LAGRANGE METHOD OF UNDETERMINED MULTIPLIER

\# Let $f(x,y,z)$ be a function of three variable x, y, z such that $\phi(x,y,z) = 0$. — (1).

By total differentiation of (1), we get

$$\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz = 0 \quad - (2).$$

And we know that $\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = 0$ — (3)

Multiplying eq (2) by λ and adding (3), we get

$$\left(\frac{\partial f}{\partial x} dx + \lambda \frac{\partial \phi}{\partial x} dx\right) + \left(\frac{\partial f}{\partial y} dy + \lambda \frac{\partial \phi}{\partial y} dy\right) +$$

$$\left(\frac{\partial f}{\partial z} dz + \lambda \frac{\partial \phi}{\partial z} dz\right) = 0$$

$$\left(\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x}\right) dx + \left(\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y}\right) dy + \left(\frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z}\right) dz = 0.$$

This equation satisfy only if

$$\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0, \quad \frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0, \quad \frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0$$

On solving, we can find the values of x, y, z and λ.

Ex. Find the volume of the largest rectangular parallelopiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Sol: we have $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

$$\Rightarrow \phi(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0. \quad -(1)$$

Let x, y, z be the length, breadth and height of the rectangular ~~parallepi~~ parallelopiped inscribed in the ellipsoid.

then Volume $=(2x)(2y)(2z)=8xyz$

$$\therefore \frac{\partial V}{\partial x}=2yz \quad \frac{\partial V}{\partial y}=2xz \quad \frac{\partial V}{\partial z}=2xy.$$

$$\frac{\partial \phi}{\partial x}=\frac{2x}{a^2} \quad \frac{\partial \phi}{\partial y}=\frac{2y}{b^2} \quad \frac{\partial \phi}{\partial z}=\frac{2z}{c^2}$$

$\therefore$ By Lagrange eqvation

$$\frac{\partial V}{\partial x}+\lambda\frac{\partial \phi}{\partial x}=0, \quad \frac{\partial V}{\partial y}+\lambda\frac{\partial \phi}{\partial y}=0, \quad \frac{\partial V}{\partial z}+\lambda\frac{\partial \phi}{\partial z}=0.$$

$$\therefore \quad 8yz + \lambda \frac{2x}{a^2} = 0 \quad - (1).$$

$$8xz + \lambda \frac{2y}{b^2} = 0 - (2).$$

$$8xy + \lambda \frac{2z}{c^2} = 0 - (3).$$

Multiply (1) by x, (2) by y, (3) by z and adding, we get.

$$24xyz + 2\lambda \left[\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right] = 0. \quad \Big| \text{ Since } \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$$24xyz + 2\lambda = 0. \Rightarrow \lambda = -12xyz$$

Putting the values of λ in (1).

$$8yz + (-12xyz)\frac{2x}{a^2} = 0. \Rightarrow 8yz - \frac{24x^2yz}{a^2} = 0.$$

$$\Rightarrow 8yz\left(1 - \frac{3x^2}{a^2}\right) = 0 \Rightarrow 1 - \frac{3x^2}{a^2} = 0 \Rightarrow x = \sqrt{\frac{a^2}{3}} = \frac{a}{\sqrt{3}}$$

Similarly on solving (2) and (3). We have

$$y = \frac{b}{\sqrt{3}}, \quad z = \frac{c}{\sqrt{3}}.$$

$\therefore$ Volume of the parallelopiped $= 8xyz$

$$= 8\left(\frac{a}{\sqrt{3}}\right)\left(\frac{b}{\sqrt{3}}\right)\left(\frac{c}{\sqrt{3}}\right)$$

$$= \frac{8abc}{3\sqrt{3}} \;//$$

Ex. Find the maximum and minimum distance of the point $(3, 4, 12)$ from the sphere $x^2+y^2+z^2=1$.

Sol. Let (x, y, z) be the coordinates of the given point, then the distance 'D' from $(3,4,12)$

$$D = \sqrt{(x-3)^2+(y-4)^2+(z-12)^2}$$

$$\therefore \quad F(x,y,z) = (x-3)^2+(y-4)^2+(z-12)^2$$

we have $x^2+y^2+z^2=1$.

$$\therefore \quad \phi(x,y,z) = x^2+y^2+z^2-1.$$

$\therefore$ By Lagrange equation

$$\frac{\partial F}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 2(x-3)+2\lambda x = 0 \quad -(1)$$
$$= (x-3)+\lambda x = 0$$

$$\frac{\partial F}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 2(y-4)+2\lambda y = 0 \quad -(2)$$
$$= (y-4)+\lambda y = 0$$

$$\frac{\partial F}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 2(z-12)+2\lambda z = 0 \quad -(3)$$
$$(z-12)+\lambda z = 0$$

Multiplying (1) by x, (2) by y, (3) by z and adding, we get.

$\cancel{2x}$ $(x^2+y^2+z^2)-3x-4y-12z+\lambda(x^2+y^2+z^2)=0$

Since $x^2+y^2+z^2=1$.

$1-3x-4y-12z+\lambda=0.$ — (*)

From (1) $(x-3)+\lambda x=0.$ $\cancel{\lambda=}$

$x+\lambda x=3.$

$x(1+\lambda)=3 \Rightarrow x=\dfrac{3}{1+\lambda}.$ — (4).

From (2). $y=\dfrac{4}{1+\lambda}$ — (5)

From (3) $z=\dfrac{12}{1+\lambda}.$ — (6).

Putting the value of x, y, z in (*).

$1-3.\dfrac{3}{1+\lambda}-4.\dfrac{4}{1+\lambda}-12.\dfrac{12}{1+\lambda}+\lambda=0.$

$=1+\lambda-\dfrac{9}{1+\lambda}-\dfrac{16}{1+\lambda}-\dfrac{144}{1+\lambda}=0. \Rightarrow (1+\lambda)^2=169$

$1+\lambda=\pm 13.$

putting $1+\lambda$ in (4), (5), (6), we have

$x=\dfrac{3}{13}$, $y=\dfrac{4}{13}$, $z=\dfrac{12}{13}$ and

$x=-\dfrac{3}{13}$, $y=-\dfrac{4}{13}$, $z=\dfrac{-12}{13}$

$\therefore$ The minimum distance $= \sqrt{(x-3)^2+(y-4)^2+(z-12)^2}$

$$= \sqrt{\left(\frac{3}{13}-3\right)^2+\left(\frac{4}{13}-4\right)^2+\left(\frac{12}{13}-12\right)^2}$$

$$= 12.$$

The maximum distance $= \sqrt{(x-3)^2+(y-4)^2+(z-12)^2}$

$$= \sqrt{\left(\frac{3}{13}+3\right)^2+\left(\frac{4}{13}+4\right)^2+\left(\frac{12}{13}+12\right)^2}$$

$$= 14.$$ //

Try yourself. Ex. Using Lagrange multiplier, find the shortest distance from the point $(1,2,2)$ to the sphere $x^2+y^2=36$.

Ex. Find the shortest and longest distance from the point $(1,2,-1)$ to the $x^2+y^2+z^2=24$.