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Phys362 Assignment #4

(i) Plot $f(t) = h \cdot exp\left(\frac{-t^2}{2\sigma^2}\right)$ and name it:

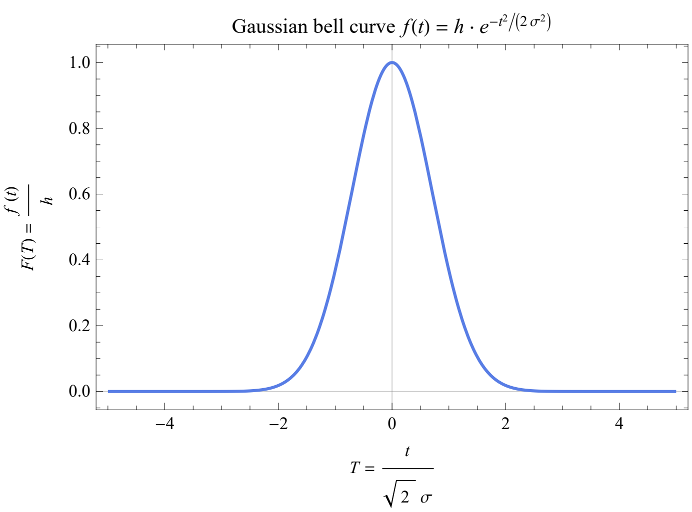
Define variable $T \equiv \frac{t}{\sqrt{2}\ \sigma}$, and function $F(T) \equiv \frac{f(t)}{h}$;

code:

In[ ]:=
```
(* Define function *)
F = Exp[-T^2];

(* Plot function *)
Plot[F, {T, -5, 5}, FrameLabel → {"T = t/(√2 σ)", "F(T) = f(t)/h"},
 PlotTheme → {"Scientific", "BoldColor"},
 PlotLabel → "Gaussian bell curve f(t) = h · e^(-t²/(2 σ²))"]
```

Out[ ]=



This is the Gaussian bell curve.

(ii) Take the Fourier transform:

$$g(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t)\, e^{i\omega t}\, dt$$

$$g(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h \cdot e^{\frac{-t^2}{2\sigma^2}}\, e^{i\omega t}\, dt$$

Substitution: $T \equiv \frac{t}{\sqrt{2}\ \sigma} \Rightarrow \frac{dT}{dt} = \frac{1}{\sqrt{2}\ \sigma} \Rightarrow dt = \sqrt{2}\ \sigma\, dT$:

$$\Rightarrow g(\omega) = \frac{1}{2\pi}\left(\sqrt{2}\,\sigma\right)\int_{-\infty}^{\infty} h\cdot e^{-T^2}\cdot e^{i\,\omega\,\sqrt{2}\,\sigma\,T}\,dT$$

Define $W = \sqrt{2}\,\sigma\,\omega$:

$$g(\omega) = (\sqrt{2}\,\sigma\cdot h\Big)\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-T^2}\cdot e^{iWT}\,dT$$

Let $G(W) \equiv \frac{g(\omega)}{\sqrt{2}\,\sigma h} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-T^2}\cdot e^{\mathrm{iWT}}\,dT$

So $G(W)$ is the Fourier transform of $F(T) = e^{-T^2}$.

The following code calculates $G(W)$:

```
(* Calculate the Fourier transformation G(W) of the function F(T) *)
In[3]:= G = Integrate[(1/(2*Pi)) * (Exp[-T^2]) * (Exp[I*W*T]), {T, -Infinity, Infinity}]
```

Out[3]= $\frac{e^{-\frac{W^2}{4}}}{2\sqrt{\pi}}$

$G(W) = \frac{1}{2\sqrt{\pi}}\cdot \exp\left(\frac{-W^2}{4}\right)$

The Fourier transform of the original function, $f(t) = h\cdot exp\left(\frac{-t^2}{2\sigma^2}\right)$, is therefore:

$$g(\omega) = G(W)\cdot\sqrt{2}\,\sigma h$$

$$= \frac{\sigma h}{\sqrt{2\pi}}\exp\left(\frac{-W^2}{4}\right)$$

Recall: $W = \sqrt{2}\,\sigma\,\omega$:

$$g(\omega) = \frac{\sigma h}{\sqrt{2\pi}}\exp\left(\frac{-2\sigma^2\omega^2}{4}\right)$$

$$g(\omega) = \frac{\sigma h}{\sqrt{2\pi}}\,e^{-\frac{\sigma^2\omega^2}{2}}$$

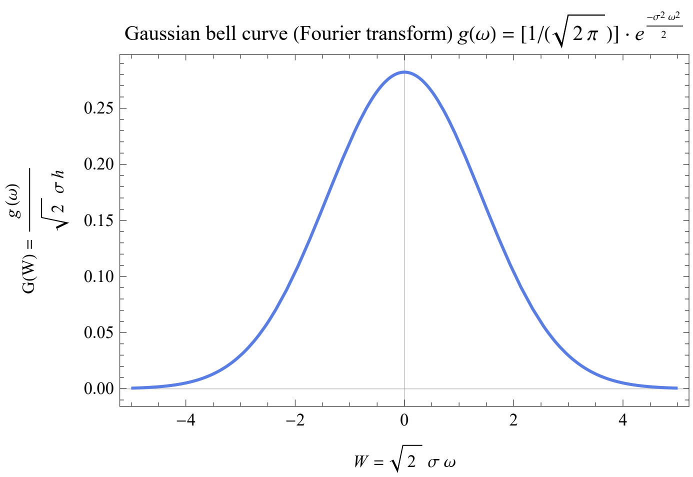
$\therefore$ The Fourier transform of the function $f(t) = h\cdot exp\left(\frac{-t^2}{2\sigma^2}\right)$ is $g(\omega) = \frac{\sigma h}{\sqrt{2\pi}}\,e^{-\frac{\sigma^2\omega^2}{2}}$. // Answer

(iii) Plot the frequency spectrum of the Fourier transform $g(\omega)$:

Code:

In[6]:=
```
(* Plot the frequency spectrum *)
Plot[G, {W, -5, 5}, FrameLabel → {"W = √2 σ ω", "G(W) = g(ω)/(√2 σ h)"},
 PlotTheme → {"Scientific", "BoldColor"},
 PlotLabel → "Gaussian bell curve (Fourier transform) g(ω) = [1/(√(2π))] · e^(-σ² ω²/2)"]
```

Out[6]=



The Fourier transform of a Gaussian bell curve is another Gaussian. The width and height of the transform $g(\omega)$ will depend on the value of σ. For example, in the original function $f(t)$, the width of the curve is σ (Pedrotti, p. 241); comparing the form of the exponential of $g(\omega)$ with that of $f(t)$, we can conclude that the width of the transform g is $1/\sigma$. Whether the transform g is wider or thinner than the original function f will depend on the value of σ. In particular, if $0 < \sigma^2 < 1$, we get a wider curve; for $\sigma^2 > 1$, a thinner curve, and for $\sigma^2 = 1$, the same width. Regarding the height, it will also depend on the value of σ and on how larger or smaller it is compared to $\sqrt{2\pi}$: $\sigma < \sqrt{2\pi}$, the curve is shorter; $\sigma > \sqrt{2\pi}$, the curve is taller (and thinner), and $\sigma = \sqrt{2\pi}$, the same height (and thinner). Notice, however, that σ cannot meet the two equality cases at the same time, so we cannot get the same width and height for both curves at the same time.