$$\frac{\partial n_i}{\partial t} = -\sum_{j<i}\{A_{ij}\beta_{ij}[n_i + W\mathcal{N}_{ij}(n_i - n_j)] + C_{ij}[n_i - n_j\exp(-h\nu_{ij}/kT)]\}$$

(2.7.1)

$$+\sum_{j>i}(g_j/g_i)\{A_{ji}\beta_{ji}[n_j + W\mathcal{N}_{ji}(n_j - n_i)] + C_{ji}[n_j - n_i\exp(-h\nu_{ji}/kT)]\},$$

$$n_i A_{ij} = \frac{n_{mol}}{g_i} X(i)\,\overbrace{WE(i)\cdot A_{ij}}^{} = \frac{n_{mol}}{g_i} x_i A'_{ij}$$

A'_{ij} is the scaled A_{ij} in molpop

$$n_j A_{ij} = \frac{n_{mol}}{g_j} X(j)\, WE(j) A_{ij} = \frac{n_{mol}}{g_j} X_j \frac{WE(j)}{WE(i)} A'_{ij}$$

$$= \frac{n_{mol}}{g_j} x_j \frac{g_j}{g_i} e^{(E_i - E_j)/kT} A'_{ij} = \frac{n_{mol}}{g_i} x_j A'_{ij}\, GAP(i,j)$$

similarly: $n_i C_{ij} = \frac{n_{mol}}{g_i} X_i C'_{ij}$; $C_{ij} n_j e^{-\Delta E/kT} = \frac{n_{mol}}{g_i} C'_{ij}$

So the terms in the sum are:

low up

$j<i$:

ISN=-1

$$-\frac{n_{mol}}{g_i}\{A'_{ij}\beta_{ij}[x_i + RAD(x_i - x_j GAP(ij))] + C'_{ij}(x_i - x_j)\}$$

$$= I_{SN}\frac{n_{mol}}{g_i}\{A_{ul}\beta_{ul}[x_u + RAD(x_u - GAP\cdot x_l)] + C_{ul}(x_u - x_l)\}$$

up low

$j>i$:

ISN=+1

$$\frac{g_j}{g_i}\cdot\frac{n_{mol}}{g_j}\{A'_{ji}\beta_{ji}[x_j + RAD(x_j - x_i GAP(j,i))] + C'_{ji}(x_j - x_i)\}$$

$$= ISN\frac{n_{mol}}{g_i}\{A'_{ul}\beta_{ul}[x_u + RAD(x_u - x_l GAP)] + C_{ul}(x_u - x_l)\}$$

so the same form.

The set of equations is defined by

$$F_i(X_1 - - X_N) = \sum_{j=1}^{N} R_{ij} X_j \qquad i=1 - N$$

$$D_{ij} = \frac{\partial F_i}{\partial x_j} = R_{ij} + \sum_{j=1}^{N} \frac{\partial R_{ij}}{\partial x_j} X_j$$

Linear: $\frac{\partial R_{ij}}{\partial x_j} = 0 \Rightarrow D_{ij} = R_{ij}$

and the equations are

$$\underline{D} \cdot \underline{X} = \underline{V} \qquad \text{where } V_i = 0 \quad i = 1, \ldots N-1$$

the N-th equation $\sum_{j=1}^{N} WE_j X_j = 1$

so: $D_{Nj} = WE_j$; $V_N = 1$ or $V \Rightarrow F = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$

Non Linear: $\underline{D} \cdot \delta \underline{X} = -F$

where:

$$D_{ij} = D_{ij}(\text{linear}) + \sum_j \frac{\partial R_{ij}}{\partial x_j} X_j$$

and $F_N = \sum_j WE_j X_j - 1$

Now add dust: $\tau_{d,\lambda} = q_\lambda \cdot 5 \cdot 10^{-22}\, n_H R$

$$= q_\lambda \cdot 10^{-21}\, n_{H_2} R$$

Will introduce dust absorption parameter p such that standard dust abundance and molecular hidrogen is $p_d = 1$.

Then $\tau_{d,\lambda} = q_\lambda \cdot p_d \cdot 10^{-21}\, n_{H_2} R$

Atomic H and standard dust will have $p_d = \frac{1}{2}$.

Introduce $q_d(i,j)$ through

$$\tau_d(i,j) = q_d(i,j)\,R$$

$$= q_d\, P_d \cdot 10^{-21}\, n_{H_2}\, R$$

So: $\boxed{q_{dust}(i,j) = q_d(i,j)\,\underbrace{P_d \cdot 10^{-21} \cdot n_{H_2}}_{X_{dust}}}$ in Input

in Optdep

$$TAU(I,J) \Rightarrow \tau_\ell(i,j) + \tau_d(i,j)$$

$$= R * \left[TAUX(i,j)\,\Delta_{ij} + q_d(i,j) \right]$$

where $\Delta_{ij} = X_j\, GAP(i,j) - X_i$

Then β and $\frac{d\beta}{d\tau}$ are calulated the same with $TAU(I,J) = \tau_\ell + \tau_d$ and the final emission retains the same expression, no change.

The only effect is on the level population equations, where

in EQ

$$\beta \rightarrow \beta(\tau_t) + X_d\left[1 - \beta(\tau_t)\right]$$

where $\boxed{X_d = \frac{\kappa_d}{\kappa_d + \kappa_\ell} = \frac{q_d(i,j)}{q_d(i,j) + TAUX(i,j) * \Delta_{ij}}}$

So in sub EQ define RATE in the radiative part:

$$\beta = ESC(UP,LOW) + X_d(1-ESC)$$

$$RATE = ISN * A(UP,LOW) * \beta$$

The non-linear addition to the derivative of the rate term in the radiative part will be

$$\frac{\partial RATE}{\partial X_k} = ISN * A * \frac{\partial \beta}{\partial X_k}\left[X(UP) - RAD(UP,LOW) * \Delta(U,L)\right]$$

and
$$\frac{\partial \beta}{\partial X_k} = \frac{\partial ESC}{\partial X_k}(1-X_d) + \frac{\partial X_d}{\partial X_k}(1-ESC)$$

$$= \frac{dESC}{d\tau}(1-X_d)\frac{\partial \tau}{\partial X_k} + (1-ES)\frac{\partial X_d}{\partial X_k}$$

Now:
$$\tau = R\left(TAUX * \Delta + q_d\right)$$

$$\frac{\partial \tau}{\partial X_k} = R \cdot TAUX \cdot \frac{\partial \Delta}{\partial X_k}$$

$$\frac{\partial X_d}{\partial X_k} = -\frac{q_d}{(q_d + TAUX \cdot \Delta)^2} \cdot TAUX * \frac{\partial \Delta}{\partial X_k}$$

$$= -\frac{X_d^2}{q_d} * TAUX * \frac{\partial \Delta}{\partial X_k}$$

$$\boxed{\frac{\partial \beta}{\partial X_k} = \left[\frac{dESC}{\underset{\substack{\uparrow \\ DBDTAU}}{d\tau}}(1-X_d) \cdot R - \frac{X_d^2}{q_d}(1-ESC)\right] * TAUX * \frac{\partial \Delta}{\partial X_k}}$$

$$\boxed{\frac{\partial \Delta}{\partial X_{low}} = GAP(U,L) \; ; \; \frac{\partial \Delta}{\partial X_{up}} = -1}$$