Malus's law

Phys 29X Experiment 7

Student's manual

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1 Theory

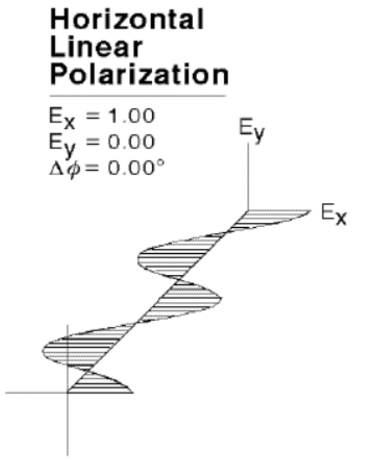
1.1 The wave equation [1]

Light is an electromagnetic wave [2]; moreover, it is a traverse wave. This means that the oscillation of its fields amplitudes is perpendicular to its direction of propagation. The fields of an electromagnetic wave are an electric and a magnetic field, each having an amplitude oscillating perpendicular to each other. When talking about the polarization of light, one refers to the general motion of its electric field. For example, a light wave travelling in the z direction with its electric field amplitude oscillating along the y axis (this implies that the magnetic field amplitude is oscillating along the x axis) is linearly polarized. See Figure 1a. If the axis of oscillation of the electric field is rotating in a plane perpendicular to the direction of propagation of the light wave (as it should be), then it is said that the light wave is either circularly polarized or elliptically polarized. See Figures 1b & 1c. [3]

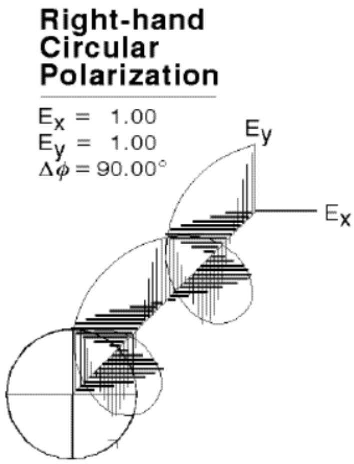
[1] Borrows from Rana [5].

[2] and a particle

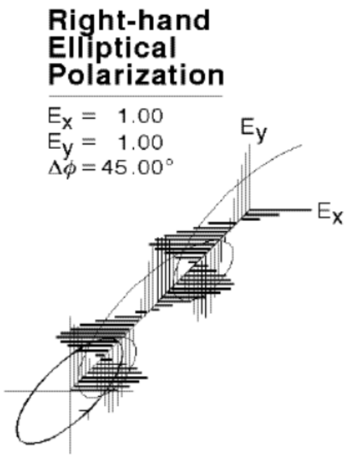
[3] For more information on polarization, visit: https://www.photonics.byu.edu/polarization.phtml.



(a) Linear polarization.



(b) Circular polarization.



(c) Elliptical polarization.

Figure 1: Types of polarization. $\Delta\phi$ denotes the phase between the x and y components of the electric field. Taken from [1]

The wave equation of an electric field propagating in vacuum as in Figure 1a, that is in one dimension, is

$$\frac{\partial^2 E_x}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 E_x}{\partial t^2}, \tag{1}$$

and in three dimensions, for any light wave,

$$\nabla^2 \vec{E} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}, \tag{2}$$

where c is the speed of light, t is time, and $\vec{E}$ is the electric vector field.

The three-dimensional wave equation is just three partial differential equations, one for each component of the electric field, x, y, z, packed in a single expression:

$$\frac{\partial^2 E_{x,y,z}}{\partial x^2} + \frac{\partial^2 E_{x,y,z}}{\partial y^2} + \frac{\partial^2 E_{x,y,z}}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 E_{x,y,z}}{\partial t^2}. \tag{3}$$

The expression for the electromagnetic wave Equation 2 can be derived from Maxwell's equations:

$$\begin{gathered}
\text{(i) } \nabla \cdot \vec{E} = \frac{1}{\epsilon_0}\rho \qquad \text{(Gauss's law)} \\
\text{(ii) } \nabla \cdot \vec{B} = 0 \\
\text{(iii) } \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \text{(Farady's law)} \\
\text{(iv) } \nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0\epsilon_0 \frac{\partial \vec{E}}{\partial t} \qquad \text{(Ampère's law)}
\end{gathered}, \tag{4}$$

where ϵ_0 is the vacuum permittivity, ρ is the current charge density of space, $\vec{B}$ is the magnetic vector field, $\vec{J}$ is the charge current density and mu_0 is the vacuum permeability. In vacuum, $\rho = 0$ and $\vec{J} = \vec{0}$. The behaviour of light in air is close to its behaviour in vacuum, and for our purposes we work the derivations in free space.

We pick Faraday's law 4 (i):

$$\nabla \times (\nabla \times \vec{E}) = -\nabla \times \left(\frac{\partial \vec{B}}{\partial t}\right) = -\frac{\partial}{\partial t}\left(\nabla \times \vec{B}\right). \tag{5}$$

Comparing with Ampère's law in free space, we notice that

$$\nabla \times (\nabla \times \vec{E}) = \mu_0\epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}.$$

Using the second derivative for curls [4], we obtain

$$\nabla(\nabla\cdot\vec{E}) - \nabla^2\vec{E} = -\mu_0\epsilon_0\frac{\partial^2\vec{E}}{\partial t^2}.$$

Since we are in vacuum, Gauss's law equals zero, and defining $c \equiv \frac{1}{\sqrt{\mu_0\epsilon_0}}$, we obtain the wave equation for electric fields propagating in vacuum:

$$\nabla^2\vec{E} = -\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}. \tag{6}$$

A similar method gives us the wave equation for magnetic fields propagating in vacuum:

$$\nabla^2\vec{E} = -\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}. \tag{7}$$

We assume that the light is propagating along the z axis [5] and that the x and y components of the fields amplitudes are independent of each other (which is the case for polarized light):

$$E_x = E_x(z,t)$$
$$E_y = E_y(z,t).$$

The electric field must be oscillating in the $x-y$ plane. Let us focus on the x component of the electric field; the y component is solved in the same way, and for linearly polarized light, as in Figure 1a, one can set $E_y = 0$. We have:

$$\frac{\partial^2 E_x}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 E_x}{\partial t^2}, \tag{8}$$

It is easy to check that expressions of the form $E_x = f(z+ct)$ and $E_x = g(z-ct)$ are solutions to the differential Equation 8, so the general solution to

[4]

$$\nabla\times(\nabla\times\vec{u}) = \nabla(\nabla\cdot\vec{u}) - \nabla^2\vec{u}$$

[5] Generality is not lost since we are only looking at a light ray propagating in vacuum.

the second-order partial differential Equation 8 is $E_x = Af(z+ct)+Bg(z-ct)$, with A and B constants; you can check this by plugging in the functions f and g.

It is useful to introduce at this point the constant (called the wave-number) $k \equiv \frac{2\pi}{\lambda}$, where λ is the wavelength of the analysed light ray. Assuming that light is a wave, we use the dispersion relation $c = \frac{\omega}{k}$ to rewrite Equation 8:

$$\frac{\partial^2 E_x}{\partial z^2} = \left(\frac{\omega}{k}\right)^2 \frac{\partial^2 E_x}{\partial t^2}, \tag{9}$$

and the solutions become $E_x = f\left(\frac{2\pi}{\lambda}(z+ct)\right) = f(kz+\omega t)$ and $E_x = g\left(\frac{2\pi}{\lambda}(z-ct)\right) = g(kz-\omega t)$. Notice that the introduction of $2\pi/\lambda$ implies no loss of generality, the functions f and g are still functions of $z+ct$ and $z-ct$, respectively.

We can set up a Cauchy problem for the wave Equation 9. For simplicity, we drop the x subscript and we change to the subscript notation for partial derivatives ($\frac{\partial^2 E_x}{\partial z^2} = E_{zz}$):[6]

$$\begin{gathered} E_{tt} - \left(\frac{k}{\omega}\right)^2 E_{zz} = 0,\ z \in R,\ t > 0, \\ E(z,0) = h_1(z),\ E_t(z,0) = h_2(z),\ z \in R. \end{gathered} \tag{10}$$

The solution to these type of problems is given by d'Alembert's formula for continuous doubly differentiable h_1 and differentiable h_2:

$$E(z,t) = \frac{1}{2}(h_1(kz-\omega t) + h_1(kz+\omega t)) + \frac{k}{2\omega}\int_{kz-\omega t}^{kz+\omega t} h_2(s)ds. \tag{11}$$

We set $h_1(z) = cos(kz)$ and $h_2(z) = \pm\omega sin(kz)$, knowing that the electric field is oscillating, that the derivative of a cosine function is the negative of a sine function, and that $E_x = f(kz+\omega t)$ and $E_x = g(kz-\omega t)$ are solutions to 9:

[6]For more methods on solving the wave equation see Logan [3]. The book is available online for free for students at the University of Alberta library. Ask your TA for instructions on how to obtain it. The method used here was taken from Logan [3], Section 2.2.

$$
\begin{aligned}
E(z,t) &= \frac{1}{2}(cos(kz-\omega t)+cos(kz+\omega t)) \pm \frac{k}{2\omega}\int_{kz-\omega t}^{kz+\omega t} sin(s)ds \\
&= \frac{1}{2}(cos(z-ct)+cos(z+ct)) \pm sin(z)sin(ct) \\
&= \frac{1}{2}(cos(z-ct)+cos(z+ct)) \pm \frac{1}{2}(cos(z-ct)-cos(z+ct)) \\
&= cos(kz \pm \omega t).
\end{aligned}
$$

Switching *cos* and *sin* in the boundary conditions is just as valid, and gives as a result $E(z,t) = sin(kz \pm \omega t)$. This is due to the fact that *cos* and *sin* only differ by a $\pi/2$ phase.

We further assume that the light wave is propagating in the positive z direction. So we pick the solution:

$$E(z,t) = E_0 sin(kz - \omega t). \tag{12}$$

A similar solution is found for the magnetic field:

$$B(z,t) = B_0 sin(kz - \omega t). \tag{13}$$

Now recall Equation 5. Still focusing on the x component of the electric field ($E_x \to E$ and $B_y \to B$), we obtain $\frac{\partial E}{\partial z} = -\frac{\partial B}{\partial t}$, which implies that $E = cB$.

1.2 Irradiance

The irradiance of a light source is defined as the **time averaged** amount of power delivered per unit area (watts per meter squared in the SI of units). The expression of the irradiance can be derived from the Poynting vector, which tells the direction of power flow in an electromagnetic field:

$$\vec{S} = \epsilon_0 c^2 \vec{E} \times \vec{B} = \epsilon_0 c E^2 \hat{z}, \tag{14}$$

the last step coming from the fact that the electric and magnetic fields are perpendicular to each other and we are assuming that the light ray is moving in the positive z direction. So the magnitude of the pointing vector is, for a single component of the electric field (or for linearly polarized light):

$$S = \epsilon_0 c E_0^2 sin^2(kz - \omega t). \tag{15}$$

This can be seen from the following. Power has units of energy and per unit time. There is an expression for the energy density in an electric field [7]: $U = \epsilon_0 E^2$, which tells us that the energy associated with an electric field contained in a volume V is $U \times V$. This volume $V = Act$ is defined by some area A and the time it takes light to travel a distance $l = ct$. See Figure 2. Then the power (or energy per unit time) delivered in the direction of motion of the light wave is $\epsilon_0 E^2 Ac$. The Poynting vector magnitude tells the power delivered per unit area, so we divide this last expression by A and obtain Equation 15.

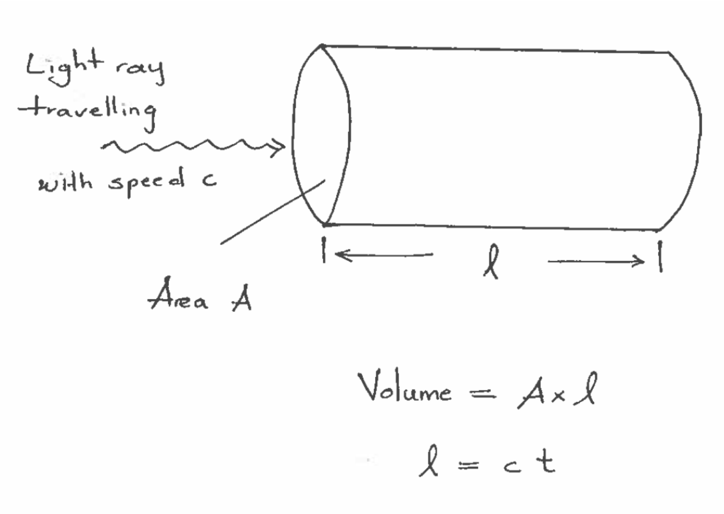


Figure 2: Adapted from Meldrum [4], Section 1b, Figure 1.

The irradiance is therefore (noticing that the time average of $sin^2(t)$ is 1/2):[8]

$$I_0 = \langle S \rangle = \frac{1}{2}\epsilon_0 c E_0^2. \tag{16}$$

The important thing to notice is here is the direct proportionality $I_0 \propto E_0^2$.

1.3 Malus's law

The way a (linear) polariser works is that it only lets one component of the electric field of the light wave to pass through it. Analysers are polarisers; the way they are called depends on the function they are serving. The term

[7] You will derive this expression in your electricity and magnetism courses.

[8] The expression here is for a single component of the electric field or a linearly polarized light ray.

polariser is used when the purpose is to give light a polarization, and the term analyser is used when the purpose is to look at light that has been previously manipulated, say through polarization. There are several light sources that are already polarized, like the screens of some computers, tv's and cellphones. Placing a linear analyser in front of these screens with the axis of transmission perpendicular to the axis of the original polarization of the light will completely block the light by absorption of its electric field energy; in this case the angle between the axis of transmission and the axis of oscillation of the originally polarized electric field is $\theta = 90°$. Notice that when the original light source is circularly polarized, or not polarized at all, there is no possible orientation of a linear analyser, or polariser, placed in front of the light source that will completely block all of the light. [9]

There is a way of determining how much of the original irradiance is transmitted when $\theta \neq 90°$. Suppose that the angle between the axis of oscillation of the electric field and the axis of transmission of the analyser is θ_0 (see Figure 3), then the component of the electric field that passes through the analyser is

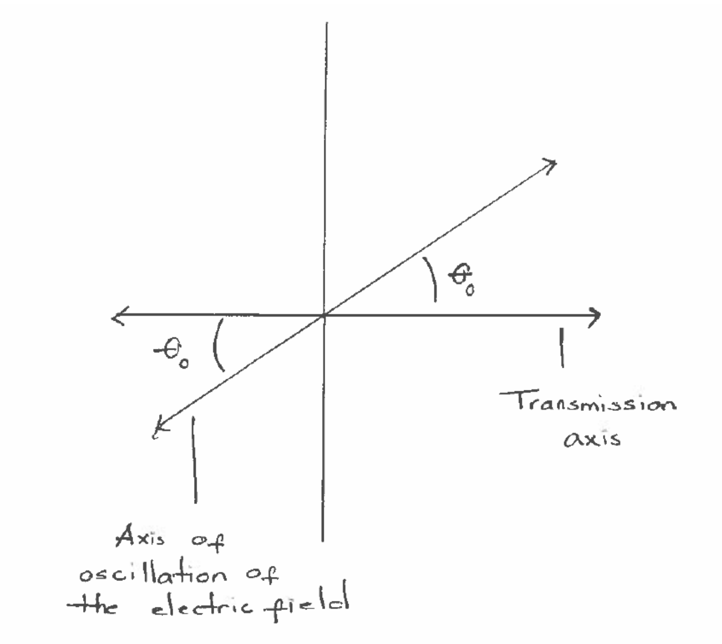


Figure 3: Angle between the axis of oscillation of the electric field of a light ray and the axis of transmission of a polariser as used in Malus's.

[9] There are special analysers with which light can be further manipulated. Half wave plates, for example, rotate the axis of polarization of light, and quarter wave plates turn circularly polarized light into linearly polarized light and vice-versa.

$$E_1 = E_0 cos(\theta). \tag{17}$$

We square this expression and remember that $I_1 \propto E_1^2$. We obtain Malus's law:

$$I_1 = I_0 cos^2(\theta_0). \tag{18}$$

2 Experimental details

In this experiment you will use two polarisers/analysers and a linear diffraction grating. You will provide explanations for several phenomena that you observe when placing the equipment in certain positions.

1) Setup 1:

a. Find a source of light that is linearly polarized. How do you know that the source you chose emitted light that is indeed linearly polarized?

b. Place one polariser in front of the source of linearly polarized light in orientation in which no light is transmitted. Then place a second polarized between the source of light and the first analyser. See Figure 4. Rotate the second polariser and describe what you observe. Which explanation would you give to this phenomenon? Which angle θ_0 between the axes of transmission between the second and first polarisers would you expect to give a maximum transmission of electric field to your eyes? Show it mathematically first and then verify it experimentally.

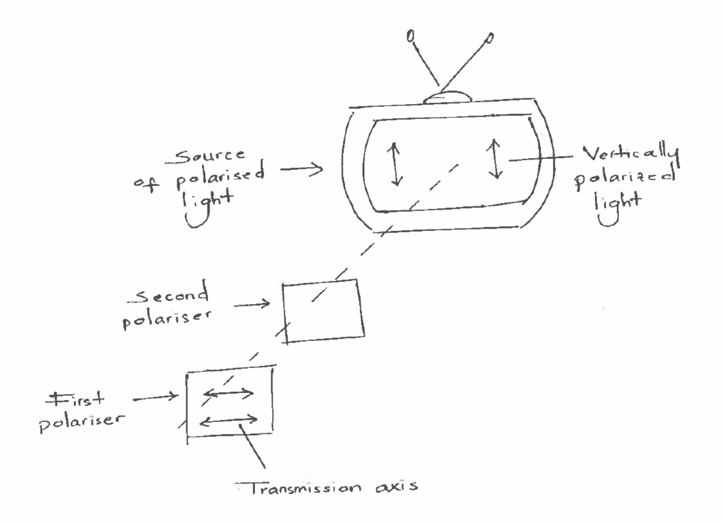


Figure 4: Experimental setup 1.

2) Setup 2:

a. Place, again, one polarizer in front of the source of linearly polarized light in the orientation in which no light is transmitted. Then you will place (do not place it yet) the linear diffraction grating in between the light source and the analyser, so that the linear diffraction grating replaces the second polariser in Figure 4. What do you expect to observe?

b. Place the linear diffraction grating in between the light source and the analyser and rotate it. What do you see? Did you expect this to happen? Why does it happen?

3) Setup 3: place one polariser in front of your eyes, directly to a general light source or anywhere inside an illuminated room. What do you expect will happen? Corroborate your expectations by using the second analyser.

Added:

Using a light meter application for smart-phones or a light meter incorporated in a camera, check Malus's law by plotting I vs. $cos^2(\theta)$. One of the two polarisers included in the lab kit are needed, plus a polarized source of light (computer, cellphone, tv screens [not Apple]). One polariser is placed in from of the (polarised) screen in such a way that a maximum amount of light is transmitted through it. A reading of the light intensity in this case is performed. This is I_0 in $I = I_0 cos^2(\theta)$. Subsequently, readings of I are taken as the polariser is rotated in front of the screen. The slope of the graph I vs. $cos^2(\theta)$ should give I_0.

A method for determining I for a certain angle θ. Download an image of a protractor in your computer and extend it over the entire screen. Then tape the polariser in front of a cellphone camera. Look at the image of the protractor through the light meter application in the cellphone. To get I for a certain θ align the edges of the cellphone screen with the different lines marking the angles in the protractor.

References

[1] The Brigham Young University Optics Lab. Polarization and Polarimetric Testing. Retrieved on Aug. 27, 2020 from : https://www.photonics.byu.edu/polarization.phtml.

[2] Griffiths, D. (2017). Introduction to electrodynamics. 4th ed. Cambridge University Press.

[3] Logan, J. D. (2015). Applied partial differential equations. 3rd ed. Springer.

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