Multiple Integrals.

* Evaluation of Double Integrals:

1(a) If a region A be given by the inequalities $a \le x \le b$, $c \le y \le d$, then the double Integral is given by

$$\iint_A f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dx\,dy = \int_a^b \left[\int_c^d f(x,y)\,dy \right] dx$$

Or

$$\iint_A f(x,y)\,dx\,dy = \int_c^d \int_a^b f(x,y)\,dy\,dx. = \int_c^d \left[\int_a^b f(x,y)\,dx \right] dy.$$

(b) If the region A is bounded by the curves $y = f_1(x)$; $y = f_2(x)$, $x = a$ and $x = b$, then

$$\iint_A f(x,y)\,dx\,dy = \int_a^b \int_{f_1(x)}^{f_2(x)} f(x,y)\,dx\,dy$$

where integration w.r.t. 'y' is performed first treating x as a constant.

Ex. Evaluate the following double integrals

(i) $$\int_0^a \int_0^b (x^2+y^2)\, dx\, dy$$

$$= \int_0^a \left[\int_0^b (x^2+y^2)\, dy \right] dx$$

$$= \int_0^a \left[x^2 y + \frac{y^3}{3} \right]_0^b dx.$$

$$= \int_0^a \left(bx^2 + \frac{b^3}{3} \right) dx.$$

$$= \left[\frac{bx^3}{3} + \frac{b^3 x}{3} \right]_0^a$$

$$= \frac{a^3 b}{3} + \frac{ab^3}{3} \Rightarrow \frac{ab}{3}\left[a^2+b^2\right] //$$

(ii) $$\int_0^{\pi/2} \int_{\pi/2}^{\pi} \cos(x+y)\, dy\, dx.$$

$$= \int_0^{\pi/2} \left[\int_{\pi/2}^{\pi} \cos(x+y)\, dx \right] dy$$

$$= \int_0^{\pi/2} \left[\sin(x+y) \right]_{\pi/2}^{\pi} dy$$

$$= \int_0^{\pi/2} \sin(\pi+y) - \sin(\pi/2+y)\, dy$$

Since $\sin(\pi+\theta) = -\sin\theta$

$\sin(\pi/2+\theta) = \cos\theta.$

$$= \int_0^{\pi/2} (-\sin y - \cos y)\, dy.$$

$$= [\cos y - \sin y]_0^{\pi/2}$$

$$= (\cos \pi/2 - \sin \pi/2) - (\cos 0 - \sin 0)$$

$$= 0 - 1 - 1 - 0 = -2,,$$

Try yourself:

Evaluate:

1. $\int_0^2 \int_0^{2x-4} \frac{2y-1}{x+1}\, dx\, dy.$ (Ans: $-36 + 42\log 3.$)

2. $\int_0^2 \int_0^{3y} y\, dy\, dx$ (Ans: 7)

3. $\int_1^2 \int_0^{x} \frac{dx\, dy}{x^2+y^2}$ (Ans: $\frac{1}{4}\pi \log 2$)

4. $\int_0^3 \int_1^2 xy(1+x+y)\, dx\, dy$ (Ans: $\frac{123}{4}$).

5. $\int_0^1 \int_0^{\sqrt{(1-y^2)}} 4y\, dy\, dx$ (Ans: $\frac{4}{3}$)

Double Integration

1. Evaluate $\iint x^2y^2\,dx\,dy$ over the region $x^2+y^2\le 1$.

Sol: Let R be the region $x^2+y^2\le 1$. Then R is the region bounded by the circle $x^2+y^2=1$. The limits of integration for this region can be expressed as

$$-1\le x\le 1,\quad -\sqrt{1-x^2}\le y\le \sqrt{1-x^2}\quad \text{or}$$

$$-\sqrt{1-y^2}\le x\le \sqrt{1-y^2},\quad -1\le y\le 1.$$

Thus for a fixed value of y, x varies from $-\sqrt{1-y^2}$ to $\sqrt{1-y^2}$ in the area bounded by the circle $x^2+y^2=1$. Also y varies from -1 to 1 to cover the whole area of the circle $x^2+y^2=1$.

$$\therefore \iint x^2y^2\,dx\,dy = \int_{-1}^{1} y^2\left[\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} x^2\,dx\right]dy$$

$$=\int_{-1}^{1} y^2\left[2\int_0^{\sqrt{1-y^2}} x^2\,dx\right]dy = \int_{-1}^{1} 2y^2\left[\frac{x^3}{3}\right]_0^{\sqrt{1-y^2}} dy$$

$$=\int_{-1}^{1}\frac{2}{3}y^2(1-y^2)^{3/2}\,dy.\quad \left[\text{Put } y=\sin\theta \text{ s.t. } dy=\cos\theta\,d\theta.\ \ y=0,\ \theta=0\quad y=1,\ \theta=\pi/2.\right.$$

$$=\int_0^{1} 2\cdot\frac{2}{3}\sin^2\theta\,(1-\sin^2\theta)^{3/2}\,\cancel{d\theta}\ \cos\theta = \int_0 \frac{4}{3}\sin^2\theta\,.\cos^3\theta\cos\theta\,d\theta.$$

$$=\int_0^{\pi/2}\frac{4}{3}\sin^2\theta\cos^4\theta\,d\theta = \frac{4}{3}\cdot\frac{1.3.1}{6.4.2}\cdot\frac{\pi}{2} = \frac{\pi}{24}.$$

$$\left[\text{Wallis formula } \int_0^{\pi/2}\sin^m x\cos^n x = \frac{(m-1)(m-3)\ldots(n-1)(n-3)(n-5)}{(m+n)(m+n-2)(m+n-4)\ldots}\right].$$

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Evaluate $\iint xy\,dx\,dy$ over the region in the positive quadrant for which $x+y \leq 1$.

Sol: The region of integration is the area bounded by the lines $x=0$, $y=0$ and $x+y=1$

To cover this region of integration R, x varies from 0 to 1 and y varies from $1-x$.

$$\therefore \iint_R xy\,dx\,dy = \int_0^1 \int_0^{1-x} xy\,dy\,dx$$

$$= \int_0^1 x \left[\frac{y^2}{2}\right]_0^{1-x} dx$$

$$= \int_0^1 x \frac{(1-x)^2}{2} dx = \int_0^1 \frac{x(1-2x+x^2)}{2} dx$$

$$= \frac{1}{2} \int_0^1 (x-2x^2+x^3)\,dx$$

$$= \frac{1}{2} \left[\frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4}\right]_0^1$$

$$= \frac{1}{2}\left[\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right] = \frac{1}{2}\left[\frac{6-8+3}{12}\right]$$

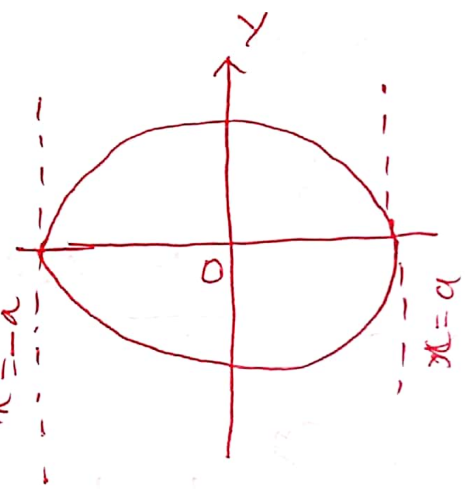
$$= \frac{1}{2} \cdot \frac{1}{12} = \frac{1}{24}$$

3. Evaluate $\iint (x+y)^2\, dx\,dy$ over the area bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Sol: For the ellipse : $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

The region of integration can be expressed as

$-a \le x \le a$ and $-b\sqrt{1-\frac{x^2}{a^2}} \le y \le b\sqrt{1-\frac{x^2}{a^2}}$



$\therefore \iint (x+y)^2\, dx\,dy$

$$= \int_{-a}^{a} \int_{-b\sqrt{1-x^2/a^2}}^{b\sqrt{1-x^2/a^2}} (x^2+y^2+2xy)\, dx\,dy.$$

But $2xy$ being an odd function of y.

its integration under the given limits of y is 0.

$$= \int_{-a}^{a} \int_{-b\sqrt{1-x^2/a^2}}^{b\sqrt{1-\frac{x^2}{a^2}}} (x^2+y^2)\, dx\,dy.$$

$$= \int_{-a}^{a} 2\left[\int_{0}^{b\sqrt{1-x^2/a^2}} (x^2+y^2)\, dy\right] dx.$$

$$= 2\int_{-a}^{a} \left[x^2 y + \frac{y^3}{3}\right]_0^{b\sqrt{1-x^2/a^2}} dx.$$

$$= 2\int_{-a}^{a} x^2 b\sqrt{1-\frac{x^2}{a^2}} + \frac{b^3}{3}\left(1-\frac{x^2}{a^2}\right)^{3/2} dx.$$

$$= 2.2\left[\int_0^a x^2 \frac{b}{a}\sqrt{a^2-x^2} + \frac{b^3}{3a^3}(a^2-x^2)^{3/2}\right] dx$$

Putting $x = a\sin\theta$ and $dx = a\cos\theta\, d\theta$.

$$= 4b\left[\int_0^a \frac{a^2\sin^2\theta}{a}\sqrt{a^2(1-\sin^2\theta)} + \frac{b^2}{3\cancel{a^3}} \cancel{a^3}(1-\sin^2\theta)^{3/2}\right]$$

$a\cos\theta\, d\theta.$

$$= 4b\left[\int_0^{\pi/2} \frac{a^2\sin^2\theta}{\cancel{a}} . \cancel{a}\cos\theta + \frac{b^2}{3}\cos^3\theta\right] a\cos\theta\, d\theta.$$

$$= 4ab\left[\int_0^{\pi/2}\left[a^2\sin^2\theta\cos^2\theta + \frac{b^2}{3}\cos^4\theta\right] d\theta\right.$$

$$= 4ab\left[a^2\int_0^{\pi/2}\sin^2\theta\cos^2\theta\, d\theta + \frac{b^2}{3}\int_0^{\pi/2}\cos^4\theta\, d\theta.\right.$$

By Walli's formula.

$$= 4ab\left[a^2\cdot\frac{1.1}{4.2}\cdot\frac{\pi}{2}+\frac{b^2}{\cancel{3}}\cdot\frac{\cancel{3}.1}{4.2}\,\frac{\pi}{2}\right]$$

$$= 4ab\left[\frac{\pi a^2}{16}+\frac{\pi b^2}{16}\right]$$

$$= \frac{1}{4}\pi ab\left(a^2+b^2\right) //$$

Evaluation of Triple Integrals:

(a) If the region V be specified by the inequalities $a \le x \le b$, $c \le y \le d$, $e \le z \le f$. then the triple integral

$$\iiint_V f(x,y,z)\,dx\,dy\,dz = \int_a^b \int_c^d \int_e^f f(x,y,z)\,dx\,dy\,dz$$

$$= \int_a^b dx \int_c^d dy \int_e^f f(x,y,z)\,dz.$$

Ex. Evaluate $\int_0^3 \int_0^2 \int_0^1 (x+y+z)\,dx\,dy\,dz.$

$$= \int_0^3 \int_0^2 \left[\int_0^1 (x+y+z)\,dz \right] dx\,dy.$$

$$= \int_0^3 \int_0^2 \left[xz + yz + \frac{z^2}{2} \right]_0^1 dx\,dy = \int_0^3 \left[\int_0^2 \left(x+y+\frac{1}{2} \right) dy \right] dx$$

$$= \int_0^3 \left[xy + \frac{y^2}{2} + \frac{y}{2} \right]_0^2 dx = \int_0^3 \left[2x + \frac{4}{2} + \frac{2}{2} \right] dx$$

$$= \int_0^3 (2x+3)\,dx = \left[\frac{2x^2}{2} + 3x \right]_0^3 = \frac{2 \cdot 9}{2} + 3 \cdot 3$$

$$= 9 + 9 = 18$$

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2. Evaluate $\int_0^4 \int_0^{2\sqrt{z}} \int_0^{\sqrt{4z-x^2}} dz\,dx\,dy$.

Sol: $I = \int_0^4 \int_0^{2\sqrt{z}} [y]_0^{\sqrt{4z-x^2}}\ \cancel{dx\,dy}.\,dz\,dx.$

$$= \int_0^4 \int_0^{2\sqrt{z}} \sqrt{4z-x^2}\ dz\,dx.$$

$$= \int_0^4 \left[\frac{1}{2}x\sqrt{4z-x^2} + \frac{1}{2}4z\sin^{-1}\left(\frac{x}{\sqrt{4z}}\right)\right]_0^{2\sqrt{z}} dz.$$

$$= \int_0^4 \left[\frac{1}{2}2\sqrt{z}\sqrt{4z-4z} + \frac{1}{2}4z\sin^{-1}\frac{2\sqrt{z}}{\sqrt{4z}}\right] dz$$

$$= \int_0^4 2z\sin^{-1}(1)\,dz. = \int_0^4 2z\sin^{-1}(\sin \pi/2)\,dz$$

$$= \int_0^4 \cancel{2}z.\,\pi/\cancel{2}\,dz. = \int_0^4 z\pi\,dz = \pi\left[\frac{z^2}{2}\right]_0^4$$

$$= \pi.\frac{16}{2}$$

$$= 8\pi$$

Triple Integral:

1. Find the volume of tetrahedron bounded by the coordinates planes and the plane $x+y+z=1$.

Sol: Here the region of integration V to cover the volume of tetrahedron can be expressed as.

$0 \le x \le 1$, $0 \le y \le 1-x$, $0 \le z \le 1-x-y$

Therefore the required volume of the tetrahedron

$$= \iiint_V dx\,dy\,dz = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} dx\,dy\,dz$$

$$= \int_0^1 \int_0^{1-x} [z]_0^{1-x-y}\, dx\,dy.$$

$$= \int_0^1 \int_0^{1-x} (1-x-y)\,dx\,dy$$

$$= \int_0^1 1-x \int_0^{1-x} dy - \left[\frac{y^2}{2}\right]_0^{1-x.} dx = \int_0^1 (1-x)(1-x) - \frac{(1-x)^2}{2} dx.$$

$$= \int_0^1 (1-x^2) - \frac{(1-x)^2}{2} dx.$$

$$= \int_0^1 \frac{2(1-x^2) - (1-x^2)}{2} dx = \int_0^1 \frac{(1-x^2)\,dx}{2.}$$

$$= \frac{1}{2} \int_0^1 dx - \int_0^1 x^2 dx = \frac{1}{2} [x]_0^1 - \left[\frac{x^3}{3}\right]_0^1$$

$$= \frac{1}{2} \cdot 1 - \frac{1}{3} = \frac{3-2}{6} = \frac{1}{6}$$

2019 Evaluate $\iiint (x+y+z)\,dx\,dy\,dz$ over the tetrahedron $x=0$, $y=0$, $z=0$ and $x+y+z=1$.

Sol: The region of integration V for the given tetrahedron can be expressed as $0 \le x \le 1$, $0 \le y \le 1-x$, $0 \le z \le 1-x-y$.

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} (x+y+z)\,dx\,dy\,dz.$$

$$= \int_0^1 \int_0^{1-x} (x+y)\left[z\right]_0^{1-x-y} + \left[\frac{z^2}{2}\right]_0^{1-x-y} dx\,dy.$$

$$= \int_0^1 \int_0^{1-x} (x+y)(1-x-y) + \frac{(1-x-y)^2}{2}\,dx\,dy$$

$$= \int_0^1 \int_0^{1-x} (1-x-y)\left\{x+y+\frac{(1-x-y)}{2}\right\} dx\,dy.$$

$$= \int_0^1 \int_0^{1-x} (1-x-y)\left\{\frac{2x+2y+1-x-y}{2}\right\} dx\,dy.$$

$$= \int_0^1 \int_0^{1-x} (1-x-y)\left(\frac{1+x+y}{2}\right) dx\,dy.$$

$$= \cancel{\int_0^1 \int_0^{1-x} (1-x-y)^2\,dx\,dy} = \frac{1}{2}\int_0^1 \int_0^{1-x} (1+x+y-\cancel{x}-xy-x^2-\cancel{y} -xy-y^2)\,dx\,dy.$$

$$= \cancel{\int_0^1 x(1-x-y)} = \frac{1}{2}\int_0^1 \int_0^{1-x} 1-(x^2+2xy+y^2).\,dx\,dy$$

$$= \frac{1}{2}\int_0^1 \int_0^{1-x} 1-(x+y)^2\,dy\,dx. = \frac{1}{2}\int_0^1 \left[y-\frac{(x+y)^3}{3}\right]_0^{1-x}$$

$$= \frac{1}{2}\int_0^1 (1-x) - \left[\frac{(x+(1-x))^3}{3} + \frac{(x+0)^3}{3}\right] dx.$$

$$= \frac{1}{2}\int_0^1 (1-x) - \frac{1}{3} + \frac{x^3}{3}\,dx = \frac{1}{2}\int_0^1 \frac{3-3x-1+x^3}{3}$$

$$= \frac{1}{2}\int_0^1 \frac{2}{3}dx - \int_0^1 x\,dx + \int_0^1 \frac{x^3}{3}dx = \frac{1}{2}\left[\frac{2}{3}-\frac{1}{2}+\frac{1}{12}\right] = \frac{1}{2}\cdot\frac{1}{4} = \frac{1}{8}.$$

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Evaluate $\iiint z^2\,dx\,dy\,dz$ over the sphere $x^2+y^2+z^2=1$.

Sol: The region of integration can be expressed as

$$-1 \le x \le 1,\quad -\sqrt{1-x^2} \le y \le \sqrt{1-x^2},\quad -\sqrt{(1-x^2-y^2)} \le z \le \sqrt{1-x^2-y^2}$$

$\therefore$ The required integral

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1+x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} z^2\,dx\,dy\,dz.$$

$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left[\frac{z^3}{3}\right]_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} dx\,dy = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{(1-x^2-y^2)^{3/2}+(1-x^2-y^2)^{3/2}}{3}\,dx\,dy$$

$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2\,(1-x^2-y^2)^{3/2}}{3}\,dx\,dy.$$

put $y = \sqrt{1-x^2}\sin\theta$ so that $dy = \sqrt{1-x^2}\cos\theta\, d\theta$

when $y=0$, $\theta = 0$, when $y = \sqrt{1-x^2}$, $\theta = \pi/2$.

$$= \frac{2}{3}\int_{-1}^{1}\int_{-\pi/2}^{\pi/2} [(1-x^2) - (1-x^2)\sin^2\theta]^{3/2} \cdot \cos\theta\sqrt{1-x^2}\,d\theta]\,dx.$$

$$= \frac{2}{3}\int_{-1}^{1}\int_{-\pi/2}^{\pi/2} [(1-x^2)\{1-\sin^2\theta\}]^{3/2} \cdot \cos\theta\sqrt{1-x^2}\,d\theta]\,dx.$$

$$= \frac{2}{3}\int_{-1}^{1}\int_{-\pi/2}^{\pi/2} [(1-x^2)\cos^2\theta]^{3/2} \cdot \sqrt{1-x^2}\cos\theta\,d\theta]\,dx.$$

$$= \frac{2}{3}\int_{-1}^{1}\int_{-\pi/2}^{\pi/2} (1-x^2)^{\frac{3}{2}+\frac{1}{2}} \cdot \cos^{2\cdot\frac{3}{2}}\theta \cdot \cos\theta\,d\theta]\,dx.$$

$$= \frac{2}{3}\int_{-1}^{1}\int_{-\pi/2}^{\pi/2} [(1-x^2)^2\cos^4\theta\,d\theta]\,dx$$

$$\left[\int_{-\pi/2}^{\pi/2}\cos^4\theta = 2\int_0^{\pi/2}\cos^4\theta\right.$$

$$= 2\cdot\frac{(n-1)(n-3)}{n(n-2)\ldots}\cdot\frac{\pi}{2}$$

$$= \frac{2}{3}\int_{-1}^{1} [(1-x^2)^2 \cdot \frac{3\cdot 1}{4\cdot 2}\cdot\frac{\pi}{2}]\,dx.$$

$$= \frac{2}{3}\int_{-1}^{1} 2\cdot(1-x^2)^2 \cdot \frac{3}{\cancel{8}}\cdot\frac{\pi}{2}]\,dx.$$

$$= \frac{2}{3}\int_{-1}^{1} [1-2x^2+x^4]\cdot\frac{3}{4}\cdot\frac{\pi}{2}]\,dx.$$

$$= \cancel{\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{\pi}{2}\,[x]_{-1}^{1} - 2\left[\frac{x^3}{3}\right]_{-1}^{1} + \left[\frac{x^5}{5}\right]_{-1}^{1}}$$

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$$= \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{\pi}{2} \int_0^1 2(1-x^2)^2 \, dx.$$

$$= \frac{\cancel{2}}{\cancel{3}} \cdot \frac{\cancel{3}}{\underset{2}{\cancel{4}}} \cdot \frac{\pi}{\cancel{2}} \cdot \cancel{2} \int_0^1 1 - 2x^2 + x^4 \, dx.$$

$$= \frac{\pi}{2} \left[[x]_0^1 - 2\left[\frac{x^3}{3}\right]_0^1 + \left[\frac{x^5}{5}\right]_0^1 \right].$$

$$= \frac{\pi}{2} \left[1 - \frac{2}{3} + \frac{1}{5} \right].$$

$$= \frac{\pi}{2} \left[\frac{15 - 10 + 3}{15} \right] = \frac{\pi}{\cancel{2}} \cdot \frac{\overset{4}{\cancel{8}}}{15} = \frac{4\pi}{15} \; //$$