Since $\vec{E} = -\vec{\nabla} V$, the above equation can be written as

$$-\vec{\nabla} V_{above} - (-\vec{\nabla} V_{below}) = \frac{\sigma}{\varepsilon_0} \hat{n}$$

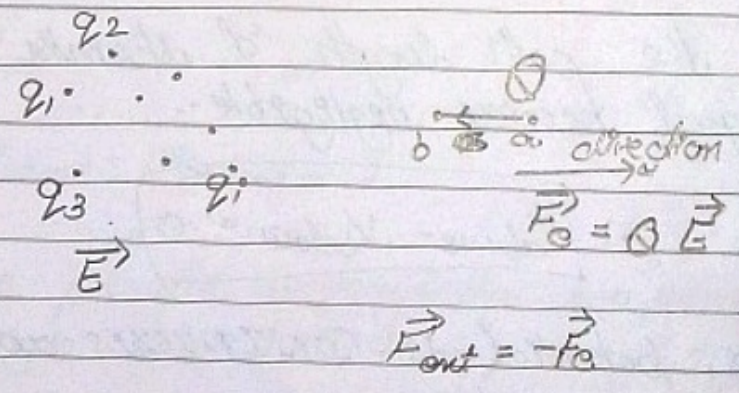
$$\vec{\nabla} V_{above} - \vec{\nabla} V_{below} = -\frac{\sigma}{\varepsilon_0} \hat{n}$$

$$\boxed{\frac{\partial V_{above}}{\partial n} - \frac{\partial V_{below}}{\partial n} = -\frac{\sigma}{\varepsilon_0}}$$

where $\frac{\partial V}{\partial n} = \vec{\nabla} \cdot V \cdot \hat{n}$, denotes the normal derivative of V i.e., the rate of change of V on the direction perpendicular to the surface.

- Work and Energy:

Consider a collection of source charges $q_1, q_2, q_3, \ldots, q_i$ having electric field $\vec{E}$. And we want to find work done on moving a test charge Q from point 'a' to point 'b', as shown in the following figure.



Since the test charge is placed in electric field $\vec{E}$, force on 'Q' due to collection of the source charges is

$$\vec{F_e} = Q\vec{E}$$

Thus, the minimum amount of external force to remove the test charge from point 'a' to point 'b' is given by

$$\vec{F}_{ext} = -Q\vec{E}$$

$\therefore$ The work done by the external force to move the test charge from point 'a' to point 'b' is

$$W = \int_a^b \vec{F}_{ext} \cdot d\vec{l}$$

$$= -\int_a^b Q\vec{E} \cdot d\vec{l}$$

$$= Q\left[-\int_a^b \vec{E} \cdot d\vec{l}\right]$$

$$= Q[V(b) - V(a)]$$

i.e. $V(b) - V(a) = \frac{W}{Q}$

$\therefore$ The potential difference between points 'a' and 'b' is equal to the work per unit charge required to carry a charge from 'a' to 'b'.

If we want to bring the test charge Q from a reference point at infinity to point 'r', the work done is given by

$$W = Q[V(r) - \overset{0}{V(\infty)}]$$

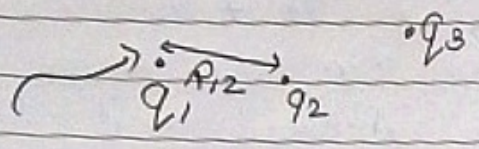
$$W = QV(r)$$

- **Energy of a point charge distribution:**

We know that a work done to bring a charge from infinity to a point r is given by

$$W = Q\,V(r)$$

In order to find work done to assemble an entire collection of point charges, imagine bringing in the charges, one by one, from infinity.

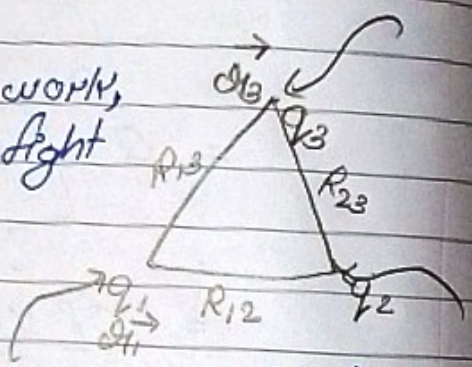


$$W_3 = q_3 \times (V_1 + V_2)$$

$$W_1 = 0 \qquad W = q_2 \times \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{R_{12}}\right)$$

The first charge q_1 takes no work, since there's no field yet to fight against,

i.e. $W_1 = 0$



Now, bring in q_2. This will cost $q_2 V_1(r_2)$.

i.e., $W_2 = q_2 \times \frac{1}{4\pi\varepsilon_0} \frac{q_1}{R_{12}}$

where R_{12} is the distance between q_1 and q_2 once they're in position.

Now, bring in q_3. This requires work $q_3 V_{1,2}(r_3)$, where $V_{1,2}$ is the potential due to charge q_1 and q_2.

Thus, $W_3 = q_3\, V_{1,2}(r_3)$

$$= q_3 \times \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{R_{13}} + \frac{q_2}{R_{23}}\right)$$

Similarly, the extra work to bring in q_4 will be

$$W_4 = q_4 \times \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{R_{14}} + \frac{q_2}{R_{24}} + \frac{q_3}{R_{34}}\right)$$

$\therefore$ The total work necessary to assemble the first four charges is

$$W = W_1 + W_2 + W_3 + W_4$$

$$= \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1 q_2}{R_{12}} + \frac{q_1 q_3}{R_{13}} + \frac{q_2 q_3}{R_{23}} + \frac{q_1 q_4}{R_{14}} + \frac{q_2 q_4}{R_{24}} + \frac{q_3 q_4}{R_{34}}\right)$$

Hence, the work done to assemble an entire collection of point charges can be written as

$$W = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\sum_{\substack{j=1 \\ j>i}}^{n}\frac{q_i q_j}{R_{ij}}$$

or

$$W = \frac{1}{2}\times\frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\sum_{\substack{j=1 \\ j\neq i}}^{n}\frac{q_i q_j}{R_{ij}}$$

$$= \frac{1}{2}\sum_{i=1}^{n} q_i\left(\sum_{\substack{j=1 \\ j\neq i}}^{n}\frac{1}{4\pi\varepsilon_0}\frac{q_j}{R_{ij}}\right)$$

$$= \frac{1}{2}\sum_{i=1}^{n} q_i V(r_i)$$

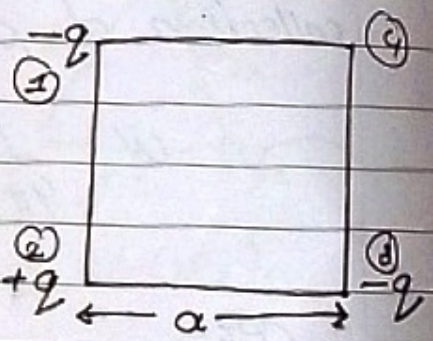
where $V(r_i)$ is the potential at point r_i

(the position of q_i) due to all other charges. This is the required amount of work done to assemble a configuration of point charges. This is also the amount of work we would get back if we dismantled the system.

Q. i) Three charges are situated at the corners of a square of side 'a', as shown in the figure. How much work does it take to bring in another charge $+q$, from far away and place it in the fourth corner P

ii) How much work does it take to assemble the whole configuration of four charges?

Ans i) Ans



$$q \times \frac{1}{4\pi\varepsilon_0}\left(\frac{-q}{a} + \frac{q}{a\sqrt{2}} - \frac{q}{a}\right)$$

$$q \times \frac{1}{4\pi\varepsilon_0}\left(\frac{-2q}{a} + \frac{q}{a\sqrt{2}}\right)$$

$$\frac{+q^2}{4\pi\varepsilon_0}\left(\frac{-2\sqrt{2}+1}{a\sqrt{2}}\right)$$

$$= \frac{+q^2}{4\pi\varepsilon_0}\left(\frac{1-2\sqrt{2}}{a\sqrt{2}}\right)$$

$$= \frac{q^2}{4\pi\varepsilon_0 a}\left(\frac{1-2\sqrt{2}}{\sqrt{2}}\right)$$

iii $W_1 = 0$

$$W_2 = q \times \frac{1}{4\pi\varepsilon_0} \times -\frac{q}{a}$$

$$= \frac{-q^2}{4\pi\varepsilon_0 a}$$

$$W_3 = -q \times \frac{1}{4\pi\varepsilon_0}\left(-\frac{q}{a\sqrt{2}} + \frac{q}{a}\right)$$

$$= +q^2 \times \frac{1}{4\pi\varepsilon_0 a}\left(\frac{1}{\sqrt{2}} - 1\right]$$

$$= \frac{q^2}{4\pi\varepsilon_0 a}\left(\frac{1-\sqrt{2}}{\sqrt{2}}\right)$$

$$W_{Total} = \frac{-q^2}{4\pi\varepsilon_0 a} - \frac{q^2}{4\pi\varepsilon_0}\left(\frac{\sqrt{2}-1}{\sqrt{2}}\right) - \frac{q^2}{4\pi\varepsilon_0 a}\left(\frac{2\sqrt{2}-1}{\sqrt{2}}\right)$$

$$= -\frac{q^2}{4\pi\varepsilon_0}\left(1 + \frac{\sqrt{2}-1}{\sqrt{2}} + \frac{2\sqrt{2}-1}{\sqrt{2}}\right)$$

$$= \frac{-q^2}{4\pi\varepsilon_0 a}\left(\frac{\sqrt{2}+\sqrt{2}+2\sqrt{2}-2}{\sqrt{2}}\right)$$

$$= \frac{-q^2}{4\pi\varepsilon_0}\left(\frac{4\sqrt{2}-2}{\sqrt{2}}\right)$$

$$= \frac{-q^2}{2\pi\varepsilon_0 a}\left(\frac{2\sqrt{2}-1}{\sqrt{2}}\right)$$

$$= \frac{q^2}{2\pi\varepsilon_0 a}\left(\frac{1-2\sqrt{2}}{\sqrt{2}}\right)$$

- Energy of a Continuous Charge Distribution:

We know that the work done for a point charge distribution is

$$W = \frac{1}{2} \sum_{i=1}^{n} q_i V(r_i)$$

Then, for a continuous charge distribution of volume charge density ρ, we can write

$$W = \frac{1}{2} \int_{\tau} \rho V \, d\tau \quad -(1)$$

[$V \rightarrow$ Potential]
[$\tau \rightarrow$ Volume]

But,

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\rho = \varepsilon_0 (\vec{\nabla} \cdot \vec{E}) \quad -(2)$$

Using eqn. (2) in (1), we get

$$W = \frac{\varepsilon_0}{2} \int_{\tau} (\vec{\nabla} \cdot \vec{E}) V \, d\tau$$

$$\left[\because \vec{\nabla} \cdot (V\vec{E}) = V(\vec{\nabla} \cdot \vec{E}) + \vec{E} \cdot (\vec{\nabla} V) \right]$$

$$W = \frac{\varepsilon_0}{2} \int_{\tau} \left[\vec{\nabla} \cdot (V\vec{E}) - \vec{E} \cdot \vec{\nabla} V \right] d\tau$$

$$= \frac{\varepsilon_0}{2} \left[\int_{\tau} \vec{\nabla} \cdot (V\vec{E}) \, d\tau - \int_{\tau} \vec{E} \cdot \vec{\nabla} V \, d\tau \right]$$

$$= \frac{\varepsilon_0}{2} \left[\oint_{S} (V\vec{E}) \cdot d\vec{S} - \int_{\tau} \vec{E} \cdot (-\vec{E}) \, d\tau \right]$$

~~[using Gauss~~

[Using Gauss's divergence theorem for 1st term and $\vec{E} = -\vec{\nabla} V$ for 2nd term]

$$= \frac{\varepsilon_0}{2}\left[\int_{\tau} E^2 d\tau + \oint_{S} (V\vec{E})\cdot d\vec{s}\right]$$

If we take larger and larger volume, the contribution from the volume integral goes up and that of the surface integral goes down in order to leave the sum intact.

$$\therefore W = \frac{\varepsilon_0}{2}\int_{\text{allspace}} E^2 d\tau$$

Q. Find the energy of a uniformly charged spherical shell of total charge q and radius R.

Ans

$\cancel{\rho = \frac{q}{\frac{4}{3}\pi R^3}}$ Inside the spherical shell, $\vec{E}_{inside} = 0$

$\cancel{\rho = \frac{3q}{4\pi R^3}}$ Outside the spherical shell, $\vec{E}_{outside} = \frac{1}{4\pi\varepsilon_0}\frac{q\hat{r}}{r^2}$

Now, $E^2 = \left(\frac{1}{4\pi\varepsilon_0}\right)^2 \frac{q^2}{r^4}$

$$\therefore \text{The required energy} = \frac{\varepsilon_0}{2}\int_R^\infty\int_0^\pi\int_0^{2\pi}\left(\frac{1}{4\pi\varepsilon_0}\right)^2 \frac{q^2}{r^4}\cdot r^2\sin\theta \, dr\, d\theta\, d\phi$$

$$= \frac{\varepsilon_0}{2}\times\frac{1}{16\pi^2\varepsilon_0^2} q^2 \int_R^\infty \frac{1}{r^2} dr \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi$$

$$= \frac{\varepsilon_0}{32\pi^2\varepsilon_0^2} q^2 \int_R^\infty \frac{1}{r^2} dr \int_0^\pi \sin\theta\, d\theta \times 2\pi$$

$$= \frac{q^2}{16\pi\varepsilon_0} \int_R^\infty \frac{1}{r^2}\, dr \left[-\cos\theta\right]_0^\pi$$

$$= \frac{q^2}{16\pi\varepsilon_0} \int_R^\infty \frac{1}{r^2}\, dr \times 2$$

$$= \frac{q^2}{8\pi\varepsilon_0} \int_R^\infty \frac{1}{r^2}\, dr$$

$$= \frac{q^2}{8\pi\varepsilon_0} \left[\frac{-1}{r}\right]_R^\infty$$

$$= \frac{q^2}{8\pi\varepsilon_0} \times \frac{1}{R}$$

$$= \frac{1}{8\pi\varepsilon_0} \frac{q^2}{R}$$

- Conductors:

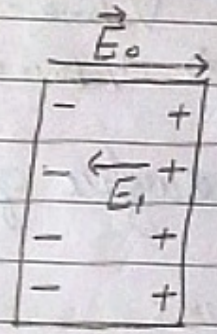
In a ~~metallic~~ metallic conductor, one or more electrons per atom free to roam about at will through the material. And in liquid conductors such as salt water, it is ions that do the moving. In fact, a perfect conductor is a material containing an unlimited supply of completely free charges.

The basic electrostatic properties of ideal conductors are:

1. $\vec{E} = 0$ inside a conductor

when we put a conductor into an external electric field $\vec{E}_0$, this will drive any free

positive charges to the right, and negative ones to the left. When they come to the edge of the material, the charges pile up such that +ve charge on the right side and -ve charge on the left. Now these induced charges produce a field of their own $\vec{E_1}$, which is in the opposite direction to $\vec{E_0}$. This means that the field of the induced charges tends to cancel off the original field. Charge will continue to flow until this cancellation is complete, and the resultant field inside the conductor is precisely zero. The whole process is practically instantaneous.



2) $\rho = 0$ inside a conductor

We know that

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

Since $\vec{E} = 0$ inside conductor,

$$\rho = 0.$$

There is still charge around, but exactly as much +ve charge as -ve charge, so the net charge density in the interior is zero.

iii Surface of a conductor is equipotential

If 'a' and 'b' are any two points within a given conductor,

$$V(b) - V(a) = -\int_a^b \vec{E} \cdot d\vec{l}$$

But $\vec{E} = 0$

$\therefore \quad V(b) - V(a) = 0$

$\Rightarrow V(b) = V(a)$

By the boundary condition of electric potential

$V_{above} - V_{below} = 0$

$\Rightarrow V_{above} = V_{below}$

Hence, surface of a conductor is equipotential.

- Capacitor:

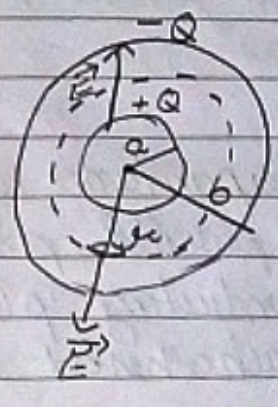
A capacitor is a device used to store electrical charge and hence electrical energy.
The amount of storage of charge in a capacitor is determined by a property called capacitance and is defined as the ratio of the maximum charge Q that can be stored in a capacitor to the applied potential difference V across the capacitor plates.

i.e. $C = \frac{Q}{V}$

- Capacitance of Spherical Capacitor:

A spherical capacitor consist of two concentric spherical shells of radii 'a' and 'b'.

A Gaussian surface of radius 'r' concentric with the two spherical shells is drawn.



According to Gauss's Law:

$$\oint_S \vec{E}\cdot d\vec{S} = \frac{1}{\varepsilon_0} q_{enc}$$

$$E \times 4\pi r^2 = \frac{1}{\varepsilon_0} Q$$

$$E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}$$

Then, potential difference between the two spherical shells is

$$V = -\int_b^a \vec{E}\cdot d\vec{l}$$

$$= -\int_b^a \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{r}\cdot(dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi})$$

$$= -\frac{Q}{4\pi\varepsilon_0}\int_b^a \frac{1}{r^2}\,dr$$

$$= \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{a} - \frac{1}{b}\right]$$

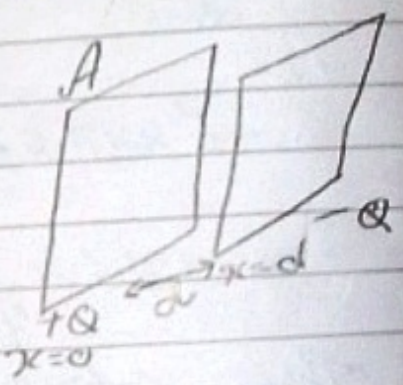
$$= \frac{Q}{4\pi\varepsilon_0}\frac{(b-a)}{ab}$$

Capacitance, $C = \frac{Q}{V}$

$$= \frac{Q}{\frac{Q}{4\pi\epsilon_0}\frac{(b-a)}{ab}}$$

$$= \frac{4\pi\epsilon_0 ab}{b-a}$$

- Capacitance of parallel plate capacitor:
A parallel plate capacitor consists of two thin conducting plates each of area 'A' held parallel to each other at a distance 'd' such that $d << A$.



The magnitude of electric field intensity at any point b/w two plates is given by
$E = \frac{\sigma}{\epsilon_0}$ [Using Gauss's Law]

$$= \frac{Q}{\epsilon_0 A}$$

Then, the potential difference b/w the plates

$$V = -\int_d^0 \vec{E}\cdot d\vec{l}$$

$$= -\int_d^0 \frac{Q}{\epsilon_0}\hat{x}\cdot(dx\hat{x} + dy\hat{y} + dz\hat{z})$$

$$= \frac{Qd}{\epsilon_0 A}$$

$\therefore$ Capacitance, $C = \frac{Q}{V} = \frac{Q}{\frac{Qd}{\epsilon_0 A}} = \frac{\epsilon_0 A}{d}$

- Energy Stored in Capacitor:

In order to find work done due to charge the capacitor up to a final amount Q, let us assume that at some intermediate stage in the process, the charge on the positive plate is q.

Then, the potential difference, $V = \frac{q}{C}$

∴ The work done to transport the next piece of charge dq is

$$dW = \frac{q}{C}\, dq$$

Hence, the total work necessary to go from q = 0 to q = Q is given by

$$W = \int dW = \int_0^Q \frac{q}{C}\, dq = \frac{Q^2}{2C}$$

$$W = \frac{Q^2}{2C}$$

$$W = \frac{(CV)^2}{2C}$$

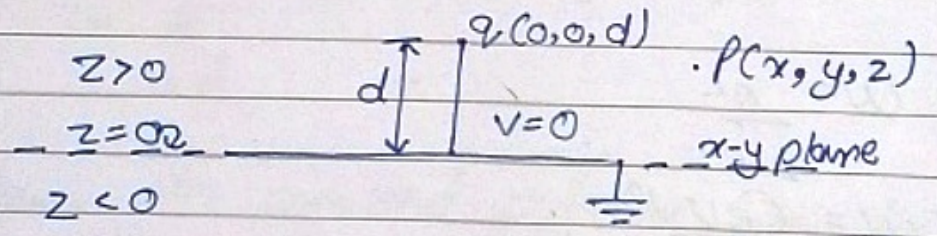
$$W = \frac{1}{2} C V^2$$; where V is the final potential of the capacitor.

- Method of Images:

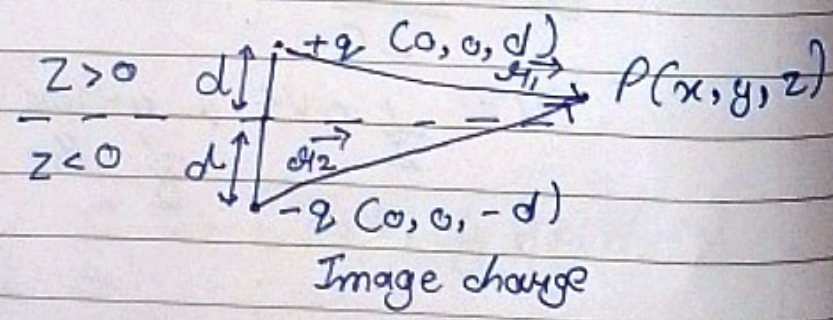
Suppose a point charge q is held at a distance 'd' above an infinite grounded conducting plane.

The potential in the region above the plane is not just $\frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}}$, for q will induce a certain amount of negative ~~energy~~ charge on the nearby surface of the conductor. The total potential is due in part to q directly, and in part to this induced charge.

Such potential in which we dont know how much charge is induced or how it is distributed can be calculated by using the method of images.



$\equiv$



$$\vec{\mathscr{r}_1} = x\hat{i} + y\hat{j} + (z-d)\hat{k}$$
$$\vec{\mathscr{r}_2} = x\hat{i} + y\hat{j} + (z+d)\hat{k}$$

$$|\vec{r_1}| = \sqrt{x^2+y^2+(z-d)^2}$$

$$|\vec{r_2}| = \sqrt{x^2+y^2+(z+d)^2}$$

$\therefore$ The net potential at point P is

$$V = V_1 + V_2$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{+q}{\sqrt{x^2+y^2+(z-d)^2}} + \frac{1}{4\pi\varepsilon_0}\frac{-q}{\sqrt{x^2+y^2+(z+d)^2}}$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

Since we know the potential, we can compute the surface charge density σ induced on the surface of conductor.

The boundary condition of $\vec{E}$ is

$$\vec{E}_{above} - \vec{E}_{below} = \frac{\sigma}{\varepsilon_0}\hat{n}$$

Because the field inside a conductor is zero, the field immediately outside is given by

$$\vec{E} = \frac{\sigma}{\varepsilon_0}\hat{n}$$

But $\vec{E} = -\vec{\nabla} V$

$$\therefore -\vec{\nabla} V = \frac{\sigma}{\varepsilon_0}\hat{n}$$

$$-\vec{\nabla} V \cdot \hat{n} = \frac{\sigma}{\varepsilon_0}\hat{n}\cdot\hat{n}$$

$$-\frac{\partial V}{\partial n} = \frac{\sigma}{\varepsilon_0}$$

Here, the normal direction is the z-direction, so we can write

$$\sigma = -\varepsilon_0 \frac{\partial V}{\partial z}\Big|_{z=0} \quad \left[\because \text{ surface is kept at } z=0\right]$$

$$\sigma = -\varepsilon_0 \frac{\partial}{\partial z}\left[\frac{q}{4\pi\varepsilon_0}\frac{1}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{1}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

$$\sigma = -\varepsilon_0 \frac{q}{4\pi\varepsilon_0}\left[\frac{-2(z-d)}{2\left(x^2+y^2+(z-d)^2\right)^{3/2}} + \frac{(z+d)}{\left(\sqrt{x^2+y^2+(z+d)^2}\right)^3}\right]$$

$$\sigma = \frac{-q}{4\pi}\left[\frac{2d}{(x^2+y^2+d^2)^{3/2}}\right]$$

$$\sigma = \frac{-q}{2\pi}\left[\frac{d}{(x^2+y^2+d^2)^{3/2}}\right],$$

So $\sigma(x,y) = \dfrac{-q}{2\pi}\dfrac{d}{(x^2+y^2+d^2)^{3/2}}$

Thus the induced charge is negative.

Now, the total induced charge is

$$Q = \int \sigma \, dS$$

This integral, over the xy-plane, could be done in cartesian coordinates, with $dS = dx\,dy$.

But it is easier to use polar coordinates (r, θ). with $r^2 = x^2 + y^2$ and $ds = r\,dr\,d\theta$.

$$\therefore \quad Q = \int_0^{\infty}\int_0^{2\pi} \frac{-qd}{2\pi (r^2+d^2)^{3/2}} \cdot r\,dr\,d\theta.$$

$$= \int_0^{\infty} \frac{-q\,d\,r}{2\pi (r^2+d^2)^{3/2}} \times 2\pi$$

$$= -qd \int_0^{\infty} \frac{r\,dr}{(r^2+d^2)^{3/2}}$$

$\cancel{r^2+d^2 = t^2}$

$\cancel{2r\,dr = 2t\,dt}$

$\cancel{r\,dr = t\,dt}$

$$\cancel{= -q \int_0^{\infty} \frac{t\,dt}{(t^2)^{3/2}}}$$

$$\cancel{= -q \int \frac{1}{t^2}}$$

$$\cancel{= \frac{q}{t}}$$

$$\cancel{= \frac{q}{\sqrt{r^2+d^2}}}$$

Let $r^2 + d^2 = u$

$2r\,dr = du$

$r\,dr = \frac{1}{2}\,du$

when $r = 0$, $u = d^2$

$r = \infty$, $u = \infty$

$$Q = -qd \int_{d^2}^{\infty} \frac{1}{2} \frac{du}{u^{3/2}}$$

$$= \frac{-qd}{2} \int_{d^2}^{\infty} u^{-3/2}\, du$$

$$= \frac{-qd}{2} \times -2 \times \left[\frac{1}{u^{1/2}}\right]_{d^2}^{\infty}$$

$$= qd \times \left[\frac{-1}{d}\right]$$

$$= -q$$

Hence the total charge induced on the conducting plane is $-q$.

It is obvious that the charge $+q$ is attracted towards the plane, because of the negative induced charge. Let's calculate the force of attraction. Thus,

$$\vec{F} = \frac{1}{4\pi\varepsilon_0} \frac{(+q)(-q)}{(2d)^2} \hat{R}$$

$$= \frac{-1}{4\pi\varepsilon_0} \frac{q^2}{4d^2} \hat{R}$$

And,

$$\vec{E} \text{ at point } P = \vec{E_1} + \vec{E_2}$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{+q}{(x^2+y^2+(z-d)^2)} \times \frac{x\hat{i} + y\hat{j} + (z-d)\hat{k}}{\sqrt{x^2+y^2+(z-d)^2}}$$

$$+ \frac{1}{4\pi\varepsilon_0} \frac{(-q)}{(x^2+y^2+(z+d)^2)} \times \frac{x\hat{i} + y\hat{j} + (z+d)\hat{k}}{\sqrt{x^2+y^2+(z+d)^2}}$$

$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\left[\frac{x\hat{i}+y\hat{j}+(z-d)\hat{k}}{(x^2+y^2+(z-d)^2)^{3/2}} - \frac{x\hat{i}+y\hat{j}+(z+d)\hat{k}}{(x^2+y^2+(z+d)^2)^{3/2}}\right]$$

Energy of the system:

with two point charges and no conductor,

$$W = \frac{1}{4\pi\varepsilon_0}\frac{(+q)(-q)}{2d}$$

$$= -\frac{1}{4\pi\varepsilon_0}\frac{q^2}{2d}$$

But for a single charge and a conducting plane, the energy is half of this.

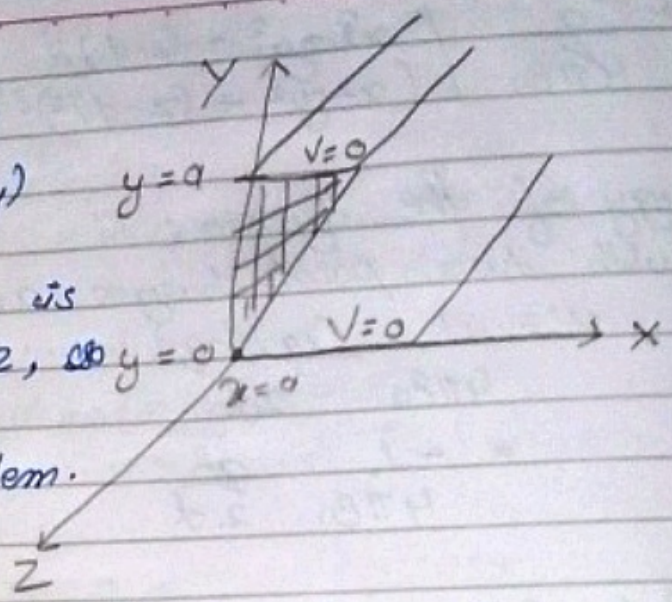
$$\therefore W = \frac{-1}{2}\frac{1}{4\pi\varepsilon_0}\frac{q^2}{2d}$$

As we move q toward the conductor, we do work only on q. It is true that induced charge is moving in over the conductor, but this costs nothing, since the whole conductor is at potential zero. By contrast, if we simultaneously bring in two point charges with no conductor, we do work on both of them, and the total is twice as great

Boundary value problems in cartesian coordinate system:

Q Two infinite grounded metal plates lie parallel to the xz plane at $y=0$, the other at $y=a$. The left end, at $x=0$, is closed off with an infinite strip insulated from the two plates and maintained at a specific potential $V_0(y)$. Find potential inside this slot.

Ans At $y = a$, $V = 0$
$y = a$, $V = 0$
$x = 0$, $V = V_0(y)$



The configuration is independent of z, so this is a two-dimensional problem.

Now, Laplace's equation in two dimension is given by

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \quad -(1)$$

~~we have~~

Subject to the boundary conditions

$$\left.\begin{array}{l} \text{i) when } y = 0,\ V = 0 \\ \text{ii) when } y = a,\ V = 0 \\ \text{iii) when } x = 0,\ V = V_0(y) \\ \text{iv) when } x \to \infty,\ V \to 0 \end{array}\right\} -(2)$$

Let $V(x,y) = X(x)Y(y)$, using method of separation of variables $-(3)$

Putting eqn. (3) in (1), we get

$$\frac{\partial^2 (XY)}{\partial x^2} + \frac{\partial^2 (XY)}{\partial y^2} = 0$$

$$Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} = 0$$

Dividing both side by XY.

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0 \quad -(4)$$

Here, the first term depends only on ~~x~~ x and the second term only on y. In other words, we have an equation of the form

$$f(x) + g(y) = 0 \quad -(5)$$

This equation will be true only if f and g must both be constant.

i.e. $\frac{1}{X}\frac{d^2X}{dx^2} = C_1$ and $\frac{1}{Y}\frac{d^2Y}{dy^2} = C_2$, with

$$C_1 + C_2 = 0$$
$$\Rightarrow C_1 = -C_2 = k^2, \text{ say}$$

Now, $\frac{1}{X}\frac{d^2X}{dx^2} = k^2$

$$\frac{d^2X}{dx^2} - k^2X = 0 \Rightarrow X(x) = ?$$

Again, $\frac{1}{Y}\frac{d^2Y}{dy^2} = -k^2$

$$\frac{d^2Y}{dy^2} + k^2Y = 0 \Rightarrow Y(y) = ?$$

Let $X = e^{\lambda x}$

So $\lambda^2 e^{\lambda x} - k^2 e^{\lambda x} = 0$

$$\lambda^2 - k^2 = 0$$
$$\lambda = \pm k$$

The solutions are e^{kx} and e^{-kx}.

The general solution is

$$X(x) = Ae^{kx} + Be^{-kx} = 0$$

Let $Y = e^{\lambda y}$

$$\lambda^2 e^{\lambda y} + k^2 e^{\lambda y} = 0$$

$$\lambda^2 + k^2 = 0$$

$$\lambda = \pm ik$$

The solutions are e^{-ik} and e^{ik}.

The general soln is

$$Y(y) = Ce^{iky} + De^{-iky}$$

$$= C\cos ik + Ci\sin ik + D\cos -ik + D\sin -ik$$

$$= C\cos ik + C$$

$$= C\sin ky + D\sin$$

$$= C\sin ky + D\cos ky$$

$\therefore$ $V(x,y) = X(x)\,Y(y)$

$$V(x,y) = (Ae^{kx} + Be^{-kx})\,(C\sin ky + D\cos ky)$$

Condition (iv)

$$0 = (Ae^{\infty} + Be^{-\infty})$$

so $A = 0$.

$$V(x,y) = Be^{-Kx}(C\sin Ky + D\cos Ky)$$

$$\cancel{V(x,y) = (C\sin Ky)}$$

$$V(x,y) = e^{-Kx}(BC\sin Ky + BD\cos Ky)$$

$$V(x,y) = e^{-Kx}(C\sin Ky + D\cos Ky)$$

Condition (iv) requires that A equals zero. Absorbing the constant B into C and D, we are left with

$$V(x,y) = e^{-Kx}(C\sin Ky + D\cos Ky)$$

Condition (i)

$y = 0$, $V = 0$

$$0 = e^{-Kx}(C\sin 0 + D\cos 0)$$

$$0 = De^{-Kx}$$

$$D = 0$$

Condition (i) now demands that D equals zero. so

$$V(x,y) = Ce^{-Kx}\sin Ky$$

Condition (ii)

$$y = a, \quad V = 0$$

$$0 = Ce^{-kx} \sin ak$$

So $\sin ak = 0$

$$ak = n\pi$$

Condition (ii) yields

$$\sin k a = 0 = \sin n\pi$$

$$ka = n\pi$$

$$k = \frac{n\pi}{a}; \quad n = 1, 2, 3 \ldots\ldots$$

$$\therefore V(x,y) = Ce^{-\frac{n\pi x}{a}} \sin n\pi/a \, y$$

Then, the general soln. is

$$V(x,y) = \sum_{n=1}^{\infty} C_n e^{-\frac{n\pi x}{a}} \sin \frac{n\pi y}{a} \quad -(6)$$

Applying condition (iii),

$$V_0(y) = \sum_{n=1}^{\infty} C_n e^{0} \sin \frac{n\pi y}{a}$$

$$V_0(y) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{a} y\right) \quad -(7)$$

In order to determine the coefficients C_n, multiply eqn. (7) by $\sin\left(\frac{n'\pi}{a} y\right)$ (where n' is is a +ve integer), and integrate from

0 to a

$$\therefore \int_0^a V_0(y) \sin\left(\frac{n'\pi}{a} y\right) dy = \sum_{n=1}^{\infty} C_n \int_0^a \sin\left(\frac{n\pi}{a} y\right) \sin\left(\frac{n'\pi y}{a}\right) dy$$

But $\int_0^a \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{n'\pi}{a} y\right) dy = \begin{cases} 0 & , n \neq n' \\ a/2 & , n = n' \end{cases}$

Now, $\int_0^a V_0(y) \sin\left(\frac{n'\pi y}{a}\right) dy = C_{n'} \times \frac{a}{2}$

$$\Rightarrow C_{n'} = \frac{2}{a} \int_0^a V_0(y) \sin\left(\frac{n'\pi y}{a}\right) dy$$

or

$$C_n = \frac{2}{a} \int_0^a V_0(y) \sin\left(\frac{n\pi y}{a}\right) dy$$

Suppose the strip at $x=0$ is a metal plate with constant potential V_0.

Then, $C_n = \frac{2V_0}{a} \int_0^a \sin\left(\frac{n\pi y}{a}\right) dy$

$$= \frac{2V_0}{n\pi} (1 - \cos n\pi)$$

$$= \begin{cases} 0 & , \text{ if } n \text{ is even} \\ \frac{4V_0}{n\pi} & , \text{ if } n \text{ is odd} \end{cases}$$

Hence, eqn. (6) becomes

$$V(x,y) = \frac{4V_0}{\pi} \sum_{n=1,3,5\ldots} e^{-\frac{n\pi}{a}} \sin\left(\frac{n\pi y}{a}\right)$$

$$V(x,y) = \frac{4V_0}{\pi} \sum_{n} \frac{1}{n} e^{-\frac{n\pi}{a}} \sin\left(\frac{n\pi y}{a}\right)$$

- **Dielectrics:**

Dielectrics are insulating materials in which all charges are attach to specific atoms or molecules. Thus, the charges are on a tight leash and all they can do is to move a bit within the atom or molecule.

- **Distinction b/w Dielectric & Insulator:**

Dielectric	Insulator
i) It can develop an internal $\vec{E}$ which nullifies the externally applied field.	i) It can't develop an internal $\vec{E}$.
ii) It can be easily polarised	ii) It can't be polarised.
iii) It contains relatively big bonds within atoms.	iii) It contains strong covalent bonds.
iv) All dielectrics are insulators.	iv) All insulators are not dielectrics.

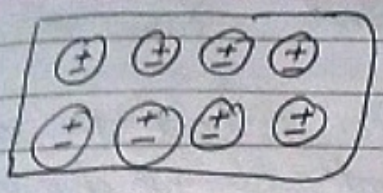
- **Types of Dielectrics:**

i) Non-Polar dielectric
ii) Polar dielectric

i) **Non-Polar Dielectric:**

Such dielectric are composed of molecules in which center of positive charge coincide with center of negative charge and are symmetric in shape.

Eg :- O_2, CO_2, CH_4, etc.



→ Non-Polar dielectric

In Non-Polar dielectrics, the dipole moment of each molecule is zero. Therefore, Net dipole moment of non-polar dielectric is,

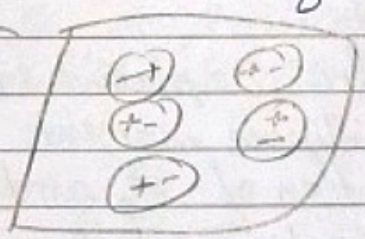
$$\vec{P}_{net} = 0$$

ii) Polar Dielectric:

Such dielectrics are composed of molecules in which center of positive charge don't coincide with center of -ve charge & are not symmetric in shape. Eg:- HCl, H_2O, NH_3, etc.

In Polar dielectrics, each molecule has non-zero dipole moment. But, the ~~dipole~~ molecules are randomly oriented due to thermal agitations. Thereby, cancelling the dipole moment of each other.

Polar Dielectric



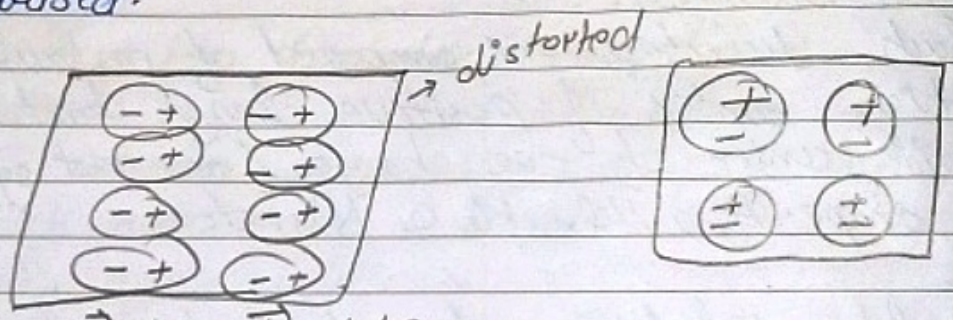
∴ Net dipole moment of a polar dielectric,

$$\vec{P}_{net} = 0.$$

- Polarisation:

1. Non-Polar Dielectric:

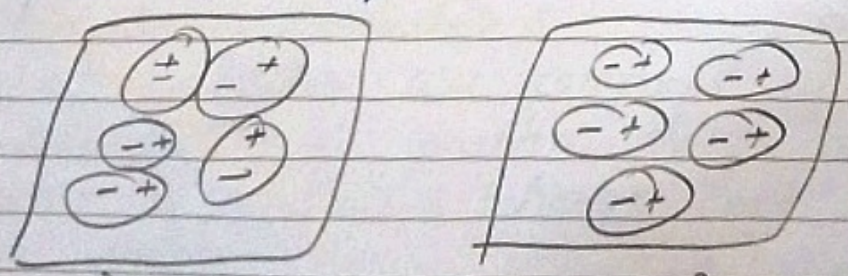
When a non-polar dielectric is held in an external $\vec{E}_0$, then the center of +ve charge is pull in the direction of $\vec{E}_0$ and the center of -ve charge is ~~pull in~~ pulled in the direction opposite to $\vec{E}_0$. In this way, the center of +ve & -ve charges are seperated. The molecules get distorted and are said to be Polarised.



$\vec{E}_0 \neq 0$, $\vec{p}_{net} \neq 0$
(deformation by stretching)

2. Polar Dielectric:

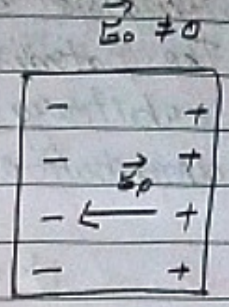
When a polar dielectric is placed in an external $\vec{E}_0$, the individual dipole moments tend to align in the direction of $\vec{E}_0$ and there will be net $\vec{p}_{net}$ in the direction of $\vec{E}_0$ that is dielectric is polarised.



$\vec{E}_0 = 0$, $\vec{p}_{net} = 0$ $\qquad$ $\vec{E}_0 \neq 0$, $\vec{p}_{net} \neq 0$

Net electric field for a polarised dielectric,

$$\vec{E} = \vec{E}_0 - \vec{E}_p$$



Thus polarization causes a decrease in the net electric field within the dielectric.

⇒ Polarisation ($\vec{P}$) is defined as the induced dipole moment per unit volume i.e.

$$\vec{P} = \frac{\vec{p}_{net}}{V}$$

and $\vec{P}$ is directly proportional to the net electric field $\vec{E}$ of the polarised dielectric

i.e. $\vec{P} \propto \vec{E}$

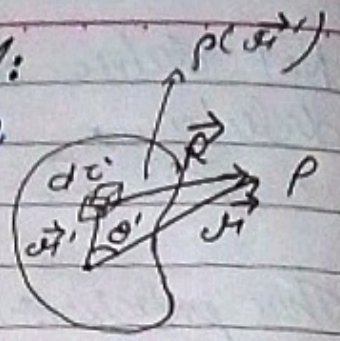
$$\Rightarrow \vec{P} = \varepsilon_0 \chi_e \vec{E}$$

where ε_0 is permittivity of free space and χ_e is called electric susceptibility

For vacuum, $\chi_e = 0$

$$\therefore \vec{P} = 0$$

- **Multipole expansion of Potential:**
The potential at P due to an arbitrary localised charge distribution is given by



$$V = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}')\,d\tau'}{R} \quad -(1)$$

Using the law of cosines,

$$R^2 = r^2 + (r')^2 - 2rr'\cos\theta'$$

$$= r^2\left[1 + \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\theta'\right]$$

$$R = r\left[1 + \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\theta'\right]^{1/2}$$

$$R = r\left[1+\varepsilon\right]^{1/2}; \quad \varepsilon = \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\theta'$$

$$\varepsilon = \left(\frac{r'}{r}\right)\left[\frac{r'}{r} - 2\cos\theta'\right]$$

$$\frac{1}{R} = \frac{1}{r}(1+\varepsilon)^{1/2}$$

For points well outside the charge distribution ($r \gg r'$),
ε is much less than 1, and this invites a binomial expansion:

$$\frac{1}{R} = \frac{1}{r}\left(1 - \frac{1}{2}\varepsilon + \frac{3}{8}\varepsilon^2 - \frac{5}{16}\varepsilon^3 + \cdots\right)$$

$$= \frac{1}{r}\left[1 - \frac{1}{2}\left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\theta'\right) + \frac{3}{8}\left(\frac{r'}{r}\right)^2 \left(\frac{r'}{r} - 2\cos\theta'\right)^2 - \frac{5}{16}\left(\frac{r'}{r}\right)^3\left(\frac{r'}{r} - 2\cos\theta'\right)^3 + \ldots\ldots\right]$$

$$\cancel{= \frac{1}{r}\left[1 + \left(\frac{r'}{r}\right)\left(\cos\theta' - \frac{r'}{2r}\right) + \left(\frac{r'}{r}\right)^2\left(\frac{3}{8}\frac{r'}{r} - \frac{3}{4}\cos\theta'\right)^2\right.}$$

$$\cancel{= \frac{1}{r}\left[1 + \frac{r'}{r}\left(\cos\theta' - \frac{r'}{2r}\right) + \left(\frac{r'}{r}\right)^2\left(\left(\frac{r'}{r}\right)^2 + 4\cos\right.\right.}$$

$$= \frac{1}{r}\left[1 + \left(\frac{r'}{r}\right)\cos\theta' + \left(\frac{r'}{r}\right)^2\left(\frac{3\cos^2\theta - 1}{2}\right) + \left(\frac{r'}{r}\right)^3\left(\frac{5\cos^3\theta' - 3\cos\theta'}{2}\right) + \ldots\right] \quad (2)$$

Using eqn. (2) in (1), we get.

$$V = \frac{1}{4\pi\epsilon_0}\left[\frac{1}{r}\int \rho(\vec{r}')\,d\tau' + \frac{1}{r^2}\int r'\cos\theta'\rho(\vec{r}')\,d\tau' + \frac{1}{r^3}\int (r')^2\left(\frac{3}{2}\cos^2\theta' - \frac{1}{2}\right)\rho(\vec{r}')\,d\tau' + \ldots\right]$$

This is the required multipole expansion of V in powers of $\frac{1}{r}$.

The first term is the monopole contribution (it goes like $\frac{1}{r}$); the second term is the dipole contribution (it goes like $\frac{1}{r^2}$); the third is quadrupole; the fourth octopole; and so on.

At large r, the multipole expansion is dominated by the monopole ~~dom~~ term.

i.e. $$V_{mon} = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\vec{r}')\, d\tau'}{r}$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{Q}{r}$$

where $Q = \int \rho(\vec{r}')\, d\tau'$ is the total charge of the configuration.

Next, the dipole contribution is

$$V_{dip} = \frac{1}{4\pi\varepsilon_0} \frac{1}{r^2} \int r' \cos\theta' \rho(\vec{r}')\, d\tau'$$

Since θ' is the angle b/w $\vec{r}$ and $\vec{r}'$,

$$r' \cos\theta = \vec{r}' \cdot \hat{r}$$
$$= \hat{r} \cdot \vec{r}'$$

$$\therefore V_{dip} = \frac{1}{4\pi\varepsilon_0} \frac{1}{r^2} \int (\hat{r} \cdot \vec{r}') \rho(\vec{r}')\, d\tau'$$

$$= \frac{1}{4\pi\varepsilon_0} \frac{1}{r^2} \hat{r} \cdot \int \vec{r}' \rho(\vec{r}')\, d\tau'$$

This integral, which doesn't depend on $\vec{r}$ at all, is called dipole moment of the distribution.

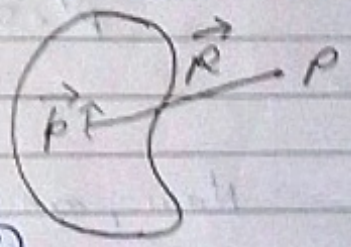
i.e. $\vec{p} \equiv \int \vec{r}' \rho(\vec{r}')\, d\tau'$

Hence, the dipole contribution to the potential simplified to

$$V_{dip} = \frac{1}{4\pi\varepsilon_0} \frac{\hat{r} \cdot \vec{p}}{r^2}$$

- Bound Charges:

Consider a polarized object containing a lot of microscopic dipoles lined up. For a volume element $d\tau'$, dipole moment is $\vec{p}$ with polarization $\vec{P}$ as the dipole moment per unit volume.



Now,

Potential at point P due to a single dipole $\vec{p}$ is

$$dV = \frac{1}{4\pi\varepsilon_0} \frac{\hat{R} \cdot \vec{p}}{R^2} \quad - (1)$$

But

$\vec{p} = \vec{P}(\vec{r}')\, d\tau'$ for each volume element $d\tau'$,

$$\therefore dV = \frac{1}{4\pi\varepsilon_0} \frac{\hat{R} \cdot \vec{P}(\vec{r}')\, d\tau'}{R^2}$$

Then, the total potential at P is

$$V = \frac{1}{4\pi\varepsilon_0} \int_{\tau'} \frac{\hat{R} \cdot \vec{P}(\vec{r}')}{R^2}\, d\tau'$$

$$= \frac{1}{4\pi\varepsilon_0} \int_{\tau'} \frac{\vec{P}(\vec{r}')\hat{R}}{R^2} \, d\tau'$$

But

$$\vec{\nabla}'\left(\frac{1}{R}\right) = \frac{\hat{R}}{R^2}$$

$$\therefore \quad V = \frac{1}{4\pi\varepsilon_0} \int_{\tau'} \vec{P} \cdot \vec{\nabla}'\left(\frac{1}{R}\right) d\tau' \quad -(2)$$

$$\left[\begin{array}{l} \vec{\nabla} \cdot (f\vec{A}) = f(\vec{\nabla} \cdot \vec{A}) + \vec{A} \cdot \vec{\nabla} f \\ \vec{A} \cdot \vec{\nabla} f = \vec{\nabla} \cdot (f\vec{A}) - f(\vec{\nabla} \cdot \vec{A}) \end{array} \right]$$

Here,

$$\vec{P} \cdot \vec{\nabla}'\left(\frac{1}{R}\right) = \vec{\nabla}\left(\frac{1}{R}\vec{P}\right) - \frac{1}{R}(\vec{\nabla} \cdot \vec{P})$$

Now, eqn. (2) can be written as

$$V = \frac{1}{4\pi\varepsilon_0} \int_{\tau'} \left[\vec{\nabla} \cdot \left(\frac{1}{R}\vec{P}\right) - \frac{1}{R}(\vec{\nabla} \cdot \vec{P}) \right] d\tau'$$

$$= \frac{1}{4\pi\varepsilon_0} \left[\int_{\tau'} \vec{\nabla} \cdot \left(\frac{1}{R}\vec{P}\right) d\tau' - \int_{\tau'} \frac{1}{R}(\vec{\nabla} \cdot \vec{P}) \, d\tau' \right]$$

By Gauss Divergence theorem

$$\int_{\tau'} \vec{\nabla} \cdot \left(\frac{1}{R}\vec{P}\right) d\tau' = \int_{S'} \frac{1}{R} \vec{P} \cdot d\vec{S}'$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\int_{S'} \frac{1}{R}\vec{P}\cdot d\vec{s}' - \int_{\tau'} \frac{1}{R}(\vec{\nabla}\cdot\vec{P})\,d\tau'\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\int_{S'} \frac{\vec{P}\cdot\hat{n}}{R}\,ds' + \frac{1}{4\pi\varepsilon_0}\int_{\tau'} \frac{(-\vec{\nabla}\cdot\vec{P})}{R}\,d\tau' \quad \text{--- (3)}$$

The first term looks like the potential of a surface charge,

$$\sigma_b \equiv \vec{P}\cdot\hat{n}$$

while the second term looks like the potential of a volume charge,

$$\rho_b \equiv -\vec{\nabla}\cdot\vec{P}$$

$\therefore$ Eqn. (3) becomes.

$$V = \frac{1}{4\pi\varepsilon_0}\int_{S'} \frac{\sigma_b}{R}\,ds' + \frac{1}{4\pi\varepsilon_0}\int_{V'} \frac{\rho_b}{R}\,d\tau'$$

Hence, the potential of a polarized object is the same as that produced by volume ~~than~~ bound charge density, $\rho_b = -\vec{\nabla}\cdot\vec{P}$ plus surface bound charge density, $\sigma_b = \vec{P}\cdot\hat{n}$.

Q. A sphere of radius R carries a polarization $\vec{P}(\vec{r}) = k\vec{r}$, where k is a constant and $\vec{r}$ is the vector from the center. Calculate the bound charge densities σ_b and ρ_b.

$$\sigma_b = (k\vec{r})\cdot\left(\frac{\vec{r}}{r}\right)$$

$$= \frac{k\,r^2}{|r|} = kr$$

$$\sigma_b = \vec{P}\cdot\hat{n}$$

$$= kR\,\hat{n}\cdot\hat{n}$$

$$= kR$$

$r = R$

$\sigma_b = kR$

Using polar coordinates.

$$\rho_b = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 P(r)\right) + \ldots \theta \ldots$$

$$\rho_b = \frac{-1}{r^2}$$

$$\rho_b = -\vec{\nabla}\cdot(k\vec{r})$$

$$= -k\vec{\nabla}\cdot(x\hat{i} + y\hat{j} + z\hat{k})$$

$\vec{\nabla}\cdot\vec{P}$

$$= -k\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot(x\hat{i} + y\hat{j} + z\hat{k})$$

$\vec{P} = k\vec{r}$

$$= -k(1+1+1)$$

$$= -3k$$

$\vec{\nabla}\cdot\vec{F} =$

$$\rho_b = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 P_r\right)$$

$$= -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 k r\right)$$

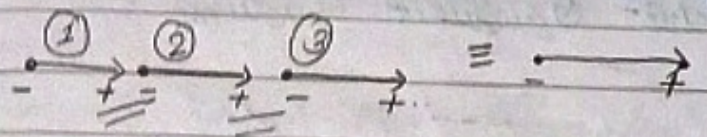
$$= -\frac{1}{r^2} \times k \frac{\partial}{\partial r}\left(r^3\right)$$

$$= -\frac{1}{r^2} \times k \times 3r^2 = -3k$$

Physical Interpretation of Bound Charges:

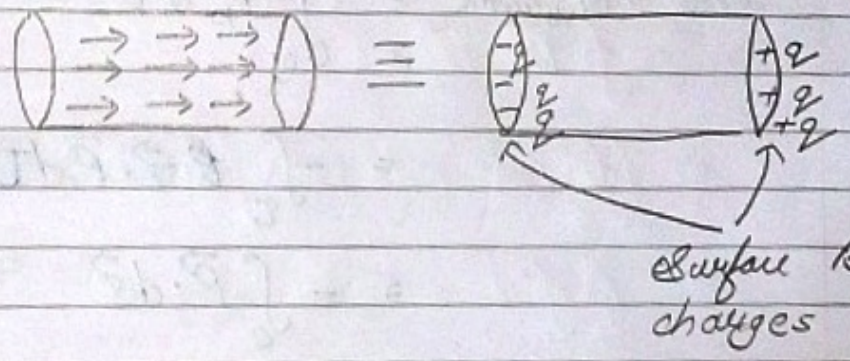
1. Surface bound charge:

Suppose we have a long string of dipoles.



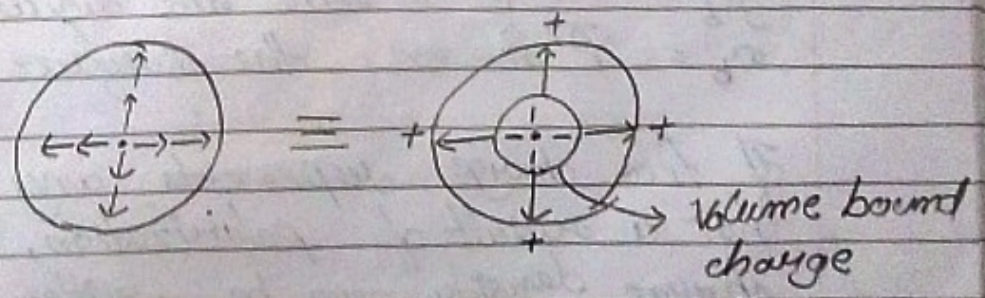
Along the line, the head of one effectively cancels the tail of its neighbour, but at the ends, there are two charges left over: +ve charge at the right end and -ve charge at the left.

Now, consider a tube of dielectric,



2. Volume bound charge:

If the polarization is nonuniform, we get accumulations of bound charge within the material as well as on the surface. In fact, a diverging $\vec{P}$ results in a pileup of negative charge.

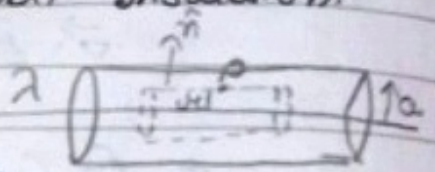


Eqn. (3) is the integral form of gauss law in presence of dielectric.

Q A long straight wire, carrying uniform line charge λ, is surrounded by rubber insulation of radius 'a'. Find the electric displacement inside and outside the rubber insulation.

Ans for $r < a$

A cylindrical Gaussian surface of radius r and length L is drawn.



By integral form of gauss's law in presence of dielectric, we get

$$\oint \vec{D} \cdot d\vec{S} = Q_{enc}$$

$$\Rightarrow \oint_S D\hat{r} \cdot \hat{n}\, ds = \int \lambda\, dl$$

$$\Rightarrow D \underbrace{\oint ds}_{\text{Curved surface area}} = \lambda L$$

$$\Rightarrow D \times 2\pi r L = \lambda L$$

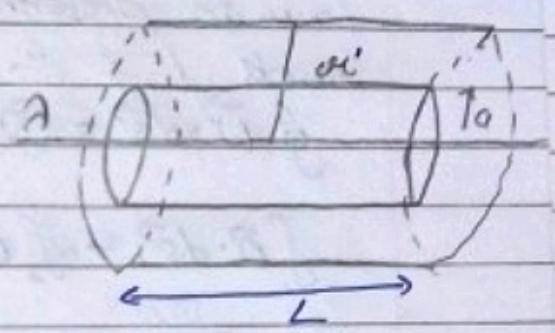
$$D = \frac{\lambda}{2\pi r}$$

$$\Rightarrow \boxed{\vec{D} = \frac{\lambda}{2\pi r}\hat{r}}$$

Inside the rubber insulation, the electric field can't be determined since we don't know $\vec{P}$.

for $r > a$:

By Gauss's law in presence of dielectric,



$$\oint \vec{D} \cdot d\vec{S} = (Q_f)_{enclosed}$$

$$\Rightarrow D \times 2\pi r' L = \lambda L$$

$$\Rightarrow \vec{D} = \frac{\lambda}{2\pi r'} \hat{r}'$$

$$\Rightarrow D = \frac{\lambda}{2\pi r'}$$

$$\Rightarrow \vec{D} = \frac{\lambda}{2\pi r'} \hat{r}'$$

Here, $\vec{P} = 0$,

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$$

$$\Rightarrow \vec{D} = \varepsilon_0 \vec{E}$$

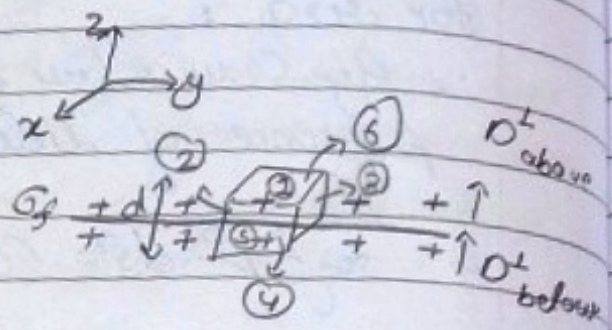
$$\Rightarrow \vec{E} = \frac{1}{\varepsilon_0} \vec{D}$$

$$\Rightarrow \vec{E} = \frac{\lambda}{2\pi \varepsilon_0 r'} \hat{r}'$$

- Boundary Conditions in dielectrics:

1. Boundary Condition for perpendicular/normal component of $\vec{D}$:

The integral form of Gauss's Law in presence of dielectric reads



$$\oint \vec{D}\cdot d\vec{s} = (Q_f)_{enc}$$

$$\int_1 \vec{D}\cdot d\vec{s} + \int_2 \vec{D}\cdot d\vec{s} + \int_3 \vec{D}\cdot d\vec{s} + \int_4 \vec{D}\cdot d\vec{s} + \int_5 \vec{D}\cdot d\vec{s} + \int_6 \vec{D}\cdot d\vec{s} = \int_S \sigma_f \, dS$$

⑤, ⑥ → front, back

As $d \to 0$, $d\vec{s} \Rightarrow 0$ for surfaces 3, 4, 5 and 6.
So, we can write

$$\int_1 \vec{D}\cdot d\vec{s} + \int_2 \vec{D}\cdot d\vec{s} = \sigma_f A$$

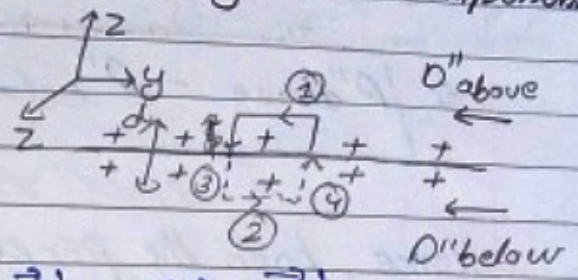
$$\Rightarrow \int_1 D^{\perp}_{above}\, \hat{z}\cdot\hat{z}\, dS + \int_2 D^{\perp}_{below}\, \hat{z}\cdot(-\hat{z})\, dS = \sigma_f A$$

$$\Rightarrow D^{\perp}_{above} \int_1 dS - D^{\perp}_{below} \int_2 dS = \sigma_f A$$

$$\Rightarrow D^{\perp}_{above} A - D^{\perp}_{below} A = \sigma_f A$$

$$\Rightarrow \boxed{D^{\perp}_{above} - D^{\perp}_{below} = \sigma_f}$$

Boundary Condition for parallel/tangential components of $\vec{D}$:



We know that
$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$

$$\Rightarrow \vec{\nabla} \times \vec{D} = \varepsilon_0 (\vec{\nabla} \times \vec{E}) + (\vec{\nabla} \times \vec{P})$$

[Taking curl on both sides]

$$\Rightarrow \vec{\nabla} \times \vec{D} = \vec{\nabla} \times \vec{P} \quad [\because \vec{\nabla} \times \vec{E} = 0]$$

Taking surface integral of both sides

$$\int_S \vec{\nabla} \times \vec{D} \cdot d\vec{S} = \int_S \vec{\nabla} \times \vec{P} \cdot d\vec{S}$$

Using Stokes's theorem,

$$\Rightarrow \oint_C \vec{D} \cdot d\vec{l} + \oint_C \vec{P} \cdot d\vec{l}$$

$$\Rightarrow \int_1 \vec{D} \cdot d\vec{l} + \int_2 \vec{D} \cdot d\vec{l} + \int_3 \vec{D} \cdot d\vec{l} + \int_4 \vec{D} \cdot d\vec{l} = \int_1 \vec{P} \cdot d\vec{l} + \int_2 \vec{P} \cdot d\vec{l} + \int_3 \vec{P} \cdot d\vec{l} + \int_4 \vec{P} \cdot d\vec{l}$$

As $d \to 0$, $dl \to 0$ for paths ③ & ④

$$\therefore \int_1 \vec{D} \cdot d\vec{l} + \int_2 \vec{D} \cdot d\vec{l} = \int_1 \vec{P} \cdot d\vec{l} + \int_2 \vec{P} \cdot d\vec{l}$$

$$\Rightarrow \int_1 D''_{above} (-\hat{y}) \cdot (-\hat{y})\, dl + \int_2 D''_{below} (-\hat{y}) \cdot (\hat{y})\, dl = \int_1 P''_{above} (-\hat{y}) \cdot (-\hat{y})\, dl + \int_2 P''_{below} (-\hat{y}) \cdot (-\hat{y})\, dl$$

$\Rightarrow D''_{above\,L} - D''_{below\,L} = P''_{above\,L} - P''_{below\,L}$

$$\Rightarrow \boxed{D''_{above} - D''_{below} = P''_{above} - P''_{below}}$$

Hence, both the perpendicular and parallel components of $\vec{D}$ are DISCONTINUOUS at the boundary.

- Linear Dielectrics :

for linear dielectrics,

$$\vec{P} \propto \vec{E}$$

$$\Rightarrow \vec{P} = \varepsilon_0 \chi_e \vec{E}$$

Then, $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$

$$= \varepsilon_0 \vec{E} + \varepsilon_0 \chi_e \vec{E}$$

$$\cancel{= \varepsilon_0 \vec{E}}$$

$$= \varepsilon_0 (1+\chi_e)\vec{E}$$

$$\vec{D} = \varepsilon \vec{E}$$

where $\varepsilon = \varepsilon_0 (1+\chi_e)$, is called permittivity of the material.

* In vacuum, where there is no matter to ~~plat~~ polarize, the electric susceptibility is zero, and the permittivity is ε_0. That's why ε_0 is called permittivity of free space.

And $\varepsilon_r = \frac{\varepsilon}{\varepsilon_0} = 1 + \chi_e$, is called RELATIVE PERMITTIVITY or DIELECTRIC CONSTANT of the material.

Date / 5 / 1

Energy In Dielectrics:

We know that the energy stored in an electrostatic system is given by :-

$$E = \frac{\varepsilon_0}{2} \int_{\text{all space}} E^2 d\tau$$

Then, energy of any dielectric system is

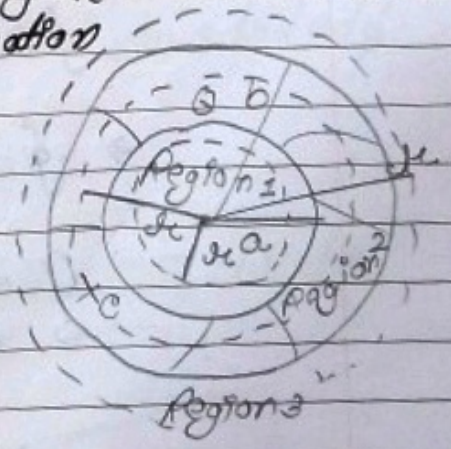
$$E = \frac{\varepsilon}{2} \int_{\text{all space}} E^2 d\tau$$

$$= \frac{\varepsilon}{2} \int_{\text{all space}} \vec{E} \cdot \vec{E} \, d\tau$$

$$= \frac{1}{2} \int_{\text{all space}} \varepsilon \, \vec{E} \cdot \vec{E} \, d\tau$$

$$\boxed{E = \frac{1}{2} \int_{\text{all space}} \vec{D} \cdot \vec{E} \, d\tau}$$, for linear dielectrics.

Q A spherical conductor of radius 'a', carrying a charge Q. It is surrounded by linear dielectric material of susceptibility χ_e and radius 'b'. Find the energy of this configuration.



We know that

$$E = \frac{1}{2} \int_{\text{all space}} \vec{D} \cdot \vec{E} \, d\tau$$

for region I $(r < a)$

Charge enclosed $(Q_f)_{enc} = 0$

Then, $\oint_S \vec{D} \cdot d\vec{S} = (Q_f)_{enc}$

$$\Rightarrow \vec{D} = 0$$

Since $\vec{D} = \varepsilon \vec{E}$ for linear dielectrics
$\Rightarrow \vec{E} = 0$

for region 2 $(a < r < b)$

$$\oint_S \vec{D} \cdot d\vec{S} = (Q_f)_{enc}$$

$$\Rightarrow \oint D \hat{n} \cdot \hat{n} \, dS = Q$$

$$\Rightarrow D \times 4\pi r^2 = Q$$

$$D = \frac{Q}{4\pi r^2}$$

$$\Rightarrow \vec{D} = \frac{Q}{4\pi r^2} \hat{r}$$

Since, $\vec{D} = \varepsilon \vec{E}$

$$\Rightarrow \vec{E} = \frac{1}{\varepsilon} \vec{D} = \frac{Q}{4\pi \varepsilon r^2} \hat{r}$$

for region ③ $(r > b)$

$$\oint \vec{D} \cdot d\vec{S} = (Q_f)_{enc}$$

$$\Rightarrow \oint \vec{D} \hat{n} \cdot \hat{n} \, dS = Q$$

$$D \times 4\pi r^2 = Q$$

$$\Rightarrow D = \frac{Q}{4\pi r^2}$$

$$\Rightarrow \vec{D} = \frac{Q}{4\pi r^2} \hat{r}$$

Then for $r > b$, $\chi_e = 0$

Then $\vec{D} = \varepsilon \vec{E} = \varepsilon_0 (1 + \chi_e) \vec{E}$

$$\vec{D} = \varepsilon_0 \vec{E}$$

$$\vec{E} = \frac{1}{\varepsilon_0} \vec{D}$$

$$\vec{E} = \frac{Q}{4\pi \varepsilon_0 r^2} \hat{r}$$

$\therefore$ The required energy

$$= \frac{1}{2} \int_{\text{all space}} \vec{D} \cdot \vec{E} \, d\tau$$

$$= \frac{1}{2} \int_{r<0} \cancel{\vec{D} \cdot \vec{E} \, d\tau}^{\,0} + \frac{1}{2} \int_{a<r<b} \vec{D} \cdot \vec{E} \, d\tau + \frac{1}{2} \int_{r>b} \vec{D} \cdot \vec{E} \, d\tau$$

$$= \frac{1}{2} \int_a^b \int_0^{\pi} \int_0^{2\pi} \frac{Q}{4\pi r^2} \hat{r} \cdot \frac{Q}{4\pi \varepsilon r^2} \hat{r} \, r^2 \sin\theta \, dr \, d\theta \, d\phi +$$

$$= \frac{1}{2} \iiint \frac{Q}{4\pi r^2} \hat{r} \cdot \frac{Q}{4\pi \varepsilon_0 r^2} \hat{r} \, r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \frac{1}{2} \times \frac{Q^2}{16\pi^2} \left[\frac{1}{\varepsilon} + \frac{1}{\varepsilon_0} \right] \int_0^{\pi} \int_a^b \frac{2\pi \sin\theta}{r^2} \, dr \, d\theta$$

$$+ \int_b^{\infty} \int_0^{\pi} \frac{2\pi \sin\theta \, dr \, d\theta}{r^2}$$

$$= \frac{1}{32\pi^2} Q^2 \left[\frac{1}{\varepsilon} + \frac{1}{\varepsilon_0}\right] \left[\int_a^b \frac{4\pi}{r^2}\, dr + \int_b^\infty \frac{4\pi}{r^2}\, dr\right]$$

$$= \frac{Q^2}{32\pi^2} \left[\frac{1}{\varepsilon} + \frac{1}{\varepsilon_0}\right] \times 4\pi \left(\left[\frac{-1}{r}\right]_a^b + \left[\frac{-1}{r}\right]_b^\infty\right)$$

$$= \frac{Q^2}{8\pi} \left[\frac{1}{\varepsilon} + \frac{1}{\varepsilon_0}\right] \left[\left[\frac{1}{a} - \frac{1}{b}\right] + \frac{1}{b}\right]$$

$$= \frac{Q^2}{8\pi a} \frac{[\varepsilon_0 + \varepsilon]}{\varepsilon\varepsilon_0} \quad \frac{Q^2}{8\pi a} \frac{[\varepsilon_0 + \varepsilon_0(1+\chi_e)]}{\varepsilon_0(1+\chi_e)\varepsilon_0}$$

$$= Q^2 \qquad \frac{Q^2}{8\pi a} \frac{(2+\chi_e)}{\varepsilon_0(1+\chi_e)}$$

$$= \frac{Q^2}{32\pi^2\varepsilon} \times 4\pi \left[\frac{1}{a} - \frac{1}{b}\right] + \frac{Q^2}{32\pi^2\varepsilon_0} \times 4\pi \left[\frac{1}{b}\right]$$

$$= \frac{Q^2}{8\pi\varepsilon} \left[\frac{1}{a} - \frac{1}{b}\right] + \frac{Q^2}{8\pi\varepsilon_0} \left[\frac{1}{b}\right]$$

$$= \frac{Q^2}{8\pi\varepsilon_0} \left[\frac{1}{a} - \frac{1}{b}\right] \times \frac{1}{1+\chi_e} + \frac{Q^2}{8\pi\varepsilon_0} \left[\frac{1}{b}\right]$$

$$= \frac{Q^2}{8\pi\varepsilon_0} \left(\left[\frac{1}{1+\chi_e}\right] \left[\frac{1}{a} - \frac{1}{b}\right] + \frac{1}{b}\right]$$

$$= \frac{Q^2}{8\pi\varepsilon_0} \left(\frac{1}{a(1+\chi_e)} + \frac{1}{b}\left[1 - \frac{1}{1+\chi_e}\right]\right)$$

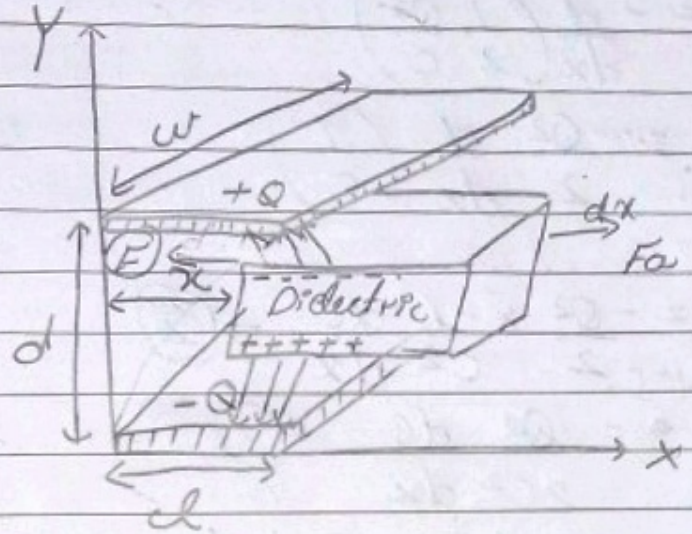
$$= \frac{Q^2}{8\pi\varepsilon_0} \left(\frac{1}{a(1+\chi_e)} + \frac{1}{b}\frac{(\chi_e)}{1+\chi_e}\right)$$

$$= \frac{Q^2}{8\pi\varepsilon_0} \left(\frac{1}{ab}\right) = \frac{Q^2}{8\pi\varepsilon_0(1+\chi_e)} \left[\frac{1}{a} + \frac{\chi_e}{b}\right]$$

$$= \frac{Q^2}{}$$

Forces on dielectrics:

Consider a slab of linear dielectric material, partially inserted between the plates of a parallel plate capacitor.



Then, there is a fringing field around the edges of the capacitor. This non-uniform fringing field is responsible for the force on the dielectric.

Let w be the energy of the system. If the dielectric is pulled out by an infinitesimal distance dx, the energy is changed by an ~~equal~~ amount equal to the work done

$$dw = F_a\,dx \quad - (1)$$

where F_a is the applied force to counteract the electric force F on the dielectric.

i.e. $F_a = -F$.

$\therefore$ The electrical force on the dielectric slab is

$$F = -F_a$$

$$= -\frac{dw}{dx} \quad [\text{Using eqn. } 1]$$

We know that the energy stored in the capacitor is

$$w = \frac{1}{2} CV^2$$

$= \frac{1}{2}\frac{Q^2}{C}$, assuming that the total charge on the plates is held constant, as the dielectric moves.

Then,

$$F = -\frac{d}{dx}\left(\frac{1}{2}\frac{Q^2}{C}\right)$$

$$= \frac{-Q^2}{2}\frac{d}{dx}\left(\frac{1}{C}\right)$$

$$= \frac{-Q^2}{2} \times \frac{-1}{C^2}\frac{dC}{dx} \quad -(2)$$

$$= \frac{Q^2}{2C^2}\frac{dC}{dx} \quad -(2)$$

Now, the capacitance C of the given system is

$$C = \frac{\varepsilon_0 w x}{d} + \frac{\varepsilon w(d-x)}{d}$$

$$= \frac{\varepsilon_0 w x}{d} + \frac{\varepsilon_0(1+\chi_e)w(d-x)}{d}$$

$$= \frac{\varepsilon_0 w}{d}\left(x + (1+\chi_e)(d-x)\right)$$

$$= \frac{\varepsilon_0 w}{d}\left[x + (1+\chi_e)d - (1+\chi_e)x\right]$$

$$= \frac{\varepsilon_0 w}{d}\left[x + \varepsilon_r d - x - \chi_e x\right]$$

$$= \frac{\varepsilon_0 w}{d}\left[\varepsilon_r d - \chi_e x\right] \quad -(3)$$

Using eqn (3) in (2), we get

$$F = \frac{Q^2}{2C^2}\frac{d}{dx}\left[\frac{\varepsilon_0 w}{d}\left(\varepsilon_r d - \chi_e x\right)\right]$$

$$= -\frac{Q^2}{2C^2} \times \frac{\varepsilon_0 w}{d} \times \chi_e$$

$$= -\frac{\varepsilon_0 \chi_e w}{2d} \times \frac{Q^2}{C^2}$$

$$F = -\frac{\varepsilon_0 \chi_e w V^2}{2d} \quad \left[\because V = \frac{Q}{C}\right]$$

The minus sign indicates that the force is in the NEGATIVE x-direction, i.e., the dielectric is pulled into the capacitor.