Laplace Transformation :-

The laplace transformation of the function f(t) is defined by integral $\int_0^\infty e^{-st} f(t)\,dt$, where S is a parameter
↳ real no.

Common notations used for laplace transform.

There are various commonly used notations for Laplace transform of f(t)

(i) $\mathcal{L}\{f(t)\}$ or $L\{f(t)\}$

(ii) $\mathcal{L}(f)$ or Lf

(iii) $\bar{F}(s)$ or $f(s)$

$$\boxed{\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt} \quad \text{—— (1)}$$

~~Linearity property~~

i.e., $\mathcal{L}\{kf(t)\} = k\mathcal{L}\{f(t)\}$. —— (2)

where k is any constant,

similarly, $\mathcal{L}\{af(t) + bg(t)\} = \int_0^\infty e^{-st}(af(t) + bg(t))\,dt$.

$= a\int_0^\infty e^{-st} f(t)\,dt + b\int_0^\infty e^{-st} g(t)\,dt$.

$= a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$.

Laplace transform of elementary funcⁿ :-

(a) $f(t) = 1$, from equation (1),

$\mathcal{L}\{1\} = \int_0^\infty e^{-st}(1)\,dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty = -\frac{1}{s}\left[e^{-s(\infty)} - e^{0}\right]$

$= -\frac{1}{s}[0-1] = \frac{1}{s}$ (provided $s>0$).

(b) $f(t) = k$. from eqⁿ (2)

$\mathcal{L}\{k\} = k\mathcal{L}\{1\}$

Hence $\mathcal{L}\{k\} = k\left(\frac{1}{s}\right) = \frac{k}{s}$ from (a) (above).

$F(t) = e^{at}$ (where a is a real constant $\neq 0$).
from eqn (i)

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} \cdot e^{at}\, dt = \int_0^\infty e^{(-s+a)t}\, dt = \left[\frac{e^{(-s+a)t}}{-s+a}\right]_0^\infty$$

$$= \frac{1}{a-s}\left[e^{-(s+a)\infty} - e^{-(s+a)0}\right] = \frac{1}{s-a}$$

provided $(s-a)>0$. i.e. $s>a$.

$f(t) = \cos at$ (where a is real constant)

from eqn.

$$\mathcal{L}\{\cos at\} = \int_0^\infty e^{-st} \cos at\, dt = \left[\frac{e^{-st}(a\sin at - s\cos at)}{s^2+a^2}\right]_0^\infty$$

$$= \frac{s}{s^2+a^2} \quad \text{(provided } s>0\text{)}.$$

$f(t) = t$.

$$\mathcal{L}\{t\} = \int_0^\infty e^{-st} \cdot t\, dt = \frac{1}{s^2} \quad \text{(provided } s>0\text{)}.$$

$f(t) = \sin at$

$$\mathcal{L}\{\sin at\} = \int_0^\infty e^{-st} \cdot \sin at\, dt = \frac{a}{s^2+a^2}$$

$f(t) = t^2$.

$$\mathcal{L}\{t^2\} = \int_0^\infty e^{-st} \cdot t^2 dt = \frac{2!}{s^3}$$

$f(t) = t^n$

$$\mathcal{L}\{t^n\} = \int_0^\infty e^{-st} \cdot t^n dt = \frac{n!}{s^{n+1}}$$

$f(t) = \cosh t$

$$\mathcal{L}\{\cosh t\} = \int_0^\infty e^{-st} \cosh t\, dt = \frac{s}{s^2-a^2}$$

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Q. Determine the following:-

① $\mathcal{L}\{1+2t-\frac{1}{3}t^4\}$. $= \mathcal{L}\{1\} + 2\mathcal{L}\{t\} - \frac{1}{3}\mathcal{L}\{t^4\}$

$= \frac{1}{s} + 2\frac{1}{s^2} - \frac{1}{3}\frac{4!}{s^5} = \frac{1}{s} + \frac{2}{s^2} - \frac{8}{s^5}$

② $\mathcal{L}\{5e^{2t} - 3e^{-t}\}$. $= \frac{5}{s-2} - \frac{3}{s-(-1)} = \frac{5}{s-2} - \frac{3}{s+1}$

$= \frac{5s+5-3s+6}{(s-2)(s+1)} = \frac{2s+11}{s^2-s-2}$

③ $\mathcal{L}\{6\sin 3t - 4\cos 5t\} = 6\mathcal{L}\{\sin 3t\} - 4\mathcal{L}\{\cos 5t\}$

$= \frac{6(3)}{s^2+9} - \frac{4s}{s^2+25} = \frac{18}{s^2+9} - \frac{4s}{s^2+25}$

④ $\mathcal{L}\{\sin^2 t\} = \mathcal{L}\{\frac{1-\cos 2t}{2}\} = \frac{1}{2}\mathcal{L}\{1\} - \frac{1}{2}\mathcal{L}\{\cos 2t\}$

$= \frac{1}{2}\times\frac{1}{s} - \frac{1}{2}\times\frac{s}{s^2+2^2} = \frac{1}{2s} - \frac{s}{2(s^2+4)}$

⑤ $\mathcal{L}\{\cosh^2 3x\} = \mathcal{L}\{\frac{(1+\cosh 6x)}{2}\}$

$= \frac{1}{2}\mathcal{L}\{1\} + \frac{1}{2}\mathcal{L}\{\cosh 6x\}$.

$= \frac{1}{2}\times\frac{1}{s} + \frac{1}{2}\cdot\frac{s}{s^2-36} = \frac{1}{2s} + \frac{s}{2(s^2-36)}$

⑥ $\mathcal{L}\{3\sin(\omega t+\alpha)\} = 3\mathcal{L}\{\sin(\omega t+\alpha)\}$

$= 3[\mathcal{L}\{\sin\omega t\cos\alpha + \sin\alpha\cos\omega t\}]$

$= 3\cos\alpha\,\mathcal{L}\{\sin\omega t\} + 3\sin\alpha\,\mathcal{L}\{\cos\omega t\}$.

$= \left[3\cos\alpha\cdot\frac{\omega}{s^2+\omega^2} + 3\sin\alpha\cdot\frac{s}{s^2+\omega^2}\right]$

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(7) (a) $5e^{3t}$ (b) $2e^{-2t}$

(8) (a) $4\sin 3t$ (b) $3\cos 2t$

(9) (a) $7\cosh 2x$ (b) $\frac{1}{3}\sinh 3t$.

(10) (a) $2\cos^2 t$ (b) $3\sin^2 2x$

(11) (a) $\cosh^2 t$ (b) $2\sinh^2 2\theta$

(12) (a) $4\sin(a+b)$, where a & b are constants.

(13) (b) $3\cos(\omega t-\alpha)$, " ω & α " "

(14) Show that $L\{\cos^2 3t - \sin^2 3t\} = \frac{s}{s^2+36}$

Properties of LAPLACE TRANSFORM :-

(1) Shifting Property :-

$$\boxed{L\{f(t)\} = f(s) = \int_0^\infty e^{-st} f(t)\,dt}$$

$$L\{e^{at} f(t)\} = \int_0^\infty e^{-st} e^{at} f(t)\,dt = \int_0^\infty e^{-(s-a)t} f(t)\,dt = f(s-a).$$

ex:- $L\{e^{at}\sin bt\}$.

$$L\{\sin bt\} = \frac{b}{s^2+b^2} = f(s).$$

$$L\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2} = f(s-a)$$

Q $L\{4e^{3t}\cos 5t\} = 4L\{e^{3t}\cos 5t\}$.

$$L\{\cos 5t\} = \frac{s}{s^2+5^2} = \frac{s}{s^2+25} = f(s)$$

$$4L\{e^{3t}\cos 5t\} = 4f(s-3) = \frac{4(s-3)}{(s-3)^2+25} = \frac{4(s-3)}{s^2-6s+34}$$

Q. $L\{e^{-2t}\sin 3t\}$

$$L\{\sin 3t\} = \frac{3}{s^2+9} = f(s)$$

$$L\{e^{-2t}\sin 3t\} = f(s-a) = f(s+2) = \frac{3}{(s+2)^2+9} = \frac{3}{s^2+4s+13}$$

Q. $L\{3e^{\theta}\cosh 4\theta\} = 3L\{e^{\theta}\cosh 4\theta\}$

$L\{\cosh 4\theta\} = \frac{\theta}{\theta^2-4^2} = \frac{\theta}{\theta^2-16} = f(\theta)$

$3L\{e^{\theta}\cosh 4\theta\} = 3f(\theta-1) = \frac{3(\theta-1)}{(\theta-1)^2-16} = \frac{3(\theta-1)}{\theta^2-2\theta-15}$

#

① $e^{at}t^n$	$n!/(s-a)^{n+1}$
$e^{at}\sin\omega t$	$\omega/(s-a)^2+\omega^2$
$e^{at}\cos\omega t$	$(s-a)/(s-a)^2+\omega^2$
$e^{at}\cosh\omega t$	$(s-a)/(s-a)^2-\omega^2$.
$e^{at}\sinh\omega t$	$\omega/(s-a)^2-\omega^2$

Q. $L\{3e^{-\frac{1}{2}x}\sin^2 x\} = 3L\{e^{-1/2x}\sin^2 x\}$.

$L\{\sin^2 x\} = L\{\frac{(1-\cos 2x)}{2}\} = L\{\frac{1}{2}\} - L\{\frac{\cos 2x}{2}\}$

$= \frac{1}{2s} - \frac{1}{2}\frac{s}{s^2+a^2} = f(s)$.

$3L\{e^{-\frac{1}{2}x}\sin^2 x\} = 3f(s-(-\frac{1}{2})) = 3f(s+\frac{1}{2})$

$= 3\left[\frac{1}{2(s+\frac{1}{2})} - \frac{1}{2}\frac{(s+\frac{1}{2})}{(s+\frac{1}{2})^2+a^2}\right] = \frac{3}{2(s+\frac{1}{2})} + \frac{3}{2}\frac{(s+\frac{1}{2})}{(s+\frac{1}{2})^2+}$

Q. ⓐ $\frac{1}{2}t^4e^{-3t}$.

ⓑ $2e^t\sin^2 t$

ⓒ $\frac{1}{2}e^{3t}\cos^2 t$

ⓓ $2e^{-t}\sinh 3t$

ⓔ $2e^t(\cos 3t-3\sin 3t)$

120×6
720×7
5040

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Q. find $L\{\sin\sqrt{t}\}$

$$\sin\sqrt{t} = \sqrt{t} - \frac{(\sqrt{t})^3}{3!} + \frac{(\sqrt{t})^5}{5!} - \frac{(\sqrt{t})^7}{7!} + \dots$$

$$= t^{1/2} - \frac{t^{3/2}}{3!} + \frac{t^{5/2}}{5!} - \frac{t^{7/2}}{7!} + \dots$$

$$L\{\sin\sqrt{t}\} = L\{t^{1/2}\} - \frac{1}{3!}L\{t^{3/2}\} + \frac{1}{5!}L\{t^{5/2}\} - \frac{1}{7!}L\{t^{7/2}\} + \dots$$

$$\because \boxed{n! = \Gamma(n+1)}$$

$$= \frac{\Gamma 3/2}{s^{3/2}} - \frac{\Gamma 5/2}{3!\, s^{5/2}} + \frac{\Gamma 7/2}{5!\, s^{7/2}} - \frac{\Gamma 9/2}{7!\, s^{9/2}} + \dots, \quad s>0.$$

also $\Gamma(n+1) = n\Gamma n$ & $\Gamma \tfrac{1}{2} = \sqrt{\pi}$

$$= \frac{\frac{1}{2}\sqrt{\pi}}{s^{3/2}} - \frac{1}{6}\frac{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{s^{5/2}} + \frac{1}{120}\frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{s^{7/2}} - \frac{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{5040\, s^{9/2}} + \dots$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}\left\{1 - \frac{1}{4s} + \frac{1}{16s^2} - \dots\right\}.$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}\, e$$

Q. Show that $L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \sqrt{\frac{\pi}{s}}\, e^{-\frac{1}{4}s}$.

$$\frac{\cos(\sqrt{t})}{\sqrt{t}} = \frac{1}{\sqrt{t}}\left[1 - \frac{(\sqrt{t})^2}{2!} + \frac{(\sqrt{t})^4}{4!} - \frac{(\sqrt{t})^6}{6!} + \dots\right]$$

$$= t^{-1/2} - \frac{t^{1/2}}{2!} + \frac{t^{3/2}}{4!} - \frac{t^{5/2}}{6!} + \dots$$

$$L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = L\{t^{-1/2}\} - L\left\{\frac{t^{1/2}}{2!}\right\} + \frac{1}{4!}L\{t^{3/2}\} - \frac{1}{6!}L\{t^{5/2}\} \dots$$

$$= \frac{\Gamma 1/2}{s^{1/2}} - \frac{1}{2!}\frac{\Gamma 3/2}{s^{3/2}} + \frac{1}{4!}\frac{\Gamma 5/2}{s^{5/2}} - \frac{1}{6!}\frac{\Gamma 7/2}{s^{7/2}} + \dots \quad (s>0)$$

$$= \frac{\sqrt{\pi}}{s^{1/2}} - \frac{\frac{1}{2}\cdot\sqrt{\pi}}{1\cdot 2}\frac{1}{s^{3/2}} + \frac{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{4\cdot 3\cdot 2\cdot 1}\frac{1}{s^{5/2}} -$$

LAPLACE TRANSFORM of derivatives :-

(a) first derivative :-

Let the first derivative of $f(t)$ be $f'(t)$ then,

$$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt$$

using integration by parts,

$$\int u\frac{dv}{dt}\,dt = uv - \int v\frac{du}{dt}\,dt$$

when evaluating $\int_0^\infty e^{-st} f'(t)\,dt$, Let $u = e^{-st}$ & $\frac{dv}{dt} = f'(t)$

$\therefore \frac{du}{dt} = -se^{-st}$ and $v = \int f'(t)\,dt = f(t)$

Hence, $$\int_0^\infty e^{-st} f'(t)\,dt = [e^{-st} f(t)]_0^\infty - \int_0^\infty f(t)(-se^{-st})\,dt$$

$$= [0 - f(0)] + \int_0^\infty e^{-st} f(t)\,dt$$

$$= -f(0) + s\,\mathcal{L}\{f(t)\}.$$

assuming $e^{-st} f(t) \to 0$ and $t \to \infty$ and $f(0)$

$\mathcal{L}\{f'(t)\} =$

For second derivative.

$$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0).$$

for 3^{rd} derivative.

$$\mathcal{L}\{f'''(t)\} = s^3 f(s) - s^2 f(0) - s.f'(0) - f''(0)$$

$L\{f(t)\} = f(s)$

$$L\{f^{iv}(t)\} = s^4 f(s) - s^3 f(0) - s^2 f'(0) - s f''(0) - f'''(0)$$

Theorem :- If $L\{f(t)\} = f(s)$ then $L\{\int_0^t f(u)\,du\} = \frac{f(s)}{s}$.

Theorem:- If $L\{f(t)\} = f(s)$, then $L\{\frac{f(t)}{t}\} = \int_s^\infty f(u)\,du$.

Q. Show that $L\{\int_0^t \sin u\,du\} = \frac{1}{s(s^2+1)}$

$$LHS = L\{\int_0^t \sin u\,du\} = \frac{L\{\sin u\}}{s} = \frac{1}{s(s^2+1^2)}$$

$$= \frac{1}{s(s^2+1)} = RHS$$

Hence proved.

Q. ~~Show~~ find $L\{\frac{\sin at}{t}\}$

$\Rightarrow \int_s^\infty \sin au\,du = [-\cos au]_s^\infty$

Properties:

① Linearity Property: If c_1 & c_2 are constants and f, g are funct of t then $L\{c_1 f(t) + c_2 g(t)\} = c_1 L\{f(t)\} + c_2 L\{g(t)\} = c_1 \bar{f}(s) + c_2 \bar{g}(s)$

② Shifting Property:- If $L\{f(t)\} = \bar{f}(s)$, then $L\{e^{at} f(t)\} = \bar{f}(s-a)$.

③ change of scale property:- If $L\{f(t)\} = \bar{f}(s)$, then $L\{f(at)\} = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$

Q. If $L\{f(t)\} = \frac{s^2-s+1}{(2s+1)^2(s-1)}$, using change of scale property prove that

$L\{f(2t)\} = \frac{s^2-2s+4}{4(s+1)^2(s-2)}$.

Sol^n: By using change of scale property.

$$L\{f(2t)\} = \frac{1}{2}\bar{f}\left(\frac{s}{2}\right).$$

$$\because \bar{f}(s) = \frac{s^2-s+1}{(2s+1)^2(s-1)}$$

$$\bar{f}\left(\frac{s}{2}\right) = \frac{\frac{s^2}{4} - \frac{s}{2} + 1}{(s+1)^2\left(\frac{s-2}{2}\right)} = \frac{s^2-2s+4}{\cancel{2}4(s+1)^2\frac{(s-2)}{\cancel{2}}}$$

$$L\{f(2t)\} = \frac{1}{2}\,\frac{s^2-2s+4}{2(s+1)^2(s-2)} = \frac{s^2-2s+4}{4(s+1)^2(s-2)}.$$

H/W ① using change of scale property, find $L\{\sin 5t\}$, $L\{e^{3t}\}$

• Laplace transform of derivatives :-

If $f(t)$ and its first $(n-1)$ derivatives be continuous, then

$$L\{f^n(t)\} = s^n f(s) - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-1} f''(0) - \cdots - f^{(n-1)}(0)$$

Laplace transform of integrals :-

Theorem :- If $L\{f(t)\} = f(s)$, then $L\{\int_0^t f(u)du\} = \frac{1}{s} f(s)$

or $L^{-1}\{\frac{1}{s} f(s)\} = \int_0^t f(u)du$, where L^{-1} = inverse Laplace Transform.

Q. Prove that $L\{\int_0^t \sin 2u du\} = \frac{2}{s(s^2+4)}$

Soln :-

LHS = $L\{\int_0^t \sin 2u du\} = \frac{1}{s} L\{\sin 2u\}$

$= \frac{1}{s} \frac{2}{(s^2+2^2)} = \frac{2}{s(s^2+4)}$

$=$ RHS

Hence Proved.

Division formula :- $L\{f(t)\} = f(s)$, then $L\{\frac{f(t)}{t}\} = \int_s^\infty f(s)ds$

Q Evaluate $L\{\int_0^t \frac{\sin x}{x} dx\}$

$L\{\sin x\} = \frac{1}{s^2+1} = f(s)$

$L\{\frac{\sin x}{x}\} = \int_s^\infty \frac{1}{s^2+1} ds = [\tan^{-1} s]_s^\infty = \tan^{-1}\infty - \tan^{-1} s$

$= \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s$.

Now, using laplace transform of integral :-

$L\{\int_0^t \frac{\sin x}{x} dx\} = \frac{1}{s} L\{\frac{\sin x}{x}\} = \frac{1}{s}\cot^{-1} s$. Ans.

$\frac{s}{s^2+a^2}$

Multiplication by t^n.

If $L\{f(t)\} = f(s)$, then $L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} f(s)$, $n = 1, 2, 3, \ldots$

ex:- $L\{t^3 \sin 2t\} = ?$

Sol^n $\Rightarrow$ $L\{\sin 2t\} = \frac{2}{s^2+4} = f(s)$.

$$L\{t^3 \sin 2t\} = (-1)^3 \frac{d^3}{ds^3}\left(\frac{2}{s^2+4}\right)$$

$$= 2 \frac{d^2}{ds^2}\left((s^2+4)^{-2} \times 2s\right)$$

$$= 4 \frac{d}{ds}\left[(s^2+4)^{-2} + s(-2)(s^2+4)^{-3}(2s)\right]$$

$$= 4\left[-2(s^2+4)^{-3}(2s) + (-4)\{(s^2+4)^{-3}(2s) + s^2(-3)(s^2+4)^{-4}(2s)\}\right]$$

$$= 4\left[-12(s^2+4)^{-3}s + 24s^3(s^2+4)^{-4}\right]$$

$$= -48\left[s(s^2+4)^{-3} - 2s^3(s^2+4)^{-4}\right].$$

H/W:- ① $t^2 \sin 4t$, $t > 0$.

② $t^3 e^{at}$

③ $3e^{2t}\cos at$

④ $t^3 \cosh(t/2)$.

Q. find Laplace transform of $\frac{e^{-at} - e^{-bt}}{t}$

$$L\left\{\frac{e^{-at}}{t} - \frac{e^{-bt}}{t}\right\} = L\left\{\frac{e^{-at}}{t}\right\} - L\left\{\frac{e^{-bt}}{t}\right\}$$

$L\{e^{-at}\} = \frac{1}{s-(-a)} = \frac{1}{s+a} = f(s)$ | $L\{e^{-bt}\} = \frac{1}{s+b} = f(s)$

$$= \int_s^\infty \frac{1}{s+a}\,ds - \int_s^\infty \frac{1}{s+b}\,ds$$

$$= \left[\log(s+a)\right]_s^\infty - \left[\log[s+b]\right]_s^\infty = \log(\infty+a) - \log(s+a) - \log(\infty+b) + [\log(s+b)]$$

$$= \log\left(\frac{s+b}{s+a}\right)$$

$$L\left[\frac{\cos at - \cos bt}{t}\right] = L\left\{\frac{\cos at}{t}\right\} - L\left\{\frac{\cos bt}{t}\right\}.$$

$$L\left\{\frac{e^{-t}\sin t}{t}\right\}$$

Solⁿ: $L\{\sin t\} = \frac{1}{s^2+1}.$

$$L\{e^{-t}\sin t\} = \frac{1}{(s+1)^2+1} = \frac{1}{s^2+2s+2} = f(s).$$

$$L\left\{\frac{e^{-t}\sin t}{t}\right\} = \int_s^\infty \frac{1}{s^2+2s+2}\,ds. = \int_s^\infty \frac{1}{1+(s+1)^2}$$

$$= \left[\tan^{-1}(s+1)\right]_s^\infty = \frac{\pi}{2} - \tan^{-1}(s+1) = \cot^{-1}(s+1)$$

Q) Evaluate $\int_0^\infty t^3 e^{-t}\sin t\,dt.$

$$= \int_0^\infty e^{-t}(t^3\sin t)\,dt, \quad s=1.$$

$$= L\{t^3\sin t\}, \quad s=1.$$

$$= (-1)^3\frac{d^3}{ds^3}L\{\sin t\} = -\frac{d^3}{ds^3}\frac{1}{s^2+1}$$

H/W ① $\int_0^\infty \frac{e^{-2t}\sin^2 t}{t}\,dt.$

② $L\left\{\int_0^t \frac{e^t\sin t}{t}\,dt\right\}.$

Inverse Laplace Transformation :-

Def^n:- If $L\{f(t)\} = f(s) = \int_0^\infty e^{-st} f(t)\,dt$, then $f(t)$ is called the inverse laplace transformation of $f(s)$ and is denoted by $L^{-1}\{f(s)\} = f(t)$.

Here, operator L^{-1} is called Inverse Laplace Transform.

ex:- ① $L\{1\} = \frac{1}{s}$

$L^{-1}\{\frac{1}{s}\} = 1.$

② $L^{-1}\{\frac{1}{s-a}\} = e^{at}$

③ $L^{-1}\{\frac{1}{s^n}\} = \frac{t^{n-1}}{(n-1)!}$ $(n>0)$

④ $L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}\sin at$.

⑤ $L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$

⑥ $L^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{1}{a}\sinh at$.

⑦ $L^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at$

⑧ $L^{-1}\left\{\frac{1}{(s-a)^2+b^2}\right\} = \frac{1}{b}e^{at}\sin bt$.

⑨ $L^{-1}\left\{\frac{s-a}{(s-a)^2-b^2}\right\} = e^{at}\cos bt$

⑩ $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \frac{1}{2a}t\sin at$

⑪ $L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\} = \frac{1}{2a^3}(\sin at - at\cos at).$

Q: find the inverse L.T. of :

① $\frac{3(s^2-1)^2}{2s^5}$ ② $\frac{3s+5\sqrt{2}}{s^2+8}$ ③ $\frac{4s+15}{16s^2-25}$.

sol^n- ① $= \frac{3}{2}\left[\frac{s^4-2s^2+1}{s^5}\right] = \frac{3}{2}\left[\frac{1}{s} - \frac{2}{s^3} + \frac{1}{s^5}\right].$

$\therefore L^{-1}\left\{\frac{3(s^2-1)^2}{2s^5}\right\} = \frac{3}{2}L^{-1}\left\{\frac{1}{s} - \frac{2}{s^3} + \frac{1}{s^5}\right\}$

$= \frac{3}{2}\left[1 - \frac{\cancel{2}\times t^2}{\cancel{2}!} + \frac{t^4}{4!}\right] = \frac{3}{2}\left[1 - t^2 + \frac{t^4}{24}\right]$

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Sol^n ② $L^{-1}\left\{\frac{3s+5\sqrt{2}}{s^2+8}\right\} = 3L^{-1}\left\{\frac{s}{s^2+(2\sqrt{2})^2}\right\} + 5\sqrt{2}L^{-1}\left\{\frac{1}{s^2+(2\sqrt{2})^2}\right\}$

$= 3\times\cos 2\sqrt{2}t + 5\sqrt{2}\times\frac{1}{2\sqrt{2}}\sin 2\sqrt{2}t$

$= 3\cos 2\sqrt{5}t + \frac{5}{2}\sin 2\sqrt{5}t$

Sol^n:- ③ $L^{-1}\left\{\frac{4s+15}{s^2-\frac{25}{16}}\right\} = 4L^{-1}\left\{\frac{s}{s^2-(\frac{5}{4})^2}\right\} + 15L^{-1}\left\{\frac{1}{s^2-(\frac{5}{4})^2}\right\}$

$= 4\times\cosh\frac{5}{4}t + 15\frac{1\times 4}{5}\sinh\frac{5}{4}t$

$= 4\cosh\frac{5}{4}t + 12\sinh\frac{5}{4}t.$

Q-4 $\frac{s+1}{s^2+s+1}$

$L^{-1}\left\{\frac{s+1}{s^2+s+1}\right\} = L^{-1}\left\{\frac{s+\frac{1}{2}+\frac{1}{2}}{(s^2+2s\times\frac{1}{2}+(\frac{1}{2})^2+1-\frac{1}{4})}\right\} = L^{-1}\left\{\frac{(s+\frac{1}{2})+\frac{1}{2}}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}\right\}$

$= L^{-1}\left\{\frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}\right\} + \frac{1}{2}L^{-1}\left\{\frac{1}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}\right\}$

$= e^{-\frac{1}{2}t}\cos$

Q5 $\frac{2s^2-4}{(s+1)(s-2)(s-3)}$

$= \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{s-3}$

$\therefore 2s^2-4 = A(s-2)(s-3) + B(s+1)(s-3) + C(s+1)(s-2).$

$2s^2-4 = s^2(A+B+C) + s(-5A-2B-C) + (6A-3B-2C).$

on comparing:- $A+B+C=2,\ -5A-2B-C=0,\ 6A-3B-2C=-4$

on solving, we get. $A=-\frac{1}{6},\ B=-\frac{4}{3},\ C=\frac{7}{2}.$

$L^{-1}\left\{\frac{2s^2-4}{(s+1)(s-2)(s-3)}\right\} = L^{-1}\left\{\frac{-1}{6(s+1)}\right\} + L^{-1}\left\{\frac{-4}{3(s-2)}\right\} + L^{-1}\left\{\frac{7}{2(s-3)}\right\}$

$= \frac{1}{6}e^{-t} - \frac{4}{3}e^{2t} + \frac{7}{2}e^{3t}.$

Q. 6 $\dfrac{2s^2-1}{(s^2+1)(s^2+4)}$

$$\Rightarrow \frac{2s^2-1}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}$$

$$2s^2-1 = (As+B)(s^2+4) + (Cs+D)(s^2+1).$$
$$= (A+C)s^3 + 4As + Bs^2 + 4B + Cs + Ds^2 + D$$
$$2s^2-1 = (A+C)s^3 + (4A+C)s + (B+D)s^2 + (4B+D).$$

$$\underbrace{A+C=0, \quad 4A+C=0}_{\Downarrow}, \quad \underbrace{B+D=2, \quad 4B+D=-1}_{\Downarrow}.$$

$2-B$

$3D = \ldots$

$B = -1$

$$A=0, \; C=0. \qquad B=-1, \; D=3$$

$\therefore$

$$L^{-1}\left\{\frac{2s^2-1}{(s^2+1)(s^2+4)}\right\} = L^{-1}\left\{\frac{0-1}{s^2+1^2}\right\} + L^{-1}\left\{\frac{0+3}{s^2+2^2}\right\} = -1 \times \sin t + 3 \times \frac{1}{2}\sin 2t$$

H/W Q. (7) $\dfrac{21s-33}{(s+1)(s-2)^3}$

(8) $\dfrac{5s+3}{(s-1)(s^2+2s+5)}$

(9) $\dfrac{s}{s^4+s^2+1}$

(10) $\dfrac{s}{s^4+4a^4}$

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Inverse Laplace transform of derivatives :

Theorem :- if $L^{-1}\{f(s)\} = f(t)$, then $L^{-1}\left\{\frac{d^n}{ds^n} f(s)\right\} = (-1)^n t^n f(t)$

where, $n = 1, 2, 3 \ldots$

Q. find the inverse L.T. of $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$

let, $f(s) = \frac{1}{s^2+a^2}$

$$\frac{d}{ds} f(s) = (-1) \frac{1}{(s^2+a^2)^2} (2s) = \frac{-2s}{(s^2+a^2)^2}$$

$$\therefore \frac{s}{(s^2+a^2)^2} = \frac{-1}{2} \frac{d}{ds} f(s).$$

$$\therefore L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = L^{-1}\left\{\frac{-1}{2} \frac{d}{ds} f(s)\right\}$$

$$= \frac{-1}{2} L^{-1}\left\{\frac{d}{ds}\left(\frac{1}{s^2+a^2}\right)\right\} \quad (n=1).$$

$$= \frac{-1}{2} (-1)^1 t^1 L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{t}{2a} \sin at \quad \text{Ans}.$$

Q. $L^{-1}\left\{\frac{s+1}{(s^2+2s+2)^2}\right\} =$

let, $f(s) = \frac{1}{((s+1)^2+1)}$

$$\frac{d}{ds} f(s) = \frac{-1 \cdot 2(s+1)}{[(s+1)^2+1]^2}$$

$$\therefore \frac{s+1}{((s+1)^2+1)^2} = \frac{-1}{2} \frac{d}{ds} f(s).$$

$$L^{-1}\left\{\frac{s+1}{((s+1)^2+1)^2}\right\} = \frac{-1}{2} L^{-1}\left\{\frac{d}{ds} \frac{1}{(s+1)^2+1}\right\} = \frac{-1}{2} (-1)^1 t \, L^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}$$

$$= \frac{t}{2} e^{-t} \sin t \quad \text{Ans}.$$

Properties of Inverse Laplace Transformation (L^{-1}):

① **First shifting theorem:**

Statement: If $L^{-1}\{f(s)\} = F(t)$ then $L^{-1}\{f(s-a)\} = e^{at} f(t)$

or $= e^{at} L^{-1}\{f(s)\}$.

② **Change of scale property:**

If $L^{-1}\{f(s)\} = f(t)$, then $L^{-1}\{f(as)\} = \frac{1}{a} f\left(\frac{t}{a}\right)$

Ques-: Find L^{-1} of (i) $\frac{s-a}{(s-a)^2+b^2}$

$$L^{-1}\left\{\frac{(s-a)}{(s-a)^2+b^2}\right\} = e^{at} L^{-1}\left\{\frac{s}{s^2+b^2}\right\} = e^{at}\cos bt.$$

(ii) $$L^{-1}\left\{\frac{s}{s^2+2s+5}\right\} = L^{-1}\left\{\frac{(s+1)-1}{(s+1)^2+2^2}\right\}$$

$$= L^{-1}\left\{\frac{(s+1)}{(s+1)^2+2^2}\right\} - L^{-1}\left\{\frac{1}{(s+1)^2+2^2}\right\}$$

By shifting prop.

$$= e^{-t} L^{-1}\left\{\frac{s}{s^2+2^2}\right\} - e^{-t} L^{-1}\left\{\frac{1}{(s)^2+2^2}\right\}$$

$$= e^{-t}\left(\cos 2t - \sin 2t\right) \text{ Ans.}$$

(iii) $$L^{-1}\left\{\frac{(s+1)-1}{(s+1)^5}\right\} = L^{-1}\left\{\frac{\cancel{(s+1)}}{\cancel{(s+1)^5}(s+1)^4}\right\} - L^{-1}\left\{\frac{1}{(s+1)^5}\right\}$$

$$= e^{-t} L^{-1}\left\{\frac{1}{s^4}\right\} - e^{-t} L^{-1}\left\{\frac{1}{s^5}\right\}$$

$$= e^{-t}\frac{t^3}{3!} - e^{-t}\frac{t^4}{4!}$$

If $L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \frac{1}{2} t\sin t$, find $L^{-1}\left\{\frac{32s}{(16s^2+1)^2}\right\}$

replace s by as then apply change of scale property.

Multiplication of powers by s :-

① If $L^{-1}\{F(s)\} = f(t)$ & $f(0)=0$, then $L^{-1}\{sF(s)\} = f'(t)$

② If $f(0) = f'(0) = f''(0) = \dots\dots f^{n(-1)}(0) = 0$, then $L^{-1}\{s^n F(s)\} = f^n(t)$

Evaluate $L^{-1}\left\{\frac{s^2}{(s^2+4)^2}\right\}$

Let $F(s) = \frac{1}{s^2+4} = (s^2+4)^{-1}$

then $f(t) = L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{1}{2}\sin 2t.$

$\frac{d}{ds}F(s) = \frac{-1(2s)}{(s^2+4)^2} = \frac{-2s}{(s^2+4)^2}$

$L^{-1}\left\{\frac{dF(s)}{ds}\right\} = L^{-1}\left\{\frac{-2s}{(s^2+4)^2}\right\} = (-1)\, t\, f(t) = -2L^{-1}\left\{\frac{s}{(s^2+4)^2}\right\}$

$= -2\frac{t}{2}f(t) = -2\frac{t}{4}\sin 2t$

$L^{-1}\left\{\frac{s}{(s^2+4)^2}\right\} = \frac{t}{4}\sin 2t.$

Now, $L^{-1}\left\{s \times \frac{s}{(s^2+4)^2}\right\} = \frac{d}{dt}\left(\frac{t}{4}\sin 2t\right) = \frac{1}{4}(\sin 2t + 2t\cos 2t)$

PLATE

Division of powers of S :-

Let $L^{-1}\{f(s)\} = F(t)$, if $L(t)$ is sectionally continous and of exponential order, such that $\lim_{t\to 0}\left\{\frac{f(t)}{t}\right\}$ exists, then for $s>0$.

$$L^{-1}\left\{\frac{f(s)}{s}\right\} = \int_0^t f(x)\,dx = \int_0^t f(t)\,dt.$$

Q. Evaluate $L^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\}$

we know that $L^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t. = f(t)$

$$L^{-1}\left\{\frac{1}{s}\cdot\frac{1}{(s^2+1)}\right\} = \int_0^t \sin t\,dt = [-\cos t]_0^t = -\cos t-(-\cos 0)$$
$$= (1-\cos t)$$

Now, $L^{-1}\left\{\frac{1}{s}\cdot\frac{1}{s(s^2+1)}\right\} = \int_0^t (1-\cos t)dt = \cancel{(\sin t)}\ t - \sin t.$

again, $L^{-1}\left\{\frac{1}{s}\cdot\frac{1}{s^2(s^2+1)}\right\} = \int_0^t (t-\sin t)dt = \frac{t^2}{2} - (-\cos t)_0^t$

$$= t^2 - (1-\cos t)$$

$$L^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\} = t^2+\cos t - 1.$$

Q. $L^{-1}\left\{\frac{1}{s}\log\frac{s+2}{s+1}\right\} = ?$

let

$f(s) = \log\frac{s+2}{s+1} = \log(s+2) - \log(s+1).$

$f'(s) = \frac{1}{s+2} - \frac{1}{s+1}.$

$L^{-1}\{f'(s)\} = e^{-2t} - e^{-t}$

$-t\,L^{-1}\{f(s)\} = e^{-2t} - e^{-t}$

$L^{-1}\{f(s)\} = \frac{e^{-t}-e^{-2t}}{t}$

$\therefore L^{-1}\left\{\frac{1}{s}\log\frac{s+2}{s+1}\right\} = \int_0^t \frac{e^{-t}-e^{-2t}}{t}\,dt.$

$= \int_0^t \frac{e^{-t}}{t}dt - \int_0^t \frac{e^{-2t}}{t}dt$

$= \int_0^t e^{-t}\cdot\frac{1}{t}\,dt - \int_0^t e^{-2t}\cdot\frac{1}{t}\,dt$

$= \frac{1}{t}(-e^{-t}) - \int\frac{+1}{t^2}\cdot -e^{-t}\,dt - \left[\frac{1}{t}\frac{e^{-2t}}{(-2)} - \int\frac{+1}{t^2}\frac{e^{-2t}}{-2}\,dt\right.$

$$= \frac{-e^{-t}}{t} - \int \frac{1}{t^2} e^{-t} dt + \frac{e^{-2t}}{2t} + \int \frac{1}{t^2} \frac{e^{-2t}}{2} dt.$$

Convolution Theorem :-

Let, $L\{H(t)\} = H(S) = \frac{1}{S(S^2+16)} = F(S) \times G(S)$

$L\{f(t)\} = F(S) = \frac{1}{S}$ and $G(S) = \frac{1}{(S^2+16)}$ $\Rightarrow L\{G(t)\} = \frac{1}{S^2+16}$

$\Rightarrow f(t) = L^{-1}\{\frac{1}{S}\} = 1.$

$G(t) = L^{-1}\{\frac{1}{S^2+16}\}$

$= \frac{1}{4} \sin 4t.$

$$H(t) = L^{-1}\left\{\frac{1}{S}\right\} \times L^{-1}\left\{\frac{1}{(S^2+16)}\right\}$$

$$= 1 \times \frac{1}{4} \sin 4t.$$

Statement :- If $f(t)$, $g(t)$ be piecewise continuous function on $[0, \infty]$ and be of exponential order and $L\{f(t)\} = f(s)$, $L\{g(t)\} = g(s)$. Then,

$$f(s) \cdot g(s) = L\left\{\int_0^t f(x) g(t-x) dx\right\}$$

or $$L^{-1}\{f(s) g(s)\} = \int_0^t f(x) g(t-x) dx.$$

Q. Find inverse Laplace transform of following using convolution theorem.

(1) $$\frac{1}{(S^2+\omega^2)^2} = \frac{1}{(S^2+\omega^2)(S^2+\omega^2)} = f(s).g(s).$$

$\therefore L\{f(t)\} = f(s) = \frac{1}{S^2+\omega^2}$ & $L\{g(t)\} = g(s) = \frac{1}{S^2+\omega^2}$

$L^{-1}\{f(s)\} = \frac{1}{\omega} \sin \omega t$ & $L^{-1}\{g(s)\} = \frac{1}{\omega} \sin \omega t.$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$
$= 1-\sin^2\theta - \sin^2\theta$
$= 1 - 2\sin^2\theta$

$$L^{-1}\{F(s).G(s)\} = \int_0^t f(x).g(t-x)\,dx = \int_0^t \frac{1}{\omega}\sin\omega x . \frac{1}{\omega}\sin\omega(t-x)\,dx$$

$$= \frac{1}{\omega^2}\int_0^t \sin\omega x . \sin\omega(t-x)\,dx$$

$$= \frac{1}{\omega^2}\int_0^t \sin\omega x\,(\sin\omega t . \cos\omega x - \cos\omega t \sin\omega x)\,dx$$

$$= \frac{1}{\omega^2}\left[\sin\omega t \int_0^t \frac{\sin 2\omega x}{2}\,dx - \cos\omega t \int_0^t \sin^2\omega x\,dx\right]$$

$$= \frac{1}{\omega^2}\left[\sin\omega t \times \left[\frac{-\cos 2\omega x}{2\times 2\omega}\right]_0^t - \cos\omega t \int_0^t \frac{1-\cos 2\omega x}{2}\,dx\right]$$

$$= \frac{1}{\omega^2}\left[\frac{\sin\omega t[\cos 2\omega t - (-1)]}{4\omega} - \cos\omega t\left[\frac{t}{4} - \frac{\sin 2\omega t}{4\omega}\right]\right]$$

(11) $\dfrac{1}{(s-2)(s+3)}$

$F(s) = \dfrac{1}{s-2}$ & $G(s) = \dfrac{1}{s+3}$ | $f(t) = L^{-1}\left(\dfrac{1}{s-2}\right) = e^{2t}$, $g(t) = L^{-1}\left(\dfrac{1}{s+3}\right) = e^{-3t}$

$$L^{-1}\{F(s)G(s)\} = \int_0^t f(x)g(t-x)\,dx = \cancel{\int_0^t \frac{1}{x-2} \times \frac{1}{t-x+3}\,dx}$$

$$= \int_0^t e^{2x}.e^{-3(t-x)}\,dx = \int_0^t e^{2x}.e^{-3t}.e^{3x}\,dx$$

$$= e^{-3t}\int_0^t e^{5x}\,dx = e^{-3t}.\left[\frac{e^{5x}}{5}\right]_0^t$$

$$= e^{-3t}\left[\frac{e^{5t}}{5} - \frac{1}{5}\right] = \frac{1}{5}\left[e^{2t} - e^{-3t}\right]$$

HW Use convolution theorem, Evaluate (1) $L^{-1}\left\{\dfrac{s}{(s^2+4)^2}\right\}$

(2) $L^{-1}\left\{\dfrac{1}{(s+2)^2(s-2)}\right\}$ (3) $L^{-1}\left\{\dfrac{s}{(s^2+1)(s^2+4)}\right\}$

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Q. Solve: $\frac{d^3y}{dt^3} + 2\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = 0$.

auxiliary eqn:- $m^3 + 2m^2 - m - 2 = 0$

$m^2(m+2) - 1(m+2) = 0$

$(m+2)(m^2-1) = 0$.

$\boxed{m = -2, -1, +1}$

$\therefore$ $CF = c_1 e^{-2t} + c_2 e^{-t} + c_3 e^{t}$

$\because$ $PI = 0$.

$\therefore$ $y = CF + PI$

$[y = c_1 e^{-2t} + c_2 e^{-t} + c_3 e^{t}.]$

again:- Solve: $\frac{d^3y}{dt^3} + 2\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = 0$, given $y(0) = y'(0) = 0$, $y''(0) = 6$.

Taking Laplace of both side.

$L\left\{\frac{d^3y}{dx^3}\right\} + 2L\left\{\frac{d^2y}{dt^2}\right\} - L\left\{\frac{dy}{dt}\right\} - 2L\{y\} = L\{0\} = 0$.

$= [s^3 f(s) - s^2 f(0) - s f(0) - f''(0)] + 2[s^2 f(s) - s f(0) - f'(0)]$

$- [s f(s) - f(0)] - 2f(s) = 0$

$f(s)[s^3 + 2s^2 - s - 2] - 6 = 0$

$f(s) = \frac{6}{s^3 + 2s^2 - s - 2}$

$L\{y\} = \frac{6}{s^3 + 2s^2 - s + 2} = \frac{1}{s-1} - \frac{3}{s+1} + \frac{2}{s+2}$.

Q. $\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = e^t$, with $x=2$ $\frac{dx}{dt} = -1$,

at $t=0$ $x(0)=2$, $x'(0)=-1$.

Taking Laplace transform on both side.

$L\left\{\frac{d^2x}{dt^2}\right\} - 2\left\{\frac{dx}{dt}\right\} + L\{x\} = L\{e^t\}$

$[s^2 x(s) - s x(0) - x'(0)] - 2[s x(s) - x(0)] + x(s) = \frac{1}{s-1}$

$x(s)[s^2-2s+1] - 2s + 1 - 2(-2) = \frac{1}{s-1}$

$x(s) = \left(\frac{1}{s-1} + 2s - 5\right)\frac{1}{(s^2-2s+1)} = \frac{1+2s^2-2s-5s+5}{(s-1)(s^2-2s+1)}$

$= \frac{2s^2-7s+6}{(s-1)(s-1)^2} = \frac{2s^2-4s-3s+6}{(s-1)^3} = \frac{(2s-3)(s-1)}{(s-1)^3}$

$L\{x\} = \frac{2}{s-1} - \frac{3}{(s-1)^2} + \frac{1}{(s-1)^3}$

$x = L^{-1}\left\{\frac{2}{s-1}\right\} - 3L^{-1}\left\{\frac{1}{(s-1)^2}\right\} + L^{-1}\left\{\frac{1}{(s-1)^3}\right\}$.

$x = 2e^t - 3\frac{e^t}{1!} + \frac{t^2e^t}{2!}$