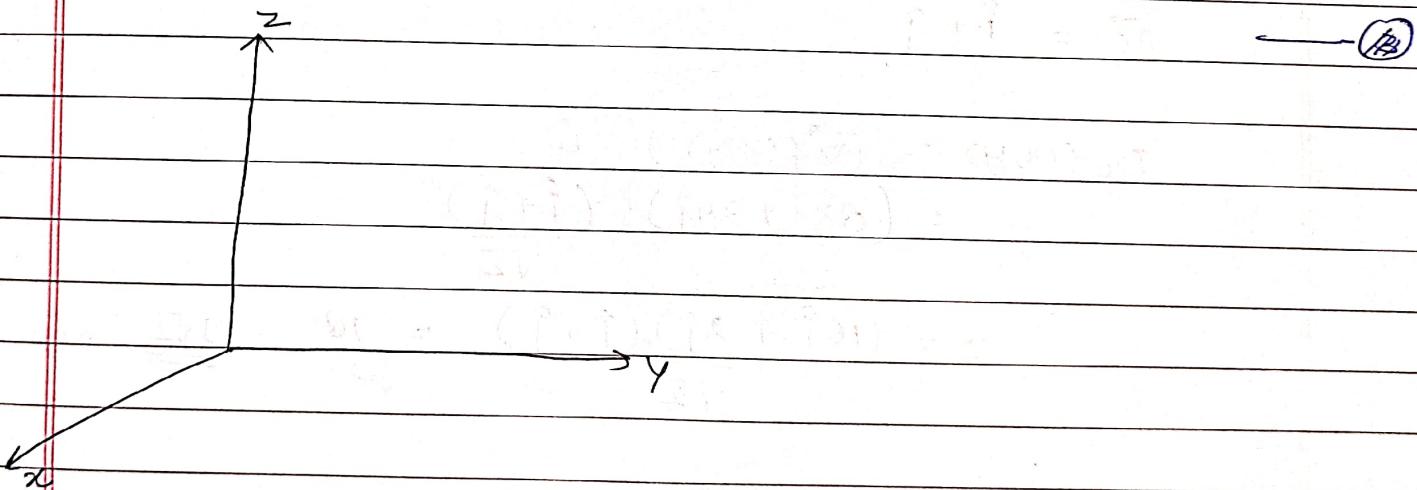


A Generalisation of Partial Differentiation:-

Suppose $u = \cos\theta \hat{i} + \sin\theta \hat{j}$ is a unit vector in the xy plane from $(x, y, 0)$ to $(x + \Delta x, y + \Delta y, 0)$. If $w = \sqrt{(\Delta x)^2 + (\Delta y)^2}$ & $h > 0$, then $\mathbf{v} = h\hat{u}$. Furthermore, let the plane \perp to xy plane that contains these points slice the surface $z = f(x, y)$ in a curve C . We ask: What is the slope of the tangent

from the figure, we see that $\Delta x = h \cos\theta$ & $\Delta y = h \sin\theta$ so that the slope of the indicated secant line is

$$\frac{f(x + \Delta x, y + \Delta y) - f(x, y)}{\Delta x} = \frac{f(x + h \cos\theta, y + h \sin\theta) - f(x, y)}{h}$$



The directional der. of $z = f(x, y)$ in the direction of a unit vector

Observe that (4) is truly a generalisation of partial differentiation, since,

$$\theta = 0, \text{ implies that, } D_1 f(x, y) = \lim_{h \rightarrow 0} \frac{f(x+h, y) - f(x, y)}{h} = \frac{\partial f}{\partial x}$$

$$\theta = \pi/2 \text{ implies that, } D_2 f(x, y) = \lim_{h \rightarrow 0} \frac{f(x, y+h) - f(x, y)}{h} = \frac{\partial f}{\partial y}$$

Computing a directional derivative :-

If $z = f(x, y)$ is a differentiable func of x, y ,
and $u = \cos\theta\hat{i} + \sin\theta\hat{j}$ then,

$$D_u f(x, y) = \vec{\nabla} f(x, y) \cdot \hat{u}$$

- Q. consider the plane \perp to xy plane and passes through $P(2, 1)$ and $Q(3, 2)$. What is the slope of the tangent line to the curve of intersection of this plane with the surface $f(x, y) = 4x^2 + y^2$ at $(2, 1)$ in the dirⁿ of Q ?

The required slope will be directional derivative along \vec{PQ} .

$$\vec{PQ} = \hat{i} + \hat{j}$$

$$\begin{aligned} D_u f(x, y) &= (\vec{\nabla} f(x, y)) \cdot \hat{u} \\ &= (\partial_x \hat{i} + \partial_y \hat{j}) \cdot \frac{(\hat{i} + \hat{j})}{\sqrt{2}} \\ &= \underbrace{(16\hat{i} + 2\hat{j}) \cdot (\hat{i} + \hat{j})}_{\sqrt{2}} = \frac{18}{\sqrt{2}} = \underline{\underline{9\sqrt{2}}} \text{ ans.} \end{aligned}$$

functions of three variables is

for the function $f(x, y, z)$ the directional derivative is defined by,

$$D_u f(x, y, z) = \lim_{h \rightarrow 0} \frac{f(x + h \cos\alpha, y + h \cos\beta, z + h \cos\gamma) - f(x, y, z)}{h}$$

where $\alpha, \beta + \gamma$ are the direction angles of unit vector u measured relative to positive x, y , and z -axes.

But in the same manner as before we can show that

$$D_u f(x, y, z) = \vec{\nabla} f(x, y, z) \cdot u.$$

Q- find the directional derivative of $f(x, y, z) = xy^2 - 4x^2y + z^2$ at $(1, -1, 2)$ in the dirⁿ of $6\hat{i} + 2\hat{j} + 3\hat{k}$.

$$\begin{aligned} D_u f(x, y, z) &= (\vec{f}(x, y, z)) \cdot \hat{u} \\ &= (y^2\hat{x}((y^2 - 8xy)\hat{i} + (2xy - 4x^2)\hat{j} + (2z)\hat{k}) \cdot \hat{u} \\ &= (9\hat{i} + 6\hat{j} + 4\hat{k}) \cdot \frac{(6\hat{i} + 2\hat{j} + 3\hat{k})}{\sqrt{36+4+9}} \\ &= \frac{(54 - 12 + 12)}{7} = \frac{54}{7} \text{ gms.} \end{aligned}$$

Q find directional derivative:-

① $f(x, y) = 5x^3y^6$; $(-1, 1)$, $\theta = \pi/6$.

$$\begin{aligned} u &= \sin \frac{\pi}{6}\hat{i} + \cos \frac{\pi}{6}\hat{j} \\ D_u f(x, y) &= (15x^2y^6\hat{i} + 30x^3y^5\hat{j}) \cdot \left(\sin \frac{\pi}{6}\hat{i} + \cos \frac{\pi}{6}\hat{j} \right) \\ &= (15\hat{i} + 30\hat{j}) \cdot \left(\frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j} \right) \\ &= \frac{15\sqrt{3}}{2}\hat{i} - 15\hat{j} \end{aligned}$$

Tangent plane :-

Let $P(x_0, y_0, z_0)$ be a point on the graph of $f(x, y, z) = c$, where $\nabla f \neq 0$. The tangent plane at P is that plane through P that is normal to ∇f evaluated at P .

Equation of tangent plane :-

Let $P(x_0, y_0, z_0)$ be a point on the graph of $f(x, y, z) = c$, where $\nabla f \neq 0$. Then equation of tangent plane at P is -

$$f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0$$

Q: find an eqn of the tangent plane to the graph of $x^2 - 4y^2 + z^2 = 16$ at $(2, 1, 4)$.

$$f_x = 2x, \quad f_y = -8y, \quad f_z = 2z$$

$$f_x(2, 1, 4) = 4, \quad f_y(2, 1, 4) = -8, \quad f_z(2, 1, 4) = 8$$

\therefore eqn :-

$$4(x-2) + (-8)(y-1) + 8(z-4) = 0$$

$$4x - 8y + 8z - 32 = 0$$

$$\boxed{x - 2y + 2z - 8 = 0}$$

Normal line: Let $P(x_0, y_0, z_0)$ be a point on the graph of $f(x, y, z) = c$, where ∇f is not 0. The line containing $P(x_0, y_0, z_0)$ that is parallel to $\nabla f(x_0, y_0, z_0)$ is called normal line to the surface at P . This line is normal to the tangent plane to the surface at P .

- Q Find parametric equations for the normal line to the surface in example $\text{if } z = \frac{1}{2}x^2 + \frac{1}{2}y^2 + 4$ at $(1, -1, 5)$.

$$\Rightarrow f = \frac{1}{2}x^2 + \frac{1}{2}y^2 - z + 4.$$

$$\nabla f = 2x\hat{i} + 2y\hat{j} - \hat{k}$$

$$\nabla f(1, -1, 5) = \hat{i} - \hat{j} - \hat{k}$$

Parametric equations of normal line are -
 $x = 1 + t ; y = -1 - t ; z = 5 - t$.

Expressed as symmetric eq's the normal line to the surface $f(x, y, z) = c$ at $P(x_0, y_0, z_0)$ is given by.

$$\frac{x - x_0}{f_x(x_0, y_0, z_0)} = \frac{y - y_0}{f_y(x_0, y_0, z_0)} = \frac{z - z_0}{f_z(x_0, y_0, z_0)}$$

Q. 15

find eqn of tangent plane to the surfaces -

(i) $x^2 + y^2 + z^2 = 9$; at $(-2, 2, 1)$.

$$\begin{aligned} & f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) \\ &= (2x)_{x=x_0}(x - x_0) + (2y)_{y=y_0}(y - y_0) + (2z)_{z=z_0}(z - z_0) = 0 \\ &\Rightarrow -4(x+2) + 4(y-2) + 2(z-1) = 0 \\ &\Rightarrow -4x + 4y + 2z - 18 = 0 \\ &\boxed{2x - 2y - z + 9 = 0} \end{aligned}$$

$$\frac{x+2}{(2x)_{x=x_0}} = \frac{y-2}{(2y)_{y=y_0}} = \frac{z-1}{(2z)_{z=z_0}}$$

$$\Rightarrow \frac{x+2}{-4} = \frac{y-2}{4} = \frac{z-1}{2}$$

Q. # $xy + yz + zx = 7$; $(1, -3, -5)$

$x^2 - y^2 - 3z^2 = 5$; $(6, 2, 3)$.

Curl:

curl of vector field \vec{F} , $\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$

Divergence:

The divergence of vector field $\vec{F} = P\hat{i} + Q\hat{j} + R\hat{k}$

is the ~~scalar~~ scalar,

$$\text{div } \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Q. $\vec{F} = (x^2y^3 - z^4)\hat{i} + 4x^5y^2\hat{j} - y^4z^6\hat{k}$ (i). curl \vec{F}

(ii) $\text{div } \vec{F}$, (iii) $\text{div}(\text{curl } \vec{F})$.

$$\begin{aligned} \text{(i). curl } \vec{F} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y^3 - z^4 & 4x^5y^2 & -y^4z^6 \end{vmatrix} \\ &= \hat{i}(-16y^3z^6) - \hat{j}(-2xy^3) + \hat{k}(20x^4y^2 - 3x^2y^2) \end{aligned}$$

(ii) $\text{div } \vec{F} = 2xy^3 + 8x^5y - 6y^4z^5$ ans.

iii) $\text{div}(\text{curl } \vec{F}) = 6xy^2 -$

Q. find out curl & divergence :

(i) $\vec{F}(x, y, z) = xz\hat{i} + yz\hat{j} + xy\hat{k}$

$\text{div } \vec{F} = z + z + 0 = 2z$

$$\begin{aligned} \text{curl } \vec{F} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & yz & xy \end{vmatrix} = (x-y)\hat{i} - (y-x)\hat{j} + (0)\hat{k} \\ &= (x-y)\hat{i} + (x-y)\hat{j} \text{ ans.} \end{aligned}$$

Q. $\mathbf{f}(x, y, z) = (x-y)^3 \hat{i} + e^{-yz} \hat{j} + xy e^{xy} \hat{k}$

$$\operatorname{div} \mathbf{f} = 3(x-y)^2 + e^{-yz}(-z) \hat{k} + 0.$$

$$\begin{aligned}\operatorname{curl} \mathbf{f} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x-y)^3 & e^{-yz} & xy e^{xy} \end{vmatrix} \\ &= [x(e^{xy} + 2ye^{xy}) - e^{-yz}(-y)] \hat{i} - (ye^{xy} - 0) \hat{j} + [x(y-1) - 0] \hat{k}\end{aligned}$$

Q. find $\operatorname{div} \vec{F}$ and $\operatorname{curl} \vec{F}$, $\vec{F} = \operatorname{grad}(x^3 + y^3 + z^3 - 3xyz)$

Q. prove that ; $\operatorname{div}(f\mathbf{v}) = f(\operatorname{div} \mathbf{v}) + (\operatorname{grad} f) \cdot \mathbf{v}$ where
 f is scalar func

Q. find the constant a, b, c so that $\vec{F} = (x+2y+az) \hat{i} + (bx-3y-z) \hat{j} + (4x+cy+2z) \hat{k}$ is irrotational.

Maxima and Minima of a function of two or more variables:-

Maxima:-

A function f of two variables is said to have a relative maximum at a point (x_0, y_0) if there is a disk centered at (x_0, y_0) such that $f(x_0, y_0) \geq f(x, y)$ for all points (x, y) that lie inside the disk, and f is said to have an absolute maximum at (x_0, y_0) if $f(x_0, y_0) \geq f(x, y)$ for all points (x, y) in the domain of f .

Minima:-

A funcⁿ f to two var. is said to have relative minimum at a point (x_0, y_0) if there is a disk centered at (x_0, y_0) such that $f(x_0, y_0) \leq f(x, y)$ for all points (x, y) that lie inside the disk, and f is said to have an absolute minimum at (x_0, y_0) if $f(x_0, y_0) \leq f(x, y)$ for all points (x, y) in the domain of f .

Extreme Value Theorem :-

If $f(x, y)$ is continuous on a closed and bounded set R , then f has both an absolute maximum and an absolute minimum.

Theorem:- The second Partial test:- Let f be funcⁿ of two variables with continuous second-order partial derivatives in some disk centred at a critical point (x_0, y_0) and let,

$$D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f_{xy}^2(x_0, y_0) \neq (f_t - B^2).$$

- ① If $D > 0$ & $f_{xx}(x_0, y_0) > 0$, then f has a rel. min. at (x_0, y_0)
- ② If $D > 0$ & $f_{xx}(x_0, y_0) < 0$, then f has rel. max. at (x_0, y_0)
- ③ If $D < 0$, f has a saddle point at (x_0, y_0)
⇒ f has no maximum value at (x_0, y_0)
- ④ If $D = 0$, then no conclusion can be drawn.

Working Rule for Maxima/Minima:

$$z = f(x, y)$$

Step I: find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

Step II: Solve for $\frac{\partial z}{\partial x} = 0$ and $\frac{\partial z}{\partial y} = 0$, simultaneously. Let $(a, b) \in (c, d)$ be the solⁿ of these equations.

Step III: find $r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 f}{\partial x^2}$ and $s = \frac{\partial^2 z}{\partial x \partial y}$ and $t = \frac{\partial^2 z}{\partial y^2}$

Step IV: (a) If $rt - s^2 > 0$, $r < 0$ for a particular solⁿ (a, b) , then $f(a, b)$ has maximum value.

(b) If $rt - s^2 > 0$, $r > 0$ for a particular solⁿ (a, b) , then $f(a, b)$ has minimum value.

(c) If $rt - s^2 < 0$, for a particular solⁿ (a, b) , then $z = f(x, y)$ has no extreme value at $(a, b) \rightarrow$ saddle point.

(d) If $rt - s^2 = 0$, at (a, b) , this case is doubtful and requires further investigation.

Q. Examine the funcⁿ $x^3 + y^3 - 3axy$ for maxima or minima.

$$z = x^3 + y^3 - 3axy$$

$$\frac{\partial z}{\partial x} = 3x^2 - 3ay = 0 \quad \frac{\partial z}{\partial y} = 3y^2 - 3ax = 0$$

$$x^2 = ay$$

$$\left(\frac{y^2}{a}\right)^2 = ay$$

$$y^3 = a^3$$

$$y^2 = ax$$

$$y = a \Rightarrow (a, a) \text{ if } (0, 0).$$

$$\Rightarrow x = a$$

$$r = \frac{\partial^2 z}{\partial x^2} = 6x - 0 = 6x$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = -3a$$

$$t = \frac{\partial^2 z}{\partial y^2} = 6y$$

$$D = rt - s^2 = 36xy - 9a^2$$

at $(0,0)$

$D = -9a^2 < 0 \Rightarrow z$ has no extreme values at $(0,0)$

~~at (a,a)~~

$$D = 36a^2 - 9a^2 = 27a^2 > 0$$

$$\text{and } r = 6a$$

case 1: when $a = +ve$, z has a minimum value at (a,a)
and min. value of $z = a^3 + a^3 - 3a^3 = -a^3$

case 2: when $a = -ve$, $\Rightarrow -6a < 0$

z has a max. value at $(a,-a)$ and maximum value
of $z = -a^3 + -a^3 + 3a(a)(-a) = -6a^3$

Q. Locate all relative extrema and saddle points of
 $f(x,y) = 4xy - x^4 - y^4$.

$$\frac{\partial f(x,y)}{\partial x} = 4y - 4x^3 \quad \frac{\partial f}{\partial y} = 4x - 4y^3$$

$$4y - 4x^3 = 0$$

or

$$y = x^3$$

$$x = y^3$$

$$x = (x^3)^3 = x^9$$

$$x - x^9 = 0$$

$$x(1-x^8) = 0.$$

$$x = 0, 1, -1 \Rightarrow y = 0, 1, -1$$

critical pts :- $(0,0); (1,1); (-1,-1)$.

$$f_{xx} = -12x^2, f_{yy} = -12y^2, f_{xy} = 4.$$

pts

$$f_{xx} \cdot f_{yy} - f_{xy}^2$$

saddle pt. $\Rightarrow (0,0)$.

$(0,0)$

$$0 \times 0 - 4^2 = -16 < 0$$

$(1,1)$

$$-12 \times 1 - 12 - 4^2 = -12 > 0$$

\Rightarrow maxima $\Rightarrow (1,1)$

$(-1,-1)$

$$-12 \times -1 - 12 - 4^2 = +12 > 0, f < 0 \Rightarrow$$

maxima $\Rightarrow (-1,-1)$

(6)

Determine the dimension of the rectangular box open at the top, having a volume of 32 ft^3 & requiring ~~at least~~ the least amount of material for its construction.

Let, x is length, y is breadth & z is height of the box.

$$S = 2xy + 2yz + 2zx$$

$$V = 32$$

$$xyz = 32 \Rightarrow z = \frac{32}{xy}$$

$$\therefore S = 2xy + \frac{64}{x} + \frac{64}{y}$$

$$\frac{\partial S}{\partial x} = y - \frac{64}{x^2} = 0$$

$$\frac{\partial S}{\partial y} = x - \frac{64}{y^2} = 0$$

$$y = \frac{64}{x^2} \Rightarrow x = \frac{64}{y^2} = \frac{64(x^2)}{(64)^2}$$

$$r = \frac{\partial^2 S}{\partial x^2} = -(-2) \frac{64}{x^3} = \frac{128}{x^3} = 2$$

$$64x = x^4$$

$$64x - x^4 = 0 \Rightarrow x(64 - x^3) = 0$$

$$x=0, x=4$$

$$t = \frac{\partial^2 S}{\partial y^2} = \frac{128}{y^3} = 2$$

$$\begin{aligned} x=4 &\Rightarrow y = \frac{64}{4^2} = 4 \\ y=4 & \\ z=2 & \end{aligned}$$

$$s = \frac{\partial^2 S}{\partial xy} = 1$$

$$rt - s^2 = 4 - 1 = 3 > 0$$

$$\therefore r > 0$$

\Rightarrow Minimum

Q. A rectangular box open at the top is to have a given capacity. Find the dimension of the box requiring least material for its construction.

$$\text{Ans i:- } x = y = (2V)^{1/3}$$

$$x = y = 2z$$

Q. Examine for maxima/minima of the function.

$$f(x, y) = \sin x + \sin y + \sin(x+y)$$

$$\frac{\partial f}{\partial x} = \cos x + \cos(x+y) \quad \left| \quad \frac{\partial f}{\partial y} = \cos y + \cos(x+y).$$

$$r = \frac{\partial^2 f}{\partial x^2} = -\sin x - \sin(x+y) \quad \left| \quad t = \frac{\partial^2 f}{\partial y^2} = -\sin y - \sin(x+y)$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = -\sin(x+y)$$

$$\frac{\partial f}{\partial x} = 0 \Rightarrow \cos x + \cos(x+y) = 0 \Rightarrow \cos(x+y) = -\cos x \quad \text{--- (1)}$$

$$\frac{\partial f}{\partial y} = \cos y + \cos(x+y) = 0 \Rightarrow \cos y - \cos x = 0. \quad \boxed{y = x}$$

Now, eqn (1)

$$\cos x + \cos(x+x) = 0.$$

$$\cos 2x = -\cos x$$

$$\cos 2x = \cos(\pi - x)$$

$$2x = \pi - x \Rightarrow \boxed{x = \pi/3} \cup \boxed{y = \pi/3}$$

$$\therefore \left(\frac{\pi}{3}, \frac{\pi}{3} \right).$$

$$r = -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} = -\sqrt{3}; t = -\sqrt{3}; s = -\frac{\sqrt{3}}{2}$$

$$\therefore r^2 - s^2 = 3 - \frac{3}{4} > 0.$$

$$\therefore r < 0$$

\Rightarrow Maxima at $(-\sqrt{3}, -\sqrt{3})$

and the max. value is $\underline{\underline{}}$.

In a plane $\triangle ABC$. Find the max. value of $\cos A \cdot \cos B \cdot \cos C$.

$$f = \cos A \cos B \cos C$$

We know that.

$$A + B + C = 180^\circ$$

$$C = 180^\circ - A - B$$

$$f = \cos A \cos B \cos(180^\circ - A - B)$$

$$\frac{\partial f}{\partial A} = -\sin A \cos B [+ \sin A \cos(180^\circ - A - B) + \cos A \sin(180^\circ - A - B)]$$

$$\frac{\partial f}{\partial B} = \cos A [+ \sin B \cos(180^\circ - A - B) + \cos B \sin(180^\circ - A - B)]$$

$$\frac{\partial f}{\partial A} = 0 \Rightarrow -\sin A \cos B (-\cos(A+B)) + \cos A \cos B (\sin(A+B)) = 0,$$

$$= \cos B (\sin A \cos(A+B) + \cos A \sin(A+B)) = 0$$

$$\Rightarrow \cos B \sin(2A+B) = 0.$$

case ① $\cos B = 0$.

$$B = \frac{\pi}{2}$$

$$\frac{\partial f}{\partial B} = 0 \Rightarrow \cos A \cdot \sin(A+2B) = 0.$$

$$\therefore B = \frac{\pi}{2}$$

$$\therefore \cos A \cdot \sin(A+\pi) = 0.$$

$$-\cos A \cdot \sin A = 0$$

$$\therefore \cos A = 0 \text{ or } \sin A = 0$$

$$\therefore A = \frac{\pi}{2} \text{ or } A = 0.$$

$$\text{for, } B = \frac{\pi}{2}, A = \frac{\pi}{2} \text{ or } 0.$$

case ② $\sin(2A+B) = 0 \text{ or so}$

$$2A + B = 0$$

Lagrange Multiplier

Lagrange method of undetermined multiplier :-

let $f(x,y,z)$ be a funcⁿ of x,y,z which is to be examined for minimum or maximum value.

Let the variables x,y,z be connected by the relation

$$\phi(x,y,z) = 0. \quad \text{--- (i)}$$

for $f(x,y,z)$ to have a maximum or minimum value,

the necessary condⁿ is $\frac{\partial f}{\partial x} = 0; \frac{\partial f}{\partial y} = 0; \frac{\partial f}{\partial z} = 0$.

$$\therefore \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial z} dz = 0. \quad \text{--- (ii)}$$

diff. partially eqn (i);

$$\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz = 0 \quad \text{--- (iii)}$$

Let, λ be the Lagrange multiplier, then

$$(ii) + \lambda (iii) \Rightarrow \left(\frac{\partial F}{\partial x} + \lambda \frac{\partial \phi}{\partial x} \right) dx + \left(\frac{\partial F}{\partial y} + \lambda \frac{\partial \phi}{\partial y} \right) dy + \left(\frac{\partial F}{\partial z} + \lambda \frac{\partial \phi}{\partial z} \right) dz = 0$$

$$\therefore \text{we have, } \frac{\partial F}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0; \frac{\partial F}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0; \frac{\partial F}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0 \quad \text{--- (iv)}$$

All these eqn (iv) along with eqn (i), gives the values of x,y,z & λ for a maximum or minimum.

Q. find the maximum or minimum distances of the point $(3, 4, 12)$ from the sphere $x^2 + y^2 + z^2 = 1$.

Let (x, y, z) be any point on the sphere. The distance of the point $A(3, 4, 12)$ from (x, y, z) is

$$\sqrt{(x-3)^2 + (y-4)^2 + (z-12)^2} = d$$

$$\Rightarrow d^2 = (x-3)^2 + (y-4)^2 + (z-12)^2.$$

$$\text{Let, } f(x, y, z) = (x-3)^2 + (y-4)^2 + (z-12)^2 \quad \text{--- (i)}$$

$$\text{& it is given, } \phi(x, y, z) = x^2 + y^2 + z^2 - 1 = 0 \quad \text{--- (ii)}$$

Consider a ^{Lagrange's} function $F(x, y, z) = f(x, y, z) + \lambda \phi(x, y, z)$

$$\therefore F(x, y, z) = (x-3)^2 + (y-4)^2 + (z-12)^2 + \lambda(x^2 + y^2 + z^2 - 1)$$

for stationary values, $dF = 0$.

$$\circ (2(x-3) + 2\lambda x) dx + (2(y-4) + 2\lambda y) dy + (2(z-12) + 2\lambda z) dz = 0.$$

$$\begin{array}{c|c|c} \therefore x-3+\lambda x=0 & y-4+\lambda y=0 & z-12+\lambda z=0 \\ x=3 & y=\frac{4}{1+\lambda} & z=\frac{12}{1+\lambda} \\ \hline 1+\lambda & 1+\lambda & 1+\lambda \end{array} \quad \text{--- (iii), (iv), (v)}$$

$$x \times (\text{iii}) + y \times (\text{iv}) + z \times (\text{v})$$

$$\Rightarrow (x^2 + y^2 + z^2) - 3x - 4y - 12z + \lambda(x^2 + y^2 + z^2) = 0$$

$$\circ 3x + 4y + 12z = \lambda + 1$$

$$\frac{3 \times 3}{1+\lambda} + \frac{4 \times 4}{1+\lambda} + \frac{12 \times 12}{1+\lambda} = \lambda + 1$$

$$9+16+144 = (\lambda + 1)^2$$

$$169 = (\lambda + 1)^2$$

$$1+\lambda = \pm 13$$

$$\boxed{\lambda = +12, -14}$$

for, $d=12$; $x = \frac{8}{13}$, $y = \frac{4}{13}$, $z = \frac{12}{13}$

for $d=-14$; $x = -\frac{3}{13}$, $y = -\frac{4}{13}$, $z = -\frac{12}{13}$

at $\left(\frac{3}{13}, \frac{4}{13}, \frac{12}{13}\right)$; $d^2 = 144$

$$\boxed{d=12}$$

at $\left(-\frac{3}{13}, -\frac{4}{13}, -\frac{12}{13}\right)$; $\boxed{d=14}$

Multiple Integral :-

$\star \quad \iint_{\substack{d \\ c \\ a}}^{b \\ b} f(x,y) dx dy = \int_c^d \left[\int_a^b f(x,y) dx \right] dy = \int_a^b \left[\int_c^d f(x,y) dy \right] dx.$

Q.
$$\begin{aligned} & \int_1^3 \left[\int_2^4 (40 - 2xy) dy \right] dx \\ &= \int_1^3 [40y - xy^2]_2^4 dx = \int_1^3 (80 - 12x) dx = [80x - 6x^2]_1^3 \\ &= [60 - 48] = 112 \text{ ans.} \end{aligned}$$

Q. Theorem (Fubini's Theorem):-

Let R be the rectangle defined by the inequalities $a \leq x \leq b$, $c \leq y \leq d$. If $f(x,y)$ is continuous on this rectangle, then.

$$\iint_R f(x,y) dA = \int_a^b \int_c^d f(x,y) dx dy = \int_a^b \int_c^d f(x,y) dy dx.$$

Q. Evaluate the double integral $\iint_R y^2 x dA$ over the rectangle

$$R = \{(x,y) : -3 \leq x \leq 2, 0 \leq y \leq 1\}. \quad \begin{matrix} (-3,1) & (0,1) \\ (-3,0) & (2,0) \end{matrix}$$

$$\begin{aligned} \iint_R y^2 x dA &= \int_{x=-3}^2 \int_{y=0}^1 y^2 x dy dx \\ &= \int_{-3}^2 \left[\frac{y^3}{3} x \right]_0^1 dx = \frac{1}{3} \int_{-3}^2 x dx \\ &= \frac{1}{3} \left[\frac{x^2}{2} \right]_{-3}^2 = \frac{1}{3} \left[\frac{-5}{2} \right] = -\frac{5}{6} = \underline{\underline{\frac{5}{6} \text{ sq unit.}}}$$

Properties

1. $\iint_R c f(x,y) dA = c \iint_R f(x,y) dA. \quad (c \text{ is a constant}).$

2. $\iint_R [f(x,y) \pm g(x,y)] dA = \iint_R f(x,y) dA \pm \iint_R g(x,y) dA.$

$$\textcircled{3} \quad \iint_R f(x, y) dA = \iint_{R_1} f(x, y) dA + \iint_{R_2} f(x, y) dA.$$

Here R is subdivided into two parts.

$$\textcircled{Q.1} \quad \int_0^2 \int_0^1 y \sin x dy dx.$$

$$\Rightarrow \int_0^2 \left[\frac{y^2}{2} \sin x \right]_0^1 dx = \int_0^2 \frac{\sin x}{2} dx = \left[-\frac{\cos x}{2} \right]_0^2 = -\cancel{\cos 2}$$

$$= -\frac{\cos 2}{2} - \left(-\frac{\cos 0}{2} \right) = -\frac{\cos 2 + 1}{2} \quad . \text{ Ans.}$$

$$\textcircled{Q.2} \quad \iint_{-1}^3 (2x - 4y) dy dx$$

$$= \int_{-1}^3 \left[2xy - \frac{4y^2}{2} \right]_1^3 dx = \int_{-1}^3 [8x] dx = \left[\frac{4x^2}{2} \right]_1^3 = \underline{16} \text{ Ans}$$

$$\textcircled{Q.3} \quad \int_3^4 \int_{-1}^2 \frac{1}{(x+y)^2} dy dx.$$

$$\Rightarrow \int_3^4 \left[\frac{(x+y)^{-1}}{-1} \right]_{-1}^2 dx =$$

$$\textcircled{Q.} \quad \iint_R 4xy^3 dA ; R = [(x, y) : -1 \leq x \leq 1, -2 \leq y \leq 2].$$

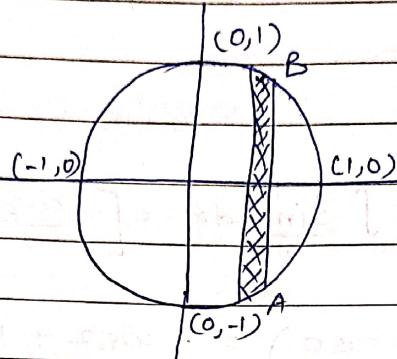
$$\Rightarrow \int_{-2}^2 \int_{-1}^1 4xy^3 dx dy = \int_{-2}^2 \left[2x^2 y^3 \right]_{-1}^1 dy = \int_{-2}^2 0 dy = \underline{0}.$$

$$\textcircled{Q.} \quad \iint_R \frac{xy}{\sqrt{x^2 + y^2 + 1}} dA ; R = [(x, y) : 0 \leq x \leq 1 ; 0 \leq y \leq 1].$$

$$= \int_{y=0}^1 \int_{x=0}^1 \frac{xy}{\sqrt{x^2 + y^2 + 1}} dx dy$$

Q. find the area of unit circle!

let the eqn of circle is; $x^2 + y^2 = 1$.



Consider a strip AB of width Δx which is parallel to y axis, then x axis

varies from -1 to 1, and

y varies from $-\sqrt{1-x^2}$ to $\sqrt{1-x^2}$

$$\begin{aligned} \text{Area} &= \int_{-1}^{1} \left(\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1 \, dy \right) dx \\ &= \int_{-1}^{1} 2\sqrt{1-x^2} \, dx = 2 \int_{-1}^{1} \sqrt{1-x^2} \, dx \end{aligned}$$

Change the order of Integration :-

Ex:-

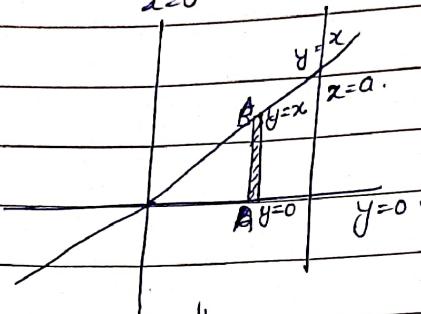
$$\int \left(\int dy \right) dx$$

$$x=0 \quad y=0$$

$$x=0, x=a$$

$$y=0, y=x$$

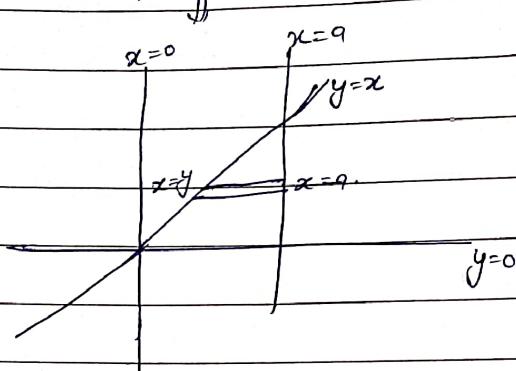
$$x=0$$



$$\int \left(\int dx \right) dy$$

$$y=0 \quad x=y$$

$$x=0$$



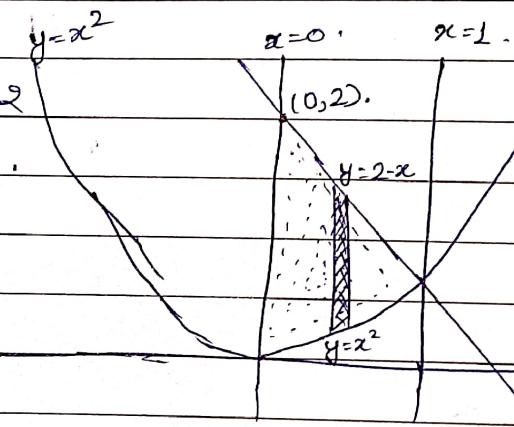
Q.

Change the order of $\iint_{0 \leq x^2} xy \, dy \, dx$ and hence evaluate the same.

Sol:-

$$\text{Here, } y = x^2, y + x = 2$$

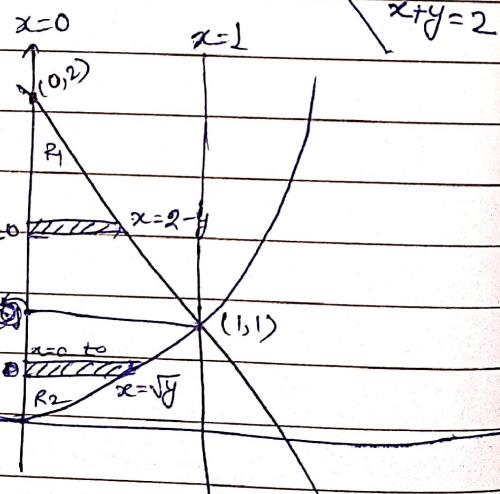
$$x = 1, x = 0$$



$$\Rightarrow \iint_{R_1} xy \, dy \, dx + \iint_{R_2} xy \, dy \, dx$$

$$R_1 \quad R_2$$

$$= \int_{y=1}^{2-y} \int_{x=0}^{x=\sqrt{y}} xy \, dx \, dy + \int_{y=0}^{1-\sqrt{y}} \int_{x=0}^{x=2-y} xy \, dx \, dy.$$



$$\begin{aligned}
 & \Rightarrow \int_0^2 \left[y \cdot \frac{2x^2}{2} \right] dy + \int_1^2 \left[\frac{x^2 \cdot y}{2} \right] dy \\
 & = \frac{1}{2} \int_0^2 [4y + y^3 - 4y^2] dy + \int_0^1 y^2 dy \\
 & = \frac{1}{2} \left[2y^2 + \frac{y^4}{4} - \frac{4y^3}{3} \right]_0^2 + \left[\frac{y^3}{6} \right]_0^1 \\
 & = \frac{1}{2} \left[(0 + 4 - 32) - \left(2 + \frac{1}{4} - \frac{4}{3} \right) \right] + \frac{1}{6} \\
 & = \frac{1}{2} \left[10 - \frac{28}{3} - \frac{1}{4} \right] + \frac{1}{6} \\
 & \Rightarrow 5 - \frac{28}{6} - \frac{1}{8} + \frac{1}{6} = 5 - \frac{1}{8} - \frac{27}{6} \\
 & = 40 - 1 - 36 = \frac{3}{8} \text{ Ans}
 \end{aligned}$$

Q.1 change the order of $\int_0^1 \int_{x-y}^x (x+y) dx dy$ and evaluate.

$$\int_0^1 \int_{x-y}^x \frac{1}{2} dy dx$$

$$\int_0^1 \int_0^{\sqrt{x^2-y^2}} \frac{xy}{2\sqrt{x^2-y^2}} dx dy$$

Q.2 Evaluate $\int_0^1 \int_0^x (x+y) dy dx$

$$\begin{aligned}
 & \int_0^1 \int_0^x (x+y) dy dx \\
 & = \int_0^1 \left[xy + \frac{y^2}{2} \right]_0^x dx \\
 & = \int_0^1 \left[x^2 + \frac{x^2}{2} \right] dx \\
 & = \int_0^1 \frac{3x^2}{2} dx \\
 & = \left[\frac{x^3}{2} \right]_0^1 = \frac{3}{2}
 \end{aligned}$$

Change of variables b/w cartesian and polar coordinates.

Ex: Evaluate $\int \int_{x=0}^2 \frac{xy \, dy \, dx}{\sqrt{x^2+y^2}}$ by changing into polar coordinates.

Solⁿ

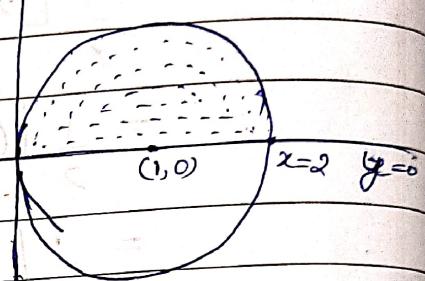
$$\text{Here, } (x=0, x=2)$$

$$y=0, y=\sqrt{2x-x^2}$$

$$y^2+x^2-2x=0 \quad (x-1)^2=1$$

$$(y-0)^2+(x-1)^2=1^2$$

circle of radius 1' &
centre $(1, 0)$



In polar coordinates,

$$x=r\cos\theta, \quad y=r\sin\theta.$$

$$r^2 - 2r\cos\theta = 0$$

$$r(r-2\cos\theta) = 0$$

$$r=0 \quad \text{or} \quad r=2\cos\theta$$

$$\pi/2 \quad 2\cos\theta$$

$$\int \int_{\theta=0}^{\pi/2} \frac{r\cos\theta}{\sqrt{r^2}} |J| dr d\theta$$

Jacobian of polar coordinate = r

$$\begin{aligned} \therefore \int \int_{\theta=0}^{\pi/2} \frac{r\cos\theta}{\sqrt{r^2}} r \, dr \, d\theta &= \int_{\theta=0}^{\pi/2} \left[\frac{r^2 \cos\theta}{2} \right]_0^{2\cos\theta} d\theta \\ &= \int_{\theta=0}^{\pi/2} 2\cos^3\theta \, d\theta \end{aligned}$$

for $\int_{0}^{\pi/2} \cos^m \theta \sin^n \theta d\theta$ type of problem, use
gamma & Beta function.

$\Gamma \rightarrow \text{gamma}$ & $\beta \rightarrow \text{Beta}$.

$$5! = \Gamma 6$$

$$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120.$$

$$\begin{aligned}\Gamma 6 &= \Gamma 5+1 = 5 \Gamma 5 = 5 \Gamma 4+1 = 5 \times 4 \times \Gamma 4 = 5 \times 4 \times 3 \Gamma 3 \\ &= 5 \times 4 \times 3 \times \sqrt{3} = 5 \times 4 \times 3 \times \sqrt{2+1} = 5 \times 4 \times 3 \times 2 \sqrt{2} \\ &= 5 \times 4 \times 3 \times 2 \sqrt{1+1} = 5 \times 4 \times 3 \times 2 \times 1 = 120\end{aligned}$$

some result:-

$$\Gamma 1 = 1, \Gamma 0 = \infty, \Gamma -1 = \infty$$

$$\Gamma n \Gamma 1-n = \pi \quad ; \quad \Gamma \frac{1}{2} = \sqrt{\pi}$$

$$\int_{0}^{\pi/2} \cos^m \theta \sin^n \theta d\theta = \frac{1}{2} \frac{\Gamma \frac{m+1}{2} \Gamma \frac{n+1}{2}}{\Gamma \frac{m+n+2}{2}} \quad \left| \begin{array}{l} m > -1 \\ n > -1 \end{array} \right.$$

A simple polar region in a polar coordinate system is a region that is enclosed b/w two rays $\theta = \alpha$ & $\theta = \beta$, & two continuous polar curves, $r = r_1(\theta)$ & $r = r_2(\theta)$, where equations of the rays and polar curves satisfy the following conditions.

$$(i) \alpha \leq \beta \quad (ii) \beta - \alpha \leq 2\pi \quad (iii) 0 \leq r_1(\theta) \leq r_2(\theta) \leq$$

A polar rectangle is a simple polar region for which the bounding polar curves are circular arcs. for example, fig 14.3.3 shows the polar rectangle R given by: $1.5 \leq r \leq 2$.

Theorem: If R is a simple polar region whose boundaries are the rays $\theta = \alpha$ and $\theta = \beta$ and the curves $r = r_1(\theta)$ and $r = r_2(\theta)$ shown in figure 14.3.8 and if $f(r, \theta)$ is continuous on R , then,

$$\iint_R f(r, \theta) dA = \iint_{\alpha \leq \theta \leq \beta} f(r, \theta) r dr d\theta$$

$\iint_R \sin \theta dA = ?$, where R is region in first quadrant that is outside circle $r=2$ and inside the cardioid $r = 2(1 + \cos \theta)$.

$$\begin{aligned} \iint_R \sin \theta dA &= \int_{\theta=0}^{\pi/2} \int_{r=2}^{2(1+\cos\theta)} \sin \theta r dr d\theta = \int_{\theta=0}^{\pi/2} \left[\frac{\sin \theta r^2}{2} \right]_{2}^{2(1+\cos\theta)} d\theta \\ &= \int_{\theta=0}^{\pi/2} \left(\frac{\sin \theta (2+2\cos\theta)^2}{2} - 2\sin \theta \right) d\theta \end{aligned}$$

$\pi/2$

$$\Rightarrow 2 \int_0^{\pi/2} (\sin\theta(1 + \cos^2\theta + 2\cos\theta) - \sin\theta) d\theta$$

$$\text{ans} = \frac{8}{3}$$

$$= 2 \int_0^{\pi/2} (\cos^2\theta \sin\theta + 2\sin\theta \cos\theta) d\theta$$

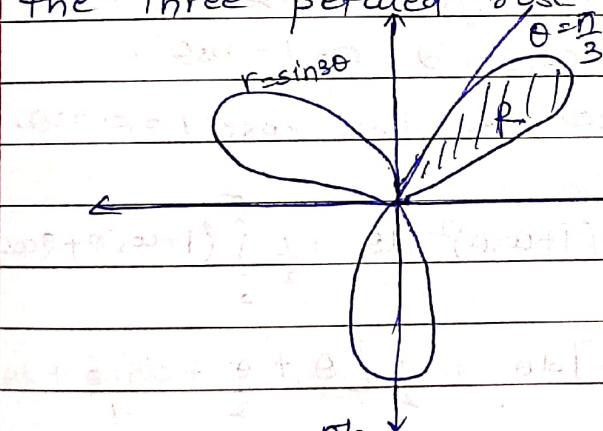
$$= 2 \int_0^{\pi/2} (\cos^2\theta \sin\theta + \sin 2\theta) d\theta$$

~~Integration by parts~~

~~Let $u = \cos\theta$, $v = \sin\theta$~~

- Q find the area b/w coordinates $r = 2(1 + \cos\theta)$ and $r = 2(1 - \cos\theta)$ $\Rightarrow \boxed{\iint_R \sin\theta dA}$

- Q Use double integral to find the area enclosed by the three petalled rose $r = \sin 3\theta$.



$$A = 3 \int_0^{\pi/3} \int_0^{r(\theta)} dA$$

$$= 3 \int_0^{\pi/3} \int_0^{\sin 3\theta} r dr d\theta$$

$$= \frac{3}{2} \int_0^{\pi/3} \sin^2 3\theta d\theta$$

$$= \frac{3}{2} \int_0^{\pi/3} (1 - \cos 6\theta) d\theta = \frac{3}{4} \left[\theta - \frac{\sin 6\theta}{6} \right]_0^{\pi/3}$$

$$= \frac{3}{4} \left[\frac{\pi}{3} - 0 \right] = \frac{\pi}{4}$$

Q

Evaluate $\int_{x^2=1}^1 \int_{y=0}^{\sqrt{1-x^2}} (x^2+y^2)^{3/2} dy dx.$

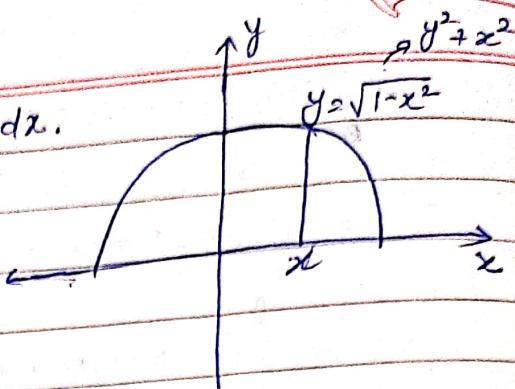
$$= \iint_R (x^2+y^2)^{3/2} dA$$

$$\because x = r\cos\theta, y = r\sin\theta$$

$$x^2+y^2 = 1$$

$$r^2(\cos^2\theta + \sin^2\theta) = 1$$

$$r^2 = 1 \Rightarrow r = \pm 1$$



Q ①

$$\int_0^{\pi} \int_0^{1+\cos\theta} r dr d\theta$$

$$Q ② \cdot \int_0^{\pi/6} \int_0^{1-\cos\theta} r dr d\theta$$

Q ③

$$\int_0^{\pi} \int_0^{r^3} r^3 dr d\theta$$

Q ④ The region enclosed by the cardioid $r = 1 - \cos\theta$.

Q ⑤ The area of region enclosed by the rose $r = \sin n\theta$.

$$SOL \text{ ⑤ } \int_0^{\pi} \left[\frac{r^2}{2} \right]_0^{1+\cos\theta} d\theta = \frac{1}{2} \int_0^{\pi} (1+\cos\theta)^2 d\theta = \frac{1}{2} \int_0^{\pi} (1+\cos^2\theta + 2\cos\theta) d\theta$$

$$\begin{aligned} \cos 2\theta &= \cos^2\theta - \sin^2\theta \\ &\Rightarrow \cos 2\theta = 1 + \cos 2\theta \\ &\Rightarrow \frac{1+\cos 2\theta}{2} = \frac{1}{2} \int_0^{\pi} \left[\frac{1+1+\cos 2\theta}{2} + 2\cos\theta \right] d\theta = \frac{1}{2} \left[\theta + \frac{\theta}{2} + \sin 2\theta + 2\sin\theta \right] \end{aligned}$$

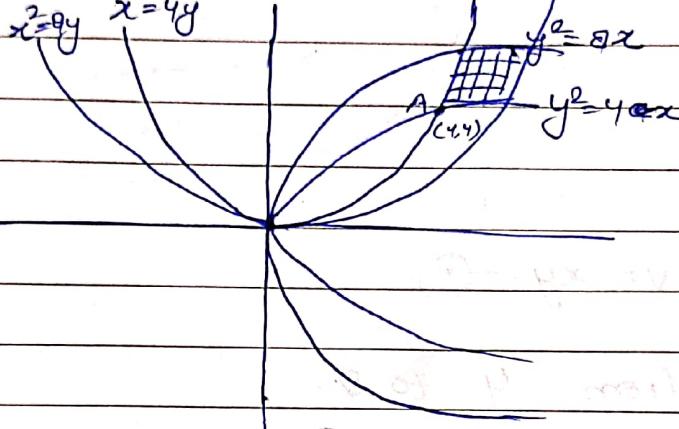
$$= \frac{1}{2} \left[\pi + \frac{\pi}{2} \right] = \frac{3\pi}{4} \text{ square units}$$

$$\beta(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$$

$$\beta(m, n) = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)} = \beta(n, m)$$

Q. Evaluate $\iint_R xy dxdy$ over the region bounded by the parabolas $y^2 = 4x$, $y^2 = 8x$, $x^2 = 4y$, $x^2 = 8y$.

sol'n



solving for pt. A.

$$y^2 = 4x, x^2 = 4y \\ x = y^2$$

$$\frac{y^4}{16} = 4y$$

$$A = (4,4)$$

$$y^4 = 64y$$

$$\text{let, } u = \frac{y^2}{x} \Rightarrow v = \frac{x^2}{y}$$

$$y(y^3 - 64) = 0$$

$$y=0, 4$$

then u varies from 4 to 8.

But, $y \neq 0 \therefore y=4$

& v " " 4 to 8,

$$x = \frac{y^2}{4} = 4$$

$$y/x = x = \frac{y^2}{u} \Rightarrow v = \frac{y^4}{u^2 y} \Rightarrow y = u^{2/3} v^{1/3}$$

$$\& x = u^{1/3} v^{2/3}$$

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{1}{3} (u^{-2/3} v^{2/3}) x \frac{1}{3} (u^{2/3} v^{-2/3}) - \frac{2}{3} (u^{1/3} v^{-1/3}) x \frac{2}{3} (u^{-1/3} v^{1/3})$$

$$= \frac{1}{9} - \frac{4}{9} = -\frac{3}{9} = -\frac{1}{3} =$$

$$\Rightarrow \iint_R xy dxdy = \iint_{u=4, v=4}^8 u^{1/3} v^{2/3} u^{2/3} v^{1/3} |J| dv du =$$

$$= \frac{1}{3} \int_4^8 \int_4^8 [uv] dv du = \frac{1}{3} \left[\int_4^8 u \cdot \frac{v^2}{2} \right]_4^8 du = \frac{192}{3} = \underline{\underline{64}}$$

Q.

Evaluate $\iint_R dx dy$, where R is the region bounded by the curves $y^2 = 8x$, $y^2 = 4x$, $xy = 25$, $xy = 16$.

SOL

let

$$u = \frac{y^2}{x} \quad \text{(i)} \quad v = xy \quad \text{(ii)}$$

u varies from 4 to 8.

v " " " 16 to 25

\Rightarrow from (i) $x = \frac{y^2}{u}$, putting in (ii)

$$v = \frac{y^2}{u} \cdot y \Rightarrow y = u^{1/3} u^{1/3} \therefore x = \frac{u^{2/3} u^{2/3}}{u}$$

$$x = u^{2/3} u^{-1/3}$$

$$\begin{aligned} J &= \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{-1}{3} \left[v^{2/3} \cdot u^{-4/3} \right] \times \frac{1}{3} v^{-4/3} u^{1/3} - \\ &\quad \frac{2}{3} v^{-1/3} \cdot u^{-1/3} \times \frac{1}{3} v^{1/3} u^{-2/3} \end{aligned}$$

$$= -\frac{1}{9} u^{-1} - \frac{2}{9} u^{-1} = -\frac{1}{3u}$$

$$\therefore \iint_R dx dy = \iint_{R'} |J| du dv = \int_{v=16}^{25} \int_{u=v}^{8} \frac{1}{3u} du dv$$

$$\Rightarrow \frac{1}{3} \int_{v=16}^{25} \left[\log u \right]_4^8 dv = \frac{1}{3} \int_{v=16}^{25} (3 \log 2 - 2 \log 2) dv$$

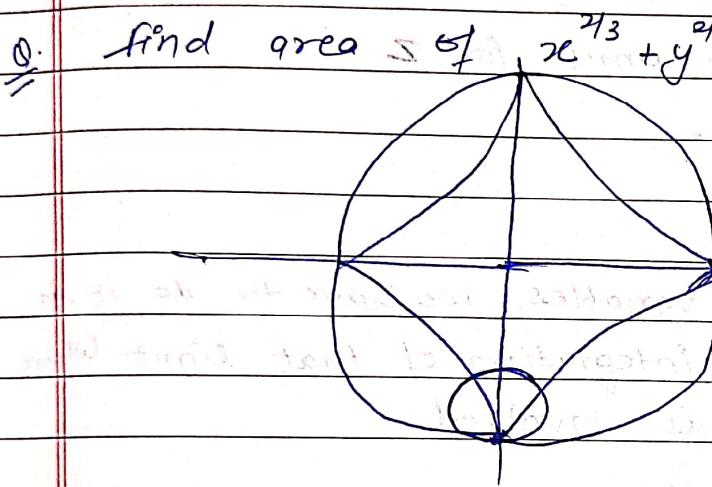
$$= \frac{1}{3} \times \log 2 [v]_{16}^{25} = 3 \log 2 \cancel{dv}$$

Now

- Q Evaluate $\iint \sqrt{a^2 - x^2 - y^2} dx dy$ over the semicircle $x^2 + y^2 = a^2$ in the first quadrant.

$$\text{Q} \quad \iint e^{-(x^2+y^2)} dy dx$$

- Q $\iint r \sin \theta dr d\theta$ over area of the cardioid $r = a(1 + \cos \theta)$ about the initial line.



#

TRIPLE INTEGRATION I

• $\int_{z=4}^8 \int_{y=2}^3 \int_{x=0}^1 dx dy dz \Rightarrow$ do any one of them first
i.e., no effect of order change

• $\int_0^{1-x} \int_0^{1-x-y} \int_0^{1-x} dx dy dz.$

First find out which limit has more no. of variable. i.e., $0 \rightarrow 1-x-y$, since it has x & y .

so this ~~$1-x-y$~~ is the limit for z

$\Rightarrow \int_{z=0}^{1-x-y} \int_{x=0}^{1-x} \int_{y=0}^{1-x} dx dy dz.$

When limits carry variables, we have to do it in order \Rightarrow do the integration of that limit where maximum variable is involved.

$$\text{Q. } \int_0^1 \int_0^1 \int_0^1 e^{x+y+z} dx dy dz = \int_0^1 \int_0^1 \int_0^1 e^x \cdot e^y \cdot e^z dx dy dz \\ = \int_0^1 \int_0^1 (e^x e^y e^z) dy dz = \int_0^1 (e^x e^y) e^z dz \\ \Rightarrow \underbrace{(e^x e^y)}_{z=1}^3 = (e^{x+y})^3$$

$$\text{Q. } \int_0^9 \int_0^9 \left(\int_0^9 (xy + yz + zx) dx \right) dy dz$$

$$\int_0^9 \int_0^9 \left[\frac{yzx^2}{2} + xyz^2 + \frac{zx^2}{2} \right]_0^9 dy dz = \int_0^9 \left[\frac{y^2 z^2}{2} + \frac{9y^2 z}{2} + \frac{9z^2}{2} \right]_0^9 dy dz$$

$$\Rightarrow \left[\frac{q^2 \cdot q^2}{2} + \frac{q \cdot q^2}{2} \cdot \frac{z^2}{2} + \frac{z^2 \cdot q^2 \cdot q}{2} \right]_0^9$$

$$\therefore \frac{q^4}{4} + \frac{q^5}{4} + \frac{q^5}{4} = \frac{3q^5}{4}$$

Q. $\int_{z=0}^{\sqrt{xy}} \int_{y=\frac{1}{x}}^1 \int_{x=1}^3 xyz \, dz \, dy \, dx$

$$\int_{x=1}^3 \int_{y=\frac{1}{x}}^1 \left(\int_{z=0}^{\sqrt{xy}} xyz \, dz \right) dy \, dx = \int_{x=1}^3 \int_{y=\frac{1}{x}}^1 \left[xyz \frac{z^2}{2} \right]_0^{\sqrt{xy}} dy \, dx$$

$$= \int_{x=1}^3 \int_{y=\frac{1}{x}}^1 \frac{xy \cdot (xy)}{2} dy \, dx = \frac{1}{2} \int_{x=1}^3 \left[x^2 \cdot \frac{y^3}{3} \right]_0^1 dx$$

$$= \frac{1}{6} \int_{x=1}^3 \left(-\frac{1}{x} + x^4 \right) dx$$

$$= \frac{1}{6} \left[-\log x + \frac{x^5}{5} \right]_1^3$$

$$= \frac{1}{6} \left[-\log 3 + 9 - \left(-\log 1 + \frac{1}{5} \right) \right]$$

$$\Rightarrow \frac{1}{6} \left[-\log 3 + 9 + \frac{1}{5} \right]$$

$$\Rightarrow -3 \log 3 + 26$$

Q. $\int_{y=0}^1 \int_{x=0}^1 \int_{z=0}^{1-x} x \, dz \, dx \, dy$

• when 2 or more limit have only one variable then do integration as it is given.

$$\Rightarrow \int_{y=0}^1 \int_{x=0}^1 \left(\int_{z=0}^{1-x} x \, dz \right) dx \, dy = \int_{y=0}^1 \int_{x=0}^1 \left[xz \right]_0^{1-x} dx \, dy = \int_{y=0}^1 \int_{x=0}^1 (x - x^2) dx \, dy$$

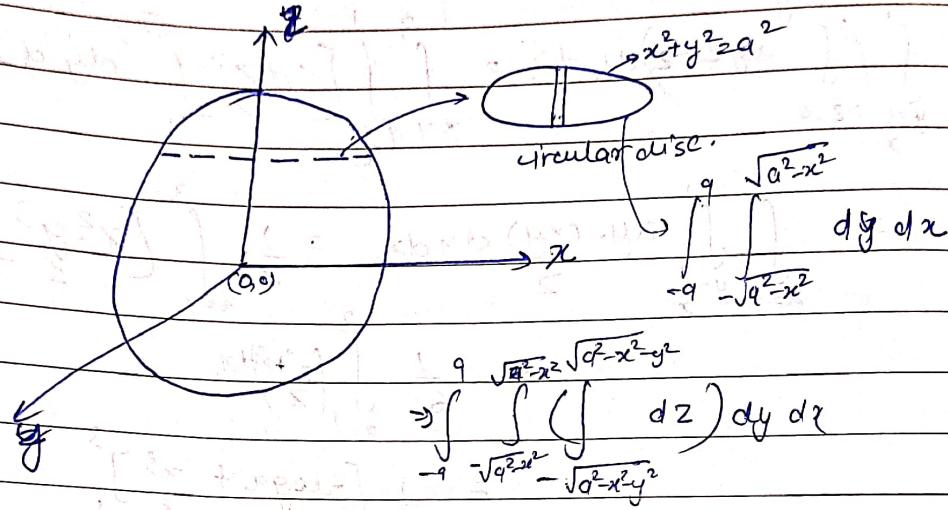
$$= \int_{y=0}^1 \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 dy = \int_{y=0}^1 \left[\left(\frac{1}{2} - \frac{1}{3} \right) - \left(\frac{y^2}{2} - \frac{y^3}{3} \right) \right] dy$$

$$= \int_{y=0}^1 \left(\frac{1}{6} - \frac{y^4}{2} + \frac{y^6}{3} \right) dy = \left[\frac{y}{6} - \frac{y^5}{5 \times 2} + \frac{y^7}{7 \times 3} \right]_0^1$$

$$= \frac{1}{6} - \frac{1}{10} + \frac{1}{21} = \frac{35 - 21 + 10}{210} = \frac{24}{210} = \frac{4}{35}$$

Q.

Volume of a sphere $x^2 + y^2 + z^2 = a^2$ is $\frac{4}{3}\pi a^3$
 find it using triple integration.



$$\Rightarrow \int_{-a}^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} dz dy dx$$

$$\Rightarrow 8 \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dz dy dx$$

$$= 8 \int_0^a \int_0^{\sqrt{a^2-x^2}} (\sqrt{a^2-x^2-y^2}) dy dx$$

$$= 8 \int_0^a \left[\frac{y}{2} \sqrt{a^2-x^2-y^2} + a^2-x^2 \sin^{-1} \frac{y}{\sqrt{a^2-x^2}} \right]_0^{\sqrt{a^2-x^2}} dx$$

$$= 8 \int_0^a \left(\sqrt{a^2-x^2} \sqrt{a^2-x^2-a^2+x^2} + \frac{a^2-x^2}{2} \sin^{-1} 1 \right) dx$$

$$\Rightarrow 48x \int_0^a \left(\frac{a^2-x^2}{2} \right) dx \Rightarrow 4\pi \left[\frac{a^2 \cdot x - x^3}{2} \right]_0^a$$

$$= 4\pi \left[\frac{a^3}{2} - \frac{a^3}{6} \right]$$

$$\Rightarrow \frac{4}{3}\pi a^3$$

Evaluation by Iterated Integrals :-

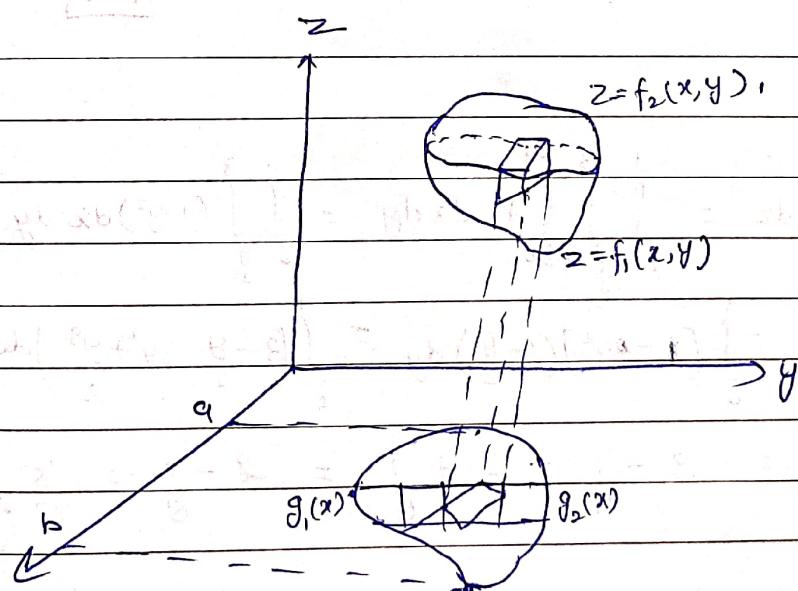
If the region D is bounded above by graph of $z = f_2(x, y)$ & bounded below by graph $z = f_1(x, y)$ then it can be shown that triple integral $\iiint_D F(x, y, z) dV$ can be expressed as double integral of partial integration

$$\int_{f_1(x,y)}^{f_2(x,y)} F(x, y, z) dz$$

$$\iiint_D F(x, y, z) dV = \iint_R \left[\int_{f_1(x,y)}^{f_2(x,y)} F(x, y, z) dz \right] dA.$$

where R is the orthogonal projection of D onto the x - y plane. In particular if R is a type I region then the triple integration of F over D can be written as an iterated integral,

$$\iiint_D F(x, y, z) dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{f_1(x,y)}^{f_2(x,y)} F(x, y, z) dz dy dx.$$



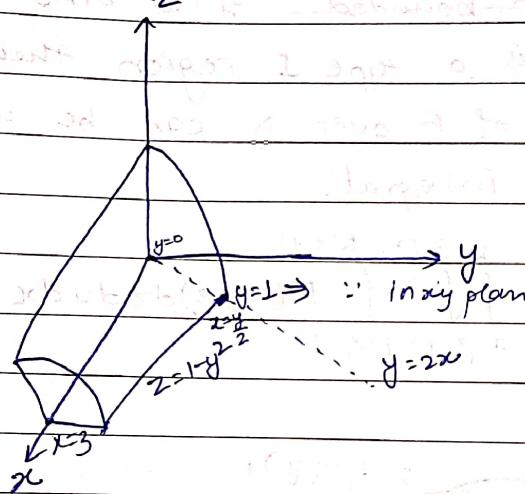
Q

find out the volume of solid in the first octant bounded by the graphs of $z = 1 - y^2$; $y = 2x$; $x = 3$.

Sol:

As indicated in fig., the first integration w.r.t. z is from 0 to $1 - y^2$. furthermore, we see that the projection of solid D in the xy -plane is a region type II. Hence, we will now integrate w.r.t. x from 0 to $\frac{y^2}{2}$. The last integration is w.r.t. y from 0 to 3 .

from 0 to $\frac{y^2}{2}$



$$\therefore \boxed{y=3}$$

$$\begin{aligned}
 V &= \iiint_D dx dy dz = \int_0^3 \int_0^{\frac{y^2}{2}} \int_{z=0}^{1-y^2} dz dx dy = \int_0^3 \int_0^{\frac{y^2}{2}} (1-y^2) dx dy \\
 &= \int_0^3 [x(1-y^2)] \Big|_0^{\frac{y^2}{2}} dy = \int_0^3 \left(\frac{y^2}{2} - 3y^2 + \frac{y^3}{2} \right) dy \\
 &= 3 - \frac{1}{4} - 1 + \frac{1}{8} = 2 - \frac{1}{8} = \frac{15}{8} \text{ cu. units}
 \end{aligned}$$

changing the order of integration:-

$\int_0^6 \int_0^{4-2x/3} \int_{3-x/2}^{3-x/2 - 3y/4} f(x, y, z) dz dy dx$, to convert $dz dy dx$
 to $dy dz dx$.

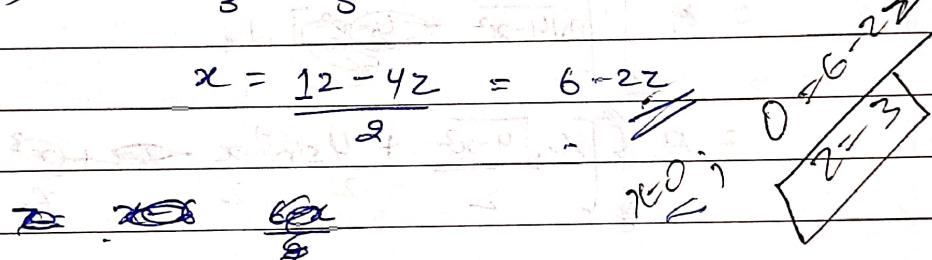
$$z = 3 - \frac{x}{2} - \frac{3y}{4} \Rightarrow 4z = 12 - 2x - 3y$$

$$y = \frac{12 - 2x - 4z}{3} = 4 - \frac{2x}{3} - \frac{4z}{3}$$

Now, for dx , we need the exp. of x in terms of z

We know that, in xz -plane $y=0$

$$\Rightarrow 4 - \frac{2x}{3} - \frac{4z}{3} = 0$$



Volume of Ellipsoid:-

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \Rightarrow \int_{-a}^a \int_{-\sqrt{1-x^2/a^2}}^{\sqrt{1-x^2/a^2}} \int_{-\sqrt{1-x^2/a^2-y^2/b^2}}^{\sqrt{1-x^2/a^2-y^2/b^2}} dz dy dx$$

$$z = \pm c \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$$

$$y = \pm b \sqrt{1 - \frac{x^2}{a^2}}$$

$$x = \pm a$$

$$\text{Let. } t = b \sqrt{1 - \frac{x^2}{a^2}} \Rightarrow \frac{1-x^2}{a^2} = \frac{t^2}{b^2}$$

$$\therefore 2 \int_{-a}^a \left(\int_t^{\sqrt{1-x^2/a^2}} \int_{-\sqrt{1-t^2/b^2}}^{\sqrt{1-t^2/b^2}} dy dz \right) dx$$

$$\Rightarrow 2\pi x \int_{-a}^a \frac{c}{b} \left[\frac{y\sqrt{t^2-y^2}}{2} + \frac{t^2 \sin^{-1} y}{2} \right]_{-t}^t dx = 4 \int_{-a}^a c t \sin^{-1} \frac{y}{t} dy = 4c \left(\frac{b^2 - b^2 x^2}{a^2} \right) \frac{\pi}{2} dx$$

$$\Rightarrow 2\pi x c \left[\frac{b^2 x}{6} - \frac{b^2 x^3}{3a^2} \right]_{-a}^a \Rightarrow 2\pi x \left[2ba - \frac{b^3 a^8}{3a^2} - \left(\frac{-b^3 (-a)^8}{3a^2} \right) \right] = 2\pi x \frac{4}{3} b^2 a x c$$

- Q. find the volume of solid bounded by the cylinder $x^2 + y^2 = 4$ & planes $x \pm y + z = 4$ and $z = 0$.

$$z = 0 \text{ to } z = 4 - y$$

$$y = \pm \sqrt{4 - x^2}$$

$$x = \pm 2$$

$$V = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^{4-y} dz dy dx$$

$$= 8 \int_{-2}^2 \int_0^{4-x^2} (4-y) dy dx = 8 \int_{-2}^2 \int_0^{4-x^2} (4y - \frac{y^2}{2}) dy dx$$

$$= 8 \int_{-2}^2 \left[8\sqrt{4-x^2} - \frac{(4-x^2)^2}{2} \right] dx$$

$$= 8 \left[\frac{x\sqrt{4-x^2}}{2} + \frac{4\sin^{-1}x}{2} - \frac{x^2+4}{6} \right]_0^2$$

$$= 8 \left[2x_0 + \frac{4}{2} x \sin^{-1} x |_0^2 + \frac{2\sin^{-1} 2}{2} \right]$$

$$= 8 \left[4x - 4 \left(\frac{3}{2} \right) \right]$$

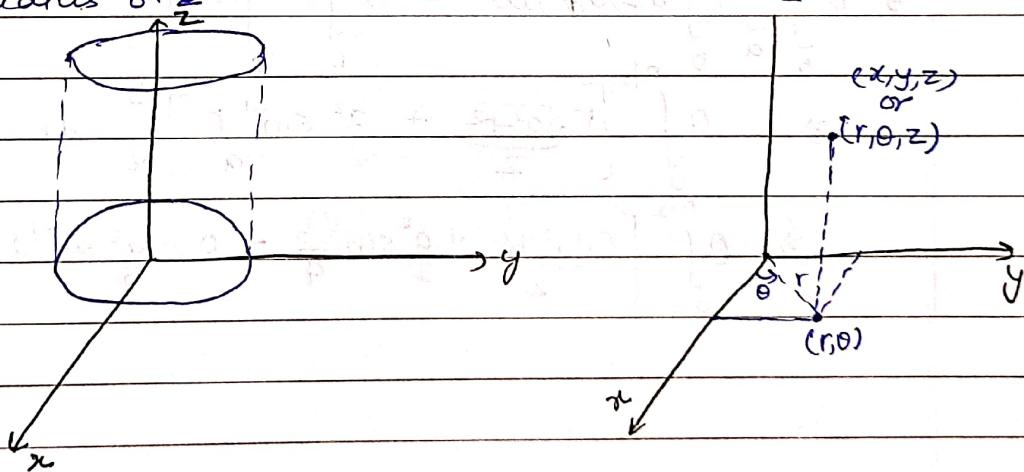
$$= 4 \cancel{8} \cancel{16} + \frac{32 \cancel{16}}{\cancel{8} \cancel{3}} \cancel{4} \cancel{3} \cancel{2} \cancel{4} \cancel{8}$$

Q Use double integral to find the volume of tetrahedron bounded by the coordinate planes and plane $z = 4 - 9x - 2y$.

Q: $\iint_R (x+1) dA$; $y = x$, $x+y = 4$, $x=0$.

Q $\iint_R 2xy dA$; $y = x^3$, $y = 8$, $x = 0$.

cylindrical coordinates: The cylindrical coordinate system combines the polar description of a point in the plane with the rectangular description of z component of a point in space. As seen in fig., cylindrical coordinates of a point P are denoted by the ordered triple (r, θ, z) . The coordinate cylindrical arises from the fact that a point P in space is determined by the intersection of planes $z = \text{constant}$ and $\theta = \text{constant}$ with a cylinder of radius r .

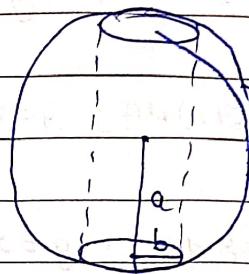


Q.

A cylindrical hole of radius 'b' is bored through sphere of radius 'a' such that $b < a$. Find the volume of remaining solid.

$$V_{\text{remaining}} = \frac{4}{3}\pi a^3 - \text{Vol. of hollow part}$$

(cylindrical part).



Let the vol. eqn of sphere :-

$$x^2 + y^2 + z^2 = a^2$$

for cylindrical coordinates,

$$x = r \cos \theta, y = r \sin \theta, z = z$$

limits of z are from 0 to $\sqrt{a^2 - r^2}$ or 0 to $\sqrt{a^2 - r^2}$

" " r are from a to b

" " θ are from 0 to $\pi/2$.

$$\Rightarrow \int_0^{\pi/2} \left(\int_a^b \left(\int_r^{\sqrt{a^2 - r^2}} dz \right) dr \right) d\theta = \int_0^{\pi/2} \left(\int_a^b \sqrt{a^2 - r^2} dr \right) d\theta$$

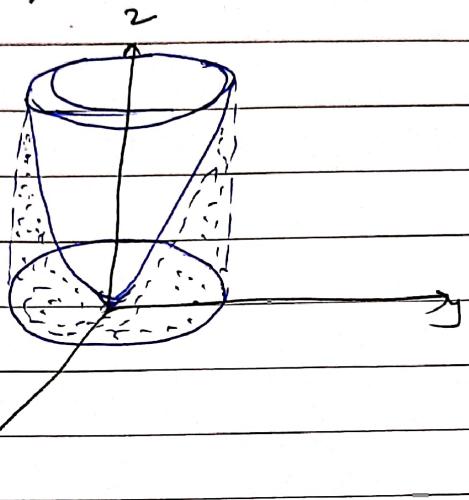
$$\Rightarrow \int_0^{\pi/2} \left[r \frac{\sqrt{a^2 - r^2}}{2} + \frac{a^2 \sin^{-1} r}{2} \right]_a^b d\theta$$

$$\Rightarrow \int_0^{\pi/2} \left[r \frac{\sqrt{a^2 - b^2}}{2} + \frac{a^2 \sin^{-1} b}{2} - 0 - \frac{a^2 \sin^{-1} a}{2} \right] d\theta$$

(Ans)

Q. find the volume ^ of the paraboloid and the cylinder bounded by the paraboloid and the cylinder

$$x^2 + y^2 = a^2$$



thus- $\frac{\pi a^3}{2}$