Differential equation of first order and first degree

The given differential can be expressed in the form $f(x)\,dx = g(y)\,dy$, then

$\int f(x)\,dx = \int g(y)\,dy + c$ is the solution of differential equation with variables seperable.

Ex. solve. $\dfrac{dy}{dx} = \dfrac{x-1}{y+2}$ $(y \neq -2)$

$(y+2)\,dy = (x-1)\,dx$. [Seperating the variables]

$\Rightarrow \int (y+2)\,dy = \int (x-1)\,dx$.

$$\frac{y^2}{2} + 2y = \frac{x^2}{2} - x + c_1$$

$$y^2 + 4y = x^2 - 2x + 2c_1$$

$y^2 + 4y - x^2 + 2x = c$, where $c = 2c_1$ is the general solution of the differential equation.

Ex. $xy^2\,dy - y^3\,dx + y^2\,dy = dx$.

$xy^2\,dy + y^2\,dy = dx + y^3\,dx$.

$dy\,(x+1)\,y^2 = (1+y^3)\,dx$

Dividing the product by $(1+x)(1+y^3)$

$\dfrac{y^2}{1+y^3}\,dy = \dfrac{1}{1+x}\,dx$. [variables seperable].

$\int \dfrac{y^2}{1+y^3}\,dy = \int \dfrac{1}{1+x}\,dx$.

put $1+y^3 = u$.

$3y^2 dy = du$.

$dy = \frac{du}{3y^2}$

$\Rightarrow \int \frac{du}{3u} = \log(1+x) + c$.

$\frac{1}{3}\int \frac{du}{u} = \log(1+x) + c$.

$\frac{1}{3}\log u = \log(1+x) + c$.

$\frac{1}{3}\log(1+y^3) = \log(1+x) + c$.

. $(x+y)^2 \frac{dy}{dx} = a^2$

put $x+y = z$.

$1 + \frac{dy}{dx} = \frac{dz}{dx}$. $\frac{dy}{dx} = \frac{dz}{dx} - 1$.

$(z)^2\left(\frac{dz}{dx} - 1\right) = a^2$

$\frac{dz}{dx} = \frac{a^2}{z^2} + 1 \Rightarrow \frac{dz}{dx} = \frac{a^2 + z^2}{z^2}$

$\Rightarrow dx = \frac{z^2}{a^2+z^2} dz$.

$\int \tan^{-1} x = \frac{1}{1+x^2}$.

$\tan^{-1}\left(\frac{z}{a}\right) = \frac{1}{1+\frac{z^2}{a^2}}$

$= \frac{1}{\frac{a^2+z^2}{a^2}}$

$= \frac{a^2}{a^2+z^2}$

$\Rightarrow \int dx = \int \frac{z^2}{a^2+z^2} dz$.

$\Rightarrow x = \int\left(1 - \frac{a^2}{a^2+z^2}\right) dz$

$\Rightarrow x = z - \tan^{-1}\left(\frac{z}{a}\right) + c$.

$\Rightarrow x = x+y - a\tan^{-1}((x+y)/a) + c$. $\Rightarrow y = \tan^{-1}\left(\frac{x+y}{a}\right) - c$.

Homogeneous Equation

The differential equation $M\,dx + N\,dy = 0$. is said to be Homogeneous when M and N are homogeneous function of the same degree in x and y.

And can be expressed in the form

$$\frac{dy}{dx} = \phi(y/x) \quad \text{or} \quad \frac{dy}{dx} = \frac{f(x,y)}{\phi(x,y)}.$$

is called a Homogeneous eq., if each term of $f(x,y)$ and $\phi(x,y)$ are homogeneous eq. of the same degree.

To solve the eq. put $\boxed{y = vx}$ or $\boxed{\frac{y}{x} = v}$

where $\frac{d}{dx}(vx) = \phi(v)$.

$$\boxed{v + x\frac{dv}{dx} = \phi v.}$$

Separating the variables

$$x\frac{dv}{dx} = \phi(v) - v.$$

$$\frac{dx}{x} = \frac{dv}{\phi(v) - v}.$$

After integrating the values we replace v by its value $\frac{y}{x}$.

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Ex. $y^2 + x^2\frac{dy}{dx} = xy\frac{dy}{dx}$.

Sol: $y^2 = xy\frac{dy}{dx} - x^2\frac{dy}{dx}$.

$$y^2 = \frac{dy}{dx}(xy - x^2).$$

$$\frac{dy}{dx} = \frac{y^2}{xy - x^2}.$$

Now Sus. $y = vx$.

$$\frac{d}{dx}(vx) = \frac{v^2x^2}{xvx - \cancel{x}x^2}.$$

$$v + x\frac{dv}{dx} = \frac{v^2\cancel{x^2}}{\cancel{x^2}(v-1)}$$

$$v + x\frac{dv}{dx} = \frac{v^2}{v-1}.$$

$$x\frac{dv}{dx} = \frac{v^2}{v-1} - v.$$

$$x\frac{dv}{dx} = \frac{v^2 - v(v-1)}{v-1}$$

$$x\frac{dv}{dx} = \frac{\cancel{v^2} - \cancel{v^2} + v}{v-1}$$

$$x\frac{dv}{dx} = \frac{v}{v-1}$$

$$\frac{dx}{x} = \frac{v-1}{v}\,dv$$

$$\int\frac{dx}{x} = \int\frac{v-1}{v}\,dv.$$

$$\log x + c = \left(\int 1 - \int\frac{1}{v}\right)dv$$

$$\log x + c = v - \log v.$$

$$\log x + \log v = v - c.$$

$$\log vx = v - c.$$

$$\log y/x \cdot x = y/x - c.$$

$$\Rightarrow \boxed{\log y = \frac{y}{x} - c}$$

$\log ab = \log a + \log b.$

$\log\frac{a}{b} = \log a - \log b.$

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Solve : 1. $\frac{dy}{dx} = \sin(x+y) + \cos(x+y)$

2. $\frac{dy}{dx} = (4x+y+1)^2$, if $y(0) = 1$.

3. $\frac{y}{x}\frac{dy}{dx} + \frac{x^2+y^2-1}{2(x^2+y^2)+1} = 0.$

(Hint: put $x^2+y^2 = t$ then

$$2x + 2y\frac{dy}{dx} = \frac{dt}{dx} \text{ or}$$

$$\frac{y}{x}\frac{dy}{dx} = \frac{1}{2x}\frac{dt}{dx} - 1 \text{)}$$

Solve using Homogeneous Equation

1. $(x^2-y^2)\,dx - xy\,dy = 0$

2. $\left(x\tan\frac{y}{x} - y\sec^2 y/x\right)dx - x\sec^2 y/x\,dy = 0.$

3. $(x^2-y^2)\,dx = 2xy\,dy.$

Linear Equation

A differential equation of the form

$\frac{dy}{dx} + Py = Q$ or $\frac{dy}{dx} + P(x)y = Q(x)$ where P and Q are function of x or constant is called a linear differential equation.

* Working rule for $\frac{dy}{dx} + Py = Q$.

(i) Find $e^{\int P dx} = I.F$

(ii) The solution is given by

$$y \times I.F = \int Q\,(I.F)\,dx + c.$$

Ex. Solve $(x+1)\frac{dy}{dx} - y = e^{3x}(x+1)^2$

Solution: Dividing $(x+1)$ throughout

$$\frac{dy}{dx} - \frac{y}{(x+1)} = e^{3x}(x+1).$$

Here $P = -\frac{1}{(x+1)}$ and $\int P\,dx = -\int \frac{dx}{(x+1)} = -\log(x+1)$

$\int P\,dx = -\log(x+1) = \log(x+1)^{-1}$ $\quad\left|\because e^{\log u} = u.\right.$

$\therefore I.F = e^{\int P dx} = e^{\log(x+1)^{-1}} = (x+1)^{-1} = \frac{1}{x+1}$

Thus the solution is

$$y \times I.F = \int (Q \times I.F)\, dx + c.$$

$$y \times \frac{1}{(x+1)} = \int e^{3x} \cancel{(x+1)} \times \frac{1}{\cancel{(x+1)}}\, dx + c.$$

$$\frac{y}{(x+1)} = \int e^{3x}\, dx + c.$$

$$\frac{y}{(x+1)} = e^{3x} \cdot \frac{1}{3} + c.$$

$$y = \left(\frac{e^{3x}}{3} + c\right)(x+1)$$

Ex. $(1+y^2)\,dx = (\tan^{-1}y - x)\,dy.$

$$(1+y^2)\frac{dx}{dy} = \tan^{-1}y - x.$$

$$\frac{dx}{dy} = \frac{\tan^{-1}y}{1+y^2} - \frac{x}{1+y^2}.$$

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2}.$$

The given eavation is in the form of $\frac{dx}{dy} + Px = Q$

$$I.F = e^{\int P dy} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}.$$

$\therefore$ The Complete Solution is

$$x \times I.F = \int Q \times I.F \, dy + c.$$

$$x \times e^{\tan^{-1} y} = \int \frac{\tan^{-1} y}{1+y^2} \times e^{\tan^{-1} y} dy + c.$$

Put $\tan^{-1} y = t$

$$\frac{dy}{1+y^2} = dt.$$

$$x e^{\tan^{-1} y} = \int t e^t dt + c.$$

$$= t . e^t + \int \frac{d(t)}{dt} \int e^t dt$$

$$= te^t - e^t + c$$

$$x e^{\tan^{-1} y} = e^t [(t-1) + c].$$

$$x e^{\tan^{-1} y} = e^{\tan^{-1} y} [(\tan^{-1} y - 1) + c.]$$

$$x = \tan^{-1} y - 1 + c e^{\tan^{-1} y}$$

~~Verify~~:

Solve 1. $\sin x \frac{dy}{dx} + 2y = \tan^3\left(\frac{x}{2}\right)$

2. $\cos^2 x \frac{dy}{dx} + y = \tan x$.

3. $(1-x^2)\frac{dy}{dx} - xy = 1$.

Bernoulli Equation

The equation $\frac{dy}{dx} + Py = Qy^n$ where P and Q are functions of x.

To solve the above linear equations we divide by y^n.

$$y^{-n}\frac{dy}{dx} + Py^{1-n} = Q \quad -(1).$$

Put $y^{1-n} = z$ so that $(1-n)y^{-n}\frac{dy}{dx} = \frac{dz}{dx}$.

then eq (1) becomes.

$$\cancel{n+1} \quad \frac{1}{1-n}\frac{dz}{dx} + Pz = Q \text{ or}$$

$$\frac{dz}{dx} + P(1-n)z = Q(1-n).$$

Ex. Solve $x\frac{dy}{dx} + y = x^3y^6$

Dividing xy^6 throughout

$$y^{-6}\frac{dy}{dx} + \frac{y^{-5}}{x} = x^2 \quad -(1)$$

Put $y^{-5} = z$, so that $-5y^{-6}\frac{dy}{dx} = \frac{dz}{dx}$

∴ Eq ① becomes

$$-\frac{1}{5}\frac{dz}{dx} + \frac{z}{x} = x^2$$

$$\frac{dz}{dx} - \frac{5z}{x} = -5x^2$$

The given eq. reduces to linear ~~df~~ differential equation.

Hence $I.F = e^{-\int P dx}$

$$= e^{-\int \frac{5}{x} dx}$$

$$= e^{-5\log x} = e^{\log x^{-5}} = x^{-5}$$

The solution of ① is

$$z \times I.F = \int Q \times I.F\, dx + C$$

$$z \times x^{-5} = \int -5x^2 \times x^{-5}\, dx + c.$$

$$y^{-5}x^{-5} = -5\int x^{-3} dx + C$$

$$= -5 . \frac{x^{-2}}{-2} + C.$$

$$y^{-5}x^{-5} = \frac{5}{2} x^{-2} + C$$

$1 = \frac{5}{2} x^3 y^5 + C x^5 y^5$ which is the required equation.

Solve: $\tan y \frac{dy}{dx} + \tan x = \cos y \cos^2 x$

Divide $\cos y$ throughout

$$\frac{1}{\cos y} \tan y \frac{dy}{dx} + \frac{1}{\cos y} \tan x = \cos^2 x.$$

$$\sec y \tan y \frac{dy}{dx} + \sec y \tan x = \cos^2 x \quad -(1).$$

put $z = \sec y$, then $\sec y \tan y \frac{dy}{dx} = \frac{dz}{dx}$.

then from (1). | $e^{\log u} = u$.

$$\frac{dz}{dx} + z \tan x = \cos^2 x$$

$$\therefore I.F = e^{\int P\,dx} = e^{\int \tan x\,dx} = e^{\log \sec x} = \sec x.$$

$\therefore$ The solution of eq. (1) is

$$z.(I.F) = \int Q \times I.F\,dx + C.$$

$$z.\sec x = \int \cos^2 x \times \sec x\,dx + c.$$

$$\sec y \sec x = \int \cos x\,dx + C$$

$$\sec y \sec x = \sin x + C.$$

$$\sec y = (\sin x + c).\frac{1}{\sec x} \Rightarrow (\sin x + c).\cos x$$

Solve :

1. $\dfrac{dy}{dx} + x \sin 2y = x^3 \cos^2 y.$

2. $(1+y^2)\, dx = (\tan^{-1} y - x)\, dy.$

3. $x^2 y - x^3 \dfrac{dy}{dx} = y^4 \cos x.$

Exact Differential Equation:

A differential equation of the form $M\,dx + N\,dy = 0$ is said to be exact differential equation, if it satisfies the condition.

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Working Rule:

1. Integrate M w.r.t. 'x' and y as constant
2. Integrate the terms of N ~~too~~ not containing 'x' w.r.t. 'y'
3. The complete solution Result I + Result II = Constant

Ex. Solve $(y^2 e^{xy^2} + 4x^3)\,dx + (2xy\,e^{xy^2} - 3y^2)\,dy = 0$.

Here, $M = y^2 e^{xy^2} + 4x^3 \qquad N = 2xye^{xy^2} - 3y^2$

$$\frac{\partial M}{\partial y} = 2ye^{xy^2} + y^2 e^{xy^2}.\,2xy \qquad \frac{\partial N}{\partial x} = 2ye^{xy^2} + 2xye^{xy^2}.\,y^2$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

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Thus the equation is exact and the solution is

$\int M dx + \int (\text{terms of } N \text{ not containing } x)\, dy = \text{Constant}.$

$\int (y^2 e^{xy^2} + 4x^3)\, dx + \int -3y^2 dy = c.$

$= \cancel{y^2} \frac{e^{xy^2}}{\cancel{y^2}} + \frac{\cancel{4}x^4}{\cancel{4}} - \frac{\cancel{3}y^3}{\cancel{3}} = c$

$= e^{xy^2} + x^4 - y^3 = c$ //

Ex. Solve $(1 + 2xy\cos x^2 - 2xy)\, dx + (\sin x^2 - x^2)\, dy = 0$

Here, $M = 1 + 2xy\cos x^2 - 2xy$ $\quad N = \sin x^2 - x^2$

$\frac{\partial M}{\partial y} = 2x\cos x^2 - 2x$ $\quad \frac{\partial N}{\partial x} = \cos x^2 . 2x - 2x$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial y}.$ the equation is exact and

its solution is

$\int M dx + \int (\text{terms of } N \text{ not containing } x)\, dy = c.$

$\int (1 + 2xy\cos x^2 - 2xy)\, dx + \int 0\, dy = c.$

$x + y [\int 2x\cos x^2 dx - \int 2x\, dx = c.$

$\Rightarrow x + y[\sin x^2 - g x^2] = c.$

$\Rightarrow x + y \sin x^2 - y x^2 = c$ //

Solve:- 1. $(y^2 - x^2)\,dx + 2xy\,dy = 0$

2. $(x^2 - ay)\,dx = (ax - y^2)\,dy.$

3. $(x^2 + y^2 - a^2)x\,dx + (x^2 - y^2 - b^2)y\,dy = 0.$

* Equation Reducible to Exact Equation:

When the differential equation is not exact, we multiply be a suitable function known as the integrating factor.

Working Rule. If $\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial y}}{N}$ is a function of 'x' alone, say $f(x)$, then $I.F = e^{\int f(x)\,dx}$.

Solve $(xy^2 - e^{1/x^3})dx - x^2y\,dy = 0$. — (1)

Sol, Here $M = xy^2 - e^{1/x^3}$ $\qquad N = -x^2y$

$$\frac{\partial M}{\partial y} = 2xy \qquad \frac{\partial N}{\partial x} = -2xy.$$

Now, $$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2xy - (-2xy)}{-x^2y} = \frac{4xy}{-x^2y} = -\frac{4}{x}$$

which is a function of x only. i.e $f(x) = -\frac{4}{x}$

$$\therefore \quad I.F = e^{\int f(x)dx} = e^{\int -\frac{4}{x}dx} = e^{-4\log x} = e^{\log x^{-4}} = x^{-4}$$

Multiply eq. (1) by x^{-4} throughout, we get

$$\left(\frac{y^2}{x^3} - \frac{1}{x^4}e^{1/x^3}\right)dx - \frac{y}{x^2}dy = 0.$$

which is an exact equation.

$\therefore$ the complete solution is given by.

$\int M dx + \int$(terms of N not containing x)$dy = c$

$$= \int\left(\frac{y^2}{x^3} - \frac{1}{x^4}e^{1/x^3}\right)dx + \int 0\,dy = c.$$

$$\Rightarrow -\frac{y^2x^{-2}}{2} + \frac{1}{3}\int e^{x^{-3}}(-3x^{-4})\,dx = c$$

$$\Rightarrow -\frac{y^2x^{-2}}{2} + \frac{1}{3}e^{x^{-3}} = c.$$

$$\Rightarrow \frac{1}{3}e^{x^{-3}} - \frac{y^2}{2x^2} = c.$$

Ex. Solve $(y\log y)\,dx + (x - \log y)\,dy = 0$

Ex. Solve $(xy^3 + y)\,dx + 2(x^2y^2 + x + y^4)\,dy = 0$