Kater pendulum

Phys 294/5 Experiment 2

Student's manual

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1 Theory

One of the most important physical constants is the acceleration due to gravity. It defines the unit of force in mechanics, and consequently, underlines all mechanical measurements. In 1817, Kater (1777-1835), following the suggestion of Bessel (1784-1846), developed a reversible pendulum that made possible the accurate measurement of the acceleration due to gravity. A reversible pendulum is one that can be swung from either of two pivot points. When the mass distribution of the pendulum is adjusted so that the periods are the same form either pivot, then this period is the same as a simple pendulum having a length equal to the distance between the pivots. While this may seem to be a simple property, it has great power in determining the value of "g" to high accuracy.

The Kater pendulum consists of a long bar in which weights can be distributed at different positions. See Figure 1. The holes on the bar allow for bolts to be attached to it and create different distributions of masses. Kater developed this form of pendulum with which he was able to adjust the period with the pendulum swinging from the top or bottom pivot point. Using pendulums of this type, the National Bureau of Standards determined the value of "g" to be 980.080 $\pm$ .003 cm/sec^2 in Washington in 1936. Similar precise measurements were made in England at Teddington and at Potsdam in Germany at about the same time.

To start with, let us take a look at a compound pendulum (see Figure **??**). The pivot of the pendulum is at O, and the center of gravity (also called the center of mass), at G. If this pendulum is moved from rest, the restoring couple is $Mgh_1 sin(\theta)$. Upon release, the motion of the pendulum is then described by, according to Newton's second law for rotational motion

$$I_0 \frac{d^2\theta}{dt^2} = -Mgh_1 sin(\theta), \tag{1}$$

where I_0 is the moment of inertia of the pendulum about the axis passing through O and perpendicular to the direction of the acceleration due to gravity, $\vec{g}$; M is the mass of the compound pendulum, and h_1 is the distance between the point O and the axis passing through G and perpendicular to $\vec{g}$. θ is the displacement angle from the equilibrium position of the pendulum. If θ is very small, then $sin(\theta) \approx \theta$, and we obtain a second-order-linear differential equation:

$$\frac{d^2\theta}{dt^2} = -\frac{Mgh_1}{I_0}\theta. \tag{2}$$

Curiously enough, it results that the term

$$T \equiv 2\pi\sqrt{\frac{I_0}{Mgh_1}} \tag{3}$$

is the period of (small) oscillations of the pendulum about the pivot O.

If we make the pendulum swing about its center of gravity, then its moment of inertia is given by I_G. However, if we make it swing from a point different from its center of gravity, then its resistance to undergo rotational motion changes, and thus its moment of inertia is affected. The new moment if inertia for the compound pendulum swinging about the point O a distance h_1 from its center of gravity is given by Steiner's theorem (better know as the parallel axis theorem):

$$I_0 = I_G + Mh_1^2. \tag{4}$$

I_G itself is given by the formula

$$Mk^2, \tag{5}$$

where k is the radius of gyration of the pendulum about an axis through G and perpendicular to $\vec{g}$. Then one can express the period of the pendulum about the pivot O as

$$T = 2\pi\sqrt{\frac{h_1 + k^2/h_1}{g}}. \tag{6}$$

Now consider a third pivot point, S, such that the distance between this point S and G is equal to $\overline{GS} \equiv k^2/h_1$. The period then becomes (noticing that $\overline{OG} = h_1$):

$$T = 2\pi\sqrt{\frac{\overline{OG} + \overline{GS}}{g}} = 2\pi\sqrt{\frac{\overline{OS}}{g}}, \tag{7}$$

where $\overline{OS}$ is the distance between the points O and S. Defining $h_2 \equiv \overline{GS}$, pivoting the pendulum at S gives the following period:

$$T = 2\pi\sqrt{\frac{I_S}{Mgh_2}}, \tag{8}$$

where I_S is the moment of inertia of the compound pendulum about the point S.

By the parallel axis theorem, and also looking back at Equation 5 and $\overline{GS} = h_2 \equiv k^2/h_1$, one can see that

$$I_S = I_G + Mh_2^2 = Mk^2 + M(k^2/h_1)^2. \tag{9}$$

Then, looking back at Equation 8 and the definition of h_2 just mentioned, the period for the pendulum pivoted at S becomes (work out the simplification yourself; it helps to understand the derivation better)

$$T = 2\pi\sqrt{\frac{h_1 + k^2/h_1}{g}}. \tag{10}$$

But $\overline{OS} = h_1 + k^2/h_1$, so the period is the same whichever pivot is used, and it is the same as a simple pendulum of length $\overline{OS}$! Later in this experiment, you will learn how to determine a value for $\overline{OS}$ such that the compound (Kater) pendulum can be reduced to a simple pendulum. From this you will determine the acceleration due to gravity.

2 Experimental details and procedure

The necessary materials are shown in Figure 1. The pendulum is to be hung from the provide clamp (Figure 2). Make sure to secure well the clamp unto the table. It is very important that the clamp does not move at any moment;

the slightest movement of it would give an incorrect period of oscillation in your measurements.

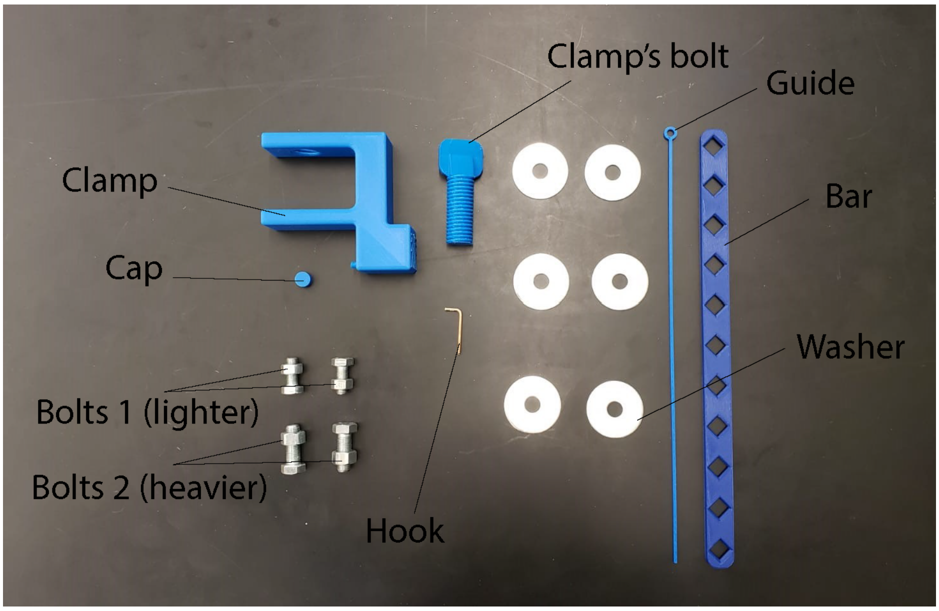
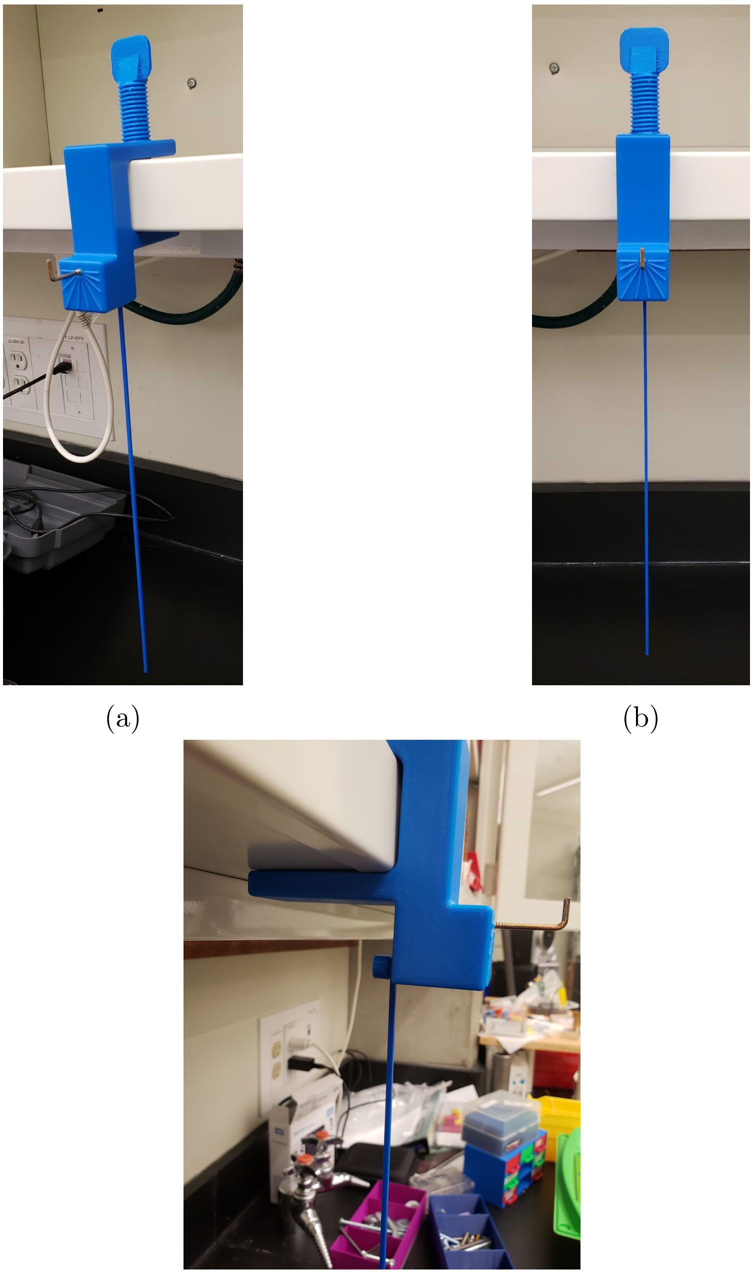


Figure 1: The assembled clamp is shown in Figure 2. The guide is placed at the back of the clamp, so that the student can see when the swinging pendulum crosses the equilibrium line (i.e. the along along which the pendulum rests when it is now swinging); the guide is secured in place, with some movement allowance (so that gravity does the straightening) by a small cap (see Figure 2c, also shown here).



(a) (b) (c)

Figure 2: Assembled clamp.

The pendulum bar itself is made of PLA plastic and has 11 holes. Be careful when handling it, specially when you attach weights on it, because it can snap quite easily. Figure 1 shows a picture of the bar. Other necessary information is provided in Table 1. You will also need moments of inertia for different shapes; please refer to Appendix **??**.

Item	mass [g]	mass unc [g]
Bar	15	$\pm$ 1
Washer	12.17	$\pm$ 0.01
Bolt 1	15.77	$\pm$ 0.01
Bolt 2	25.94	$\pm$ 0.01

Table 1

It is important to mention that at the place where you hang the pendulum, the clamp has been marked with different angles (Figure 3): 10, 30, 50, 70, and 90 degrees. You will use this marking to make sure that you are always making the pendulum swing with the same amplitude.

Figure 3: Protractor printed on the clamp. From the center to either side, each line signals 10, 30, 50, 70 and 90 degrees from the pendulum's equilibrium position.

2.1 Moment of inertia of the rod and measurement of g

Work with the bar alone. Starting by hanging the bar from one of the outermost holes (it does not matter which side you choose), you will measure its period of oscillation. To do this, you will displace the pendulum from its equilibrium position by about 20 degrees. This is quite a big number, but because the metal hook from which the pendulum swings is thick, friction makes the amplitude of oscillation decrease quite fast; for this same reason you will be measuring oscillation periods of 5 cycles (or less when you get to the innermost holes; do not go below 2 cycles). To determine the value of the period of oscillation and its uncertainty, use the techniques of Experiment 1: Gaussian statistics.

You will take at least 25 measurements of the period of oscillation of the pendulum for each hole. You will likely start encountering problems on the fourth outermost hole. You can stop taking measurements at this point if you want.

To measure the value of the acceleration due to gravity, you will work with Equations 3 & 4, which, for the the purposes of this section can be rewritten, respectively, as

$$I = m_R a g \frac{T^2}{4\pi^2}, \tag{11}$$

$$I = I_R + m_R a^2, \tag{12}$$

where I_R is the moment of inertia of the bar about its center of mass, m_R is the mass of the bar, a is the distance from the center of the bar to the pivot point, I is the moment of inertia of the about that pivot point, T is the corresponding period, and g is the acceleration due to gravity.

If you divide both Equations 11 & 12 by g, then from measurements you know their left-hand side. Take measurements of the a values, the distances from the center of the pendulum to the point of contact between the pendulum and the metal hook (this is the outermost edge of the rhombus-shaped holes; see Figure 4). If you plot the I/g vs a^2, then you obtain a straight line whose slope is given by m_R/g and its intercept with the vertical axis, by I_R/g. You know m_R, so from the slope you can determine the value of the acceleration due to gravity, and the from this value of g, you can determine the value of I_R experimentally, and compare with the value predicted by the formula for the moment of inertia for a rectangular parallelepiped (see Appendix **??**; the error introduced by ignoring the holes in the bar is small).

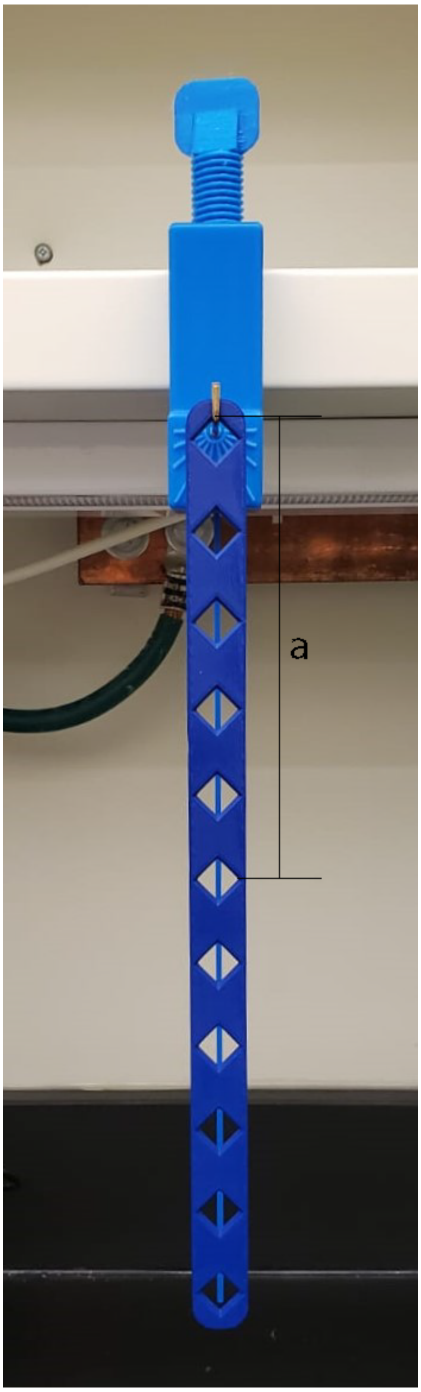
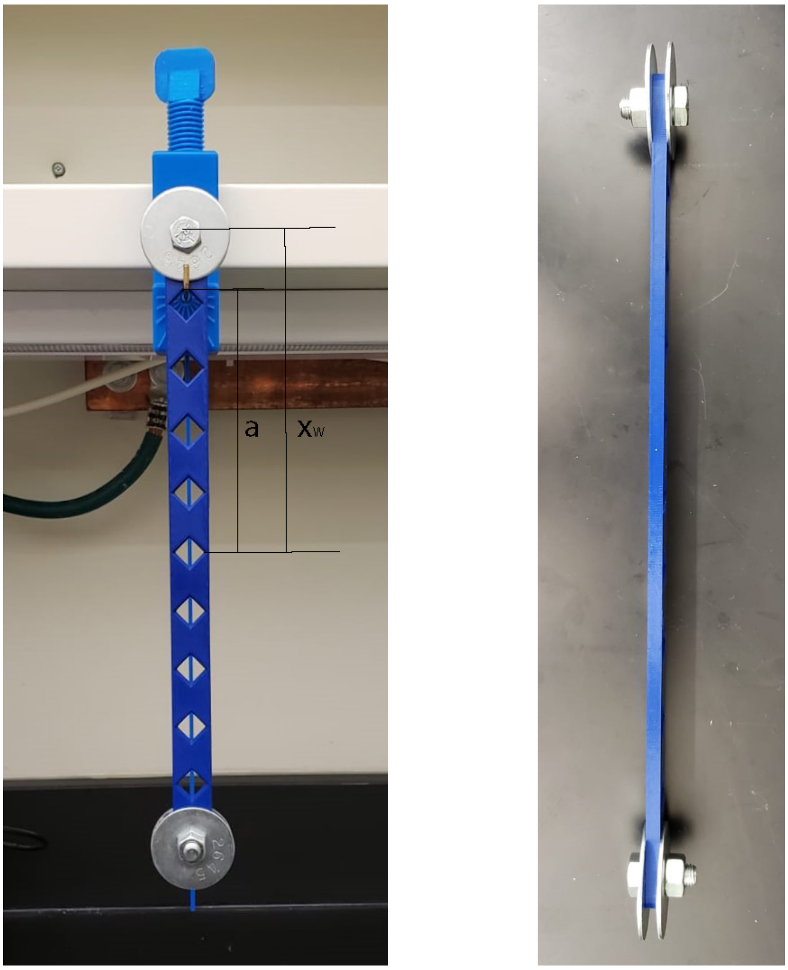


Figure 4

2.2 Symmetrical mass distribution and corroboration of consistency



(a) (b)

Figure 5

This part of the experiment is very similar to the previous one, with the difference that you will attach two washers and a bolt (one of the lighter ones) at each end of the bar, at the outermost holes. See Figure **??**. The you will measure the period of the pendulum for about 5 to 2 cycles, as needed. Because the center of gravity of the bar is still located at its center, Equation 11 is still valid, but with the change $m_R \to M$, the total mass of the pendulum system

$$I = Mag\frac{T^2}{4\pi^2}. \tag{13}$$

Using the parallel axis theorem, one finds that I is given by

$$I = I_G + Ma^2, \tag{14}$$

where I_G is the moment of inertia of the **pendulum system** (i.e. the bad and the attached weights) about its center of gravity. I_G can be determined by calculating the individual moments of inertia of the components of the pendulum system about the center of the bar:

$$I_G = I_R + 4(I_W + m_W h_W^2) + 2m_{B1}h_{B1}^2, \tag{15}$$

where I_R is, as before, the moment of inertia of the bar about its center of mass, I_W is the moment of inertia of the washers (see Appendix **??**), m_W is the mass of the washers $h_W = h_{B1}$ are the distances between the center of the rod and the weights, and m_{B1} is the mass of the lighter bolts. $m_{B1}h_{B1}^2$ is the moment of inertia of a point-like mass a distance h_{B1} from the center of the bar; this is the moment of inertia of a bolt. The factors of 4 and 2 come from the fact that there are, in total, four washers and two bolts attached, symmetrical, to the bar.

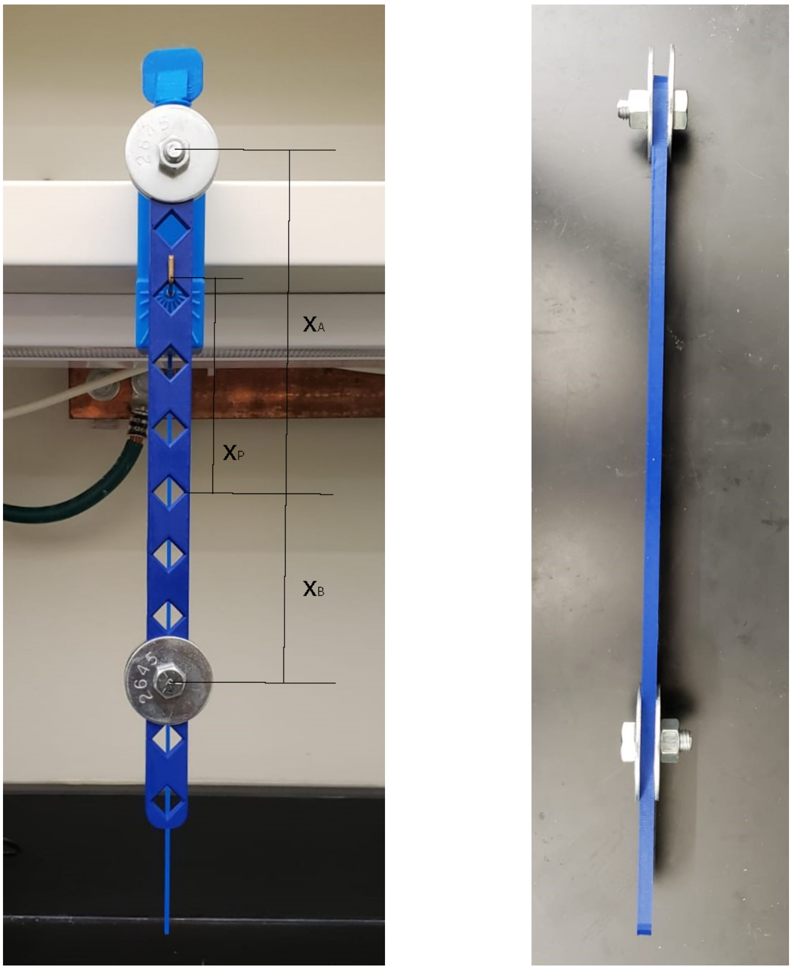
Using the linearized Equation 14, you can determine the value of I_G. You know the left-hand side of Equation 14 from measurements of the period of the pendulum at different pivot points and the value of g determined in Section 2.1. Compare the value of I_G obtained from the linearization technique with the value predicted by Equation 15; in this equation, use the value I_R determined in Section 2.1.

2.3 Arbitrary mass distribution (matching experimental results with theoretical predictions)

Choose an asymmetrical mass distribution for your pendulum and a pivot point. For example, you can place one weight [1] on one of the outermost holes in the bar; the pivot point could be the third outermost point on this same side of the bar; the other weight could be located in the third outermost hole in the opposite side of the bar. See Figure 6. With an arbitrary mass distribution, the center of gravity of the pendulum system is not anymore located at the center of the bar. To find the new center of gravity, define a coordinate system, with an axis parallel to the length of the bar. Set the

[1] For this section, by weight, I refer to the lightest bolt and two washers

center of the bar at the origin of the coordinate system. Points above the center of the bar will be positive, and **points below the center, negative**. Let x_p be the distance from the center of the bar to the pivot point, x_1 be the position of the weights (lightest bolt and two washers) above the origin and x_2 the distance of the weights (lightest bold and two washers) below the origin.



(a) (b)

Figure 6

The position of the center of mass in our coordinate system is given by

$$x_G = \frac{m_A x_A + m_B x_B}{m_A + m_B}, \tag{16}$$

where m_A and x_A are the mass of the weight A and the position between the weight A and the center of the bar, respectively; m_B and $x_B < 0$ are the mass of the weight B and the position between the weight B and the center of the bar, respectively. $x_R =$ is just the position of the center of the bar in the coordinate system. Using the parallel axis theorem, you can determine the pendulum's moment of inertia about the chosen pivot point:

$$I = I_R + m_R x_p^2 + 2I_W + m_A(x_p - x_A)^2 + 2I_W + m_B(x_p - x_B)^2, \qquad (17)$$

where I_R and m_R are the moment of inertia of the bar about its center and the mass of the bar, respectively; I_W is the moment of inertia of a washer, $m_A = m_B$ are the mass of the weight (two washers plus one of the lighter bolts); and as mentioned before, x_p, x_A, x_B, are the position of the pivot point, weight A and weight B in the coordinate system with the center of the bar at the origin.

Then using Equation 3, rewritten more appropriately as

$$T = 2\pi\sqrt{\frac{I}{Mag}}, \qquad (18)$$

where $a = x_p - x_G$ is the distance between the pivot point and the center of mass of the pendulum system, you can determine a theoretical prediction of the period of the pendulum system configuration that you chose. Before determining this value, perform the experiment by measuring the period of oscillation of your pendulum system configuration. Use a statistical method (i.e. take at least 25 measurements of the period and proceed with a statistical determination of the experimental value of the period). Then calculate T from Equation 18; doing the experiment in the reverse order could bias your measurements. Does the experimental value agree with the theoretical prediction?

2.4 Bessel pendulum and determination of g

Here we explore another method for determining the acceleration due to gravity. This time, place four washers and one of the lighter bolts at one end of the bar, and two washers and one of the lighter bolts at the other end. You will place the two heavier bolts at equal distances from the center of the bar. There are three symmetrical configurations in which you can do this if the bar hangs from one of the holes immediately next to on of the last one, where the weight is attached. See Figure **??**. For each symmetrical configuration of the heavier bolts, you will measure the period of the pendulum when the heavier weight is on top of the pendulum system, and then when the lighter weight is on top.

(a) (b)

Figure 7

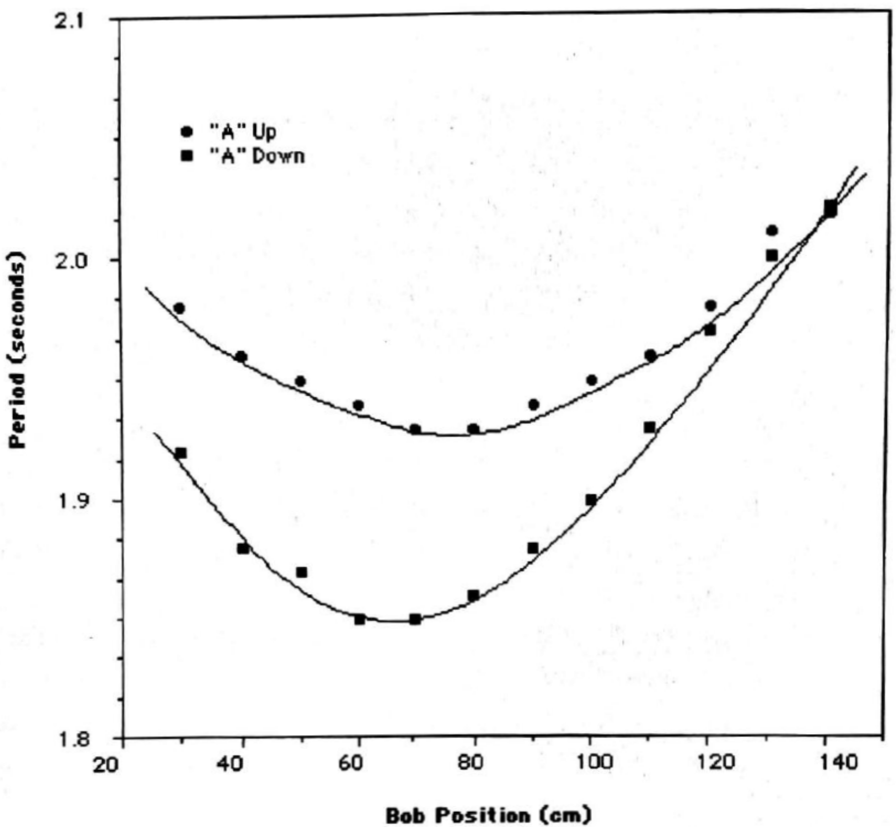


Figure 8: The point of interception between the two graphs gives the period of a simple pendulum that obeys Equation 10. Taken from [2], Figure 3.

When plotting the two periods obtained from reversing the pendulum at the different symmetrical configurations of the heavier bolts versus the distance from these heavier bolts to the center of the bar, you should obtain a graph that looks like the one shown in Figure 8. In order to find the point of intersection of the two lines, plot the data in excel and add a trendline to each data series. You can then find their common x value, call it x_0. Now you can determine the value of the acceleration due to gravity. The heavier bolts placed at this value of x, make the compound pendulum behave like a simple pendulum, and so, you can use Equation 10 to determine g:

$$g = L\left(\frac{2\pi}{T}\right)^2, \tag{19}$$

where T is determined by the point of intersection of the two lines mentioned above and L is the distance between the outermost edges of the second outermost holes in the bar (see Figure **??**). It happens that for Kater's pendulum

$L = \left|\overline{OS}\right|$, with $\overline{OS}$ as defined in the Theory section! What is the radius of gyration of your pendulum? (Hint: $k = \sqrt{[L - (x_P - x_G)](x_P - x_G)}$, with x_G as defined in Section 2.3. Caution: because the bolts are located symmetrically about the center of the bar, they do not contribute to x_G.)

3 Notes [2]

'[...] The greatest variation of "g" is ude to the latitude of the location. This latitude variation is due to two causes. The first is the difference in the radius of the earth with latitude, the earth being fatter at the equator. Since the gravitational force is inversely proportional to the square of the distance between the centers of mass, gravity is smaller at the equator. The second cause is the centrifugal force of the earth's rotation, which is a maximum at the equation and zero at the poles. Both effects are in the same direction, so "g" increases with latitude, being smallest at the equator. The Handbook of the American Institute of Physics gives the value of the acceleration as $g = 978.0490(1 + 0.0052884 sin^2(\phi) - 0.0000059 sin^2(2\phi)$, where ϕ is the latitude of the measurement point. The latitude of the location can be found accurately enough from a topographic map area. Local effects may cause small differences from the values predicted by this equation but to the level of accuracy to be expected from a normal laboratory experiment, this equation will predict the local value of "g" correctly.'

References

[1] Manual for physical pendulum/Bessel pendulum. Frederiksen. Søren Frederiksen.

[2] Physics Laboratory Manual: Phys 292/294/295/297. University of Alberta, Department of Physics. (2006). Coordinator: David Leonard.

A Moments of inertia [1]

- Rectangle with dimensions $b \times d$, mass m:

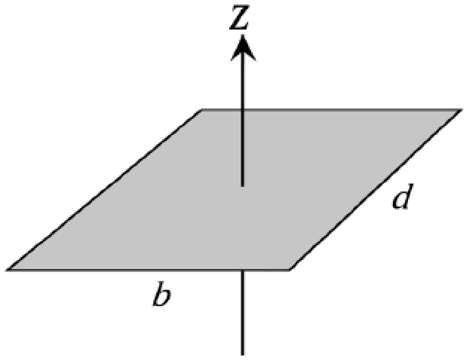


Figure 9: Taken from [1], p. 2.

$$I = \frac{m}{12}(b^2 + d^2) \tag{20}$$

- Cylinder with inner and outer radii r_1 and r_2, respectively, and mass m:

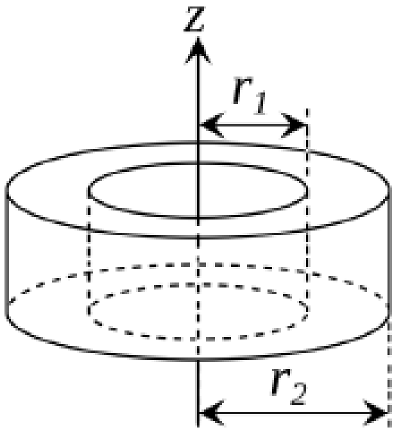


Figure 10: Taken from [1], p. 2.

$$I = \frac{m}{2}(r_1^2 + r_2^2) \tag{21}$$

- Point mass m to undergo rotational motion about the circumference of a circle (or surface of a sphere) of radius r (approximate the bolts to a point mass):

$$I = mr^2 \tag{22}$$