The particular Integral (P.I)

case 1. To find the P.I, when Q is of the form e^{ax}

Ex. $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{5x}$

$\Rightarrow (D^2 - 3D + 2)y = e^{5x}$ where $d/dx = D$.

Here $F(D) = D^2 - 3D + 2$ and $Q = e^{5x}$

$\therefore$ The auxiliary eq. is $m^2 - 3m + 2 = 0$.

$(m-1)(m-2) = 0$.

$m = 1$, or $m = 2$.

$\therefore$ Roots of the auxiliary eq. are real and unequal.

$\therefore y = c_1 e^{1x} + c_2 e^{2x}$.

$\therefore P.I = \frac{1}{F(D)} Q$

$= \frac{1}{D^2 - 3D + 2} \cdot e^{5x}$.

Working Rule: If $P.I = \frac{1}{F(D)} e^{ax}$ then put a for D in $F(D)$ and we get the P.I provided $F(a) \neq 0$.

$\therefore P.I = \frac{1}{5^2 - 3.5 + 2} e^{5x} = \frac{e^{5x}}{12}$.

$\therefore$ The Complete Solution is

$y = C.F + P.I$

$= C_1 e^{x} + C_2 e^{2x} + \frac{1}{12} e^{5x}$

Ex. $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 6y = e^{2x}$ and determine the constant so that $y=0$ when $x=0$.

$D^2 - 7D + 6 = e^{2x}$

Here $F(D) = D^2 - 7D + 6$ $D = k$

$\therefore$ The auxiliary eq. is

$m^2 - 7m + 6 = 0.$

$m^2 - (6+1)m + 6 = 0$

$m^2 - 6m - 1m + 1 = 0$

$m(m-6) - 1(m-6) = 0$

$m - 6 = 0$ or $m - 1 = 0$

$m = 6$ $m = 1$.

$\therefore$ The Roots are real and unequal.

$\therefore C.F = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$C.F = c_1 e^{x} + c_2 e^{6x}$

$P.I = \frac{1}{F(D)} e^{2x} = \frac{1}{2^2 - 7.2 + 6} e^{2x} = -\frac{e^{2x}}{4}$

$\therefore$ complete Solution is

$y = C.F + P.I$

$y = c_1 e^{x} + c_2 e^{6x} + \frac{e^{2x}}{4}$ — (1)

Now when $y = 0$, $x = 0$.

$0 = c_1 e^0 + c_2 e^0 - \frac{e^0}{4}$

$= c_1 + c_2 - \frac{1}{4}$

$c_2 = \left(\frac{1}{4} - c_1\right)$

$\therefore$ From (1)

$y = c_1 e^{x} + \left(\frac{1}{4} - c_1\right) e^{6x} - \frac{e^{2x}}{4}$

$= c_1 (e^{x} - e^{6x}) + \frac{1}{4} e^{2x} (e^{4x} - 1)$

Case II To find P.I when θ is of the form $\sin ax$ or $\cos ax$ and $F(-a^2) \neq 0$.

Working Rule: If $P.I = \left\{\frac{1}{F(D)}\right\} \underline{\sin ax}$ or $\underline{\cos ax}$, put $-a^2$ for D^2, $-a^2 D$ for D^3, $(-a^2)^2$ for D^4, $a^4 D$ for D^5,

Ex. $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = \sin 2x.$

$(D^2 - D - 2)y = \sin 2x.$

$F(D) = D^2 - D - 2 \quad \theta = \sin 2x$

$\therefore$ Auxiliary eq is

$C.F \Rightarrow m^2 - m - 2 = 0.$

$(m+1)(m-2) = 0.$

$m = 1, 2.$

Roots are real and unequal

$C.F = c_1 e^x + c_2 e^{2x}.$

Now $P.I = \frac{1}{F(D)} \theta \Rightarrow \frac{1}{D^2 - D - 2} \sin 2x$

$\Rightarrow \frac{1}{-(2)^2 - D - 2} \sin 2x.$

$\Rightarrow \frac{1}{-4 - D - 2} \sin 2x.$

$\Rightarrow -\frac{1}{(D+6)} \sin 2x. \Rightarrow -\frac{D-6}{(D+6)(D-6)} \sin 2x.$

$\Rightarrow -\frac{D-6}{D^2 - 36} \sin 2x \Rightarrow \frac{-D-6}{-4-36} \sin 2x \Rightarrow +\frac{1}{40} (D-6) \sin 2x$

$\Rightarrow \frac{1}{40} \{ D \sin 2x - 6 \sin 2x \}$ (Here D implies differentiation).

$\Rightarrow \frac{1}{40} \{ \cos 2x \cdot 2 - 6 \sin 2x \} = \frac{1}{20} \cos 2x - \frac{3}{20} \sin 2x.$

Case III To find P.I when Q is of the form x^m, m is +ve

Working Rule: In order to evaluate $\{1/F(D)\}x^m$, bring out common the lowest degree term in D from F(D) so that remaining factor in the denominator is of the form [1+F(D)] or [1-F(D)] which is taken in numerator form as $[1 \pm F(D)]$ in powers of D by the binomial theorem and operate upon x^m with the expansion obtained.

The following binomial expansion should be remembered.

(i) $(1-x)^{-1} = 1+x+x^2+x^3+x^4 \ldots$

(ii) $(1+x)^{-1} = 1-x+x^2-x^3+x^4 \ldots$

(iii) $(1-x)^{-2} = 1+2x+3x^2+4x^3 \ldots$

(iv) $(1+x)^{-2} = 1-2x+3x^2-4x^3 \ldots$

Ex. $\frac{d^2y}{dx^2} - 4y = x^2$

$(D^2-4)y = x^2$

$F(D) = D^2-4 \quad Q = x^2$

$\therefore$ The auxiliary eq is

$m^2-4=0$

$m^2 = 4$

$m = \sqrt{4} = \pm 2$

$\therefore$ The roots are real and unequal

$C.F = e^{mx}(c_1 + c_2 x)$

$= c_1 e^{2x} + c_2 e^{-2x}$

$$P.I = \frac{1}{D^2-4}x^2 = \frac{1}{-4\left[1-\frac{1}{4}D^2\right]}x^2 = -\frac{1}{4}\left[1-\frac{1}{4}D^2\right]^{-1}x^2.$$

$$= -\frac{1}{4}\left[1+\frac{1}{4}D^2+\ldots\right]x^2$$

$$= -\frac{1}{4}\left[x^2+\frac{1}{4}D^2(x^2)\right]$$ because all the term vanishes

$$= -\frac{1}{4}\left[x^2+\frac{1}{4}D.D(x^2)\right] = -\frac{1}{4}\left[x^2+\frac{1}{4}D(2x)\right]$$

$$= -\frac{1}{4}\left[x^2+\frac{1}{4}.2\right] = -\frac{1}{4}\left[x^2+\frac{1}{2}\right].$$

$\therefore$ The Complete Sol is $y = C.F + P.I$

$$y = c_1e^{2x}+c_2e^{-2x} + \frac{1}{4}\left[x^2+\frac{1}{2}\right].$$

Ex. $(D^2+D-2)y = x+\sin x$

Auxiliary eq. is

$$m^2+m-2 = 0.$$

$$(m-1)(m+2) = 0$$

$$m = 1, -2.$$

$$\frac{1}{F(D)}\sin ax.$$

$$C.F = c_1e^{x}+c_2e^{-2x}.$$

$$P.I = \frac{1}{D^2+D-2}(x+\sin x) = \frac{1}{D^2+D-2}x + \frac{1}{D^2+D-2}\sin x.$$

(**).

$$= \frac{1}{-2\left(1-\frac{1}{2}D-\frac{1}{2}D^2\right)}x + \frac{1}{-1+D-2}\sin x.$$

$$= -\frac{1}{2}\left[1-\left(\frac{1}{2}D+\frac{1}{2}D^2\right)\right]^{-1}x + \frac{1}{D-3}\sin x.$$

$$= -\frac{1}{2}\left[1+\frac{1}{2}D+\dots\right]x + \frac{(D+3)}{(D-3)(D+3)}\sin x$$

$$= -\frac{1}{2}\left[x+\frac{1}{2}Dx\right] + \frac{D+3}{D^2-3^2}\sin x.$$

$$= -\frac{1}{2}\left[x+\frac{1}{2}Dx\right] + \frac{D+3}{-1-9}\sin x$$

$$= -\frac{1}{2}\left[x+\frac{1}{2}Dx\right] - \frac{1}{10}\{D\sin x - 3\sin x\}.$$

$$= -\frac{1}{2}x + \frac{1}{\cancel{2}4}\cdot 1 + \frac{1}{10}\{\cos x + 3\sin x\}.$$

$$= -\frac{1}{2}x - \frac{1}{4} - \frac{1}{10}\{\cos x + 3\sin x\}.$$

$\therefore$ Complete Solution is

$$y = C.F + P.I$$

$$= c_1 e^{x} + c_2 e^{-2x} - \frac{1}{2}x - \frac{1}{4} - \frac{1}{10}\cos x - \frac{3}{10}\sin x.$$

* Case IV : To Find P.I when $Q = e^{ax}.V$, where V is any function of x.

Working Rule : Replace D by $(D+a)$ and take out e^{ax} before the operator $\frac{1}{F(D)}$. Then determine $\left\{\frac{1}{F(D+a)}\right\}V$ by the previous method.

Ex. $(D^2-2D+1)y=x^2e^{3x}$

the auxiliary eq is

$$m^2-2m+1=0$$

$$(m-1)^2=0$$

$$m=1,1$$

the roots are real and equal

$$C.F=(C_1+C_2x)e^x$$

$$P.I=\frac{1}{D^2-2D+1}x^2e^{3x}=\frac{1}{(D-1)^2}x^2e^{3x}$$

$$=e^{3x}\cdot\frac{1}{(D-1)^2}x^2$$

$$=e^{3x}\frac{1}{\{D+3-1\}^2}x^2$$ [putting D+3 for D and bringing e^{3x} before the operator].

$$=e^{3x}\frac{1}{(D+2)^2}x^2$$

$$=e^{3x}\frac{1}{4(1+\frac{D}{2})^2}x^2=\frac{1}{4}e^{3x}(1+\frac{1}{2}D)^{-2}x^2$$

$$=\frac{1}{4}e^{3x}[1-2\frac{1}{2}D+3\cdot\frac{1}{4}D^2+\dots]x^2$$

$$=\frac{1}{4}e^{3x}[1-D+\frac{3}{4}D^2]x^2$$ all other vanishes

$$=\frac{1}{4}e^{3x}[x^2-D.x^2+\frac{3}{4}D^2(x^2)]$$

$$=\frac{1}{4}e^{3x}\{x^2-2x+\frac{3}{4}2\}=\frac{1}{4}e^{3x}\{x^2-2x+\frac{3}{2}\}.$$

$$\therefore y=(C_1+C_2x)e^x+\frac{1}{4}e^{3x}\{x^2-2x+3/2\}.$$

Ex. $(D^2-2D+5)y = e^{2x}\sin x$

The auxiliary eq^n is

$m^2-2m+5=0.$

$$\therefore \quad m = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{2 \pm \sqrt{4-5.4.1}}{2.1} = \frac{2 \pm \sqrt{4-20}}{2}$$

$$= \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm \sqrt{16}i}{2} = \frac{2 \pm 4i}{2}$$

$$= \frac{\cancel{2}(1 \pm 2i)}{\cancel{2}}$$

$$= 1 \pm 2i$$

The roots are real and Complex

$$C.F = e^{x}(c_1 \cos 2x + c_2 \sin 2x)$$

$$P.I = \frac{1}{D^2-2D+5} e^{2x} \sin x.$$

$$= e^{2x} \frac{1}{D^2-2D+5} \sin x.$$

$$= e^{2x} \frac{1}{(D+2)^2-2(D+2)+5} \sin x \quad [\text{putting } D+2 \text{ for } D]$$

$$= e^{2x} \frac{1}{D^2+4D+4-2D-\cancel{4}+5} \sin x$$

$$= e^{2x} \frac{1}{D^2+2D+5} \sin x$$

$$= e^{2x} \frac{1}{-1^2+2D+5} \sin x.$$

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$$\therefore e^{2x}\frac{1}{2D+4}\sin x = \frac{1}{2}e^{2x}\frac{1}{D+2}\sin x.$$

$$\Rightarrow \frac{1}{2}e^{2x}\cdot\frac{D-2}{(D+2)(D-2)}\sin x$$

$$\Rightarrow \frac{1}{2}e^{2x}\cdot\frac{D-2}{D^2-4}\sin x$$

$$\Rightarrow \frac{1}{2}e^{2x}\cdot\frac{D-2}{-1^2-4}\sin x$$

$$\Rightarrow \frac{1}{2}e^{2x}\frac{D-2}{-5}\sin x$$

$$\Rightarrow -\frac{1}{10}e^{2x}\cdot(D\sin x - 2\sin x)$$

$$\Rightarrow -\frac{1}{10}e^{2x}(\cos x - 2\sin x).$$

$\therefore$ The Complete Solution is

$$y = e^{x}(c_1\cos 2x + c_2\sin x) - \frac{1}{10}e^{2x}(\cos x - 2\sin x)$$

Case V: To find P.I when $Q = e^{ax}$ and $F(a) = 0$.

Now this is of the form $e^{ax}.v$. Therefore put $(D+a)$ for D in the operator.

$$P.I = e^{ax}\cdot\frac{1}{F(D+a)}1.$$

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Ex. $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^x$

$m^2 - 3m + 2 = 0.$

$(m-1)(m-2) = 0 \quad \therefore m = 1, 2.$

$C.F = c_1 e^x + c_2 e^{2x}.$

$P.I = \frac{1}{D^2 - 3D + 2} e^x = \frac{1}{(D-2)(D-1)} e^x$

{Here put D for 1 in the factor D-2 but not for D-1 because by putting 1 for D it becomes 0}

$\therefore \Rightarrow \frac{1}{(1-2)(D-1)} e^x \Rightarrow \frac{1}{-1(D-1)} e^x$

$\Rightarrow -\frac{1}{D-1} e^x \Rightarrow -e^x \frac{1}{(D+1)-1} \cdot 1$

[D stands for differentiation & $\frac{1}{D}$ stands for integration]

$\Rightarrow -e^x \cdot \frac{1}{D} \cdot 1 = -e^x \cdot x = -e^x \cdot x$

Ex// $(D^2 + 4D + 4)y = e^{2x} - e^{-2x}$

Auxiliary equation is

$m^2 + 4m + 4 = 0$

$(m+2)^2 = 0.$

$m = -2, -2$

The roots are real and equal

$C.F = (c_1 + c_2 x) e^{-2x}$

$$P.I = \frac{1}{(D+2)^2}(e^{2x}-e^{-2x}) = \frac{1}{(D+2)^2}e^{2x} - \frac{1}{(D+2)^2}e^{-2x}$$

Here $\frac{1}{(D+2)^2}e^{2x} = \frac{\cancel{e^{2x}}}{(2+2)^2}e^{2x} = \frac{e^{2x}}{16}$ [here $F(a) \neq 0$].

$\frac{1}{(D+2)^2}e^{-2x} = \frac{1}{(D+2)^2}e^{-2x}.1$ [$F(a)=0$ if we put $a=-2$].

$$= e^{-2x}\frac{1}{((D-2)+2)^2}.1 = e^{-2x}\frac{1}{D^2}.1 = e^{-2x}.\frac{1}{D}:x = \frac{e^{-2x}x^2}{2!}.$$

complete Solution is $y = (c_1 + c_2x)e^{-2x} + \frac{1}{16}e^{2x} - \frac{1}{2}x^2e^{-2x}$

Ex. $D^2y - 3Dy + 2y = \cosh x.$

$(D^2 - 3D + 2)y = \cosh x$ [put $\cosh x = \frac{e^x + e^{-x}}{2}$].

Case VI. To Find P.I when $\theta = \sin ax$ or $\cos ax$.

and $F(-a^2) = 0$.

To find out these types of P.I. replace $\sin ax$ or $\cos ax$ by the exponential value and apply.

Thus P.I of $\cos ax$.

$\frac{1}{F(D^2)}\cos ax$ when $F(-a^2)=0$ $e^{iax} = \cos ax + i\sin x$

$= \frac{1}{F(D^2)}$ (Real part of e^{iax})

Ily For $\sin ax$

$= \frac{1}{F(D^2)}$ imaginary part of e^{iax}.

Ex. $\frac{d^2y}{dx^2} + a^2 y = \cos ax$ Ex.

Auxiliary eq. is $m^2 + a^2 = 0$

$m = \pm ai$

$C.F = e^{0x}(c_1 \cos ax + c_2 \sin ax) = c_1 \cos ax + c_2 \sin ax$

$P.I = \frac{1}{D^2+a^2} \cos ax \Rightarrow$ the real part in $\frac{1}{D^2+a^2}(\cos ax + i \sin ax)$

$= \frac{1}{D^2+a^2} e^{iax} \Rightarrow \frac{1}{D^2-(ia)^2} e^{iax}$

$\Rightarrow \frac{1}{(D+ia)(D-ia)} e^{iax} = \frac{1}{(ia+ia)(D-ia)} e^{iax}$ [put $D = ia$ since $F(a) \neq 0$].

$\Rightarrow \frac{1}{2ia(D-ia)} e^{iax} \cdot 1 = \frac{1}{2ia} e^{iax} \cdot \frac{1}{(D+ia)-ia} \cdot 1$

$= \frac{1}{2ia} e^{iax} \frac{1}{D} \cdot 1 = \frac{1}{2ia} e^{iax} \cdot x$

$= \frac{1}{2ia} x (\cos ax + i \sin ax)$

$= \frac{-ix}{2a} (\cos ax + i \sin ax)$

$= \frac{-ix}{2a} \cos ax + \frac{x}{2a} \sin ax.$

$\therefore \frac{1}{D^2+a^2} \cos x = \frac{x}{2a} \sin ax - i\frac{x}{2a} \cos ax.$

$P.I = \frac{x}{2a} \sin ax.$

Complete sol $=$ C.F + P.I

Ex. $(D^2+4)y = \sin^2 x$.

Auxiliary eq. is $m^2+4=0 \Rightarrow m = 0 \pm 2i$

$C.F = c_1 \cos 2x + c_2 \sin 2x$.

$$P.I = \frac{1}{D^2+4}\sin^2 x = \frac{1}{D^2+4}\cdot\frac{1}{2}(2\sin^2 x) = \frac{1}{2}\frac{1}{D^2+4}(1-\cos 2x)$$

$$= \frac{1}{2}\frac{1}{D^2+4}\cdot 1 - \frac{1}{2}\frac{1}{D^2+4}\cos 2x. \quad \text{— (1)}.$$

Now $\frac{1}{2}\frac{1}{D^2+4}\cdot 1 = \frac{1}{2}\cdot\frac{1}{4}\frac{1}{\left(1+\frac{D^2}{4}\right)} = \frac{1}{8}\left(1+\frac{D^2}{4}\right)^{-1}$

$$= \frac{1}{8}\cdot\left(1-\frac{1}{4}D^2+\frac{1}{4}D^3\ldots\right)\cdot 1 = \frac{1}{8}$$

Again $\frac{1}{2}\frac{1}{D^2+4}\cos 2x = \frac{1}{2}\cdot\left[\frac{1}{D^2+2^2}\cos 2x\right]$

$$= \frac{1}{2}\left[\frac{1}{D^2-(2i)^2}\cos 2x + i\sin 2x\right] = \frac{1}{2}\left[\frac{1}{D^2-(2i)^2}e^{i2x}\right]$$

$\therefore$ From previous Ex. $\left[\frac{1}{D^2-(2i)^2}e^{i2x} = \frac{x}{2a}\sin ax\right]$.

$$= \frac{1}{2}\cdot\frac{x}{2\cdot 2}\sin 2x = \frac{x}{8}\sin 2x$$

$\therefore y = C.F + P.I$

$$= c_1\cos 2x + c_2\sin 2x + \frac{1}{8} - \frac{x}{8}\sin 2x.$$

Case (vii) To find P.I when $Q = xV$, where V is any function of x.

Here $P.I = \frac{1}{F(D)}(xV)$. To find the P.I

we transpose by

$$\frac{1}{F(D)} \cdot (xV) = x \frac{1}{F(D)} V - \frac{F'D}{\{F(D)\}^2}$$

$$= x \frac{1}{F(D)} V + \left\{ \frac{d}{dD} \frac{1}{F(D)} \right\} V.$$

Ex. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x \sin x.$

$\Rightarrow m^2 - 2m + 1 = 0.$

$$P.I = \frac{1}{(D-1)^2} x \sin x$$

$$= x \frac{1}{(D-1)^2} \sin x + \left\{ \frac{d}{dD} \frac{1}{(D-1)^2} \right\} \cdot \sin x$$

$$= x \frac{1}{(D-1)^2} \sin x + \left\{ \frac{d}{dD} (D-1)^{-2} \right\} \sin x.$$

$$= x \frac{1}{(D-1)^2} \sin x + \left(\frac{-2 \cdot}{(D-1)^3} \right) \sin x$$

$$= x \frac{1}{(D-1)^2} \sin x - \frac{2}{(D-1)^3} \sin x.$$

$$= x \frac{1}{D^2 - 2D + 1} \sin x - \frac{2}{D^2 - 3D^2 + 3D - 1} \sin x.$$

$$= x \frac{1}{-1^2 - 2D + 1} \sin x - \frac{2}{-1^2 D - 3(-1^2) + 3D - 1} \sin x.$$

$$= \frac{x}{-2D} \sin x - \frac{2}{-D + 3 + 3D - 1} \sin x.$$

$$= \frac{x}{-2} \frac{1}{D} \sin x - \frac{\cancel{2}}{\cancel{2}(D+1)} \sin x$$

$$= -\frac{x(-\cos x)}{2} - \frac{1}{(D+1)} \sin x$$

$$= \frac{x}{2} \cos x - \frac{1}{(D+1)} \sin x.$$

$$= \frac{x}{2} \cos x - \frac{D-1}{(D+1)(D-1)} \sin x \Rightarrow \frac{x}{2} \cos x - \frac{D-1}{D^2 - 1^2} \sin x.$$

$$\Rightarrow \frac{x}{2} \cos x - \frac{(D-1)}{-1-1} \sin x.$$

$$\Rightarrow \frac{x}{2} \cos x + \frac{1}{2} D \sin x - \frac{1}{2} \sin x$$

$$\Rightarrow \frac{x}{2} \cos x - \frac{1}{2}(\cos x - \sin x) \,//$$

Solve:

1. $(D^2+5D+6)y=e^x$

2. $(D^3+2D^2-D-2)y=e^x.$

3. $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+2y=\sinh x$

4. $(D^2-4D+3)y=x^3.$

5. $\frac{d^4y}{dx^4}+4y=x^4$

6. $\frac{d^2y}{dx^2}+6y=\sin t.$

7. $(D^4+2a^2D^2+a^4)y=8\cos ax$

8. $(D^2-5D+6)y=e^x\sin x.$

9. $\frac{d^2y}{dx^2}-7\frac{dy}{dx}+10y=e^{2x}\sin x$

10. $(D^2+4)y=3x\sin x$

11. $(D^4+2D^2+1)y=x^2\cos x$