$= \int_{\theta=0}^{\pi} \left(\frac{\sin\theta(2+2\cos\theta)^2}{2} - \frac{2\cancel{4}\sin\theta}{\cancel{2}} \right) d\theta$$

$$\Rightarrow 2\int_0^{\pi/2} \left(\sin\theta\,(1+\cos^2\theta+2\cos\theta) - \sin\theta\right) d\theta$$

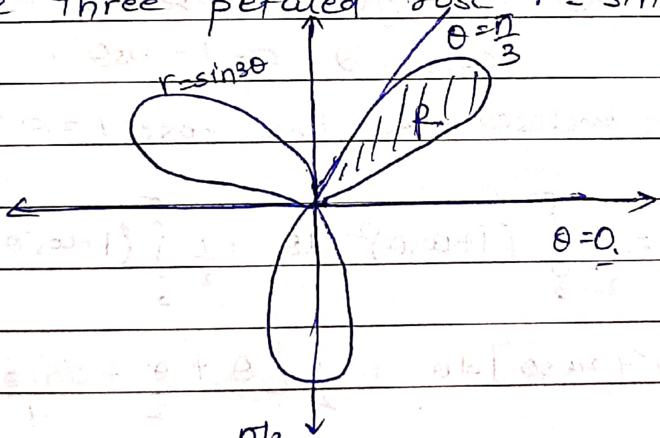
$\boxed{Ans = \frac{8}{3}}$

$$= 2\int_0^{\pi/2} (\cos^2\theta \sin\theta + 2\sin\theta\cos\theta)\, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} (\cos^2\theta \sin\theta + \sin 2\theta)\, d\theta$$

Q find the area b/w coordinates $r = 2(1+\cos\theta)$ and $r = 2(1-\cos\theta)$ & $\boxed{\iint_R \sin\theta\, dA}$

Q Use double integral to find the area enclosed by the three petaled rose $r = \sin 3\theta$.



$$A = 3\iint_R dA$$

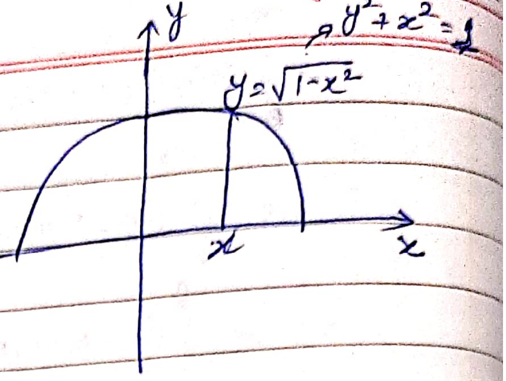
$$= 3\int_0^{\pi/3}\int_0^{\sin 3\theta} r\,dr\,d\theta$$

$$= \frac{3}{2}\int_0^{\pi/3} \sin^2 3\theta\, d\theta$$

$$= \frac{3}{4}\int_0^{\pi/3} (1-\cos 6\theta)\, d\theta = \frac{3}{4}\left[\theta - \frac{\sin 6\theta}{6}\right]_0^{\pi/3}$$

$$= \frac{3}{4}\left[\frac{\pi}{3} - 0\right] = \frac{\pi}{4} \text{ Ans}$$

Q Evaluate $\int_{x=-1}^{1}\int_{y=0}^{\sqrt{1-x^2}} (x^2+y^2)^{3/2}\, dy\, dx.$



$= \iint_R (x^2+y^2)^{3/2}\, dA$

$x = r\cos\theta, \quad y = r\sin\theta$

$x^2+y^2=1$

$r^2(\cos^2\theta+\sin^2\theta)=1$

$r^2 = 1 \Rightarrow r = \pm 1$

Q ① $\int_0^{\pi}\int_0^{1+\cos\theta} r\,dr\,d\theta$ Q② $\int_0^{\pi/6}\int_0^{\cos 3\theta} r\,dr\,d\theta$

Q③ $\int_0^{\pi/2}\int_0^{\cos\theta} r^3\,dr\,d\theta$ Q④ The region enclosed by the cardioid $r = 1-\cos\theta$.

Q⑤ The area of region enclosed by the rose $r=\sin 2\theta$.

Sol^n ① $\int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{1+\cos\theta} d\theta = \frac{1}{2}\int_0^{\pi}(1+\cos\theta)^2\, d\theta = \frac{1}{2}\int_0^{\pi}(1+\cos^2\theta+2\cos\theta)$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$
$\Rightarrow \cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$
$\frac{1+\cos 2\theta}{2} = \cos^2\theta$

$\Rightarrow \frac{1}{2}\int_0^{\pi}\left[1+\frac{1+\cos 2\theta}{2}+2\cos\theta\right]d\theta = \frac{1}{2}\left[\theta+\frac{\theta}{2}+\frac{\sin 2\theta}{4}+2\sin\theta\right]_0^{\pi}$

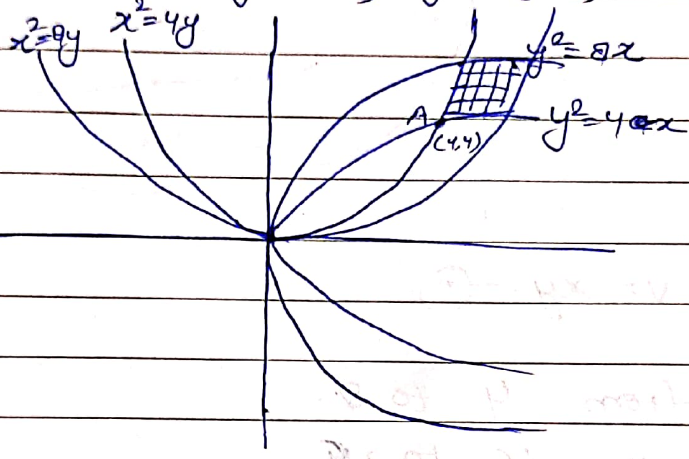
$= \frac{1}{2}\left[\pi+\frac{\pi}{2}\right] = \frac{3\pi}{4}$ Ans

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

$$\beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma m+n} = \beta(n,m)$$

Q. Evaluate $\iint_R xy\,dx\,dy$ over the region bounded by the parabolas $y^2=4x$, $y^2=8x$, $x^2=4y$, $x^2=8y$.

Sol^n



solving for pt. A.

$y^2=4x$, $x^2=4y$

$x=\frac{y^2}{4}$

$\frac{y^4}{16}=4y$

$y^4=64y$

$y(y^3-64)=0$

$\boxed{y=0,4}$

But, $y\neq 0$ ∴ $\boxed{y=4}$

$\boxed{x=\frac{4^2}{4}=4}$

~~A = (4,4)~~

let, $u=\frac{y^2}{x}$, $v=\frac{x^2}{y}$

then u varies from 4 to 8.

& v " " 4 to 8.

$x=\frac{y^2}{u}$ ⇒ $v=\frac{y^4}{u^2\cdot y}$ ⇒ $\boxed{y=u^{2/3}v^{1/3}}$

& $\boxed{x=u^{1/3}v^{2/3}}$

$$J=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix} = \frac{1}{3}(u^{-2/3}v^{2/3})\times\frac{1}{3}(u^{2/3}\cdot v^{-2/3}) - \frac{2}{3}(u^{1/3}v^{-1/3})\times\frac{2}{3}(u^{-1/3}\cdot v^{1/3})$$

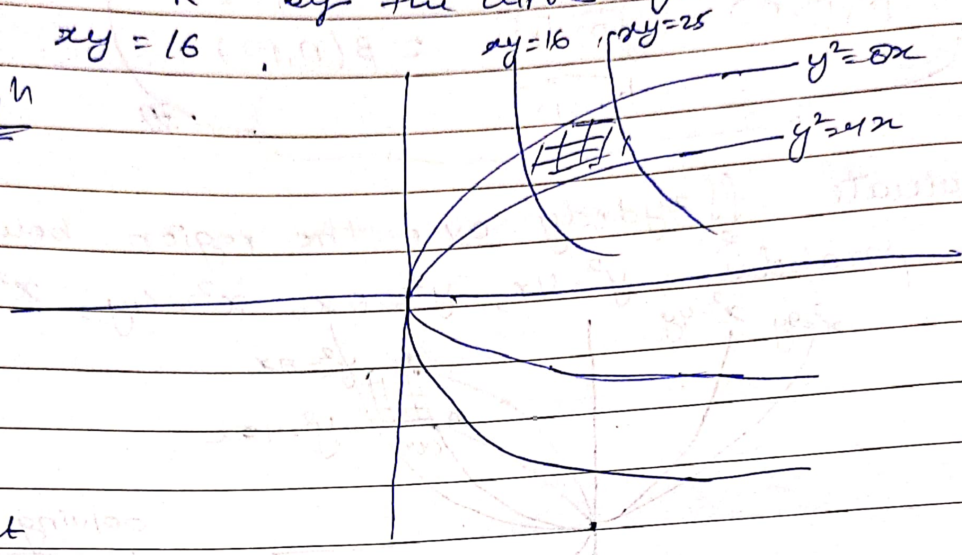
$$=\frac{1}{9}-\frac{4}{9}=\frac{-3}{9}=\frac{-1}{3}$$

$$\Rightarrow \iint_R xy\,dx\,dy = \int_{u=4}^{8}\int_{v=4}^{8} u^{1/3}v^{2/3}u^{2/3}v^{1/3}\,|J|\,dv\,du$$

$$=\frac{1}{3}\int\left[\int_4^8[uv\,dv\right]du=\frac{1}{3}\left[\int_{u=4}^{8}\left[u\cdot\frac{v^2}{2}\right]_4^8 du=192\right.$$

Q. Evaluate $\iint_R dx\,dy$, where R is the region bounded by the curves $y^2 = 8x$, $y^2 = 4x$, $xy = 25$, $xy = 16$.

Sol^n



let

$u = \frac{y^2}{x}$ — (i) & $v = xy$ — (ii)

u varies from 4 to 8.

v " " 16 to 25

$\Rightarrow$ from (i) $x = \frac{y^2}{u}$, putting in (ii)

$v = \frac{y^2}{u} \cdot y \Rightarrow \boxed{y = v^{1/3} u^{1/3}}$ $\therefore x = \frac{v^{2/3} u^{2/3}}{u}$

$\boxed{x = v^{2/3} u^{-1/3}}$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{-1}{3}\left[v^{2/3} \cdot u^{-4/3}\right) \times \frac{1}{3} v^{-2/3} \cdot u^{1/3} - \frac{2}{3} v^{-1/3} \cdot u^{-1/3} \times \frac{1}{3} v^{1/3} u^{-2/3}$$

$$= -\frac{1}{9} u^{-1} - \frac{2}{9} u^{-1} = -\frac{1}{3u}$$

$$\therefore \iint_R dx\,dy = \int_{v=16}^{25} \int_{u=4}^{8} |J|\, du\, dv = \int_{v=16}^{25} \int_{u=4}^{8} \frac{1}{3u}\, du\, dv$$

$$\Rightarrow \frac{1}{3} \int_{v=16}^{25} \left[\log u\right]_4^8 dv = \frac{1}{3} \int_{v=16}^{25} (3\log 2 - 2\log 2)\, dv$$

$$= \frac{1}{3} \times \log 2\, [v]_{16}^{25} = 3\log 2 \text{ Ans}$$

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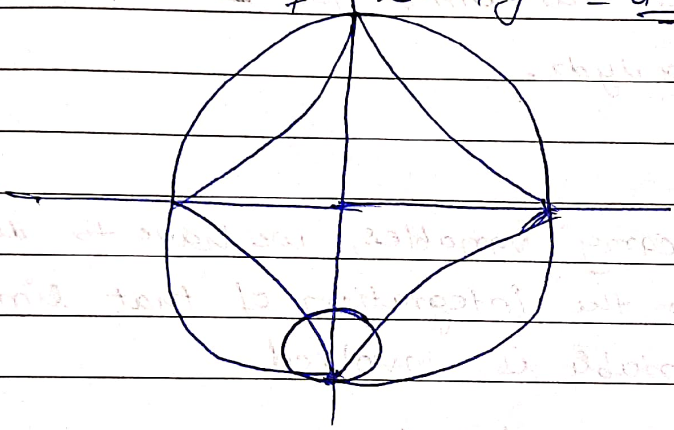
H/w

Q. Evaluate $\iint \sqrt{a^2-x^2-y^2}\, dx\, dy$ over the semicircle $x^2+y^2=ax$ in the first quadrant.

Q. $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy$

Q. $\iint r \sin\theta \, d\theta$ over area of the cardioid $r=a(1+\cos\theta)$ about the initial line.

Q. find area of $x^{2/3}+y^{2/3}=a^{2/3}$.



TRIPPLE INTEGRATION :-

- $\int_{z=4}^{0}\int_{y=2}^{3}\int_{x=0}^{1} dx\,dy\,dz$ ⇒ do any one of them first
i.e., no effect of order change

- $\int_{0}^{1-x-y}\int_{0}^{1}\int_{0}^{1-x} dx\,dy\,dz.$

first find out which limit has max no. of variable. i.e., $0 \to 1-x-y$ since it has x & y
so this ~~to~~ is the limit for z

$$\Rightarrow \int_{z=0}^{1-x-y}\int_{x=0}^{1}\int_{y=0}^{1-x} dx\,dy\,dz.$$

When limits carry variables, we have to do it in order ⇒ do the integration of that limit where maximum variable is involved.

Q. $\int_0^1\int_0^1\int_0^1 e^{x+y+z}\,dx\,dy\,dz = \int_0^1\int_0^1\int_0^1 e^x.e^y.e^z\,dx\,dy\,dz$

$$= \int_0^1\int_0^1 (e-1)e^y e^z\,dy\,dz = \int_0^1 (e-1)(e-1)e^z\,dz$$

$$\Rightarrow \underline{\underline{(e-1)^3}}\ \text{Ans}$$

Q. $\int_0^a\int_0^a\left(\int_0^a (xy+yz+zx)\,dx\right)dy\,dz$

$$\int_0^a\int_0^a\left[\frac{yx^2}{2}+xyz+\frac{zx^2}{2}\right]_0^a dy\,dz = \int_0^a\left[\frac{y^2}{2}.\frac{a^2}{2}+\frac{ay^2}{2}z+\frac{za^2y}{2}\right]$$

$$\Rightarrow \left[\frac{a^2}{2}.\frac{a^2}{2}z+\frac{a.a^2}{2}\frac{z^2}{2}+\frac{z^2}{2}.\frac{a^2}{2}.a\right]_0^a$$

$$= \frac{a^5}{4}+\frac{a^5}{4}+\frac{a^5}{4} = \frac{3a^5}{4}\ \text{Ans}$$

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Q. $\int_{z=0}^{\sqrt{xy}} \int_{y=\frac{1}{x}}^{1} \int_{x=1}^{3} xyz \, dx \, dy \, dz$

$\int_{x=1}^{3} \int_{y=\frac{1}{x}}^{1} \left(\int_{z=0}^{\sqrt{xy}} xyz \, dz \right) dy \, dz = \int_{x=1}^{3} \int_{y=\frac{1}{x}}^{1} \left[xy \frac{z^2}{2} \right]_0^{\sqrt{xy}} dy \, dx$

$= \int_{x=1}^{3} \int_{y=\frac{1}{x}}^{1} \frac{xy \cdot (xy)}{2} \, dy \, dx = \frac{1}{2} \int_{x=1}^{3} \left[x^2 \cdot \frac{y^3}{3} \right]_{\frac{1}{x}}^{1} dx$

$= \frac{1}{6} \int_{x=1}^{3} \left(\frac{-1+x^2}{x} \right) dx$

$= \frac{1}{6} \left[-\log x + \frac{x^3}{3} \right]_1^3$

$= \frac{1}{6} \left[-\log 3 + 9 - \left(-\cancel{\log 1} + \frac{1}{3} \right) \right]$

$\Rightarrow \frac{1}{6} \left[-\log 3 + 9 + \frac{1}{3} \right]$

$\Rightarrow \frac{-3\log 3 + 26}{18}$ Ans

Q. $\int_{0}^{1} \int_{y^2}^{1} \int_{0}^{1-x} x \, dz \, dx \, dy.$

- when 2 or more limit have only one variable then do integration as it is given. ✓.

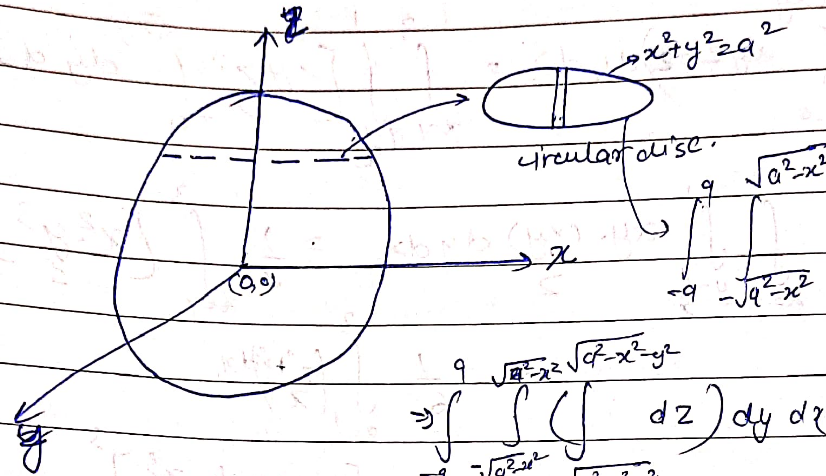
$\Rightarrow \int_{y=0}^{1} \int_{x=y^2}^{1} \left(\int_{z=0}^{1-x} x \, dz \right) dx \, dy = \int_0^1 \int_{y^2}^1 [xz]_0^{1-x} dx \, dy = \int_0^1 \int_{y^2}^1 (x - x^2) \, dx \, dy$

$= \int_0^1 \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_{y^2}^1 dy = \int_0^1 \left[\left(\frac{1}{2} - \frac{1}{3} \right) - \left(\frac{y^4}{2} - \frac{y^6}{3} \right) \right] dy$

$= \int_0^1 \left(\frac{1}{6} - \frac{y^4}{2} + \frac{y^6}{3} \right) dy = \left[\frac{y}{6} - \frac{y^5}{5 \times 2} + \frac{y^7}{7 \times 3} \right]_0^1$

$= \frac{1}{6} - \frac{1}{10} + \frac{1}{21} = \frac{35 - 21 + 10}{210} = \frac{24}{210} = \frac{4}{35}$ Ans

Q. Volume of a sphere $x^2+y^2+z^2=a^2$ is $\frac{4}{3}\pi a^3$ find it using triple integration.



$$\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy\, dx$$

$$\Rightarrow \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\left(\int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} dz\right) dy\, dx$$

$$\Rightarrow 8\int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2-y^2}} dz\, dy\, dx$$

$$= 8\int_0^a\int_0^{\sqrt{a^2-x^2}}\left(\sqrt{a^2-x^2-y^2}\right) dy\, dx$$

$$= 8\int_0^a\left[\frac{y}{2}\sqrt{a^2-x^2-y^2} + \frac{a^2-x^2}{2}\sin^{-1}\frac{y}{\sqrt{a^2-x^2}}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= 8\int_0^a\left(\sqrt{a^2-x^2}\,\cancel{\sqrt{a^2-x^2-(a^2-x^2)}} + \frac{a^2-x^2}{2}\sin^{-1}1\right) dx$$

$$\Rightarrow \cancel{8}4 \times \frac{\pi}{\cancel{2}}\int_0^a\left(\frac{a^2}{2}-\frac{x^2}{2}\right) dx \Rightarrow 4\pi\left[\frac{a^2}{2}\cdot x - \frac{x^3}{6}\right]_0^a$$

$$= 4\pi\left[\frac{a^3}{2}-\frac{a^3}{6}\right]$$

$$\Rightarrow \frac{4}{3}\pi a^3 \text{ Ans}$$

Evaluation by Iterated Integrals :-

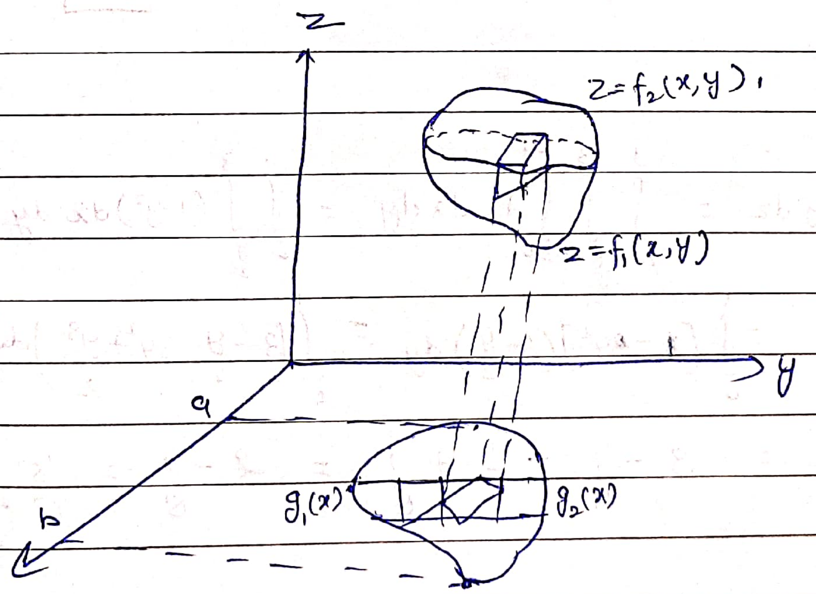
If the region D is bounded above by graph of $z = f_2(x,y)$ & bounded below by graph $z = f_1(x,y)$ then it can be shown that triple integral can be expressed as double integral of partial integration $\int_{f_1(x,y)}^{f_2(x,y)} F(x,y,z)\,dz$; that is,

$$\iiint_D F(x,y,z)\,dV = \iint_R \left[\int_{f_1(x,y)}^{f_2(x,y)} F(x,y,z)\,dz\right] dA.$$

where R is the ~~region bounded~~ orthogonal projection of D onto the x-y plane.

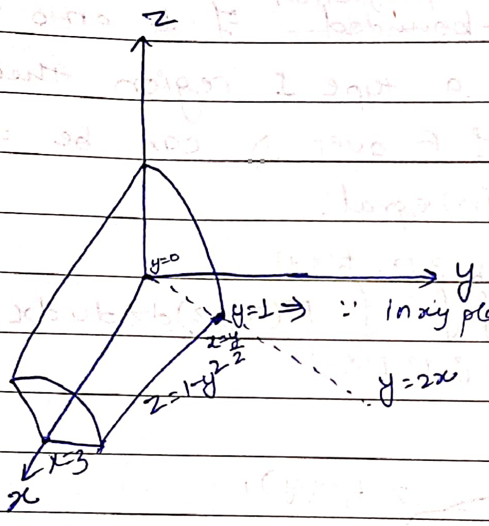
In particular if R is a type I region then the triple integration of F over D can be written as an iterated integral,

$$\iiint_D F(x,y,z)\,dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{f_1(x,y)}^{f_2(x,y)} F(x,y,z)\,dz\,dy\,dx.$$



Q find out the volume of solid in the first octant bounded by the graphs of $z = 1-y^2$; $y = 2x$; $x = 3$.

Sol^n: As indicated in fig., the first integration w.r.t. z is from 0 to $1-y^2$. furthermore, we see that the projection of solid D in the xy plane is a region type II. Hence, we will now integrate w.r.t. x from $\frac{y}{2}$ to 3. The last integration is w.r.t. y from 0 to 1



$\because$ In xy plane $z=0 \Rightarrow 1-y^2=0 \Rightarrow y=\pm 1$

but $y \to +ve$

$\therefore \boxed{y=1}$

$$V = \iiint_D dx\,dy\,dz = \int_0^1 \int_{\frac{y}{2}}^{3} \int_{z=0}^{1-y^2} dz\,dx\,dy = \int_0^1 \int_{\frac{y}{2}}^{3} (1-y^2)\,dx\,dy$$

$$= \int_0^1 [1-y^2](3-\frac{y}{2})\,dy = \int_0^1 \left(3 - \frac{y}{2} - 3y^2 + \frac{y^3}{2}\right)dy$$

$$= 3 - \frac{1}{4} - 1 + \frac{1}{8} = 2 - \frac{1}{8} = \frac{15}{8} \text{ Ans}$$

Changing the order of integration:-

$$\int_0^6 \int_0^{4-2x/3} \int_0^{3-\frac{x}{2}-\frac{3y}{4}} f(x,y,z)\,dz\,dy\,dx \text{, to convert } dz\,dy\,dx$$

to $dy\,dx\,dz$.

$$z = 3 - \frac{x}{2} - \frac{3y}{4} \Rightarrow 4z = 12 - 2x - 3y$$

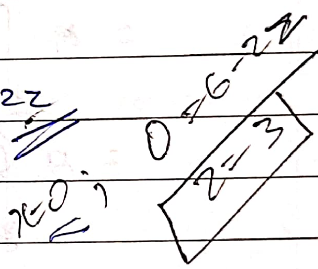
$$y = \frac{12-2x-4z}{3} = 4 - \frac{2}{3}x - \frac{4}{3}z$$

Now, for dx, we need the exp. of x in terms of z

we know that, in xz plane $y=0$.

$$\Rightarrow 4 - \frac{2x}{3} - \frac{4z}{3} = 0$$

$$x = \frac{12-4z}{2} = 6-2z$$



Volume of Ellipsoid:- $\iiint dz\,dy\,dx$.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$$z = \pm c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$$

$$y = \pm b\sqrt{1-\frac{x^2}{a^2}}$$

$$x = \pm a$$

$$= \int_{-a}^{a}\int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}}\int_{-c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\,dy\,dx$$

$$= 2\int_{-a}^{a}\int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}} c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\,dy\,dz$$

let. $t = b\sqrt{1-\frac{x^2}{a^2}} \Rightarrow 1-\frac{x^2}{a^2} = \frac{t^2}{b^2}$

$$\therefore 2\int_{-a}^{a}\left(\int_{-t}^{t} \frac{c}{b}\sqrt{t^2-y^2}\,dy\right)dx$$

$$\Rightarrow 2\times 2\int_{-a}^{a}\frac{c}{b}\left[\frac{y\sqrt{t^2-y^2}}{2} + \frac{t^2}{2}\sin^{-1}\frac{y}{t}\right]_{-t}^{t} dx = 4\int_{-a}^{a}\frac{c}{b}t^2\sin^{-1}1\,dx = 4\frac{c}{b}\int_{-a}^{a}\left(b^2 - \frac{b^2x^2}{a^2}\right)\frac{\pi}{2}\,dx$$

$$\Rightarrow \frac{2\pi\times c}{b}\left[b^2x - \frac{b^2x^3}{3a^2}\right]_{-a}^{a} \Rightarrow \frac{2\pi\times c}{b}\left[2b^2a - \frac{b^2a^3}{3a^2} - \left(\frac{-b^2(-a)^3}{3a^2}\right)\right] = 2\pi\times\frac{4}{3}b^2a\times\frac{c}{b}$$

Q. find the volume of solid bounded by the cylinder $x^2+y^2=4$ & planes $y+z=4$ and $z=0$.

$z=0$ to $z=4-y$

$y=\pm\sqrt{4-x^2}$

$x=\pm 2$

$$V=\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{4-y} dz\,dy\,dx$$

$$=\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{4-y} dz\,dy\,dx = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(4-y)\,dy\,dx$$

$4y-\frac{y^2}{2}$

$$=\int_{-2}^{2}\left[8\sqrt{4-x^2}\right]dx$$

$$=8\left[\frac{x\sqrt{4-x^2}}{2}+\frac{4}{2}\sin^{-1}\frac{x}{2}\right]_{-2}^{2}$$

$$=8\left[2\times 0+\frac{4}{2}\times\sin^{-1}\frac{2}{2}\right]$$

$2\sin^{-1}1$

$$=8\left[\frac{2\times\pi}{2}\right]$$

$8\times 4\times\frac{\pi}{2}=16\pi$

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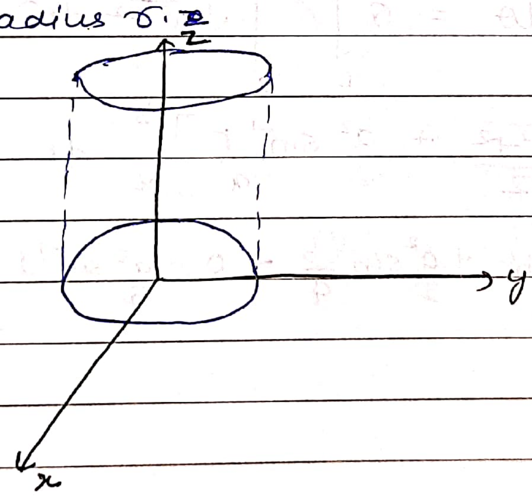
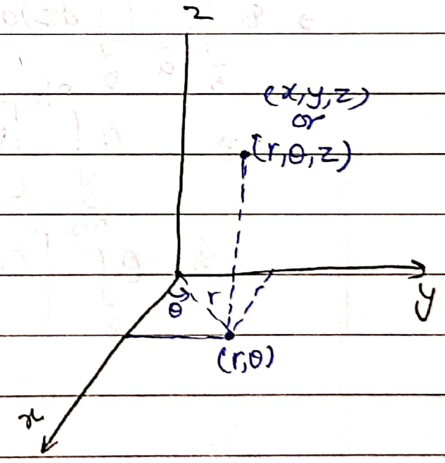
Q Use double integral to find the volume of tetrahedron bounded by the coordinate planes and plane $z = 4 - 4x - 2y$.

Q: $\iint_R (x+1)\, dA$; $y = x$, $x + y = 4$, $x = 0$.

Q $\iint_R 2xy\, dA$; $y = x^3$, $y = 8$, $x = 0$.

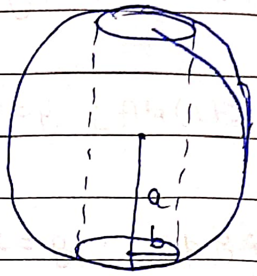
Cylindrical Coordinates :- The cylindrical coordinate system combines the polar description of a point in the plane with the rectangular description of z component of a point in space. As seen in fig., cylindrical coordinates of a point P are denoted by the ordered triple (r, θ, z). The coord cylindrical arises from the fact that a point P in space is determined by the intersection of planes z = constant and θ = constant with a cylinder of radius r.





Q. A cylindrical hole of radius 'b' is bored through a sphere of radius a such that $b < a$. find the volume of remaining solid.

$$V_{remaining} = \frac{4}{3}\pi a^3 - \text{Vol. of hollow part (cylindrical part).}$$



Let the ~~vol~~. eq^n of sphere :-

$$x^2+y^2+z^2=a^2$$

for cylindrical coordinates,

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

Limit of z are:- from 0 to $\sqrt{a^2+(x^2+y^2)}$ or 0 to $\sqrt{a^2-r^2}$

" " r are:- from a to b

" " θ are:- from 0 to $\pi/2$.

$$\Rightarrow 8\int_0^{\pi/2}\left(\int_a^b\left(\int_0^{\sqrt{a^2-r^2}} dz\right)dr\right)d\theta = 8\int_0^{\pi/2}\left(\int_a^b \sqrt{a^2-r^2}\,dr\right)d\theta$$

$$\Rightarrow 8\int_0^{\pi/2}\left[\frac{r\sqrt{a^2-r^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{r}{a}\right]_a^b d\theta$$

$$\Rightarrow 8\int_0^{\pi/2}\left[\frac{r\sqrt{a^2-b^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{b}{a} - 0 - \frac{a^2}{2}\sin^{-1} 1\right]d\theta$$

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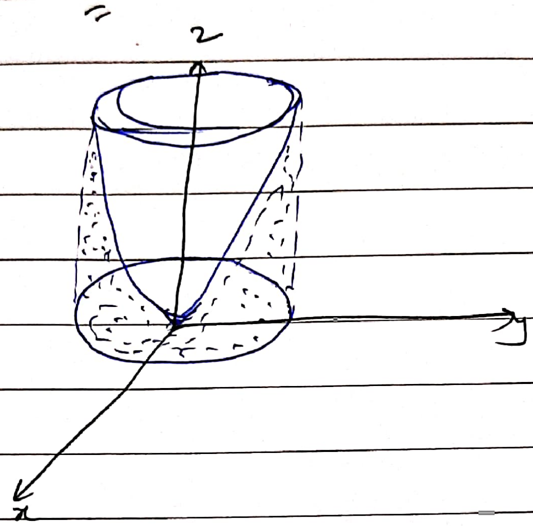
Q. find the volume bounded ~~of~~ by the paraboloid and the cylinder $x^2+y^2=a^2$.



Ans- $\frac{\pi a^3}{2}$