Tangent planes and normal

Equation of tangent plane is given by

$$(x-x_1)\frac{\partial F}{\partial x} + (y-y_1)\frac{\partial F}{\partial y} + (z-z_1)\frac{\partial F}{\partial z} = 0$$

and the equation of the normal to the plane is given by

$$\frac{x-x_1}{\frac{\partial F}{\partial x}} = \frac{y-y_1}{\frac{\partial F}{\partial x}} = \frac{z-z_1}{\frac{\partial F}{\partial z}}.$$

Ex. Find the equation of the tangent plane and normal line to the surface

$$x^2+2y^2+3z^2=12 \text{ at } (1,2,-1)$$

Sol: $F(x,y,z) = x^2+2y^2+3z^2-12.$

$$\frac{\partial F}{\partial x} = 2x \quad \frac{\partial F}{\partial y} = 4y \quad \frac{\partial F}{\partial z} = 6z$$

$\therefore$ At the point $(1,2,-1)$

$$\frac{\partial F}{\partial x} = 2 \quad \frac{\partial F}{\partial y} = 8 \quad \frac{\partial F}{\partial z} = -6$$

Hence the eq. of the tangent plane at $(1, 2, -1)$ is

$$2(x-1) + 8(y-2) - 6(z+1) = 0$$

$$2x - 2 + 8y - 16 - 6z - 6 = 0.$$

$$2x + 8y - 6z = 24$$

$$\Rightarrow \quad x + 4y - 3z = 12$$

$\therefore$ Equation of normal is

$$\frac{x-1}{2} = \frac{y-2}{8} = \frac{z+1}{6}$$

Ex. Find the eq. of the ~~plane~~ tangent plane and the normal to the surface

$$xyz = 6 \text{ at } (1, 2, 3)$$

Sol: $F(x, y, z) = xyz$.

$$\frac{\partial F}{\partial x} = yz \quad \frac{\partial F}{\partial y} = xz \quad \frac{\partial F}{\partial z} = xy.$$

At the points $(1, 2, 3)$

$$\frac{\partial f}{\partial x} = 6 \quad \frac{\partial F}{\partial y} = 3 \quad \frac{\partial F}{\partial z} = 2.$$

Hence eq. of the tangent plane at $(1,2,3)$ is

$$6(x-1)+3(y-2)+2(z-3)=0.$$

$$6x-6+3y-6+2z-6=0.$$

$$6x+3y+2z=18.$$

$\therefore$ Equation of normal is

$$\frac{x-1}{6}=\frac{y-2}{3}=\frac{z-3}{2}.$$ //

Ex. Show that the surface $x^2-2yz+y^3=4$ is perpendicular to any number of the family of surfaces $x^2+1=(2-4a)y^2+az^2$ at the pt. of intersection $(1,-1,2)$.

Sol: Given $f(x,y,z)=x^2-2yz+y^3-4=0$ — ①

$$F(x,y,z)=x^2+1-(2-4a)y^2-az^2=0 \quad -②$$

$\therefore \frac{\partial f}{\partial x}=2x \quad \frac{\partial f}{\partial y}=-2z+3y^2, \quad \frac{\partial f}{\partial z}=-2y.$

Direction ratios to the normal of the tangent plane to (1) are

$2x, -2z+3y^2, -2y.$

Direction ratio at the pt. $(1,-1,2)$ are

$2\times1, -2\times2+3\times1, +2\times1$

$= 2, -1, +2$

Now differentiating (2) ~~w.r.t.~~

$\frac{\partial F}{\partial x} = 2x$, $\frac{\partial F}{\partial y} = -2(2-4a)y$, $\frac{\partial F}{\partial z} = -2az$

$\therefore$ Direction ratio at the pt. $(1,-1,2)$ are

$2, 4-8a, -4a$

Now to be perpendicular we need to show

$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$.

$\therefore = 2.2 + -1.(4-8a) + (+2)(-4a) = 0$.

$= \cancel{4} - \cancel{4} + \cancel{8a} - \cancel{8a}$

$= 0$.

Hence the given surface are perpendicular at $(1,-1,2)$.

JACOBIANS

If u and v functions of the two independent variables x and y, then the determinant

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

is called the Jacobian of u, v w.r.t. x, y and is written as $\frac{\partial(u,v)}{\partial(x,y)}$ or $J\left(\frac{u,v}{x,y}\right)$

Ily the Jacobian of u, v, w w.r.t. x, y, z is

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}.$$

Ex. If $x = r\cos\theta$, $y = r\sin\theta$ evaluate

$$\frac{\partial(x,y)}{\partial(r,\theta)} \quad \text{and} \quad \frac{\partial(r,\theta)}{\partial(x,y)}$$

Sol: We have, $x = r\cos\theta$ $\qquad y = r\sin\theta$

$$\frac{\partial x}{\partial r} = \cos\theta \qquad \frac{\partial y}{\partial r} = \sin\theta$$

$$\frac{\partial x}{\partial \theta} = -r\sin\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta.$$

$$\therefore \frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r.$$

Now $r^2 = x^2 + y^2$ and $\cancel{\frac{x}{y} = \frac{\sin\theta}{\cos\theta}}$ $\quad \frac{y}{x} = \frac{\sin\theta}{\cos\theta} \Rightarrow \theta = \tan^{-1}\frac{y}{x}$

Now $2r\frac{\partial r}{\partial x} = 2x$, $\qquad \frac{\partial \theta}{\partial x} = \frac{1}{1-\frac{y^2}{x^2}} \cdot \frac{\partial}{\partial x}\left(\frac{y}{x}\right)$

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

$$= \frac{x^2}{x^2-y^2} \cdot -yx^{-2}$$

$$= \frac{x^2}{x^2-y^2} \cdot \frac{-y}{x^2}$$

$$\frac{\partial \theta}{\partial x} = \frac{-y}{x^2-y^2} = \frac{-y}{r^2}$$

$$\frac{\partial r}{\partial y} = \frac{y}{r} \qquad \frac{\partial \theta}{\partial y} = \frac{x}{x^2+y^2} = \frac{x}{r^2}$$

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{x}{r} & \frac{y}{r} \\ \frac{-y}{r^2} & \frac{x}{r^2} \end{vmatrix}$$

$$= \frac{x^2}{r^3} + \frac{y^2}{r^3} = \frac{x^2+y^2}{r^3} = \frac{r^2}{r^3} = \frac{1}{r}.$$

Ex. If $y_1 = \frac{x_2 x_3}{x_1}$, $y_2 = \frac{x_3 x_1}{x_2}$, $y_3 = \frac{x_1 x_2}{x_3}$.

Show that the Jacobian of ~~$y_1 y_2$~~ y_1, y_2, y_3 w.r.t. x_1, x_2, x_3 is 4.

Sol: Here, we have $y_1 = \frac{x_2 x_3}{x_1}$, $y_2 = \frac{x_3 x_1}{x_2}$, $y_3 = \frac{x_1 x_2}{x_3}$

$$\frac{\partial(y_1,y_2,y_3)}{\partial(x_1,x_2,x_3)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \frac{\partial y_1}{\partial x_3} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \frac{\partial y_2}{\partial x_3} \\ \frac{\partial y_3}{\partial x_1} & \frac{\partial y_3}{\partial x_2} & \frac{\partial y_3}{\partial x_3} \end{vmatrix} = \begin{vmatrix} -\frac{x_2 x_3}{x_1^2} & \frac{x_3}{x_1} & \frac{x_2}{x_1} \\ \frac{x_3}{x_2} & -\frac{x_3 x_1}{x_2^2} & \frac{x_1}{x_2} \\ \frac{x_2}{x_3} & \frac{x_1}{x_3} & -\frac{x_1 x_2}{x_3^2} \end{vmatrix}$$

$$= \frac{1}{x_1^2 x_2^2 x_3^2} \begin{vmatrix} -x_2x_3 & x_3x_1 & x_1x_2 \\ x_2x_3 & -x_3x_1 & x_1x_2 \\ x_2x_3 & x_3x_1 & -x_1x_2 \end{vmatrix}$$

$$= \frac{x_1^2 \cancel{x_2^2} \cancel{x_3^2}}{x_1^2 \cancel{x_2^2} x_3^2} \{ 4 \} = \frac{4 x_1^2 \cancel{x_2^2} x_3^2}{x_1^2 x_2^2 \cancel{x_3^2}} = 4.$$

Ex. If $u = \dfrac{x}{y-z}$, $v = \dfrac{y}{z-x}$, $w = \dfrac{z}{x-y}$ show that

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0.$$

Ex. If $x = r\cos\theta$, $y = r\sin\theta$, $z = z$

find $\dfrac{\partial(x,y,z)}{\partial(r,\theta,z)}$.

Properties of Jacobian.

If u and v are functions of x and y, then

$$\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = 1 \quad - (1)$$

Ex. If $u = xyz$, $v = x^2+y^2+z^2$, $w = x+y+z$,

Find $J = \dfrac{\partial(x,y,z)}{\partial(u,v,w)}$.

Sol: We have, $J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$

$$= \begin{vmatrix} yz & zx & xy \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= yz(2y-2z) - zx(2x-2z) + xy(2x-2y)$$

$$= 2y^2z - 2yz^2 - 2x^2z + 2xz^2 + 2x^2y - 2xy^2$$

$$= 2[x^2(y-z) - x(y^2-z^2) + yz(y-z)]$$

$$= 2(y-z)[x^2 - x(y+z) + yz]$$

$$= 2(y-z)[x^2 - xy - xz + yz]$$

$$= 2(y-z)[y(z-x) - x(z-x)]$$

$$= 2(y-z)(z-x)(y-x) \Rightarrow -2(x-y)(y-z)(z-x)$$

Hence, by $JJ' = 1$.

$$J' = \frac{1}{J}$$

$$= \frac{-1}{2(x-y)(y-z)(z-x)}$$

Ex: If $u = x^2 - y^2$, $v = 2xy$ Calculate $\frac{\partial(x,y)}{\partial(u,v)}$

Ex: If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$,

$z = r\cos\theta$, Find $\frac{\partial(r,\theta,\phi)}{\partial(x,y,z)}$.

(Hint: Ans: $\frac{1}{r^2\sin\theta}$)

Second property:

If u, v are the functions of r, s where r, s are functions of x, y then

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \times \frac{\partial(r,s)}{\partial(x,y)}$$

Ex. Find the value of the Jacobian $\frac{\partial(u,v)}{\partial(r,\theta)}$, where $u = x^2 - y^2$, $v = 2xy$ and $x = r\cos\theta$, $y = r\sin\theta$.

Sol÷ $$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} = 4(x^2+y^2) = 4r^2$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r.$$

$$\therefore \frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(r,\theta)}$$

$$= 4r^2 \cdot r = 4r^3$$

Ex. If $u^3 + v^3 = x + y$, $u^2 + v^2 = x^3 + y^3$, show that

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{2} \frac{(y^2 - x^2)}{uv(u-v)}.$$

Third property:

If functions u, v, w of three independent variables x, y, z are not independent, then

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$$

Ex. Verify whether the given functions are functionally dependent.

~~Sol~~: $u = \frac{x+y}{1-xy}$ $\quad v = \tan^{-1}x + \tan^{-1}y$.

Sol: $$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix}$$

$$= \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2} = 0.$$

Hence u, v are functionally related.

$$\therefore \tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}$$

$$v = \tan^{-1}u$$

$$u = \tan v.$$

Try yourself:

Ex. If $x^2+y^2+u^2-v^2=0$, and $uv+xy=0$, prove that $\dfrac{\partial(u,v)}{\partial(x,y)} = \dfrac{x^2-y^2}{u^2+v^2}$

Ex: If $u^3+v^3=x+y$, $u^2+v^2=x^3+y^3$, then prove that $\dfrac{\partial(u,v)}{\partial(x,y)} = \dfrac{1}{2}\,\dfrac{y^2-x^2}{2uv(u-v)}$

* Taylor series of two variables

If $f(x,y)$ and all its partial derivatives upto the n^{th} order are finite and continuous for all points (x,y) where $a \le x \le a+h$, $b \le y \le b+k$, then

$$f(a+h,b+k) = f(a,b) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f + \frac{1}{3!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f + \cdots$$

Ex. Expand $e^x \cdot \sin y$ in powers of x and y, $x=0$, $y=0$ as far as terms of third degree.

Sol: We have $f(x,y) = e^x \cdot \sin y$. at $x=0$ $y=0$

$$f(0,0) = e^0 \cdot \sin 0 = 0.$$

		$x=0,\ y=0$
$\therefore\ f(x,y)$	$e^x \sin y$	$0.$
$f_x(x,y)$	$e^x \cdot 1 . \sin y$	$0.$
$f_y(x,y)$	$e^x . \cos y$	$1.$
$f_{xx}(x,y)$	$e^x . 1 . 1 . \sin y$	$0.$
$f_{xy}(x,y)$	$e^x . 1 . \cos y$	$1.$
$f_{yy}(x,y)$	$-e^x \sin y.$	$0.$
$f_{xxx}(x,y)$	$e^x . 1 . 1 . 1 \sin y$	$0.$
$f_{xxy}(x,y)$	$e^x \cos y$	$1.$
$f_{xyy}(x,y)$	$-e^x \sin y.$	$0.$
$f_{yyy}(x,y)$	$-e^x \cos y$	$-1.$

By Taylor's theorem,

$$f(x,y) = f(0,0) + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) f(0,0) + \frac{1}{2!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2$$

$$f(0,0) + \frac{1}{3!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^3 f(0,0) + \dots$$

$$= f(0,0) + x f_x(0,0) + y f_y(0,0) + \frac{x^2}{2!} f_{xx}(0,0) +$$

$$+ \frac{2xy}{2!} f_{xy}(0,0) + \frac{y^2}{2!} f_{yy}(0,0) + \frac{1}{3!} x^3 f_{xxx}(0,0) +$$

$$\frac{3x^2y}{3!} f_{xxy}(0,0) + \frac{3xy^2}{3!} f_{xyy}(0,0) + \frac{1}{3!} y^3 f_{yyy}(0,0) + \dots$$

$$e^x \sin y = 0 + x(0) + y(1) + \frac{x^2}{2}(0) + xy(1) + \frac{y^2}{2}(0) +$$

$$\frac{x^3}{6}(0) + \frac{3x^2y}{6}(1) + \frac{3xy^2}{6}(0) + \frac{y^3}{6}(-1) + \dots$$

$$= y + xy + \frac{x^2y}{2} - \frac{y^3}{6} + \dots$$

Ex. Find the expansion for $\cos x \cos y$ in powers of x, y upto fourth order terms.

Ex. Find the first six terms of the expansion of the function $e^x \log(1+y)$ in a Taylor's series in the neighbourhood of the point $(0,0)$

Sol: Given.

~~x=0, y=0.~~

$$f(x) = e^x \log(1+y)$$

		$x=0,\ y=0.$
$f(x,y)$	$e^x \log(1+y)$	$0.$
$\frac{\partial f}{\partial x}$	$e^x \log(1+y)$	$0.$
$\frac{\partial f}{\partial y}$	$\frac{e^x}{(1+y)}$	$1.$
$\frac{\partial^2 f}{\partial x^2}$	$e^x \log(1+y)$	$0.$
$\frac{\partial^2 f}{\partial y^2}$	$-\frac{e^x}{(1+y)^2}$	$-1.$
$\frac{\partial^2 f}{\partial x \partial y}$	$\frac{e^x}{(1+y)} \cdot 1$	$1.$

$\therefore$ Taylor series is given by

$$f(x,y) = f(0,0) + \left(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\right) \cancel{f(0,0)} +$$

$$\frac{1}{2!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 f(0,0) + \ldots$$

$$= f(0,0) + \left(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\right) \cancel{f(0,0)}) +$$

$$\frac{1}{2!}\left(x^2\frac{\partial^2 f}{\partial x} + 2xy\frac{\partial f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2}\right) + \ldots$$

$$= 0 + x.0 + y.1 + \frac{1}{2!}\left[(x^2.0 + 2xy.1 + y^2(-1))\right]$$

$$e^x \log(1+y) = y + \frac{2xy}{2!} - \frac{y^2}{2}$$

$$e^x \log(1+y) = y + xy - \frac{y^2}{2}.$$

Ex: Expand $e^x \cos y$ at $(0,0)$ upto three term.

Ans: $1 + x + \frac{1}{2}(x^2 - y^2) + \ldots$