4.3 ECONOMIC INTERPRETATION OF DUALITY

LPP as a resource allocation model, seeks to maximize revenue under limited resources. Associated DUAL LPP can also be analyzed for interesting economic interpretations.

Consider the following simple case of Primal & Dual LPPs.

Primal LPP	Dual LPP
Maximize $z = \sum_{j=1}^{n} c_j x_j$	Minimize $W = \sum_{i=1}^{m} b_i y_i$
Subject to $\sum_{j=1}^{n} a_{ij} x_j \le b_i$; $i = 1, 2, \cdots m$	Subject to $\sum_{i=1}^{m} a_{ij} y_i \ge c_j$; $j = 1, 2, \cdots n$
$x_j \ge 0$; $j = 1, 2, \cdots n$	$y_i \ge 0$; $i = 1, 2, \cdots m$

$n \to$ Economic activities $(j = 1, 2, \cdots n)$

$m \to$ resources $(i = 1, 2, 3 \cdots m)$

$c_j \to$ coefficient in Primal objective represents the revenue per unit of activity j

$a_{ij} \to$ consumption rate of resource i per unit of activity j

$b_i \to$ maximum availability for resource i $(i = 1, 2, \cdots m)$

$y_i \to$ Dual variables $(i = 1, 2 \cdots m)$

$\hookrightarrow$ represents "worth per unit" of resource i (Also called shadow price of resource i)

$\sum_{i=1}^{m} b_i y_i$ = Total worth of all i resources $(i = 1, 2, 3 \cdots m)$

NOTE:

$$\underbrace{z = \sum_{j=1}^{n} c_j x_j}_{\text{(Total Revenue)}} \le \underbrace{\sum_{i=1}^{m} b_i y_i = W}_{\text{(Total worth of resources)}}$$ for any pair of feasible primal and dual solutions.

$\boxed{\text{optimum}}$

Maximize z $\longrightarrow$ $\longleftarrow$ minimize W

$$z = \sum_{j=1}^{n} c_j x_j = \sum_{i=1}^{m} b_i y_i = W$$ for optimum feasible primal and dual solutions

Prob: BagCo Produces leather Jackets and handbags. A Jacket requires 8 m^2 of leather and a handbag only 2 m^2. The labor requirements for the two Products are 12 and 5 hours respectively. The current weekly supplies of leather and labor are limited to 600 m^2 and 925 hours. The company sells the Jackets and handbags at \$350 and \$120 respectively. The objective is to determine the Production schedule that maximizes the net revenue.

	Leather (m^2)	Labor (hrs)	Revenue
Jackets	8	12	\$350/u
Handbags	2	5	\$120/u
Availabi	600/week	925/week	

a) Determine the optimum solution.

b) BagCo considering expansion of Production. What is the maximum purchase Price the company should pay for additional leather? For additional labor?

Sol:

$x_1 \rightarrow$ Number of Jackets/week ; $x_2 \rightarrow$ Number of handbags/week.

Obj. fn: maximize $Z = 350x_1 + 120x_2$

subject to

$8x_1 + 2x_2 \le 600$ (leather constraint)

$12x_1 + 5x_2 \le 925$ (labour hr. constraint)

$x_1, x_2 \ge 0$ (non-negativity constraint)

Simplex Iterations

Basic	x_1	x_2	x_3	x_4	solution	Fixed ratio
Z	-350	-120	0	0	0	
x_3	8	2	1	0	600	75
x_4	12	5	0	1	925	77.1
Z	0	-32.5	43.75	0	26250	
x_1	1	1/4	1/8	0	75	300
x_4	0	2	-1.5	1	25	12.5
Z	0	0	19.375	16.250	26656.25	
x_1	1	0	0.3125	-1/8	71.875	
x_2	0	1	-0.75	0.5	12.5	

Primal LPP (standard form)

$\max Z = 350x_1 + 120x_2 + 0x_3 + 0x_4$

S.t

$8x_1 + 2x_2 + x_3 = 600$

$12x_1 + 5x_2 + x_4 = 925$

$x_1, x_2, \underbrace{x_3, x_4}_{\text{slack variables}} \ge 0$

(a) Optimum solution

$x_1 = 71.875$, $x_2 = 12.5$

max $Z = 26656.25$ dollars

(b) Shadow (Dual) Prices for resources

\$ 19.375 $\longrightarrow$ Leather/m^2

\$ 16.250 $\longrightarrow$ Labour/h

BagCo should not pay more than \$19.375/m^2 of leather and \$16.25/h of labor time

Economic Interpretation of Dual Constraints

$$\sum_{i=1}^{m} a_{ij} y_i$$ represents the imputed cost of all resources $(i = 1,2,3 \cdots m)$ needed to produce one unit of activity j

LHS of Dual Constraint

c_j represents revenue from one unit of activity j

RHS of Dual Constraint

So for maximization optimality;

It is economically advantageous to increase the level of an activity if its unit revenue exceeds its unit imputed cost.

(ie) $\left\{ \begin{array}{l} \text{Imputed cost of} \\ \text{resources used} \\ \text{by one unit of} \\ \text{activity } j \end{array} \right\} < \left(\begin{array}{c} \text{Revenue per unit} \\ \text{of activity } j \end{array} \right)$

Prob: In the Bagco Problem, check whether it is economically viable to produce if labour cost increases by 10% per hour and leather cost decreases by 20% per m^2.

$y_1 = \$19.375/m^2$
$y_2 = \$16.250/hr$

Dual LPP

$\min W = 600 y_1 + 925 y_2$

Sot

$8y_1 + 12y_2 \geq 350$

$2y_1 + 5y_2 \geq 120$

$y_1, y_2 \geq 0$

Sol:

$8(1-0.2)y_1 + 12(1+0.1)y_2 < 350$ — ①

$2(1-0.2)y_1 + 5(1+0.1)y_2 < 120$ — ②

① => $338.5 < 350 \longrightarrow$ valid

② => $120.375 \not< 120 \longrightarrow$ not valid

After the change of resources cost, the Hand bag production becomes economically unviable and it becomes viable if the Price/unit of Hand bag is increased atleast to 120.375

Economic Interpretation of Dual Constraints

$\sum_{i=1}^{m} a_{ij} y_i$ represent Imputed Cost of activity j to Produce one unit

TRANSPORTATION MODEL

Objective: MINIMIZE TRANSPORTATION COST

Consider the following Transportation Problem involving m-sources and n-destinations, where each source is capable of supplying some fixed number of a Product to any of the destinations, while destinations have there own levels of demand for that Product.

Sources	Destinations D_1	D_2	D_3	. . .	D_n	SUPPLY
S_1	x_{11} c_{11}	x_{12} c_{12}	x_{13} c_{13}	. . .	x_{1n} c_{1n}	a_1
S_2	x_{21} c_{21}	x_{22} c_{22}	x_{23} c_{23}	. . .	x_{2n} c_{2n}	a_2
S_3	x_{31} c_{31}	x_{32} c_{32}	x_{33} c_{33}	. . .	x_{3n} c_{3n}	a_3
⋮				. . .		⋮
S_m	x_{m1} c_{m1}	x_{m2} c_{m2}	x_{m3} c_{m3}		x_{mn} c_{mn}	a_m
DEMAND	b_1	b_2	b_3	. . .	b_n	

Here c_{ij} : The cost of transportation of one unit of the Product from source S_i to destination D_j

Total supply $= \sum_{i=1}^{m} a_i$

Total demand $= \sum_{j=1}^{n} b_j$

(*) If $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$ then the Prob is a BALANCED one ELSE it is UNBALANCED

Solving this Prob. means finding how many units need to be ~~transported~~ from each source to each destination under the supply and demand constraints, that minimizes the total transportation cost.

Hence we need $m \times n$ decision var. x_{ij} ($i = 1, 2, \ldots m$, $j = 1, 2, 3 \ldots n$) representing the number of units transported from S_i to D_j that minimize the total transpor. cost.

Hence the Problem can be modelled as a minimization LPP.

$$\boxed{\begin{aligned} &\min\ z = \begin{Bmatrix}\text{Total} \\ \text{Transportation cost}\end{Bmatrix} \\ &\text{Subject to} \\ &\quad m\text{-(supply constraints)} \\ &\quad n\text{-(Demand constraints)} \end{aligned}}$$

(ie) $\min\ z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} \cdot x_{ij}$

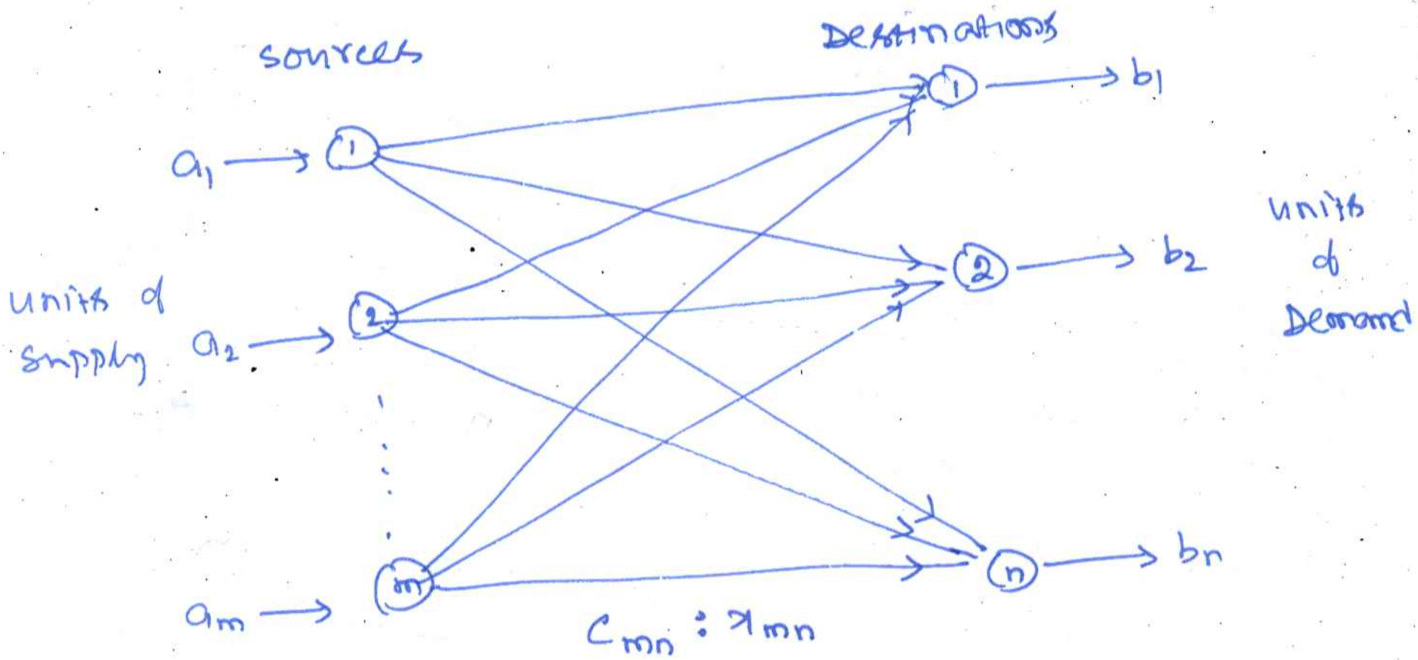
Subject to

$$\sum_{j=1}^{n} x_{ij} = a_i \quad ; \ i = 1, 2, \cdots m$$

$$\sum_{i=1}^{m} x_{ij} = b_j \quad ; \ j = 1, 2, \cdots n$$

$$x_{ij} \geq 0 \quad \forall\ i, j$$

Diagram representing Transportation model



Eg (1)

	D_1	D_2	Supply
S_1	80 x_{11}	215 x_{12}	1000
S_2	100 x_{21}	108 x_{22}	1500
S_3	102 x_{31}	68 x_{32}	1200
Demand	2300	1400	3700

$m = 3$ sources

$n = 2$ destinations.

Given cost matrix and Demand/supply.

$$\boxed{\Sigma S_i = \Sigma D_i = 3700}$$

$\boxed{\text{Balanced T-P}}$

Obj. fn: $\min z = 80x_{11} + 215x_{12} + 100x_{21} + 108x_{22} + 102x_{31} + 68x_{32}$

s.t

$$\left.\begin{aligned} x_{11} + x_{12} &= 1000 \\ x_{21} + x_{22} &= 1500 \\ x_{31} + x_{32} &= 1200 \end{aligned}\right\} \text{supply constraints}$$

$$\text{Demand constraints} \left\{\begin{aligned} x_{11} + x_{21} + x_{31} &= 2300 \\ x_{12} + x_{22} + x_{32} &= 1400 \end{aligned}\right.$$

$x_{ij} \geq 0 \quad i = 1,2,3, \quad j = 1,2.$

Solving using Excel solver, the optimal sol for the above LPP (Transportation Problem) is

$x_{11} = 1000, \quad x_{12} = 0$

$x_{21} = 1300, \quad x_{22} = 200$

$x_{31} = 0, \quad x_{32} = 1200$

$\min z = (80)(1000) + (215)(0) + (100)(1300) + (100)(1300) + (108)(200) + (102)(0) + (68)(1200)$

$= \underline{\underline{313200}}$

BALANCING THE TRANSPORTATION PROBLEMS

If Total supply ≠ Total Demand then the Problem is an unbalanced one.

To balance the Problem use the following:

case(I): If total supply > total Demand add a DUMMY Destination to absorb the extra supply (ie) Add a NEW column

case(II) If total Demand > total supply add a DUMMY source to generate the required extra supply (ie) Add a NEW row.

Both Dummy destination / source are added with zero transportation cost as these were added only to balance the problem.

Eg(2): (Eg(1) Continuation) Suppose the supply of 2nd source is reduced to 1300 (instead of 1500) then the total supply (= 3500) and total demand remains at 3700. total supply < total demand

Hence add a dummy source generating exactly (3700 − 3500) number of Product to augment the shortfall.

	D_1	D_2	Supply
S_1	x_{11} 80	x_{12} 215	1000
S_2	x_{21} 100	x_{22} 108	1300
S_3	x_{31} 102	x_{32} 68	1200
S_D	x_{41} 0	x_{42} 0	200
Demand	2300	1400	3700

Solving by Excel solver

$x_{11} = 1000$ $x_{12} = 0$

$x_{21} = 1300$ $x_{22} = 0$

$x_{31} = 0$ $x_{32} = 1200$

$x_{32} = 0$ $x_{33} = 200$

min cost : (80)(1000) + (1300)(100) + (1200)(68) + (0)(200)

$\boxed{= 291600}$

NOTE: ① The amounts shipped to a dummy destination represent surplus at the shipping source.

② The amounts shipped from a dummy source represent shortages at the receiving destinations.

Eg: (3) (Eg (1) Continuation)

Suppose if the Demand at D_1 reduces to 1900 units (instead of 2300) then total demand becomes 3300 (less than the total supply of 3700). Hence add a dummy destination to absorb the extra supply of 400 units and there by converting the Problem into a balanced one.

	D_1	D_2	D_{dummy}	Supply
S_1	80 x_{11}	215 x_{12}	0 x_{13}	1000
S_2	100 x_{21}	108 x_{22}	0 x_{23}	1500
S_3	102 x_{31}	68 x_{32}	0 x_{33}	1200
Demand	1900	1400	400	3700

Solving it by EXCEL solver

$x_{11} = 1000 \quad x_{12} = 0 \quad x_{13} = 0$

$x_{21} = 900 \quad x_{22} = 200 \quad x_{23} = 400$

$x_{31} = 0 \quad x_{32} = 1200, \ x_{33} = 0$

Min Transportation Cost

$= (1000)(80) + (900)(100) + (200)(108) + (400)(0) + (1200)(68)$

$= 273200$

Eg. (4)

In Ex (3), where a dummy destination is added, suppose that the source-2 (S_2) must ship out all its supply. How can this restriction be implemented?

Sol: By assigning a very high cost say M to the route from S_2 to Dummy destination (D_{dummy})

Eg (3) shows that S_2 has surplus supply (indicated by 400 units shipped from S_2 to D_{dummy}).

New solution:

$x_{11} = 600 \quad x_{13} = 400$ → Represent surplus at source-1 (and not a real supply)

source-2 supply fully utilised ← $x_{21} = 1300 \quad x_{22} = 200$

$x_{32} = 1200$

	D_1	D_2	D_{dummy}	Supply
S_1	80 600	215	0 400	1000
S_2	100 1300	108 200	M	1500
S_3	102	68 1200	0	1200
Demand	1900	1400	400	3700

Trans. Cost:

Prob: Three electric power plants with capacities of 25, 40 and 30 million KWh supply electricity to three cities. The maximum demands at these cities are 30, 35 and 25 million KWh. The price per million KWh at the three cities is given in the table.

		City 1	City 2	City 3
Plant	1	600	700	400
	2	320	300	350
	3	500	480	450

price in \$/million KWh

During the month of August, there is a 20% increase in demand at each of the three cities which can be met by purchasing electricity from another network at a premium rate of \$1000 per million KWh. The network is not linked to city 3 however. The utility company wishes to determine the most economical plan for the distribution and purchase of additional energy.

a) Formulate the problem as a transportation model.

b) Determine an optimal distribution plan for the utility company.

c) Determine the cost of the additional power purchased by each of the three cities.

Sol: Total supply (95) ≠ Total demand (90)

Add Dummy city to consume the surplus 5 million KWh

	1	2	3	D	Supply
1	600 x_{11}	700 x_{12}	400 x_{13}	0 x_{14}	25
2	320 x_{21}	300 x_{22}	350 x_{23}	0 x_{24}	40
3	500 x_{31}	480 x_{32}	450 x_{33}	0 x_{34}	30
Demand	30	35	25	5	95
20% increase means	36	42	30		=108

→ For Problem-c

minimize

$$z = 600x_{11} + 700x_{12} + 400x_{13} + 320x_{21} + 300x_{22} + 350x_{23} + 500x_{31} + 480x_{32} + 450x_{33}$$

s.t

~~600 x11 + 700 x12 + 400 x13 + 0 x14 = 25~~

$$\left.\begin{aligned} x_{11} + x_{12} + x_{13} + x_{14} &= 25 \\ x_{21} + x_{22} + x_{23} + x_{24} &= 40 \\ x_{31} + x_{32} + x_{33} + x_{34} &= 30 \end{aligned}\right\} \text{ supply constraints}$$

$$\left.\begin{aligned} x_{11} + x_{21} + x_{31} &= 30 \\ x_{12} + x_{22} + x_{32} &= 35 \\ x_{13} + x_{23} + x_{33} &= 25 \\ x_{14} + x_{24} + x_{34} &= 5 \end{aligned}\right\} \text{ Demand constraints}$$

$$x_{ij} \geq 0 \quad i = 1,2,3, \quad j = 1,2,3,4.$$

(b) Using Excel solver

$x_{11} = 0$, $x_{12} = 0$, $x_{13} = 25$, $x_{14} = 0$

$x_{21} = 30$, $x_{22} = 10$, $x_{23} = 0$, $x_{24} = 0$

$x_{31} = 0$, $x_{32} = 25$, $x_{33} = 0$, $x_{34} = 5$

Min $Z = \$34600$

(c)

After hiking the Demand by 20% for each city: Cost from new network = $1000/kwh

	1	2	3	Supply
1	600 (x_{11})	700 (x_{12})	400 (x_{13})	25
2	320 (x_{21})	300 (x_{22})	350 (x_{23})	40
3	500 (x_{31})	480 (x_{32})	450 (x_{33})	30
NEW NETWORK	1000 (x_{41})	1000 (x_{42})	2000 (x_{43})	13
	36	42	30	108

Since there is no link with city-3, cell(3,3) was given a penalty of $2000

High penalty of $2000 prevents any allocation to that cell.

Answer: (c)

CITY-1 excess

Cost: $(x_{41})(c_{41})$
$= (13)(1000)$
$= 13000$

CITY-2 : NIL

CITY-3 : NIL

Optimal Sol (Excel solver)

$x_{11} = 0$; $x_{12} = 0$, $x_{13} = 25$

$x_{21} = 0$, $x_{22} = 40$, $x_{23} = 0$

$x_{31} = 23$, $x_{32} = 02$, $x_{33} = 5$

$x_{41} = 13$, $x_{42} = 0$, $x_{43} = 0$

Min Z $= \$49710$

5.3 THE TRANSPORTATION ALGORITHM

Transportation Problems are special type of Linear Programming Problems and so they can be solved by the following procedure by taking advantage of their special structure.

Step I : Find a starting basic feasible solution

Step II : check the optimality of the initial basic feas. sol.
If optimal, stop. Else go to step. 3

Step III : Revise the starting basic feasible solution by identifying a entering and a leaving variable and find a new basic solution.
Again check the optimality of the new solution.
If optimal stop. Else again revise the solution until optimality reached.

NOTE: Before applying the above procedure, we need to ensure that the Transportation Problem is a Balanced one. If not add dummy source / destination and make it into a balanced Trans. Prob.

Determination of starting solution (For step-I)

Use any one of the following methods to find initial sol.

a) North-west corner method
b) Least-cost method
3) Vogel approximation method. ⟶ better method than the other two.

Prob: Four Factories A, B, C and D Produce sugar and sell their Product in three markets X, Y and Z. The following table gives the transportation cost of 1 ton of sugar from each factory to the destinations and also supply and demand values. Find the initial basic solution

FACTORIES	COST in $ per ton (x100) MARKETS X	Y	Z	AVAILABILITY IN TONS
A	4	3	2	10
B	5	6	1	8
C	6	4	3	5
D	3	5	4	6
REQUIREMENT IN TONS	7	12	5	24 / 29

UNBALANCED

Sol:

Since total supply > total demand

Add a dummy destination (MARKET) to absorb the excess supply of 5 units and then start solving the problem.

We have allocations $m+n-1$
$4+4-1=7$

a) North-west Corner method:

	X	Y	Z	D	supply
A	(7) 4	(3) 3	× 2	× 0	10/3/0
B	× 5	(8) 6	× 1	× 0	8/0
C	× 6	(1) 4	(4) 3	× 0	5/4/0
D	× 3	× 5	(1) 4	(5) 0	6/5
Demand	7/0	12/9/1/0	5/1/0	5/0	29

North west corner. Start allocation from that cell always.

$x_{11}=7$
$x_{12}=3$
$x_{22}=8$
$x_{32}=1$
$x_{33}=4$
$x_{43}=1$
$x_{44}=5$

so it is a basic sol.

Total cost:
$=7\times4+3\times3+8\times6+1\times4+4\times3+1\times4+5\times0=105$

(b) Solution by least cost cell method. Availability

	X	Y	Z	Dummy	Availability
A	x [4]	(8) [3]	(2) [2]	x [0]	10/8/0
B	x [5]	x [6]	(3) [1]	(5) [0]	8/3/0
C	(1) [6]	(4) [4]	x [3]	x [0]	5/1/0
D	(6) [3]	x [5]	x [4]	x [0]	6/0
Demand	7/1	12/4/0	5/2/0	5/0	

Start your allocation by locating least cost cell. If two cells with the same least cost occur then resolve this tie arbitrarily (ie) choose any one

Allocation: $x_{12}=8$ $x_{13}=2$; $x_{33}=3$ $x_{34}=5$
$x_{32}=4$ $x_{41}=6$
$x_{31}=1$

No. of allocation: $m+n-1 = 4+4-1 = 7$.

So the solution is feasible basic solution

$$\text{Cost} = 8\times3+2\times2+3\times1+5\times0+1\times6+4\times4+6\times3$$
$$= 71$$

(c) Solution by vogel approximation method
(opportunity cost method)

Step I: In this we use the concept of opportunity cost. It is the penalty for not taking correct decision. To find row opportunity cost, deduct the smallest element in the row from the next smallest element. Similarly calculate column opportunity cost. Write these penalties just by the end of row or column

Step II: Identify the highest opportunity cost among all the opportunity costs and make a ✓ mark.

If two or more opportunity costs are equal then select any one of them arbitrarily.

Step III: Make allocation in the ticked row/ column and (choose a least cost cell in that row / column)
if supply exhausted, notionally remove that row from further consideration for allocation.
If demand get exhausted, notionally remove that column from further consideration for allocation

Step IV: Stop the allocation process when the number of allocation becomes $(m+n-1)$. These $(m+n-1)$- allocations give the initial basic feasible solution to the problem.

NOTE: In step III, while making allocation, some time if both demand and supply both simultaneously get exhausted then we will not get the required $(m+n-1)$ allocation. Instead we will get $(m+n-2)$ allocation only. Then the corresponding solution is not considered a valid basic solution. In this situation we need to use another technique to resolve this situation.

NOTE: For a T.P problems with **m-source** and **n-destination** the maximum number of allocated cells should be $(m+n-1)$. If number of allocated cells in any solution is less than $(m+n-1)$ then the solution is called **Degenerate solution**. To resolve degeneracy allocate ϵ (a very small quantity) to a **least cost cell with the** independent position in the cost matrix

	X	Y	Z	Dummy	supply	Row Differences				
A	4 (3)	3 (7)	2	0	10/3/0	2	1	1	1	1
B	5 (3)	6	1 (5)	0	8/3/0	1	4	1	1	1
C	6	4 (5)	3	0	5/0	3	1	2	2	—
D	3 (1)	5	4	0 (5)	6/1/0	3	1	2	—	—
Demand	7/6/0	12/7/0	5/0	5/0						

Column Differen:

X	Y	Z	Dummy
1	1	1	0
1	1	—	—
1	1	—	—
1	1	—	—
1	3	—	—

No. of Allocations : 7 $= (m+n-1)$ since $m=4, n=4$.

So it gives the Basic solution.

viz., $x_{11}=3$, $x_{12}=7$, $x_{21}=3$, $x_{23}=5$, $x_{32}=5$

$x_{41}=1$, $x_{44}=0$

Transportation Cost: $3\times4 + 7\times3 + 3\times5 + 5\times1 + 5\times4 + 1\times3 + 5\times0$

$= \underline{\underline{76}}$

OPTIMAL SOLUTION TO TRANSPORTATION PROBLEM

After obtaining Initial basic feasible solution, the next step is to find optimal solution. we use the following method for this.

MODIFIED DISTRIBUTION METHOD (method of multipliers)

Start with Initial basic feasible solution obtained by any method. Check the number of allocation (loaded cells) is equal to $(m+n-1)$. Otherwise the solution is degenerate solution. A different approach need to be used in this case.

In this method, we associate the multipliers u_i and v_j with row i and column j of the transportation table.

The multipliers u_i and v_j need to be computed using the rule. $u_i + v_j = c_{ij}$ for all $(m+n-1)$ allocated cells. (ie) $u_i + v_j - c_{ij} = 0$ for all allocated cells.

The solution is optimal if $(u_i + v_j) - c_{ij} \leq 0$ for all unallocated cells. $[\ \Delta_{ij} = (u_i + v_j) - c_{ij}\ ]$

In case $(u_i + v_j) - c_{ij} \nleq 0$ for all unallocated cells then the current allocation is not optimal and need to be revised. [This revision procedure is explained along with an example]

NOTE:

Opportunity cost of a cell = Implied cost − Actual cost of cell

$$\Delta_{ij} = (u_i + v_j) - c_{ij}$$

Δ_{ij} = Opportunity cost of a cell with allocation $= 0$

Once the opportunity costs of all empty cells are negative the solution is said to be optimal. In case unallocated cell with +ive opportunity cost occurs then the program is to be modified.

Prob: Find the optimal solution to the previous sugar factories Prob (use initial basic solution given by vogel's approximation method)

Solution:

	X	Y	Z	D
A	4 (3)	3 (7)	2	0
B	5 (3)	6	1 (5)	0
C	6	4 (5)	3	0
D	3 (1)	5	4	0 (5)

Allocation as given by vogel's method

To check optimality of the initial sol, compute the multipliers u_i's and v_j's $(i = 1, 2, 3, 4,\ j = 1, 2, 3, 4)$

(1) Arbitrarily assign the value 0 for any one u_i or v_j and compute the remaining u_i's and v_j's using the condition $u_i + v_j = c_{ij}$ for allocated cells (loaded cell)

(2) After the computation of all u_i's and v_j's check the optimality using the condition $(u_i + v_j) - c_{ij} \leq 0$ for all unallocated cells. If for some cells $(u_i + v_j) - c_{ij} \nleq 0$ then the solution is not optimal and it has to be revised.

(3) Locate the cell with most positive value for $(u_i + v_j) - c_{ij}$ and start drawing a loop starting from the above identified cell. Use straight lines to form the loop and end the lines only at the allocated cells and draw lines in a suitable way such that the corner points of loop always falls at an allocated cell and complete the loop by returning back to the ~~identified~~ started cell.

I - Iteration

	X	Y	Z	D	u_i
A	4 (3) (−)	3 (7) (+)	2 −2	0 1	0
B	5 (3)	6 −2	1 (5)	0 +2	1
C	6 −1	4 (5) (−)	3 −2	0 (2) (+)	1
D	3 (1) (+)	5 −3	4 −5	0 (5) (−)	−1
v_j	4	3	0	1	

Find u_i's & v_j's using $u_i + v_j = c_{ij}$ for allocated cell. Start this computation by arbitrarily putting $u_1 = 0$. After finding all u_i's & v_j's find $u_i + v_j - c_{ij}$ value for all unallocated cells. If all $u_i + v_j - c_{ij} \leq 0$ the solution is optimal. Else revise the solution. Enter the values of $(u_i + v_j) - c_{ij}$ for all unallocated cells at the left bottom corner.

II Iteration:

	X	Y	Z	D	u_i
A	4 −2	3 (10)	2 0	0 −1	2
B	5 (3) (−)	6 0	1 (5)	0 2 (+)	5
C	6 −3	4 (2)	3 −4	0 (3)	3
D	3 (4) (+)	5 −1	4 −5	0 (2) (−)	3
v_j	0	1	−4	−3	

I - Iteration: Refer next page

II - Iteration: After I-Iteration revision of allocation, again reevaluate u_i's & v_j's and again check the optimality of the revised solution given by I - Iteration. If optimal, stop. Else again repeat a new revision of allocation as II iteration, by drawing a new loop from most positive $(u_i + v_j - c_{ij})$ value cell. Here it is at cell (2,4) and $u_i + v_j - c_{ij} = 2$ for it. Construct a loop as we did in Iteration-I and revise the allocations. Again check for optimality. If optimal, stop else repeat Iteration.

Again check

III Iteration

	X	Y	Z	D	u_i
A	4 0	3 (10)	2 −2	0 −1	−1
B	5 (1)	6 −2	1 (5)	0 (2)	0
C	6 −1	4 (2)	3 −2	0 (3)	0
D	3 (6)	5 −3	4 −5	0	−2
v_j	5	+4	1	0	

Since all $(u_i - v_j) - c_{ij} \leq 0$ for all unallocated cell the sol given in III iteration is optimal.

Optimal solution to the Transportation

Allocation		Transportation cost		
$x_{12} = 10$	(X)	3	=	30
$x_{21} = 1$	(X)	5	=	05
$x_{23} = 5$	(X)	1	=	05
$x_{24} = 2$	(X)	0	=	00
$x_{32} = 2$	(X)	4	=	08
$x_{34} = 3$	(X)	0	=	00
$x_{41} = 6$	(X)	3	=	18
				66

NOTE: when optimal solution is asked use Vogel's method (for Initial Basic sol) and then find optimal sol using modified distribution method.

Explanation: I-Iteration: since for cell (3, 4), $u_i + v_j - c_{ij} \neq 0 \ (=2)$ the maximum positive value for $(u_i + v_j) - c_{ij}$ among the all unallocated cells, we start drawing the loop from (3, 4) using other allocated cells as corners for the loop (as shown). Now put (+) sign at cell (3, 4) and (−) sign in the next corner and (+) sign in the next corner and continue like this by alternatively entering (+) and (−) signs in all corner (allocated) cells. Then find the minimum allocation among all (−) sign marked cells in the loop. Next, add this minimum allocation value with all those corner (allocated cells) marked with (+) sign, subtract this minimum allocation value with all those corner-allocated cells marked (−) sign. Here corner with (−) sign marked with least value allocated, is (cell (1,1)) with the value 3.

The start cell (3, 4) is considered as a cell with 0 allocation, and since it is marked with (+) sign add 3 with 0 and the new value of allocation in cell (3, 4) is 3 now. Similarly subtract 3 from the allocation at the cell (3, 2) as it is marked with (−) sign and similarly revise all allocation in the loop.

Prob: Consider the transportation model given below (Table a, b, c).

a) Use North-west corner method to find the starting sol.

b) Develop the iterations that lead to the optimum sol.

Sol:

0	2	1	6
2	1	5	9
2	4	3	5
5	5	10	

Table-a (Prob-a)

10	4	2	8
2	3	4	5
1	2	0	6
7	6	6	

Table-b (Prob-b)

–	3	5	4
7	4	9	7
1	8	6	19
5	6	19	

Table-c (Prob-c)

Sol:

Prob (a):

0	2	1	6
2	1	5	9
2	4	3	5
5	5	10	

$\Rightarrow$

0 (5)	2 (1)	1 ×	6/1/0
2 ×	1 (4)	5 (5)	9/5/0
2 ×	4 ×	3 (5)	5/0
5/0	5/4	10	

Using North-west corner method

No. of allocation : 5 $(= m+n-1)$ Here $m = 3$
$= 5$ $n = 3$

So the problem is a non-degenerate one.

Check the initial sol for optimality using modified dis. method

				u_i
	0 (5)	2 (1) (–)	1 (+) 5	0
	2 -3	1 (4) (+)	5 (5) (–)	–1
	2 -5	4 -5	3 (5)	–3
v_j	0	2	6	

[find u_i and v_j and check whether $(u_i+v_j) - c_{ij} \leq 0$ for all unallocated cells]

For cell (1, 3) : $(u_i+v_j) - c_{ij} = 5 \nleq 0$

So sol is not optimal.

So redistribute the allocation using loop method.

				u_i
	[0] (5) (−)	[2] −5	[1] (1) (+)	0
	✓ [2] 2 (+)	[1] (5)	[5] (4) (−)	4
	[2] 0	[4] −5	[3] (5)	2
v_j	0	−3	1	

Check the optimality again.

Find again $(u_i + v_j) - c_{ij}$ values of un-allocated cells.

For cell (2,1) $(u_i + v_j) - c_{ij} \nleq 0$ (= 2)

So again redistribute.

	[0] (1)	[2] −3	[1] (5)	0
	[2] (4)	[1] (5)	[5] −2	2
	[2] 0	[4] −3	[3] (5)	2
	0	−1	1	

Check the optimality again

All $(u_i + v_j) - c_{ij} \leq 0$

So the solution is optimal

Optimal Sol : $x_{11} = 1$, $x_{13} = 5$, $x_{21} = 4$, $x_{22} = 5$, $x_{33} = 5$ (Basic variables)

$x_{12} = x_{23} = x_{31} = x_{32} = 0$ (non-basic variab)

Minimum Transportation Cost $= (1 \times 0) + (5 \times 1) + (4 \times 2) + (5 \times 1) + (5 \times 3)$

$= 33$

Prob (6)

				Supply
	[10] 7	[4] 1	[2]	8/1/0
	[2]	[3] 5	[4]	5/0
	[1]	[2]	[0] 6	6
Demand	7/0	6/5/0	6	

Initial Basic feasible sol by north-west corner method

no of allocation (4) $\neq$ $m+n-1$ (5)

so sol is a degenerate sol.

To check optimality we need $m+n-1 = 5$ allocation. But since we have only 4 allocation

				Supply
	[10] 7	[4] 1	[2]	8
	[2]	[3] 5	[4]	5
	[1] ϵ	[2]	[0] 6	6
Demand	7	6	6	

To resolve degeneracy we assign a very small quantity ϵ to a unallocated cell with minimum cost - Here cell (3,1). With this ϵ we have 5 allocation. So degeneracy is resolved.

Now check the optimality using these 5 allocations

I - Iteration

				u_i
	[10] (-) 7	[4] 1 (+)	[2] +7	0
	[2] (+) 7 ✓	[3] 5 (-)	[4] 4	-1
	[1] ϵ	[2] -7	[0] 6	-9
v_j	10	4	9	

check $(u_i+v_j) - c_{ij} \leq 0$ for all unallocated cells.

cell (2,1) $(u_i+v_j) - c_{ij} \neq 0$ (=7

so redistribute the allocation

				u_i
	[10] 2 (+)	[4] 6	[2] -1	0
	[2] 5	[3] 3	[4] -3	0
	[1] ϵ	[2] 3	[0] 6 -6	-1
v_j	2	6	1	

~~Again check the optimality~~

~~for cell (2,2)~~

~~$(u_i+v_j)-c_{ij} \neq 0$ (= 3)~~

~~so redistribute allocation.~~

II - Iteration

				u_i
	10 / 2 (−)	4 / 6	2 / ✓ (+) 7	10
	2 / 5	3 / −6	4 / −5	2
	1 / ∈ (+)	2 / −7	0 / 6 (−)	1
v_j	0	−6	−1	

Again check optimality by computing u_i's + v_j's afresh.

$(u_i + v_j) - c_{ij} \not\leq 0$ for cell (1,3)

so again redistribute the allocation, by drawing a loop, starting at (1,3).

III - Iteration

				u_i
	10 / −11	4 / 6	2 / 2	0
	2 / 5	3 / 0	4 / −3	−1
	1 / 2	2 / 0	0 / 4	−2
v_j	3	4	2	

Again check optimality by computing new sets of u_i's + v_j's.

$(u_i + v_j) - c_{ij} \leq 0$ for all cells

So sol is optimal

∈ is very small. so ∈ + 2 = 2 only.

$x_{12} = 6, \quad x_{13} = 2, \quad x_{21} = 5, \quad x_{31} = 2, \quad x_{33} = 4$ → Basic var.

$x_{11} = 0, \quad x_{22} = 0, \quad x_{23} = 0, \quad x_{32} = 0$ → Non basic var.

Minimum Transportation Cost

$$= 6 \times 4 + 2 \times 2 + 5 \times 2 + 2 \times 1 + 4 \times 0$$

$$= \underline{40}$$

NOTE: In II - Iteration, the least value allocation with (−) sign over the four corners of the loop is 2. so redistribute the allocation as shown: { cell (1,3) : old allocation = 0, ∴ new allocation = 0 + 2 = 2

cell (3,3) : 6 − 2 = 4; cell (3,1) ; ∈ + 2 = 2 ; cell (1,1) : 2 − 2 = 0.

Problem (c)

				Supply
	15 (4)	3	5	4/0
	7 (1)	4 (6)	9	7/6/0
	1	8	6 (19)	19
Demand	5/4/0	6/0	19	

cell (1,1) is not available for allocation. so make the cost of the cell very high. (higher than all other costs in the table). so put $C_{11} = 15$

No. of allocation = 4 $\neq m+n-1$ (5)

So sol. is degenerate solution.

Allocate 'ϵ' to a least cost unallocated cell and make number of allocation to 5 (= m+n-1) (cell (3,1))

				u_i
	15 (4) (−)	3 [9]	5 [15] (+)	15
	7 (1)	4 (6)	9 [3]	7
	1 ϵ (+)	8 [−10]	6 (19) (−)	1
v_j	0	−3	5	

Note: cell (1,3) :- $(u_i + v_j) - c_{ij} \neq 0$ (=15)

Redistribute the allocations :

				u_i
	15 [−15]	3 [−6]	5 (4)	0
	7 (1) (−)	4 (6)	9 [+3] (+)	7
	1 (4) (+)	8 [−10]	6 (15) (−)	1
v_j	0	−3	5	

Check the optimality again $(u_i + v_j) - c_{ij} \neq 0$ for cell (2,3)

Draw a new loop, redistribute the allocations again

$\epsilon + 4 = 4$ in cell (3,1)

In cell (1,1)

new allocation = (old allocation) − 4

= 4 − 4 = 0

In cell (1,3)

new allocation = old allocation + 4

= 0 + 4 = 4

In cell (3,3) : 19 − 4 = 15

				u_i
	15 (−15)	3 (−3)	5 (4)	−
	7 (−3)	4 (6)	9 (1)	3
	1 (5)	8 (−7)	6 (14)	0
v_j	1	[illegible]	6	

Again check the optimality

All $(u_i+v_j)-c_{ij} \leq 0$ for all the unallocated cells.

$\therefore$ The solution is optimal

optimal sol : $x_{13}=4$, $x_{22}=6$, $x_{23}=1$

$x_{31}=5$ $x_{33}=14$

Min. Transportation Cost $= 4\times5+6\times4+1\times9$
$+5\times1+14\times6$

$=\underline{\underline{142}}$

---

(2) In the following transportation Problem total demand exceeds the total supply. Suppose that the penalty costs per unit of unsatisfied demand are \$2, \$5 and \$3 for destinations 1, 2 and 3 respectively. Use Least-Cost starting solution and compute the iterations leading to the optimal solution.

Sol:

				Supply
	5	1	7	10
	6	4	6	80
	3	2	5	15
Demand	75	20	50	

Total Demand : 145
Total Supply : 105

Excess Demand : $\underline{\underline{40}}$

$\therefore$ Create a dummy supply with a capacity 40 units

Least cost method

				Supply
	5 x	(10) 1	7 x	10/0
	(30) 6	4 x	(50) 6	80
	(5) 3	(10) 2	5 x	15/5/0
Dummy supply	(40) 2	5 x	3 x	40/0
Demand	75/35/30	20/10/0	50	145

Add dummy row to augment the shortage in supply (40)
Enter the penalty cost in the dummy row ($2, $5, $3).
(But the transportation cost is zero for this dummy row)

No of allocation = 6 = $\underline{(m+n-1) \; (=6)}$: valid for optimality check

				u_i
	$\boxed{5}$ (-3)	(10) $\boxed{1}$	$\boxed{7}$ (-5)	-4
	(30) (-) $\boxed{6}$	(1) ✓ (+) $\boxed{4}$	(50) $\boxed{6}$	0
	(5) (+) $\boxed{3}$	(10) (-) $\boxed{2}$	$\boxed{5}$ (-2)	-3
	(40) $\boxed{2}$	$\boxed{5}$ (-4)	$\boxed{3}$	-4
v_j	6	5	6	

				u_i
	$\boxed{5}$ (-2)	(10) $\boxed{1}$	$\boxed{7}$ (-4)	-3
	(20) $\boxed{6}$	(10) $\boxed{4}$	(50) $\boxed{6}$	0
	(15) $\boxed{3}$	$\boxed{2}$ (-1)	$\boxed{5}$ (-2)	-3
	(40) $\boxed{2}$	$\boxed{5}$ (-5)	$\boxed{3}$ (-1)	-4
v_j	6	4	6	

Check optimality again using u-v method.

$(u_i + v_j) - c_{ij} > 0 \;\forall i, j$

So sol. is optimal

$x_{12} = 10$
$x_{21} = 20$
$x_{22} = 10$
$x_{23} = 50$
$x_{31} = 15$
$x_{41} = 40$

Optimal Basic Solution

($x_{41} = 40$ is not a real supply).

min. Transportation cost : $10 + 120 + 40 + 300 + 45 + \underline{80}$ → penalty cost

$= \underline{595}$

{Desti-1 is getting only 35 units and a short supply of 40 units} so penalty cost : $40 \times 2 = 80$.

(3) Solve Prob-(2) assuming that the demand at destination 1 must be satisfied completely. Find the optimal solution

Optimal sol (Final Answer)

[Allot a penalty of $M = 100$ in cell (3,1). which prevents Destination-1 from getting any supply from dummy supplier.]

	[5]	10 [1]	[7]	10
	60 [6]	10 [4]	10 [6]	80
	15 [3]	[2]	[5]	15
(Dummy supplier)	[100]	[5]	40* [3]	40
	75	20	50	

Min. trans. cost + Penalty Cost

$= 10 \times 1 + (60)(6) + (10)(4) + (10)(6) + (15)(3) + \underline{(40)(3)}$

Penalty Cost

$= \underline{\underline{635}}$

[*: 40 in cell (3,3) is imaginary supply only infact it is shortage for Destination-3]

Demand at Destination-3 is not fully satisfied. It is 40 units are short supplied to destin-3. It is getting only 10 units though its demand is 50 units.

A penalty cost of \$100 is given to cell (4,1) to force the demand of destin-1 is fully satisfied from real suppliers (S_1, S_2 & S_3)

(5) In a 3×3 transportation problem, let x_{ij} be the amount shipped from source i to destination j and let c_{ij} be the corresponding transportation cost per unit. The amounts of supply at sources 1, 2 and 3 are 15, 30 and 85 units resp. and the demands at destinations 1, 2 and 3 are 20, 30 and 80 units respectively. Assume that the starting North-west corner solution is optimal and that the associated values of the multipliers are given as $u_1 = -2$, $u_2 = 3$, $u_3 = 5$, $v_1 = 2$, $v_2 = 5$ and $v_3 = 10$.

(a) Find the associated optimal cost

(b) Determine the smallest value of c_{ij} for each non-basic variable that will maintain the optimality of North-west corner solution.

Sol:

	1	2	3	supply
1	c_{11} 15	c_{12} x	c_{13} x	15/0
2	c_{21} 5	c_{22} 25	c_{23} x	30/25/0
3	c_{31} x	c_{32} 5	c_{33} 80	85/0
Demand	20/5/0	30/5	80/0	

Find the solution using North-west corner method

Basic sol is

$x_{11}=15$, $x_{21}=5$, $x_{22}=25$

$x_{32}=5$, $x_{33}=80$

$\left.\begin{array}{l} x_{12}=0,\ x_{13}=0 \\ x_{23}=0,\ x_{31}=0 \end{array}\right\}$ Non-Basic variables

Since it is given that the solution given by North-west method is optimal, we can use optimality conditions for both allocated cells (Basic variable cells) and unallocated cells (Non-Basic variable cells). [Associated u_i's and v_j's are given]

	1	2	3	u_i
1	c_{11} 15	c_{12}	c_{13}	-2
2	c_{21} 5	c_{22} 25	c_{23}	3
3	c_{31}	c_{32} 5	c_{33} 80	5
v_j	2	5	10	

(a) For Basic cells optimality condition is

$u_i + v_j = c_{ij}$

Cell (1,1): $u_1 + v_1 = c_{11}$

$-2 + 2 = c_{11}$

$\boxed{c_{11} = 0}$

cell (2,1): $u_2 + v_1 = c_{21}$

$3 + 2 = c_{21}$

$\boxed{c_{21} = 5}$

III^{ly} $c_{22}=8$, $c_{32}=10$, $c_{33}=15$

$\therefore$ optimal Trans. cost

$= (15)(0) + (5)(5) + (25)(8) + (5)(10) + (80)(15)$

$= \underline{1475}/-$

(b) For non-Basic cells (to retain optimality of the solution)

$u_i + v_j - c_{ij} \le 0$

For cell (1,2): $u_1 + v_2 - c_{12} \le 0 \Rightarrow -2+5-c_{12} \le 0$

$\Rightarrow \underline{c_{12} \ge 3}$

III^{ly} $c_{13} \ge 8$, $c_{23} \ge 13$, $c_{31} \ge 7$

Maximization case in Transportation Problems

In the sugar factory Prob. discussed earlier, if the matrix gives the returns the factory can get, by selling the sugar in each market, then the problem is a transportation model with a maximization objective. [obj: maximize Transportation Returns by finding optimal allocation]

	X	Y	Z	Supply
A	4	3	2	10
B	5	6	1	8
C	6	4	3	5
D	3	5	4	6
Demand	7	12	5	

Sol: Balance the Problem by adding a dummy column

	X	Y	Z	Dy	Supply
A	4	3	2	0	10
B	5	6	1	0	8
C	6	4	3	0	5
D	3	5	4	0	6
Demand	7	12	5	5	29

	X	Y	Z	Dy	supply
A	2	3	4	6	10
B	1	0	5	6	8
C	0	2	3	6	5
D	3	1	2	6	6
Dem	7	12	5	5	29

Find the largest entry in the matrix and subtract all the entries of the matrix from this largest entry. Supply and demand values remain same. Then apply the regular transportation algorithm

Use vogel's method and find IBFS

	X	Y	Z	D	Supply
A	(2) 2	(3) 3	x 4	(5) 6	10/0
B	x 1	(8) 0	x 5	x 6	8/0
C	(5) 0	x 2	x 3	x 6	5/0
D	x 3	(1) 1	(5) 2	x 6	6/1/0
Demand	7/2	12/1/0	5/0	5	29

Row diff

1 1 1 1 1

1 1 1 1 —

2 ← — — — —

1 1 2 ← — —

Col. diff:

1 1 1 0

1 1 2↑ 0

1 1 — 0

1 3↑ — 0

$x_{11} = 2$

$x_{12} = 3$

$x_{14} = 5$

$x_{22} = 8$

$x_{31} = 5$

$x_{42} = 1$

$x_{43} = 5$

$m+n-1 = 7$

so valid sol.

Max returns

$= 2\times 4 + 3\times 3 + 5\times 6 + 8\times 6 + 5\times 6 + 1\times 5 + 5\times 4$

$= 8 + 9 + 48 + 30 + 5 + 20$

$= \underline{\underline{120}}$

Optimal allocations

					u_i
	(2) 2	(3) 3	-4 4	(5) 6	0
	-2 1	(8) 0	-8 5	-4 6	-3
	(5) 0	-1 2	-5 3	-3 6	-2
	-3 3	(1) 1	(5) 2	-3 6	-2
v_j	2	3	4	5	

optimal

	x	y	z	Dummy	Supply	Row diff.
A	2	3	4	6	10	
B	1	0	5	6	8	
C	0	2	3	6	5	
D	3	1	2	6	6	
Demand	7	12	5	5		

Solve the above modified transportation problem using own regular minimization procedure. Find Initial sol by vogel's method

					Supply	Row difference
A	2 (2)	3 (3)	4 X	6 (5)	10/0	1 1 1 1
B	1 X	0 (8)	5 X	6 X	8/0	1 1 1 1
C	0 (5)	2 X	3 X	6 X	5/0	← $\boxed{2}$ – – –
D	3 X	1 (1)	2 (5)	6 X	6/1/0	1 1 ← $\boxed{2}$ –
Demand	7/2	12/11/3	5/0	5		
Column Difference	1	1	1	0		
	1	1	$\boxed{2}$	0		
	1	1	–	0		
	1	$\boxed{3}$	–	0		

Initial solution: $x_{11} = 2$, $x_{12} = 3$, $x_{14} = 5$, $x_{22} = 8$, $x_{31} = 5$, $x_{42} = 1$, $x_{43} = 5$

$$\left.\begin{array}{r}\text{Returns}\\ \text{as given by Initial}\\ \text{solution}\end{array}\right\} = (2)(4) + (3)(3) + (5)(0) + (8)(6) + (5)(6) + (1)(5) + (5)(4)$$

$$= \underline{\underline{120}}$$

(Use the returns in the original matrix in the corresponding cell)

Check the optimality of Initial solution by modified distribution method

					u_i
	cost 2, (2)	cost 3, (3)	cost 4, 0	cost 6, (5)	0
	cost 1, −2	cost 0, (8)	cost 5, −4	cost 6, −4	−3
	cost 0, (5)	cost 2, −1	cost 3, −1	cost 6, −3	−2
	cost 3, −3	cost 1, (1)	cost 2, (5)	cost 6, −3	−2
v_j	2	3	4	5	

Compute $u_i : i = 1, 2, 3, 4$
$v_j - j = 1, 2, 3, 4$

find $\Delta_{ij} = u_i + v_j - c_{ij}$ for all unallocated cell and enter it in the bottom left side of the cell.

Since $\Delta_{ij} \leq 0$ for all unallocated cell the solution is optimal.

∴ The Initial solution itself is an optimal sol for the given problem.

ASSIGNMENT MODEL

Consider a task of assigning m-workers to m-Jobs with the following matrix giving the cost of assignments.

		Jobs				
		1	2	3	- - -	m
Workers	1	c_{11}	c_{12}	c_{13}		c_{1m}
	2	c_{21}	c_{22}	c_{23}		c_{2m}
	3	c_{31}	c_{32}	c_{33}		c_{3m}
	⋮					
	m	c_{m1}	c_{m2}	c_{m3}		c_{mm}

Always assign one worker to only one Job and vice versa.

Objective:

minimize (total cost of assigning m-workers to m-Jobs)

It is also a special form of LPP. And a particular case of Transportation problem where the Each m-source supply capacity is 1 and each m-destination requirement is 1

Balanced assignment problem means: Number of workers = Number of Jobs.

HUNGARIAN METHOD to solve the Problem

Prob: Solve the assignment problem by Hungarian method to find minimum cost.

		Jobs				
		1	2	3	4	5
Workers	1	9	8	2	10	3
	2	6	5	2	7	5
	3	6	3	2	7	5
	4	8	4	12	3	5
	5	7	8	6	7	7

Step I: Subtract the least elt of each row from all the elements of that row.

After completing the above procedure subtract the least elt of each column from all the elements of that column.

7	6	0	8	1
4	3	0	5	3
4	1	0	5	3
5	1	9	0	2
1	2	0	1	1

row-wise

6	5	0	8	0
3	2	0	5	2
3	0	0	5	2
4	0	8	0	1
0	1	0	1	0

column-wise

Step II. The revised matrix is called reduced cost matrix

In the reduced cost matrix attempt to find a feasible assignment among all the resulting zero entries, as given below:

(a) Start from I row and check ~~for~~ all the rows sequentially, whether any row contains a single zero. If any row contains a single zero, enclose that zero by a box and cross off (X) all zeros in the corresponding column.

(b) Start from I column and check all columns sequentially whether any column contains a single zero. If any column contains a single zero enclose it by a box and cross off (X) all zeros in the corresponding row.

(c) Repeat steps (a) + (b) until possibilities are exhausted.

Now count the number of boxes and if it is equal to the order of the cost matrix then optimum assignment solution is obtained.

Boxes represent the assignment made. (ie) which worker is assigned to which Job.

If number of boxes is less than the order of the cost matrix given, then proceed to step III

Step III : If Number of assignments < Number of workers (or Job). revise the assignment as follows.

(a) Tick(✓) mark the rows not having any assignment.

(b) Tick(✓) mark all those columns which have zeros in the ticked rows

(c) Tick(✓) mark all rows having assignment in ticked columns.

(d) Repeat (a), (b) + (c) until all possibilities of Ticking is exhausted.

(e) Now draw horizontal lines through unticked rows and vertical lines through ticked columns. These line are called covering lines (Theoretially, we need to draw minimum number of lines to cover all zeros)
Identify the minimum cost among all the cost values which are not crossed by any line.

(f) Subtract the minimum cost value from all uncovered cost values of the matrix. Add this minimum value with all those cost values crossed by both horizontal and vertical lines. Rest all cost values are not changed. This procedure would have increased number of zeros available in the reduced cost-matrix.

(g) Again repeat step II. If the number of assignments is equal to the order of cost matrix, then stop.
Else repeat step III.

(h) step II and III are repeated until we get number of assignment = order of the cost matrix.

<u>Step IV</u>: After obtaining exact number of assignment equal to the order of cost matrix, shift the assignment boxes to the same cell positions of the original given cost matrix. Then add all assigned costs to find minimum cost of assignment for the problem and the assignment schedule.

(Continuing the Problem-1)

Step-II

$$
\begin{bmatrix}
6 & 5 & \cancel{0} & 8 & \boxed{0} \\
3 & 2 & \boxed{0} & 5 & 2 \\
3 & \boxed{0} & \cancel{0} & 5 & 2 \\
4 & \cancel{0} & 8 & \boxed{0} & 1 \\
\boxed{0} & 1 & \cancel{0} & 1 & \cancel{0}
\end{bmatrix}
$$

No: of assignments = 5

= order of the cost matrix

So optimal assignment is reached.

Optimal assignment schedule

Job	Worker	Cost
1	5	\$3
2	3	\$2
3	2	\$3
4	4	\$3
5	1	\$7
		\$18

Minimum Total assignment cost = \$18

(No need for other steps in this problem)

② Find optimal assignment for the following assignment problem

$$
\begin{array}{cc}
 & \text{Worker} \\
\text{Job} &
\begin{bmatrix}
3 & 12 & 2 & 2 & 7 \\
6 & 1 & 5 & 8 & 6 \\
9 & 4 & 7 & 13 & 3 \\
2 & 5 & 4 & 2 & 1 \\
10 & 6 & 1 & 4 & 6
\end{bmatrix}
\end{array}
$$

∴ Optimal assignment schedule

Job	Worker	Cost
1	4	\$2
2	2	\$1
3	5	\$3
4	1	\$2
5	3	\$1
		\$9 → minimum cost

Required optimal sol.

Sol:

$$
\begin{bmatrix}
1 & 10 & 0 & 0 & 7 \\
5 & 0 & 4 & 7 & 5 \\
6 & 1 & 4 & 10 & 0 \\
1 & 4 & 3 & 1 & 0 \\
9 & 5 & 0 & 3 & 5
\end{bmatrix}
$$

Row wise

$$
\begin{bmatrix}
\cancel{0} & 10 & \cancel{0} & \boxed{0} & 7 \\
4 & \boxed{0} & 4 & 7 & 5 \\
5 & 1 & 4 & 10 & \boxed{0} \\
\boxed{0} & 4 & 3 & 1 & \cancel{0} \\
8 & 5 & \boxed{0} & 3 & 5
\end{bmatrix}
$$

Column wise

③ Solve the following Assignment Problem

	A	B	C	D	E
m_1	9	11	15	10	11
m_2	12	9	–	10	9
m_3	–	11	14	11	7
m_4	14	8	12	7	8

Sol: Problem is unbalanced since rows = 4, column = 5 and so Add one more dummy row (with zero cost) (to Balance the Problem) and solve the Problem in the regular way

m_1	9	11	15	10	11
m_2	12	9	20	10	9
m_3	20	11	14	11	7
m_4	14	8	12	7	8
Dummy	0	0	0	0	0

Replace '–' position by high penalty cost to prevent any allocation in those cells. The high penalty cost that we select should be bigger than all other costs in the matrix. Here it is taken as 20

$\boxed{0}$	2	6	1	2
3	$\boxed{0}$	11	1	$\cancel{0}$
13	4	7	4	$\boxed{0}$
7	1	5	$\boxed{0}$	1
$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$

No. of allocation = 5 = order of matrix

so optimal sol is reached.

$m_1 \to A$: \$9

$m_2 \to B$: \$9

$m_3 \to E$: \$7

$m_4 \to D$: \$7

Minimum assignment cost \$32

Job C assigned to Dummy m/c means it has not been assigned and left out

(2) Joshop needs to assign 4 Jobs to 4 workers. The cost of performing a Job is a function of the skills of the workers. Table given gives the cost of assignments. Worker-1 cannot do Job-3 and worker-3 cannot do Job-4. Find optimal assignment using Hungarian method.

Worker	1	2	3	4
1	50	50	–	20
2	70	40	20	30
3	90	30	50	–
4	70	20	60	70

Assign a high cost (penalty) M in those cells where we need to avoid assignment

Sol. Replace '–' position by ~~very~~ very high penalty cost so that these cells would not get any assignment. Here very high penalty cost means a value higher than any other cost in that matrix.

Here $M=100$
a value greater than all other values

Worker	1	Job 2	3	4
	50	50	100	20
	70	40	20	30
	90	30	50	100
	70	20	60	70

→

	1	2	3	4
1	30	30	80	0
2	50	20	0	10
3	60	0	20	70
4	50	0	40	50

Row deduction

→

$\boxed{0}$	30	80	$\cancel{0}$
20	20	$\boxed{0}$	10
30	$\boxed{0}$	20	70 ✓
20	$\cancel{0}$	40	50 ✓

✓ Least elt: 20

$\cancel{0}$	50	80	$\boxed{0}$
20	40	$\boxed{0}$	10
10	$\boxed{0}$	$\cancel{0}$	50
$\boxed{0}$	$\cancel{0}$	20	30

Optimal sol reached:
(as no. of assignment = order of matrix)

Worker	Job	Cost (from original matrix)
1	4	\$ 20
2	3	\$ 20
3	2	\$ 30
4	1	\$ 70

Total cost: 140 ← minimum cost of assignment

Maximization IN ASSIGNMENT PROBLEM

An airline Company has drawn up a new flight schedule involving five flights. To assist in allocating five pilots to the flights, it has asked them to state their preference scores by giving each flight a number out of 10. The highest the number, the greater is the preference. Certain of these flights are unsuitable to some pilots owing to domestic reasons. These have been marked as X. What should be the allocation of the pilots to flights in order to meet as many preferences as possible?

Flight number

PILOT	A	B	C	D	E
A	8	2	X	5	4
B	10	9	2	8	4
C	5	4	9	6	X
D	3	6	2	8	7
E	5	6	10	4	3

In cell (1,3) & (3,5) assign a high negative preference (say) −M to prevent assignment in those cells

Sol: Since the prob. is to MAXIMIZE the total preference score, in order to apply HUNGARIAN method the matrix need to be revised. The revised matrix is obtained by subtracting all the elements of the given matrix from the largest element (Here = 10) including itself.

Revised matrix for applying Hungarian method (M: High penalty)

$10-(-M)$
$=10+M$
$=M$

2	8	M	5	6
0	1	8	2	6
5	6	1	4	M
7	4	8	2	3
5	4	0	6	7

0	6	M	3	4
0	1	8	2	6
4	5	0	3	M
5	2	6	0	1
5	4	0	6	7

$\boxed{0}$	5	M	3	3
~~0~~	$\boxed{0}$	8	2	5
4	4	$\boxed{0}$	3	M
5	1	6	$\boxed{0}$	~~0~~
5	3	~~0~~	6	6

$\boxed{0}$	5	M	3	3
~~0~~	$\boxed{0}$	11	2	5
1	1	~~0~~	$\boxed{0}$	M
5	1	9	~~0~~	$\boxed{0}$
2	~~0~~	$\boxed{0}$	3	3

Stop the Hungarian method as we get 5 assignment equaling the order of matrix.
Use the same assignment position in the given (maximization) matrix to know the assignment and maximum preference score (Total)

$A \to 1,\ B \to 2,\ C \to 4,\ D \to 5,\ E \to 3$; Max score:
$8 + 9 + 6 + 7 + 10 = \boxed{40}$

HW: (a) Find the minimum cost solution for the 5x5 assignment problem whose cost coefficients are as given below.

	1	2	3	4	5
A	-2	-4	-8	-6	-1
B	0	-9	-5	-5	-4
C	-3	-8	-9	-2	-6
D	-4	-3	-1	0	-3
E	-9	-5	-8	-9	-5

(b) Find the maximum Profit solution for the 5x5 assignment problem whose Profit coefficients are as given below

2	4	8	6	1
0	9	5	5	4
3	8	9	2	6
4	3	1	0	3
9	5	8	9	5