Photon-Bunching in Quantum Memory

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1 Introduction

There is a simple analogy between a beamsplitter and a quantum memory. If one thinks of a beamsplitter as two distinct modes, a and b, the action of the beamsplitter is to mix these modes in a certain way. For instance, a 50-50 beamsplitter means that a single photon in mode a will have a 50% chance of staying in mode a and a 50% of becoming mode b.

In the same way, a quantum memory deals with two modes: photon and atomic excitation. The action of the quantum memory will be to mix the two modes in a certain way. Usually, one wants to completely convert the photon mode into an atomic mode, or vis versa. However, if one used a 50% efficient quantum memory, we obtain a very similar situation as that of a beamsplitter.

This is interesting, as there is a phenomena that occurs in beamsplitters called 'photon-bunching', where–due to the boson statistics– if you have a photon incident on the beamsplitter in mode a, and another photon incident in mode b, you have a 50% chance of seeing two photons in mode a, and a 50% chance of seeing two photons in mode b.

The purpose of this report is to quantify and illustrate the extent of similarities between a beamsplitter and a quantum memory.

2 CD Memory

We can demonstrate photon bunching explicitly in the case of the CD memory[**?**]. This quantum memory is used because the author was familiar with the equations of motion.

We have previously derived the Heisenberg equations of motion for the operators of interest, E_{out} and $\sigma_{z?}$.

The heisenburg state we prepared for photon bunching is:

$$|\Psi\rangle = \sigma(t=0)^{\dagger} \int d\omega \psi(w) E_0^{\dagger}(w)|0\rangle \tag{1}$$

Where $E_0(\omega)$ is the creation operator of a photon of frequency ω, $\psi(\omega)$ is the single-photon envelope in frequency space, and $\sigma(t=0)$ is the initial creation operator of a atomic excitation. In what follows, we will be assuming that a photon is already stored in the memory at $t=0$, and that another pulse is incident.

The three terms we are concerned with are:

$$\langle\psi_{20}|\Psi\rangle = \langle 0|\frac{1}{\sqrt{2}}\sigma(t)\sigma(t)|\Psi\rangle \tag{2}$$

$$\langle\psi_{11}|\Psi\rangle = \langle 0|\sigma(t')\int^{t'} dt E_{\text{out}}(t)|\Psi\rangle \tag{3}$$

$$\langle\psi_{02}|\Psi\rangle = \langle 0|\frac{1}{\sqrt{2}}\int dt \int dt' E_{\text{out}}(t)E_{\text{out}}(t')|\Psi\rangle \tag{4}$$

Where $|\psi_{20}\rangle$ is a double excitation in the photon field, $|\psi_{11}\rangle$ is photon and an atomic excitation, and $|\psi_{02}\rangle$ is a double atomic excitation.

3 $|\psi_{11}\rangle$

First, we consider the term that deals with a single excitation in the field and a single excitation in the atom. If photon-bunching is seen in this memory, then this term should equal zero.

$$|\langle\psi_{11}|\Phi\rangle|^2 = \int^{t'} dt \left|\langle 0|E_{\text{out}}(t)\sigma(t')\int d\omega\phi(\omega)E_0^{\dagger}(\omega)\sigma^{\dagger}(0)|0\rangle\right|^2 \tag{5}$$

(6)

Physically (in terms of detectors), the above equation corresponds to having two measurements. In the first, you are directly measuring the atomic ensemble at time $t = t'$. In the second, you have left a photon detector on to measure the resaviour field from time $t = -\infty, t'$. Concisely, this measures the probability of both: seeing a photon in the resovior field, and seeing an atomic excitation.

We also know the equations for $E_{\text{out}}(t)$ and $\sigma(t)$ from our previous work.

$$E_{\text{out}}(t) = E_{\text{in}}(t) + i\sqrt{\frac{2}{\kappa}}g(t)\sigma(t) \tag{7}$$

$$\sigma(t) = \sigma(0)e^{-\tau} + i\sqrt{2}e^{-\tau}\int_{w}^{\tau} dt' e^{\tau} E_{\text{in}}(t)\frac{g(t)}{\sqrt{\kappa}} \tag{8}$$

Where $\tau = \int^{t} dt g^{2}(t)/\kappa$, $g(t)$ is a time-dependant coupling between the atomic ensemble and the cavity field, and κ is the decay rate of the cavity. Inserting these equations we obtain:

$$|\langle\psi_{11}|\Phi\rangle|^{2} = \int^{t'} dt \left|\langle 0| \left(E_{\text{in}}(t) + i\sqrt{\frac{2}{\kappa}}g(t)\sigma(t)\right) \times \right. \tag{9}$$

$$\left. \left(\sigma(0)e^{-\tau} + i\sqrt{2}e^{-\tau}\int_{w}^{\tau} dt' e^{\tau} E_{\text{in}}(t)\frac{g(t)}{\sqrt{\kappa}}\right)|\Psi\rangle\right|^{2} \tag{10}$$

$$\tag{11}$$

Expanding this out, we get the following terms:

$$|\langle\psi_{11}|\Phi\rangle|^{2} = \int^{t'} dt \left|\langle 0|E_{\text{in}}(t)\left(\sigma(0)e^{-\tau(t')} + i\sqrt{2}e^{-\tau(t')}\int^{t'} dt'' e^{\tau(t'')}E_{\text{in}}(t'')\frac{g(t'')}{\sqrt{\kappa}}\right)\right. \tag{12}$$

$$+ i\sqrt{\frac{2}{\kappa}}g(t)\left(\sigma(0)e^{-\tau} + i\sqrt{2}e^{-\tau}\int^{t} dt''' e^{\tau(t''')}E_{\text{in}}(t''')\frac{g(t''')}{\sqrt{\kappa}}\right) \times \tag{13}$$

$$\left. \left(\sigma(0)e^{-\tau(t')} + i\sqrt{2}e^{-\tau(t')}\int^{t'} dt'' e^{\tau}(t'')E_{\text{in}}(t'')\frac{g(t'')}{\sqrt{\kappa}}\right)|\Psi\rangle\right|^{2} \tag{14}$$

If we apply the commutation relations, we can get rid of any terms that don't have $E_{\text{in}}(t)$ and $\sigma(0)$ in them, as any term in the above equation that doesn't have both of these operators will be annihilated through simple commutations. Ie. if a term only has $E_{\text{in}}E_{\text{in}}$, we know that $|\Psi\rangle$ contains a $\sigma^\dagger(0)$, and that $\sigma^\dagger(0)$ and E_{in} commute, so $\sigma^\dagger(0)$ can move through and act as $\langle 0|\sigma^\dagger(0) = 0$. So any term that survives must contain terms that don't commute with E_{in} as well as $\sigma(0)$.

This leaves us with the following terms:

$$|\langle\psi_{11}|\Phi\rangle|^2 = \int^{t'} dt \Bigg| \langle 0|E_{\text{in}}(t)\sigma(0)e^{-\tau(t')}\sigma^\dagger(0)\int d\omega E_0(\omega)\phi(\omega)|0\rangle - \tag{15}$$

$$\langle 0|\frac{2}{\sqrt{\kappa}}g(t)\sigma(0)e^{-\tau(t)-\tau(t')}\int^{t'} dt'' e^{\tau(t'')}E_{\text{in}}(t'')\frac{g(t'')}{\sqrt{\kappa}}\sigma^\dagger(0)\int d\omega E_0(\omega)\phi(\omega)|0\rangle - \tag{16}$$

$$\langle 0|\frac{2}{\sqrt{\kappa}}g(t)\sigma(0)e^{-\tau(t)-\tau(t')}\int^{t} dt''' e^{\tau(t''')}E_{\text{in}}(t''')\frac{g(t''')}{\sqrt{\kappa}}\sigma^\dagger(0)\int d\omega E_0(\omega)\phi(\omega)|0\rangle\Bigg|^2 \tag{17}$$

We know that E_{in} is the input-pulse, and can also be defined in frequency space as $E_{\text{in}} = \int dw e^{-iwt}E_0(\omega)$ and furthermore, $[E_0(\omega), E_0^\dagger(\omega')] = \delta(\omega - \omega')$, thus we know that

$$\langle 0|E_{\text{in}}(t)\int d\omega\phi(\omega)E_0^\dagger(\omega)|0\rangle \tag{18}$$

$$= \langle 0|\int d\omega' \int d\omega E_0(\omega')E_0(\omega)\phi(\omega)e^{-i\omega t}|0\rangle \tag{19}$$

$$= \int d\omega' \int d\omega\delta(\omega' - \omega)\phi(\omega)e^{-i\omega t} \tag{20}$$

$$= \mathcal{F}\{\phi(\omega)\} \equiv \tilde{\phi}(t) \tag{21}$$

using this property, the equation reduces to:

$$|\langle\psi_{11}|\Phi\rangle|^2 = \int^{t'} dt \left|\tilde{\phi}(t)e^{-\tau(t')} - \frac{2}{\sqrt{\kappa}}g(t)e^{-\tau(t)-\tau(t')}\int^{t'}\tilde{\phi}(t'')\frac{g(t'')}{\sqrt{\kappa}}dt'' - \right. \tag{22}$$

$$\left. \frac{2}{\sqrt{\kappa}}g(t)e^{-\tau(t)-\tau(t')}\int^{t}\tilde{\phi}(t''')\frac{g(t''')}{\sqrt{\kappa}}dt''' \right|^2 \tag{23}$$

Note, we haven't shown how the $\sigma(0)\sigma^{\dagger}(0)$ term vanishes, but using the commutator properties, you can show that in this case you can simply replace this term with unity.

This is a completely general equation, as we have not assumed anything about the shape of the incoming field. To simply this equation however, we can make the assumption that the incoming field meets the optimal condition discussed in the original paper.

This allows us to do two things. First, because the field is optimally conditioned, all of the efficiency is controlled by the value of τ_w, so we only have one variable to deal with. Second, it will allow the above equation to be integrated very simply.

The optimal condition requires that: $\tilde{\phi}(t) = \sqrt{\frac{2}{\kappa}}g(t)e^{\tau(t)}$

However, we also have to normalize the incoming field, such that:

$$\int_{\infty}^{\infty} |\tilde{\phi}(t)|^2 dt' = 1 \tag{24}$$

Plugging the optimal field condition in, the normalization condition reduces to:

$$\int_{\infty}^{\infty} A^2|\tilde{\phi}(t)|^2 dt' = 1 \tag{25}$$

$$\int_{\infty}^{\infty} A^2\frac{2}{\kappa}g^2(t)e^{2\tau(t)} = 1 \tag{26}$$

$$A^2\left(e^{2\tau_w} - 1\right) = 1 \tag{27}$$

Thus, the value of τ when $t = \infty$ defines the normalization constant.

$$A = \sqrt{\frac{1}{(e^{2\tau_w} - 1)}} \tag{28}$$

We can also note that in the specific case where $\eta = 1/2$, i.e. when the efficiency is one-half, then $\tau_w = \ln(\sqrt{2})$ and our normalization coefficient becomes:

$$A = \sqrt{\frac{1}{\left(e^{2\ln(\sqrt{2})} - 1\right)}} \tag{29}$$

$$A = \sqrt{\frac{1}{(2 - 1)}} \tag{30}$$

$$A = 1 \tag{31}$$

Inserting this into our equation, we obtain:

$$|\langle\psi_{11}|\Phi\rangle|^2 = A^2 \int^{t'} dt \left| \sqrt{\frac{2}{\kappa}} g(t) e^{\tau(t)-\tau(t')} - 2\sqrt{\frac{2}{\kappa}} g(t) e^{-\tau(t)-\tau(t')} \int^{t'} \frac{g(t'')^2}{\kappa} e^{2\tau(t'')} dt'' \right. \tag{32}$$

$$\left. -2\sqrt{\frac{2}{\kappa}} g(t) e^{\tau(t')-\tau(t)} \int^{t} \frac{g(t''')^2}{\kappa} e^{2\tau(t''')} dt''' \right|^2 \tag{33}$$

and, knowing that $\frac{g^2(t)}{\kappa} dt = d\tau$, the above can be integrate to obtain:

$$|\langle\psi_{11}|\Phi\rangle|^2 = A^2 \int^{t'} dt \left| \sqrt{\frac{2}{\kappa}} g(t) e^{\tau(t)-\tau(t')} - \sqrt{\frac{2}{\kappa}} g(t) \left(e^{\tau(t)-\tau(t')} - e^{-\tau(t')-\tau(t)} \right) \right. \tag{34}$$

$$\left. -\sqrt{\frac{2}{\kappa}} g(t) \left(e^{\tau(t')-\tau(t)} - e^{-\tau(t')-\tau(t)} \right) \right|^2 \tag{35}$$

which can be factored to obtain:

$$|\langle\psi_{11}|\Phi\rangle|^2 = A^2 \int^{t'} dt \left|\sqrt{\frac{2}{\kappa}} g(t) e^{-\tau(t)} \left(-e^{\tau(t')} + 2e^{-\tau(t')}\right)\right|^2 \tag{36}$$

If we use the condition that $\eta = 0.5$, then $\tau(t') = \ln(\sqrt{2})$. It should be noted that $\tau(t = \infty) = \tau(t')$. That is, we have let an infinite time pass when we do the measurement. We are preforming a measurement on the atomic ensemble at time t', but we have left out photon detectors on from time t_0 to t'. However, because we know what $\tau(t')$ is equal to, we can evaluate the term in brackets in the above equation.

$$|\langle\psi_{11}|\Phi\rangle|^2 = A^2 \int^{t'} dt \left|\sqrt{\frac{2}{\kappa}} g(t) e^{-\tau(t)} \left(e^{\ln(\sqrt{2})} - 2\frac{1}{e^{\ln(\sqrt{2})}}\right)\right|^2 = 0 \tag{37}$$

This shows that there is zero chance to measure a photon in the reservoir field and an atomic excitation when the quantum is operating at 50% efficiency. This is precisely what is expected for a beam-splitter.

4 $|\psi_{20}\rangle$

Next, we deal with the case term that measures the amplitude of obtaining a double excitation in the atomic ensemble.

$$\langle\psi_{20}|\Psi\rangle = \langle 0|\frac{1}{\sqrt{2}}\sigma(t)\sigma(t)|\Psi\rangle \tag{38}$$

$$\langle\psi_{20}|\Psi\rangle = \langle 0|\frac{1}{\sqrt{2}}\sigma\sigma\sigma(t=0)^\dagger \int d\omega\psi(w)E_0^\dagger(w)|0\rangle \tag{39}$$

and the heisenburg equations for the operators E_{out} and σ are:

$$E_{\text{out}}(t) = E_{\text{in}}(t) + i\sqrt{\frac{2}{\kappa}}g(t)\sigma(t) \tag{40}$$

$$\sigma(t) = \sigma(0)e^{-\tau} + i\sqrt{2}e^{-\tau}\int^{t_w} dt'e^{\tau}E_{\text{in}}(t)\frac{g(t)}{\sqrt{\kappa}} \tag{41}$$

Where $\tau = \int^t dtg^2(t)/\kappa$, and $g(t)$ is a time-dependant coupling between the atomic ensemble and the cavity field, and κ is the decay rate of the cavity. Inserting these equations we obtain:

$$\begin{aligned}\langle\phi_{20}|\Psi\rangle = \langle 0|\frac{1}{\sqrt{2}}&\left(\sigma(0)e^{-\tau} + i\sqrt{2}e^{-\tau}\int^{t_w} dt'e^{\tau}E_{\text{in}}(t)\frac{g(t)}{\sqrt{\kappa}}\right) \\ &\left(\sigma(0)e^{-\tau} + i\sqrt{2}e^{-\tau}\int^{t_w} dt'e^{\tau}E_{\text{in}}(t)\frac{g(t)}{\sqrt{\kappa}}\right)\sigma(t=0)^{\dagger}\int d\omega\psi(w)E_0^{\dagger}(w)|0\rangle\end{aligned} \tag{42}$$

As shown before, we know the following relation,

$$E_{\text{in}}\int dw'\phi(\omega')E_0^{\dagger}(\omega') = \mathcal{F}[\phi(\omega)] \tag{43}$$

And we know that the only terms that will survive in $\langle\phi|$ must contain $E_{\text{in}}(t)\sigma(t)$. This leaves the following expansion:

$$\langle\psi_{20}|\Psi\rangle = 2ie^{-2\tau}\sigma(0)\int^{t} E_{in}(t')\frac{g(t')}{\sqrt{\kappa}}e^{\tau'}\sigma^{\dagger}(0)\int dw'E_0^{\dagger}(\omega')\phi(\omega') \tag{44}$$

$$= 2ie^{-2\tau}\int d\tau'e^{\tau'}\mathcal{F}(\phi(\omega))\frac{\sqrt{\kappa}}{g(\tau')} \tag{45}$$

Now, we can also make a simplifying assumption, by noting that the optimal read-in condition is met when $\mathcal{F}(\psi(\omega)) = A\sqrt{\frac{2}{\kappa}}g(t)e^{\tau(t)}$ and under this condition is met, the above equation reduces to:

$$\langle\psi_{20}|\Psi\rangle = A2\sqrt{2}ie^{-2\tau}\int d\tau' e^{2\tau} \tag{46}$$

$$= A\sqrt{2}i\left(1 - e^{-2\tau}\right) \tag{47}$$

With the probability equal to:

$$|\langle\psi_{20}|\Psi\rangle|^2 = |A2\sqrt{2}ie^{-2\tau}\int d\tau' e^{2\tau'}|^2 \tag{48}$$

$$= |A\sqrt{2}i(1 - e^{-2\tau_w})|^2 \tag{49}$$

Also, as previously shown, we know that τ_w is equal to $\tau_w = \ln\sqrt{2}$, so we can evaluate the above equation to obtain (remembering that $A = 1$:

$$|\langle\psi_{20}|\Psi\rangle|^2 = |\sqrt{2}i\frac{1}{2}|^2 \tag{50}$$

$$= \frac{1}{2} \tag{51}$$

5 $|\psi_{02}\rangle$

We could invoke a conservation of probability to show that since this is the only other state that is possible in the double excitation space, it must have probability 1/2, however it is good to do the calculations out to double check.

The quantity that we are interested in is the probability of measuring a double excitation in the reservoir field:

$$|\langle\psi_{02}|\Psi\rangle|^2 = \int dt \int dt' \left|\langle 0|\frac{1}{\sqrt{2}}E_{\text{out}}(t)E_{\text{out}}(t')\sigma^\dagger(0)\int d\omega\phi(\omega)E_0(\omega)|0\rangle\right|^2 \tag{52}$$

Using the equations for $E_{\text{out}}(t)$, this becomes:

$$|\langle\psi_{02}|\Phi\rangle|^2 = \int dt \int dt' \left| \frac{1}{\sqrt{2}} \left(E_{\text{in}}(t) + i\sqrt{\frac{2}{\kappa}} g(t) \left(\sigma(0)e^{-\tau} + i\sqrt{2}e^{-\tau} \int^t dt''' e^\tau(t''') E_{\text{in}}(t''') \frac{g(t''')}{\sqrt{\kappa}} \right) \right) \times \right. \tag{53}$$

$$\left. \left(E_{\text{in}}(t') + i\sqrt{\frac{2}{\kappa}} g(t') \left(\sigma(0)e^{-\tau(t')} + i\sqrt{2}e^{-\tau(t')} \int^{t'} dt'' e^\tau(t'') E_{\text{in}}(t'') \frac{g(t'')}{\sqrt{\kappa}} \right) \right) \Psi\rangle \right|^2 \tag{54}$$

As before, we know that only terms that contain E_{in} and $\sigma(0)$ will survive the expansion. With this in mind, we can reduce the above equation to:

$$|\langle\psi_{02}|\Phi\rangle|^2 = \frac{1}{2} \int dt \int dt' \left| \langle 0| E_{\text{in}}(t) i\sqrt{\frac{2}{\kappa}} g(t') e^{-\tau(t')} \sigma(0) + E_{\text{in}}(t') i\sqrt{\frac{2}{\kappa}} g(t) e^{-\tau(t)} \sigma(0) + \right. \tag{55}$$

$$- i\sqrt{2} \frac{2}{\sqrt{\kappa}} g(t') g(t) e^{-\tau(t')-\tau(t)} \sigma(0) \int^{t'} dt'' E_{\text{in}}(t'') \frac{g(t'')}{\sqrt{\kappa}} e^{\tau(t'')} \tag{56}$$

$$\left. -i\sqrt{2} \frac{2}{\sqrt{\kappa}} g(t') g(t) e^{-\tau(t')-\tau(t)} \sigma(0) \int^t dt''' E_{\text{in}}(t''') \frac{g(t''')}{\sqrt{\kappa}} e^{\tau(t''')} |\Psi\rangle \Psi\rangle \right|^2 \tag{57}$$

Again, knowing that

$$\langle 0| E_{\text{in}}(t) \int d\omega E_0^\dagger(\omega) |0\rangle = \mathcal{F}\{\phi(\omega)\} \equiv \tilde{\phi}(t) \tag{58}$$

and again using the optimization condition: $\tilde{\phi}(t) = A\sqrt{\frac{2}{\kappa}} g(t) e^\tau$, we can obtain the following equation:

$$|\langle\psi_{02}|\Phi\rangle|^2 = A\frac{1}{2}\int dt \int dt' \left|\frac{2}{\kappa} i g(t) g(t') \left(e^{\tau-\tau'} + e^{-(\tau-\tau')}\right) - \right. \tag{59}$$

$$\left. i\frac{4}{\kappa} g(t) g(t') e^{-\tau-\tau(t')} \left(\int^{t'} d\tau(t'') e^{2\tau(t'')} + \int^{t} d\tau(t''') e^{2\tau(t''')}\right)\right|^2 \tag{60}$$

Integrating this, we obtain:

$$|\langle\psi_{02}|\Phi\rangle|^2 = A^2\frac{1}{2}\int dt \int dt' \left|\frac{2}{\kappa} i g(t) g(t') \left(e^{\tau-\tau'} + e^{-(\tau-\tau')}\right) - \right. \tag{61}$$

$$\left. i\frac{2}{\kappa} g(t) g(t') e^{-\tau-\tau(t')} \left(e^{2\tau(t')} - 1 - e^{2\tau(t)} - 1\right)\right|^2 \tag{62}$$

$$= A^2\frac{1}{2}\int dt \int dt' \left|\frac{2}{\kappa} i g(t) g(t') \left(e^{\tau-\tau'} + e^{-(\tau-\tau')}\right) - \right. \tag{63}$$

$$\left. i\frac{2}{\kappa} g(t) g(t') \left(e^{\tau(t')-\tau(t)} + e^{\tau(t)-\tau(t')} - 2e^{-\tau(t)-\tau(t')}\right)\right|^2 \tag{64}$$

There is some cancellation, leaving:

$$|\langle\psi_{02}|\Phi\rangle|^2 = A^2\int dt \int dt' \left|i2\frac{\sqrt{2}}{\kappa} g(t) g(t') e^{-\tau(t)-\tau(t')}\right|^2 \tag{65}$$

$$= A^2\frac{8}{4}\left(e^{-2\tau_w} - 1\right)^2 \tag{66}$$

$$\tag{67}$$

Now, if we require that $\tau_w = \ln(\sqrt{2})$, this further reduces to:

$$|\langle\psi_{02}|\Phi\rangle|^2 = 2(-\frac{1}{2})(-\frac{1}{2}) = \frac{1}{2} \tag{68}$$

6 Full-Equivalence

We have shown that the CD memory behaves like a beamsplitter under the very specific condition that $\eta = 1/2$, however, this suggests that there may be some mapping whereupon the equations that describe the CD memory are exactly mapped to the equations that describe a beamsplitter.

First, if we plot Equations (36),(65),(48) (taking into account that the normalization factor A is dependant on $\tau(t = \infty)$), then we obtain the following graph:

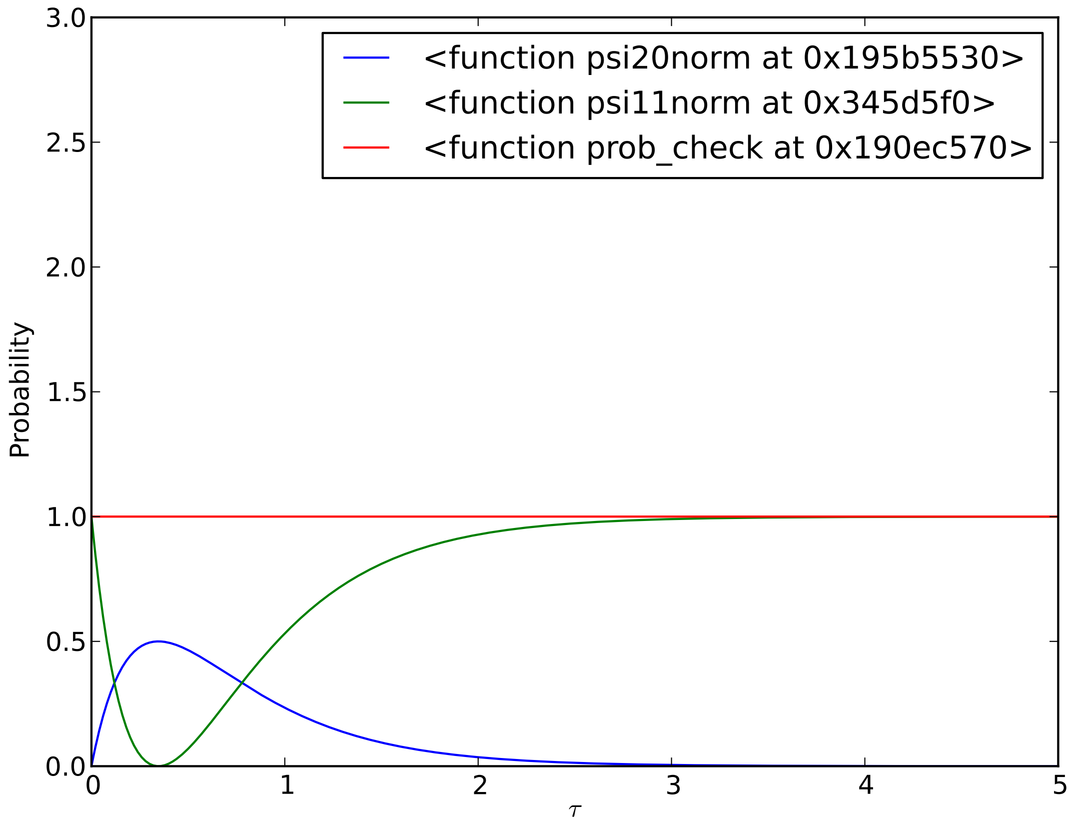

Figure 1: The function prob-check sums up all the probabilities. Note: the function psi20norm is equal to psi02norm, so I only show one of them. Also psi20norm is the normalize function $|\langle\psi_2 0|\Psi\rangle|^2$

However, if we plot the same functions vs the efficiency, the graph of the probabilities becomes:

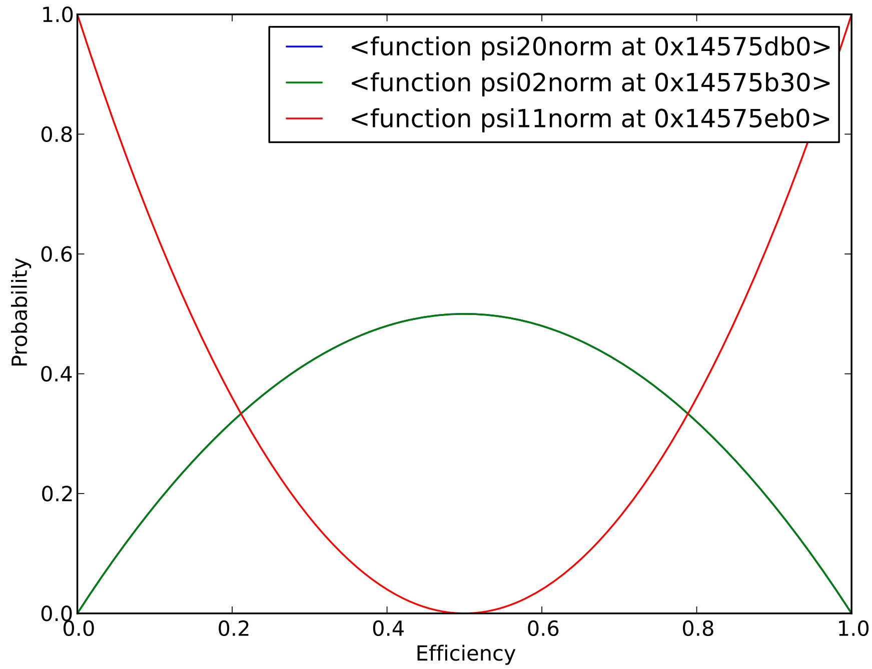

Figure 2: The same functions, plotted as a function of the efficiency of the system. Intuitively, we can interpret the efficiency as equivalent to the transmission co-efficient of the beamsplitter equations, as it is a measure of how much a single photon will come out of the system into the other 'mode' (atom-to-photon or photon-to-atom (atom-to-photon or photon-to-atom)). In this paradigm, the transmission co-efficient would be a measure of how much a single photon was converted into an atomic excitation

The equations that define the beamsplitter system can be solved to obtain:

$$U_{BS} a U_{BS}^{\dagger} = a_{out} = a \cos|\theta| + b \sin|\theta| \tag{69}$$

$$U_{BS} b U_{BS}^{\dagger} = b_{out} = b \cos|\theta| - a \sin|\theta| \tag{70}$$

Where a is the destruction operator for a mode. In the CD memory, we can think of this as a destruction operator acting on the photon field. Then b is the destruction operator acting on the other mode coming into the beamsplitter. In the CD case, this would act on the atomic ensemble. Note, in this case we are in the Schrodinger picture, which is why we are looking at the dynamics corresponding to the unitary operator U_{BS}.

Now, if we find the probability of a state $\langle psi_{20}| = \frac{1}{\sqrt{2}} a^{\dagger}_{out} a^{\dagger}_{out} |0\rangle$, after preparing an initial state $|\Psi_1 1\rangle = a^{\dagger} b^{\dagger}\rangle$ this is equal to:

$$\langle \psi_{20} | \Psi_{11} \rangle = \sqrt{2} \cos|\theta| \sin|\theta| \tag{71}$$

However, using the equation for the transmission coefficient: $T = \cos|\theta|$, we can recast this equation into:

$$\langle \psi_{20} | \Psi_{11} \rangle = \sqrt{2} T \sqrt{(1 - T^2)} \tag{72}$$

Square to get probability:

$$|\langle \psi_{20} | \Psi_{11} \rangle|^2 = T^2 (1 - T^2) \tag{73}$$

And using a similar treatment, we can render the equation for the amplitude of a state $|\psi_{11}\rangle = a^{\dagger}_{\text{out}} b^{\dagger}_{\text{out}} |0\rangle$

$$\langle \psi_{11} | \Psi_{11} \rangle = \langle 0 | a^{\dagger}_{out} b^{\dagger}_{out} a^{\dagger} b^{\dagger} |0\rangle \tag{74}$$

$$= \cos^2|\theta| - \sin^2|\theta| \tag{75}$$

$$= -1 + 2T^2 \tag{76}$$

and then square to get the probability:

$$|\langle \psi_{11} | \Psi_{11} \rangle|^2 = |-1 + 2T^2|^2 \tag{77}$$

If we graph these equations, we can see that they are equivalent to the graph of the CD memory equations that were plotted as a function of the effiency.

6.1 Mapping of CD to Beamsplitter

If we take the equations derived that describe the dynamics of the CD-Memory in the beamsplitter regime, Eqs. (36),(65),(48) and rewrite them in terms of the efficiency, η, we can see a full analytical mapping between the equations for beamsplitters and the equations for CD.

Starting with a double excitation equation:

$$|\langle\psi_{20}|\Psi\rangle|^2 = |A\sqrt{2}i(1 - e^{-2\tau_w}) \tag{78}$$

We also know that the efficiency η, is equal to:

$$\eta = 1 - e^{-2\tau_w} \tag{79}$$

Additionally, we know that our normalization constant, A is:

$$A = \sqrt{\frac{1}{(e^{2\tau_w} - 1)}} \tag{80}$$

And this can be rewritten in terms of *eta* in the following way:

$$A = \sqrt{\frac{e^{-2\tau_w}}{1 - e^{-2\tau_w}}} \tag{81}$$

$$A = \sqrt{\frac{1-\eta}{\eta}} \tag{82}$$

And rewriting Eq (48) in the same we, we obtain:

$$|\langle\psi_{20}|\Psi\rangle|^2 = |A\sqrt{2}i(1-e^{-2\tau})|^2 \tag{83}$$
$$= |\sqrt{\frac{1-\eta}{\eta}}\sqrt{2}\eta|^2 \tag{84}$$
$$= |\sqrt{2}i\sqrt{\eta(1-\eta)}|^2 \tag{85}$$

We make the connection between the transmission coefficient and the efficiency by noting that:

$$T^2 = \eta$$

as T^2 will deal with the intensity of the field going through the beamsplitter, which is the analogy of how much of the atomic (photon) field is converted into the photon (atomic): this is what the efficiency measures.

Thus, the equation becomes:

$$|\langle\psi_{20}|\Psi\rangle|^2 = |\sqrt{2}T\sqrt{1-T^2}|^2 \tag{86}$$
$$= 2T^2(1-T^2) \tag{87}$$

which, as we have shown earlier is the beamsplitter equation, exactly.

Similarly, we can show the equivalence between the $\langle\psi_{11}|\Psi\rangle$ states in the beamsplitter picture and the CD memory.

$$|\langle\psi_{11}|\Phi\rangle|^2 = A^2 \int^{t'} dt \left| \sqrt{\frac{2}{\kappa}} g(t) e^{-\tau(t)} \left(-e^{\tau(t')} + 2e^{-\tau(t')} \right) \right|^2 \tag{88}$$

$$= \frac{e^{-2\tau_w}}{1 - e^{-2\tau_w}} \left(1 - e^{-2\tau_w}\right) \left(e^{tau_w} - 2e^{-\tau_w}\right)^2 \tag{89}$$

$$= \left(1 - 2e^{-2\tau_w}\right)^2 \tag{90}$$

$$= (-1 + 2 * \eta)^2 \tag{91}$$

$$= \left(-1 + 2 * T^2\right)^2 \tag{92}$$

7 AFC

The afc memory has some differences from the CD-Memory, but it is also equivalent to a beamsplitter. Instead of the two channels being a memory storage and an emitted photon, you can think of the two channels as being two photons, but time delayed.

The section is laid out as follows:

Section 7.1 : I lay out the equations of motion

Section 7.2 : A high-level description of what is happening

Section 7.3.3 : A walk through of the calculations for the beamsplitting process.

Section 7.5 : A presentation of the general case, where I plot the probabilites of the three states as a function of the efficiency/

7.1 Equations of Motion

The equations of motion for the AFC memory in a cavity are:

$$\dot{\sigma}_\omega(t) = -i\omega\sigma - \gamma_h\sigma + iPE \tag{93}$$

$$\dot{E}(t) = -\kappa E + \sqrt{2\kappa}E_{in} + i\tilde{P}\int d\omega n(\omega)\sigma_\omega \tag{94}$$

$$E_{out} = -E_{in} + \sqrt{2\kappa}E \tag{95}$$

We can easily solve the equation for sigma, and then inserting it into the equation for the cavity field. If we additionaly make the leaky cavity assumption, we can set $\dot{E} = 0$, and and we have a solution for E, given by:

$$0 = -\kappa E + \sqrt{2\kappa}E_{in} + iP\tilde{P}\int dt'\tilde{n}(t-t')E(t') \tag{96}$$

You can see the AFC paper for more details, but because of the nature of the frequency comb, the function $\tilde{n}(t-t')$ can be approximated by a chain of delta functions, which are spaced apart by $\frac{2\pi}{\Delta}$ in time.

Because of this, we can solve for local solutions for E around peaks in the comb. In fact, because the integral is over all time, we need to use a recursive strategy to solve for the E field at later times. This is because the E field at some time $t \approx n\frac{2\pi}{\Delta}$ will depend on every previous E field, due to the integral term $iP\tilde{P}\int_{-\infty}^{t}\tilde{n}(t-t')E(t')dt'$

7.2 High-Level

The protocol works in the following way. There is an AFC memory in a cavity. The cavity has a decay rate of κ that can be changed somehow. There is a pulse of light sent in with a time-envelope of $\Theta(t)$, such that is incident upon the cavity at around $t \sim 0$. Because of assumptions made in the AFC cavity protocol, the incident light pulse will overlap with a narrow gaussian function, that can be approximated as a delta function. We additionally assume that the impedence matching condition is on, such that $\kappa = \Gamma$, where Γ is the absorption rate for the atoms within the gaussian spectral density. It has been shown that this can result in arbitrary close to unity efficiency.

This means that there is one pulse now stored in the medium. Each 'tooth', or narrow gaussian function within the atomic frequency comb will gain phase at different rates. However, because of the periodicity of the comb, each 'tooth' will be in phase at $t = \frac{2\pi}{\Delta}$. This means that the AFC is primed to re-emit at that time.

However, right when the AFC is going to re-emit, we send in a second pulse of light, with time-envelope Φ. Additionally, we change the cavity decay rate κ, such that the emission efficiency is 50%. This results in an interference effect that I will show results in a $\frac{1}{2}$ probability of seeing two photons at that time.

If, however, we don't observe the photons at that time, we know that they must have both been stored in the cavity. We wait until an additional rephasing has occured at time $t = \frac{4\pi}{\Delta}$ time, and change the emission efficiency back to 100% by setting $\kappa = \Gamma$. In this way, we 'flush out' the memory. The equations show that there is a $\frac{1}{2}$ chance of seeing two photons there.

Additionally, I will also show that there is a zero chance of first seeing a photon at $t = \frac{2\pi}{\Delta}$ and then seeing an additional photon at time $t = \frac{4\pi}{\Delta}$.

A rough equivalence can be made to a physical beam splitter system, with two channels a and b. If we have a time delay setup for any photons coming out in channel b, such that we introduce a delay of $\frac{2\pi}{\Delta}$, and then we recombine the two channels so they overlap in space. . If we have a photon detector set-up, we will still be able to see photon bunching. There will be a 1/2 chance of seeing two photons in channel a, which will arrive at the photon detector at some time t_1. There will also be a 1/2 chance of seeing two photons in channel b, which will arrive later at some time $t_1 + \frac{2\pi}{\Delta}$.

7.3 AFC Photon Bunching

This subsection is laid out as follows. I will first define the initial state, as there are some sublities involved. Then, I will prove that the initial state will only allow certain terms in any expansion of the oppurators of interest. Then, I will show the equations for each of the three states:

1. $|\langle\psi_{20}|\Psi\rangle|^2$, two photons detected around $t \sim \frac{2\pi}{\Delta}$

2. $|\langle\psi_{02}|\Psi\rangle|^2$, two photons detected around $t \sim \frac{4\pi}{\Delta}$

3. $|\langle\psi_{11}|\Psi\rangle|^2$, one photon detected around $t \sim \frac{2\pi}{\Delta}$ and one photon detected around $t \sim \frac{4\pi}{\Delta}$

7.3.1 Initial State

Note: This is a pedantic section, and it is fairly dull. You can skip over it, as it isn't that important. I just define the initial state, and then show a couple of specific results that I will reference later.

We first have to define the initial state. As previously laid out in the overview, we have two photons, one coming in after the other, with time envelopes Θ and Φ, respectively. Additionally, the two pulses are seperated by a time $\frac{2\pi}{\Delta}$. I believe that we can define our initial state like:

$$|\Psi\rangle = \int dt' \int dt \Theta(t') E^{\dagger}_{in}(t') \Phi(t) E^{\dagger}_{in}(t) \tag{97}$$

Because the two pulses are far apart, we don't need to include a normalization factor of $\frac{1}{\sqrt{2}}$.

$E^{\dagger}_{in}(t)$ will create a photon incident on the cavity at some time t. The probability of this is weighted by the envelope function, Θ or Φ.

If we try to find the probability of seeing two photons exiting the cavity at some time t, we find the overlap between the two states:

$$|\rangle 0|\frac{1}{\sqrt{2}}E_{out}(t)E_{out}(t')|Psi|rangle|^2 \tag{98}$$

where E_{out} is a heisenburg operator.

7.3.2 Applying Operators to the Initial State

I am going to use a simplified equation for E_{out} in order to prove that only cross terms will survive the expansion. This also lets me elaborate on the notation.

Assume that $E_{out}(t) = aE_{in}(t) + bE_{in}(t - \frac{2\pi}{\Delta})$.

Then, our probability for seeing two photons is at some time t and t':

$$P2 = |\langle 0|\frac{1}{\sqrt{2}}(a^2E_{in}(t)E_{in}(t') + b^2E_{in}(t - \frac{2\pi}{\Delta})E_{in}(t' - \frac{2\pi}{\Delta}) + abE_{in}(t)E_{in}(t' - \frac{2\pi}{\Delta}) \tag{99}$$

$$+ baE_{in}(t - \frac{2\pi}{\Delta})E_{in}(t))|\Psi\rangle|^2 \tag{100}$$

Note, for this equation to be valid $t \sim t'$. They both have to be within the same neighborhood. Thus, the result I am proving is not a general one. Thus, if we are trying to detect two photons at around time $t = \frac{2\pi}{\Delta}$, then the above equation will hold.

If we look at the a^2 term, we can see that it won't survive when applied to the initial state.

$$P2_1 = \langle 0|a^2E_{in}(t)E_{in}(t')\int dt''\int dt'''\Theta(t')E^\dagger_{in}(t'')\Phi(t)E^\dagger_{in}(t''')|0\rangle \tag{101}$$

Earlier, when the input-output equations were derived, there was a communtation relation, defined as:

$$[E_{in}(t), E_{in}^{\dagger}(t')] = \delta(t-t')$$

Now, we know that any creation operator acting on a vacuum state like $\langle 0|a^{\dagger} = 0$, so we can use the commutation relations to 'shuffle' a creation operator to the very left, where it will destroy the vacuum.

Therefore, the a^2 term of $P2$ (called $P2_1$ from now on) becomes:

$$P2_1 = \langle 0| \int dt'' \int dt''' \Theta(t'')\Phi(t''')a^2 E_{in}(t) \left(E_{in}^{\dagger}(t'')E_{in}(t') + \delta(t'-t'')\right) E_{in}^{\dagger}(t''')|0\rangle \tag{102}$$

$$= \langle 0| \int dt'' \int dt''' \Theta(t'')\Phi(t''')a^2 \left(E_{in}^{\dagger}(t'')E_{in}(t) + \delta(t-t'')\right) E_{in}(t')E\dagger_{in}(t''') + E_{in}(t)\delta(t'-t'')E_{in}^{\dagger}(t''') \tag{103}$$

$$= \langle 0| \int dt'' \int dt''' \Theta(t'')\Phi(t''')a^2 \left(0 + \delta(t-t'')\left(E_{in}^{\dagger}(t''')E_{in}(t') + \delta(t'-t''')\right) + E_{in}(t)\delta(t'-t'')E_{in}^{\dagger}(t''')\right) \tag{104}$$

$$= \int dt'' \int dt''' \Theta(t'')\Phi(t''')a^2 \left(\delta(t-t'')\delta(t'-t''') + \delta(t'-t'')\delta(t-t''')\right) \tag{105}$$

$$= a^2\Theta(t)\Phi(t') + a^2\Theta(t')\Phi(t) \tag{106}$$

So, in order to be more general, we would integrate this term over the local time of one of the teeth. More specifically, we would integrate this probability equation all over the areas where the equation is valid– this is constrained to times $t \sim \frac{2\pi}{\Delta}$. However, because we are integrating both t and t' at around the same values, the $P2_1$ term is zero, as we know that there is a time delay between both pulses– if one of them is non-zero, the other has to be zero. . This is a subtle point, and I am not explaining it well. In order for the $P2_1$ term to have been valid, it would have needed something like:

$$P2_1 = a^2\Theta(t\Delta)\Phi(t' - \frac{2\pi}{\Delta}) + a^2\Theta(t' - \frac{2\pi}{\Delta})\Phi(t) \tag{107}$$

So that even though Θ and Φ don't overlap, the above function will evaluate to some non-zero. This only comes through in the cross terms.

In my future calculations, I will take it as given that we can drop the a^2 and the b^2 terms. Ie, everything but the cross terms.

We can also investigate the action of the cross-terms here, which will allow me to quote this result in the future. For the case of the cross terms, a similar methodology is used as above. In fact, one can simply take the above result and substitute $a^2 \Rightarrow ab, \qquad t' \Rightarrow t' - \frac{2\pi}{\Delta}$

For instance, the cross term ab will become:

$$P2_3 = \langle 0|abE_{in}(t)E_{in}(t' - \frac{2\pi}{\Delta}) \int dt' \int dt \Theta(t')E_{in}^\dagger(t'')\Phi(t)E_{in}^\dagger(t''')|0\rangle \tag{108}$$

If we use exactly the same algorithm, we obtain as a result:

$$P2_3(t, t') = ab\Theta(t)\Phi(t' - \frac{2\pi}{\Delta}) + ab\Theta(t' - \frac{2\pi}{\Delta})\Phi(t) \tag{109}$$

I will quote the result later, as it saves me the work of expanding out and using the commutators each time. However, I think it would be worthwile to discuss the results. Keep in mind while reading this that the entire proof and discussion up to this moment is predicated on the assumption that $t \sim t'$, ie. they are both within the same neighborhood.

If you consider two gaussian pulses, $\Theta(t)$ and $\Phi(t)$, such that $\Theta(t - \frac{2\pi}{\Delta}) = \Phi(t)$, then we can see that $P2_1$ is always zero. For instance, if $t \sim 0$, then $\Theta(t) \neq 0$, but the gaussian pulse Φ hasn't started yet so $\Phi(t') = 0$. If $t \sim \frac{2\pi}{\Delta}$, then it is the other way around: $\Theta(t) = 0$ as this pulse was centered around zero, and $\Phi(t') \neq 0$ because it is centered around $\frac{2\pi}{\Delta}$.

Using the same argument, we can see that $P2_3 \neq 0$, because we have introduced a retarding factor of $\frac{2\pi}{\Delta}$ which brings the two pulses Θ and Φ back together. This will only occur for times $t \sim \frac{2\pi}{\Delta}$. If $t \sim 0$, then $\Theta \neq 0$, but $\Phi(-\frac{2\pi}{\Delta}) = 0$, because the pulse hasn't even started yet. The other term $\Theta(-\frac{2\pi}{\Delta}) = 0$, because the Θ pulse is centered around 0.

So for times $t \sim t' \sim \frac{2\pi}{\Delta}$ $P2_3 \neq 0$.

7.3.3 The Three States

In order to calculate the evolution of the states of interest, we first need to mathematically define our states of interest:

$$P20(t,t') = |\langle 0|\frac{1}{\sqrt{2}}E_{out}(t)E_{out}(t')|\Psi\rangle|^2 \tag{110}$$

where $t \sim \frac{2\pi}{\Delta}$ and $t' \sim \frac{2\pi}{\Delta}$

$$P11(t,t') = |\langle 0|E_{out}(t)E_{out}(t')|\Psi\rangle|^2 \tag{111}$$

where $t \sim \frac{2\pi}{\Delta}$ and $t' \sim \frac{4\pi}{\Delta}$

$$P02(t,t') = |\langle 0|\frac{1}{\sqrt{2}}E_{out}(t)E_{out}(t')|\Psi\rangle|^2 \tag{112}$$

where $t \sim \frac{4\pi}{\Delta}$ and $t' \sim \frac{4\pi}{\Delta}$

So, the first deals with the probabilty of observing two photons around the neighborhood of $t \sim \frac{2\pi}{\Delta}$, the second term deals with first observing a photon in the neighborhood of $t \sim \frac{2\pi}{\Delta}$ and then observing a photon in the neighborhood of $t \sim \frac{4\pi}{\Delta}$, and the third term deals with observing two photons in the neighborhood of $t \sim \frac{4\pi}{\Delta}$

7.4 Photon Bunching

In this section, I will deal with each of the probability terms defined in section 7.3.3, and calculate them out. I will be dealing with the specific condition where the impedence matching condition is perfect for the very first pulse, and then set to 50% efficiency for the second pulse.

7.4.1 P20

First, we must calculate what the E_{out} operator is during this time. The physical situation is that we have a pulse read in with near unit efficiency, but just as it is about to rephase, we send another pulse in. The general equation for E_{out} is:

$$E_{out} = \sqrt{2\kappa} - E_{in} \tag{113}$$

So we must first calculate what the cavity field E is doing. The equation of motion for E is given by:

$$\dot{E} = -\kappa E - iP\tilde{P}\int_{-\infty}^{t} dt'\tilde{n}(t-t')E(t') + \sqrt{2\kappa}E_{in}(t) \tag{114}$$

We can divide the integral up into two segments. The first segment $-\infty < t \leq 0$ is where the first pulse is being read in with unit efficiency. In the impedence matching paper, this condition was already derived, and the equation for E during this time is $E = \frac{1}{\sqrt{2\kappa}}E_{in}(t)$, with $\kappa = \Gamma$, where κ is the cavity decay rate, and Γ is the absorption by the atomic frequency comb. However, in order to simplfy the equations that will come later, in the unit efficiency case, I will be rewriting the equations to include Γ instead of κ. I can do this because of their equivalence, and it means that I won't have to worry about having different κ's. This is because we will be changing the cavity decay rate, so κ is not a constant in time. By renaming it to Γ, it will hopefully make the end equations more managable. So when ever you see a κ, you can assume it is a cavity decay rate that sets the absorption efficiency to 50%.

So, the second sequence is when $t' \sim t$, and in that case, $E(t')$ is just $E(t')$, as it is in the neighborhood of the time that we are evaluating at. Broken up this way, the equation becomes

$$\dot{E} = -\kappa E - 2\Gamma \frac{1}{\sqrt{2\Gamma}} E_{in}(t - \frac{2\pi}{\Delta}) - \Gamma E(t) + \sqrt{2\kappa} E_{in}(t) \tag{115}$$

Using the usual trick of setting $\dot{E} = 0$, we can solve for E:

$$E = \frac{1}{\kappa + \Gamma} \left(-\sqrt{2\Gamma} E_{in}(t - \frac{2\pi}{\Delta}) + \sqrt{2\kappa} E_{in}(t) \right) \tag{116}$$

So, using this equation in our equation for E_{out}, we get:

$$E_{out} = \frac{\sqrt{2\kappa}}{\kappa + \Gamma} \left(-\sqrt{2\Gamma} E_{in}(t - \frac{2\pi}{\Delta}) + \sqrt{2\kappa} E_{in}(t) \right) - E_{in}(t) \tag{117}$$

$$= -\frac{2\sqrt{\kappa\Gamma}}{\kappa + \Gamma} E_{in}(t - \frac{2\pi}{\Delta}) + \frac{\kappa - \Gamma}{\kappa + \Gamma} E_{in}(t) \tag{118}$$

So, to set κ so that the absorption efficiency is 50% for a single pulse. The absorption efficiency for a single pulse is given by:

$$\eta = |\frac{\kappa - \Gamma}{\kappa + \Gamma}|^2 \tag{119}$$

Setting $\eta = .5$, we can derive the following relation between κ and Γ:

$$\frac{1}{\sqrt{2}} = \frac{\kappa - \Gamma}{\kappa + \Gamma} \tag{120}$$

$$\frac{1}{\sqrt{2}}(\kappa + \Gamma) = \kappa - \Gamma \tag{121}$$

$$(\frac{1}{\sqrt{2}} - 1)\kappa = (\frac{1}{\sqrt{2}} + 1)\Gamma \tag{122}$$

$$\kappa = \Gamma \frac{\frac{1}{\sqrt{2}} + 1}{\frac{1}{\sqrt{2}} - 1} \tag{123}$$

So, letting:

$$\alpha = \frac{\frac{1}{\sqrt{2}} + 1}{\frac{1}{\sqrt{2}} - 1}, \tag{124}$$

then Eq (117) becomes:

$$E_{out} = -\frac{2\sqrt{\alpha}}{1+\alpha} E_{in}(t - \frac{2\pi}{\Delta}) + \frac{1}{\sqrt{2}} E_{in}(t) \tag{125}$$

$$E_{out} = -\frac{1}{\sqrt{2}} E_{in}(t - \frac{2\pi}{\Delta}) + \frac{1}{\sqrt{2}} E_{in}(t) \tag{126}$$

And this is exactly the beamsplitter relation. To calculate the probability, we take:

$$P20(t,t') = |\langle 0| \frac{1}{\sqrt{2}} E_{out}(t) E_{out}(t') |\Psi\rangle|^2 \tag{127}$$

We have shown in Section 7.3.2 that only the cross terms will survive when we expand Eq. (127) out. Using this knowledge, Eq. (127) becomes:

$$P20(t,t') = \left| \langle 0 \left(\frac{1}{2} E_{in}(t - \frac{2\pi}{\Delta}) E_{in}(t') + \frac{1}{2} E_{in}(t' - \frac{2\pi}{\Delta}) E_{in}(t) \right) \Psi \rangle \right|^2 \tag{128}$$

$$\tag{129}$$

Now, we can use the result we calculated in Section 7.3.2, with $a = -\frac{1}{\sqrt{2}}$ and $b = \frac{1}{\sqrt{2}}$.

$$P20(t,t') = \frac{1}{2}\left|-\frac{1}{2}\Theta(t - \frac{2\pi}{)}\Delta\Phi(t') - \frac{1}{2}\Theta(t' - \frac{2\pi}{\Delta})\Phi(t)\right|^2 \tag{130}$$

$$P20(t,t') = \frac{1}{2}\frac{1}{4}\left(\Theta^2(t - \frac{2\pi}{\Delta})\Phi^2(t') + \Theta^2(t' - \frac{2\pi}{\Delta})\Phi^2(t) + 2\Theta(t - \frac{2\pi}{\Delta})\Phi(t)\Theta(t' - \frac{2\pi}{\Delta})\Phi(t')\right) \tag{131}$$

Where $\Theta(t)$ is the envelope of the first pulse, incident around $t \sim 0$ and $\Phi(t)$ is the second pulse, incident around $t \sim \frac{2\pi}{\Delta}$.

Now, we are interested in detecting photons in the neighborhood of $t = \frac{2\pi}{\Delta}$, so we need to integrate $P20(t,t')$. We are just integrating in the neighborhood of $t \sim \frac{2\pi}{\Delta}$. $\int \Theta(t - \frac{2\pi}{\Delta})^2(t)dt = 1$ and $\int \Phi^2(t')dt' = 1$ due to normalization. Even though we are not integrating over all time, we are assuming the the incoming pulse is short enough that it is entirely included in the neighborhood of $t \sim \frac{2\pi}{\Delta}$, so this assumption is valid. So, we can write our integral as:

$$P20 = \frac{1}{8}\left(\int dt \int dt' \left(\Theta^2(t - \frac{2\pi}{\Delta})\Phi^2(t') + \Theta^2(t' - \frac{2\pi}{\Delta})\Phi^2(t)\right) + 2\left(\int dt \int dt'\Theta(t' - \frac{2\pi}{\Delta})\Phi(t)\right)^2\right) \tag{132}$$

If we make the additional assumption that the two pulses have the same temporal shape, then the overlap integral in the third term in Eq (132) will evaluate to 1, and we get:

$$P20 = \frac{1}{8}(1 + 1 + 2) \tag{133}$$

$$P20 = \frac{1}{2} \tag{134}$$

Therefore, the probabilty of detecting two photons in the neighborhood of $\frac{2\pi}{\Delta}$ is 1/2

7.4.2 Two Photons at $t \sim \frac{4\pi}{\Delta}$

The formalism of finding this out is very similar. First we need to calculate E_{out} and then we can calculate the probability.

First, we are back to the impedence matching condition $\kappa = \Gamma$, as we have reset our mirror so we can 'flush' out our AFC. In order to avoid confusion with the κ mirror term that was moved to get 50% efficiency previously, I will preemptively change all κ terms to Γ, so that the only κ terms will refer to the 50% efficiency κ. Under this change, the equation for E_{out} goes from::

$$E_{out}(t) = \sqrt{2\kappa}E - E_{in} \tag{135}$$

to

$$E_{out}(t) = \sqrt{2\Gamma}E - E_{in} \tag{136}$$

However, we are not sending in any new pulses, so around the time that we are evaluating $t \sim \frac{4\pi}{\Delta}$, we can reduce this equation to:

$$E_{out}(t) = \sqrt{2\Gamma}E \tag{137}$$

And the equation for E is given by:

$$\dot{E} = -\Gamma E + \sqrt{2\Gamma}E_{in} - iP\tilde{P}\int_{-\infty}^{t} \tilde{n}(t-t')E(t') \tag{138}$$

(Note: it used to be $\dot{E} = -\kappa E + \sqrt{2\kappa} + ...$, but since $\kappa = \Gamma$ in this case we can rewrite it with no loss of generality.)

As noted before, $E_{in}(t \sim \frac{4\pi}{\Delta}) = 0$. Additionally, as we did before, we can divide the intergral up into several segments:

$$\dot{E} = -\Gamma E - iP\tilde{P}\left(\int_{-\infty}^{0+\delta} \tilde{n}(t-t')E(t')dt' + \int_{0+\delta}^{\frac{2\pi}{\Delta}+\delta} \tilde{n}(t-t')E(t')dt' + \int_{\frac{2\pi}{\Delta}+\delta}^{t} \tilde{n}(t-t')E(t')dt'\right) \tag{139}$$

This highlights the use of the recursive strategy, as we need to know the values of E for the different values of t:

$$E(t' \sim 0) = \frac{1}{\sqrt{2\Gamma}} E_{in}(t') \tag{140}$$

$$E(t' \sim \frac{2\pi}{\Delta}) = -\frac{\sqrt{2\Gamma}}{\kappa+\Gamma} E_{in}(t' - \frac{2\pi}{\Delta}) + \sqrt{t2\kappa}\kappa + \Gamma E_{in}(t') \tag{141}$$

$$\tag{142}$$

Inserting these identities in Eq (139) as appropriate, we obtain:

$$\dot{E} = -\Gamma E - iP\tilde{P}\left(\int_{-\infty}^{0+\delta} \tilde{n}(t-t')\left(\frac{1}{\sqrt{2\Gamma}}E_{in}(t')\right)dt'\right. \tag{143}$$

$$\left. + \int_{0+\delta}^{\frac{2\pi}{\Delta}+\delta} \tilde{n}(t-t')\left(-\frac{\sqrt{2\Gamma}}{\kappa+\Gamma}E_{in}(t'-\frac{2\pi}{\Delta}) + \frac{\sqrt{2\kappa}}{\kappa+\Gamma}E_{in}(t')\right)dt' + \int_{\frac{2\pi}{\Delta}+\delta}^{t} \tilde{n}(t-t')E(t')dt'\right) \tag{144}$$

We can now evaluate each integral in turn. The first term in the integral expansion looks like:

$$I1 \propto \int dt'\tilde{n}(t-t')E_{in}(t') \tag{145}$$

with $t \sim \frac{4\pi}{\Delta}$ and $t' \sim 0$. This means that $t - t' \sim \frac{4\pi}{\Delta}$. Since $\tilde{n}(x)$ is a comb of approximate delta functions spaced by $\frac{2\pi}{\Delta}$, we know that there will be an approximate delta function that occurs when $x \sim \frac{4\pi}{\Delta}$. Therefore, our integral term $I1$ becomes:

$$I1 \propto 2\Gamma\sqrt{\eta_{F2}}E_{in} \tag{146}$$

The term $\sqrt{\eta_{F2}}$ is due to the irreversible dephasing that is present due to the finite width of the approximate delta functions. It is actually very important, and I take a brief pause in my derivation of the probability P02 to discuss it now.

The Important Subtleties of Dephasing Terms The original impedence matching paper pointed out that there would be irriversible atomic dephasing that is due to the fact that the approximate delta functions in $\tilde{n}(t-t')$ have some non-zero width. So, for a single pulse being stored with unit efficiency, the read-out equations will look like:

$$\dot{E} = -\kappa E - iP\tilde{P}\int_{-\infty}^{t} \tilde{n}(t-t')E \tag{147}$$

$$\dot{E} = -\kappa E(t) - 2\Gamma\sqrt{\eta_F}\frac{1}{2\Gamma}E_{in}(t-\frac{2\pi}{\Delta}) - \Gamma E(t) \tag{148}$$

Setting $\dot{E} = 0$, we obtain:

$$E(t) = -\frac{\sqrt{2\Gamma}}{\kappa+\Gamma}E_{in}(t-\frac{2\pi}{\Delta})\sqrt{\eta_F} \tag{149}$$

and under the impedence matching condition($\kappa = \Gamma$), this becomes:

$$E(t) = -\frac{1}{\sqrt{2\kappa}}\sqrt{\eta_F}E_{in}(t-\frac{2\pi}{\Delta}) \tag{150}$$

Therefore, $E_{out} = -\sqrt{\eta_F}E_{in}(t-\frac{2\pi}{\Delta})$.

It is reasonable to expect that this irreversible dephasing has some time dependence– the longer that the pulse stays in the atom, we would expect it to become more and more randomly dephased. Above, the pulse stayed in the atom for an effective time of $t = \frac{2\pi}{\Delta}$, and it became randomally dephased by a factor of $\sqrt{\eta_F}$.

However, we are now dealing with pulses that are staying in the atomic memory for times

$t = \frac{4\pi}{\Delta}$, so how what is the factor of random dephasing now?

We can use a conservation of energy argument to prove that the random dephasing that occurs when a pulse stays in for $t = \frac{4\pi}{\Delta}$ is the square of the random dephasing that occurs when a pulse stays in for $t = \frac{2\pi}{\Delta}$. The argument is as follows.

If we consider the situation we have mathematically described above, we have the following:

1. A pulse is incident on an AFC quantum memory and is read in with perfect efficiency
2. The pulse stays in the atomic memory for a time $t = \frac{2\pi}{\Delta}$ where it undergoes random dephasing
3. The pulse is read out completely, expect for the fraction lost to the random dephasing in the memory– represented by the factor $\sqrt{\eta_F}$.

.

So, if we look at subsequent echoes– say at $t = \frac{4\pi}{\Delta}$, we know that $E_{out}(t)$ has to equal zero. The entire pulse was read out at time $t = \frac{2\pi}{\Delta}$, so there is nothing left to read out.

So, looking at solution of $E(t)$ when $t \sim \frac{4\pi}{\Delta}$, we get:

$$\dot{E} = -\kappa - iP\tilde{P}\left(\int_{-\infty}^{0+\delta} \tilde{n}(t-t')E(t')dt' + \int_{0+\delta}^{\frac{2\pi}{\Delta}+\delta} \tilde{n}(t-t')E(t')dt' + \int_{\frac{2\pi}{\Delta}+\delta}^{t} \tilde{n}(t-t')E(t')dt'\right) \tag{151}$$

and we know the values of E for different times:

$$E(t \sim 0) = \frac{1}{\sqrt{2\kappa}} E_{in}(t) \tag{152}$$

$$E(t \sim \frac{2\pi}{\Delta}) = -\frac{1}{\sqrt{2\kappa}}\sqrt{\eta_F} E_{in}(t - \frac{2\pi}{\Delta}) \tag{153}$$

So, our integral becomes:

$$\dot{E} = -\kappa E - iP\tilde{P}\left(\int_{-\infty}^{0+\delta} \tilde{n}(t-t')\frac{1}{\sqrt{2\kappa}}E_{in}(t)dt'\right. \tag{154}$$

$$\left. - \int_{0+\delta}^{\frac{2\pi}{\Delta}+\delta} \tilde{n}(t-t')\frac{1}{\sqrt{2\kappa}}\sqrt{\eta_F}E_{in}(t-\frac{2\pi}{\Delta})dt' + \int_{\frac{2\pi}{\Delta}+\delta}^{t} \tilde{n}(t-t')E(t')dt'\right) \tag{155}$$

As before, the $\tilde{n}(t-t')$ has a peak around $t-t' = \frac{4\pi}{\Delta}$, but as discussed before we need to include an additional dephasing term due to the fact that the pulse has stayed in the atomic memory for $t = \frac{4\pi}{\Delta}$. We will call this term $\sqrt{\eta_{F2}}$.

For the second term in the integral expasion, when $t' \sim \frac{2\pi}{\Delta}$, the $\tilde{n}(t-t')$ term has a peak, as $t-t' \sim \frac{2\pi}{\Delta}$. However, we will get the usual dephasing term $\sqrt{\eta_F}$, because all that matters in the $\tilde{n}(t-t')$ is the change in time. As this change in time is $\frac{2\pi}{\Delta}$, we get the random dephasing term η_F.

Under these assumptions the above equation for the cavity field becomes:

$$\dot{E} = -\kappa E - \Gamma E - \frac{2\Gamma\sqrt{\eta_{F2}}}{\sqrt{2\kappa}}E_{in}(t-\frac{4\pi}{\Delta}) + \frac{2\Gamma\eta_F}{\sqrt{2\kappa}}E_{in}\left((t-\frac{2\pi}{\Delta})-\frac{2\pi}{\Delta}\right) \tag{156}$$

Notice, in the above equation we have a term $E_{in}\left((t-\frac{2\pi}{\Delta})-\frac{2\pi}{\Delta}\right)$, which is due to the fact that upon applying the approximate delta function to E_{in}, the delta function will act on the argument of $E_{in}(x)$ to retard it by another $\frac{2\pi}{\Delta}$.

Again, using the fact that $\dot{E} = 0$ and $\Gamma = \kappa$ and $E_{out} = \sqrt{2\kappa}E$, we have a solution for E_{out} given by:

$$E_{out} = -\sqrt{\eta_{F2}}E_{in}(t-\frac{4\pi}{\Delta}) + \eta_F E_{in}(t-\frac{4\pi}{\Delta}) \tag{157}$$

We can interpret the terms in the equation in the following way. The first term with $\sqrt{\eta_{F2}}$, is due to an actual photon-echo. The atoms in the various 'teeth' in the AFC continue to gain phase, coming back in phase at the period condition $t_1 + \frac{2\pi}{\Delta}$, where t_1 is the time of the last echo.

However, at time t_1, the AFC emitted a photon pulse. This gave rise to a non-zero cavity field. I can only explain the second term by assuming that a part of this cavity field was reabsorbed, by the AFC.

We also know that $E_{out}(t \sim \frac{4\pi}{\Delta}) = 0$, as we read out the entire pulse at time $\frac{2\pi}{\Delta}$.

Therefore, $\sqrt{\eta_{F2}} = \eta_F$, and there is an interference between the rephasing of the original pulse, and an re-absorption/re-emission pulse that perfectly cancels out.

Now that we have this result, I can continue with my derivation of the $P02$ probability.

Back to Derivation of P02 The equation we had before our detour looked like:

$$\dot{E} = -\Gamma E - iP\tilde{P}\left(\int_{-\infty}^{0+\delta} \tilde{n}(t-t')\left(\frac{1}{\sqrt{2\Gamma}}E_{in}(t')\right)dt' \right. \tag{158}$$

$$\left. + \int_{0+\delta}^{\frac{2\pi}{\Delta}+\delta} \tilde{n}(t-t')\left(-\frac{\sqrt{2\Gamma}}{\kappa+\Gamma}E_{in}(t-\frac{2\pi}{\Delta}) + \frac{\sqrt{2\kappa}}{\kappa+\Gamma}E_{in}(t')\right)dt' + \int_{\frac{2\pi}{\Delta}+\delta}^{t} \tilde{n}(t-t')E(t')dt'\right) \tag{159}$$

And we were going to evaluate each integral term in turn. The first term in the integral expansion looks like:

$$I1 \propto \int dt'\tilde{n}(t-t')E_{in}(t') \tag{160}$$

Which becomes:

$$I1 \propto 2\Gamma\sqrt{\eta_{F2}}E_{in} \tag{161}$$

And now we know that $\sqrt{\eta_{F2}} = \eta_F$.

The second term in the integral expansion looks like:

$$I2 \propto \int dt'\left(\tilde{n}(t-t')E_{in}(t'-\frac{2\pi}{\Delta}) - \tilde{n}(t-t')E_{in}(t')\right)dt', \tag{162}$$

where $t' \sim \frac{2\pi}{\Delta}$ We know that $\tilde{n}(x)$ has a peak when $x \sim \frac{2\pi}{\Delta}$, and the action of $\tilde{n}(x)$ in that regime is to multiply $E_{in}(y)$ by a factor, $\sqrt{\eta_F}\beta$, and retard it by $\frac{2\pi}{\Delta}$ to get $\sqrt{\eta_F}\beta E_i n(y - \frac{2\pi}{\Delta})$

This gives $I2$ as:

$$I2 \propto \sqrt{\eta_F} 2\beta E_i n(t - \frac{4\pi}{\Delta}) - \sqrt{\eta_F}\beta E_i n(t - \frac{2\pi}{\Delta}) \tag{163}$$

And then $I3 \propto \Gamma E(t)$ as usual.

Now, doing the bookkeeping properly for all of the factors, our equation for the cavity field E Eq. (158) has become:

$$\dot{E} = -\Gamma E(t) - \Gamma E(t) - \eta_F \frac{2\Gamma}{\sqrt{2\Gamma}} E_{in}(t - \frac{4\pi}{\Delta}) + \eta_F 2\Gamma \frac{\sqrt{2\Gamma}}{\kappa + \Gamma} E_{in}(t - \frac{4\pi}{\Delta}) - \sqrt{\eta_F} 2\Gamma \frac{\sqrt{2\kappa}}{\kappa + \Gamma} E_{in}(t - \frac{2\pi}{\Delta}) \tag{164}$$

So, setting $\dot{E} = 0$, and using the equation for $E_{out} = \sqrt{2\Gamma}E$, we have our equation for $E_{out}(t)$, given by:

$$E_{out}(t) = \frac{\sqrt{2\Gamma}}{2\Gamma} \left(\left(-\eta_F \frac{2\Gamma}{\sqrt{2\Gamma}} + \eta_F 2\Gamma \frac{\sqrt{2\Gamma}}{\kappa + \Gamma} \right) E_{in}(t - \frac{4\pi}{\Delta}) - \sqrt{\eta_F} \frac{\sqrt{2\kappa} 2\Gamma}{\kappa + \Gamma} E_{in}(t - \frac{2\pi}{\Delta}) \right) \tag{165}$$

$$E_{out}(t) = \left(\left(-\eta_F + \eta_F \frac{2\Gamma}{\kappa + \Gamma} \right) E_{in}(t - \frac{4\pi}{\Delta}) - \sqrt{\eta_F} 2 \frac{\sqrt{\kappa\Gamma}}{\kappa + \Gamma} E_{in}(t - \frac{2\pi}{\Delta}) \right) \tag{166}$$

This is the general equation for E_{out}, where κ is the value of κ at time $t \sim \frac{2\pi}{\Delta}$. If we want the beamsplitter relationship, we already have an equation that relates κ and Γ, given by Eq. (124).

$$\kappa = \alpha\Gamma \tag{167}$$

where $\alpha = \frac{\sqrt{2}+1}{\sqrt{2}-1}$.

Using this relation, Eq. (166) becomes:

$$E_{out}(t) = \left(-\eta_F + \eta_F \frac{2}{\alpha+1}\right) E_{in}(t - \frac{4\pi}{\Delta}) - \sqrt{\eta_F}\frac{2\sqrt{\alpha}}{\alpha+1} \tag{168}$$

$$E_{out}(t) = \eta_F \frac{-\alpha+1}{\alpha+1} E_{in}(t - \frac{4\pi}{\Delta}) - \sqrt{\eta_F}\frac{2\sqrt{\alpha}}{\alpha+1} \tag{169}$$

$$\tag{170}$$

and under evaluation, this becomes:

$$E_{out}(t) = -\eta_F \frac{1}{\sqrt{2}} E_{in}(t - \frac{4\pi}{\Delta}) - \sqrt{\eta_F}\frac{1}{\sqrt{2}} E_{in}(t - \frac{2\pi}{\Delta}) \tag{171}$$

Now, there are definitely some imperfections that arise from the η_F terms, but if we neglect those, we have a perfect beamsplitter relation.

We can find the probability using a very similar method as in the previous Section.

$$P02(t,t') = |\langle 0| \frac{1}{\sqrt{2}} E_{out}(t) E_{out}(t') |\Psi\rangle|^2 \tag{172}$$

when both t and t' are in the neighborhood of $\frac{4\pi}{\Delta}$. i.e. $t \sim t' \sim \frac{4\pi}{\Delta}$. And by using the same arguments, only terms that contain both $E_{in}(t - \frac{4\pi}{\Delta})$ and $E_{in}(t - \frac{2\pi}{\Delta})$ will survive.

The only thing that changes is that instead of having $E_{in}(t - \frac{2\pi}{\Delta})$ and $E_{in}(t)$, both of these operators are retarded by an additional $\frac{2\pi}{\Delta}$, but it makes no difference to the arguments I used to show that only the cross terms will survive.

Additionally, we can re-use a lot of the same methodology from the previous section.

$$P02(t,t') = \frac{1}{2}\left|\frac{1}{2}\Theta(t-\frac{4\pi}{\Delta})\Phi(t'-\frac{2\pi}{\Delta}) + \frac{1}{2}\Theta(t'-\frac{4\pi}{\Delta})\Phi(t-\frac{2\pi}{\Delta})\right|^2 \tag{173}$$

$$P02(t,t') = \frac{1}{8}\Bigg(\Theta^2(t-\frac{4\pi}{\Delta})\Phi^2(t'-\frac{2\pi}{\Delta}) + \Theta^2(t'-\frac{4\pi}{\Delta})\Phi^2(t-\frac{2\pi}{\Delta}) \tag{174}$$

$$+2\Theta(t-\frac{4\pi}{\Delta})\Phi(t'-\frac{2\pi}{\Delta})\Theta(t'-\frac{4\pi}{\Delta})\Phi(t-\frac{2\pi}{\Delta})\Bigg) \tag{175}$$

Now, we are concered with finding the probability of detecting two photons around the neighborhood of $t \sim \frac{4\pi}{\Delta}$ and $t' \sim \frac{4\pi}{\Delta}$, so we need to integrate $P20(t,t')$ over that neighborhood of time for t and t'.

We also know that $\int_{\frac{4\pi}{\Delta}-\delta}^{\frac{4\pi}{\Delta}+\delta} \Theta^2(t-\frac{4\pi}{\Delta})dt = 1$ and$\int_{\frac{4\pi}{\Delta}-\delta}^{\frac{4\pi}{\Delta}+\delta} \Phi^2(t-\frac{2\pi}{\Delta})dt = 1$. Additionally, if the Θ and Φ have the same temporal shape, then the overlap integral between them will also evaluate to 1. so our equation for detecting two photons around time $t \sim \frac{4\pi}{\Delta}$ is given by:

$$P02 = \int dt \int dt' P02(t,t') = \frac{1}{8}\left(\int dt \int dt' \Theta^2(t-\frac{4\pi}{\Delta})\Phi^2(t'-\frac{2\pi}{\Delta}) + \int dt \int dt' \Theta^2(t'-\frac{4\pi}{\Delta})\Phi^2(t-\frac{2\pi}{\Delta})\right. \tag{176}$$

$$\left. +2\left(\int dt \Theta(t-\frac{4\pi}{\Delta})\Phi(t'-\frac{2\pi}{\Delta})\right)^2\right) \tag{177}$$

$$P02 = \frac{1}{8}(1+1+2) \tag{178}$$

$$P02 = \frac{1}{2} \tag{179}$$

So, the probability of detecting two photons around $\frac{4\pi}{\Delta}$ is $\frac{1}{2}$, in agreement with the beam-splitter.

7.4.3 P11: Detecting one photon at time $t_1 \sim \frac{2\pi}{\Delta}$ and then a subsequent photon at time $t_2 \sim \frac{4\pi}{\Delta}$

We have already defined $P11(t_1, t_2)$ to be:

$$P11(t_1, t_2) = |\langle 0|E_{out}(t_1)E_{out}(t_2)|\Psi\rangle|^2 \tag{180}$$

And, We have already derived the equations for $E_{out}(t_1)$ and $E_{out}(t_2)$ in the previous two sections. Their result is shown below for reference:

$$E_{out}(t_1) = -\sqrt{\eta_F}\frac{1}{\sqrt{2}}E_{in}(t_1 - \frac{2\pi}{\Delta}) + \frac{1}{\sqrt{2}}E_{in}(t_1) \tag{181}$$

$$E_{out}(t_2) = -\eta_F\frac{1}{\sqrt{2}}E_{in}(t_2 - \frac{4\pi}{\Delta}) - \sqrt{\eta_F}\frac{1}{\sqrt{2}}E_{in}(t_2 - \frac{2\pi}{\Delta}) \tag{182}$$

Now, the only terms that will survive when we mutiply $E_{out}(t_1)$ and $E_{out}(t_2)$ will be terms that contain either: $E_{in}(t_1 - \frac{2\pi}{\Delta})E_{in}(t_2 - \frac{2\pi}{\Delta})$, or $E_{in}(t_1)E_{in}(t_2 - \frac{4\pi}{\Delta})$. The reason becomes more apparent if we relabel the arguments of each term. Both factors can be written as $E_{in}(x_1)E_{in}(x_2)$, where:

$$x_1 \sim 0 \tag{183}$$

$$x_2 \sim \frac{2\pi}{\Delta} \tag{184}$$

Then, it becomes more transparent how application of the arguments I made at the beginning of this Section about the cross-terms will apply to the case we are discussing now.

If we only focus on the arguments x_1 and x_2, then the above equation has the same form as we discussed in Section 7.3.2, and we can conclude that again, only the cross terms will survive.

If only the cross terms survive, then $P11(t_1, t_2)$ becomes:

$$P11(t_1, t_2) = \left|\eta_F \frac{1}{2}\Theta(t_1 - \frac{2\pi}{\Delta})\Phi(t_2 - \frac{2\pi}{\Delta}) - \eta_F \frac{1}{2}\Phi(t_1)\Theta(t_2 - \frac{4\pi}{\Delta})\right|^2 \tag{185}$$

$$P11(t_1, t_2) = \frac{1}{4}\eta_F^2 \left(\Theta^2(t_1 - \frac{2\pi}{\Delta})\Phi^2(t_2 - \frac{2\pi}{\Delta}) + \Phi^2(t_1)\Theta^2(t_2 - \frac{4\pi}{\Delta})\right. \tag{186}$$

$$\left. -2\Theta(t_1 - \frac{2\pi}{\Delta})\Phi(t_2 - \frac{2\pi}{\Delta})\Phi(t_1)\Theta(t_2 - \frac{4\pi}{\Delta})^2\right) \tag{187}$$

$$\tag{188}$$

Again, as before, we need to integrate over the neighborhood where $t_1 \sim \frac{2\pi}{\Delta}$ and then where $t_2 \sim \frac{4\pi}{\Delta}$. Then our full probability $P11$ is:

$$P11 = \int_{\frac{4\pi}{\Delta}-\delta}^{\frac{4\pi}{\Delta}+\delta} dt_2 \int_{\frac{2\pi}{\Delta}-\delta}^{\frac{2\pi}{\Delta}+\delta} dt_1 P11(t_1, t_2) \tag{189}$$

$$\tag{190}$$

And using similar arguments as the previous sections, this can be reduced to:

$$P11 = \frac{1}{4}(1 + 1 - 2) \tag{191}$$

$$P11 = 0 \tag{192}$$

7.5 AFC Photon Interference

It is very easy to generalize the equations above for any effiency. The only thing that will change is the factor of α that relates κ to Γ. We can find the relationship between α and the efficiency by using the equation for the effiency of a single pulse being absorbed.

$$\frac{\kappa - \Gamma}{\kappa + \Gamma} = \sqrt{\text{ef}} \tag{193}$$

$$\kappa\left(1 - \sqrt{\text{ef}}\right) = \Gamma\left(\sqrt{\text{ef}} + 1\right) \tag{194}$$

$$\kappa = \frac{\sqrt{\text{ef}} + 1}{1 - \sqrt{\text{ef}}}\Gamma \tag{195}$$

$$\kappa = \alpha\Gamma \tag{196}$$

So, keeping the factor of α instead of evaluating it for a specific efficiency, we can obtain general equations for all three probabilites:

$$P20 = 8\alpha\frac{(\alpha - 1)^2}{(\alpha + 1)^4} \tag{197}$$

$$P02 = 8\alpha\frac{(1 - \alpha)^2}{(1 + \alpha)^4} \tag{198}$$

$$P11 = 16\frac{\alpha^2}{(1 + \alpha)^4} + \frac{(\alpha - 1)^2(1 - \alpha)^2}{(1 + \alpha)^4} + 8\alpha\frac{(\alpha - 1)(\alpha + 1)}{(\alpha + 1)^4} \tag{199}$$

$$\tag{200}$$

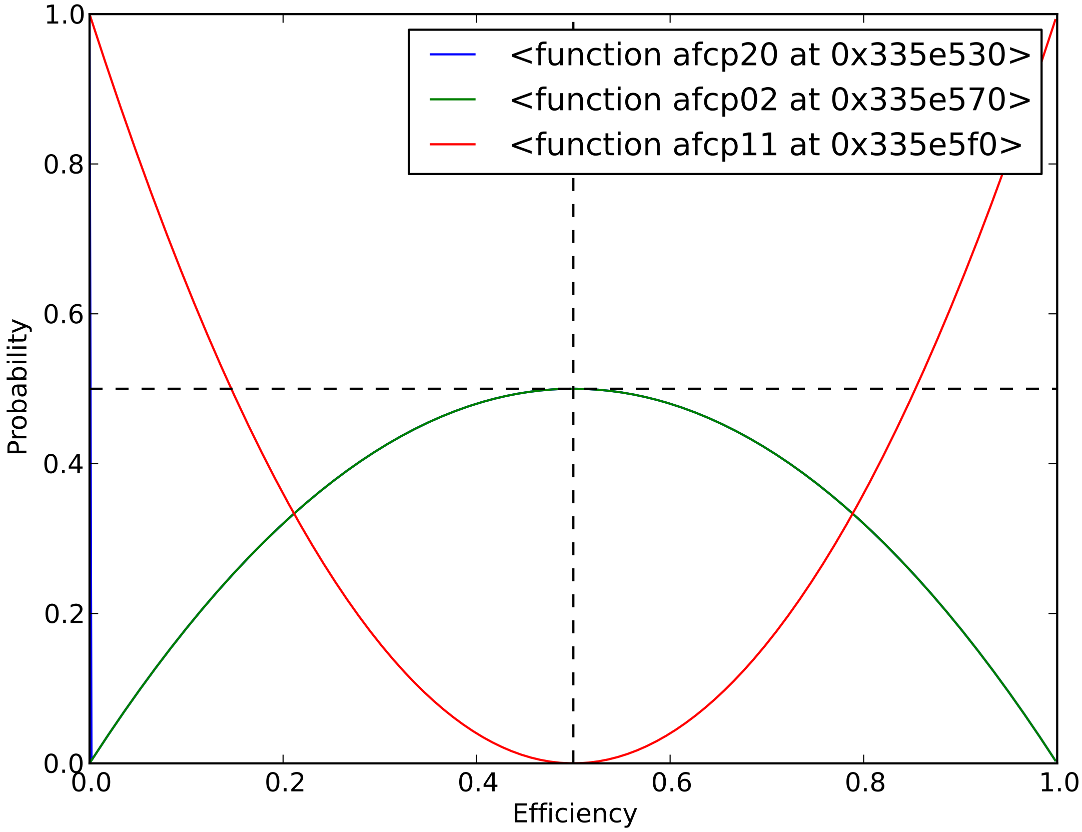

Figure 3: The probability functions of two photons seperated by a time $\frac{2\pi}{\Delta}$, incident on an AFC memory in a cavity. P20 and P02 give the same values. Note the equivalence to the beamsplitter curve.

8 Conclusion

I have shown full analytical equivalence between the CD-memory in the two photon regime, and a beam-splitter.