LAPLACE TRANSFORMS

INTRODUCTION

Laplace Transformations were introduced by Pierre Simmon Marquis De Laplace (1749-1827), a French Mathematician known as a Newton of French. Laplace Transformations is a powerful Technique; it replaces operations of calculus by operations of Algebra. Suppose an Ordinary (or) Partial Differential Equation together with Initial conditions is reduced to a problem of solving an Algebraic Equation.

USES:

- Particular Solution is obtained without first determining the general solution
- Non-Homogeneous Equations are solved without obtaining the complementary Integral
- Solutions of Mechanical (or) Electrical problems involving discontinuous force functions (R.H.S function) (or) Periodic functions other than and are obtained easily.

Applications:

- L.T is applicable not only to continuous functions but also to piece-wise continuous functions, complicated periodic functions, step functions, Impulse functions.

Definition:

Let f (t) be a function of't' defined for all positive values of t. Then Laplace transforms of f (t) is denoted by L {f (t)} is defined by $L\{f(t)\} = \int_{0}^{\infty} e^{-st} f(t)dt = \overline{f}(s)$ (1)

Provided that the integral exists. Here the parameter's' is a real (or) complex number.

The relation (1) can also be written as $f(t) = L^{-1}\{\overline{f}(s)\}$

In such a case the function f(t) is called the inverse Laplace transform of $\overline{f}(s)$.The symbol 'L' which transform f(t) in to $\overline{f}(s)$ is called the Laplace transform operator. The symbol 'L^{-1}' which transforms $\bar{f}$(s) to f (t) can be called the inverse Laplace transform operator.

Conditions for Laplace Transforms

Exponential order: A function f (t) is said to be of exponential order 'a' If $\underset{t\to\infty}{lt}\, e^{-st} f(t) =$ a finite quantity.

Ex: (i). The function t^2 is of exponential order

(ii). The function e^{t^3} is not of exponential order (which is not limit)

Piece – wise Continuous function: A function f (t) is said to be piece-wise continuous over the closed interval [a,b] if it is defined on that interval and is such that the interval can be divided in to a finite number of sub intervals, in each of which f (t) is continuous and has both right and left hand limits at every end point of the subinterval.

Sufficient conditions for the existence of the Laplace transform of a function:

The function f (t) must satisfy the following conditions for the existence of the L.T.

(i).The function f (t) must be piece-wise continuous (or) sectionally continuous in any limited interval $0 < a \le t \le b$

(ii).The function f (t) is of exponential order.

Laplace Transforms of standard functions:

1. Prove that $L\{1\} = \frac{1}{s}$

Proof: By definition

$$L\{1\} = \int_0^\infty e^{-st}.1dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{e^{-\infty}}{-s} - \frac{e^0}{-s} = 0 + \frac{1}{s}\; if\; s > 0$$

$$L\{1\} = \frac{1}{s}\; \left(\therefore e^{-\infty} = 0\right)$$

2. Prove that $L\{t\} = \frac{1}{s^2}$

Proof: By definition

$$L\{t\} = \int_0^\infty e^{-st}.tdt = \left[t.\left(\frac{e^{-st}}{-s}\right) - \int 1.\frac{e^{-st}}{-s}dt\right]_0^\infty$$

$$= \left[t.\frac{e^{-st}}{-s} - \frac{e^{-st}}{(-s)^2}\right]_0^\infty = \frac{1}{s^2}$$

3. Prove that $L\left\{t^{n}\right\}=\frac{n!}{s^{n+1}}$ where n is a +ve integer

Proof: By definition

$$L\left\{t^{n}\right\}=\int_{0}^{\infty} e^{-st}.t^{n}dt=\left[t^{n}.\frac{e^{-st}}{-s}\right]_{0}^{\infty}-\int_{0}^{\infty} n.t^{n-1}.\frac{e^{-st}}{-s}dt$$

$$=0-0+\frac{n}{s}\int_{0}^{\infty} e^{-st}t^{n-1}dt$$

$$=\frac{n}{s}L\left\{t^{n-1}\right\}$$

$$Similarly\quad L\left\{t^{n-1}\right\}=\frac{n-1}{s}L\left\{t^{n-2}\right\}$$

$$L\left\{t^{n-2}\right\}=\frac{n-2}{s}L\left\{t^{n-3}\right\}$$

By repeatedly applying this, we get

$$L\left\{t^{n}\right\}=\frac{n}{s}.\frac{n-1}{s}.\frac{n-2}{s}.....\frac{2}{s}.\frac{1}{s}L\left\{t^{n-n}\right\}$$

$$=\frac{n!}{s^{n}}L\{1\}=\frac{n!}{s^{n}}.\frac{1}{s}=\frac{n!}{s^{n+1}}$$

Note: $L\left\{t^{n}\right\}$ can also be expressed in terms of Gamma function.

$$i.e.,\mathrm{L}\left\{t^{n}\right\}=\frac{n!}{s^{n+1}}=\frac{\Gamma(n+1)}{s^{n+1}}\left(\because \Gamma(n+1)=n!\right)$$

Def: If n>0 then Gamma function is defined by $\Gamma(n)=\int_{0}^{\infty} e^{-x}x^{n-1}dx$

We have $L\left\{t^{n}\right\}=\int_{0}^{\infty} e^{-st}.t^{n}dt$

Putting x=st on R.H.S, we get

$$L\left\{t^{n}\right\}t=\int_{0}^{\infty} e^{-x}.\frac{x^{n}}{s^{n}}.\frac{1}{s}dx \qquad \begin{pmatrix} x=st \\ \frac{1}{s}dx=dt \end{pmatrix}$$

$$=\frac{1}{s^{n+1}}\int_{0}^{\infty} e^{-x}.x^{n}dx \qquad \begin{pmatrix} When & t=0, x=0 \\ When & t=\infty, x=\infty \end{pmatrix}$$

$$L\left\{t^{n}\right\}=\frac{1}{s^{n+1}}.\Gamma(n+1)$$

$If\ 'n'\text{is } a \text{ +ve integer then } \Gamma(n+1)=n!$

$$\therefore L\left\{t^{n}\right\}=n!\big/s^{n+1}$$

Note: The following are some important properties of the Gamma function.

1. $\Gamma(n+1) = n.\Gamma(n) \, if \; n > 0$

2. $\Gamma(n+1) = n!$ if n is a +ve integer

3. $\Gamma(1) = 1, \Gamma(1/2) = \sqrt{\pi}$

Note: Value of $\Gamma(n)$ in terms of factorial

$$\Gamma(2) = 1 \times \Gamma(1) = 1!$$
$$\Gamma(3) = 2 \times \Gamma(2) = 2!$$
$$\Gamma(4) = 3 \times \Gamma(3) = 3!$$

In general $\Gamma(n+1) = n!$ provided 'n' is a +ve integer.

Taking n=0, it defined 0! = $\Gamma(1) = 1$

4. Prove that $L\{e^{at}\} = \dfrac{1}{s-a}$

Proof: By definition,

$$L\{e^{at}\} = \int_0^\infty e^{-st} e^{at} dt = \int_0^\infty e^{-(s-a)t} dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)} \right]_0^\infty$$

$$= \frac{-e^{-\infty}}{s-a} + \frac{e^0}{s-a} = \frac{1}{s-a} \, if \; s > a$$

Similarly $L\{e^{-at}\} = \dfrac{1}{s+a} \, if \; s > -a$

5. Prove that $L\{\sinh at\} = \dfrac{a}{s^2 - a^2}$

Proof: $L\{\sinh at\} = L\left\{ \dfrac{e^{at} - e^{-at}}{2} \right\} = \dfrac{1}{2}\left[L\{e^{at}\} - L\{e^{-at}\} \right]$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a} \right] = \frac{1}{2}\left[\frac{s+a-s+a}{s^2-a^2} \right] = \frac{2a}{2(s^2-a^2)} = \frac{a}{s^2-a^2}$$

6. Prove that $L\{\cosh at\} = \dfrac{s}{s^2 - a^2}$

Proof: $L\{\cosh at\} = L\left\{ \dfrac{e^{at} + e^{-at}}{2} \right\}$

$$=\frac{1}{2}\left[L\left\{e^{at}\right\}+L\left\{e^{-at}\right\}\right]=\frac{1}{2}\left\{\frac{1}{s-a}+\frac{1}{s+a}\right\}$$

$$=\frac{1}{2}\left[\frac{s+a+s-a}{s^2-a^2}\right]=\frac{2s}{2\left(s^2-a^2\right)}=\frac{s}{s^2-a^2}$$

7. Prove that $L\{\sin at\}=\dfrac{a}{s^2+a^2}$

Proof: By definition,

$$L\{\sin at\}=\int_0^\infty e^{-st}\sin at\,dt$$

$$=\left[\frac{e^{-st}}{s^2+a^2}\left(-s\sin at-a\cos at\right)\right]_0^\infty$$

$$\left[\because \int e^{ax}\sin bx\,dx=\frac{e^{ax}}{a^2+b^2}\left(a\sin bx-b\cos bx\right)\right]$$

$$=\frac{a}{s^2+a^2}$$

8. Prove that $L\{\cos at\}=\dfrac{s}{s^2+a^2}$

Proof: We know that $L\left\{e^{at}\right\}=\dfrac{1}{s-a}$

Replace 'a' by 'ia' we get

$$L\left\{e^{iat}\right\}=\frac{1}{s-ia}=\frac{s+ia}{(s-ia)(s+ia)}$$

$$i.e., \mathrm{L}\left\{cosat+i\sin at\right\}=\frac{s+ia}{s^2+a^2}$$

Equating the real and imaginary parts on both sides, we have

$$L\{\cos at\}=\frac{s}{s^2+a^2}\ and\ L\{\sin at\}=\frac{a}{s^2+a^2}$$

Problems

1. Find the Laplace transforms of $(t^2+1)^2$

Sol: Here $f(t)=(t^2+1)^2=t^4+2t^2+1$

$$L\{(t^2+1)^2\}=L\{t^4+2t^2+1\}=L\{t^4\}+2L\{t^2\}+L\{1\}$$

$$=\frac{4!}{s^{4+1}}+2.\frac{2!}{s^3}+\frac{1}{s}=\frac{4!}{s^5}+2.\frac{2!}{s^3}+\frac{1}{s}$$

$$=\frac{24}{s^5}+\frac{4}{s^3}+\frac{1}{s}=\frac{1}{s^5}(24+4s^2+s^4)$$

2. Find the Laplace transform of $L\left\{\dfrac{e^{-at}-1}{a}\right\}$

Sol: $L\left\{\dfrac{e^{-at}-1}{a}\right\}=\dfrac{1}{a}L\left\{e^{-at}-1\right\}=\dfrac{1}{a}\left[L\left\{e^{-at}\right\}-L\left\{1\right\}\right]$

$$=\frac{1}{a}\left[\frac{1}{s+a}-\frac{1}{s}\right]=-\frac{1}{s(s+a)}$$

3. Find the Laplace transform of Sin2tcost

Sol: W.K.T $\sin 2t\cos t=\frac{1}{2}[2\sin 2t\cos t]=\frac{1}{2}[\sin 3t+\sin t]$

$\therefore L\{\sin 2t\cos t\}=L\left\{\frac{1}{2}[\sin 3t+\sin t]\right\}=\frac{1}{2}\left[\mathrm{L}\left\{\sin 3t\right\}+L\left\{\sin t\right\}\right]$

$$=\frac{1}{2}\left[\frac{3}{s^2+9}+\frac{1}{s^2+1}\right]=\frac{2(s^2+3)}{(s^2+1)(s^2+9)}$$

4. Find the Laplace transform of Cosh²2t

Sol: W.K.T $\cosh^2 2t=\frac{1}{2}\left[1+\cosh 4t\right]$

$L\left\{\cosh^2 2t\right\}=\frac{1}{2}\left[L(1)+L\{\cosh 4t\}\right]$

$$=\frac{1}{2}\left[\frac{1}{s}+\frac{s}{s^2-16}\right]=\frac{s^2-8}{s(s^2-16)}$$

5. Find the Laplace transform of Cos³3t

Sol: Since cos9t=cos3(3t)

cos9t=4cos³3t-3cos3t (or) $\cos^3 3t=\frac{1}{4}\left[\cos 9t+3\cos 3t\right]$

$L\{\cos^3 3t\}=\frac{1}{4}L\{\cos 9t\}+\frac{3}{4}L\{\cos 3t\}$

$$\therefore \quad =\frac{1}{4}\cdot\frac{s}{s^2+81}+\frac{3}{4}\cdot\frac{s}{s^2+9}$$

$$=\frac{\mathrm{s}}{4}\left[\frac{1}{s^2+81}+\frac{3}{s^2+9}\right]=\frac{s\left(s^2+63\right)}{\left(s^2+9\right)\left(s^2+81\right)}$$

6. Find the Laplace transforms of $\left(\sin t+\cos t\right)^2$

Sol: Since $\left(\sin t+\cos t\right)^2=\sin^2 t+\cos^2 t+2\sin t\cos t\;=1+\sin 2t$

$$L\{(\sin t+\cos t)^2\}=L\{1+\sin 2t\}$$

$$=L\{1\}+L\{\sin 2t\}$$

$$=\frac{1}{s}+\frac{2}{s^2+4}=\frac{s^2+2s+4}{s\left(s^2+4\right)}$$

7. Find the Laplace transforms of cost cos2t cos3t

Sol: $\cos t\cos 2t\cos 3t=\frac{1}{2}.\cos t\left[2.\cos 2t.\cos 3t\right]$

$$=\frac{1}{2}\cos t[\cos 5t+\cos t]=\frac{1}{2}[\cos t\cos 5t+\cos^2 t]$$

$$=\frac{1}{4}\left[2\cos t\cos 5t+2\cos^2 t\right]=\frac{1}{4}\left[\left(\cos 6t+\cos 4t\right)+\left(1+\cos 2t\right)\right]$$

$$=\frac{1}{4}[1+\cos 2t+\cos 4t+\cos 6t]$$

$$\therefore L\{\cos t\cos 2t\cos 3t\}=\frac{1}{4}L\{1+\cos 2t+\cos 4t+\cos 6t\}$$

$$=\frac{1}{4}[L\{1\}+L\{\cos 2t\}+L\{\cos 4t\}+L\{\cos 6t\}]$$

$$=\frac{1}{4}\left[\frac{1}{s}+\frac{s}{s^2+4}+\frac{s}{s^2+16}+\frac{s}{s^2+36}\right]$$

8. Find L.T. of $\mathbf{Sin^2 t}$

Sol: $L\{\sin^2 t\}=L\left\{\frac{1-\cos 2t}{2}\right\}$

$$=\frac{1}{2}[\mathrm{L}\{1\}-L\{\cos 2t\}]=\frac{1}{2}\left[\frac{1}{s}-\frac{s}{s^2+4}\right]$$

9. Find $\mathbf{L\left(\sqrt{t}\right)}$

Sol: $L\left\{\sqrt{t}\right\}=L\left[t^{1/2}\right]=\frac{\Gamma\left(\frac{1}{2}+1\right)}{s^{\frac{1}{2}+1}}$ where n is not an integer

$$=\frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{s^{\frac{3}{2}}}=\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}\quad \because \Gamma\left(n+1\right)=n.\Gamma\left(n\right)$$

10. Find $\boldsymbol{L\ \{sin(\omega t+\alpha)\}}$, where α a constant is

Sol: $L\{sin(\omega t+\alpha)\}=L\{sin\omega t cos\alpha+cos\omega t sin\alpha\}$

$$=cos\alpha\ L\{sin\omega t\}+sin\alpha\ L\{cos\omega t\}$$

$$=cos\alpha\frac{\omega}{s^2+\omega^2}+sin\alpha\frac{\omega}{s^2+\omega^2}$$

Properties of Laplace transform:

Linearity Property:

Theorem1: The Laplace transform operator is a Linear operator.

i.e. $(i).L\{cf(t)\} = c.\mathrm{L}\{f(t)\}$ $(ii).L\{f(t)+g(t)\} = L\{f(t)\}+L\{g(t)\}$ Where 'c' is constant

Proof: (i) By definition

$$L\{cf(t)\} = \int_0^\infty e^{-st}cf(t)\,dt = c\int_0^\infty e^{-st}f(t)\,dt = cL\{f(t)\}$$

(ii) By definition

$$L\{f(t)+g(t)\} = \int_0^\infty e^{-st}\{f(t)+g(t)\}dt$$

$$= \int_0^\infty e^{-st}f(t)dt + \int_0^\infty e^{-st}g(t)dt = L\{f(t)\}+L\{g(t)\}$$

Similarly the inverse transforms of the sum of two or more functions of 's' is the sum of the inverse transforms of the separate functions.

Thus, $L^{-1}\{\overline{f}(s)+\overline{g}(s)\} = L^{-1}\{\overline{f}(s)\}+L^{-1}\{\overline{g}(s)\} = f(t)+g(t)$

Corollary: $L\{c_1f(t)+c_2g(t)\} = c_1L\{f(t)\}+c_2L\{g(t)\}$, where c_1, c_2 are constants

Theorem2: If a, b, c be any constants and f, g, h any functions of t, then

$$L\{af(t)+bg(t)-ch(t)\} = a.L\{f(t)\}+b.L\{g(t)\}-cL\{h(t)\}$$

Proof: By the definition

$$L\{af(t)+bg(t)-ch(t)\} = \int_0^\infty e^{-st}\{af(t)+bg(t)-ch(t)\}dt$$

$$= a.\int_0^\infty e^{-st}f(t)\,dt + b\int_0^\infty e^{-st}g(t)\,dt - c\int_0^\infty e^{-st}h(t)\,dt$$

$$= a.L\{f(t)\}+bL\{g(t)\}-cL\{h(t)\}$$

Change of Scale Property:

If $L\{f(t)\} = \bar{f}(s)$ *then* $L\{f(at)\} = \frac{1}{a}.\bar{f}\left(\frac{s}{a}\right)$

Proof: By the definition we have

$$L\{f(at)\} = \int_0^\infty e^{-st} f(at)dt$$

$$Put \quad at = u \Rightarrow dt = \frac{du}{a}$$

$$when \;\; t \to \infty \;\; then \;\; u \to \infty \;\; and \; t = 0 \;\; then \; u = 0$$

$$\therefore L\{f(at)\} = \int_0^\infty e^{-\frac{su}{a}} f(u)\frac{du}{a} = \frac{1}{a}.\int_0^\infty e^{-\left(\frac{s}{a}\right).u} f(u)du = \frac{1}{a}.\overline{f}\left(s/_a\right)$$

1. Find $L\{\sinh 3t\}$

Sol: $L\{\sinh t\} = \frac{1}{s^2-1} = \overline{f}(s)$

$$\therefore L\{\sinh 3t\} = \frac{1}{3}\overline{f}(s/_3) \text{(Change of scale property)}$$

$$= \frac{1}{3}\frac{1}{\left(s/_3\right)^2 - 1} = \frac{3}{s^2 - 9}$$

2. Find $L\{\cos 7t\}$

Sol: $L\{\cos t\} = \frac{s}{s^2+1} = \overline{f}(s)\ (say)$

$$L\{\cos 7t\} = \frac{1}{7}\overline{f}\left(s/_7\right) \text{ (Change of scale property)}$$

$$L\{\cos 7t\} = \frac{1}{7}\frac{s/_7}{(s/_7)^2+1} = \frac{s}{s^2+49}$$

First shifting property:

If $L\{f(t)\} = \overline{f}(s)$ then $L\{e^{at} f(t)\} = \overline{f}(s - a)$

Proof: By the definition

$$L\{e^{at} f(t)\} = \int_0^\infty e^{-st} e^{at} f(t)dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)dt$$

$$= \int_0^\infty e^{-ut} f(t)dt \; where \, u = s - a$$

$$= \overline{f}(u) = \overline{f}(s-a)$$

Note: Using the above property, we have $L\{e^{-at} f(t)\} = \overline{f}(s+a)$

Applications of this property, we obtain the following results

1. $L\{e^{at} t^n\} = \dfrac{n!}{(s-a)^{n+1}} \left[\because L(t^n) = \dfrac{n!}{s^{n+1}}\right]$

2. $L\{e^{at}\sin bt\} = \dfrac{b}{(s-a)^2+b^2}\left[\because L(\text{sinbt}) = \dfrac{b}{s^2+b^2}\right]$

3. $L\{e^{at}\cos bt\} = \dfrac{s-a}{(s-a)^2+b^2}\left[\because L(\text{cosbt}) = \dfrac{s}{s^2+b^2}\right]$

4. $L\{e^{at}\sinh bt\} = \dfrac{b}{(s-a)^2-b^2}\left[\because L(\text{sinhbt}) = \dfrac{b}{s^2-b^2}\right]$

5. $L\{e^{at}\cosh bt\} = \dfrac{s-a}{(s-a)^2-b^2}\left[\because L(\text{coshbt}) = \dfrac{s}{s^2-b^2}\right]$

1. Find the Laplace Transforms of t^3e^{-3t}

Sol: Since $L\{t^3\} = \dfrac{3!}{s^4}$

Now applying first shifting theorem, we get

$$L\{t^3 e^{-3t}\} = \frac{3!}{(s+3)^4}$$

2. Find the L.T. of $e^{-t}\cos 2t$

Sol: Since $L\{\cos 2t\} = \frac{s}{s^2+4}$

Now applying first shifting theorem, we get

$$L\{e^{-t}\cos 2t\} = \frac{s+1}{(s+1)^2+4} = \frac{s+1}{s^2+2s+5}$$

3. Find L.T of $\boldsymbol{e^{2t}cos^2 t}$

Sol: - $L[e^{2t}cos^2 t] = L[e^{2t}(\frac{1+cos2t}{2})]$

$$= \frac{1}{2}\{L[e^{2t}] + L[e^{2t}cos2t]\}$$

$$= \frac{1}{2}\left(\frac{1}{s-2}\right) + \frac{1}{2}\{L[cos2t]\}_{s\to s-2}$$

$$= \frac{1}{2}\left(\frac{1}{s-2}\right) + \frac{1}{2}\frac{s-2}{(s-2)^2+2^2}$$

$$= \frac{1}{2}\left(\frac{1}{s-2}\right) + \frac{1}{2}\frac{s-2}{(s^2-4s+8)}$$

Second translation (or) second Shifting theorem:

If $L\{f(t)\} = \overline{f}(s) and\ g(t) = \begin{cases} f(t-a), t>a \\ 0 \quad\quad t<a \end{cases} then\ L\{g(t)\} = e^{-as}\overline{f}(s)$

Proof: By the definition

$$L\{g(t)\} = \int_0^\infty e^{-st}\, g(t)dt = \int_0^a e^{-st}\, g(t)dt + \int_a^\infty e^{-st}\, g(t)dt$$

$$= \int_0^\infty e^{-st}.\, odt + \int_a^\infty e^{-st}\, f(t-a)dt = \int_a^\infty e^{-st}\, f(t-a)dt$$

Let t-a = u so that dt = du And also u = 0 when t = a and $u \to \infty$ when $t \to \infty$

$\therefore L\{g(t)\} = \int_0^\infty e^{-s(u+a)} f(u)du = e^{-as}\int_0^\infty e^{-su} f(u)du = e^{-as}\int_a^\infty e^{-st} f(t)dt$

$= e^{-as}L\{f(t)\} = e^{-as}\overline{f}(s)$

Another Form of second shifting theorem:

If $L\{f(t)\} = \overline{f}(s)$ *and* $a > 0$ *then* $L\{F(t-a)H(t-a)\} = e^{-as}\overline{f}(s)$

where H (t) = $\begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$ and H(t) is called Heaviside unit step function.

Proof: By the definition

$$L\{F(t-a)H(t-a)\} = \int_0^\infty e^{-st} F(t-a)H(t-a)dt \to (1)$$

Put t-a=u so that dt= du and also when t=0, u=-a when t $\to \infty$, u$\to \infty$

Then $L\{F(t-a)H(t-a)\} = \int_a^\infty e^{-s(u+a)} F(u)H(u)du.$ $[by\ eq(1)]$

$$= \int_{-a}^{0} e^{-s(u+a)} F(u)H(u)du + \int_0^\infty e^{-s(u+a)} F(u)H(u)du$$

$$= \int_{-a}^{0} e^{-s(u+a)} F(u).0du + \int_0^\infty e^{-s(u+a)} F(u).1du$$

[Since By the definition of H (t)]

$$= \int_0^\infty e^{-s(u+a)} F(u)du = e^{-as}\int_a^\infty e^{-su} F(u)du$$

$= e^{-sa}\int_0^\infty e^{-st}F(t)dt$ by property of Definite Integrals

$$= e^{-as}L\{F(t)\} = e^{-as}\overline{f}(s)$$

Note: $H(t-a)$ *is also denoted by* $u(t-a)$

1. **Find the L.T. of g (t) when** $g(t) = \begin{cases} \cos\left(t - \pi/3\right) & if\ t > \pi/3 \\ 0 & if\ t < \pi/3 \end{cases}$

Sol. Let $f(t) = \cos t$

$\therefore L\{F(t)\} = L\{cost\} = \frac{s}{s^2+1} = \overline{f}(s)$

$$g(t) = \begin{cases} f(t - \pi/3) = cos(t - \pi/3), & if\ t > \pi/3 \\ 0 & , if\ t < \pi/3 \end{cases}$$

Now applying second shifting theorem, then we get

$$L\{g(t)\} = e^{\frac{-\pi s}{3}}\left(\frac{s}{s^2+1}\right) = \frac{s.e^{\frac{-\pi s}{3}}}{s^2+1}$$

2. **Find the L.T. of (ii)$(t-2)^3u(t-2)$ (ii) $e^{-3t}u(t-2)$**

Sol:

(i). Comparing the given function with f(t-a) u(t-a), we have a=2 and f(t)=t^3

$$\therefore L\{f(t)\} = L\{t^3\} = \frac{3!}{s^4} = \frac{6}{s^4} = \overline{f}(s)$$

Now applying second shifting theorem, then we get

$$L\{(t-2)^3 u(t-2)\} = e^{-2s}\frac{6}{s^4} = \frac{6e^{-2s}}{s^4}$$

(ii). $L\{e^{-st}u(t-2)\} = L\left\{e^{-s(t-2)}.e^{-6}u(t-2)\right\} = e^{-6}L\left\{e^{-3(t-2)}u(t-2)\right\}$

$f(t) = e^{-3t}$ then $\overline{f}(s) = \frac{1}{s+3}$

Now applying second shifting theorem then, we get

$$L\{e^{-3t}u(t-2)\} = e^{-6}.e^{-2s}\frac{1}{s+3} = \frac{e^{-2(s+3)}}{s+3}$$

Multiplication by‘t’:

Theorem: If $\boldsymbol{L\{f(t)\} = \overline{f}(s)}$ ***then*** $\boldsymbol{L\{tf(t)\} = \frac{-d}{ds}\overline{f}(s)}$

Proof: By the definition $\overline{f}(s) = \int_0^\infty e^{-st} f(t)\,dt$

$$\frac{d}{ds}\left\{\overline{f}(s)\right\} = \frac{d}{ds}\int_0^\infty e^{-st} f(t)\,dt$$

By Leibnitz’s rule for differentiating under the integral sign,

$$\frac{d}{ds}\overline{f}(s) = \int_0^\infty \frac{\partial}{\partial s} e^{-st} f(t)\,dt$$

$$= \int_0^\infty -te^{-st} f(t)\,dt$$

$$= -\int_0^\infty e^{-st}\{tf(t)\}dt = -L\{tf(t)\}$$

Thus $L\{tf(t)\} = \frac{-d}{ds}\overline{f}(s)$

$$\therefore L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} = \overline{f}(s)$$

Note: Leibnitz’s Rule

If $f(x,\alpha)$ *and* $\frac{\partial}{\partial\alpha} f(x,\alpha)$ be continuous functions of x and α then

$$\frac{d}{d\alpha}\left\{\int_a^b f(x,\alpha)dx\right\} = \int_a^b \frac{\partial}{\partial\alpha} f(x,\alpha)dx$$

Where a, b are constants independent of α

Problems:

1. Find L.T of tcosat

Sol: Since $L\{tcos\ at\} = \frac{s}{s^2+a^2}$

$$L\{tcos\ at\} = -\frac{d}{ds}\left[\frac{s}{s^2+a^2}\right]$$

$$= \frac{-s^2+a^2-s.2s}{(s^2+a^2)^2} = \frac{s^2-a^2}{(s^2+a^2)^2}$$

2. Find t^2sin at

Sol: Since $L\{sin\ at\} = \frac{a}{s^2+a^2}$

$$L\{t^2.sin\ at\} = (-1)^2\frac{d^2}{ds^2}\left(\frac{a}{s^2+a^2}\right)$$

$$= \frac{d}{ds}\left(\frac{-2as}{\left(s^2+a^2\right)^2}\right) = \frac{2a\left(3s^2-a^2\right)}{\left(s^2+a^2\right)^3}$$

3. Find $L.T\ of\ te^{-t}\sin 3t$

Sol: Since $L\{sin\ 3t\} = \frac{3}{s^2+3^2}$

$$\therefore L\{tsin\ 3t\} = \frac{-d}{ds}\left[\frac{3}{s^2+3^2}\right] = \frac{6s}{(s^2+9)^2}$$

Now using the shifting property, we get

$$L\{te^{-t}sin\ 3t\} = \frac{6(s+1)}{((s+1)^2+9)^2} = \frac{6(s+1)}{(s^2+2s+10)^2}$$

4. Find $L\{te^{2t}sin\ 3t\}$

Sol: Since $L\{sin\ 3t\} = \frac{3}{s^2+9}$

$$\therefore \mathrm{L}\left\{e^{2t}\sin 3t\right\} = \frac{3}{\left(s-2\right)^2+9} = \frac{3}{s^2-4s+13}$$

$$L\{te^{2t}sin\ 3t\} = (-1)\frac{d}{ds}\left[\frac{3}{s^2-4s+13}\right] = (-1)\left[\frac{0-3(2s-4)}{(s^2-4s+13)^2}\right]$$

$$= \frac{3\left(2s-4\right)}{\left(s^2-4s+13\right)^2} = \frac{6\left(s-2\right)}{\left(s^2-4s+13\right)^2}$$

5. Find the L.T. of $\left(1+te^{-t}\right)^2$

Sol: Since $(1+te^{-t})^2 = 1+2te^{-t}+t^2e^{-2t}$

$$\therefore \mathrm{L}\left(1+te^{-t}\right)^2 = \mathrm{L}\left\{1\right\}+2\mathrm{L}\left\{te^{-t}\right\}+\mathrm{L}\left\{t^2e^{-2t}\right\}$$

$$= \frac{1}{s}+2(-1)\frac{d}{ds}\left(\frac{1}{s+1}\right)+(-1)^2\frac{d^2}{ds^2}\left(\frac{1}{s+2}\right)$$

$$=\frac{1}{s}+\frac{2}{(s+1)^2}+\frac{d}{ds}\left(\frac{-1}{(s+2)^2}\right)$$

$$=\frac{1}{s}+\frac{2}{(s+1)^2}+\frac{2}{(s+2)^3}$$

6. Find the L.T of t^3e^{-3t} (already we have solved by another method)

Sol: $L\{t^3e^{-3t}\}=(-1)^3\frac{d^3}{ds^3}L\{e^{-3t}\}$

$$=-\frac{d^3}{ds^3}\left(\frac{1}{s+3}\right)=\frac{-3!(-1)^3}{(s+3)^4}$$

$$=\frac{3!}{(s+3)^4}$$

7. Find $L\{\cosh at \sin at\}$

Sol. $L\{\cosh at\sin at\}=L\left\{\frac{e^{at}+e^{-at}}{2}.\sin at\right\}$

$$=\frac{1}{2}[L\{e^{at}\sin at\}+L\{e^{-at}\sin at\}]$$

$$=\frac{1}{2}\left[\frac{a}{(s-a)^2+a^2}+\frac{a}{(s+a)^2+a^2}\right]$$

8. Find the L.T of the function $\begin{aligned} f(t)&=(t-1)^2, \quad t>1\\ &=0 \qquad 0<t<1\end{aligned}$

Sol: By the definition

$$L\{f(t)\}=\int_0^\infty e^{-st}f(t)dt=\int_0^1 e^{-st}f(t)dt+\int_1^\infty e^{-st}f(t)dt$$

$$=\int_0^1 e^{-st}0dt+\int_1^\infty e^{-st}(t-1)^2dt$$

$$=\int_1^\infty e^{-st}(t-1)^2dt=\left[(t-1)^2\frac{e^{-st}}{-s}\right]_1^\infty-\int_1^\infty 2(t-1)\frac{e^{-st}}{-s}dt$$

$$=0+\frac{2}{s}\int_1^\infty e^{-st}(t-1)dt$$

$$=\frac{2}{s}\left[\left\{(t-1)\left(\frac{e^{-st}}{-s}\right)\right\}_1^\infty-\int_1^\infty\frac{e^{-st}}{-s}dt\right]$$

$$=\frac{2}{s}\left[0+\frac{1}{s}\int_1^\infty e^{-st}dt\right]=\frac{2}{s^2}\left(\frac{e^{-st}}{-s}\right)_1^\infty=\frac{-2}{s^3}\left(e^{-st}\right)_1^\infty$$

$$=\frac{-2}{s^3}\left(0-e^{-s}\right)=\frac{2}{s^3}e^{-s}$$

9. **Find the L.T of f (t) defined as** $f(t)=3$, **t >2**

$=0$, **0<t<2**

Sol: $L\{f(t)\} = \int_0^{\infty} e^{-st} f(t)dt$

$$=\int_0^2 e^{-st} f(t)dt+\int_2^{\infty} e^{-st} f(t)dt$$

$$= \int_0^2 e^{-st}.0dt + \int_2^{\infty} e^{-st}\, 3dt$$

$$=0+\int_2^{\infty} e^{-st} 3dt=\frac{-3}{s}\left(e^{-st}\right)_2^{\infty}=\frac{-3}{s}\left(0-e^{-2s}\right)$$

$$=\frac{3}{s}e^{-2s}$$

10. **Find** $\boldsymbol{L\{t\,cos(at+b)\}}$

Sol: $L\{cos(at+b)\} = L\{\cos at \cos b - \sin at \sin b\}$

$$= \cos b. L\{\cos at\} - \sin b\, L\{\sin at\}$$

$$= \cos b.\frac{s}{s^2+a^2} - \sin b.\frac{a}{s^2+a^2}$$

$$L\{t.cos(at+b)\} = \frac{-d}{ds}\left[\cos b.\frac{s}{s^2+a^2} - \sin b.\frac{a}{s^2+a^2}\right]$$

$$=-\cos b.\left(\frac{s^2+a^2.1-s.2s}{\left(s^2+a^2\right)^2}\right)+\sin b\left(\frac{\left(s^2+a^2\right).0-a.2s}{\left(s^2+a^2\right)^2}\right)$$

$$=\frac{1}{\left(s^2+a^2\right)^2}\left[\left(s^2-a^2\right)^2\cos b-2as\sin b\right]$$

11. **Find L.T of L [t**$\boldsymbol{e^t sint}$**]**

Sol: - We know that L[sint] = $\frac{1}{s^2+1}$

L[tsint] = (-1) $\frac{d}{ds}$L[sint] = - $\frac{d}{ds}(\frac{1}{s^2+1})$= - $\frac{(-1)2s}{(s^2+1)^2}$

$$=\frac{2s}{(s^2+1)^2}$$

By First Shifting Theorem

L [t$e^t sint$] $= \left[\frac{2s}{(s^2+1)^2}\right]_{s\to s-1} = \frac{2(s-1)}{((s-1)^2+1)^2} = \frac{2(s-1)}{(s^2-2s+2)^2}$

Division by‘t’:

Theorem: If $\boldsymbol{L\{f(t)\} = \overline{f}(s)\ then\ L\left\{\frac{1}{t}f(t)\right\} = \int_s^{\infty}\overline{f}(s)ds}$

Proof: We have $\overline{f}(s)=\int_0^{\infty} e^{-st} f(t)dt$

Now integrating both sides w.r.t s from s to ∞, we have

$$\int_0^{\infty}\overline{f}(s)ds = \int_s^{\infty}\left[\int_0^{\infty} e^{-st} f(t)dt\right]ds$$

$$= \int_0^{\infty}\int_s^{\infty} f(t)e^{-st}dsdt \text{ (Change the order of integration)}$$

$$= \int_0^{\infty} f(t)\left[\int_s^{\infty} e^{-st}ds\right]dt \text{ ($\because$ t is independent of‘s’)}$$

$$= \int_0^{\infty} f(t)\left(\frac{e^{-st}}{-t}\right)_s^{\infty} dt$$

$$=\int_0^{\infty} e^{-st}\frac{f(t)}{t}dt (or) \mathcal{L}\left\{\frac{1}{t}f(t)\right\}$$

Problems:

1. **Find** $\boldsymbol{L\left\{\frac{\sin t}{t}\right\}}$

Sol: Since $L\{sint\} = \frac{1}{s^2+1} = \overline{f}(s)$

Division by‘t’, we have

$$L\left\{\frac{\sin t}{t}\right\} = \int_s^{\infty}\overline{f}(s)ds = \int_s^{\infty}\frac{1}{s^2+1}ds$$

$$= [Tan^{-1}s]_s^{\infty} = Tan^{-1}\infty - Tan^{-1}s$$

$$= \pi/2 - Tan^{-1}s = \cot^{-1}s$$

2. **Find the L.T of** $\frac{\sin at}{t}$

Sol: Since $L\{\sin at\} = \frac{a}{s^2+a^2} = \overline{f}(s)$

Division by t, we have

$$L\left\{\frac{\sin a t}{t}\right\} = \int_s^{\infty}\overline{f}(s)ds = \int_s^{\infty}\frac{a}{s^2+a^2}ds$$

$$= a.\frac{1}{a}\left[Tan^{-1} s/a\right]_s^{\infty} = Tan^{-1}\infty - Tan^{-1} s/a$$

$$= \pi/2 - Tan^{-1}\left(s/a\right) = \cot^{-1} s/a$$

3. **Evaluate** $\boldsymbol{L\left\{\frac{1-\cos a t}{t}\right\}}$

Sol: Since $L\{1-\cos a t\} = L\{1\} - L\{\cos at\} = \frac{1}{s} - \frac{s}{s^2+a^2}$

$$L\left\{\frac{1-\cos a t}{t}\right\} = \int_s^{\infty}\left(\frac{1}{s} - \frac{s}{s^2+a^2}\right)ds$$

$$= \left[\log s - \frac{1}{2}\log\left(s^2+a^2\right)\right]_s^{\infty}$$

$$=\frac{1}{2}\left[2\log s-\log\left(s^2+a^2\right)\right]_s^\infty=\frac{1}{2}\left[\log\left(\frac{s^2}{s^2+a^2}\right)\right]_s^\infty$$

$$=\frac{1}{2}\left[l\log\left(\frac{1}{1+a^2/s^2}\right)\right]_s^\infty=\frac{1}{2}\left[\log 1-\log\frac{s^2}{s^2+a^2}\right]$$

$$=-\frac{1}{2}l\log\left(\frac{s^2}{s^2+a^2}\right)=\log\left(\frac{s^2}{s^2+a^2}\right)^{\frac{-1}{2}}=\log\sqrt{\frac{s^2+a^2}{s^2}}$$

Note: $L\left\{\frac{1-\cos t}{t}\right\}=\log\sqrt{\frac{s^2+1}{s}}$ (Putting a=1 in the above problem)

4. Find $L\left\{\frac{e^{-at}-e^{-bt}}{t}\right\}$

Sol: $L\left\{\frac{e^{-at}-e^{-bt}}{t}\right\}=\int_s^\infty\left(\frac{1}{s+a}-\frac{1}{s+b}\right)ds$

$$=\left[\log(s+a)-\log(s+b)\right]_s^\infty=\left[\log\left(\frac{s+a}{s+b}\right)\right]_s^\infty$$

$$=\lim_{s\to\infty}\left\{\log\frac{1+\frac{a}{s}}{1+\frac{b}{s}}\right\}-\log\left(\frac{s+a}{s+b}\right)$$

$$=\log 1-\log(s+a)+\log(s+b)=\log\left(\frac{s+b}{s+a}\right)$$

5. Find $\mathrm{L}\left\{\frac{1-\cos t}{t^2}\right\}$

Sol: $\mathrm{L}\left\{\frac{1-\cos t}{t^2}\right\}=\mathrm{L}\left\{\frac{1}{t}.\frac{1-\cos t}{t}\right\}$.....(1)

Now $\mathrm{L}\left\{\frac{1-\cos t}{t}\right\}=\int_s^\infty\left(\frac{1}{s}-\frac{s}{s^2+1}\right)ds=\left[\log s-\frac{1}{2}\log\left(s^2+1\right)\right]_s^\infty$

$$=\frac{1}{2}\left[\log\frac{s^2}{s^2+1}\right]_s^\infty=\frac{-1}{2}\left[\log\frac{s^2}{s^2+1}\right]=\frac{1}{2}\log\frac{s^2+1}{s^2}$$

$$\therefore \mathrm{L}\left[\frac{1-\cos t}{t^2}\right]=\int_s^\infty\frac{1}{2}\log\frac{s^2+1}{s^2}ds$$

$$=\frac{1}{2}\left[\left\{\log\left(\frac{s^2+1}{s^2}\right)\right\}.s\right]_s^\infty-\int_s^\infty\frac{s^2}{s^2+1}\left(\frac{-2}{s^3}\right).sds$$

$$=\frac{1}{2}\left[\left\{\underset{s\to\infty}{lt}\ s.\log\left(1+\frac{1}{s^2}\right)\right\}-s\log\left(\frac{s^2+1}{s^2}\right)+2\int_s^\infty\frac{ds}{s^2+1}\right]$$

$$=\frac{1}{2}\left[\left\{\underset{s\to\infty}{lt}\ s\left(\frac{1}{s^2}-\frac{1}{2s^4}+\frac{1}{3s^6}+....\right)-s\log\frac{s^2+1}{s^2}\right\}+2Tan^{-1}s\right]_s^\infty$$

$$=\frac{1}{2}\left[\left\{0-s\log\left(1+\frac{1}{s^2}\right)+2\left(\frac{\pi}{2}-Tan^{-1}s\right)\right\}\right]\quad\because\left(\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+.....\right)$$

$$=\cot^{-1}s-\frac{1}{2}s\log\left(1+\frac{1}{s^2}\right)$$

3. Find L.T of $\frac{e^{-at}-e^{-bt}}{t}$

Sol: W.K.T L $[e^{-at}]=\frac{1}{s+a}$, L $[e^{-bt}]=\frac{1}{s+b}$

L$[\frac{f(t)}{t}]=\int_s^\infty \bar{f}(s)ds$

$$\therefore\quad L\left[\frac{e^{-at}-e^{-bt}}{t}\right]=\int_s^\infty\left(\frac{1}{s+a}-\frac{1}{s+b}\right)ds$$

$$=[\log(s+a)-\log(s+b)]_s^\infty$$

$$=\log\left(\frac{s+a}{s+b}\right)_s^\infty$$

$$=\log\left(\frac{1+\frac{a}{s}}{1+\frac{b}{s}}\right)_s^\infty$$

$$=\log(1)-\log\left(\frac{s+a}{s+b}\right)$$

$$=0-\log\left(\frac{s+a}{s+b}\right)=\log\left(\frac{s+b}{s+a}\right)$$

Laplace transforms of Derivatives:

If $f^1(t)$ be continuous and $L\{f(t)\}=\overline{f}(s)$ $then$ $L\{f^1(t)\}=s\overline{f}(s)-f(0)$

Proof: By the definition

$$L\{f^1(t)\}=\int_0^\infty e^{-st}f^1(t)dt$$

$$=\left[e^{-st}f(t)\right]_0^\infty-\int_0^\infty(-s)e^{-st}f(t)dt\qquad\text{(Integrating by parts)}$$

$$=\left[e^{-st}f(t)\right]_0^\infty+s\int_0^\infty e^{-st}f(t)dt$$

$$=\underset{t\to\infty}{lt}\ e^{-st}f(t)-f(0)+s.L\{f(t)\}$$

Since f (t) is exponential order

$$\therefore\underset{t\to\infty}{lt}\ e^{-st}f(t)=0$$

$$\therefore L\{f^1(t)\}=0-f(0)+sL\{f(t)\}$$

$$= s\overline{f}(s) - f(0)$$

The Laplace Transform of the second derivative $f^{11}(t)$ is similarly obtained.

$$\therefore L\{f^{11}(t)\} = s.L\{f^{1}(t)\} - f^{1}(0)$$

$$= s.\left[s\overline{f}(s) - f(0)\right] - f^{1}(0)$$

$$= s^{2}\overline{f}(s) - sf(0) - f^{1}(0)$$

$$\therefore L\{f^{111}(t)\} = s.L\{f^{11}(t)\} - f^{11}(0)$$

$$= s[s^{2}L\{f(t)\} - sf(0) - f^{1}(0)] - f^{11}(0)$$

$$= s^{3}L\{f(t)\} - s^{2}f(0) - sf^{1}(0) - f^{11}(0)$$

Proceeding similarly, we have

$$L\{f^{n}(t)\} = s^{n}L\{f(t)\} - s^{n-1}f(0) - s^{n-2}f^{1}(0) \ldots \ldots f^{n-1}(0)$$

Note 1: $L\{f^{n}(t)\} = s^{n}\overline{f}(s)\ if\ f(0) = 0\ and\ f^{1}(0) = 0, f^{11}(0) = 0 \ldots\ f^{n-1}(0) = 0$

Note 2: Now $|f(t)| \leq M.e^{at}\ for\ all\ t \geq 0\ and\ for\ some\ constants\ a\ amd\ M.$

We have $|e^{-st}f(t)| = e^{-st}|f(t)| \leq e^{at}.Me^{at}$

$=\text{M}.\ e^{-(s-a)t} \rightarrow 0\ as\ t\ \rightarrow \infty$ if s>a

$$\therefore \underset{t\rightarrow\infty}{lt}\ e^{-st}f(t) = 0\ for\quad s > a$$

Problems:

Using the theorem on transforms of derivatives, find the Laplace Transform of the following functions.

(i). e^{at} (ii). cosat (iii). t sin at

(i). $Let\ f(t) = e^{at}\ Then\ f^{1}(t) = a.e^{at}\ and\ f(0) = 1$

$$Now\ L\{\ f^{1}(t)\} = s.L\{\ f(t)\} - f(0)$$

$$i.e., L\{ae^{at}\} = s.L\{e^{at}\} - 1$$

$$i.e., L\{e^{at}\} - s.L\{e^{at}\} = -1$$

$$i.e., (a - s)L\{e^{at}\} = -1$$

$$\therefore L\{e^{at}\} = \frac{1}{s-a}$$

(ii). $Let\ f(t) = cosat\ then\ f^{1}(t) = -asinat\ and\ f^{11}(t) = -a^{2}cosat$

$$\therefore L\{\ f^{11}(t)\} = s^{2}L\{\ f(t)\} - s.f(0) - f^{1}(0)$$

$$Now\ f(0) = \cos 0 = 1\ and\ \ f^{1}(0) = -a\sin 0 = 0$$

$$Then\ L\{-a^{2}\cos at\} = s^{2}L\{\cos at\} - s.1 - 0$$

$$\Rightarrow -a^{2}L\{\cos at\} - s^{2}L\{\cos at\} = -s$$

$$\Longrightarrow -(s^2+a^2)L\{\cos at\} = -s \Rightarrow L\{\cos at\} = \frac{s}{s^2+a^2}$$

(iii). $Let\ f(t) = t\sin at\ then\ f^1(t) = \sin at + at\cos at$

$$f^{11}(t) = a\cos at + a[\cos at - at\sin at] = 2a\cos at - a^2 t\sin at$$

Also $f(0) = 0$ and $f^1(0) = 0$

$Now\ L\{f^{11}(t)\} = s^2L\{f(t)\} - sf(0) - f^1(0)$

$i.e., L\{2a\cos at - a^2t\sin at\} = s^2L\{t\sin at\} - 0 - 0$

$i.e., 2a\ L\{\cos at\} - a^2L\{t\sin at\} - s^2L\{t\sin at\} = 0$

$i.e., -(s^2+a^2)L\{t\sin at\} = \frac{-2as}{s^2+a^2} \Rightarrow L\{t\sin at\} = \frac{2as}{(s^2+a^2)^2}$

Laplace Transform of Integrals:

If $L\{f(t)\} = \overline{f}(s)\ then\ L\left\{\int_0^t f(x)\,dx\right\} = \frac{\overline{f}(s)}{s}$

Proof: $Let\ g(t) = \int_0^t f(x)\,dx$

Then $g^1(t) = \frac{d}{dt}\left[\int_0^t f(x)\,dx\right] = f(t)\ and\ g(0) = 0$

Taking Laplace Transform on both sides

$L\{g^1(t)\} = L\{f(t)\}$

But $L\{g^1(t)\} = sL\{g(t)\} - g(0) = sL\{g(t)\} - 0$ $[Since\ g(0) = 0]$

$\therefore L\{g^1(t)\} = L\{f(t)\}$

$\Longrightarrow sL\{g(t)\} = L\{f(t)\} \Rightarrow L\{g(t)\} = \frac{1}{s}L\{f(t)\}$

$But\ g(t) = \int_0^t f(x)\,dx$

$\therefore L\left\{\int_0^t f(x)\,dx\right\} = \frac{\overline{f}(s)}{s}$

Problems:

1. Find the L.T of $\int_0^t \sin at\,dt$

Sol: $L\{\sin at\} = \frac{a}{s^2+a^2} = \overline{f}(s)$

Using the theorem of Laplace transform of the integral, we have

$L\left\{\int_0^t f(x)\,dx\right\} = \frac{\overline{f}(s)}{s}$

$\therefore L\left\{\int_0^t \sin at\right\} = \frac{a}{s(s^2+a^2)}$

2. Find the L.T of $\int_0^t \frac{\sin t}{t}\,dt$

Sol: $L\{\sin t\} = \frac{1}{s^2+1}\ also\ \underset{t\to 0}{lt}\ \frac{\sin t}{t} = 1\ exists$

$$\therefore L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty L\{\sin t\}ds = \int_s^\infty \frac{1}{s^2+1}ds$$

$$= \left[Tan^{-1}s\right]_s^\infty = Tan^{-1}\infty - Tan^{-1}s = \pi/2 - Tan^{-1}s = \cot^{-1}s\,(or)\,Tan^{-1}\left(1/s\right)$$

$$i.e., L\left\{\frac{\sin t}{t}\right\} = Tan^{-1}(1/s)(or)cot^{-1}s$$

$$\therefore\ L\left\{\int_0^t \frac{\sin t}{t}dt\right\} = \frac{1}{s}Tan^{-1}(1/s)\ (or)\ \frac{1}{s}cot^{-1}s$$

3. Find L.T of $e^{-t}\int_0^t \frac{sint}{t}dt$

Sol: L [$e^{-t}\int_0^t \frac{sint}{t}dt$]

We know that

$$L\{sint\} = \frac{1}{s^2+1} = \bar{f}(s)$$

$$L\left\{\frac{sint}{t}\right\} = \int_s^\infty \bar{f}(s)ds = \int_s^\infty \frac{1}{s^2+1}\ ds$$

$$=(tan^{-1}s)_s^\infty$$

$$=tan^{-1}\infty - tan^{-1}s = \frac{\pi}{2} - tan^{-1}s\ \ = cot^{-1}s$$

$$\therefore\ L\left\{\frac{sint}{t}\right\} = cot^{-1}s$$

Hence $L\left\{\int_0^t \frac{sint}{t}dt\right\} = \frac{1}{s}cot^{-1}s$

By First Shifting Theorem

$$L\left[e^{-t}\int_0^t \frac{sint}{t}dt\right] = \bar{f}(s+1) = \left(\frac{cot^{-1}s}{s}\right)_{s\to s+1}$$

$$\therefore L\left[e^{-t}\int_0^t \frac{sint}{t}dt\right] = \frac{1}{s+1}cot^{-1}(s+1)$$

Laplace transform of Periodic functions:

If f (t) is a periodic function with period ‘a’. i.e, $f(t+a)=f(t)$ then

$$L\{f(t)\} = \frac{1}{1-e^{-sa}}\int_0^a e^{-st}f(t)dt$$

Eg: sin x is a periodic function with period 2π

$i.e.,\ \sin x = \sin(2\pi + x) = \sin(4\pi + x)$............

Problems:

1. A function f (t) is periodic in (0,2b) and is defined as $f(t) = 1\ if\ 0<t<b$

$= -1\ if\ b<t<2b$

Find its Laplace Transform.

Sol: $$L\{f(t)\} = \frac{1}{1-e^{-2bs}} \int_0^{2b} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2bs}} \left[\int_0^{b} e^{-st} f(t)\,dt + \int_b^{2b} e^{-st} f(t)\,dt \right]$$

$$= \frac{1}{1-e^{-2bs}} \left[\int_0^{b} e^{-st}\,dt - \int_b^{2b} e^{-st}\,dt \right]$$

$$= \frac{1}{1-e^{-2bs}} \left[\left(\frac{e^{-st}}{-s} \right)_0^{b} - \left(\frac{e^{-st}}{-s} \right)_b^{2b} \right]$$

$$= \frac{1}{s\left(1-e^{-2bs}\right)} \left[-\left(e^{-sb}-1\right) + \left(e^{-2bs}-e^{-sb}\right) \right]$$

$$L\{f(t)\} = \frac{1}{s\left(1-e^{-2bs}\right)} \left[1-2e^{-sb}+e^{-2bs} \right]$$

2. **Find the L.T of the function** $f(t) = \sin\omega t \; if \; 0 < t < \dfrac{\pi}{\omega}$

$$= 0 \; if \; \frac{\pi}{\omega} < t < \frac{2\pi}{\omega} \; where \; f(t) \; has \; period \; \frac{2\pi}{\omega}$$

Sol: Since f (t) is a periodic function with period $\dfrac{2\pi}{\omega}$

$$L\{f(t)\} = \frac{1}{1-e^{-sa}} \int_0^{a} e^{-st} f(t)\,dt$$

$$L\{f(t)\} = \frac{1}{1-e^{-s2\pi/\omega}} \int_0^{2\pi/\omega} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2s\pi/\omega}} \left[\int_0^{\pi/\omega} e^{-st} \sin\omega t\,dt + \int_{\pi/\omega}^{2\pi/\omega} e^{-st}.0dt \right]$$

$$= \frac{1}{1-e^{-2s\pi/\omega}} \left[\frac{e^{-st}\left(-s\sin\omega t - \omega\cos\omega t\right)}{s^2+\omega^2} \right]_0^{\pi/\omega}$$

$$\because \int_a^b e^{at} \sin bt = \frac{e^{at}}{a^2+b^2}\left(a\sin bt - b\cos bt\right)$$

$$= \frac{1}{1-e^{-2\pi s/\omega}} \left[\frac{1}{s^2+\omega^2} \left(e^{-s\pi/\omega}.\omega + \omega \right) \right]$$

Laplace Transform of Some special functions:

1. The Unit step function or Heaviside's Unit functions:

It is defined as $u(t-a) = \begin{cases} 0 & t < a \\ 1 & t > a \end{cases}$

Laplace Transform of unit step function:

To prove that $L\{u(t-a)\} = \frac{e^{-as}}{s}$

Proof: Unit step function is defined as $u(t-a) = \begin{cases} 0 & t < a \\ 1 & t > a \end{cases}$

Then $L\{u(t-a)\} = \int_0^\infty e^{-st} u(t-a)\, dt$

$$= \int_0^a e^{-st} u(t-a)\, dt + \int_a^\infty e^{-st} u(t-a)\, dt$$

$$= \int_0^a e^{-st}.0 dt + \int_a^\infty e^{-st}.1 dt$$

$$= \int_a^\infty e^{-st} dt = \left[\frac{e^{-st}}{-s}\right]_a^\infty = -\frac{1}{s}.\left[e^{-\infty} - e^{-as}\right] = \frac{e^{-as}}{s}$$

$\therefore L\{u(t-a)\} = \frac{e^{-as}}{s}$

Laplace Transforms of Dirac Delta Function:

The Dirac delta function or Unit impulse function $f_\in(t) = \begin{cases} 1/\in & 0 \le t \le \in \\ 0 & t > \in \end{cases}$

2. **Prove that $L\{f_\in(t)\} = \frac{1-e^{-s\in}}{s\in}$ hence show that $L\{\delta(t)\} = 1$**

Proof: By the definition $f_\in(t) = \begin{cases} 1/\in & 0 \le t \le \in \\ 0 & t > \in \end{cases}$

And Hence $L\{f_\in(t)\} = \int_0^\infty e^{-st} f_\in(t)\, dt$

$$= \int_0^\in e^{-st} f_\in(t)\, dt + \int_\in^\infty e^{-st} f_\in(t)\, dt$$

$$= \int_0^\in e^{-st} \frac{1}{\in} dt + \int_\in^\infty e^{-st}.0\, dt$$

$$= \frac{1}{\in}\left[\frac{e^{-st}}{-s}\right]_0^\in = -{}^{1}\!/_{\in\, s}\left[e^{-s\in} - e^0\right] = \frac{1-e^{-s\in}}{s\in}$$

$\therefore L\{f_\in(t)\} = \frac{1-e^{-s\in}}{s\in}$

Now $L\{\delta(t)\} = \underset{\in\to 0}{lt} L\{f_\in(t)\} = \underset{\in\to 0}{lt} \frac{1-e^{-s\in}}{s\in}$

$\therefore L\{\delta(t)\} = 1$ *using* L-Hospital rule.

Properties of Dirac Delta Function:

1. $\int_0^\infty \delta(t)\,dt = 0$

2. $\int_0^\infty \delta(t)G(t)\,dt = G(0)$ where G(t) is some continuous function.

3. $\int_0^\infty \delta(t-a)G(t)\,dt = G(a)$ where G(t) is some continuous function.

4. $\int_0^\infty G(t)\delta^1(t-a) = -G^1(a)$

Problems

1. Prove that $L\{\delta(t-a)\} = e^{-as}$

Sol: By Translation theorem

$L\{\delta(t-a)\} = e^{-as}L\{\delta(t)\}$

$= e^{-as}$ $[\sin ce\, L\{\delta(t)\} = 1]$

2. Evaluate $\int_0^\infty \cos 2t\,\delta\left(t-{}^{\pi}/_{3}\right)dt$

Sol: By using property (3) then we get

$\int_0^\infty \delta(t-a)G(t)dt = G(a)$

Here $a = {}^{\pi}/_{3}, G(t) = \cos 2t$

$\therefore G(a) = G\left({}^{\pi}/_{3}\right) = \cos 2{}^{\pi}/_{3} = {}^{-1}/_{2}$

$\therefore \int_0^\infty \cos 2at\,\delta\left(t-{}^{\pi}/_{3}\right)dt = \cos 2\,{}^{\pi}/_{3} = {}^{-\pi}/_{2}$

3. Evaluate $\int_0^\infty e^{-4t}\delta^1(t-2)dt$

Sol: By the 4th Property then we get

$\int_0^\infty \delta^1(t-a)G(t)dt = -G^1(a)$

$G(t) = e^{-4t}$ *and* $a = 2$

$G^1(t) = -4.e^{-4t}$

$\therefore G^1(a) = G^1(2) = -4.e^{-8}$

$\therefore \int_0^\infty e^{-4t}\delta^1(t-2)dt = -G^1(a) = 4.e^{-8}$

Inverse Laplace Transforms:

If $\overline{f}(s)$ is the Laplace transforms of a function of f (t) i.e. $L\{f(t)\} = \overline{f}(s)$ then f (t) is called the inverse Laplace transform of $\overline{f}(s)$ and is written as $f(t) = L^{-1}\{\overline{f}(s)\}$

$\therefore L^{-1}$ is called the inverse L.T operator.

Table of Laplace Transforms and Inverse Laplace Transforms

S.No.	$L\{f(t)\} = \overline{f}(s)$	$L^{-1}\{\overline{f}(s)\} = f(t)$
1.	$L\{1\} = {}^{1}/_{s}$	$L^{-1}\{{}^{1}/_{s}\} = 1$
2.	$L\{e^{at}\} = \frac{1}{s-a}$	$L^{-1}\{{}^{1}/_{s-a}\} = e^{at}$
3.	$L\{e^{-at}\} = \frac{1}{s+a}$	$L^{-1}\{{}^{1}/_{s+a}\} = e^{at}$
4.	$L\{t^n\} = \frac{n!}{s^{n+1}} n \text{ is a} + ve \text{ integer}$	$L^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{n!}$
5.	$L\{t^{n-1}\} = \frac{(n-1)!}{s^n}$	$L^{-1}\{{}^{1}/_{s^n}\} = \frac{t^{n-1}}{(n-1)!}, n = 1,2,3 \ldots$
6.	$L\{\sin at\} = \frac{a}{s^2+a^2}$	$L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}.\sin at$
7.	$L\{\cos at\} = \frac{s}{s^2+a^2}$	$L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$
8.	$L\{\sinh at\} = \frac{a}{s^2-a^2}$	$L^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{1}{a}\sinh at$
9.	$L\{\cosh at\} = \frac{s}{s^2-a^2}$	$L^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at$
10.	$L\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2}$	$L^{-1}\left\{\frac{1}{(s-a)^2+b^2}\right\} = \frac{1}{b}.e^{at}\sin bt$
11.	$L\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2}$	$L^{-1}\left\{\frac{(s-a)}{(s-a)^2+b^2}\right\} = e^{at}\cos bt$
12.	$L\{e^{at}\sinh bt\} = \frac{b}{(s-a)^2-b^2}$	$L^{-1}\left\{\frac{1}{(s-a)^2-b^2}\right\} = \frac{1}{b}.e^{at}\sinh bt$
13.	$L\{e^{at}\cosh bt\} = \frac{s-a}{(s-a)^2-b^2}$	$L^{-1}\left\{\frac{(s-a)}{(s-a)^2-b^2}\right\} = e^{at}\cosh bt$
14.	$L\{e^{-at}\sin bt\} = \frac{b}{(s+a)^2+b^2}$	$L^{-1}\left\{\frac{1}{(s+a)^2+b^2}\right\} = \frac{1}{b}.e^{-at}\sin bt$
15.	$L\{e^{-at}\cos bt\} = \frac{s+a}{(s+a)^2+b^2}$	$L^{-1}\left\{\frac{s+a}{(s+a)^2+b^2}\right\} = e^{-at}\cos bt$
16.	$L\{e^{at}f(t)\} = \overline{f}(s-a)$	$L^{-1}\{\overline{f}(s-a)\} = e^{at}L^{-1}\{\overline{f}(s)\}$
17.	$L\{e^{-at}f(t)\} = \overline{f}(s+a)$	$L^{-1}\{\overline{f}(s+a)\} = e^{-at}f(t)e^{-at}L^{-1}\{\overline{f}(s)\}$

Problems

1. Find the Inverse Laplace Transform of $\dfrac{s^2-3s+4}{s^3}$

Sol: $L^{-1}\left\{\frac{s^3-3s+4}{s^3}\right\} = L^{-1}\left\{1/s - 3.\,1/s^2 + 4/s^3\right\}$

$$= L^{-1}\left\{1/s\right\} - 3L^{-1}\left\{1/s^2\right\} + L^{-1}\left\{4/s^3\right\}$$

$$= 1 - 3t + 4.\frac{t^2}{2!} = 1 - 3t + 2t^2$$

2. Find the Inverse Laplace Transform of $\frac{s+2}{s^2-4s+13}$

Sol: $L^{-1}\left\{\frac{s+2}{s^2-4s+13}\right\} = L^{-1}\left\{\frac{s+2}{(s-2)^2+9}\right\} = L^{-1}\left\{\frac{s-2+4}{(s-2)^2+3^2}\right\}$

$$= L^{-1}\left\{\frac{s-2}{(s-2)^2+3^2}\right\} + 4.\,L^{-1}\left\{\frac{1}{(s-2)^2+3^2}\right\}$$

$$= e^{2t}\cos 3t + \frac{4}{3}e^{2t}\sin 3t$$

3. Find the Inverse Laplace Transform of $\dfrac{2s-5}{s^2-4}$

Sol: $L^{-1}\left\{\frac{2s-5}{s^2-4}\right\} = L^{-1}\left\{\frac{2s}{s^2-4} - \frac{5}{s^2-4}\right\}$

$$= 2L^{-1}\left\{\frac{s}{s^2-4}\right\} - 5L^{-1}\left\{\frac{1}{s^2-4}\right\}$$

$$= 2.\cosh 2t - 5.\frac{1}{2}\sinh 2t$$

4. Find $L^{-1}\left\{\frac{2s+1}{s(s+1)}\right\}$

Sol: $L^{-1}\left\{\frac{s+s+1}{s(s+1)}\right\} = L^{-1}\left\{\frac{1}{s+1} + \frac{1}{s}\right\}$

$$= L^{-1}\left\{\frac{1}{s+1}\right\} + L^{-1}\left\{\frac{1}{s}\right\} = e^{-t} + 1$$

5. Find $L^{-1}\left\{\frac{3s-8}{4s^2+25}\right\}$

Sol: $L^{-1}\left\{\frac{3s-8}{4s^2+25}\right\} = L^{-1}\left\{\frac{3s}{4s^2+25}\right\} - 8L^{-1}\left\{\frac{1}{4s^2+25}\right\}$

$$= \frac{3}{4}L^{-1}\left\{\frac{s}{s^2+\left(5/2\right)^2}\right\} - \frac{8}{4}L^{-1}\left\{\frac{1}{s^2+\left(5/2\right)^2}\right\}$$

$$= \frac{3}{4}.\cos\frac{5}{2}t - \frac{8}{4}.\frac{2}{5}\sin\frac{5}{2}t$$

$$=\frac{3}{4}\cos\frac{5}{2}t-\frac{4}{5}\sin\frac{5}{2}t$$

6. Find the Inverse Laplace Transform of $\frac{s}{(s+a)^2}$

Sol: $L^{-1}\left\{\frac{s}{(s+a)^2}\right\}=L^{-1}\left\{\frac{s+a-a}{(s+a)^2}\right\}=e^{-at}L^{-1}\left\{\frac{s-a}{s^2}\right\}$

$$=e^{-at}L^{-1}\left\{\frac{1}{s}-\frac{a}{s^2}\right\}$$

$$=e^{-at}\left[L^{-1}\left\{\frac{1}{s}\right\}-a.L^{-1}\left\{\frac{1}{s^2}\right\}\right]$$

$$=e^{-at}\left[1-at\right]$$

7. Find $\boldsymbol{L^{-1}\left\{\frac{3s+7}{s^2-2s-3}\right\}}$

Sol: $Let\ \frac{3s+7}{s^2-2s-3}=\frac{A}{s+1}+\frac{B}{s-3}$

$A(s-3)+B(s+1)=3s+7$

$put\ s=3, 4B=16\Rightarrow B=4$

$put\ s=-1,-4A=4\Rightarrow A=-1$

$\therefore\frac{3s+7}{s^2-2s-3}=\frac{-1}{s+1}+\frac{4}{s-3}$

$$L^{-1}\left\{\frac{3s+7}{s^2-2s-3}\right\}=L^{-1}\left\{\frac{-1}{s+1}+\frac{4}{s-3}\right\}=-1L^{-1}\left\{\frac{1}{s+1}\right\}+4L^{-1}\left\{\frac{1}{s-3}\right\}$$

$$=-e^{-t}+4.e^{3t}$$

8. Find $\boldsymbol{L^{-1}\left\{\frac{s}{(s+1)^2(s^2+1)}\right\}}$

Sol: $\frac{s}{(s+1)^2(s^2+1)}=\frac{A}{s+1}+\frac{B}{(s+1)^2}+\frac{Cs+D}{s^2+1}$

$A(s+1)(s^2+1)+B(s^2+1)+(Cs+D)(s+1)^2=s$

Equating Co-efficient of s^3, A+C=0……..(1)

Equating Co-efficient of s^2, A+B+2C+D=0…….(2)

Equating Co-efficient of s, A+C+2D=1…….(3)

$put\ s=-1, 2B=-1\Rightarrow B=-\frac{1}{2}$

Substituting (1) in (3) $2D=1\Rightarrow D=\frac{1}{2}$

Substituting the values of B and D in (2)

i.e. $A-\frac{1}{2}+2C+\frac{1}{2}=0\Rightarrow A+2C=0, also\ A+C=0\Rightarrow A=0, C=0$

$$\therefore \frac{s}{(s+1)^2(s^2+1)} = \frac{\frac{-1}{2}}{(s+1)^2} + \frac{\frac{1}{2}}{s^2+1}$$

$$L^{-1}\left\{\frac{s}{(s+1)^2(s^2+1)}\right\} = \frac{1}{2}\left[L^{-1}\left\{\frac{1}{s^2+1}\right\} - L^{-1}\left\{\frac{1}{(s+1)^2}\right\}\right]$$

$$= \frac{1}{2}\left[\sin t - e^{-t}L^{-1}\left\{\frac{1}{s^2}\right\}\right]$$

$$= \frac{1}{2}\left[\sin t - te^{-t}\right]$$

9. Find $L^{-1}\left\{\frac{s}{s^4+4a^4}\right\}$

Sol: Since $s^4+4a^4 = \left(s^2+2a^2\right)^2 - (2as)^2$

$$= (s^2+2as+2a^2)(s^2-2as+2a^2)$$

$$\therefore Let \quad \frac{s}{s^4+4a^4} = \frac{As+B}{s^2+2as+2a^2} + \frac{Cs+D}{s^2-2as+2a^2}$$

$$(As+B)(s^2-2as+2a^2)+(Cs+D)(s^2+2as+2a^2) = s$$

Solving we get $A=0, C=0, B=\frac{-1}{4a}, D=\frac{1}{4a}$

$$L\left\{\frac{s}{s^4+4a^4}\right\} = L^{-1}\left\{\frac{-\frac{1}{4a}}{s^2+2as+2a^2}\right\} + L^{-1}\left\{\frac{\frac{1}{4a}}{s^2-2as+2a^2}\right\}$$

$$= \frac{-1}{4}a.\mathrm{L}^{-1}\left\{\frac{1}{(s+a)^2+a^2}\right\} + \frac{1}{4a}..\mathrm{L}^{-1}\left\{\frac{1}{(s-a)^2+a^2}\right\}$$

$$= \frac{-1}{4a}.\frac{1}{a}.e^{-at}\sin at + \frac{1}{4a}.\frac{1}{a}e^{at}\sin at$$

$$= \frac{1}{4a^2}\sin at\left(e^{at}-e^{-at}\right) = \frac{1}{4a^2}.\sin at.2\sinh at = \frac{1}{2a^2}\sin at \sinh at$$

10. Find i. $L^{-1}\left\{\frac{s^2-3s+4}{s^3}\right\}$ ii. $L^{-1}\left\{\frac{3\left(s^2-2\right)^2}{2s^5}\right\}$

Sol:

i. $L^{-1}\left\{\frac{s^2-3s+4}{s^3}\right\} = L^{-1}\left\{\frac{s^2}{s^3} - \frac{3s}{s^3} + \frac{4}{s^3}\right\} = L^{-1}\left\{\frac{1}{s} - \frac{3}{s^2} + \frac{4}{s^3}\right\}$

$$= L^{-1}\left\{\frac{1}{s}\right\} - 3L^{-1}\left\{\frac{1}{s^2}\right\} + 4L^{-1}\left\{\frac{1}{s^3}\right\}$$

$$= 1 - 3t + 4\frac{t^2}{2!} = 1 - 3t + 2t^2$$

ii. $L^{-1}\left\{\frac{3\left(s^2-2\right)^2}{2s^5}\right\} = \frac{3}{2}L^{-1}\left\{\frac{\left(s^2-2\right)^2}{s^5}\right\} = \frac{3}{2}L^{-1}\left\{\frac{s^4-4s^2+4}{s^5}\right\}$

$$= \frac{3}{2}L^{-1}\left\{\frac{1}{s} - \frac{4}{s^3} + \frac{4}{s^5}\right\} + \frac{3}{2}\left\{L^{-1}\left\{\frac{1}{s}\right\} - 4L^{-1}\left\{\frac{1}{s^3}\right\} + 4L^{-1}\left\{\frac{1}{s^5}\right\}\right\}$$

$$=\frac{3}{2}\left[1-4\frac{t^2}{2!}+\frac{4t^4}{4!}\right]=\frac{3}{2}\left[1-2t^2+\frac{t^4}{6}\right]=\frac{1}{4}\left[t^4-6t^2+6\right]$$

11. Find $L^{-1}\left[\frac{s}{s^2-a^2}\right]$

Sol:

$$L^{-1}\left[\frac{s}{s^2-a^2}\right]=L^{-1}\left[\frac{2s}{2(s^2-a^2)}\right]=\frac{1}{2}L^{-1}\left[\frac{2s}{(s-a)(s+a)}\right]=\frac{1}{2}L^{-1}\left[\frac{1}{s-a}+\frac{1}{s+a}\right]$$

$$=\frac{1}{2}\left[e^{at}+e^{-at}\right]=\cosh at$$

12. Find $L^{-1}\left[\frac{4}{(s+1)(s+2)}\right]$

Sol: $L^{-1}\left[\frac{4}{(s+1)(s+2)}\right]=4L^{-1}\left[\frac{1}{(s+1)(s+2)}\right]=4\,L^{-1}\left[\frac{1}{s+1}-\frac{1}{s+2}\right]=4[\mathrm{e}^{-t}-\mathrm{e}^{-2t}]$

13. Find $L^{-1}\left\{\frac{1}{(s+1)^2(s^2+4)}\right\}$

Sol: $\frac{1}{(s+1)^2(s^2+4)}=\frac{A}{s+1}+\frac{B}{(s+1)^2}+\frac{Cs+D}{s^2+4}$

$$A=\frac{2}{25},B=\frac{1}{5},C=\frac{-2}{25},D=\frac{-3}{25}$$

$$\therefore L^{-1}\left\{\frac{1}{(s+1)^2(s^2+4)}\right\}=\frac{2}{25}L^{-1}\left\{\frac{1}{s+1}\right\}+\frac{1}{5}L^{-1}\left\{\frac{1}{(s+1)^2}\right\}-\frac{2}{25}L^{-1}\left\{\frac{s}{s^2+4}\right\}-\frac{3}{25}L^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$=\frac{2}{25}e^{-t}\,L^{-1}\left\{\frac{1}{s}\right\}+\frac{1}{5}e^{-t}L^{-1}\left\{\frac{1}{s^2}\right\}-\frac{2}{25}\cos 2t-\frac{3}{25}.\frac{1}{2}\sin 2t$$

$$=\frac{2}{25}e^{-t}+\frac{1}{5}e^{-t}.t-\frac{2}{25}\cos 2t-\frac{3}{50}\sin 2t$$

14. Find $L^{-1}\left[\frac{s^2+s-2}{s(s+3)(s-2)}\right]$

Sol: $\frac{s^2+s-2}{s(s+3)(s-2)}=\frac{A}{s}+\frac{B}{s+3}+\frac{C}{s-2}$

Comparing with s^2, s, constants, we get

$$A=1/3,B=4/15,C=2/5$$

$$L^{-1}\left[\frac{s^2+s-2}{s(s+3)(s-2)}\right]=L^{-1}\left[\frac{1}{3s}+\frac{4}{15(s+3)}+\frac{2}{5(s-2)}\right]$$

$$= L^{-1}\left[\frac{1}{3s}\right] + L^{-1}\left[\frac{4}{15(s+3)}\right] + L^{-1}\left[\frac{2}{5(s-2)}\right]$$

$$= \frac{1}{3} + \frac{4}{15}e^{-3t} + \frac{2}{5}e^{2t}$$

15. **Find** $L^{-1}\left[\frac{s^2+2s-4}{(s^2+9)(s-5)}\right]$

Sol: $\frac{s^2+2s-4}{(s^2+9)(s-5)} = \frac{A}{s-5} + \frac{Bs+C}{s^2+9}$

Comparing with s^2, s, constants, we get

$$A = 31/34, B = 3/34, C = 83/34$$

$$L^{-1}\left[\frac{s^2+2s-4}{(s^2+9)(s-5)}\right] = L^{-1}\left[\frac{s^2+2s-4}{(s^2+9)(s-5)}\right]$$

$$= L^{-1}\left[\frac{31}{34(s-5)}\right] + L^{-1}\left[\frac{3}{34(s^2+9)}\right] + L^{-1}\left[\frac{83}{34(s^2+9)}\right]$$

$$= \frac{31}{34}e^{5t} + 1/34\left[3\cos 3t + \frac{83}{3}\sin 3t\right]$$

First Shifting Theorem:

If $L^{-1}\left\{\overline{f}(s)\right\} = f(t), \text{then} L^{-1}\left\{\overline{f}(s-a)\right\} = e^{at}f(t)$

Proof: We have seen that $L\left\{e^{at}f(t)\right\} = \overline{f}(s-a) \therefore L^{-1}\left\{\overline{f}(s-a)\right\} = e^{at}f(t) = e^{at}L^{-1}\left\{\overline{f}(s)\right\}$

1. **Find** $L^{-1}\left\{\frac{1}{(s+2)^2+16}\right\} = L^{-1}\left\{\overline{f}(s+2)\right\}$

Sol: $L^{-1}\left\{\frac{1}{(s+2)^2+16}\right\} = e^{-2t}L^{-1}\left\{\frac{1}{s^2+16}\right\}$

$$= e^{-2t}.\frac{1}{4}\sin 4t = \frac{e^{-2t}\sin 4t}{4}$$

2. **Find** $L^{-1}\left\{\frac{3s-2}{s^2-4s+20}\right\}$

Sol: $L^{-1}\left\{\frac{3s-2}{s^2-4s+20}\right\} = L^{-1}\left\{\frac{3s-2}{(s-2)^2+16}\right\} = L^{-1}\left\{\frac{3(s-2)+4}{(s-2)^2+4^2}\right\}$

$$= 3L^{-1}\left\{\frac{s-2}{(s-2)^2+4^2}\right\} + 4L^{-1}\left\{\frac{1}{(s-2)^2+4^2}\right\}$$

$$= 3e^{2t}L^{-1}\left\{\frac{s}{s^2+4^2}\right\} + 4e^{2t}L^{-1}\left\{\frac{1}{s^2+4^2}\right\}$$

$$= 3e^{2t}\cos 4t + 4e^{2t}\frac{1}{4}\sin 4t$$

3. **Find** $L^{-1}\left\{\frac{s+3}{s^2-10s+29}\right\}$

Sol: $L^{-1}\left\{\frac{s+3}{s^2-10s+29}\right\} = L^{-1}\left\{\frac{s+3}{(s-5)^2+2^2}\right\} = L^{-1}\left\{\frac{s-5+8}{(s-5)^2+2^2}\right\}$

$$= e^{5t}\mathrm{L}^{-1}\left\{\frac{s+8}{s^2+2^2}\right\} = e^{5t}\left\{\cos 2t + 8.\frac{1}{2}\sin 2t\right\}$$

Second shifting theorem:

If $L^{-1}\left\{\overline{f}(s)\right\} = f(t), then \quad L^{-1}\{e^{-as}\overline{f}(s)\} = G(\mathrm{t})$, where $\mathrm{G}(t) = \begin{cases} f\{t-a\} & if\ t > a \\ 0 & if\ t < a \end{cases}$

Proof: We have seen that $\mathrm{G}(t) = \begin{cases} f\{t-a\} & if\ t > a \\ 0 & if\ t < a \end{cases}$

$$then \quad L\{G(t)\} = e^{-as}.\overline{f}(s)$$

$$\therefore L^{-1}\left\{e^{-as}\overline{f}(s)\right\} = G(t)$$

1. **Evaluate (i)** $L^{-1}\left\{\frac{1+e^{-\pi s}}{s^2+1}\right\}$ **(ii)** $L^{-1}\left\{\frac{e^{-3s}}{(s-4)^2}\right\}$

Sol: (i) $L^{-1}\left\{\frac{1+e^{-\pi s}}{s^2+1}\right\} = L^{-1}\left\{\frac{1}{s^2+1}\right\} + L^{-1}\left\{\frac{e^{-\pi s}}{s^2+1}\right\}$

Since $L^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t = f(t)$, say

$\therefore$ By second Shifting theorem, we have $L^{-1}\left\{\frac{e^{-\pi s}}{s^2+1}\right\} = \begin{cases} \sin(\mathrm{t}-\pi) & , if\ t > \pi \\ 0 & , if\ t < \pi \end{cases}$

or $L^{-1}\left\{\frac{e^{-\pi s}}{s^2+1}\right\}$ =sin(t-π)H(t-π)= -sint. H(t-π)

Hence $L^{-1}\left\{\frac{1+e^{-\pi s}}{s^2+1}\right\}$ =sint-sint. H (t-π) =sint [1- H (t-π)]

Where H (t-π) is the Heaviside unit step function

(ii) Since $L^{-1}\left\{\frac{1}{(s-4)^2}\right\} = e^{4t}L^{-1}\left\{\frac{1}{s^2}\right\}$

$= e^{4t}.t = f(t)$, say

∴ By second Shifting theorem, we have $L^{-1}\left\{\frac{e^{-3s}}{(s-4)^2}\right\} = \begin{cases} e^{4(t-3)}.(t-3) & , if\ t>3 \\ 0 & , if\ t<3 \end{cases}$

or $L^{-1}\left\{\frac{e^{-3s}}{(s-4)^2}\right\} = e^{4(t-3)}.(t-3)\,\mathrm{H(t\text{-}3)}$

Where H (t-3) is the Heaviside unit step function

Change of scale property:

If $L\{f(t)\} = \bar{f}(s)$, Then $L^{-1}\{\bar{f}(as)\} = \frac{1}{a}f\left(\frac{t}{a}\right), a>0$

Proof: We have seen that $L\{f(t)\} = \bar{f}(s)$

$$Then\quad \bar{f}(as) = \frac{1}{a}L\left\{f\left(\frac{t}{a}\right)\right\}, a>0$$

$$\therefore L^{-1}\{\bar{f}(as)\} = \frac{1}{a}f\left(\frac{t}{a}\right), a>0$$

1. **If** $L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \frac{1}{2}t\sin t$ **, find** $L^{-1}\left\{\frac{8s}{(4s^2+1)^2}\right\}$

Sol: We have $L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \frac{1}{2}t\sin t$,

Writing as for s,

$L^{-1}\left\{\frac{as}{(a^2s^2+1)^2}\right\} = \frac{1}{2}.\frac{1}{a}.\frac{t}{a}\sin\frac{t}{a} = \frac{t}{2a^2}.\sin\frac{t}{a}$, by change of scale property.

Putting a=2, we get

$L^{-1}\left\{\frac{2s}{(4s^2+1)^2}\right\} = \frac{t}{8}\sin\frac{t}{2}$ **or** $L^{-1}\left\{\frac{8s}{(4s^2+1)^2}\right\} = \frac{1}{2}\sin\frac{t}{2}$

Inverse Laplace Transform of derivatives:

Theorem: $L^{-1}\{\bar{f}(s)\} = f(t)$, then $L^{-1}\{\bar{f}^n(s)\} = (-1)^n t^n f(t)$ where $\bar{f}^n(s) = \frac{d^n}{ds^n}\left[\bar{f}(s)\right]$

Proof: We have seen that $L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\bar{f}(s)$

$$\therefore L^{-1}\left\{\overline{f}^{n}(s)\right\} = (-1)^n t^n f(t)$$

1. Find $L^{-1}\left\{\log\frac{s+1}{s-1}\right\}$

Sol: Let $L^{-1}\left\{\log\frac{s+1}{s-1}\right\} = f(t)$

$$L\{f(t)\} = \log\frac{s+1}{s-1}$$

$$L\{tf(t)\} = \frac{-d}{ds}\left\{\log\frac{s+1}{s-1}\right\}$$

$$L\{tf(t)\} = \frac{-1}{s+1} + \frac{1}{s-1}$$

$$tf(t) = L^{-1}\left\{\frac{-1}{s+1} + \frac{1}{s-1}\right\}$$

$$tf(t) = -1.\mathrm{L}^{-1}\left\{\frac{1}{s+1}\right\} + L^{-1}\left\{\frac{1}{s-1}\right\}$$

$$= e^{-t} + e^{t}$$

$$t\,f(t) = 2\sinh t \Rightarrow f(t) = \frac{2\sinh t}{t}$$

$$\therefore L^{-1}\left\{\log\frac{s+1}{s-1}\right\} = \frac{2\sinh t}{t}$$

Note: $L^{-1}\left\{\log\frac{1+s}{s}\right\} = \frac{1-e^{-t}}{t}$

2. Find $L^{-1}\left\{\cot^{-1}(s)\right\}$

Sol: Let $L^{-1}\left\{\cot^{-1}(s)\right\} = f(t)$

$$L\{f(t)\} = \cot^{-1}(s)$$

$$L\{tf(t)\} = \frac{-d}{ds}[\cot^{-1}(s) = -\left[\frac{-1}{1+s^2}\right] = \frac{1}{1+s^2}$$

$$tf(t) = L^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$$

$$f(t) = \frac{\sin t}{t}$$

$$\therefore L^{-1}\left\{\cot^{-1}(s)\right\} = \frac{1}{\mathrm{t}}\sin t$$

Inverse Laplace Transform of integrals:

Theorem: $L^{-1}\left\{\overline{f}(s)\right\}=f(t)$, then $L^{-1}\left\{\int_s^\infty \overline{f}(s)ds\right\}=\frac{f(t)}{t}$

Proof: we have seen that $L\left\{\frac{f(t)}{t}\right\}=\int_s^\infty \overline{f}(s)ds$

$$\therefore L^{-1}\left\{\int_s^\infty \overline{f}(s)ds\right\}=\frac{f(t)}{t}$$

1. **Find** $L^{-1}\left\{\frac{s+1}{(s^2+2s+2)^2}\right\}$

Sol: **Let** $\overline{f}(s)=\frac{s+1}{(s^2+2s+2)^2}$

Then $L^{-1}\left\{\overline{f}(s)\right\}=L^{-1}\left\{\int_s^\infty \frac{s+1}{(s^2+2s+2)^2}ds\right\}$

$$=L^{-1}\left\{\frac{s+1}{[(s+1)^2+1]^2}\right\}$$

$$=e^{-t}L^{-1}\left\{\frac{s}{(s^2+1)^2}\right\}, \text{ by First Shifting Theorem}$$

$$=e^{-t}\frac{t}{2}\sin t=\frac{t}{2}e^{-t}\sin t \because L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}=\frac{t}{2a}\sin at$$

Multiplication by power of's':

Theorem: $L^{-1}\left\{\overline{f}(s)\right\}=f(t)$, and f (0),then $L^{-1}\left\{s\overline{f}(s)\right\}=f^1(t)$

Proof: we have seen that $L\left\{f^1(t)\right\}=s\overline{f}(s)-f(0)$

$$\therefore L\left\{f^1(t)\right\}=s\overline{f}(s) \quad [\because f(0)=0] \text{ or}$$

$$L^{-1}\left\{s\overline{f}(s)\right\}=f^1(t)$$

Note: $L^{-1}\left\{s^n\overline{f}(s)\right\}=f^n(t), \text{if } f^n(0)=0 \text{ for } n=1,2,3.........\text{n}-1$

1. **Find (i)** $L^{-1}\left\{\frac{s}{(s+2)^2}\right\}$ **(ii)** $L^{-1}\left\{\frac{s}{(s+3)^2}\right\}$

Sol: Let $\overline{f}(s)=\frac{1}{(s+2)^2}$ **Then**

$$L^{-1}\left\{\overline{f}(s)\right\}=L^{-1}\left\{\frac{1}{(s+2)^2}\right\}=e^{-2t}L^{-1}\left\{\frac{1}{s^2}\right\}=e^{-2t}.t=f(t),$$

Clearly f (0) =0

Thus $L^{-1}\left\{\frac{s}{(s+2)^2}\right\}=L^{-1}\left\{s.\frac{s}{(s+2)^2}\right\}=L^{-1}\left\{s.\overline{f}(s)\right\}=f^1(t)$

$$=\frac{d}{dt}(te^{-2t})=\mathrm{t}(-2e^{-2t})+e^{-2t}.1=e^{-2t}(1-2\mathrm{t})$$

Note: in the above problem put 2=3, then $L^{-1}\left\{\frac{s}{(s+3)^2}\right\}=e^{-3t}(1-3\mathrm{t})$

Division by S:

Theorem: If $L^{-1}\left\{\overline{f}(s)\right\}=f(t)$, Then $L^{-1}\left\{\frac{\overline{f}(s)}{s}\right\}=\int_0^t f(u)du$

Proof: We have seen that $L\left\{\int_0^t f(u)du\right\}=\frac{\overline{f}(s)}{s}$

$$\therefore L^{-1}\left\{\frac{\overline{f}(s)}{s}\right\}=\int_0^t f(u)du$$

Note: If $L^{-1}\left\{\overline{f}(s)\right\}=f(t)$, then $L^{-1}\left\{\frac{\overline{f}(s)}{s^2}\right\}=\int_0^t\int_0^t f(u)du.du$

1. Find the inverse Laplace Transform of $\frac{1}{s^2(s^2+a^2)}$

Sol: Since $L^{-1}\left[\frac{1}{(s^2+a^2)}\right]=\frac{1}{a}\mathrm{sinat}$, we have

$$L^{-1}\left[\frac{1}{\mathrm{s}(s^2+a^2)}\right]=\int_0^t\frac{1}{a}\sin at dt$$

$$=\frac{1}{a}\left(\frac{-\cos at}{a}\right)_0^t=-\frac{1}{a^2}(\mathrm{cosat}-1)=\frac{1}{a^2}(1-\mathrm{cosat})$$

Then $L^{-1}\left[\frac{1}{s^2(s^2+a^2)}\right]=\int_0^t\frac{1}{a^2}(1-\cos at)dtdt$

$$=\frac{1}{a^2}\left(t-\frac{\sin at}{a}\right)_0^t=\frac{1}{a^2}\left(t-\frac{\sin at}{a}\right)$$

$$\therefore L^{-1}\left[\frac{1}{s^2(s^2+a^2)}\right]=\frac{1}{a^2}\left(t-\frac{\sin at}{a}\right)$$

Convolution Definition:

If f (t) and g (t) are two functions defined for $t \geq 0$ then the convolution of f (t) and g (t) is defined as $f(t) * g(t) = \int_0^t f(u) g(t-u)\,du$

$f(t) * g(t)$ *can also be written as* $(f * g)(t)$

Properties:

The convolution operation * has the following properties

1. Commutative i.e. $(f * g)(t) = (g * f)(t)$

2. Associative $\left[f * (g * h)\right](t) = \left[(f * g) * h\right](t)$

3. Distributive $\left[f * (g + h)\right](t) = (f * g)(t) + (f * h)(t)$ *for* $t \geq 0$

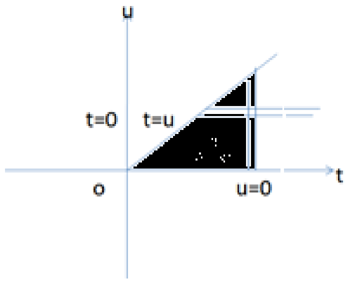
Convolution Theorem: If $f(t)$ and $g(t)$ are functions defined for $t \geq 0$ then

$$L\{f(t) * g(t)\} = L\{f(t)\}.L\{g(t)\} = \bar{f}(s).\bar{g}(s)$$

i.e., The L.T of convolution of f(t) and g(t) is equal to the product of the L.T of f(t) and g(t)

Proof: WKT $L\{\phi(t)\} = \int_0^\infty e^{-st} \left\{\int_0^t f(u) g(t-u)\,du\right\} dt$

$$= \int_0^\infty \int_0^t e^{-st} f(u) g(t-u)\,du\,dt$$



The double integral is considered within the region enclosed by the line u=0 and u=t

On changing the order of integration, we get

$$L\{\phi(t)\} = \int_0^\infty \int_u^\infty e^{-st} f(u) g(t-u)\,dt\,du$$

$$= \int_0^\infty e^{-su} f(u) \left\{\int_u^\infty e^{-s(t-u)} g(t-u)\,dt\right\} du$$

$$= \int_0^\infty e^{-su} f(u) \left\{\int_0^\infty e^{-sv} g(v)\,dv\right\} du \quad put\ t-u=v$$

$$= \int_0^\infty e^{-su} f(u) \left\{\bar{g}(s)\right\} du = \bar{g}(s) \int_0^\infty e^{-su} f(u)\,du = \bar{g}(s).\bar{f}(s)$$

$$L\{f(t) * g(t)\} = L\{f(t)\}.L\{g(t)\} = \bar{f}(s).\bar{g}(s)$$

Problems:

1. Using the convolution theorem find $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$

Sol: $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}=L^{-1}\left\{\frac{s}{s^2+a^2}.\frac{1}{s^2+a^2}\right\}$

$$Let\ \overline{f}(s)=\frac{s}{s^2+a^2}\ and\ \overline{g}(s)=\frac{1}{s^2+a^2}$$

So that $\mathrm{L}^{-1}\left\{\overline{f}(s)\right\}=\mathrm{L}^{-1}\left\{\frac{s}{s^2+a^2}\right\}=\cos at=f(t)-say$

$$L^{-1}\left\{\overline{g}(s)\right\}=L^{-1}\left\{\frac{s}{s^2+a^2}\right\}=\frac{1}{a}\sin at=g(t)\to say$$

$\therefore$ By convolution theorem, we have

$$L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}=\int_0^t \cos au.\frac{1}{a}.\sin a(t-u)du$$

$$=\frac{1}{2a}\int_0^t\left[\sin(\mathrm{au}+at-au)-\sin(au-at+au)\right]du$$

$$=\frac{1}{2a}\int_0^t\left[\sin at-\sin(2au-at)\right]du$$

$$=\frac{1}{2a}\left[\sin at.u+\frac{1}{2a}.\cos(2au-at)\right]_0^t$$

$$=\frac{1}{2a}\left[t\sin at+\frac{1}{2a}\cos\left(2at-at\right)-\frac{1}{2a}\cos\left(-at\right)\right]$$

$$=\frac{1}{2a}\left[t\sin at+\frac{1}{2a}\cos at-\frac{1}{2a}\cos at\right]$$

$$=\frac{t}{2a}\sin at$$

2. Use convolution theorem to evaluate $L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}$

Sol: $L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}=L^{-1}\left\{\frac{s}{s^2+a^2}.\frac{s}{s^2+b^2}\right\}$

$$Let\ \overline{f}(s)=\frac{s}{s^2+a^2}\ and\ \overline{g}(s)=\frac{s}{s^2+b^2}$$

So that $L^{-1}\left\{\overline{f}(s)\right\}=L^{-1}\left\{\frac{s}{s^2+a^2}\right\}=\cos at=f(t)\to say$

$$L^{-1}\left\{\overline{g}(s)\right\}=L^{-1}\left\{\frac{s}{(s^2+b^2)}\right\}=\cos bt=g(t)\to say$$

$\therefore$ By convolution theorem, we have

$$L^{-1}\left\{\frac{s}{s^2+a^2}\cdot\frac{s}{s^2+b^2}\right\}=\int_0^t \cos au.\cos b(t-u)du$$

$$=\frac{1}{2}\int_0^t\left[\cos(au-bu+bt)+\cos(au+bu-bt)\right]du$$

$$=\frac{1}{2}\left[\frac{\sin(au-bu+bt)}{a-b}+\frac{\sin(au+bu-bt)}{a+b}\right]_0^t$$

$$=\frac{1}{2}\left[\frac{\sin at-\sin bt}{a-b}+\frac{\sin at+\sin bt}{a+b}\right]=\frac{a\sin at-b\sin bt}{a^2-b^2}$$

3. Use convolution theorem to evaluate $L^{-1}\left\{\frac{1}{s(s^2+4)^2}\right\}$

Sol: $$L^{-1}\left\{\frac{1}{s(s^2+4)^2}\right\}=L^{-1}\left\{\frac{1}{s^2}\cdot\frac{s}{(s^2+4)^2}\right\}$$

$$Let\ \overline{f}(s)=\frac{1}{s^2}\ and\ \overline{g}(s)=\frac{s}{\left(s^2+4\right)^2}$$

So that $L^{-1}\left\{\overline{g}(s)\right\}=L^{-1}\left\{\frac{1}{s^2}\right\}=t=g(t)\to say$

$$L^{-1}\left\{\overline{f}(s)\right\}=L^{-1}\left\{\frac{s}{(s^2+4)^2}\right\}=\frac{t.\sin 2t}{4}=f(t)-say\left[\therefore L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}=\frac{t\sin 2t}{2a}\right]$$

$$\therefore L^{-1}\left\{\frac{1}{s^2}\cdot\frac{s}{(s^2+4)^2}\right\}=\int_0^t\frac{u}{4}\sin 2u(t-u)du$$

$$=t/4\int_0^t u\sin 2udu-\frac{1}{4}\int_0^t u^2\sin 2udu$$

$$=t/4\left(-\frac{u}{2}\cos 2u+\frac{1}{4}\sin 2u\right)_0^t$$

$$=-\frac{1}{4}\left[\frac{-u^2}{2}\cos 2u+\frac{u}{2}\sin 2u+\frac{1}{4}\cos 2u\right]_0^t$$

$$=\frac{1}{16}[1-t\sin 2t-\cos 2t]$$

4. **Find** $L^{-1}\left[\frac{1}{(s-2)(s^2+1)}\right]$

Sol: $L^{-1}\left[\frac{1}{(s-2)(s^2+1)}\right]=L^{-1}\left[\frac{1}{s-2}.\frac{1}{s^2+1}\right]$

$$Let\ \overline{f}(s)=\frac{1}{s-2}\ and\ \overline{g}(s)=\frac{1}{s^2+1}$$

So that $L^{-1}\left\{\overline{f}(s)\right\}=L^{-1}\left\{\frac{1}{s-2}\right\}=e^{2t}=f(t)\rightarrow say$

$L^{-1}\left\{\overline{g}(s)\right\}=L^{-1}\left\{\frac{1}{s^2+1}\right\}=\sin t=g(t)\rightarrow say$

$\therefore L^{-1}\left\{\frac{1}{s-2}.\frac{1}{s^2+1}\right\}=\int_0^t f(u).\mathrm{g(t}-u)\,\mathrm{du}$ (By Convolution theorem)

$$=\int_0^t e^{2u}\sin(\mathrm{t-u})\,\mathrm{du}\ \ (\mathrm{or})\ \int_0^t \mathrm{sinu}.e^{2(t-u)}\,\mathrm{du}$$

$$=e^{2t}\int_0^t \sin ue^{-2u}du$$

$$=e^{2t}\left[\frac{e^{-2u}}{2^2+1}\left[-2\sin u-\cos u\right]\right]_0^t$$

$$=e^{2t}\left[\frac{1}{5}e^{-2t}\left(-2\sin t-\cos t\right)-\tfrac{1}{5}(-1)\right]$$

$$=\frac{1}{5}\left(e^{2t}-2\sin t-\cos t\right)$$

5. **Find** $L^{-1}\left\{\frac{1}{(s+1)(s-2)}\right\}$

Sol: $L^{-1}\left\{\frac{1}{(s+1)(s-2)}\right\}=L^{-1}\left\{\frac{1}{s+1}.\frac{1}{s-2}\right\}$

$$Let\ \overline{f}(s)=\frac{1}{s+1}\ and\ \overline{g}(s)=\frac{1}{s-2}$$

So that $L^{-1}\left\{\overline{f}(s)\right\}=L^{-1}\left\{\frac{1}{s+1}\right\}=e^{-t}=f(t)\rightarrow say$

$L^{-1}\left\{\overline{g}(s)\right\}=L^{-1}\left\{\frac{1}{s-2}\right\}=e^{2t}=g(t)\rightarrow say$

$\therefore$ By using convolution theorem, we have

$$L^{-1}\left\{\frac{1}{(s+1)(s-2)}\right\} = \int_0^t e^{-u}e^{2(t-u)}du$$

$$= \int_0^t e^{2t}e^{-3u}du = e^{2t}\int_0^t e^{-3u}du = e^{2t}\left[\frac{e^{-3u}}{-3}\right]_0^t = \frac{1}{3}\left[e^{2t} - e^{-t}\right]$$

6. **Find** $L^{-1}\left\{\frac{1}{s^2(s^2-a^2)}\right\}$

Sol: $L^{-1}\left\{\frac{}{s\ \ (s\ \ -a\ \ )}\right\} = L^{-1}\left\{\frac{1}{s^2}.\frac{}{s\ \ -a\ \ }\right\}$

$$Let\ \overline{f}(s) = \frac{1}{s^2}\ and\ \overline{g}(s) = \frac{1}{s^2-a^2}$$

So that $L^{-1}\left\{\overline{f}(s)\right\} = L^{-1}\left\{\frac{1}{s^2}\right\} = t = f(t) - say$

$$L^{-1}\left\{\overline{g}(s)\right\} = L^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{1}{a}\sinh at = g(t) - say$$

By using convolution theorem, we have

$$L^{-1}\left\{\frac{1}{s^2(s^2-a^2)}\right\} = \int_0^t u.\frac{1}{a}\sinh a(t-u)du$$

$$= \frac{1}{a}\int_0^t u\sinh(at-au)du$$

$$= \frac{1}{a}\left[\frac{-u}{a}\cosh(at-au) - \frac{\sin(at-au)}{a^2}\right]_0^t$$

$$= \frac{1}{a}\left[\frac{-t}{a}\cosh(at-at) - 0 - \frac{1}{a^2}[0-\sinh at]\right]$$

$$= \frac{1}{a}\left[\frac{-t}{a} + \frac{1}{a^2}\sinh at\right]$$

$$= \frac{1}{a^3}\left[-at + \sinh at\right]$$

3. **Using Convolution theorem, evaluate** $\boldsymbol{L^{-1}\{\frac{s}{(S+2)(s^2+9)}\}}$

Sol: $L^{-1}\left\{\frac{1}{s+2}.\frac{s}{s^2+9}\right\} = L^{-1}\{\frac{1}{s+2}.\frac{s}{s^2+3^2}\} = L^{-1}\{\bar{f}(s).\bar{g}\ (s)\}$

$\bar{f}(s) = \frac{1}{s+2} = L\{f(t)\} \Rightarrow f(t) = L^{-1}\left\{\frac{1}{s+2}\right\} = e^{-2t}$----------------- (1)

$\bar{g}(s) = \frac{s}{s^2+3^2} = L\{g(t)\} \Rightarrow g(t) = L^{-1}\left\{\frac{s}{s^2+3^2}\right\} = cos3t$----------------- (2)

By Convolution theorem we have

$$L^{-1}\{\bar{f}(s).\bar{g}\,(s)\} = f(t) * g(t)$$

Where $f(t) * g(t) = \int_0^t g(u)f(t-u)du$

$\therefore \quad L^{-1}\left\{\frac{1}{s+2}.\frac{s}{s^2+9}\right\} = \int_0^t e^{-2(t-u)}cos3udu$

$$= e^{-2t}\int_0^t e^{2u}cos3udu$$

$$= e^{-2t}.\frac{1}{2^2+3^2}[2cos3u - 3sin3u]_0^t$$

$$= \frac{e^{-2t}}{13}[2cos3t - 2 - 3sin3t]$$

$$= \frac{1}{13}[e^{-2t}(2cos3t - 3sin3t)] - \frac{2e^{-2t}}{13}$$

Application of L.T to ordinary differential equations:

(Solutions of ordinary DE with constant coefficient):

1. **Step1:** Take the Laplace Transform on both the sides of the DE and then by using the formula $L\{f^n(t)\} = s^n L\{f(t)\} - s^{n-1}f(0) - s^{n-1}f^1(0) - s^{n-2}f^2(0) - \ldots\ldots\ldots\ldots f^{n-1}(0)$ and apply given initial conditions. This gives an algebraic equation.
2. **Step2:** replace f (0), $f^1(0)$, $f^2(0)$,……… $f^{n-1}(0)$ with the given initial conditions.

   Where $f^1(0) = s\overline{f}(0) - f(0)$

   $f^2(0) = s^2\overline{f}(s) - sf(0) - f^1(0)$, and so on
3. **Step3:** solve the algebraic equation to get derivatives in terms of s.
4. **Step4:** take the inverse Laplace transform on both sides this gives f as a function of t which gives the solution of the given DE

Problems:

1. Solve $y^{111} + 2y^{11} - y^1 - 2y = 0$ using Laplace Transformation given that $y(0) = y^1(0) = 0$ and $y^{11}(0) = 6$

Sol: Given that $y^{111} + 2y^{11} - y^1 - 2y = 0$

Taking the Laplace transform on both sides, we get

$L\{y^{111}(t)\} + 2L\{y^{11}(t)\} - L\{y^1\} - 2L\{y\} = 0$

$\Rightarrow s^3L\{y(t)\} - s^2y(0) - sy^1(0) - y^{11}(0) + 2\{s^2L\{y(t)\} - sy(0) - y^1(0)\} -$

$\{sL\{y(t)\} - y(0)\} - 2L\{y(t)\} = 0$

$$\Rightarrow \left\{s^3+2s^2-s-2\right\} \mathrm{L}\left\{y(t)\right\} = s^2 y(0) + sy^1(0) + y^{11}(0) + 2sy(0) + 2y^1(0) - y(0)$$

$$= 0+0+6+2.0+2.0-0$$

$$\Rightarrow \left\{s^3+2s^2-s-2\right\} \mathrm{L}\left\{y(t)\right\} = 6$$

$$\mathrm{L}\left\{y(t)\right\} = \frac{6}{s^3+2s^2-s-2} = \frac{6}{(s-1)(s+1)(s+2)}$$

$$= \frac{A}{s-1} + \frac{B}{s+1} + \frac{C}{s+2}$$

$$\Rightarrow A(s+1)(s+2) + B(s-1)(s+2) + C(s-1)(s+1) = 6$$

$$\Rightarrow A(s^2+3s+2) + B(s^2-s-2) + C(s^2-1) = 6$$

Comparing both sides s^2,s,constants,we have

$$\Rightarrow A+B+C=0, 3A-B=0, 2A-2B-C=6$$

$$\mathrm{A}+\mathrm{B}+\mathrm{C}=0$$
$$2\mathrm{A}-2\mathrm{B}-\mathrm{C}=6$$

$$3A-B=6$$
$$3A+B=0$$

$$6A=6 \Rightarrow A=1$$
$$3A+B=0 \Rightarrow B=-3A \Rightarrow B=-3$$

$$\therefore A+B+C=0 \Rightarrow C=-A-B=-1+3=2$$

$$\therefore \mathrm{L}\left\{y(t)\right\} = \frac{1}{s-1} - \frac{3}{s+1} + \frac{2}{s+2}$$

$$y(t) = \mathrm{L}^{-1}\left\{\frac{1}{s-1}\right\} - 3.\mathrm{L}^{-1}\left\{\frac{1}{s+1}\right\} + 2.\mathrm{L}^{-1}\left\{\frac{1}{s+2}\right\} = e^t - 3e^{-t} + 2.e^{-2t}$$

Which is the required solution

2. **Solve** $y^{11}-3y^1+2y=4t+e^{3t}$ **using Laplace Transformation given that** $y(0)=1 \, and \; y^1(0)=-1$

Sol: Given that $y^{11}-3y^1+2y=4t+e^{3t}$

Taking the Laplace transform on both sides, we get

$$\mathrm{L}\left\{y^{11}(t)\right\} - 3\mathrm{L}\left\{y^1(t)\right\} + 2\mathrm{L}\left\{y(t)\right\} = 4\mathrm{L}\left\{t\right\} + \mathrm{L}\left\{e^{3t}\right\}$$

$$\Rightarrow s^2\mathrm{L}\left\{y(t)\right\} - sy(0) - y^1(0) - 3\left[s\mathrm{L}\left\{y(t)\right\} - y(0)\right] + 2\mathrm{L}\left\{y(t)\right\} = \frac{4}{s^2} + \frac{1}{s-3}$$

$$\Rightarrow (s^2-3s+2)\mathrm{L}\{y(t)\} = \frac{4}{s^2} + \frac{1}{s-3} + s - 4$$

$$\Rightarrow (s^2-3s+2)\mathrm{L}\{y(t)\} = \frac{4s-12+s^4+s^2-3s^3-4s^3+12s^2}{s^2(s-3)}$$

$$\Rightarrow \mathrm{L}\{y(t)\} = \frac{s^4-7s^3+13s^2+4s-12}{s^2(s-3)(s^2-3s+2)}$$

$$\Rightarrow \mathrm{L}\{y(t)\} = \frac{s^4-7s^3+13s^2+4s-12}{s^2(s-3)(s-1)(s-2)}$$

$$\Rightarrow \frac{s^4-7s^3+13s^2+4s-12}{s^2(s-3)(s-1)(s-2)} = \frac{As+B}{s^2} + \frac{C}{s-3} + \frac{D}{s-1} + \frac{E}{s-2}$$

$$= \frac{(As+B)(s-1)(s-2)(s-3) + C(s^2)(s-1)(s-2) + D(s^2)(s-2)(s-3) + E(s^2)(s-1)(s-3)}{s^2(s-3)(s-1)(s-2)}$$

$$\Rightarrow s^4-7s^3+13s^2+4s-12 = (As+B)(s^3-6s^2+11s-6) +$$
$$C(s^2)(s^2-3s+2) + D(s^2)(s^2-5s+6) + E.s^2(s^2-4s+3)$$

Comparing both sides s^4,s^3,we have

$A+C+D+E=1$......................(1)

$-6A+B-3C-5D-4E=-7$......................(2)

$put\ s=1, 2D=-1 \Rightarrow D=\frac{-1}{2}$

$put\ s=2, -4E=8 \Rightarrow E=-2$

$put\ \ s=3, 18C=9 \Rightarrow C=\frac{1}{2}$

$from\,eq.(1)\ A = 1-\frac{1}{2}+\frac{1}{2}+2 \Rightarrow A=3$

from eq.(2) B= -7+18+$\frac{3}{2}-\frac{5}{2}-8=3-1=2$

$$y(t) = \mathrm{L}^{-1}\left\{\frac{3}{s} + \frac{2}{s^2} + \frac{1}{2(s-3)} - \frac{1}{2(s-1)} - \frac{2}{s-2}\right\}$$

$$y(t) = 3+2t+\frac{1}{2}e^{3t} - \frac{1}{2}e^{t} - 2.e^{2t}$$

3. Using Laplace Transform Solve $\frac{d^2y}{dt^2}+2\frac{dy}{dt}-3y=\sin t$, **given that** $y=\frac{dy}{dt}=0$ **when t=0**

Sol: Given equation is $\frac{d^2y}{dt^2}+2\frac{dy}{dt}-3y=\sin t.$

$$L\left\{y^{11}(t)\right\}+2L\left\{y^{1}(t)\right\}-3L\left\{y(t)\right\}=L\left\{\sin t\right\}$$

$$s^2L\left\{y(t)\right\}-sy(0)-y^1(0)+2\left[sL\left\{y(t)\right\}-y(0)\right]-3.L\left\{y(t)\right\}=\frac{1}{s^2+1}$$

$$\Rightarrow\left(s^2+2s-3\right)L\left\{y(t)\right\}=\frac{1}{s^2+1}$$

$$\Rightarrow L\left\{y(t)\right\}=\left(\frac{1}{\left(s^2+1\right)\left(s^2+2s-3\right)}\right)$$

$$\Rightarrow y(t)=L^{-1}\left(\frac{1}{(s-1)(s+3)\left(s^2+1\right)}\right)$$

Now consider

$$\frac{1}{(s-1)(s+3)\left(s^2+1\right)}=\frac{A}{s-1}+\frac{B}{s+3}+\frac{Cs+D}{s^2+1}$$

$$A(s+3)\left(s^2+1\right)+B(s-1)\left(s^2+1\right)+(Cs+D)(s-1)(s+3)=1$$

Comparing both sides s^3,we have

$$put\ s=1, 8A=1\Rightarrow A=\frac{1}{8}$$

$$put\ s=-3,-40B=1\Rightarrow B=\frac{-1}{40}$$

$$A+B+C=0\Rightarrow C=0-\frac{1}{8}+\frac{1}{40}$$

$$C=\frac{-5+1}{40}=\frac{-4}{40}=\frac{-1}{10}$$

$$3A-B+2C+D=0\Rightarrow D=-\frac{3}{8}-\frac{1}{40}+\frac{1}{5}$$

$$D=\frac{-15-1+8}{40}=\frac{-8}{40}=\frac{-1}{5}$$

$$\therefore y(t)=L^{-1}\left\{\frac{\frac{1}{8}}{s-1}+\frac{\frac{-1}{40}}{s+3}+\frac{\frac{-1}{10}s-\frac{1}{5}}{s^2+1}\right\}$$

$$=\frac{1}{8}L^{-1}\left\{\frac{}{s-}\right\}-\frac{1}{40}L^{-1}\left\{\frac{}{s+}\right\}-\frac{1}{10}L^{-1}\left\{\frac{s}{s^2+}\right\}-\frac{1}{5}L^{-1}\left\{\frac{}{s^2+}\right\}$$

$$\therefore y(t)=\frac{1}{8}e^{t}-\frac{1}{40}e^{-3t}-\frac{1}{10}\cos t-\frac{1}{5}\sin t$$

4. Solve $\frac{dx}{dt}+x=\sin\omega \mathrm{t}, x(0)=2$

Sol: Given equation is $\frac{dx}{dt}+x=\sin\omega \mathrm{t}$

$$\mathrm{L}\left\{x^{1}(t)\right\}+\mathrm{L}\left\{x(t)\right\}=\mathrm{L}\left\{\sin\omega \mathrm{t}\right\}$$

$$\Rightarrow \mathrm{s.L}\left\{x(t)\right\}-x(0)+\mathrm{L}\left\{x(t)\right\}=\frac{\omega}{s^2+\omega^2}$$

$$\Rightarrow \mathrm{s.L}\left\{x(t)\right\}-2+\mathrm{L}\left\{x(t)\right\}=\frac{\omega}{s^2+\omega^2}$$

$$\Rightarrow (s+1)\mathrm{L}\left\{x(t)\right\}=\frac{\omega}{s^2+\omega^2}+2$$

$$\Rightarrow x(t)=\mathrm{L}^{-1}\left\{\frac{\omega}{(s+1)(s^2+\omega^2)}+\frac{2}{s+1}\right\}$$

$$=2\mathrm{L}^{-1}\left\{\frac{1}{s+1}\right\}+\mathrm{L}^{-1}\left\{\frac{\omega}{(s+1)(s^2+\omega^2)}\right\}$$ (By using partial fractions)

$$=2e^{-t}+\mathrm{L}^{-1}\left\{\frac{\frac{\omega}{\omega^2+1}}{s+1}-\frac{\frac{s\omega}{1+\omega^2}}{s^2+\omega^2}+\frac{\frac{\omega}{1+\omega^2}}{s^2+\omega^2}\right\}$$

$$=2e^{-t}+\frac{\omega}{\omega^2+1}e^{-t}-\frac{\omega}{1+\omega^2}\cos\omega t+\frac{\omega}{1+\omega^2}.\frac{1}{\omega}\sin\omega t$$

5. Solve $(D^2+n^2)x=a\sin(nt+\alpha)$ **given that x=Dx=0, when t=0**

Sol: Given equation is $(D^2+n^2)x=a\sin(nt+\alpha)$

$$x^{11}(t)+n^2x(t)=a\sin(nt+\alpha)$$

$$\mathrm{L}\left\{x^{11}(t)\right\}+n^2\mathrm{L}\left\{x(t)\right\}=\mathrm{L}\left\{a\sin nt\cos\alpha+a\cos nt\sin\alpha\right\}$$

$$\Rightarrow s^2\mathrm{L}\left\{x(t)\right\}-sx(0)-x^1(0)+n^2\mathrm{L}\left\{x(t)\right\}=a\cos\alpha\mathrm{L}\left\{\sin nt\right\}+a\sin\alpha\mathrm{L}\left\{\cos nt\right\}$$

$$\Rightarrow (s^2+n^2)\mathrm{L}\left\{x(t)\right\}=a\cos\alpha\frac{n}{s^2+n^2}+a\sin\alpha.\frac{s}{s^2+n^2}$$

$$\Rightarrow \mathrm{L}\left\{x(t)\right\}=a\cos\alpha\frac{n}{(s^2+n^2)^2}+a\sin\alpha\frac{s}{(s^2+n^2)^2}$$

(By using convolution theorem I –part, partial fraction in II-part)

$$= na\cos\alpha \int_0^t \frac{1}{n}.\sin nx.\frac{1}{n}\sin n(t-x)\,dx - \frac{a\sin\alpha}{2} L^{-1}\left\{\frac{d}{ds}\frac{1}{(s^2+n^2)}\right\}$$

$$= \frac{a\cos\alpha}{2n}\int_0^t \{\cos(nt-2nx)-\cos nt\}\,dx + \frac{a\sin\alpha}{2} t \frac{1}{n}\sin nt$$

$$= \frac{a\cos\alpha}{2n}\left[\int_0^t \{\cos n(t-2x)-\cos nt\}\,dx + \frac{a}{2n}\sin\alpha t \sin nt\right]$$

$$= \frac{a\cos\alpha}{2n}\left[\frac{-1}{2n}.\sin n(t-2x) - x\cos nt\right]_0^t + \frac{at\sin\alpha}{2n}\sin nt$$

$$= \frac{a\cos\alpha}{2n}\left[\frac{\sin nt}{2n} - t\cos nt\right] + \frac{at\sin\alpha}{2n}\sin nt$$

$$= \frac{a\cos\alpha\sin nt}{2n^2} - \frac{at}{2n}[\cos\alpha\cos nt - \sin\alpha\sin nt]$$

$$= \frac{a\cos\alpha\sin nt}{2n^2} - \frac{at}{2n}\cos(\alpha+nt)$$

6. Solve $y^{11} - 4y^1 + 3y = e^{-t}$ using L.T given that y (0) = y^1 (0) = 1.

Sol: Given equation is $y^{11} - 4y^1 + 3y = e^{-t}$

Applying L.T on both sides we get $L(y^{11}) - 4L(y^1) + 3L(y) = L(e^{-t})$

$\Rightarrow \{s^2L[y] - s\,y(0) - y^1(0)\} - 4\{s\,L[y] - y(0)\} + 3L\{y\} = \frac{1}{s+1}$

$\Rightarrow (s^2 + 4s + 3)\,L\{y\} - s - 1 - 4 = \frac{1}{s+1}$

$\Rightarrow (s^2 + 4s + 3)\,L\{y\} = \frac{1}{s+1} + s + 5$

$\Rightarrow (s^2 + 4s + 3)\,L\{y\} = \frac{1}{s+1} + s + 5$

$$L\{y\} = \frac{1}{(s+1)(s^2+4s+3)} + \frac{s+5}{(s^2+4s+3)}$$

$$y = L^{-1}\left[\frac{1}{(s+1)(s^2+4s+3)}\right] + L^{-1}\left[\frac{s+5}{(s^2+4s+3)}\right]$$

Let us consider

$$L^{-1}\left[\frac{1}{(s+1)(s^2+4s+3)}\right] = L^{-1}\left[\frac{1}{(s+1)^2(s+3)}\right]$$

$$\frac{1}{(s+1)(s^2+4s+3)} = \frac{1}{(s+1)^2(s+3)}$$

$$= \frac{A}{s+1} + \frac{B}{(S+1)^2} + \frac{C}{S+3}$$

$$= \frac{(-\frac{1}{4})}{s+1} + \frac{(\frac{1}{2})}{(S+1)^2} + \frac{(\frac{1}{4})}{S+3}$$

$$= L^{-1}\left[\frac{(-\frac{1}{4})}{s+1} + \frac{(\frac{1}{2})}{(S+1)^2} + \frac{(\frac{1}{4})}{S+3}\right]$$

$=L^{-1}[\frac{(-\frac{1}{4})}{s+1}+\frac{(\frac{1}{2})}{(S+1)^2}+\frac{(\frac{1}{4})}{S+3}]$

$=-\frac{1}{4}L^{-1}[\frac{1}{s+1}]+\frac{1}{2}L^{-1}[\frac{1}{(S+1)^2}]+\frac{1}{4}L^{-1}[\frac{1}{S+3}]$

$$L^{-1}\left[\frac{1}{(s+1)(s^2+4s+3)}\right]=-\frac{1}{4}e^{-t}+\frac{1}{2}te^{-t}+\frac{1}{4}e^{-3t}----\to(1)$$

$L^{-1}\left[\frac{s+5}{(s^2+4s+3)}\right]=L^{-1}[\frac{s+2}{((s+2)^2-1)}]+L^{-1}[\frac{3}{((s+2)^2-1)}]$

$=\boldsymbol{e^{-2t}}L^{-1}[\frac{s}{(s^2-1)}]+L^{-1}+\boldsymbol{3e^{-2t}}L^{-1}[\frac{1}{(s^2-1)}]$

$L^{-1}\left[\frac{s+5}{(s^2+4s+3)}\right]=\boldsymbol{cost}+\boldsymbol{3e^{-2t}sint}---\to(2)$

From (1) & (2)

$\therefore y=-\frac{1}{4}e^{-t}+\frac{1}{2}te^{-t}+\frac{1}{4}e^{-3t}+\boldsymbol{e^{-2t}cost}+\boldsymbol{3e^{-2t}sint}$

7. Solve $\frac{d^2x}{d^2t}+9x=cos2t$ using L.T. given x (0) =1, x ($\frac{\pi}{2}$) = -1.

Sol: Given $x^{11}+9x=cos2t$

L $[x^{11}]+9[x]=L[cos2t]$

$\Rightarrow\ s^2L[x]-sx(0)-x^1(0)+9L[x\}=\frac{s}{s^2+4}$

$\Rightarrow(s^2+9)L[x]-s-a=\frac{s}{s^2+4}$

$\Rightarrow(s^2+9)L[x]=\ \frac{s}{s^2+4}+(s+a)$

$L[x]=\ \frac{s}{(s^2+4((s^2+9)}+\frac{s}{(s^2+9)}+\frac{a}{(s^2+9)}$

X$=L^{-1}[\frac{s}{(s^2+4((s^2+9)}]+L^{-1}[\frac{s}{(s^2+9)}]+L^{-1}[\frac{a}{(s^2+9)}]$

$=\frac{1}{5}L^{-1}[\frac{s}{s^2+4}-\frac{s}{s^2+9}]+cos3t+\frac{a}{3}sin3t$

$=\frac{1}{5}L^{-1}[\frac{s}{s^2+4}]-\frac{1}{5}L^{-1}[\frac{s}{s^2+9}]+cos3t+\frac{a}{3}sin3t$

$=\frac{1}{5}cos2t-\frac{1}{5}cos3t+cos3t+\frac{a}{3}sin3t$ ------------------→ (1)

Given x ($\frac{\pi}{2}$) = -1.

$\therefore\ -1=\frac{1}{5}cos2\left(\frac{\pi}{2}\right)-\frac{1}{5}cos\frac{3\pi}{2}+cos\frac{3\pi}{2}+cos\frac{3\pi}{2}+\frac{a}{3}sin\frac{3\pi}{2}$

$\Rightarrow\quad$ -1= $-\frac{1}{5}-0+0-\frac{a}{3}$

$\frac{a}{3}=-\frac{1}{5}+1$

$\frac{a}{3}=\frac{4}{5}$

$\therefore\quad$ x = $\frac{1}{5}cos2t+\frac{4}{5}cos3t+\frac{4}{5}sin3t$ From (1)

8. $\boldsymbol{Solve(D^3-3D^2+3D-1)y=t^2e^t}$ **Using L.T given y (0) =1,** $\boldsymbol{y^1=0, y^{11}(0)=-2}$

Sol: Given $y^{111}-3y^{11}+3y^1-y=t^2e^t$

$L[y^{111}]-3L[y^{11}]+3L[y^1]-L[y]=L[t^2e^t]$

$\Rightarrow \{s^3L[y]-s^2y(0)-sy^1(0)-y^{11}(0)\}-3\{s^2L[y]-sy^1(0)-y(0)\}+$
$3\{sL[y]-y(0)\}-L[y]=L[t^2e^t]$

$$\Rightarrow (s^3-3s^2+3s-1)L[y]-s^2-0+2+0+3(1)-3(1)=(-1)^2\frac{d^2}{ds^2}L[e^t]$$

$\Rightarrow (s-1)^3$L[y]-$s^2+2=\frac{d^2}{ds^2}\left(\frac{1}{s-1}\right)$

$=\frac{2}{(s-1)^3}$

$\Rightarrow (s-1)^3$L[y] $=\frac{2}{(s-1)^3}+s^2-2$

$$L[y]=\frac{2}{(s-1)^6}+\frac{s^2}{(s-1)^3}-\frac{2}{(s-1)^3}$$

$y=L^{-1}[\frac{2}{(s-1)^6}]+L^{-1}[\frac{s^2}{(s-1)^3}]-L^{-1}[\frac{2}{(s-1)^3}]$

$=2L^{-1}[\frac{1}{(s-1)^6}]+L^{-1}[\frac{s^2}{(s-1)^3}]-2L^{-1}[\frac{1}{(s-1)^3}]$

$=2e^tL^{-1}\left[\frac{1}{(s)^6}\right]+L^{-1}\frac{s^2}{(s-1)^3}-2e^tL^{-1}\left[\frac{1}{s^3}\right]$

$=2e^t\frac{t^5}{5!}-2e^t\frac{t^2}{2!}+L^{-1}\left[\frac{s^2}{(s-1)^3}\right]$

Consider $L^{-1}\left[\frac{s^2}{(s-1)^3}\right]$

W.K.T $L^{-1}\left[\frac{1}{(s-1)^3}\right]=e^tL^{-1}\left[\frac{1}{s^3}\right]=e^t\frac{t^2}{2!}=\frac{e^tt^2}{2}$

$$L^{-1}\left[\frac{s^2}{(s-1)^3}\right]=\frac{d^2}{ds^2}\left(\frac{e^tt^2}{2}\right)=\frac{1}{2}\frac{d}{dt}(2te^t+t^2e^t)=\frac{1}{2}(2e^t+2te^t+2te^t+t^2e^t)$$

$=\frac{1}{2}(2e^t+4te^t+t^2e^t)$

$\therefore \quad y=2e^t\frac{t^5}{5!}-2e^t\frac{t^2}{2!}-\frac{1}{2}(2e^t+4te^t+t^2e^t)$