Millikan experiment for detection of fractional charges via horizontal deflection due to electric field in vertical fall - report 1

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I. DESCRIPTION

This study explores the possibility of detecting fractional charges through a modified Millikan experiment. The experiment consists in dropping identical 25-micron diameter glass spheres between an electric field of 500-1000 volts/mm produced by 10 meter long vertically oriented parallel plate capacitors. The charge of a sphere would be calculated indirectly by the amount of horizontal deflection it achieves by the end of the 10 meter fall. The landing plane would have tiny boxes placed at specific landing locations for each charge as predicted by a computer model. See Figure 1. The effects of Brownian motion are taken into consideration. The main advantage of this experiment over the classical and other Millikan experiments is that it would allow the measurement of the charge of a large amount of tiny glass spheres in a short period of time. The results shown here were generated with computer simulations.

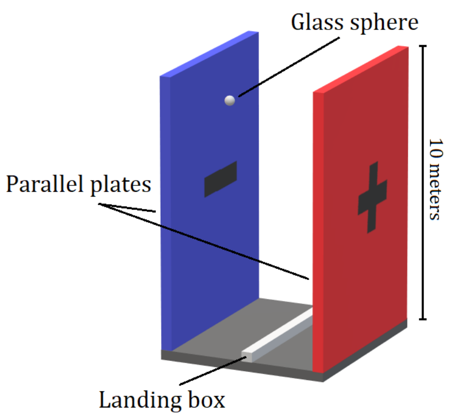

FIG. 1: Experiment setup (not to scale).

II. METHOD

A study of the possibility of performing the experiment via computer simulations was performed. The study focused on two parts: (1) the effects of air pressure for maximizing the fall time and horizontal displacements of the glass spheres, (2) the deviations from the predicted path due to Brownian motion. The equations used are the following. (The full derivations can be found in Appendix B.)

The equation of motion for the vertical displacement y is given by:

$$\frac{d^2y}{dt^2} = -K\frac{dy}{dt} - \frac{F_{b_y}}{m_{eff}} + g, \tag{1}$$

where y is the vertical displacement from zero, t is time, and g is the acceleration due to gravity. $F_{b_y} = \rho V g$, where $V \approx$ *volume of the sphere* is the displaced fluid and ρ is the air density, is the buoyancy force. $K = \frac{9\eta}{2a^2(\sigma-\rho)}k$ is given by Stokes' law [1]; η is the air viscosity and $k = (1+\frac{B}{aP})^{-1}$, where P is the pressure and $B = 0.100$, is a correction factor provided by McKeehan [1]. $m_{eff} = V(\sigma-\rho)$, where σ is the density of glass, is the effective mass of the sphere. The solution is:

$$y(t) = \frac{g}{K}[t + \frac{1}{K}(e^{-Kt}-1)](1-\frac{\rho}{\sigma-\rho}), \tag{2}$$

where a is the radius of the sphere.

For times long enough, the exponential term is negligible, and the sphere reaches a terminal velocity of:

$$v_{y-terminal} = \frac{g}{K}(1-\frac{\rho}{\sigma-\rho}) \tag{3}$$

The equation of motion for the horizontal displacement x is:

$$\frac{d^2x}{dt} = -K\frac{dx}{dt} - \frac{F_{b_x}}{m_{eff}} + \frac{qE}{m_{eff}}, \tag{4}$$

where q is the charge of the sphere and E is the electric field. The solution is:

$$x(t) = \frac{qE}{V(\sigma-\rho)K}[t + \frac{1}{K}(e^{-Kt}-1)](1-\frac{\rho}{\sigma-\rho}). \tag{5}$$

Using Einstein's theory of Brownian motion as presented by Millikan [2], one derives the equation that describes the contribution to the x and y displacements of random motion:

$$\overline{\Delta y} = \sqrt{\frac{4}{\pi K}[\frac{RT}{N} - (F_{b_y}-F_g)(\tau - \frac{m_{eff}}{K})\sqrt{\frac{RT}{Nm_{eff}}}]\tau}, \tag{6}$$

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$$\overline{\Delta x} = \sqrt{\frac{4}{\pi K}[\frac{RT}{N} - (F_{b_x} - qE)(\tau - \frac{m_{eff}}{K})\sqrt{\frac{RT}{Nm_{eff}}}]\tau}, \quad (7)$$

where R is the universal gas constant per gram molecule, T, the temperature, and N, the number of air molecules in one gram. τ is the time between collisions of the air molecules against the glass sphere.

An approximation of the fall time can be calculated from the following equation:

$$\frac{10K}{g}(1 - \frac{\rho}{\sigma - \rho})^{-1} - \frac{1}{K}(e^{-Kt_{fall}} - 1) - t_{fall} = 0. \quad (8)$$

III. EQUIPMENT

- Very large amount of 25-micron diameter glass spheres
- Two 10-meter long plate capacitors for generating 500 and 1000 volts/mm electric fields
- Tiny boxes for capturing the landing spheres. The boxes will be placed at locations specified by the computer simulation

IV. RESULTS

The results that follow were obtained using the parameters listed in Appendix A, unless otherwise stated.

Figure 2 shows the results for the maximal horizontal displacement of a sphere of charge $q = e$ in electric fields of 500 and 1000 volts/mm for different fall times and pressures. The fall time as a function of pressure is also shown. Here I assumed that the viscosity of the air is approximately constant at all pressures.

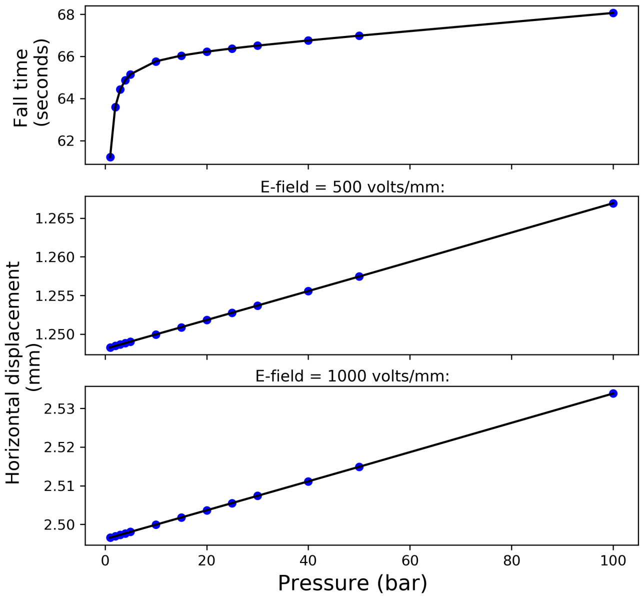

FIG. 2: Fall time and horizontal displacements of a sphere of charge $q = e$ in horizontal electric field.

As the air pressure is increased, both the fall time and the horizontal displacements increase. The fall time is independent of the charge for a horizontal electric field. The fall trajectory of a sphere of charge $q = 1.0e$ in an electric field of 1000 volts/mm and air pressure of 10 bar is shown in Figure 3.

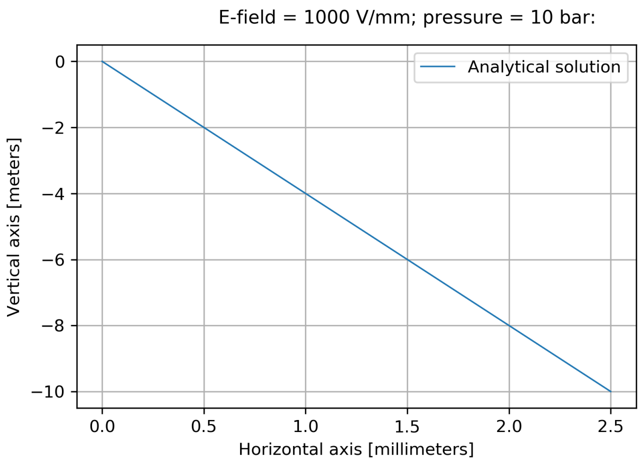

FIG. 3: Fall trajectory of sphere of charge $q = 1.0e$.

Tables I & II show the maximal horizontal displacements for identical spheres of different charges falling at different pressures in different horizontal electric fields. The expected deviation from the unperturbed path landing position due to Brownian motion is shown for 40 and 400 collisions per second.

It seems that as the number of collisions per second increase, the deviation from unperturbed path landing position due to Brownian motion decreases. One would expect that the real number of collisions per second is larger than the numbers used here by several factors of ten, and so that the deviation from the unperturbed path would get smaller.

Electric filed E = 500 volts/mm

Charge	Pressure [bar]	Horizontal displacement [mm]	[diameters]	Brownian motion deflection [mm] 40 collisions/sec	400 collisions/sec
1.0e	1.00	1.2482903262	49.93	3e-09	5e-10
	2.00	1.2484758106	49.94	‖	
	5.00	1.2490325944	49.96	‖	
	10.0	1.2499616719	50.00	‖	
	20.0	1.2518239796	50.07	‖	
	50.0	1.2574443479	50.30	‖	
	100	1.2669246250	50.68	‖	
0.1e	1.00	0.1248290326	4.993	2e-09	5e-10
	2.00	0.1248475811	4.995	‖	
	5.00	0.1249032594	4.996	‖	
	10.0	0.1249961672	5.000	‖	
	20.0	0.1251823980	5.007	‖	
	50.0	0.1257444348	5.030	‖	
	100	0.1266924625	5.068	‖	
0.01e	1.00	0.0124829033	0.4993	1e-09	5e-10
	2.00	0.0124847581	0.4994	1e-09	4e-10
	5.00	0.0124903259	0.4996	‖	
	10.0	0.0124996167	0.5000	‖	
	20.0	0.0125182398	0.5007	‖	
	50.0	0.0125744435	0.5030	‖	
	100	0.0126692462	0.5068	‖	
0.001e	1.00	0.0012482903	0.04993	1e-09	5e-10
	2.00	0.0012484758	0.04994	1e-09	4e-10
	5.00	0.0012490326	0.04996	‖	
	10.0	0.0012499617	0.05000	‖	
	20.0	0.0012518240	0.05007	‖	
	50.0	0.0012574443	0.05030	‖	
	100	0.0012669246	0.05068	‖	

TABLE I: Maximal horizontal displacements in millimeters and sphere diameters for spheres of 25-micron diameters of different charges falling in air at different pressures and in an electric field of 500 volts/mm. Deflection from unperturbed path landing position due to Brownian motion shown for 40 and 400 collisions per second of the glass sphere against particular air molecules.

Electric filed E = 1000 volts/mm

Charge	Pressure [bar]	Horizontal displacement [mm]	[diameters]	Brownian motion deflection [mm] 40 collisions/sec	400 collisions/sec
1.0e	1.00	2.4965806525	99.86	4e-09	6e-10
	2.00	2.4969516211	99.88	‖	
	5.00	2.4980651888	99.92	‖	
	10.0	2.4999233439	100.0	‖	
	20.0	2.5036479593	100.1	‖	
	50.0	2.5148886957	100.6	‖	
	100	2.5338492499	101.4	‖	
0.1e	1.00	0.2496580652	9.986	4e-09	5e-10
	2.00	0.2496951621	9.988	‖	
	5.00	0.2498065189	9.992	‖	
	10.0	0.2499923344	10.00	‖	
	20.0	0.2503647959	10.01	‖	
	50.0	0.2514888696	10.06	‖	
	100	0.2533849250	10.14	‖	
0.01e	1.00	0.0249658065	0.9986	2e-09	5e-10
	2.00	0.0249695162	0.9988	1e-09	4e-10
	5.00	0.0249806519	0.9992	‖	
	10.0	0.0249992334	1.000	‖	
	20.0	0.0250364796	1.001	‖	
	50.0	0.0251488870	1.006	‖	
	100	0.0253384925	1.014	‖	
0.001e	1.00	0.0024965807	0.09986	1e-09	5e-10
	2.00	0.0024969516	0.09988	1e-09	4e-10
	5.00	0.0024980652	0.09992	‖	
	10.0	0.0024999233	0.1000	‖	
	20.0	0.0025036480	0.1001	‖	
	50.0	0.0025148887	0.1006	‖	
	100	0.0025338492	0.1014	‖	

TABLE II: Maximal horizontal displacements in millimeters and sphere diameters for spheres of 25-micron diameters of different charges falling in air at different pressures and in an electric field of 1000 volts/mm. Deflection from unperturbed path landing position due to Brownian motion shown for 40 and 400 collisions per second of the glass sphere against particular air molecules.

Further exploration:

Designing a setup that allows for the generation of a diagonally-oriented electric field of magnitude 500 and 1000 volts/mm would not increase the horizontal displacement of the spheres. With the presence of a vertical component of electric field, the acceleration due to gravity becomes, effectively, $g_{eff} = g - (qE_y)/m_{eff} \approx 9.8$ m/s^2 for all charges q and at practically any angle from the horizontal at which the electric field E is generated. See Figure 4.

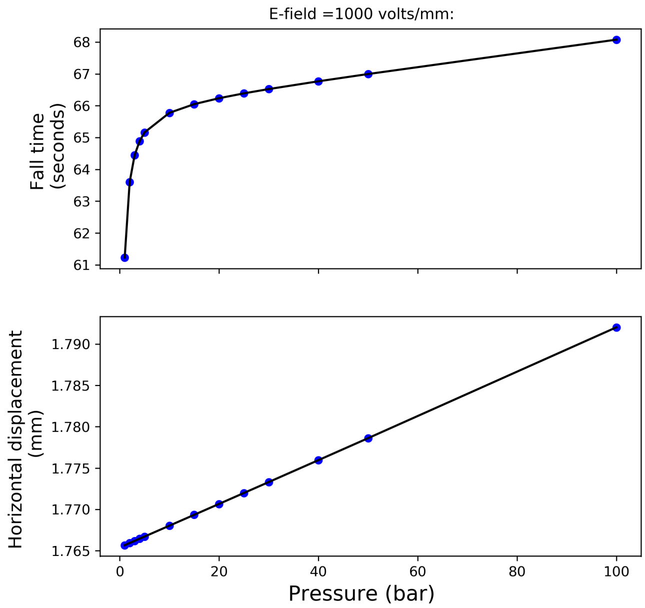

FIG. 4: Fall time and horizontal displacements of a sphere of charge $q = e$ in diagonally-oriented electric field 45° from the horizontal.

Allowing the sphere to fall in vacuum does not increase the landing position significantly, plus there exists the possibility of the sphere shattering upon landing with a speed of $\approx$ 14 m/s. See Figure 5.

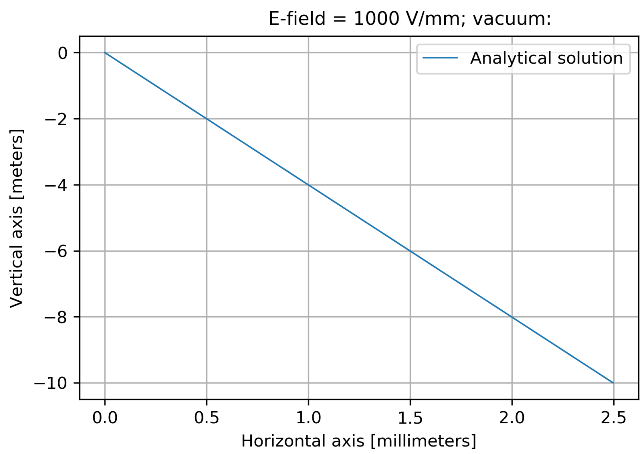

FIG. 5: Fall trajectory of sphere of charge $q = 1.0e$ in vacuum.

V. CONCLUSIONS

One can see in Tables I & II that the horizontal displacements are larger than the diameter of the sphere for the charges $q = [1.0e, 0.1e]$, and smaller for the rest. One could only hope to detect displacements corresponding to an elementary charge and of a tenth of a charge. A diagonally-oriented electric field does not improve the results, it only makes them less desirable and it complicates the setup of the experiment.

The displacements obtain when letting the sphere fall in vacuum are similar to the ones one obtain by letting it fall in air at a pressure of 10 bar. Compare Figures 3 & 5. The main disadvantage of vacuum is that the glass spheres could shatter upon reaching the ground with a speed of $\approx$ 50 km/h.

A further exploration could involve the measurement of the charges of the spheres by letting them fall through a vertically-oriented solenoid. In this case, however, the charges would be determined from the fall time, which would vary according to the amount of spiralling the sphere would suffer throughout the fall.

VI. APPENDIX

A.

Parameters list:

Electric field: $E = [500, 1000]$ volts/mm
Acceleration due to gravity: $g = 9.806$ m/s^2
Effective acceleration due to gravity (used in exploration of diagonally-oriented E-field): $g_{eff} = g - (qE_y)/m_{eff}$
Charge of the electron: 1.602^{-19} C
Fall height: $h = 10$ meters

Air:

Pressure: $P = [1, 2, 5, 10, 20, 50, 100]$ bar
Individual gas constant: $R_{air} = 287.15$ J/(kg mol)
Temperature: $T = 20°$ C
Density: $\rho = P/(R_{air}T)$ kg/m^3
(Dynamic) viscosity: $\eta = 1.8 \times 10^{-5}$

Glass sphere:

Diameter: $a = 25$ microns
Density (of glass): $\sigma = 8000$ kg/m^3
Mass: $m = 6.544 \times 10{-}11$ kg
Effective mass: $m_{eff} = volume \cdot (\sigma - \rho)$
Charge: $q = [1.0e, 0.1e, 0.01e, 0.001e]$

Thermodynamics:

Universal gas constant: $R = 8.314$ J/(mol K)
Boltzmann constant: $k_B = 1.38064852 \times 10^{-23}$ m^2 kg/(s^2 K)
Number of air molecules in one gram: $N = R/k_B$

Other constants and factors:

$B = 0.100$
$k = (1 + B/(aP))^{-1}$
$K = 9\eta/[2a^2(\sigma - rho)]k$

B. Brownian motion

Full derivation of equations:

- Vertical displacement

  Equation of motion (positive down):

$$m_{eff}\frac{d^2y}{dt^2} = -6\pi\eta ak\frac{dy}{dt} - F_{b_y} + m_{eff}g$$

$$\frac{d^2y}{dt^2} = -K\frac{dy}{dt} - \frac{F_{b_y}}{m_{eff}} + g$$

But $F_{b_y} = \rho Vg$ and $m_{eff} = V(\sigma-\rho)$

$$\rightarrow \frac{d^2y}{dt^2} = -K\frac{dy}{dt} - \frac{\rho g}{\sigma-\rho} + g$$

This is a linear equation. Using $t_0 = 0 \rightarrow v_y(0) = 0$ and $t_f = t \rightarrow v_y(t_f) = v_y(t)$:

$$v_y(t) = \frac{g}{K}(1-e^{-Kt})(1-\frac{\rho}{\sigma-\rho})$$

$$\Rightarrow v_{y-terminal} = \frac{g}{K}(1-\frac{\rho}{\sigma-\rho})$$

Integrating the expression for $v_y(t)$ and using $t_0 = 0 \rightarrow y(0) = 0$ and $t_f = t \rightarrow y(t_f) = y(t)$, one gets

$$y(t) = \frac{g}{k}[t + \frac{1}{K}(e^{-Kt}-1)](1-\frac{\rho}{\sigma-\rho})$$

- Fall time

  For $y(t_{fall}) = 10$, one finds the fall time from

$$\frac{10K}{g}(1-\frac{\rho}{\sigma-\rho})^{-1} - \frac{1}{K}(e^{-Kt_{fall}}-1) - t_{fall} = 0$$

- Horizontal displacement

  Equation of motion (positive right):

$$m_eff\frac{d^2x}{dt^2} = -Km_{eff}\frac{dx}{dt} - F_{b_x} + qE$$

$$\frac{d^2x}{dt^2} = -K\frac{dx}{dt} - \frac{F_{b_x}}{m_{eff}} + \frac{qE}{m_{eff}}$$

Using $F_{b_x} = \rho V\frac{qE}{m_{eff}}$ and $m_{eff} = V(\sigma-\rho)$:

$$\frac{d^2x}{dt^2} = -K\frac{dx}{dt} + \frac{qE}{V(\sigma-\rho)}(1-\frac{\rho}{\sigma-\rho})$$

This is a linear equation. Using $t_0 = 0 \rightarrow v_x(0) = 0$ and $t_f = t \rightarrow v_x(t_f) = v_x(t)$:

$$v_x(t) = \frac{qE}{\frac{4}{3}\pi a^3(\sigma-\rho)K}(1-e^{-Kt})(1-\frac{\rho}{\sigma-\rho})$$

Integrating the expression for $v_x(t)$ and using $t_0 = 0 \rightarrow x(0) = 0$ and $t_f = t \rightarrow x(t_f) = x(t)$, one gets

$$x(t) = \frac{qE}{\frac{4}{3}\pi a^3(\sigma-\rho)K}(1-\frac{\rho}{\sigma-\rho})[t + \frac{1}{K}(e^{-Kt}-1)]$$

- Horizontal-direction (derivation based on (Millikan, Appendix C) [2])

  Equation of motion (positive right):

$$m_{eff}\frac{d^2x}{dt^2} = Km_{eff}\frac{dx}{dt} - F_{b_x} + F_e + X$$

$$m_{eff}\frac{d^2x}{dt^2} = Km_{eff}\frac{dx}{dt} - F_{b_x} + qE + X$$

X is the force imparted to the particle (in this case the sphere) by the surrounding (air) molecules

We are interested in the absolute values of displacement. Multiplying by x and also noticing that $x\frac{d^2x}{dt} = \frac{1}{2}\frac{d^2(x^2)}{dt^2} - (\frac{dx}{dt})^2$ and that $x\frac{dx}{dt} = \frac{1}{2}\frac{d(x^2)}{dt}$, one gets:

$$\frac{m_{eff}}{2} - m_{eff}(\frac{dx}{dt})^2 = -\frac{K}{2}\frac{d(x^2)}{dt} - xF_{b_x} + xqE + xX$$

Consider the mean result arising from applying this equation to a large number of different particles all just alike:

$$\frac{m_{eff}}{2}\frac{d^2(\overline{x^2})}{dt^2} - m_{eff}\overline{(\frac{dx}{dt})}^2 = -\frac{K}{2}\frac{d(\overline{x^2})}{dt} - \overline{xF_{b_x}} + \overline{xqE} + \overline{xX}$$

With $\overline{xX} = 0$ (xX is as likely positive as negative), $z(t) \equiv \frac{d\overline{x^2}}{dt}$, $\frac{RT}{N} = m_{eff}\overline{(\frac{dx}{dt})}^2$, and noticing that, for a fixed dt, $\overline{v_x} = \overline{(\frac{dx}{dt})} \rightarrow \overline{v_x} = \frac{\overline{dx}}{dt} \Rightarrow \frac{\overline{dx}}{dt} = \sqrt{\frac{RT}{Nm_{eff}}}$ and also that, for small displacements from the undisturbed path, $\overline{dx} \rightarrow \overline{\Delta x} = \overline{(x-0)} = \overline{x}$, one gets

$$z(t) = \frac{2RT}{NK}(1-e^{-\frac{K}{m_{eff}}t}) - \frac{2}{K}(F_{b_x}-qE)\sqrt{\frac{RT}{Nm_{eff}}}[t - \frac{m_{eff}}{K}(1-e^{-\frac{K}{m_{eff}}t})] + z_0e^{-\frac{K}{m_{eff}}t}$$

For t long enough, $e^{-\frac{K}{m_{eff}}t} \approx 0$, and

$$z(t) = \frac{2RT}{NK} - \frac{2}{K}(F_{b_x}-qE)\sqrt{\frac{RT}{Nm_eff}}(t - \frac{m_eff}{K})$$

If one assumes t is long enough and short enough at the same time, one can use $dt \rightarrow \Delta t - 0 = t = \tau$, and one arrives at

$$\Delta\overline{x^2} = \frac{2}{K}[\frac{RT}{N} - (F_{b_x}-qE)(\tau - \frac{m_{eff}}{K})\sqrt{\frac{RT}{Nm_{eff}}}]\tau$$

Assuming $\overline{\Delta x} = \sqrt{\frac{2}{\pi}\Delta\overline{x^2}}$:

$$\overline{\Delta x} = \sqrt{\frac{4}{\pi K}[\frac{RT}{N} - (F_{b_x}-qE)(\tau - \frac{m_eff}{K})\sqrt{\frac{RT}{Nm_eff}}]\tau}$$

- Vertical displacement

Equation of motion (positive down):

$$m_{eff}\frac{d^2y}{dt^2} = -Km_{eff}\frac{dy}{dt} - F_{b_y} + F_g + Y$$

It is easy to see that the vertical displacement due to Brownian motion from the undisturbed path is given by:

$$\overline{\Delta y} = \sqrt{\frac{4}{\pi K}[\frac{RT}{N} - (F_{b_y} - m_{eff}g)(\tau - \frac{m_{eff}}{K})\sqrt{\frac{RT}{Nm_{eff}}}]\tau}$$

[1] L. W. McKeehan. (1911). The terminal velocity of fall of small spheres in air ate reduced pressures. University of Minesota.

[2] R. A. Millikan. (1935). Electrons (+ and -), protons, photons, neutrons, and cosmic rays. Chicago: University of Chicago Press.