Linear Differential Equation:

The general linear differential equation of n^{th} order is

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1}}{dx^{n-1}} y + P_2 \frac{d^{n-2}}{dx^{n-2}} y + \dots P_n y = X$$

Where X and the coefficient $P_1, P_2, \dots P_n$ are constant.

* Linear equation with Constant Coefficients

The linear eq. with constant Coefficients has the form

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1}}{dx^{n-1}} y + P_2 \frac{d^{n-2}}{dx^{n-2}} y + \dots P_n y = X$$

$$y_n + P_1 y_{n-1} + P_2 y_{n-2} + \dots + P_n y = X \quad \text{— (1)}$$

where $y_1, y_2 \dots y_n$ denote the differentiation of y w.r.t. x.

Now the Complete Solution of (1) contain as part of itself, the Complete Solution of

$$y_n + P_1 y_{n-1} + P_2 y_{n-2} + \dots + P_n y = 0 \quad \text{— (2)}$$

* The Complementary function:

The C.F. of the differential eq. of n^{th} order is

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1}}{dx^{n-1}} y + P_2 \frac{d^{n-2}}{dx^{n-2}} y + \dots + P_n y = 0 \quad -(1)$$

or $D^n y + P_1 D^{n-1} y + P_2 D^{n-2} y + \dots + P_n y = 0.$

or $(D^n + P_1 D^{n-1} + P_2 D^{n-2} + \dots + P_n) y = 0.$

where $D = \frac{d}{dx}$ and $P_1 P_2 \dots P_n$ are constant.

* Equation of the First Order:

Let us first discuss.

$$\frac{dy}{dx} + P_1 y = 0. \quad \text{or} \quad \frac{dy}{y} + P_1 dx = 0.$$

or. $(D + P_1) y = 0.$

=> Now integrating.

$$\int \frac{dy}{y} + \int P_1 dx = \cancel{\log} c.$$

$$\log y + P_1 x = c$$

$$\log y = c - P_1 x.$$

$$y = e^{c - P_1 x}.$$

$$y = e^c * e^{-P_1 x}$$

Taking $e^c = A.$

$y = A \cdot e^{-P_1 x}$ where A is an arbitrary constant.

* Equation of Second Order.

The complementary function is given by

$$\frac{d^2y}{dx^2} + P_1\frac{dy}{dx} + P_2 y = 0 \quad - (3)$$

$$(D^2 + P_1 D + P_2)y = 0$$

In order to find the particular solution of this equation.

put $y = e^{mx} \neq 0$. where m is constant

Substituting $y = e^{mx}$ in (3).

$$\frac{d}{dx}\left(\frac{d}{dx}e^{mx}\right) + P_1\frac{d}{dx}e^{mx} + P_2 e^{mx} = 0$$

$$\frac{d}{dx}(e^{mx}.m) + P_1 e^{mx}.m + P_2 e^{mx} = 0$$

$$e^{mx}.m.m + P_1 e^{mx}.m + P_2 e^{mx} = 0.$$

$$e^{mx}(m^2 + P_1 m + P_2) = 0.$$

Since e^{mx} cannot be zero, therefore

$$\underline{m^2 + P_1 m + P_2 = 0} \quad - (4)$$

equation (4) is known as the characteristic or auxiliary eq.

Now the auxiliary eq. being simple and quadratic eq. may have.

(i) roots, real and unequal

(ii) roots. conjugate

(iii) roots, real and equal

(iv)

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* Roots of the auxiliary eq. are real and unequal.

Let the auxiliary eq. (4) have two distinct roots m_1 and m_2, then the two solution

$y = e^{m_1 x}$ and $y = e^{m_2 x}$ can be constructed by

$$\underline{y = C_1 e^{m_1 x} + C_2 e^{m_2 x}} \quad (y = C_1 u_1 + C_2 u_2)$$

becomes the general solution of the given eq (3).

* Roots of the auxiliary eq. are complex.

when eq. (4) has complex roots $\alpha \pm i\beta$, taking $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$, the general sol. of (3) is

$$y = C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x}$$

$$= C_1 e^{\alpha x} . e^{i\beta x} + C_2 e^{\alpha x} . e^{-i\beta x}$$

$$= e^{\alpha x} \{ (C_1 + C_2) \cos \beta x + i(C_1 - C_2) \sin \beta x \}$$

$$y = \underline{e^{\alpha x} \{ A \cos \beta x + B \sin \beta x \}}$$

Where $C_1 + C_2 = A$ · $i(C_1 - C_2) = B$

* Roots of the auxiliary eq. are real and equal.

Let the roots of eq (4) be real and equal to α. then $m_1 = \alpha = m_2$ then the solution is

$$y = C_1 e^{\alpha x} + C_2 e^{\alpha x} = (C_1 + C_2) e^{\alpha x} = C e^{\alpha x}$$

. Since it cannot be regarded as general solution. contain only one arbitrary constant.

However eq (4) having two roots being equal to α, the differential eq ③ takes the form

$$\frac{d^2y}{dx^2} - 2\alpha\frac{dy}{dx} + \alpha^2 y = 0.$$

$$(D^2 - 2\alpha D + \alpha^2)y = 0.$$

or $(D-\alpha)(D-\alpha)y = 0.$

put $(D-\alpha)y = z$ then $(D-\alpha)z = 0.$

Comparing eq ② {i.e $(D+P_1)y = 0$ the general sol is $y = Ae^{-P_1x}$}.

So its solution is $z = c_2 e^{\alpha x}$

$$\therefore (D-\alpha)y = c_2 e^{\alpha x}$$

$$\frac{dy}{dx} - \alpha y = c_2 e^{\alpha x}$$

Multiplying $e^{-\alpha x}$

$$\frac{dy}{dx}e^{-\alpha x} - e^{-\alpha x}\alpha y = c_2 e^{\alpha x} \cdot e^{-\alpha x}$$

$$\frac{d}{dx}(ye^{-\alpha x}) = c_2$$

Integrating

$$d(ye^{-\alpha x}) = c_2\,dx.$$

$$ye^{-\alpha x} = c_2 x + c_1$$

$y = e^{\alpha x}(c_1 + c_2 x)$ as the general solution

Ex. Solve $\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 15y = 0.$

$(D^2 - 8D + 15)y = 0.$

Let $y = e^{mx} \neq 0$ then

$m^2 - 8m + 15 = 0.$

$\Rightarrow m^2 - (5+3)m + 15 = 0.$

$\Rightarrow m^2 - 5m - 3m + 15 = 0$

$\Rightarrow m(m-5) - 3(m-5) = 0$

$\Rightarrow (m-3)(m-5) = 0.$

$m = 3$ or $m = 5$

real and unequal

$\therefore$ the general solution is

$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$= c_1 e^{3x} + c_2 e^{5x}$

Ex. $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0.$

$\Rightarrow (D^2 - 6D + 9)y = 0.$

Let $y = e^{mx} \neq 0$, then $M^2 - 6M + 9 = 0$

$\cancel{D^2 - 6D + 9} = 0.$

$(m-3)^2 = 0.$ when $m = 3, 3.$

Roots are real and equal, therefore general sol. is

$\therefore y = e^{3x}(c_1 + c_2 x)$

Solve $(D^3-1)y=0$.

Let $y=e^{mx}\neq 0$, then the auxiliary eqn. becomes

$$m^3-1=0.$$

$$(m-1)(m^2-m+1)=0.$$

$\therefore$ the roots are 1, and for m^2-m+1

To find the a.e. of m^2-m+1, it is in the standard form $ax^2+bx+c=0$. where

$$a=1,\ b=1,\ c=1.$$

$$\therefore\ x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-1\pm\sqrt{1-4.1.1}}{2.1}$$

$$=\frac{-1\pm\sqrt{-3}}{2}=\frac{-1\pm\sqrt{3}i}{2}$$

$\therefore$ the roots are 1, $-\frac{1}{2}\pm\frac{i\sqrt{3}}{2}$, the roots are complex.

$\therefore$ the general solution is

$$y=c_1e^{x}+e^{-\frac{1}{2}x}\{A\cos\frac{\sqrt{3}}{2}x+B\sin\frac{\sqrt{3}}{2}x$$