Photon-Bunching in Quantum Memory

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July 12, 2012

1 Introduction

There is a simple analogy between a beamsplitter and a quantum memory. If one thinks of a beamsplitter as two distinct modes, a and b, the action of the beamsplitter is to mix these modes in a certain way. For instance, a 50-50 beamsplitter means that a single photon in mode a will have a 50% chance of staying in mode a and a 50% of becoming mode b.

In the same way, a quantum memory deals with two modes: photon and atomic excitation. The action of the quantum memory will be to mix the two modes in a certain way. Usually, one wants to completely convert the photon mode into an atomic mode, or vis versa. However, if one used a 50% efficient quantum memory, we obtain a very similar situation as that of a beamsplitter.

This is interesting, as there is a phenomena that occurs in beamsplitters called 'photon-bunching', where–due to the boson statistics– if you have a photon incident on the beamsplitter in mode a, and another photon incident in mode b, you have a 50% chance of seeing two photons in mode a, and a 50% chance of seeing two photons in mode b.

The purpose of this report is to quantify and illustrate the extent of similarities between a beamsplitter and a quantum memory.

2 CD Memory

We can demonstrate photon bunching explicitly in the case of the CD memory[**?**]. This quantum memory is used because the author was familiar with the equations of motion.

We have previously derived the Heisenberg equations of motion for the operators of interest, E_{out} and $\sigma_{z?}$.

The heisenburg state we prepared for photon bunching is:

$$|\Psi\rangle = \sigma(t=0)^{\dagger} \int d\omega \psi(w) E_0^{\dagger}(w)|0\rangle \tag{1}$$

Where $E_0(\omega)$ is the creation operator of a photon of frequency ω, $\psi(\omega)$ is the single-photon envelope in frequency space, and $\sigma(t=0)$ is the initial creation operator of a atomic excitation. In what follows, we will be assuming that a photon is already stored in the memory at $t=0$, and that another pulse is incident.

The three terms we are concerned with are:

$$\langle\psi_{20}|\Psi\rangle = \langle 0|\frac{1}{\sqrt{2}}\sigma(t)\sigma(t)|\Psi\rangle \tag{2}$$

$$\langle\psi_{11}|\Psi\rangle = \langle 0|\sigma(t') \int^{t'} dt E_{\text{out}}(t)|\Psi\rangle \tag{3}$$

$$\langle\psi_{02}|\Psi\rangle = \langle 0|\frac{1}{\sqrt{2}} \int dt \int dt' E_{\text{out}}(t) E_{\text{out}}(t')|\Psi\rangle \tag{4}$$

Where $|\psi_{20}\rangle$ is a double excitation in the photon field, $|\psi_{11}\rangle$ is photon and an atomic excitation, and $|\psi_{02}\rangle$ is a double atomic excitation.

3 $|\psi_{11}\rangle$

First, we consider the term that deals with a single excitation in the field and a single excitation in the atom. If photon-bunching is seen in this memory, then this term should equal zero.

$$|\langle\psi_{11}|\Phi\rangle|^2 = \int^{t'} dt \left| \langle 0|E_{\text{out}}(t)\sigma(t') \int d\omega \phi(\omega) E_0^{\dagger}(\omega)\sigma^{\dagger}(0)|0\rangle \right|^2 \tag{5}$$

$$\tag{6}$$

Physically (in terms of detectors), the above equation corresponds to having two measurements. In the first, you are directly measuring the atomic ensemble at time $t = t'$. In the second, you have left a photon detector on to measure the resaviour field from time $t = -\infty, t'$. Concisely, this measures the probability of both: seeing a photon in the resovior field, and seeing an atomic excitation.

We also know the equations for $E_{\text{out}}(t)$ and $\sigma(t)$ from our previous work.

$$E_{\text{out}}(t) = E_{\text{in}}(t) + i\sqrt{\frac{2}{\kappa}}g(t)\sigma(t) \tag{7}$$

$$\sigma(t) = \sigma(0)e^{-\tau} + i\sqrt{2}e^{-\tau}\int_w^\tau dt' e^\tau E_{\text{in}}(t)\frac{g(t)}{\sqrt{\kappa}} \tag{8}$$

Where $\tau = \int^t dt g^2(t)/\kappa$, $g(t)$ is a time-dependant coupling between the atomic ensemble and the cavity field, and κ is the decay rate of the cavity. Inserting these equations we obtain:

$$|\langle\psi_{11}|\Phi\rangle|^2 = \int^{t'} dt \left|\langle 0| \left(E_{\text{in}}(t) + i\sqrt{\frac{2}{\kappa}}g(t)\sigma(t)\right) \times\right. \tag{9}$$

$$\left.\left(\sigma(0)e^{-\tau} + i\sqrt{2}e^{-\tau}\int_w^\tau dt' e^\tau E_{\text{in}}(t)\frac{g(t)}{\sqrt{\kappa}}\right)|\Psi\rangle\right|^2 \tag{10}$$

$$\tag{11}$$

Expanding this out, we get the following terms:

$$|\langle\psi_{11}|\Phi\rangle|^2 = \int^{t'} dt \left|\langle 0|E_{\text{in}}(t)\left(\sigma(0)e^{-\tau(t')} + i\sqrt{2}e^{-\tau(t')}\int^{t'} dt'' e^{\tau(t'')}E_{\text{in}}(t'')\frac{g(t'')}{\sqrt{\kappa}}\right)\right. \tag{12}$$

$$+ i\sqrt{\frac{2}{\kappa}}g(t)\left(\sigma(0)e^{-\tau} + i\sqrt{2}e^{-\tau}\int^t dt''' e^{\tau(t''')}E_{\text{in}}(t''')\frac{g(t''')}{\sqrt{\kappa}}\right)\times \tag{13}$$

$$\left.\left(\sigma(0)e^{-\tau(t')} + i\sqrt{2}e^{-\tau(t')}\int^{t'} dt'' e^\tau(t'')E_{\text{in}}(t'')\frac{g(t'')}{\sqrt{\kappa}}\right)|\Psi\rangle\right|^2 \tag{14}$$

If we apply the commutation relations, we can get rid of any terms that don't have $E_{\text{in}}(t)$ and $\sigma(0)$ in them, as any term in the above equation that doesn't have both of these operators will be annihilated through simple commutations. Ie. if a term only has $E_{\text{in}}E_{\text{in}}$, we know that $|\Psi\rangle$ contains a $\sigma^\dagger(0)$, and that $\sigma^\dagger(0)$ and E_{in} commute, so $\sigma^\dagger(0)$ can move through and act as $\langle 0|\sigma^\dagger(0) = 0$. So any term that survives must contain terms that don't commute with E_{in} as well as $\sigma(0)$.

This leaves us with the following terms:

$$|\langle\psi_{11}|\Phi\rangle|^2 = \int^{t'} dt \left| \langle 0|E_{\text{in}}(t)\sigma(0)e^{-\tau(t')}\sigma^\dagger(0)\int d\omega E_0(\omega)\phi(\omega)|0\rangle - \right. \tag{15}$$

$$\langle 0|\frac{2}{\sqrt{\kappa}}g(t)\sigma(0)e^{-\tau(t)-\tau(t')}\int^{t'} dt'' e^{\tau(t'')}E_{\text{in}}(t'')\frac{g(t'')}{\sqrt{\kappa}}\sigma^\dagger(0)\int d\omega E_0(\omega)\phi(\omega)|0\rangle - \tag{16}$$

$$\left. \langle 0|\frac{2}{\sqrt{\kappa}}g(t)\sigma(0)e^{-\tau(t)-\tau(t')}\int^{t} dt''' e^{\tau(t''')}E_{\text{in}}(t''')\frac{g(t''')}{\sqrt{\kappa}}\sigma^\dagger(0)\int d\omega E_0(\omega)\phi(\omega)|0\rangle \right|^2 \tag{17}$$

We know that E_{in} is the input-pulse, and can also be defined in frequency space as $E_{\text{in}} = \int dw e^{-iwt}E_0(\omega)$ and furthermore, $[E_0(\omega), E_0^\dagger(\omega')] = \delta(\omega - \omega')$, thus we know that

$$\langle 0|E_{\text{in}}(t)\int d\omega\phi(\omega)E_0^\dagger(\omega)|0\rangle \tag{18}$$

$$= \langle 0|\int d\omega' \int d\omega E_0(\omega')E_0(\omega)\phi(\omega)e^{-i\omega t}|0\rangle \tag{19}$$

$$= \int d\omega' \int d\omega \delta(\omega' - \omega)\phi(\omega)e^{-i\omega t} \tag{20}$$

$$= \mathcal{F}\{\phi(\omega)\} \equiv \tilde{\phi}(t) \tag{21}$$

using this property, the equation reduces to:

$$|\langle\psi_{11}|\Phi\rangle|^2 = \int^{t'} dt \left|\tilde{\phi}(t)e^{-\tau(t')} - \frac{2}{\sqrt{\kappa}}g(t)e^{-\tau(t)-\tau(t')}\int^{t'}\tilde{\phi}(t'')\frac{g(t'')}{\sqrt{\kappa}}dt'' - \right. \tag{22}$$

$$\left. \frac{2}{\sqrt{\kappa}}g(t)e^{-\tau(t)-\tau(t')}\int^{t}\tilde{\phi}(t''')\frac{g(t''')}{\sqrt{\kappa}}dt'''\right|^2 \tag{23}$$

Note, we haven't shown how the $\sigma(0)\sigma^\dagger(0)$ term vanishes, but using the commutator properties, you can show that in this case you can simply replace this term with unity.

This is a completely general equation, as we have not assumed anything about the shape of the incoming field. To simply this equation however, we can make the assumption that the incoming field meets the optimal condition discussed in the original paper.

This allows us to do two things. First, because the field is optimally conditioned, all of the efficiency is controlled by the value of τ_w, so we only have one variable to deal with. Second, it will allow the above equation to be integrated very simply.

The optimal condition requires that: $\tilde{\phi}(t) = \sqrt{\frac{2}{\kappa}}g(t)e^{\tau(t)}$

However, we also have to normalize the incoming field, such that:

$$\int_{\infty}^{\infty} |\tilde{\phi}(t)|^2 dt' = 1 \tag{24}$$

Plugging the optimal field condition in, the normalization condition reduces to:

$$\int_{\infty}^{\infty} A^2|\tilde{\phi}(t)|^2 dt' = 1 \tag{25}$$

$$\int_{\infty}^{\infty} A^2\frac{2}{\kappa}g^2(t)e^{2\tau(t)} = 1 \tag{26}$$

$$A^2\left(e^{2\tau_w} - 1\right) = 1 \tag{27}$$

Thus, the value of τ when $t = \infty$ defines the normalization constant.

$$A = \sqrt{\frac{1}{(e^{2\tau_w} - 1)}} \tag{28}$$

We can also note that in the specific case where $\eta = 1/2$, i.e. when the efficiency is one-half, then $\tau_w = \ln(\sqrt{2})$ and our normalization coefficient becomes:

$$A = \sqrt{\frac{1}{\left(e^{2\ln(\sqrt{2})} - 1\right)}} \tag{29}$$

$$A = \sqrt{\frac{1}{(2-1)}} \tag{30}$$

$$A = 1 \tag{31}$$

Inserting this into our equation, we obtain:

$$|\langle\psi_{11}|\Phi\rangle|^2 = A^2 \int^{t'} dt \left| \sqrt{\frac{2}{\kappa}} g(t) e^{\tau(t)-\tau(t')} - 2\sqrt{\frac{2}{\kappa}} g(t) e^{-\tau(t)-\tau(t')} \int^{t'} \frac{g(t'')^2}{\kappa} e^{2\tau(t'')} dt'' \right. \tag{32}$$

$$\left. -2\sqrt{\frac{2}{\kappa}} g(t) e^{\tau(t')-\tau(t)} \int^{t} \frac{g(t''')^2}{\kappa} e^{2\tau(t''')} dt''' \right|^2 \tag{33}$$

and, knowing that $\frac{g^2(t)}{\kappa} dt = d\tau$, the above can be integrate to obtain:

$$|\langle\psi_{11}|\Phi\rangle|^2 = A^2 \int^{t'} dt \left| \sqrt{\frac{2}{\kappa}} g(t) e^{\tau(t)-\tau(t')} - \sqrt{\frac{2}{\kappa}} g(t) \left(e^{\tau(t)-\tau(t')} - e^{-\tau(t')-\tau(t)} \right) \right. \tag{34}$$

$$\left. -\sqrt{\frac{2}{\kappa}} g(t) \left(e^{\tau(t')-\tau(t)} - e^{-\tau(t')-\tau(t)} \right) \right|^2 \tag{35}$$

which can be factored to obtain:

$$|\langle\psi_{11}|\Phi\rangle|^2 = A^2 \int^{t'} dt \left|\sqrt{\frac{2}{\kappa}} g(t) e^{-\tau(t)} \left(-e^{\tau(t')} + 2e^{-\tau(t')}\right)\right|^2 \tag{36}$$

If we use the condition that $\eta = 0.5$, then $\tau(t') = \ln(\sqrt{2})$. It should be noted that $\tau(t = \infty) = \tau(t')$. That is, we have let an infinite time pass when we do the measurement. We are preforming a measurement on the atomic ensemble at time t', but we have left out photon detectors on from time t_0 to t'. However, because we know what $\tau(t')$ is equal to, we can evaluate the term in brackets in the above equation.

$$|\langle\psi_{11}|\Phi\rangle|^2 = A^2 \int^{t'} dt \left|\sqrt{\frac{2}{\kappa}} g(t) e^{-\tau(t)} \left(e^{\ln(\sqrt{2})} - 2\frac{1}{e^{\ln(\sqrt{2})}}\right)\right|^2 = 0 \tag{37}$$

This shows that there is zero chance to measure a photon in the reservoir field and an atomic excitation when the quantum is operating at 50% efficiency. This is precisely what is expected for a beam-splitter.

4 $|\psi_{20}\rangle$

Next, we deal with the case term that measures the amplitude of obtaining a double excitation in the atomic ensemble.

$$\langle\psi_{20}|\Psi\rangle = \langle 0|\frac{1}{\sqrt{2}}\sigma(t)\sigma(t)|\Psi\rangle \tag{38}$$

$$\langle\psi_{20}|\Psi\rangle = \langle 0|\frac{1}{\sqrt{2}}\sigma\sigma\sigma(t=0)^\dagger \int d\omega\psi(w)E_0^\dagger(w)|0\rangle \tag{39}$$

and the heisenburg equations for the operators E_{out} and σ are:

$$E_{\text{out}}(t) = E_{\text{in}}(t) + i\sqrt{\frac{2}{\kappa}}g(t)\sigma(t) \tag{40}$$

$$\sigma(t) = \sigma(0)e^{-\tau} + i\sqrt{2}e^{-\tau}\int^{t_w} dt' e^{\tau} E_{\text{in}}(t)\frac{g(t)}{\sqrt{\kappa}} \tag{41}$$

Where $\tau = \int^t dtg^2(t)/\kappa$, and $g(t)$ is a time-dependant coupling between the atomic ensemble and the cavity field, and κ is the decay rate of the cavity. Inserting these equations we obtain:

$$\begin{aligned}\langle\phi_{20}|\Psi\rangle = \langle 0|\frac{1}{\sqrt{2}}&\left(\sigma(0)e^{-\tau} + i\sqrt{2}e^{-\tau}\int^{t_w} dt' e^{\tau} E_{\text{in}}(t)\frac{g(t)}{\sqrt{\kappa}}\right)\\ &\left(\sigma(0)e^{-\tau} + i\sqrt{2}e^{-\tau}\int^{t_w} dt' e^{\tau} E_{\text{in}}(t)\frac{g(t)}{\sqrt{\kappa}}\right)\sigma(t=0)^{\dagger}\int d\omega\psi(w)E_0^{\dagger}(w)|0\rangle\end{aligned} \tag{42}$$

As shown before, we know the following relation,

$$E_{\text{in}}\int dw'\phi(\omega')E_0^{\dagger}(\omega') = \mathcal{F}[\phi(\omega)] \tag{43}$$

And we know that the only terms that will survive in $\langle\phi|$ must contain $E_{\text{in}}(t)\sigma(t)$. This leaves the following expansion:

$$\langle\psi_{20}|\Psi\rangle = 2ie^{-2\tau}\sigma(0)\int^{t} E_{in}(t')\frac{g(t')}{\sqrt{\kappa}}e^{\tau'}\sigma^{\dagger}(0)\int dw' E_0^{\dagger}(\omega')\phi(\omega') \tag{44}$$

$$= 2ie^{-2\tau}\int d\tau' e^{\tau'}\mathcal{F}(\phi(\omega))\frac{\sqrt{\kappa}}{g(\tau')} \tag{45}$$

Now, we can also make a simplifying assumption, by noting that the optimal read-in condition is met when $\mathcal{F}(\psi(\omega)) = A\sqrt{\frac{2}{\kappa}}g(t)e^{\tau(t)}$ and under this condition is met, the above equation reduces to:

$$\langle\psi_{20}|\Psi\rangle = A2\sqrt{2}ie^{-2\tau}\int d\tau' e^{2\tau} \tag{46}$$

$$= A\sqrt{2}i\left(1-e^{-2\tau}\right) \tag{47}$$

With the probability equal to:

$$|\langle\psi_{20}|\Psi\rangle|^2 = |A2\sqrt{2}ie^{-2\tau}\int d\tau' e^{2\tau'}|^2 \tag{48}$$

$$= |A\sqrt{2}i(1-e^{-2\tau_w})|^2 \tag{49}$$

Also, as previously shown, we know that τ_w is equal to $\tau_w = \ln\sqrt{2}$, so we can evaluate the above equation to obtain (remembering that $A = 1$:

$$|\langle\psi_{20}|\Psi\rangle|^2 = |\sqrt{2}i\frac{1}{2}|^2 \tag{50}$$

$$= \frac{1}{2} \tag{51}$$

5 $|\psi_{02}\rangle$

We could invoke a conservation of probability to show that since this is the only other state that is possible in the double excitation space, it must have probability 1/2, however it is good to do the calculations out to double check.

The quantity that we are interested in is the probability of measuring a double excitation in the reservoir field:

$$|\langle\psi_{02}|\Psi\rangle|^2 = \int dt\int dt'\left|\langle 0|\frac{1}{\sqrt{2}}E_{\rm out}(t)E_{\rm out}(t')\sigma^\dagger(0)\int d\omega\phi(\omega)E_0(\omega)|0\rangle\right|^2 \tag{52}$$

Using the equations for $E_{\text{out}}(t)$, this becomes:

$$|\langle\psi_{02}|\Phi\rangle|^2 = \int dt \int dt' \left| \frac{1}{\sqrt{2}} \left(E_{\text{in}}(t) + i\sqrt{\frac{2}{\kappa}} g(t) \left(\sigma(0)e^{-\tau} + i\sqrt{2}e^{-\tau} \int^t dt''' e^\tau(t''') E_{\text{in}}(t''') \frac{g(t''')}{\sqrt{\kappa}} \right) \right) \times \right. \tag{53}$$

$$\left. \left(E_{\text{in}}(t') + i\sqrt{\frac{2}{\kappa}} g(t') \left(\sigma(0)e^{-\tau(t')} + i\sqrt{2}e^{-\tau(t')} \int^{t'} dt'' e^\tau(t'') E_{\text{in}}(t'') \frac{g(t'')}{\sqrt{\kappa}} \right) \right) \Psi\rangle \right|^2 \tag{54}$$

As before, we know that only terms that contain E_{in} and $\sigma(0)$ will survive the expansion. With this in mind, we can reduce the above equation to:

$$|\langle\psi_{02}|\Phi\rangle|^2 = \frac{1}{2} \int dt \int dt' \left| \langle 0| E_{\text{in}}(t) i\sqrt{\frac{2}{\kappa}} g(t') e^{-\tau(t')} \sigma(0) + E_{\text{in}}(t') i\sqrt{\frac{2}{\kappa}} g(t) e^{-\tau(t)} \sigma(0) + \right. \tag{55}$$

$$- i\sqrt{2} \frac{2}{\sqrt{\kappa}} g(t') g(t) e^{-\tau(t')-\tau(t)} \sigma(0) \int^{t'} dt'' E_{\text{in}}(t'') \frac{g(t'')}{\sqrt{\kappa}} e^{\tau(t'')} \tag{56}$$

$$\left. -i\sqrt{2} \frac{2}{\sqrt{\kappa}} g(t') g(t) e^{-\tau(t')-\tau(t)} \sigma(0) \int^{t} dt''' E_{\text{in}}(t''') \frac{g(t''')}{\sqrt{\kappa}} e^{\tau(t''')} |\Psi\rangle\Psi\rangle \right|^2 \tag{57}$$

Again, knowing that

$$\langle 0| E_{\text{in}}(t) \int d\omega E_0^\dagger(\omega) |0\rangle = \mathcal{F}\{\phi(\omega)\} \equiv \tilde{\phi}(t) \tag{58}$$

and again using the optimization condition: $\tilde{\phi}(t) = A\sqrt{\frac{2}{\kappa}} g(t) e^\tau$, we can obtain the following equation:

$$|\langle\psi_{02}|\Phi\rangle|^2 = A\frac{1}{2}\int dt\int dt'\left|\frac{2}{\kappa}ig(t)g(t')\left(e^{\tau-\tau'}+e^{-(\tau-\tau')}\right)-\right. \tag{59}$$

$$\left. i\frac{4}{\kappa}g(t)g(t')e^{-\tau-\tau(t')}\left(\int^{t'}d\tau(t'')e^{2\tau(t'')}+\int^{t}d\tau(t''')e^{2\tau(t''')}\right)\right|^2 \tag{60}$$

Integrating this, we obtain:

$$|\langle\psi_{02}|\Phi\rangle|^2 = A^2\frac{1}{2}\int dt\int dt'\left|\frac{2}{\kappa}ig(t)g(t')\left(e^{\tau-\tau'}+e^{-(\tau-\tau')}\right)-\right. \tag{61}$$

$$\left. i\frac{2}{\kappa}g(t)g(t')e^{-\tau-\tau(t')}\left(e^{2\tau(t')}-1-e^{2\tau(t)}-1\right)\right|^2 \tag{62}$$

$$= A^2\frac{1}{2}\int dt\int dt'\left|\frac{2}{\kappa}ig(t)g(t')\left(e^{\tau-\tau'}+e^{-(\tau-\tau')}\right)-\right. \tag{63}$$

$$\left. i\frac{2}{\kappa}g(t)g(t')\left(e^{\tau(t')-\tau(t)}+e^{\tau(t)-\tau(t')}-2e^{-\tau(t)-\tau(t')}\right)\right|^2 \tag{64}$$

There is some cancellation, leaving:

$$|\langle\psi_{02}|\Phi\rangle|^2 = A^2\int dt\int dt'\left|i2\frac{\sqrt{2}}{\kappa}g(t)g(t')e^{-\tau(t)-\tau(t')}\right|^2 \tag{65}$$

$$= A^2\frac{8}{4}\left(e^{-2\tau_w}-1\right)^2 \tag{66}$$

$$\tag{67}$$

Now, if we require that $\tau_w = \ln(\sqrt{2})$, this further reduces to:

$$|\langle\psi_{02}|\Phi\rangle|^2 = 2(-\frac{1}{2})(-\frac{1}{2}) = \frac{1}{2} \tag{68}$$

6 Full-Equivalence

We have shown that the CD memory behaves like a beamsplitter under the very specific condition that $\eta = 1/2$, however, this suggests that there may be some mapping whereupon the equations that describe the CD memory are exactly mapped to the equations that describe a beamsplitter.

First, if we plot Equations (36),(65),(48) (taking into account that the normalization factor A is dependant on $\tau(t = \infty)$), then we obtain the following graph:

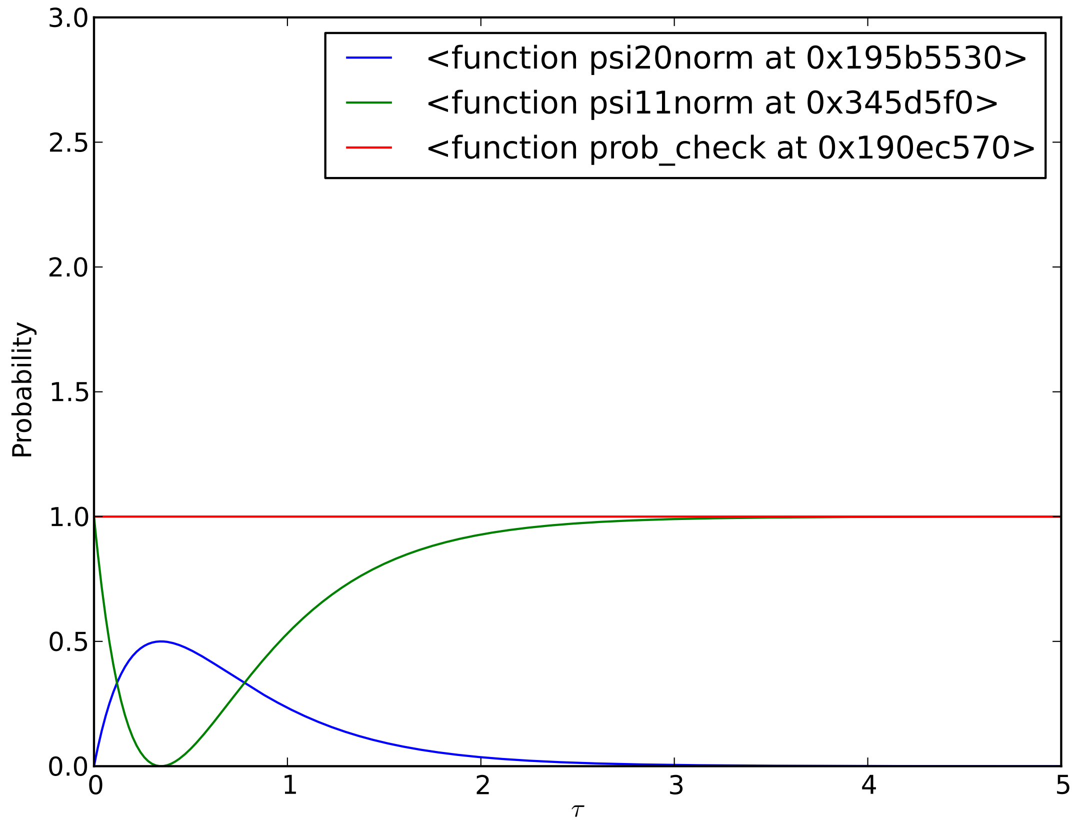

Figure 1: The function prob-check sums up all the probabilities. Note: the function psi20norm is equal to psi02norm, so I only show one of them. Also psi20norm is the normalize function $|\langle\psi_2 0|\Psi\rangle|^2$

However, if we plot the same functions vs the efficiency, the graph of the probabilities becomes:

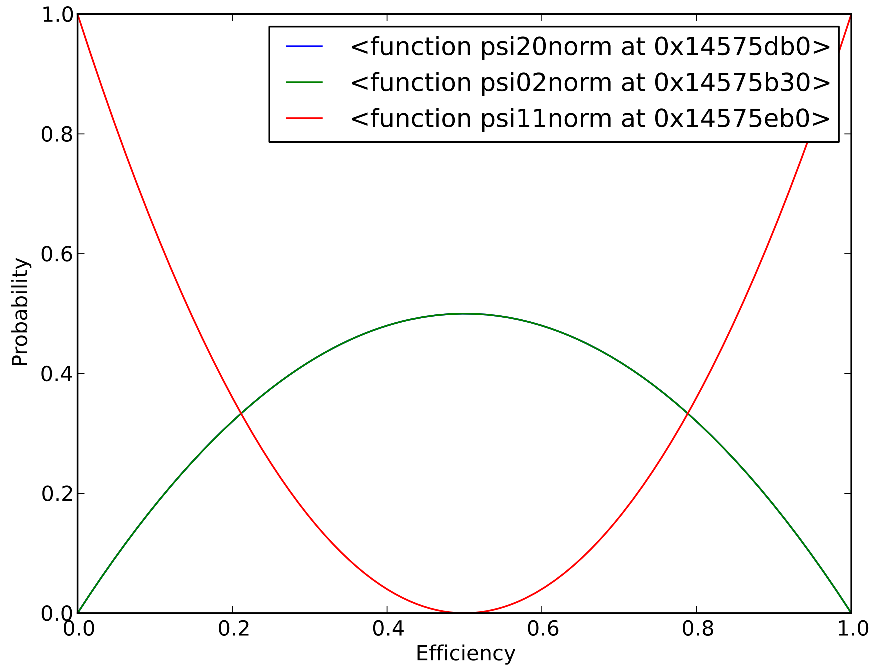

Figure 2: The same functions, plotted as a function of the efficiency of the system. Intuitively, we can interpret the efficiency as equivalent to the transmission co-efficient of the beamsplitter equations, as it is a measure of how much a single photon will come out of the system into the other 'mode' (atom-to-photon or photon-to-atom (atom-to-photon or photon-to-atom)). In this paradigm, the transmission co-efficient would be a measure of how much a single photon was converted into an atomic excitation

The equations that define the beamsplitter system can be solved to obtain:

$$U_{BS} a U_{BS}^{\dagger} = a_{out} = a \cos|\theta| + b \sin|\theta| \tag{69}$$

$$U_{BS} b U_{BS}^{\dagger} = b_{out} = b \cos|\theta| - a \sin|\theta| \tag{70}$$

Where a is the destruction operator for a mode. In the CD memory, we can think of this as a destruction operator acting on the photon field. Then b is the destruction operator acting on the other mode coming into the beamsplitter. In the CD case, this would act on the atomic ensemble. Note, in this case we are in the Schrodinger picture, which is why we are looking at the dynamics corresponding to the unitary operator U_{BS}.

Now, if we find the probability of a state $\langle psi_{20}| = \frac{1}{\sqrt{2}} a^{\dagger}_{out} a^{\dagger}_{out} |0\rangle$, after preparing an initial state $|\Psi_1 1\rangle = a^{\dagger} b^{\dagger}\rangle$ this is equal to:

$$\langle \psi_{20} | \Psi_{11} \rangle = \sqrt{2} \cos|\theta| \sin|\theta| \tag{71}$$

However, using the equation for the transmission coefficient: $T = \cos|\theta|$, we can recast this equation into:

$$\langle \psi_{20} | \Psi_{11} \rangle = \sqrt{2} T \sqrt{(1 - T^2)} \tag{72}$$

Square to get probability:

$$|\langle \psi_{20} | \Psi_{11} \rangle|^2 = T^2 (1 - T^2) \tag{73}$$

And using a similar treatment, we can render the equation for the amplitude of a state $|\psi_{11}\rangle = a^{\dagger}_{\text{out}} b^{\dagger}_{\text{out}} |0\rangle$

$$\langle \psi_{11} | \Psi_{11} \rangle = \langle 0 | a^{\dagger}_{out} b^{\dagger}_{out} a^{\dagger} b^{\dagger} | 0 \rangle \tag{74}$$

$$= \cos^2|\theta| - \sin^2|\theta| \tag{75}$$

$$= -1 + 2T^2 \tag{76}$$

and then square to get the probability:

$$|\langle \psi_{11} | \Psi_{11} \rangle|^2 = |-1 + 2T^2|^2 \tag{77}$$

If we graph these equations, we can see that they are equivalent to the graph of the CD memory equations that were plotted as a function of the effiency.

6.1 Mapping of CD to Beamsplitter

If we take the equations derived that describe the dynamics of the CD-Memory in the beamsplitter regime, Eqs. (36),(65),(48) and rewrite them in terms of the efficiency, η, we can see a full analytical mapping between the equations for beamsplitters and the equations for CD.

Starting with a double excitation equation:

$$|\langle\psi_{20}|\Psi\rangle|^2 = |A\sqrt{2}i(1 - e^{-2\tau_w}) \tag{78}$$

We also know that the efficiency η, is equal to:

$$\eta = 1 - e^{-2\tau_w} \tag{79}$$

Additionally, we know that our normalization constant, A is:

$$A = \sqrt{\frac{1}{(e^{2\tau_w} - 1)}} \tag{80}$$

And this can be rewritten in terms of eta in the following way:

$$A = \sqrt{\frac{e^{-2\tau_w}}{1 - e^{-2\tau_w}}} \tag{81}$$

$$A = \sqrt{\frac{1-\eta}{\eta}} \tag{82}$$

And rewriting Eq (48) in the same we, we obtain:

$$|\langle\psi_{20}|\Psi\rangle|^2 = |A\sqrt{2}i(1-e^{-2\tau})|^2 \tag{83}$$
$$= |\sqrt{\frac{1-\eta}{\eta}}\sqrt{2}\eta|^2 \tag{84}$$
$$= |\sqrt{2}i\sqrt{\eta(1-\eta)}|^2 \tag{85}$$

We make the connection between the transmission coefficient and the efficiency by noting that:

$$T^2 = \eta$$

as T^2 will deal with the intensity of the field going through the beamsplitter, which is the analogy of how much of the atomic (photon) field is converted into the photon (atomic): this is what the efficiency measures.

Thus, the equation becomes:

$$|\langle\psi_{20}|\Psi\rangle|^2 = |\sqrt{2}T\sqrt{1-T^2}|^2 \tag{86}$$
$$= 2T^2(1-T^2) \tag{87}$$

which, as we have shown earlier is the beamsplitter equation, exactly.

Similarly, we can show the equivalence between the $\langle\psi_{11}|\Psi\rangle$ states in the beamsplitter picture and the CD memory.

$$|\langle\psi_{11}|\Phi\rangle|^2 = A^2 \int^{t'} dt \left| \sqrt{\frac{2}{\kappa}} g(t) e^{-\tau(t)} \left(-e^{\tau(t')} + 2e^{-\tau(t')} \right) \right|^2 \tag{88}$$

$$= \frac{e^{-2\tau_w}}{1 - e^{-2\tau_w}} \left(1 - e^{-2\tau_w}\right) \left(e^{tau_w} - 2e^{-\tau_w}\right)^2 \tag{89}$$

$$= \left(1 - 2e^{-2\tau_w}\right)^2 \tag{90}$$

$$= (-1 + 2 * \eta)^2 \tag{91}$$

$$= \left(-1 + 2 * T^2\right)^2 \tag{92}$$

7 Conclusion

I have shown full analytical equivalence between the CD-memory in the two photon regime, and a beam-splitter.