Function of Several Variables

Limits: See textbook:

* Working rule to find the limits:

Step 1: Find the value of $f(x,y)$ along $x \to a$ and $y \to b$.

Step 2: Find the value of $f(x,y)$ along $y \to b$ and $x \to a$

Step 3: If $a \to 0$, $b \to 0$ find the limit along $y = mx$ or $y = mx^n$.

Note: put $x = 0$ and then $y = 0$. Find its value f_1.
put $y = 0$ and then $x = 0$. Find the value f_2.

If $f_1 \neq f_2$ limit does not exist.

If $f_1 = f_2$ then put $y = mx$ and find the limit f_3.

If $f_1 = f_2 \neq f_3$, the limit does not exist.

If $f_1 = f_2 = f_3$ then put $y = mx^2$ and find the limit f_4.

If $f_1 = f_2 = f_3 \neq f_4$ then limit does not exist.

If $f_1 = f_2 = f_3 = f_4$ then limit exists.

Ex.

a. $$\lim_{(x,y)\to(0,1)} \frac{x - xy + 3}{x^2y + 5xy - y^3}$$

$$= \frac{0 - 0.1 + 3}{0^2.1 + 5.0.1 - 1^3} = -3.$$

b. $$\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}.$$

As $\sqrt{x} - \sqrt{y} \to 0$. as $(x,y) \to (0,0)$ we cannot use the Quotient rule.

$$\therefore \lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} = \lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} \times \frac{\sqrt{x} + \sqrt{y}}{\sqrt{x} + \sqrt{y}}.$$

$$= \lim_{(x,y)\to(0,0)} \frac{x(x-y)\sqrt{x} + \sqrt{y}}{(\sqrt{x})^2 - (\sqrt{y})^2}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x\cancel{(x-y)}\sqrt{x} + \sqrt{y}}{\cancel{(x-y)}}$$

$$= 0. \;//$$

Limits and Continuity

Exercise :

1. $$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2 y}{x^4+y^2}$$

(i) $\lim_{y\to 0} \dfrac{0.y}{0+y^2} = 0$. ($f_1$ say)

(ii) $\lim_{x\to 0} \dfrac{x^2.0}{x^4+0} = 0$ (f_2 say)

Here $f_1 = f_2$ therefore put $y = mx$.

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2 mx}{x^4+m^2x^2} = \frac{\cancel{x^3} m}{\cancel{x^2}(x^2+m^2)} = \frac{mx}{(x^2+m^2)} = 0 \ (f_3 \text{ say})$$

(iii) Put $y = mx^2$

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2 mx^2}{x^4+m^4x^4} = \frac{\cancel{x^4} m}{\cancel{x^4}(1+m^4)} = \frac{m}{1+m^4} = f_4$$

Here $f_1 = f_2 = f_3 \neq f_4$

Thus limit does not exist.

Ex.

1. Evaluate $\lim_{\substack{x \to 0 \\ y \to 0}} (x^3 + y^3)$

2. Evaluate $\lim_{\substack{x \to 1 \\ y \to 2}} \dfrac{3x^2 y}{x^2 + y^2 + 5}$

3. $\lim_{(x,y) \to (0,0)} \dfrac{3x^2 - y^2 + 5}{x^2 + y^2 + 2}$

4. $\lim_{(x,y) \to (0,0)} \sec x \tan y$.

* Continuity

Def: See textbook.

Working Rule:

Step 1: $f(a,b)$ should be well defined.

Step 2: $\lim_{(x,y) \to (a,b)} f(x,y)$ should exist.

Step 3. $\lim_{(x,y) \to (a,b)} f(x,y) = f(a,b)$.

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Ex. Test the function $f(x,y) = \begin{cases} \frac{x^3-y^3}{x^2+y^2} & \text{when } x \neq 0, y \neq 0 \\ 0, & \text{when } x=0, y=0 \end{cases}$

for Continuity

Sol. Step 1. The function is well defined at $(0,0)$

Step 2.

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2}$$

$$= \lim_{x\to 0}\left[\lim_{y\to mx} \frac{x^3+y^3}{x^2+y^2}\right]$$

$$= \lim_{x\to 0} \frac{x^3-m^3x^3}{x^2+m^2x^2} = \lim_{x\to 0} \frac{x(1-m^3)}{1+m^2} = 0$$

Thus limit exist at $(0,0)$.

Step 3. limit of $f(x)$ at origin = value of the function at origin.

$$\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2} = f(0,0) = 0.$$

Hence the function f is continuous at the origin.

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Ex. Discuss the continuity of $f(x,y) = \begin{cases} \dfrac{x}{\sqrt{x^2+y^2}}, & x \neq 0, y \neq 0 \\ 2, & x=0, y=0 \end{cases}$

at the origin.

Sol:- Here, we have $f(x,y) = \begin{cases} \dfrac{x}{\sqrt{x^2+y^2}}, & x \neq 0, y \neq 0 \\ 2, & x=0, y=0 \end{cases}$

Step 1. The function $f(x,y)$ at $(0,0)$ is well-defined.

Step 2. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x}{\sqrt{x^2+y^2}} = \lim\limits_{x\to 0}\left[\lim\limits_{y\to mx} \dfrac{x}{\sqrt{x^2+y^2}}\right]$

$$= \lim_{x\to 0} \frac{x}{\sqrt{x^2+x^2\cancel{y}^2}} = \lim_{x\to 0} \frac{x}{\sqrt{x^2(1+m^2)}}$$

$$= \lim_{x\to 0} \frac{x}{x\sqrt{1+m^2}} = \frac{1}{\sqrt{1+m^2}}$$

For different ~~values of m~~ values of m the limit f is not unique.

So the $\lim\limits_{(x,y)\to(0,0)} \dfrac{x}{\sqrt{x^2+y^2}}$ does not exist.

Hence $f(x,y)$ is not continuous at origin.

Ex. Test the Continuity

1. $f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2}, & \text{when } x \neq 0, y \neq 0 \\ 0, & \text{when } x = 0 \; y = 0 \end{cases}$

at origin.

2. $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{when } x \neq 0, y \neq 0 \\ 0, & \text{when } x = 0, y = 0 \end{cases}$

at origin.

3. $f(x,y) = \begin{cases} \dfrac{x^2-y^2}{x^2+y^2}, & \text{when } x \neq 0, y \neq 0 \\ 0, & \text{when } x = 0, y = 0 \end{cases}$

at origin.

Partial Derivatives.

Let $z = f(x,y)$ be function of two independent variables x and y. If we keep y constant and x varies then z becomes a function of x only. The derivatives of z w.r.t. x, keeping y as constant is called partial derivatives of 'z' w.r.t. 'x' and is denoted by the symbol.

$$\frac{\partial z}{\partial x}, \frac{\partial f}{\partial x}, f_x(x,y) \text{ etc.}$$

IIIy, the partial derivatives of 'z' w.r.t. 'y' keeping x as constant is denoted by

$$\frac{\partial z}{\partial y}, \frac{\partial f}{\partial y}, f_y(x,y) \text{ etc.}$$

Ex. Find $\frac{\partial f}{\partial y}$ if $f(x,y) = y \sin xy$.

Sol: $\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(y \sin xy)$

$= y \frac{\partial}{\partial y}(\sin xy) + \sin xy \frac{\partial}{\partial y}(y)$

$= y \cos xy \frac{\partial}{\partial y}(xy) + \sin xy \cdot 1$

$= y \cos xy \cdot x + \sin xy = xy \cos xy + \sin xy$.

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Ex. If $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$. Find the value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$.

Sol:- $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}} \cdot \frac{\partial}{\partial x}\left(\frac{x}{y}\right) + \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{\partial}{\partial x}\left(\frac{y}{x}\right)$$

$$= \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}} \cdot \frac{1}{y} + \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \left(-\frac{y}{x^2}\right)$$

$$= \frac{\cancel{y}}{\frac{\sqrt{y^2-x^2}}{\cancel{y}}} + \frac{1}{\frac{x^2+y^2}{\cancel{x^2}}}\left(-\frac{y}{\cancel{x^2}}\right)$$

$$= \frac{1}{\sqrt{y^2-x^2}} - \frac{y}{x^2+y^2}$$

$$x\frac{\partial u}{\partial x} = \frac{x}{\sqrt{y^2-x^2}} - \frac{xy}{x^2+y^2} \quad \text{—(1)}$$

Now $$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}} \cdot \frac{-x}{y^2} + \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x}$$

$$= \frac{-x}{y\sqrt{y^2-x^2}} + \frac{x}{x^2+y^2}$$

$$y\frac{\partial u}{\partial y} = \frac{-x}{\sqrt{y^2-x^2}} + \frac{xy}{x^2+y^2} \quad \text{— ②}$$

Adding ① and ②.

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0. \;//$$

Ex. If $u = (1-2xy+y^2)^{-1/2}$. prove that $x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = y^2u^3$.

Sol: $u = (1-2xy+y^2)^{-\frac{1}{2}}$

Diff. partially w.r.t. x

$$\frac{\partial u}{\partial x} = -\frac{1}{2}(1-2xy+y^2)^{-3/2}(-2y)$$

$$x\frac{\partial u}{\partial x} = xy(1-2xy+y^2)^{-3/2} \quad \text{— ①}$$

Diff. partially w.r.t. y.

$$\frac{\partial u}{\partial y} = -\frac{1}{2}(1-2xy+y^2)^{-3/2}(-2x+2y)$$

$$y\frac{\partial u}{\partial y} = +(xy-y^2)(1-2xy+y^2)^{-3/2} \quad \text{— ②}.$$

Subtracting ② from ①,

$$x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = xy(1-2xy+y^2)^{-3/2} - (xy-y^2)(1-2xy+y^2)^{-3/2}.$$

$$= y^2(1-2xy+y^2)^{-3/2}$$

$$= y^2u^3. \;//$$

Ex. If $u = x^y$ show that $\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial^3 u}{\partial x \partial y \partial x}$

Ex. If $u = \log(x^3 + y^3 + z^3 - 3xyz)$ show that

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}$$

Ex. If $z^3 - 3yz - 3x = 0$, show that

$$z\frac{\partial z}{\partial \cancel{x}\,y} = \frac{\partial z}{\partial y} \quad \text{and} \quad \frac{\partial^2 z}{\partial x \partial y} + \left(\frac{\partial z}{\partial x}\right)^2 = \frac{\partial^2 z}{\partial y^2}$$

Ex Find $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ if

1. $f(x, y, z) = (x^2 + y^2 + z^2)^{-1/2}$

2. $f(x, y, z) = x - \sqrt{y^2 + z^2}$

3. $f(x, y, z) = \sinh(xy - z^2)$

Partial derivatives of Higher Order

Let $z = f(x, y)$, then $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ being the function of x and y can be further differentiated partially with respect to x and y.

Symbolically, $$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y}$$

Ex. If $z = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$

Sol:- $$z = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$$

$$\frac{\partial z}{\partial x} = 2x \tan^{-1}\left(\frac{y}{x}\right) + x^2 \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) - y^2 \cancel{+} \frac{1}{1+\frac{x^2}{y^2}} \cdot \left(\frac{1}{y}\right)$$

$$= 2x \tan^{-1}\left(\frac{y}{x}\right) - \frac{x^2 y}{x^2+y^2} - \frac{y^3}{y^2+x^2}$$

$$= 2x \tan^{-1}\left(\frac{y}{x}\right) - \frac{y(x^2+y^2)}{x^2+y^2}$$

$$= 2x \tan^{-1}\left(\frac{y}{x}\right) - y$$

$$\frac{\partial^2 z}{\partial y \partial x} = 2x \frac{1}{1+\frac{y^2}{x^2}} \cdot \left(\frac{1}{x}\right) - 1$$

$$= \frac{2x^2}{x^2+y^2} - 1$$

$$= \frac{2x^2 - x^2 + y^2}{x^2+y^2} = \frac{x^2 - y^2}{x^2+y^2} \;//$$

Ex. If $u = e^{xyz}$, find the value of $\frac{\partial^3 u}{\partial x \partial y \partial z}$

Ex. If $u = \log(x^3 + y^3 + z^3 - 3xyz)$. Show that

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{9}{(x+y+z)^2}$$

* Homogenous Function:

A function $f(x,y)$ is said to be homogenous function in which the powes of each term is the same.

A function $f(x,y)$ is a homogenous function of order n, if the degree of each of its terms in x and y is equal to n. Thus

$$a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + \dots + a_{n-1} x y^{n-1} + a_n y^n \quad \text{—①}$$

is a homogenous function of order n.

Example: The function $x^3\left[1+\frac{y}{x}+3\left(\frac{y}{x}\right)^2+5\left(\frac{y}{x}\right)^3\right]$ is a homogeneous function of order 3.

* Euler's Theorem on Homogeneous Function

Statement: If z is a homogeneous function of x, y of order n, then

$$x.\frac{\partial z}{\partial x}+y.\frac{\partial z}{\partial y}=nz.$$

Ex. Verify Euler's theorem for $z=\frac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}$

Sol: we have

$$z=\frac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}=\frac{x^{1/3}\left[1+\left(\frac{y}{x}\right)^{1/3}\right]}{x^{1/2}\left[1+\left(\frac{y}{x}\right)^{1/2}\right]}=x^{-1/6}\phi\left(\frac{y}{x}\right)$$

Thus z is a homogeneous function of degree $-\frac{1}{6}$

By Euler's theorem $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=-\frac{1}{6}z$

Diff 'z' w.r.t. 'x' we get

$$\frac{\partial z}{\partial x}=\frac{\left(x^{\frac{1}{2}}+y^{1/2}\right)\left(\frac{1}{3}x^{-\frac{2}{3}}\right)-\left(x^{1/3}+y^{1/3}\right)\left(\frac{1}{2}x^{-\frac{1}{2}}\right)}{\left(x^{1/2}+y^{1/2}\right)^2}$$

$$= \frac{\frac{1}{3}x^{-\frac{1}{6}} + \frac{1}{3}x^{-\frac{2}{3}}y^{\frac{1}{2}} - \frac{1}{2}x^{-\frac{1}{6}} - \frac{1}{2}x^{-\frac{1}{2}}y^{1/3}}{(x^{\frac{1}{2}}+y^{1/2})^2}$$

$$x\frac{\partial z}{\partial x} = \frac{\frac{1}{3}x^{\frac{5}{6}} + \frac{1}{3}x^{\frac{1}{3}}y^{\frac{1}{2}} - \frac{1}{2}x^{\frac{5}{6}} - \frac{1}{2}x^{\frac{1}{2}}y^{1/3}}{(x^{1/2}+y^{1/2})^2} \quad -①$$

Diff 'z' w.r.t. y.

$$\frac{\partial z}{\partial y} = \frac{(x^{\frac{1}{2}}+y^{1/2})(\frac{1}{3}y^{-\frac{2}{3}}) - (x^{\frac{1}{3}}+y^{\frac{1}{3}})(\frac{1}{2}y^{-\frac{1}{2}})}{(x^{1/2}+y^{1/2})^2}$$

$$= \frac{\frac{1}{3}x^{\frac{1}{2}}y^{-\frac{2}{3}} + \frac{1}{3}y^{-\frac{1}{6}} - \frac{1}{2}x^{\frac{1}{3}}y^{-\frac{1}{2}} - \frac{1}{2}y^{-\frac{1}{6}}}{(x^{\frac{1}{2}}+y^{1/2})^2}$$

$$y\frac{\partial z}{\partial y} = \frac{\frac{1}{3}x^{\frac{1}{2}}y^{\frac{1}{3}} + \frac{1}{3}y^{\frac{5}{6}} - \frac{1}{2}x^{\frac{1}{3}}y^{\frac{1}{2}} - \frac{1}{2}y^{\frac{5}{6}}}{(x^{1/2}+y^{1/2})^2} \quad -②$$

Adding ① and ②

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{\frac{1}{3}x^{\frac{5}{6}} + \frac{1}{3}x^{\frac{1}{3}}y^{1/2} - \frac{1}{2}x^{5/6} - \frac{1}{2}x^{1/2}y^{1/3} + \frac{1}{3}x^{\frac{1}{2}}y^{1/3} + \frac{1}{3}y^{5/6} - \frac{1}{2}x^{1/3}y^{\frac{1}{2}} - \frac{1}{2}y^{5/6}}{(x^{1/2}+y^{1/2})^2}$$

$$= -\frac{1}{6}\frac{[x^{5/6}+y^{5/6}+x^{1/3}y^{1/2}+x^{1/2}y^{1/3}]}{(x^{1/2}+y^{1/2})^2}$$

$$= -\frac{1}{6}\frac{\left[x^{\frac{1}{2}}\left(x^{\frac{1}{3}}+y^{\frac{1}{3}}\right)+y^{\frac{1}{2}}\left(x^{\frac{1}{3}}+y^{1/3}\right)\right]}{\left(x^{1/2}+y^{1/2}\right)^2}$$

$$= -\frac{1}{6}\frac{\left(x^{\frac{1}{3}}+y^{1/3}\right)\left(x^{1/2}+y^{1/2}\right)}{\left(x^{1/2}+y^{1/2}\right)^2}$$

$$= -\frac{1}{6}\,\frac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}} = -\frac{1}{6}z$$

Hence Euler theorem is verified.

Ex. If $u = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}\cos u = 0.$$

Ex If $u = \sin^{-1}\left[\frac{x+y}{\sqrt{x}+\sqrt{y}}\right]$ prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{-\sin u \cos 2u}{4\cos^3 u}$$

Ex. If $u = \frac{x^2y^2z^2}{x^2+y^2+z^2} + \cos\frac{xy+yz}{x^2+y^2+z^2}$ show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \frac{4x^2y^2z^2}{x^2+y^2+z^2}$$

Vector differential operator Del i.e ∇

The vector differential operator Del is denoted by ∇. It is defined as

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

* Gradient of a scalar function

If $\phi(x, y, z)$ be a scalar function then $\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$ is called the gradient of the scalar function ϕ. and is denoted by grad ϕ.

Thus,

$$\text{grad } \phi = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi(x, y, z)$$

$$\text{grad } \phi = \nabla \phi$$

* Normal and directional derivatives:

(i) Normal: If $\phi(x,y,z) = c$ represents a family of surface for different values of the constant c on differentiating ϕ. we get $d\phi = 0$.

But $d\phi = \nabla\phi \cdot d\vec{r}$, so

$$\nabla\phi \cdot d\vec{r} = 0$$

Thus $\nabla\phi$ is a vector normal to the surface $\phi(x,y,z) = c$.

(ii) Directional derivatives.

The components of $\nabla\phi$ in the direction of a vector $\vec{d}$ is equal to $\nabla\phi \cdot \hat{d}$ and is called the directional derivatives of ϕ at P in the direction $\vec{d}$.

$$\frac{\partial\phi}{\partial r} = \lim_{\delta r \to 0} \frac{\delta\phi}{\delta r}, \quad \text{where } \delta r = PQ.$$

$\frac{\partial\phi}{\partial r}$ is called the directional derivatives of ϕ at P in the direction of PQ.

Ex. If $\phi = 3x^2y - y^3z^2$, Find grad ϕ at the point $(1,-2,-1)$.

Sol: grad $\phi = \nabla\phi$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(3x^2y - y^3z^2)$$

$$= \hat{i}\,6xy + \hat{j}\,(3x^2 - 3y^2z^2) + \hat{k}\,(-2y^3z)$$

grad ϕ at $(1,-2,-1)$.

$$= \hat{i}\,6.1.(-2) + \hat{j}\,(3.1^2 - 3.(-2)^2(-1)^2) + \hat{k}\,(-2(-2)^3.(-1))$$

$$= -12\hat{i} - 9\hat{j} - 16\hat{k}$$

Ex: Find the directional derivatives of $x^2y^2z^2$ at the point $(1,1,-1)$ in the direction of the tangent of the curve $x = e^t$, $y = \sin 2t + 1$, $z = 1 - \cos t$ at $t = 0$.

Sol: Let $\phi = x^2y^2z^2$

Directional derivatives of ϕ

$$= \nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2y^2z^2)$$

$$= 2xy^2z^2\hat{i} + 2yx^2z^2\hat{j} + 2zx^2y^2\hat{k}$$

D.D. of ϕ at $(1, 1, -1)$.

$= 2(1)(1)^2(-1)^2\hat{i} + 2(1)(1)^2(-1)^2\hat{j} + 2(-1)(1)^2(1)^2\hat{k}$

$= 2\hat{i} + 2\hat{j} + 2\hat{k}$ — ①.

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$= e^t\hat{i} + (\sin 2t + 1)\hat{j} + (1 - \cos t)\hat{k}$

Tangent vector

$\vec{T} = \frac{d\vec{r}}{dt} = e^t\hat{i} + 2\cos 2t\,\hat{j} + \sin t\,\hat{k}$

tangent at $t = 0$.

$= e^0\hat{i} + 2\cos 2.0\,\hat{j} + \sin 0\,\hat{k}$

$= \hat{i} + 2\hat{j}$ — ②.

Required directional derivatives along tangent

$= (2\hat{i} + 2\hat{j} + 2\hat{k}) \frac{\hat{i} + 2\hat{j}}{\sqrt{1+4}}$

$= \frac{2 + 4 + 0}{\sqrt{5}} = \frac{6}{\sqrt{5}}$

Ex. Find the rate of change $\phi = xyz$ in the direction normal to the surface $x^2y + y^2x + yz^2 = 3$ at the point $(1,1,1)$.

Sol: Rate of change of $\phi = \nabla\phi$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(xyz)$$

$$= \hat{i}yz + \hat{j}xz + \hat{k}xy$$

Rate of change of ϕ at $(1,1,1) = \hat{i} + \hat{j} + \hat{k}$

Normal to the surface $(\psi) = x^2y + y^2x + yz^2 - 3$

$$\nabla\psi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2y + y^2x + yz^2 - 3)$$

$$= \hat{i}(2xy + y^2) + \hat{j}(x^2 + 2yx + z^2) + \hat{k}\,2yz$$

$\nabla\psi$ at $(1,1,1)$

$$= 3\hat{i} + 4\hat{j} + 2\hat{k}$$

Unit normal $= \dfrac{3\hat{i} + 4\hat{j} + 2\hat{k}}{\sqrt{3^2 + 4^2 + 2^2}} = \dfrac{3\hat{i} + 4\hat{j} + 2\hat{k}}{\sqrt{29}}$

Ex. Find the constant m and n such that the surface $mx^2 - 2myz = (m+4)x$ will be orthogonal to the surface $4x^2 + z^3 = 4$ at the point $(1,-1,2)$

Sol: The point $P(1,-1,2)$ lies on both surface.
At this point lies in $mx^2 - 2nyz = (m+4)x$, so

$$m(1)^2 - 2n(-1)(2) = (m+4)(1)$$

$$\cancel{m} + 4n = \cancel{m} + 4$$

$$\Rightarrow \quad n = 1.$$

Let $\phi_1 = mx^2 - 2yz - (m+4)x$ and $\phi_2 = 4x^2 y + z^3 - 4$

Normal to $\phi_1 = \nabla \phi_1$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(mx^2 - 2yz - (m+4)x)$$

$$= i(2mx - m - 4) - \hat{j}2z - \hat{k}2y$$

Normal to $(1,-1,2)$

$$= \hat{i}(2m - m - 4) - 4\hat{j} + 2\hat{k}$$

$$= (m-4)\hat{i} - 4\hat{j} + 2\hat{k}$$

Normal to $\phi_2 = \nabla \phi_2$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(4x^2y + z^3 - 4)$$

$$= \hat{i}\, 8xy + \hat{j}\, 4x^2 + 3z^2\hat{k}$$

Normal to ϕ_2 at $(1,-1,2) = -8\hat{i} + 4\hat{j} + 12\hat{k}$

Since ϕ_1 and ϕ_2 are orthogonal, then normal are perpendicular to each other.

$$\nabla\phi_1 \cdot \nabla\phi_2 = 0.$$

$$[(m-4)\hat{i} - 4\hat{j} + 2\hat{k}] \cdot [-8\hat{i} + 4\hat{j} + 12\hat{k}] = 0.$$

$$-8(m-4) - 4.4 + 2.12 = 0.$$

$$-8m + 32 - 16 + 24 = 0.$$

$$-8m + 40 = 0.$$

$$m = 5$$

Hence $m = 5$, $n = 1$.

Ex. Find the values of constant λ and μ so that the surface $\lambda x^2 - \mu yz = (\lambda+2)x$ and $4x^2y + z^3 = 4$ intersect orthogonally at the pt. $(1,-1,2)$.

Ex. Find the directional derivatives of $\nabla(\nabla f)$ at the pt. $(1,-2,1)$ in the direction of the normal to the surface $xy^2z = 3x + z^2$, where $f = 2x^3y^2z^4$.

Ex. Find the D.D of the scalar function of $f(x,y,z) = xyz$ in the direction of the outer normal to the surface $z = xy$ at the pt. $(3,1,3)$