Ordinary Differential Equation :- (ODE)

Linear Differential Eqⁿ of second order with variable coefficients

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x).y = F(x).$$

Linear DE of second order with constant coefficients

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} + by = 0. , \quad a,b \to \text{constant}.$$

ex:- ① $\frac{d^2y}{dx^2} \cancel{+} 5\frac{dy}{dx} + 6y = 0.$

Auxiliary eqⁿ :- $m^2 \cancel{+} 5m + 6 = 0.$

$m = \cancel{2,3}\ 2,3.$

$\therefore y = C_1e^{3x} + C_2e^{3x} = C_1e^{3x} + C_2e^{2x}$;

② $\frac{d^2y}{dx^2} \cancel{+} - 4\frac{dy}{dx} + 4y = 0.$

auxiliary eqⁿ :- $m^2 - 4m + 4 = 0$

$m = 2,2.$

$\therefore y = (C_1 + C_2x)e^{2x}$

suppose $m = 2,2,2.$

$y = (C_1 + C_2x + C_3x^2)e^{2x}.$

Nature of roots :-

Nature of roots of Aux. eqⁿ	set of linear independent solⁿ	General solⁿ / Particular solⁿ.
① Distinct real roots ; m_1 & m_2	e^{m_1x}, e^{m_2x}	$y = C_1e^{m_1x} + C_2e^{m_2x}$
② Real repeated roots ; $m_1 = m_2 = m$	e^{mx}, xe^{mx}	$y = C_1e^{mx} + C_2xe^{mx}$
③ Complex conjugate ; $m_1 = p+iq,\ m_2 = p-iq$	$e^{px}\cos qx$, $e^{px}\sin qx$	$e^{px}(C_1\cos qx + C_2\sin qx)$
④		

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Solve : $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 12y = 0$

auxiliary eqn : $m^2 - m - 12 = 0$.

$m^2 - 4m + 3m - 12 = 0$.

$m(m-4) + 3(m-4) = 0$

$\boxed{m = -3, 4}$

$\therefore$ General soln :-

$[y = c_1 e^{-3x} + c_2 e^{4x}]$.

verification :- $\frac{dy}{dx} = c_1 e^{-3x}(-3) + c_2 e^{4x}(4)$

$\frac{d^2y}{dx^2} = -3c_1 e^{-3x}(-3) + 4c_2 e^{4x}(4)$

$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 12y = (+3c_1 e^{-3x} - 4c_2 e^{4x}) + 9c_1 e^{-3x} + 16c_2 e^{4x} - 12c_1 e^{-3x} - 12c_2 e^{4x}$

$\Rightarrow 0$

Hence verified

Solve $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$, $y(0) = -1$, $y'(0) = 0$.

auxiliary eqn $m^2 - 3m + 2 = 0$

$\boxed{m = 1, 2}$

general soln :- $y = c_1 e^{x} + c_2 e^{2x} = y(x)$.

it is given that $y(0) = -1$

i.e. $c_1 e^0 + c_2 e^{2(0)} = -1$

$c_1 + c_2 = -1$ — (i)

also, $y'(0) = 0$.

$y'(x) = c_1 e^x + 2c_2 e^{2x}$

$y'(0) = c_1 + 2c_2 = 0$.

$c_1 = -2c_2$ — (ii)

from (i) & (ii)

$-2c_2 + c_2 = -1$

$-c_2 = -1 \Rightarrow \boxed{c_2 = 1}$

$\boxed{c_1 = -2}$

therefore -

$\boxed{y = -2e^{x} + e^{2x}}$

particular soln

Q. solve - (1) $4\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 5y = 0$

(2) $2\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$

(3) $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 8y = 0$; $y(0) = 1$, $y'(0) = 6$.

(4) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + y = 0$.

sol^n (1) $4\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 5y = 0$

auxilary eq^n :- $4m^2 - 12m + 5 = 0$

$4m^2 - 10m - 2m + 5 = 0$

$2m(2m-5) - 1(2m-5) = 0$

$m = \frac{5}{2}, \frac{1}{2}$

general eq^n :- $y = c_1 e^{\frac{5}{2}x} + c_2 e^{\frac{1}{2}x}$.

(2) auxiliary eq^n :- $2m^2 + m - 6 = 0$

$2m^2 + 4m - 3m - 6 = 0$

$2m(m+2) - 3(m+2) = 0$

$m = \frac{3}{2}, -2$

general eq^n :- $y = c_1 e^{3/2 x} + c_2 e^{-2x}$.

(3) auxiliary eq^n :- $m^2 - 6m + 8 = 0$

$m^2 - 4m - 2m + 8 = 0$

$(m-4)(m-2) = 0$.

$m = 2, 4$

general eq^n :- $y = c_1 e^{2x} + c_2 e^{4x}$, $= y(x)$

$y(0) = 1$

$c_1 + c_2 = 1$ — (i)

$y'(x) = 2c_1 e^{2x} + 4c_2 e^{4x}$

$y'(0) = 2c_1 + 4c_2 = 6$

$c_1 + 2c_2 = 3$ — (ii)

from (i) + (ii)

$c_2 + 3 - 2c_2 = 1$

$-c_2 = -2 \Rightarrow \boxed{c_2 = 2}$

$\boxed{c_1 = -1}$

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(4) auxiliary eq^n :- $m^2 - 4m + 4 = 0$

$m = 2, 2.$

general eq^n :- $y = (c_1 + xc_2)e^{2x}$ Ans.

solve:- $\frac{d^4y}{dx^4} + 6\frac{d^3y}{dx^3} + 9\frac{d^2y}{dx^2} = 0.$

Auxiliary Eq^n :- $m^4 + 6m^3 + 9m^2 = 0$

$m^2(m^2 + 6m + 9) = 0$

$m^2(m+3)^2 = 0.$

$[m = 0, 0, -3, -3]$

General sol^n :-

$y = (c_1 e^{0x} + c_2 x e^{0x}) + (c_3 e^{-3x} + c_4 x e^{-3x})$

$= c_1 + c_2 x + c_3 e^{-3x} + c_4 x e^{-3x}.$

$4\frac{d^4y}{dx^4} - 8\frac{d^3y}{dx^3} - 7\frac{d^2y}{dx^2} + 11\frac{dy}{dx} + 6y = 0.$

auxiliary eq^n :-

$4m^4 - 8m^3 - 7m^2 + 11m + 6 = 0.$

$4m^4 - 8m^3 - 5m^2 - (2m^2 - 11m - 6) = 0.$

$m^2(4m^2 - 8m - 5) - (2m^2 - 11m - 6) = 0.$

$m^2(4m^2 - 10m + 2m - 5) - (2m^2 - 12m + m - 6) = 0.$

$m^2(2m+1)(2m-5) - (2m+1)(m-6) = 0.$

$(2m+1)[m^2(2m-5) - (m-6)] = 0.$

$(2m+1)(2m^3 - 5m^2 - m + 6) = 0.$

$(m = -1)$

$(2m+1)(m+1)(2m^2$

$m = -\frac{1}{2}, -1, \frac{3}{2}, 2.$

gen. $eq^n \Rightarrow c_1 e^{-x} + c_2 e^{-\frac{1}{2}x} + c_3 e^{3/2 x} + c_4 e^{2x}.$

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Q $\frac{d^4y}{dx^2} + 18\frac{d^2y}{dx^2} + 81y = 0.$

Auxiliary eqn:

$m^4 + 18m^2 + 81 = 0.$

$(m^2+9)^2 = 0.$

$m = \pm 3i$ (double)

$= 0 \pm 3i$ (double)

$y = e^{0x}[(c_1 + c_2x)\cos 3x + (c_3 + c_4x)\sin 3x]$

HW solve:

① $\frac{d^3y}{dx^3} + 4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4 = 0.$

② $\frac{d^4y}{dx^4} - 4\frac{d^3y}{dx^3} + 14\frac{d^2y}{dx^2} - 20\frac{dy}{dx} + 25y = 0.$

③ $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 4y = 0$

④ $\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 18y = 0$

Method of finding Particular Integral (PI):-

Notations:- $\frac{d}{dx} = D$; $\frac{d}{dy} = D'$

$\frac{d^2}{dx^2} = D^2$; $\frac{1}{D} = \int dx$.

$y = CF + PI$

ex:- $\frac{d^2y}{dx^2} + \frac{4dy}{dx} + 4y = e^x$

aux. eq^n:- $m^2 + 4m + 4 = 0$.

$\Rightarrow m = -2, -2$.

CF $\Rightarrow c_1 e^{-2x} + c_2 x e^{-2x}$.

$$PI = \frac{1}{D^2+4D+4} e^{1.x}$$

$$= \frac{1}{(1)^2+4(1)+4} = \frac{1}{9} e^x.$$

\# When $y = e^{ax}$, $a \to$ constant.

(i) $PI = \frac{1}{f(D)} e^{ax} = \frac{e^{ax}}{f(a)}$; provided $f(a) \neq 0$.

(ii) $PI = \frac{1}{f(D-a)} e^{ax} = \frac{x^n}{n!} e^{ax}$; $n = 1, 2, 3, \ldots\ldots$

To solve:- ex:- $\frac{1}{D^2-4} e^{ax} = \frac{\int 1 dx}{\frac{d}{dD}(D^2-4)} e^{ax}$

$$= \frac{x}{2D} e^{ax} = \frac{x^2}{2} e^{ax}.$$

Solve: $D^2(D+1)^2(D^2+D+1)^2 y = e^x$.

auxilliary eq^n:- $m^2(m+1)^2(m^2+m+1)^2 = 0$.

$m = 0, 0, -1, -1, \left(-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}\right)$ double.

$$CF = (c_1 + c_2x) + (c_3 + c_4x)e^{-x} + e^{-\frac{1}{2}x}\left[(c_5 + c_6x)\cos\frac{\sqrt{3}}{2}x + (c_7 + c_8x)\sin\frac{\sqrt{3}}{2}x\right]$$

$$PI = \frac{1}{D^2(D+1)^2(D^2+D+1)^2} e^{1.x} = \frac{e^x}{(1)^2(2)^2(3)^2} = \frac{1}{36} \cdot e^x.$$

H/W: (i) $(D+2)(D-1)^3 y = e^x$

(ii) $(D^2+4D+4)y = 2\sinh 2x. = e^{2x} - e^{-2x}.$

(iii) $(D^3-1)y = (e^x+1)^2$

When $y = \sin ax / \cos ax.$

ex:- ~~PI~~ $(D^2+4D+4)y = \sin 2x$

$$PI = \frac{1}{(D^2+4D+4)} \cdot \sin 2x = \sin 2x$$

$$= \frac{1}{-(2^2)+4D+4} \sin 2x = \frac{1}{4} \cdot \frac{1}{D} \sin 2x$$

$$= \frac{1}{4}\int \sin 2x \, dx.$$

when $f(x)$ can be expressed as $\phi(D^2)$ and $\phi(-a^2) \neq 0$

we shall have the following formula.

$$\frac{1}{\phi(D^2)} \sin ax = \frac{1}{\phi(-a^2)} \sin ax$$

Similarly for $\cos ax$.

provided $\phi(-a^2) \neq 0.$

when $f(D)$ can be expressed as $\phi(D^2)$ where $\phi(-a^2) = 0.$

then, $\frac{1}{f(D)} e^{ax} V = e^{ax} \frac{1}{f(D+a)} V$, $V = V(x).$

$\frac{1}{D^2+4} e^{2x} \cdot x = e^{2x} \frac{1}{(D+2)^2+4} x.$

Q. Solve:- $(D^2+1)y = \cos 2x$.

A.E.:- $(m^2+1) = 0$

$m = \pm i = 0 \pm i$

$\therefore$ CF $\Rightarrow e^{0x}[c_1\cos x + c_2\sin x]$.

PI :- $\frac{1}{D^2+1}\cos 2x = \frac{1}{-2^2+1}\cos 2x = \frac{1}{-3}\cos 2x$.

$\therefore$ The general sol^n is := CF + PI

$y = [c_1\cos x + c_2\sin x - \frac{1}{3}\cos 2x]$

Q. $(D^2 - 3D + 2)y = \sin 3x$.

AE:- $m^2 - 3m + 2 = 0$.

$(m-2)(m-1) = 0$.

$m = 1, 2. \Rightarrow$ CF $= c_1e^x + c_2e^{2x}$.

PI :- $\frac{1}{D^2-3D+2}\sin 3x = \frac{1}{-3^2-3D+2}\sin 3x = \frac{1}{-3D-7}\sin 3x$.

$\Rightarrow \frac{1\,(3D-7)\sin 3x}{-(3D+7)(3D-7)} = \frac{(3D-7)}{-(9D^2-49)}\sin 3x = \frac{3D-7\sin 3x}{(9(-3^2)-49)}$

$\Rightarrow \frac{3D-7}{-130}\sin 3x = \frac{1}{130}\left[3\frac{d}{dx}\sin 3x - 7\sin 3x\right]$

$\Rightarrow \frac{1}{130}[9\cos 3x - 7\sin 3x]$

$\therefore$ The general solution :$\Rightarrow$ CF + PI.

$\therefore y = c_1e^x + c_2e^{2x} + \frac{1}{130}[9\cos 3x - 7\sin 3x]$

Q. $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 9y = 40\sin 5x.$

$\frac{1}{40}\frac{d^2y}{dx^2} - \frac{8}{40}\frac{dy}{dx} + \frac{9}{40}y = \sin 5x.$

AE:- $m^2 - 8m + 9 = 0.$

$m = \frac{-(-8) \pm \sqrt{64-36}}{2} = \frac{8 \pm \sqrt{28}}{2}$

$= 4 \pm \sqrt{7} \Rightarrow 4+\sqrt{7},\ 4-\sqrt{7}.$

$CF = C_1 e^{4+\sqrt{7}} + C_2 e^{4-\sqrt{7}}.$

PI :- $\frac{1}{\frac{1}{40}(D^2-8D+9)} \sin 5x = \frac{40\sin 5x}{D^2-8D+9}.$

$= \frac{40\sin 5x}{-5^2-8D+9} = \frac{40\sin 5x}{-16-8D} = \frac{\overset{5}{\cancel{40}}\sin 5x}{-\cancel{8}(2+D)}$

D^2-4 $= \frac{1(D-2)\,5\sin 5x}{(D+2)(D-2)} = \frac{-5(D-2)\sin 5x}{-5^2-4}$

$\Rightarrow \frac{5}{29}\left[\frac{d}{dx}\sin 5x - 2\sin 5x\right] = \frac{5}{29}\left[5\cos 5x - 2\sin 5x\right]$

Finding the PI when $y = x^m$ ($m \to$ +ve integer).

step-1: Bring out the ~~largest~~ lowest degree term from $f(D)$, so that the remaining factor in the denominator is of the form $[1+\phi(D)]^n$ or $[1-\phi(D)]^n$, $n \to$ +ve integer

step-II: We take $[1+\phi(D)]^n$ or $[1-\phi(D)]^n$ in the numerator so that it takes the form $[1+\phi(D)]^{-n}$ or $[1-\phi(D)]^{-n}$

step-III: Expand ↑ in Binomial theorem.

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solve:- $(D^4-D^2)y = x^2$

$\cancel{D^2(D^2-1)}$ AE $\Rightarrow$ $m^4 - m^2 = 0$.

$m = 0, 0, -1, 1.$

PI :- $\frac{1}{D^4-D^2} x^2 = \frac{1}{D^2} \cdot \frac{1}{[D^2-1]} x^2 = \cancel{\frac{-1}{D^2}}$

$= -\frac{1}{D^2}[1-D^2]^{-1} x^2 = \frac{-1}{D^2}[1+D^2+\dots] x^2$

$= \frac{-1}{D^2}[x^2+2] = \frac{-1}{D}\left[\frac{x^3}{3} + 2x\right]$

$= \frac{-1}{1}\left[\frac{x^4}{12} + x^2\right] = -\left[\frac{x^4}{12} + x^2\right].$

Q solve:- $(D^2+D)y = x^2+2x+4.$

AE :- $m^2+m = 0$.

$m = 0, -1.$

CF $= c_1 + c_2 e^{-x}.$

PI $= \frac{1}{D^2+D}(x^2+2x+4)$

$= \frac{1}{D}\left[\frac{1}{D+1}\right](x^2+2x+4).$

$\Rightarrow \frac{1}{D}[1+D]^{-1}(x^2+2x+4) = \frac{1}{D}\left[1 - D + \frac{D^2}{2!} \dots\right](x^2+2x+4)$

$= \frac{1}{D}\left[x^2+\cancel{2x}+4-\cancel{2x}-2+\frac{1}{\cancel{2}}\times\cancel{2}\right] = \frac{1}{D}(x^2+3).$

$= \frac{x^3}{3} + 3x.$

H/w

(1) $(D^3 - D^2 - 6D)y = x^2+1.$

(2) $(D^2+2D+1)y = 2x+x^2.$

(3) $(D^4 - 2D^3 + 5D^2 - 8D + 2)y = x^2.$

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polynomial.

Method of Finding PI when $y = e^{ax} \cdot v(x)$.

$$\frac{1}{f(D)} e^{ax} v(x) = e^{ax} \frac{1}{f(D+a)} v(x).$$

solve: $(D^2-2D+1)y = x^2e^{3x}$.

$$PI :- \frac{1}{(D^2-2D+1)} x^2 e^{3x} = e^{3x} \cdot \frac{1}{(D+3)^2-2(D+3)+1} \cdot x^2$$

$$= e^{3x} \frac{1}{D^2+6D+9-2D-6+1} \cdot x^2$$

$$= e^{3x} \cdot \frac{1}{D^2+4D+4} \cdot x^2.$$

$$= e^{3x} \cdot \frac{1}{(D+2)^2} x^2 = \frac{e^{3x}}{4} \left(1+\frac{D}{2}\right)^{-2} x^2$$

$$= \frac{1}{8} e^{x}[2x^2-2x+3].$$

solve: $(D^2+3D+2)y = e^{2x}\sin x$.

A.E :- $m^2+3m+2=0$.

$(m+2)(m+1)=0$

$m=-1,-2$

$CF \Rightarrow c_1e^{-x}+c_2e^{-2x}$.

$$AI :- \frac{1}{(D^2+3D+2)} e^{2x} \cdot \sin x$$

$$= e^{2x} \cdot \frac{1}{(D+2)^2+3(D+2)+2} \sin x$$

$$= e^{2x} \cdot \frac{1}{D^2+7D+12} \sin x = e^{2x} \cdot \frac{1}{-1^2+7D+12} \sin x$$

$$= e^{2x} \cdot \frac{1}{(7D+11)(7D-11)} (7D-11) \sin x = e^{2x} \frac{(7D-11)}{49D^2-121} \sin x.$$

$$= \frac{e^{2x}(7\cos x - 11\sin x)}{49(-1^2)-121} = \frac{-1}{170} e^{2x}(7\cos x - 11\sin x)$$

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Exact Differential Equation :-

A differential equation, $f\left(d^n y/dx^n, d^{n-1}y/dx^{n-1}, \ldots dy/dx, y\right) = \phi(x)$, ① is said to be exact when it can be derived by differentiation only & without any further process, from an eqn of next lower order of the form.

$$f\left(d^{n-1}y/dx^{n-1}, d^{n-2}y/dx^{n-2}, \ldots dy/dx, y\right) = \int \phi(x)dx + C \quad (2)$$

Remark-1:- Eqn ② is said to be first integral of ①. If ② is also exact, then it can be obtained from an equation of next lower order of form.

$$f\left(d^{n-2}y/dx^{n-2}, d^{n-3}y/dx^{n-3}, \ldots dy/dx, y\right) = \iint \phi(x)(dx)^2 + C \quad (3)$$

as before eqn ③ is said to be a second integral of ①. In general there will be n integrals for a differential eqn of n^{th} order.

Condition of exactness of a linear differential equation of order n:-

Let the linear diff. eqn of order n be :

$P_0(dy^n/dx^n) + P_1(d^{n-1}y/dx^{n-1}) + \ldots P_n y = \phi(x)$ ①, where $P_0, P_1, \ldots P_n$ & ϕ are functions of x alone. Let ① be exact i.e., it can be obtained from an eqn of lower order simply by differention. In what follows the successive derivatives will be denoted by dashes. Since $P_0(d^ny/dx^n)$ can be easily obtained by simply differentiating once $P_0(d^ny/dx^n)$

$Q_1 = P_1 - P_0'$, $Q_2 = P_2 - P_1' + P_0''$, $Q_3 = P_3 - P_2' + P_1'' - P_0'''$.

$Q_{n-1} = P_{n-1} - P_{n-2}' + P_{n-3}'' - P_{n-4}''' - \dots + (-1)^{n-1} P_0^{(n-1)}$

Ex: solve $(1+x+x^2)(d^3y/dx^3) + (3+6x)(d^2y/dx^2) + 6(dy/dx) = 0$

standard form:

$(1+x+x^2)(d^3y/dx^3) + (3+6x)(d^2y/dx^2) + 6(dy/dx) + 0.(y) = 0$ — ①

comparing with ① general eqn

$P_0 = 1+x+x^2$, $P_1 = 3+6x$ & $P_2 = 6$, $P_3 = 0$, $\phi(x) = 0$ — ②

The given eqn will be exact if,

$P_3 - P_2' + P_1'' - P_0''' = 0$ — ③

LHS:- $0 - \frac{d}{dx}(6) + \frac{d^2}{dx^2}(3+6x) - \frac{d^3}{dx^3}(1+x+x^2) = 0-0+0-0$
$= 0$

$\boxed{LHS = 0}$ ⇒ the given eqn is exact.

and the first integral is:

$P_0(d^2y/dx^2) + (P_1 - P_0')dy/dx + (P_2 - P_1' + P_0'')y = C_1$ — ④

using ②

$(1+x+x^2)(d^2y/dx^2) + (2+4x)dy/dx + (+2)y = C_1$ — ⑤

Now, we shall repeat the whole process for ⑤.

on comparing ⑤ with general eqn:

$P_0 = 1+x+x^2$, $P_1 = 2+4x$, $P_2 = 2$, $\phi(x) = C_1$.

Here, $P_2 - P_1' + P_0'' = 2 - 4 + 2 = 0$

⇒ eqn ⑤ is also exact its integral will be

second integral of ①.

$P_0(dy/dx) + (P_1 - P_0')y = \int C_1 dx + C_2$ — ⑥

⇒ $(1+x+x^2)dy/dx + (1+2x)y = C_1x + C_2$ — ⑦.

again examine for ⑦

$P_0 = 1+x+x^2$, $P_1 = 1+2x$, $\phi(x) = C_1x + C_2$.

$P_1 - P_0' = 1+2x-(1+2x) = 0$ ⇒ ⑦ is exact & its integral

$P_0 y = \int (C_1x + C_2)dx + C_3 = \frac{C_1x^2}{2} + C_2x + C_3$.

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Q. solve $x(d^3y/dx^3) + (x^2+x+3)(d^2y/dx^2) + (4x+2)dy/dx + 2y = 0$ — (1)

on comparing with general eq^n:

$P_0 = x$, $P_1 = x^2+x+3$, $P_2 = 4x+2$, $P_3 = 2$, $\phi(x) = 0$.

$P_3 - P_2' + P_1'' - P_0''' = 2 - 4 + 2 - 0 = 0$

$\Rightarrow$ (1) is exact, & its first integral will be:

$P_0\, d^2y/dx^2 + (P_1 - P_0')\frac{dy}{dx} + (P_2 - P_1' + P_0'')y = C_1$

$$\Rightarrow x\frac{d^2y}{dx^2} + (x^2+x+2)\frac{dy}{dx} + (2x+1)y = C_1 \quad \text{— (2)}$$

again: $P_0 = x$, $P_1 = x^2+x+2$, $P_2 = 2x+1$ & $\phi(x) = C_1$

$P_2 - P_1' + P_0'' = 2x+1-(2x+1)+0 = 0 \Rightarrow$ (2) is exact

& its first integral will be —

$$P_0\frac{dy}{dx} + (P_1 - P_0')y = \int C_1\,dx + C_2$$

$$x\frac{dy}{dx} + (x^2+x+1)y = C_1x + C_2 \quad \text{— (3)}$$

again: $P_0 = x$, $P_1 = x^2+x+1$, $\phi(x) = C_1x + C_2$

$P_1 - P_0' = x^2+x+1-1 = x^2+x$

(3) is not exact: dividing by x:-

$$\frac{dy}{dx} + \left(x+1+\frac{1}{x}\right)y = C_1 + \frac{C_2}{x} \quad \text{— (4)}$$

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Q. Show that $\cos x\, y'' + 2\sin x\, y' + 3\cos x . y = \tan^2 x$ is exact.

$P_0 = \cos x, \quad P_1 = 2\sin x, \quad P_2 = 3\cos x \quad \& \phi(x) = \tan^2 x$. (1)

$P_2 - P_1' + P_0'' = 3\cos x - 2\cos x - \cos x = \underline{\underline{0}}$

$\therefore$ It is an exact.

Q. Show that $(1+x^2)y'' + 3xy' + y = 1+3x^2$ is exact & hence solve — (1)

$P_0 = 1+x^2, \quad P_1 = 3x, \quad P_2 = 1$

$P_2 - P_1' + P_0'' = 1 - 3 + 2 = \underline{\underline{0}} \Rightarrow$ (1) is an exact.

Its first integral is —

$P_0 \frac{dy}{dx} + (P_1 - P_0')y = \int (1+3x^2)\,dx + C_1$

$(1+x^2)\frac{dy}{dx} + (x)y = x + x^3 + C_1$

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Q Test for exactness & solve $(1+x^2)y'' + 4xy' +$

Method of Grouping:-

We shall now solve the differential equation of this example by grouping of terms in such a way that its left member appears as the sum of certain exact differentials. We write the differential equation,

$$(3x^2 + 4xy)dx + (2x^2 + 2y)dy = 0.$$

$$3x^2dx + (4xy\,dx + 2x^2dy) + 2ydy = 0$$

$$d(x^3) + d(2x^2y) + d(y^2) = 0.$$

on integrating :-

$$x^3 + 2x^2y + y^2 = C.$$

Q. $(2x\cos y + 3x^2y)\,dx + (x^3 - x^2\sin y - y)\,dy = 0$, $\{y(0)=2$

$$2x\cos y\,dx + 3x^2y\,dx + x^3dy - x^2\sin y\,dy - ydy$$

$$\Rightarrow d(x^2\cos y) + d(x^3y) - d\left(\frac{y^2}{2}\right) = 0.$$

on integrating, we get -

$$x^2\cos y + x^3y - \frac{y^2}{2} = C$$

For $x=0$, $y=2$. [C → const.]

$$\therefore 0^2(\cos 2) + 0^3(2) - \frac{2^2}{2} = C \Rightarrow [C = -2]$$

$$\therefore \left[x^2\cos y + x^3y - \frac{y^2}{2} + 2 = 0\right]$$

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Integrating factor:-

If the differential eqn :- $M(x,y)\,dx + N(x,y)\,dy = 0$ is not exact in a domain D but the diff. eqn $\mu(x,y)\,M(x,y)\,dx + \mu(x,y)\,N(x,y)\,dy = 0$ is exact in D then, $\mu(x,y)$ is called integrating factor of the differential equation.

Q. Consider the diff. eqn :- $(3y+4xy^2)\,dx + (2x+3x^2y)\,dy = 0$.

$M = 3y + 4xy^2$ | $N = (2x + 3x^2y)$

$\frac{\partial M}{\partial y} = 3 + 8xy \neq \frac{\partial N}{\partial x} = (2 + 6xy)$

$\therefore$ this is not an exact.

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 1 + 2xy$$

Let $\mu(x,y) = x^2y$. Then the corresponding diff. eqn. of form is -

$$(3x^2y^2 + 4x^3y^3)\,dx + (2x^3y + 3x^4y^2)\,dy = 0$$

$$d(x^3y^2) + d(x^4y^3) = 0$$

Integrating both side -

$$[x^3y^2 + x^4y^3 = c]$$

Q - $(3x+2y)\,dx + (2x+y)\,dy = 0$.

Q. $(3x^2y + 2)\,dx + (x^3+y)\,dy = 0$.

$\frac{dM}{dy} = 3x^2 \neq \frac{dM}{dx} = 3x^2$

$\Rightarrow$ not an exact.

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variable separable :-

$$(x-4)y^4\,dx + (y^2-3)x^3\,dy = 0.$$

$$\frac{(x-4)}{x^3}dx - \frac{(y^2-3)}{y^4}dy = 0.$$

$$(x^{-2} - 4x^{-3})\,dx - (y^{-2} - 3y^{-4})\,dy = 0.$$

$$\frac{-1}{x} - 4\frac{1}{x^3(-2)} - \frac{1}{y(-1)} + 3\frac{1}{y^3(-3)} = C$$

$$\left[-\frac{1}{x} + \frac{2}{x^3} + \frac{1}{y} - \frac{1}{y^3} = c\right]$$

$c \rightarrow$ constant

Q- $x\sin y\,dx + (x^2+1)\cos y\,dy = 0$ & $y(1) = \frac{\pi}{2}$.

$$\frac{x}{x^2+1}dx + \frac{\cos y}{\sin y}dy = 0.$$

$$\frac{x}{x^2+1}dx + \cot y\,dy = 0.$$

Linear Equations & Bernoulli's Equation :-

The linear differential equation, $\frac{dy}{dx} + P(x).y = Q(x)$.

has an integrating factor of form :- $\boxed{e^{\int P(x)dx}}$.

A one parameter family of soln of this equation is :-

$$y.(IF) = \int e^{\int P(x)dx} . Q(x)\, dx + C.$$

$$\Rightarrow \boxed{y\, e^{\int P(x)dx} = \int e^{\int P(x)dx} . Q(x)\, dx + C}$$

Q. $\frac{dy}{dx} + \left(\frac{2x+1}{x}\right) y = e^{-2x}.$

$P(x) = \frac{2x+1}{x}$, $Q(x) = e^{-2x}$

$IF = e^{\int P(x)\, dx}.$

$\int \frac{2x+1}{x}\, dx = \int \left(2 + \frac{1}{x}\right) dx = (2x + \ln x)$

$\therefore IF = e^{2x + \ln x} = e^{2x} . e^{\ln x} = x . e^{2x}.$

$\therefore$ Soln :-

$$y . xe^{2x} = \int x.\cancel{e^{2x}} \times \cancel{e^{-2x}}\, dx + C.$$

$$xye^{2x} = \frac{x^2}{2} + C$$

$$y = \frac{1}{2} x e^{-2x} + C$$

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Q. $(x^2+1)\frac{dy}{dx} + 4xy = x$. & $y(2) = 1$

$$\frac{dy}{dx} + \left(\frac{4x}{x^2+1}\right)\cdot y = \frac{x}{x^2+1}$$

$$p(x) = \frac{4x}{x^2+1} \quad ; \quad Q(x) = \frac{x}{x^2+1}$$

$$IF = e^{\int p(x)\,dx}$$

$$\int p(x)\,dx = \int \frac{4x}{x^2+1}\,dx = \int \frac{2\,dt}{t}$$

$$x^2+1 = t$$

$$2x\,dx = dt$$

$$\int p(x)\,dx = 2\ln(x^2+1) = \ln(x^2+1)^2$$

$$IF = e^{\ln(x^2+1)^2} = (x^2+1)^2$$

sol^n:

$$(x^2+1)^2\cdot y = \int (x^2+1)^{\cancel{2}}\cdot\frac{x}{\cancel{(x^2+1)}}\,dx + C$$

$$\boxed{(x^2+1)^2 \times y = \frac{x^4}{4} + \frac{x^2}{2} + c}$$

for $x = 2,\ y = 1$

$$25 \times 1 = 4 + 2 + c$$

$$c = 19$$

$$\therefore \boxed{(x^2+1)^2 y = \frac{x^4}{4} + \frac{x^2}{2} + 19} \text{ Ans}$$

Q.

Q $y^2 dx + (3xy-1)dy = 0.$

$$\frac{dy}{dx} = \frac{8-y^2}{3xy-1}$$

$$\frac{dy}{dx} = \frac{y^2}{1-3xy}$$

This is not in standard form

$$\frac{dx}{dy} = \frac{1-3xy}{y^2} = \frac{1}{y^2} - \frac{3x}{y}$$

$$\frac{dx}{dy} + \left(\frac{3}{y}\right)x = \frac{1}{y^2}$$

$$P(y) = \frac{3}{y}, \quad Q(y) = \frac{1}{y^2}$$

$$IF = e^{\int P(y)dy} = e^{3\ln y} = \underline{\underline{y^3}}$$

$\therefore$ Sol^n:- $x.y^3 = \int y^{\cancel{3}} \times \frac{1}{\cancel{y^2}} dy + c$

$$\boxed{x.y^3 = \frac{y^2}{2} + c}$$

Q **Bernoulli's Equation** :-

An eq^n of form :- $\frac{dy}{dx} + P(x)y = Q(x)y^n$ is called as bernoulli's equation.

Theorem :- Suppose $n \neq 0$ or 1, then the transformation $v = y^{1-n}$ reduces the bernoulli equation to a linear in v.

$\int 1 \cdot 11\, dx = 1 \int 11\, dx - \int d(1) \cdot \int 11\, dx$

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Bernoulli' eq^n:

Proof :
$$\frac{dy}{dx} + P(x)\cdot y = Q(x)\cdot y^n$$

$$y^{-n}\frac{dy}{dx} + P(x)\, y^{1-n} = Q(x). \quad \text{--- (I)}$$

let,
$$V = y^{1-n}$$

$$\frac{dV}{dx} = (1-n)\, y^{-n}\frac{dy}{dx}. \quad \text{--- (II)}$$

from (I) & (II).

$$\frac{1}{(1-n)}\frac{dV}{dx} + P(x)V = Q(x)$$

$$\frac{dV}{dx} + (1-n)P(x)V = (1-n)Q(x).$$

let,
$$(1-n)P(x) = P_1(x)$$
&
$$(1-n)Q(x) = Q_1(x).$$

then,
$$\frac{dV}{dx} + P_1(x)V = Q_1(x)$$

is a linear eq^n in $\underline{V}$

$-\frac{2x}{-2}\left[\frac{dt}{dx}\right] = dt$

Q
$$\frac{dy}{dx} + y = xy^3$$

$$y^{-3}\frac{dy}{dx} + y^{-2} = x$$

let $V = y^{-2}$

$$\frac{dV}{dx} = -2y^{-3}\frac{dy}{dx}.$$

then, $-\frac{1}{2}\frac{dV}{dx} + V = x.$

$$\frac{dV}{dx} + (-2)V = -2x$$

$P(x) = -2$, $Q(x) = -2x$.

$$If = e^{\int -2dx} = e^{-2x}$$

$$V \cdot e^{-2x} = \int e^{-2x}\cdot(-2x)dx + c,$$

$$y^{-2}e^{-2x} = \int e^{-2x}(-2x)dx + c$$

$$y^{-2}e^{-2x} = -2\int e^{-2x}\cdot x\, dx + c.$$

$$\cancel{y^{-2}e^{-2x} = -2\left[e^{-2x}\times\frac{x^2}{2} - \int e^{-2x}(-2)\frac{x^2}{2}dx\right]}$$

$$y^{-2}e^{-2x} = -2\left[x\frac{e^{-2x}}{-2} - \int\frac{e^{-2x}}{-2}dx\right] + c$$

$$y^{-2}e^{-2x} = -2\left[\frac{xe^{-2x}}{-2} - \frac{e^{-2x}}{4}\right] + c$$

$$\boxed{y^{-2} = x + \frac{1}{2} + c}$$

Q. $\frac{dz}{dx} + \frac{z}{x}\log z = \left[\frac{z}{x^2}\cdot(\log z)^2\right]$.

$\frac{1}{z(\log z)^2}\frac{dz}{dx} + \frac{\cancel{z}\,\cancel{(\log z)}}{x\cdot\cancel{z}(\log z)^{\cancel{2}}} = \frac{1}{x^2}$

let $\frac{1}{\log z} = V$

$\Rightarrow (\log z)^{-1} = V$

$\frac{-1}{(\log z)^2}\frac{1}{z}\frac{dz}{dx} = \frac{dV}{dx}.$

$\therefore$

$-\frac{dV}{dx} + \frac{1}{x}\cdot V = \frac{1}{x^2}$

$P(x) = \frac{1}{x}, \quad Q(x) = \frac{1}{x^2}$

$IF = e^{\int P(x)dx} = e^{\ln x} = x.$

Sol^n:-

$\therefore V\cdot x = \int \cancel{x}\times\frac{1}{x^{\cancel{2}}}dx + C$

$\boxed{\frac{1}{\log z}x = \log x + C.}$

Q. $(\sec x\tan y\tan x - e^x)dx + \sec x\sec^2 y\,dy = 0.$

$\frac{dy}{dx} = -\frac{(\sec x\tan y\tan x - e^x)}{\sec x\sec^2 y}$

$\frac{dy}{dx} + \frac{\tan y\tan x}{\sec^2 y} = \frac{e^x}{\sec^2 y\sec x}$

$\sec^2 y\frac{dy}{dx} + \tan y\tan x = e^x\cos x$

let $\tan y = V.$

$\sec^2 y\frac{dy}{dx} = \frac{dV}{dx}$

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$\Rightarrow \frac{dV}{dx} + V\tan x = e^{x}\cos x$

$P(x) = \tan x, \quad Q(x) = e^{x}\cos x$

$If = e^{\int \tan x\, dx} = e^{\log\sec x} = \sec x.$

Sol^n

$\therefore V\cdot\sec x = \int e^{x}\cancel{\cos x}\cdot\cancel{\sec x}\, dx + C$

$$\boxed{\tan y\cdot\sec x = e^{x} + C}$$

Q $(xy^2 + e^{-1/x^3})\,dx - x^2y\,dy = 0$

$\frac{dy}{dx} = \frac{xy^2 + e^{-1/x^3}}{x^2y}$

$\frac{dy}{dx} + \frac{e^{-1/x^3}}{x^2y}$ $\quad x^2y\frac{dy}{dx} = xy^2 + e^{-1/x^3}.$

$x^2y\frac{dy}{dx} - xy^2 = e^{-1/x^3}$

Let $V = y^2$

$\frac{dV}{dx} = 2y\frac{dy}{dx}$

$\Rightarrow x^2\frac{1}{2}\frac{dV}{dx} - xV = e^{-1/x^3}$

$\frac{dV}{dx} - \frac{2}{x}V = \frac{2}{x^2}(e^{-1/x^3}),$

$If = e^{\int -\frac{2}{x}dx} = e^{-2\log x} = x^{-2}$

Sol^n,

$Vx^{-2} = \int x^{-2}\cdot\frac{2}{x^2}\cdot e^{-1/x^3}\,dx = 2\int x^{-4}\cdot e^{-x^{-3}}\,dx.$

$-x^{-3} = t$

$3x^{-4}\,dx = dt$

The Existence and uniqueness theorem:-

Let D be a domain in x-y plane and let (x_0, y_0) be an (interior) point of D. Consider the differential equation:

$$\frac{dy}{dx} = f(x,y).$$

where f is continuous real function defined on D. Consider the following problem.

We wish to determine:

1. an interval $\alpha \le x \le \beta$ of the real x axis such that $\alpha < x_0 < \beta$ and.

2. a differentiable real func^n ϕ defined on interval $[\alpha, \beta]$ and satisfying the following three requirements:-

(i) $[x, \phi(x)] \in D$, & thus $f[x, \phi(x)]$ is defined for all $x \in [\alpha, \beta]$.

(ii) $\frac{d\phi(x)}{dx} = f[x, \phi(x)]$, & thus ϕ satisfies the differential equation, for all $x \in [\alpha, \beta]$.

(iii) $\phi(x_0) = y_0$.

We shall call this problem the initial-value problem associated with the differential eq^n & the point (x_0, y_0). We shall denote it briefly by

$$\frac{dy}{dx} = f(x,y)$$

$$y(x_0) = y_0,$$

theorem:-

and let

D. Consider

defined on

is such

interval [α,β]

requirements:-

defined

the differential

value

& the

fly by

and call a function ϕ satisfying above requirements on an interval $[\alpha,\beta]$ a solution of the problem on the interval $[\alpha,\beta]$.

Hypothesis :-

1. Let D be a domain of xy plane & f be a real function satisfying following 2 requirements.

(i) f is continuous in D.

(ii) f satisfies Lipschitz $cond^n$ (with r.t. y) in D; that is there exists a constant $K>0$ such that:

$$\boxed{|f(x,y_1)-f(x,y_2)|\le K|y_1-y_2|}$$

for all $(x,y_1),(x,y_2)\in D$.

2. Let (x_0,y_0) be an (interior) point of D; let a & b be such that the rectangle $R: |x-x_0|\le a, |y-y_0|\le b$

Q. Show that the function $y=cx^2+x+3$ is a sol^n, though not unique, of the initial value problem $x^2y''-2xy'+2y=6$ with $y'(0)=1$, $y(0)=3$ on $(-\infty,\infty)$.

Sol^n

$$x^2y''-2xy'+2y=6 \quad \text{—— (i)}$$

$$y=cx^2+x+3 \quad \text{—— (ii)}$$

$$y'=2cx+1 \; \& \; y''=2c \quad \text{—— (iii)}$$

Putting in LHS of (i)

$$x^2(2c)-2x(2cx+1)+2(cx^2+x+3)$$

$$\Rightarrow \cancel{2x^2c}-\cancel{4x^2c}-\cancel{2x}+\cancel{2cx^2}+\cancel{2x}+6$$

$$=6=RHS$$

$\Rightarrow$ (ii) is sol^n of (i) :- again from (ii) & (iii), we get

$y(0)=c\times0+0+3=0$; $y'(0)=2c\times0+1=1$.

comparing (i) with standard eq^n:- $a_0(x)y''+a_1(x)y'+a_2(x)y=r(x)$

$a_0(x)=x^2$; $a_1(x)=2x$; $a_2(x)=2$ & $r(x)=6$, which are continuous on $(-\infty,\infty)$.

Since $a_0(x) = x^2 = 0$ for $x = 0 \in (-\infty, \infty)$, therefore the sol^n $y = cx^2 + x + 3$ is not unique. We see that $y = cx^2 + x + 3$ is a sol^n for any real value c. for example $y = 2x^2 + x + 3$ & $y = 3x^2 + x + 3$ are both sol^n of (i) with $y(0) = 3$ & $y'(0) = 1$.

Q. Show that $y = 3e^{2x} + e^{-2x} - 3x$ is the unique sol^n of the initial value problem $y'' - 4y = 12x$ where $y(0) = 4$, $y'(0) = 1$.

$$y'' - 4y = 12x \quad \text{— (i)}$$
$$y = 3e^{2x} + e^{-2x} - 3x \quad \text{— (ii)}$$
$$y' = 6e^{2x} - 2e^{-2x} - 3 \; ; \; y'' = 12e^{2x} + 4e^{-2x} \quad \text{— (iii)}$$

From (i), (ii) & (iii).

$$12e^{2x} + 4e^{-2x} - 4(3e^{2x} + e^{-2x} - 3x)$$
$$\cancel{12e^{2x}} \cancel{+4e^{-2x}} - \cancel{12e^{2x}} - \cancel{4e^{-2x}} + 12x$$
$$= 12x = RHS.$$

$\Rightarrow$ (ii) is the sol^n of (i)

again using (iii) & (ii)

$$y(0) = 3 \cdot e^0 + e^{-0} - 3(0) = 4.$$
$$y'(0) = 6e^0 - 2e^{-0} - 3 = 1.$$

Comparing (i) with $a_0(x) y'' + a_1(x) y' + a_2(x) y = r(x)$

$a_0(x) = 1$, $a_1(x) = 0$, $a_2(x) = -4$, $r(x) = 12x$.

Since $a_0(x) = 1 \neq 0$ for each $x \in (-\infty, \infty)$ therefor by existence & uniqueness theorem, (ii) is a unique sol^n of (i) satisfying $y(0) = 4$, $y'(0) = 1$

Q. find the unique solⁿ of $y''=1$ satisfying $y(0)=1$ & $y'(0)=2$.

$$y''=1 \quad \text{--- (I)}$$

$$\Rightarrow \frac{d^2y}{dx^2}=1$$

$$y'=\frac{dy}{dx}=x+c_1 \quad \text{--- (II)}$$

$$y=\frac{x^2}{2}+c_1x+c_2 \quad \text{--- (III)}$$

$y'(0)=0+c_1=c_1$ | $y(0)=\frac{0^2}{2}+c_1(0)+c_2=1$

$\boxed{2=c_1}$ | $\boxed{c_2=1}$

unique solⁿ $\Rightarrow \boxed{y=\frac{x^2}{2}+2x+1.}$ --- (IV)

Now comparing (I) with $a_0(x)y''+a_1(x)y'+a_2(x)y=r(x)$

$a_0(x)=1$, $a_1(x)=0$, $a_2(x)=0$, $r(x)=1$.

since $a_0(x)=1\neq 0$ for all $x\in(-\infty,\infty)$, so by uniqueness & existence theorem (IV) is the unique solⁿ of (I) with $y(0)=1$ & $y'(0)=2$.

Q. S.T. $y=x+x\log x-1$, is the unique solⁿ of $xy''-1=0$ satisfying $y(1)=0$ & $y'(1)=2$.

$$xy''-1=0 \quad \text{--- (I)}$$

$$y=x+x\log x-1 \quad \text{--- (II)}$$

$$y'=1+\log x+1 \;;\; y''=\frac{1}{x}+0=\frac{1}{x}. \quad \text{--- (III)}$$

from (I) & (III)

LHS $=\cancel{x}\times\frac{1}{\cancel{x}}-1=1-1=0=$ RHS

also from (I) & (II) $y(1)=1+\log 1-1=0$ | $y'(1)=2+\log 1=2$

Comparing ① with $a_0(x)y'' + a_1(x)y' + a_2(x)y = r(x)$

$a_0(x) = x$, $a_1(x) = 0$, $a_2(x) = 0$, $r(x) = 1$, which are ~~since all a~~ continuous for all $x \in (-\infty, \infty)$, therefore by existence & uniqueness theorem, solⁿ of ① is unique.

Q② Show that $y = \frac{1}{4}\sin 4x$ is unique solⁿ of problem $y'' + 16y = 0$ with $y(0) = 0$ & $y'(0) = 1$

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Simultaneous first order linear equations with constant coefficient :-

Q) Solve the simultaneous eqn $\frac{dx}{dt} - 7x + y = 0$, $\frac{dy}{dx} - 2x - 5y = 0$.

Let $D = \frac{d}{dt}$.

$\therefore$ $Dx - 7x + y = 0$, $Dy - 2x - 5y = 0$

$(D-7)x + y = 0$ (i), $-2x + (D-5)y = 0$. (ii)

(i) $\times (D-5)$ - (ii) $\Rightarrow (D-7)(D-5)x + \cancel{(D-5)y} + 2x - \cancel{(D-5)y} = 0$

$$(D^2 - 12D + 35)x + 2x = 0.$$

$$(D^2 - 12D + 37)x = 0.$$

$$\frac{d^2x}{dt^2} - 12\frac{dx}{dt} + 37x = 0.$$

auxilliary eqn:- $m^2 - 12m + 37 = 0$.

$$m = \frac{12 \pm \sqrt{144 - 148}}{2} = 6 \pm i$$

$$x = e^{6t}[c_1 \cos t + c_2 \sin t].$$

from eqn (i)

$$\frac{dx}{dt} - 7x + y = 0.$$

$$6e^{6t}[c_1 \cos t + c_2 \sin t] + e^{6t}[-c_1 \sin t + c_2 \cos t] - 7e^{6t}[c_1 \cos t + c_2 \sin t] + y = 0.$$

$$y = e^{6t}[c_1 \cos t + c_2 \sin t] - e^{6t}[-c_1 \sin t + c_2 \cos t].$$

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HOMOGENEOUS LINEAR EQUATION [Cauchy-Euler Equation]:-

A differential eqn of the form

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} x' \frac{dy}{dx} + a_n y = X \quad \text{--- (1)}$$

where $a_0, a_1, \ldots a_n$ are constants and X is either constant or a function of x is called homogenous linear differential equation.

To solve (i), we introduce a new independent variable z such that $x = e^z$ or $\log x = z$

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{dy}{dz}\frac{d(\log x)}{dx}$$

$$\Rightarrow \boxed{x\frac{dy}{dx} = \frac{dy}{dz}} \Rightarrow x\frac{d}{dx} = \frac{d}{dz}$$

$$\Rightarrow \boxed{xD = D_1}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right)$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d^2y}{dz^2}\frac{1}{x}$$

$$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$$

$$x^2 D^2 y = D_1^2 y - D_1 y \Rightarrow \boxed{x^2 D y = (D_1^2 - D_1) y}$$

Similarly,

$$x^3\frac{d^3y}{dx^3} = x^3 D^3 y = D_1(D_1-1)(D_1-2)y.$$

$$x^4\frac{d^4y}{dx^4} = D_1(D_1-1)(D_1-2)(D_1-3)y$$

and so on

Q. Solve: $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - 4y = 0$ — ①

which is HLDE.

Let $x = e^z$

$\therefore x\frac{dy}{dx} = \frac{dy}{dz} = D_1 y$, $x^2\frac{d^2y}{dx^2} = D_1(D_1-1)y$; $D_1 = \frac{d}{dz}$

$\therefore$ eqⁿ ① becomes –

$D_1(D_1-1)y + D_1 y - 4y = 0$

$D_1^2 y - \cancel{D_1 y} + \cancel{D_1 y} - 4y = 0$

$(D_1^2 - 4)y = 0$

auxiliary eqⁿ

$m^2 - 4 = 0 \Rightarrow m = \pm 2$

$\therefore y = c_1 e^{2z} + c_2 e^{-2z}$

$y = c_1 e^{2\log x} + c_2 e^{-2\log x}$

$\boxed{y = c_1 x^2 + c_2 x^{-2}}$

(ii) $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = 0$

let $x = e^z$

$\therefore x\frac{dy}{dx} = \frac{dy}{dz} = D_1 y$, $x^2\frac{d^2y}{dx^2} = D_1(D_1-1)y$; $D_1 = \frac{d}{dz}$

$D_1(D_1-1)y - 3D_1 y + 4y = 0$

$D_1^2 y - 4D_1 y + 4y = 0$

auxiliary eqⁿ :- $m^2 - 4m + 4 = 0$

$\boxed{m = 2, 2}$

$y = c_1 e^{2z} + c_2 z e^{2z}$

$y = c_1 e^{2\log x} + c_2(\log x)e^{2\log x}$

$\boxed{y = c_1 e^{x^2} + c_2(\log x)x^2}$

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Q Solve: $(x^2D^2 - 3xD + 5y) = \sin(\log x)$

$x = e^z$ $x^2D^2 = D_1(D_1 - 1)y$; $xD = D_1$; $D_1 = \frac{d}{dz}$

$D = \frac{d}{dx}$

$D_1(D_1 - 1) - 3(D_1) + 5y$ $\sin(\log^z x) = \sin z$

$(D_1^2 - 4D_1 + 5)y = \sin z$.

auxiliary eqn :- $m^2 - 4m + 5 = 0$

$m = \frac{4 \pm \sqrt{-4}}{2} = 2 \pm i$

$CF = e^{2z}[c_1 \cos z + c_2 \sin z]$

$PI = ?$ $= \frac{1}{D_1^2 - 4D_1 + 5} \sin z = \frac{1}{-1 - 4D_1 + 5} \sin z$.

$= \frac{1}{-4D_1 + 4} \sin z$.

Legendre's Linear Equation :-

A Linear differential equation of the form

$$a_0(a+bx)^n \frac{d^ny}{dx^n} + a_1(a+bx)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(a+bx)\frac{dy}{dx} + a_n y = X$$

where $a, b, a_0, a_1, a_2, \ldots a_n$ are constant and X is either a constant or a function of x only, is called Legendre's Linear equation.

Method: Let $a+bx = e^z$; $\log(a+bx) = z$

$$\therefore \frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = \frac{b}{a+bx}\frac{dy}{dz}$$

$$\boxed{(a+bx)\frac{dy}{dx} = b\, D_1 y}$$

similarly, $\left[(a+bx)^2\frac{d^2y}{dx^2} = b^2 D_1(D_1-1)y\right]$

& $\left[(a+bx)^3\frac{d^3y}{dx^3} = b^3 D_1(D_1-1)(D_1-2)y\right]$.

Q. Solve $(1+x)^2\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 4\cos\log(1+x)$. — ①

Sol^n:- Let $1+x = e^z \Rightarrow \log(1+x) = z$.

$$(1+x)\frac{dy}{dx} = (1)\, D_1 y \quad ; \; D_1 = \frac{d}{dz}$$

$$(1+x)^2\frac{d^2y}{dx^2} = (1)^2 D_1(D_1-1)y \quad — ②$$

From ① & ②

$$D_1(D_1-1)y + D_1 y + y = 4\cos\log(1+x)$$

$$D_1^2 y - \cancel{D_1 y} + \cancel{D_1 y} + y = 4\cos\log(1+x)$$

$$\frac{d^2y}{dz^2} + y = 4\cos z$$

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auxiliary eqn:-

$$m^2+1=0$$

$$[m=\pm i]$$

$$C.F. \Rightarrow e^{0(z)}[c_1\cos z + c_2\sin z] = c_1\cos z + c_2\sin z.$$

$$PI = \frac{1}{D_1^2+1} 4\cos z = \frac{1}{}$$

$$\because D_1^2+1 = -(1)^2+1 = 0 \quad \therefore \quad \boxed{\frac{1}{D^2+a^2}\cos ax = \frac{x\sin ax}{2a}}$$

$$\therefore \frac{4}{D_1^2+1^2}\cos z = \frac{4z \times \sin z}{2} = 2z\sin z = \cancel{2z\sin}$$

$$= 2\log(1+x)\sin[\log(1+x)].$$

Q. $(1+2x)^2\frac{d^2y}{dx^2} - 6(1+2x)\frac{dy}{dx} + 16y = 8(1+2x)^2$ — ①

Soln let $1+2x = e^z \Rightarrow \log(1+2x) = z.$

$$(1+2x)\frac{dy}{dx} = 2D_1(y) \quad ; \quad D_1 = \frac{d}{dz}$$

$$(1+2x)^2\frac{d^2y}{dz^2} = 4D_1(D_1-1)y.$$

$\therefore$ from eqn ①

$$4D_1(D_1-1)y - 6\times 2D_1y + 16y = 8e^{2z}$$

$$4D_1^2y - 16D_1y + 16y = 8e^{2z}$$

$$D_1^2y - 4D_1y + 4y = 2e^{2z}.$$

$$(D_1^2-4D_1+4)y = 2e^{2z}.$$

$$D_1 = 2, 2$$

$$C.F. = (c_1+c_2z)e^{2z}.$$

$$PI = \frac{1}{D_1^2-4D_1+4} 2.e^{2z} = 2.$$

$$\boxed{\frac{1}{(D-a)^n}e^{ax} = \frac{x^n}{n!}e^{ax}}$$

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Q. ① $(5+2x)^2 D^2y - 6(5+2x)Dy + 8y = 0.$

② $(x+1)^2 D^2y - 3(x+1)Dy - 4y = x^2$

③ $(3x+2)^2 D^2y + 3(3x+2)Dy - 36y = 3x^2 + 4x + 1.$

Method of variation of parameters:-

A D.E. $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$, where P, Q, R are func^n of x or constant.

Step-1:- Re-write given eq^n in standard form $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$

Step 2:- Consider $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$ (A) which can be obtained by taking $R = 0$.

Step-3:- Solve (A) by any method & CF will be $y = C_1u + C_2v$, where C_1 & C_2 are constant

Step-4:- General sol^n of (A) is $y = CF + PI$ where $PI = uf(x) + vg(x)$

where $f(x) = -\int \frac{vR}{W} dx$, $g(x) = \int \frac{uR}{W} dx$, where

$W =$ wronskian of u & v $= \begin{vmatrix} u & v \\ u' & v' \end{vmatrix}$

Q. Use variation of parameters, solve $\frac{d^2y}{dx^2} + n^2y = \sec nx$.

$$\frac{d^2y}{dx^2} + n^2y = \sec nx.$$

on comparing:- $P = 0$, $Q = n^2$, $R = \sec nx$

Auxiliary eqn:- $m^2 + n^2 = 0$

$m = \pm ni$

$\therefore$ $CF = c_1 \cos nx + c_2 \sin nx$.

Here, $u = \cos nx$; $v = \sin nx$.

$$W = \begin{vmatrix} \cos nx & \sin nx \\ -n\sin nx & n\cos nx \end{vmatrix} = n\cos^2 nx + n\sin^2 nx = \underline{n}$$

$PI = u f(x) + v g(x)$

$= u\left[-\int \frac{vR}{W} dx\right] + v\left[\int \frac{uR}{W} dx.\right]$

$= \cos nx\left[-\int \frac{\sin nx \sec nx}{n} dx\right] + \sin nx\left[\int \frac{\cos nx \sec nx\, dx}{n}\right]$

$= \frac{1}{n}\left[\cos nx\left(-\int \tan nx\, dx\right) + \sin nx \int dx\right].$

$= \frac{1}{n}\left[\cos nx \frac{1}{n}\log \cos nx + x\sin nx\right].$

Q. Use variation of parameters to solve, $(D^2 - 2D + 1)y = x^2 e^{3x}$.

$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x^2e^{3x} \Rightarrow P = 2, Q = 1, R = x^2e^{3x}$.

auxilliary eqn:- $m^2 - 2m + 1 = 0$.

$m = 1, 1$. $\Rightarrow CF = c_1e^x + c_2xe^x$

$u = e^x$; $v = xe^x$

$$W = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^{2x} + \cancel{xe^{2x}} - \cancel{xe^{2x}} = \underline{e^{2x}}.$$

$PI = e^x\left[-\int \frac{xe^x x^2 e^{3x}}{e^{2x}} dx\right] + e^{2x}\left[\int \frac{e^x x^2 e^{3x}}{e^{2x}} dx\right] = e^x\left[-\int x^3 e^{2x} dx\right] + e^{2x}\left[\int x^2 e^{2x} dx\right]$

$= e^x\left[-\left[x^3\frac{e^{2x}}{2} - \int 3x^2\frac{e^{2x}}{2}dx\right]\right.$

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Q. $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = \frac{e^x}{1+e^x}$

$R = \frac{e^x}{1+e^x}$

auxiliary eqn:- $m^2 - 3m + 2 = 0.$

$m^2 - 2m - m + 2 = 0.$

$(m-1)(m-2) = 0 \Rightarrow m = 1, 2$

$CF = c_1 e^x + c_2 e^{2x}$

$u = e^x, \quad V = e^{2x}.$

$w = \begin{vmatrix} e^x & e^{2x} \\ e^x & \frac{e^{2x}}{2} \end{vmatrix} = \frac{e^x \cdot e^{2x}}{2} - e^x \cdot e^{2x} = \frac{e^x \cdot e^{2x}}{2} = \frac{e^{3x}}{2},$

$PI = u f(x) + v g(x)$

$= u\left[-\int \frac{VR\,dx}{w}\right] + V\left[\int \frac{UR}{w}dx\right]$

$= e^x\left[-\int \frac{e^{2x} e^x \, 2}{(1+e^x)\cancel{e^x} e^{2x}}\right] + e^{2x}\left[\int \frac{e^x \cdot e^x \, 2}{(1+e^x)\cancel{e^x} \cdot e^{2x}} dx\right].$

$= e^x\left[-\int \frac{2}{1+e^x} dx\right] + e^{2x}\left[\frac{2}{(1+e^x)e^x} dx\right]$

$1 + e^x = t$

$0 + e^x dx = dt$

$PI = e^x \log(1+e^{-x}) + e^{2x}(-e^{-x} - x + \log(1+e^x))$

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Use variation of parameters to solve, $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = e^x \sin x$.

auxiliary eqn $\Rightarrow m^2 - 2m = 0$ $\quad R = e^x \sin x$.

$m = 0, 2$

$CF = c_1 e^{0x} + c_2 e^{2x} = c_1 + c_2 e^{2x}$.

$u = 1, \quad v = e^{2x}$

$$W = \begin{vmatrix} 1 & e^{2x} \\ 0 & \frac{e^{2x}}{2} \end{vmatrix} = \frac{e^{2x}}{2},$$

$$PI = 1 \left[-\int \frac{e^{2x} \cdot e^x \sin x \cdot 2\,dx}{e^{2x}} \right] + e^{2x} \left[\int \frac{1 \cdot e^x \sin x \cdot 2\,dx}{e^{2x}} \right]$$

$$= \left[-2\int e^x \sin x \, dx \right] + e^{2x} \left[2\int \frac{\sin x}{e^x} dx \right],$$

(II) (I)

$$= -\frac{1}{2} \sin x \, e^{x}.$$

(I) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = e^x \cos x$

(II) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x e^x \sin x$

(III) $\frac{d^2y}{dx^2} + 4y = 4 \tan 2x$

(IV) $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x^2 e^x$.