Maxima and Minima

A function $f(x,y)$ is said to have a maximum value at $x=a$, $y=b$ if there exist a small neighbourhood of (a,b) s.t.

$$f(a,b) > f(a+h, b+k)$$

Whereas a fuction $f(x,y)$ is said to have a minimum value for $x=a$, $y=b$ if there exist a small neighbourhood of (a,b) s.t.

$$f(a,b) < f(a+h, b+k).$$

Saddle point : A point where a function is neither maximum nor minimum.

Working rule to find extremum values:

(1) Differentiate $f(x,y)$ for $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, $\frac{\partial^2 f}{\partial x \partial y}$

(2). Solve the equation for x and y when for

$\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

(3) Evaluate $r = \frac{\partial^2 f}{\partial x^2}$, $s = \frac{\partial^2 f}{\partial x \partial y}$, $t = \frac{\partial^2 f}{\partial y^2}$

for the value (a,b).

(4) If $rt - s^2 > 0$ ~~and~~ then

$f(x,y)$ has maximum value when $r < 0$

$f(x,y)$ has minimum value when $r > 0$.

(5.) If $rt - s^2 < 0$, then $f(x,y)$ has no extremum value at the point (a,b)

(6) If $rt - s^2 = 0$ then the test is inconclusive.

Ex. Discuss the maximum and minimum of $x^2+y^2+6x+12$.

Sol: We have $f(x,y) = x^2+y^2+6x+12$.

$$\frac{\partial f}{\partial x} = 2x+6, \quad \frac{\partial f}{\partial y} = 2y, \quad \frac{\partial^2 f}{\partial x^2} = 2, \quad \frac{\partial^2 f}{\partial y^2} = 2, \quad \frac{\partial^2 f}{\partial x \partial y} = 0$$

For maxima and minima, $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

$$2x+6 = 0, \quad 2y = 0$$

$$x = -3, \quad y = 0.$$

At $(-3,0)$

$$rt - s^2 = 2\times 2 - 0 = 4 > 0$$

and $r = 2 > 0$.

Hence $f(x,y)$ is minimum value when $x=-3$, $y=0$

And minimum value at $(-3,0) = (-3)^2 + (0)^2 + 6.(-3) + 12$

$$= 9 \cancel{+} 18 + 12$$

$$= 21 - 18 = 3$$

Ex. Examine $f(x,y) = x^3 + y^3 - 3axy$ for maximum and minimum value.

Sol: We have, $f(x,y) = x^3 + y^3 - 3axy$

$$\frac{\partial f}{\partial x} = 3x^2 - 3ay, \quad \frac{\partial f}{\partial y} = 3y^2 - 3ax$$

$$\frac{\partial^2 f}{\partial x^2} = 6x, \quad \frac{\partial^2 f}{\partial y^2} = 6y, \quad \frac{\partial^2 f}{\partial x \partial y} = -3a$$

For maxima and minima,

$$\frac{\partial f}{\partial x} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = 0.$$

$$3x^2 - 3ay = 0 \qquad\qquad 3y^2 - 3ax = 0$$

$$x^2 - ay = 0 \qquad\qquad y^2 - ax = 0.$$

$$x^2 = ay \qquad\qquad y^2 = ax$$

$$\left.y = \frac{x^2}{a}\right\} (1) \qquad\qquad \left.x = \frac{y^2}{a}\right\} (2)$$

Putting the value of y in (2).

$$\left(\frac{x^2}{a}\right)^2 = ax$$

$$x^4 = a^3 x$$

$$x^4 - a^3 x = 0 \Rightarrow x(x^3 - a^3) = 0$$

$$= x(x-a)(x^2+ax+a^2)=0$$

x^3-y^3
$=(x-y)(x^2-yx+y^2)$

Either ~~x-a=0~~ ~~or~~

$x=0$ or $(x-a)(x^2+ax+a^2)=0.$

$x=0$, $x-a=0 \Rightarrow x=a.$

$\therefore$ $x=0, a.$

Putting $x=0$ in (1), we get $y=0.$

When $x=a$ in (1), we get $y=a.$

$\therefore$ At $(0,0)$.

$$rt-s^2 = (6x.6y)-(-3a)^2$$
$$= 18xy - 9a^2$$
$$= 18.0.0 - 9a^2$$
$$= -9a^2 < 0.$$

At (a,a)

$$rt-s^2 = 6.a.6.a-(-3a)^2$$
$$= 36a^2 - 9a^2$$
$$= 27a^2 > 0$$

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At (0,0) there is no extremum value,

Since $rt - s^2 < 0$

At (a,a), $rt - s^2 > 0$, $r > 0$

therefore at (a,a) is a point of minimum value.

And minimum value at $f(a,a)$

$$= a^3 + a^3 - 3.a.a.a$$

$$= 2a^3 - 3a^3 = -a^3.$$

Try yourself:

1. Examine $f(x,y) = \cancel{x^2 + y^2 + 6x + 12}$ $y^2 + 4xy + 3x^2 + x^3$ for maximum and minimum.

2. Examine $f(x,y) = x^3 y^2 (1-x-y)$