Neutrino transformations in the sun: quantum oscillation and the Mikheyev-Smirnov-Wolfenstein effect

A. A. Salazar-Lobos
Department of Physics
University of Alberta
Edmonton, Alberta, T6G 2J1, Canada

Abstract

The transformation of neutrinos in the Sun is explored via a quantum oscillation process argument. We first attempt to show how the oscillation process can be explained by the Maki-Nakawaga-Sakata-Pontecorvo mixing matrix. Secondly, we attempt to explain the matter enhancement of the neutrino oscillation process as they travel through high electron density regions in the Sun via the Mikheyev-Smirnov-Wolfenstein mechanism. Our conclusions are compared with experimental data on neutrino oscillation and solar neutrino flux on Earth from different experiments, including the results from the Sudbury Neutrino Observatory collaboration, Kamiokande and KamLAND.

1 Introduction

Early experiments on solar neutrino captures on Earth [5] obtained 1/3 the number of expected electron neutrinos. No identified solar effects could cause this, for instance, day-night and seasonal annual modulations neutrino flux studies showed no significant effects. The proof that neutrinos have masses (contrary to the original assumption) in the 1990s, from the Super-Kamiokande experiment [3], with the observation of the flavour change in atmospheric neutrinos (muon to tauon), opened the path to solve this problem via a quantum oscillation argument. The idea consisted in that the different known 'flavour' neutrinos, ν_e, ν_μ, and ν_τ are linear combinations of mass eigenstates ν_1, ν_2, and ν_3. [2] As the neutrino propagates through space, the quantum mechanical phases of the three mass states advance at slightly different rates due to the difference in neutrino masses. [8] The Sudbury Neutrino Observatory (SNO) collaboration put this hypothesis of neutrino oscillation to test by designing an experiment independent of solar model calculations to search for clear indication of neutrino flavour change in solar neutrinos.

Interaction	Flux
CC	$1.59^{+0.08}_{-0.07}(\text{stat})^{+0.06}_{-0.08}(\text{syst})$
NC	$5.21 \pm 0.27(\text{stat}) \pm 0.38(\text{syst})$
ES	$2.21^{+0.31}_{-0.26}(\text{stat}) \pm 0.10(\text{syst})$

Table 1: SNO results for solar neutrino measurements on Earth. Neutrino flux in $1 \times 10^6 \text{cm}^{-2}\text{s}^{-1}$. Data taken from [5].

Neutrinos in the sun are produced in an initial state of electron flavour, and the original assumption was the they were massless (according to the Standard Model [SM] preditions), so they were suppose to remain electron neutrinos throughout time. This is not the case, however. Experiments in the second half of the 20th century found that the electron neutrino flux on Earth did not agree with neither the predictions by the SM nor the predictions from solar models. To cite two experiments: [5] Ray Davis measured a solar neutrino flux of 2.56 $\pm$0.16(stat) $\pm$0.16(syst) SNU[1], while the prediction by the SM was 8.5$\pm$1.8 SNU; the Kamiokande experiment compared the electron neutrino flux on Earth of 2.8$\pm$0.19(stat)$\pm$0.33(syst) SNU with the solar model prediction of 5.82(1$\pm$0.23) SNU. Other experiments that reached the same conclusions include SAGE, GALLEX, GNO, and Super-Kamiokande. What the SNO collaboration added in the neutrino study is that it looked at the total neutrino flux on Earth. The findings were that this total neutrino flux agreed with the predictions by the SM, and, at the same time, that approximately 1/3 of this number was in the form of electron neutrinos. Furthermore, the results obtained by the SNO collaboration agreed with the neutrino oscillation hypothesis; the 'no flavour change hypothesis' was rejected by more than 7σ [5].

The SNO collaboration looked at three types of neutrino interactions on Earth: [5]

- Charged-current reactions on deuterium:
  $\nu_e + d \rightarrow p + p + e^-$ (CC) Sensitive only to electron neutrinos
- Neutral-current reactions:
  $\nu_x + d \rightarrow p + n + \nu_x$ (NC) Sensitive to muon and tauon neutrinos)
  $\nu_x + e^- \rightarrow \nu_x + e^-$ (ES) Sensitive to all neutrino types

The results are summarized on Table 1; the total solar neutrino flux is as predicted by the SM and it is consistent with the flavour change hypothesis. For more information on the comparison between the SNO results with the neutrino oscillation predictions, please read [5], by the Nobel Laureate Arthur B. McDonald.

[1]SNU = solar neutrino unit = one electron capture per 1×10^{36} atoms of the target element per second.

2 The Maki-Nakagawa-Sakata-Pontecorvo (MNSP) matrix and mass eigenstates superposition

Neutrinos interact as flavour eigenstates ($\nu_e,\ \nu_\mu,\ \nu_\tau$) but propagate as eigentstates of the free-particle Hamiltonian (as mass eigenstates $\nu_1,\ \nu_2,\ \nu_3$) [4](Chapter 11.5). The flavour eigenstates are made up of superpositions, or mixing, of the mass eigenstates. This is better described by the Maki-Nakagawa-Sakata-Pontecorvo (MNSP) matrix, which relates these two types of eigenstates:

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} \tag{1}$$

The mixing matrix U_{fi} ($f = e,\ \mu,\ \tau$ and $i = 1,\ 2,\ 3$) can be expressed in terms of mixing angles and a phase factor δ: [4]

$$U = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}, \tag{2}$$

where $c_{ij} = \cos(\theta_{ij}),\ s_{ij} = \sin(\theta_{ij}),\ \theta_{ij},\ i,j = 1,2,3$ being mixing paramaters.

Then the electrons produced at the Sun, which are in the electron flavour, can be expressed as:

$$|\nu_e\rangle = \cos(\theta_{12})\cos(\theta_{13})\,|\nu_1\rangle + \sin(\theta_{12})\cos(\theta_{13})\,|\nu_2\rangle + \sin(\theta_{13})e^{-i\delta}\,|\nu_3\rangle\,. \tag{3}$$

The coefficients of the mass eigenstates change in time, as we will see later, and so the whole state describing the solar neutrino also evolves in time. Figure 1 shows the results from a simulation of the mixing parameters that best reproduce the experimental observations; the best fit values obtained from the merge of the SNO collaboration results and KamLAND are shown in Figure 2 for comparison.

The results of the simulation (Figure 1) on neutrino oscillation based the MNSP model that best match the observed data for solar neutrinos in an initial electron flavour state for which 2/3 convert into other flavour type neutrinos give the following mixing parameters: $\theta_{12} \approx 0.59,\ \theta_{13} \approx -0.20,\ \theta_{23} \approx 0.79,\ \delta \approx 0$, which match to a good degree with the results obtained by SNO and KamLAND together. The discrepancy in the mixing angle θ_{12} is due mainly to the fact that at the end of our derivation for the Mikheyev-Smirnov-Wolfenstein effect on the oscillation process, as we see in the next section, we do not consider the time traveled by the neutrinos in their journey to the Earth from the moment they leave the Sun. The mixing angle θ_{23} does not come into play in our argument, and θ_{13} is known to be

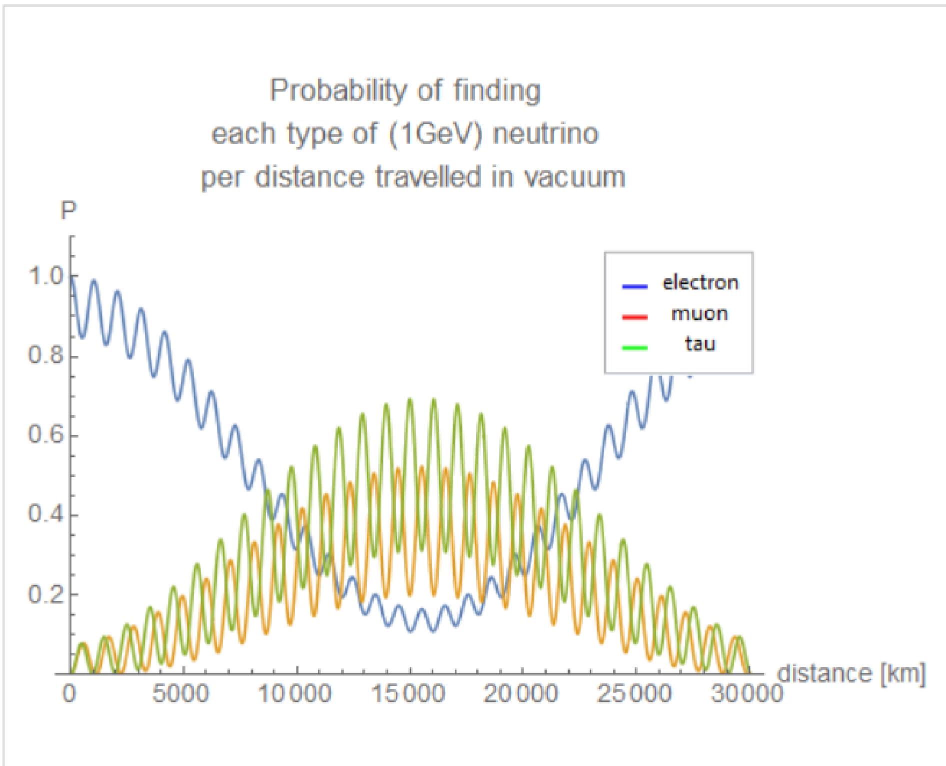

Figure 1: Neutrino oscillation simulation for electron initial flavour. Best parameters for reproducing observations on Earth: mixing parameters: $\theta_{23} \approx 0.79$, $\theta_{13} \approx -0.20$, $\theta_{12} \approx 0.59$, $\delta \approx 0$; mass parameters: $\Delta m^2_{21} \approx 8.00 \times 10^{-5}$ $(\text{eV}/c)^2$, $\Delta m^2_{32} \approx 2.40 \times 10^{-3}$ $(\text{eV}/c)^2$. The mass hierarchy is $m_3 \gg m_2 > m_1$. For comparison, with the merge of the SNO and KamLAND results, see Figure 2. Simulation based on [6].

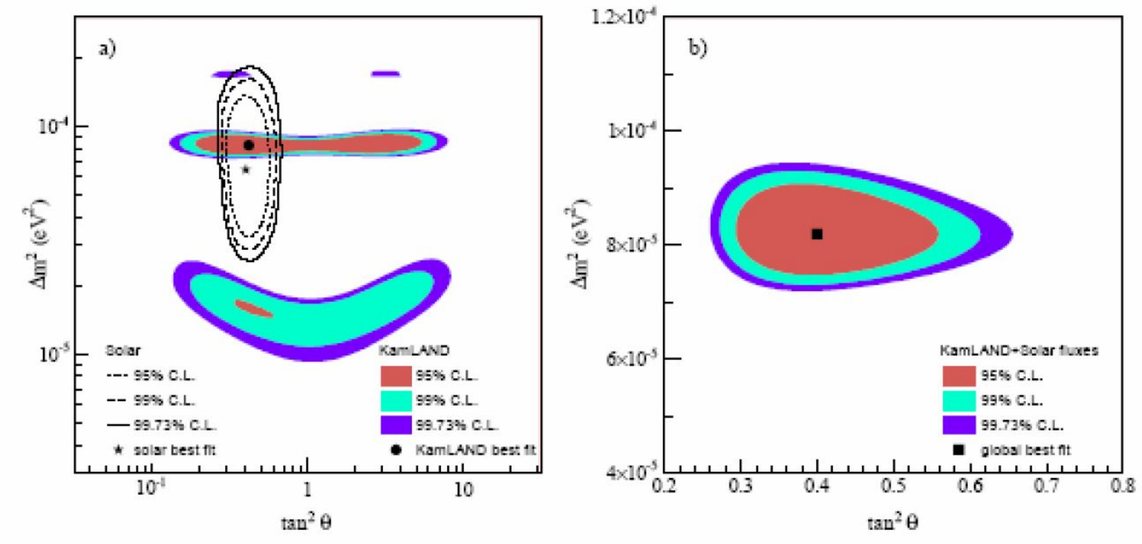

Figure 2: Merge of SNO collaboration results with KamLAND's reactor anti-neutrinos experiment. Assumption: ν_1, ν_2 mixing is dominant. The mixing angle is $\theta_{12} = 0.40^{+0.09}_{-0.07}$, and the mass parameter is $\Delta m^2_{21} = 8.2^{+0.6}_{-0.5} \times 10^{-5}(\mathrm{eV}/c)^2$, with $m_2 > m_1$. Taken from [5], [1].

very small; altering its value in the simulation does not change the probability oscillation corresponding to the electron flavour. We can chose $\delta \approx 0$ to simplify our calculations (it will become obvious later that the value of δ is an arbitrary choice; the overall phase disappears when calculating probabilities):

$$|\nu_e\rangle \approx (\cos(\theta_{12})\cos(\theta_{13})\,|\nu_1\rangle + \sin(\theta_{12})\cos(\theta_{13})\,|\nu_2\rangle + \sin(\theta_{13})\,|\nu_3\rangle)\,. \tag{4}$$

The time evolution of an eigenstate $|\psi_E\rangle$ (energy E) of the Hamiltonian is just $|\psi(t)\rangle = e^{-iEt\hbar}\,|\psi_E\rangle$ [2]. Therefore, at some time $t > 0$, the state of the electron neutrino is:

$$|\psi(t)\rangle = \left(e^{-i\frac{E_1 t}{\hbar}} cos(\theta_{12})\,|\nu_1\rangle + e^{-i\frac{E_2 t}{\hbar}} sin(\theta_{12})\,|\nu_2\rangle + e^{-i\frac{E_3 t}{\hbar} - i\delta}\,|\nu_3\rangle\right), \tag{5}$$

where E_i are the energies corresponding to the mass eigenstates $|\nu_i\rangle$, $i = 1,\ 2,\ 3$. It takes approximately 500s for the solar neutrino to reach the Earth as it travels with the speed of light. [2] This means that the neutrino undergoes approximately 4983.33 oscillations in its journey, according to the wavelength of the probability amplitude oscillation in Figure 1, which means that there is an extra 9900 km, approximately, of neutrino oscillation distance, which implies that the probability of detecting the solar neutrino on Earth in the electron flavour state is no longer 1.0. We work under the assumption that the neutrinos are very light, and so for momenta $p \gg m/c$ we can approximate the relativistic energy to

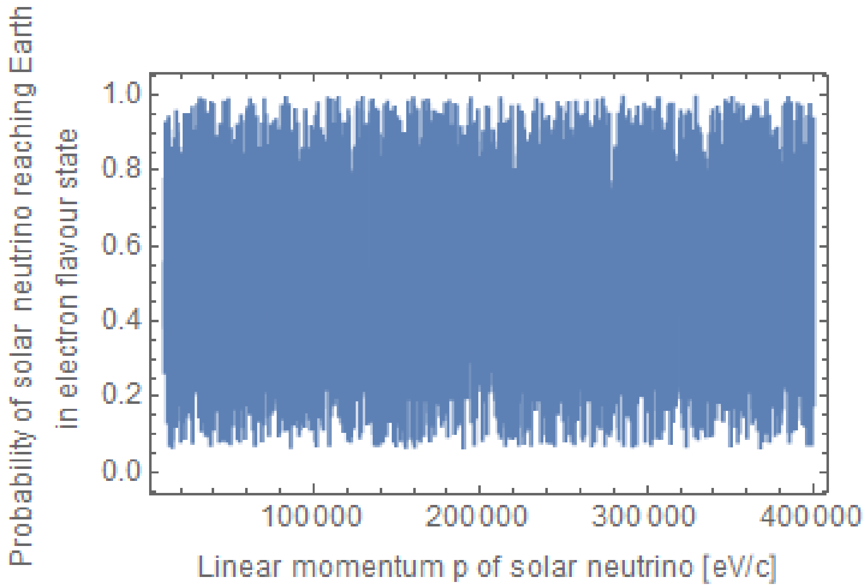

Figure 3: Solar neutrino probability of reaching the Earth in the electron flavour state as a function of linear momentum. $10^4 \leq p \leq 4 \times 10^5$ eV/c.

$E = \sqrt{pc^2 + m^2c^4} \approx pc + m^2c^3/(2p)$, which allows us to work with the mass parameters in the simulation in Figure 1:

$$\Delta E_{ij} = E_i - E_j \approx (m_i^2 - m_j^2)c^3/(2p). \tag{6}$$

We now compute the probability amplitude, $P(t)$, of detecting the solar neutrino in the electron flavour state:

$$\begin{aligned} P(t) = |\langle \nu_e|\psi(t)\rangle|^2 \approx &[(\cos^4(\theta_{12}) + \sin^4(\theta_{12}))\cos^4(\theta_{13}) \\ &+ 2\cos((E_1 - E_2)t/\hbar)\cos^2(\theta_{12})\sin^2(\theta_{12})\cos^4(\theta_{13}) \\ &+ 2\cos((E_1 - E_3)t/\hbar)\cos^2(\theta_{12})\cos^2(\theta_{13})\sin^2(\theta_{13}) \\ &+ 2\cos((E_2 - E_3)t/\hbar)\sin^2(\theta_{12})\cos^2(\theta_{13})\sin^2(\theta_{13}) + \sin^4(\theta_{13})]. \end{aligned} \tag{7}$$

Figure 3 shows the graph of the oscillation in the probability amplitude $P(500s)$ as a function of linear momentum p in the interval in the interval $[10^4, 4 \times 10^5]$ eV/c. Notice that this momentum is (approximately) the same for the different mass eigenstates. This is totally valid. The neutrino itself moves at a unique speed, however, the phases of the different mass states advance at slightly different rates; this is the oscillation process, which does not affect the speed of propagation of the neutrino in vacuum. In addition, we are working in the limit in which the neutrino masses are extremely tiny (millions of time smaller than the mass of the electron). We find the average value of the probability function that the solar neutrino arrives on Earth in the electron flavour. We use the mass and angle parameters given in Figure 1.

$$\begin{aligned}
&\Delta m_{21}^2 \equiv m_2^2 - m_1^2 \approx 8.00 \times 10^{-5} (eV/c^2)^2, \\
&\Delta m_{32}^2 \equiv \Delta m_{31}^2 \approx 2.40 \times 10^{-3} (eV/c^2)^2, \\
&\theta_{12} \approx 0.59,\ \theta_{13} \approx -0.2.
\end{aligned} \tag{8}$$

$$\begin{aligned}
\langle P(p) \rangle = &\frac{1}{4 \times 10^5 - 10^4} \int_{10^4}^{4\times 10^5} dp[(\cos^4(\theta_{12}) + \sin^4(\theta_{12})) \cos^4(\theta_{13}) \\
&+ 2\cos((E_1 - E_2)t/\hbar) \cos^2(\theta_{12}) \sin^2(\theta_{12}) \cos^4(\theta_{13}) \\
&+ 2\cos((E_1 - E_3)t/\hbar) \cos^2(\theta_{12}) \cos^2(\theta_{13}) \sin^2(\theta_{13}) \\
&+ 2\cos((E_2 - E_3)t/\hbar) \sin^2(\theta_{12}) \cos^2(\theta_{13}) \sin^2(\theta_{13}) + \sin^4(\theta_{13})] \\
\approx &\ 0.53.
\end{aligned} \tag{9}$$

This is still not our desired 1/3, but it does tell us that there is a non 100% chance of detecting solar neutrinos in the electron flavour state on Earth. Let us now consider the effects of matter enhancement of the oscillation process by the Mikheyec-Smirnov-Wolfenstein effect.

3 The Mikheyev-Smirnov-Wolfenstein (MSW) effect

So far we have assumed that the neutrinos travel only through vacuum on their journey from the Sun to the Earth, but that is not true. The solar neutrinos first travel through high electron density regions inside the Sun; interactions between the neutrinos and other particles in these regions occurs through elastic scattering of the solar neutrinos and the Z-mediated neutral current interaction. The net effect of these interactions is that they modify the mixing angle and mass splitting (mass enhancement process) of the neutrino oscillation. [4] This is known as the Mikheyev-Smirnov-Wolfenstein (MSW) effect. We will denote the enhanced mass eigenstates $\tilde{\nu}_1,\ \tilde{\nu}_2,\ \tilde{\nu}_3$.

To account for this effect in the high electron density regions in the Sun, we introduce a perturbation in the Hamiltonian in the form: [2]

$$\hat{V} = V \left|\nu_e\right\rangle \left\langle\nu_e\right| . \tag{10}$$

To see the connection of this potential V to the internal structure of the Sun, we use $V = GN_e$, with G being a positive constant and N_e being the electron number density in the medium (here, the Sun). We compute the projection operator $\left|\nu_e\right\rangle \left\langle\nu_e\right|$:

$$
\begin{aligned}
|\nu_e\rangle\langle\nu_e| &= \begin{pmatrix} \cos(\theta_{12})\cos(\theta_{13}) \\ \sin(\theta_{12})\cos(\theta_{13}) \\ \sin(\theta_{13})e^{-i\delta} \end{pmatrix} \left(\cos(\theta_{12})\cos(\theta_{13}), \sin(\theta_{12})\cos(\theta_{13}), \sin(\theta_{13})e^{i\delta}\right) \\
&= \begin{pmatrix} \cos^2(\theta_{12})\cos^2(\theta_{13}) & \cos(\theta_{12})\sin(\theta_{12})\cos^2(\theta_{13}) & \cos(\theta_{12})\cos(\theta_{13})\sin(\theta_{13}) \\ \sin(\theta_{12})\cos(\theta_{12})\cos^2(\theta_{13}) & \sin^2(\theta_{12})\cos^2(\theta_{13}) & \sin(\theta_{12})\cos(\theta_{13})\sin(\theta_{13}) \\ \cos(\theta_{12})\cos(\theta_{13})\sin(\theta_{13}) & \sin(\theta_{12})\cos(\theta_{13})\sin(\theta_{13}) & \sin^2(\theta_{13}) \end{pmatrix}.
\end{aligned}
\tag{11}
$$

As said before, we treat V as a perturbation, and so a correction to the energy of the mass eigenstates can be found from the diagonal in the (second) matrix in Equation 11: [2]

$$
\tilde{E_1} - E_1 = \langle\nu_1|\,\hat{V}\,|\nu_1\rangle = V\cos^2(\theta_{12})\cos^2(\theta_{13}), \tag{12}
$$

$$
\tilde{E_2} - E_2 = V\sin^2(\theta_{12})\cos^2(\theta_{13}), \tag{13}
$$

$$
\tilde{E_3} - E_3 = V\sin^2(\theta_{13}), \tag{14}
$$

where $\tilde{E_i}$ is the total energy of the ith mass eigenstate, and E_i is its unperturbed energy. Then in the basis $\{\nu_1, \nu_2, \nu_3\}$, the Hamiltonian becomes:

$$
\begin{aligned}
\hat{H} &= \begin{pmatrix} E_1 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & E_3 \end{pmatrix} \\
&+ V \begin{pmatrix} \cos^2(\theta_{12})\cos^2(\theta_{13}) & \cos(\theta_{12})\sin(\theta_{12})\cos^2(\theta_{13}) & \cos(\theta_{12})\cos(\theta_{13})\sin(\theta_{13}) \\ \sin(\theta_{12})\cos(\theta_{12})\cos^2(\theta_{13}) & \sin^2(\theta_{12})\cos^2(\theta_{13}) & \sin(\theta_{12})\cos(\theta_{13})\sin(\theta_{13}) \\ \cos(\theta_{12})\cos(\theta_{13})\sin(\theta_{13}) & \sin(\theta_{12})\cos(\theta_{13})\sin(\theta_{13}) & \sin^2(\theta_{13}). \end{pmatrix}
\end{aligned}
\tag{15}
$$

We perform a couple of approximations, namely $\sin(\theta_{13}) \approx \sin^2(\theta_{13}) \approx 0$ and $\cos(\theta_{13}) \approx \cos^2(\theta_{13}) \approx 1$.These approximations are very rough, but noticing that $\tilde{E_3} - E_3 = V$ $sin^2(\theta_{13}) \approx 0$ for small V (or if we want, we could scale things to V), these are not too bad, and actually give a good idea of the behaviour of the solar neutrino's propagation. Without these approximations, the calculations get EXTREMELY complicated for a term project to explore them fully. We get:

$$
\begin{aligned}
\hat{H} &= \begin{pmatrix} E_1 + V\cos^2(\theta_{12}) & V\cos(\theta_{12})\sin(\theta_{12}) & 0 \\ V\sin(\theta_{12})\cos(\theta_{12}) & E_2 + V\sin^2(\theta_{12}) & 0 \\ 0 & 0 & E_3 \end{pmatrix} \\
&= \begin{pmatrix} E1 + \frac{V}{2}(1+\cos(2\theta_{12})) & \frac{V}{2}\sin(\theta_{12}) & 0 \\ \frac{V}{2}\sin(\theta_{12}) & E_2 + \frac{V}{2}(1-\cos(\theta_{12})) & 0 \\ 0 & 0 & E_3 \end{pmatrix}
\end{aligned}
\tag{16}
$$

Now we decompose the intial solar neutrino state $|\nu_e\rangle$ on the eigenbasis of the perturbed Hamiltonian (that is we write $|\nu_e\rangle$ in terms of the mass eigenvectors of the perturbed Hamiltonian): $\{\tilde{\nu_1}, \tilde{\nu_2}, \tilde{\nu_3}\}$. Here it becomes apparent that the perturbation is just a correction due to the neutrino-matter interaction in the Sun to the first considered case (the MNSP matrix alone):

$$
|\nu_e\rangle = \langle\tilde{\nu}_1\nu_e\rangle\,|\tilde{\nu}_1\rangle + \langle\tilde{\nu}_2|\nu_e\rangle\,|\tilde{\nu}_2\rangle + \langle\tilde{\nu}_3|\nu_e\rangle\,|\tilde{\nu}_3\rangle \tag{17}
$$

The perturbed Hamiltonian in Equation 16 can be rewritten as:

$$
\hat{H} = \begin{pmatrix} \frac{E_1+E_2+V}{2} & 0 & 0 \\ 0 & \frac{E_1+E_2+V}{2} & 0 \\ 0 & 0 & E_3 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} \Delta - V\cos(2\theta_{12}) & -V\sin(2\theta_{12}) & 0 \\ -V\sin(2\theta_{12}) & -\Delta + V\cos(2\theta_{12}) & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{18}
$$

where $\Delta \equiv E_2 - E_1$. We choose a parameter $\tilde{\theta}_{12}$ such that

$$
\begin{aligned}
\hat{H} = &\begin{pmatrix} \frac{E_1+E_2+V}{2} & 0 & 0 \\ 0 & \frac{E_1+E_2+V}{2} & 0 \\ 0 & 0 & E_3 \end{pmatrix} \\
&- \frac{1}{2}\sqrt{\Delta^2 + V^2 - 2V\Delta\cos(2\theta_{12})}\begin{pmatrix} \cos(2\tilde{\theta}_{12}) & \sin(2\tilde{\theta}_{12}) & 0 \\ \sin(2\tilde{\theta}_{12}) & -\cos(2\tilde{\theta}_{12}) & 0 \\ 0 & 0 & 0 \end{pmatrix},
\end{aligned}
\tag{19}
$$

$\cos(2\tilde{\theta}_{12}) \equiv \frac{\Delta - V\cos(2\theta_{12})}{\sqrt{\Delta^2+V^2-2V\Delta\cos(2\theta_{12})}}$, $\sin(2\tilde{\theta}_{12}) \equiv \frac{-V\sin(2\theta_{12})}{\sqrt{\Delta^2+V^2-2V\Delta\cos(2\theta_{12})}}$. Its eigenvectors are (in the basis $\{\nu_1, \nu_2, \nu_3\}$)

$$
\begin{pmatrix} \cos(\tilde{\theta}_{12}) \\ \sin(\tilde{\theta}_{12}) \\ 0 \end{pmatrix}, \begin{pmatrix} -\sin(\tilde{\theta}_{12}) \\ \cos(\tilde{\theta}_{12}) \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{20}
$$

with the following corresponding eigenvalues

$$
\begin{aligned}
\tilde{E_1} &= \frac{1}{2}\left(E_1 + E_2 + V - \sqrt{\Delta^2 + V^2 - 2V\Delta\cos(2\theta_{12})}\right),\\
\tilde{E_2} &= \frac{1}{2}\left(E_1 + E_2 + V + \sqrt{\Delta^2 + V^2 - 2V\Delta\cos(2\theta_{12})}\right),\\
\tilde{E_3} &= E_1.
\end{aligned}
\tag{21}
$$

We see that for the highly energetic solar neutrinos, the MSW effect is noticeable: $V \gg \Delta \Rightarrow \cos(2\tilde{\theta}_{12}) \approx -\cos(2\theta_{12})$ & $\sin(2\tilde{\theta}_{12}) \approx -sin(2\theta_{12}) \Rightarrow \tilde{\theta}_{12} \approx \theta_{12} \pm \frac{\pi}{2}$. The effect is negligible for solar neutrinos with energy in the low limit: $V \ll \Delta \Rightarrow \cos(2\tilde{\theta}_{12}) \approx 1$ & $\sin(2\tilde{\theta}_{12}) \approx 0 \Rightarrow \tilde{\theta}_{12} \approx 0$. From this point on we consider the high energy limit then.

We develop Equation 17:

$$
\begin{aligned}
|\nu_e\rangle &= \langle\tilde{\nu_1}|\nu_e\rangle\,|\tilde{\nu_1}\rangle + \langle\tilde{\nu_2}|\nu_e\rangle\,|\tilde{\nu_2}\rangle + \langle\tilde{\nu_3}|\nu_e\rangle\,|\tilde{\nu_3}\rangle\\
&= \left(\cos(\theta_{12})\cos(\theta_{13})\cos(\tilde{\theta}_{12}) + \sin(\theta_{12})\cos(\theta_{13})\sin(\tilde{\theta}_{12})\right)|\tilde{\nu}_1\rangle\\
&\quad + \left(-\cos(\theta_{12})\cos(\theta_{13})\sin(\tilde{\theta}_{12}) + \sin(\theta_{12})\cos(\theta_{13})\cos(\tilde{\theta}_{12})\right)|\tilde{\nu}_2\rangle + \sin(\theta_{13})\,|\tilde{\nu}_3\rangle\\
&= \cos(\theta_{13})\cos(\theta_{12} - \tilde{\theta}_{12})\,|\tilde{\nu}_1\rangle + \cos(\theta_{13})\sin(\theta_{12} - \tilde{\theta}_{12})\,|\tilde{\nu}_2\rangle + \sin(\theta_{13})\,|\tilde{\nu}_3\rangle\,,
\end{aligned}
\tag{22}
$$

and the time evolution of the solar neturinos is such that as the neutrino leaves the Sun, there is a progression from $\tilde{\nu}_i \to \nu_i$. We can describe the time evolution in the state $|\psi\rangle$ of the solar neutrino with the aid of the phases ϕ_i associated with the corrected energies $\tilde{E}_i$ (we will not calculate this explicitely; this would be good for another project). Finally, under the adiabatic approximation (i.e we consider a slow transition in mass eigenstates, from $\tilde{\nu}_i$ to ν_i), the coefficients remain as in $|\nu_e\rangle$: [2]

$$
\begin{aligned}
|\psi\rangle &= e^{i\phi_1}\cos(\theta_{13})\cos(\theta_{12} - \tilde{\theta}_{12})\,|\tilde{\nu}_1\rangle + e^{i\phi_2}\cos(\theta_{13})\sin(\theta_{12} - \tilde{\theta}_{12})\,|\tilde{\nu}_2\rangle\\
&\quad + e^{i\phi_3}\sin(\theta_{13})\,|\tilde{\nu}_3\rangle\,.
\end{aligned}
\tag{23}
$$

Now we compute the probability of detecting the highly energetic solar neutrino in the electron flavour state on Earth $P = |\langle\nu_e|\psi\rangle|^2$:

$$
\begin{aligned}
\langle\nu_e|\psi\rangle &= e^{i\phi_1}\cos(\theta_{12})\cos^2(\theta_{13})\cos(\theta_{12} - \tilde{\theta}_{12}) + e^{i\phi_2}\sin(\theta_{12})\sin(\theta_{12} - \tilde{\theta}_{12})\\
&\quad + e^{i\phi_3}\sin^2(\theta_{13})
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
P = |\langle\nu_e|\psi\rangle|^2 &= \cos^2(\theta_{12})\cos^4(\theta_{13})\cos^2(\theta_{12} - \tilde{\theta}_{12})\\
&\quad + 2\cos(\phi_2 - \phi_1)\cos(\theta_{12})\sin(\theta_{12})\cos^4(\theta_{13})\cos(\theta_{12} - \tilde{\theta}_{12})\sin(\theta_{12} - \tilde{\theta}_{12})\\
&\quad + 2\cos(\phi_3 - \phi_1)\cos(\theta_{12})\cos^2(\theta_{13})\sin^2(\theta_{13})\cos(\theta_{12} - \tilde{\theta}_{12})\\
&\quad + 2\cos(\phi_3 - \phi_2)\sin(\theta_{12})\cos^2(\theta_{13})\sin^2(\theta_{13})\sin(\theta_{12} - \tilde{\theta}_{12})\\
&\quad + \sin^2(\theta_{12})\cos^4(\theta_{13})\sin^2(\theta_{12} - \tilde{\theta}_{12}) + \sin^4(\theta_{13}).
\end{aligned}
\tag{25}
$$

Now we maximize the time of oscillation associated with $\phi_2 - \phi_1 \leftrightarrow \hbar/(\tilde{E}_2 - \tilde{E}_1)$: $min(\tilde{E}_2 - \tilde{E}_1) \approx \Delta \sin(2\theta_{12})$ (this can be seen from Equation 21. We consider highly energetic solar neutrinos, so we pick $p \approx 4 \times 10^5(\mathrm{eV}/c)^2$, the largest momentum value in the interval considered before in our discussion of neutrino oscillation in the context of the MNSP matrix, which gives $\hbar/(\tilde{E}_2 - \tilde{E}_1) \approx \hbar/(\Delta \sin(2\theta_{12})) = 1.13s$, which is less than the time it takes a neutrino to cross the sun ($\approx$ 2s [2]). We can, therefore, safely assume that the $\phi_2 - \phi_1$ varies enough so that $\cos(\phi_2 - \phi_1)$ averages to zero. Looking at Figure 1, we see that the probability amplitudes of the muonic and tauon flavours are symmetrical with that of the electron flavour, so we can safely assume that $\cos(\phi_3 - \phi_1) \approx \cos(\phi_3 - \phi_2) \approx 0$. Furthermore, in the high energy limit, as seen above, $\tilde{\theta}_{12} \approx \theta_{12} \pm \pi/2 \Rightarrow \cos(\theta_{12} - \tilde{\theta}_{12}) \approx 0$ and $\sin(\theta_{12} - \tilde{\theta}_{12}) \approx 1$. We have

$$\begin{aligned} P &= \sin^2(\theta_{12})\cos^4(\theta_{13}) + \sin^4(\theta_{13}) \\ &= 0.29 \approx 1/3. \end{aligned} \tag{26}$$

This result is not exactly 1/3 because we did not compute the time-dependent phase factors associated with the perturbed Hamiltonian directly. Nevertheless, we have shown that the MSW effect does enhance the oscillation process in the flavour state of the solar neutrinos, which gives a probability of detecting them in the electron flavour state that is consistent with the experimental data.

4 Conclusion

The oscillation process in the solar neutrinos' flavour state can only be possible if they have non-zero mass. We have showned above how this provides a good explanation for the electron-neutrino capture results on Earth; we have seen that having 2/3 of the solar neutrinos converting into other active flavour types upon their arrival on Earth is totally plausible, and this agrees with experimental data. The oscillation itself is well explained by the Maki-Nakagawa-Sakata-Pontecorvo matrix, and the correction to this process for having an agreement between theory and experiment is given by the Mikheyev-Smirnov-Wolfenstein effect. However, we have not answered questions like why do neutrinos have mass?, and what are their masses? These questions are absolutely important in particle physics, mainly because their answer could provide an explanation of why matter is dominant in the universe over anti-matter. The physical properties of the neutrinos point towards the exitence physics beyond the Standard Model. Simply put, the Standard Model predicts massless neutrinos, but, as shown above, this is not the case. This suggests that the Standard Model is incomplete, or perhaps wrong, and that particle physics still is a very promising branch of physics.

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