EXAM 02 - PHYS 3210 **Amanda Ash**

(1)Types of Equations: Write an example of each type of equation listed below. Your examples do not need to be from real systems.

(a) Linear equation

$y = \alpha x$ where α is a constant

(b) Non-linear equation

$y = x^n$ where n is a constant not equal to zero or one

(c) 1st-order, linear differential equation

$\dot{y} = \frac{dy}{dt} = \alpha y(t) + t$ where α is a constant

(d) 4th-order, non-linear differential equation

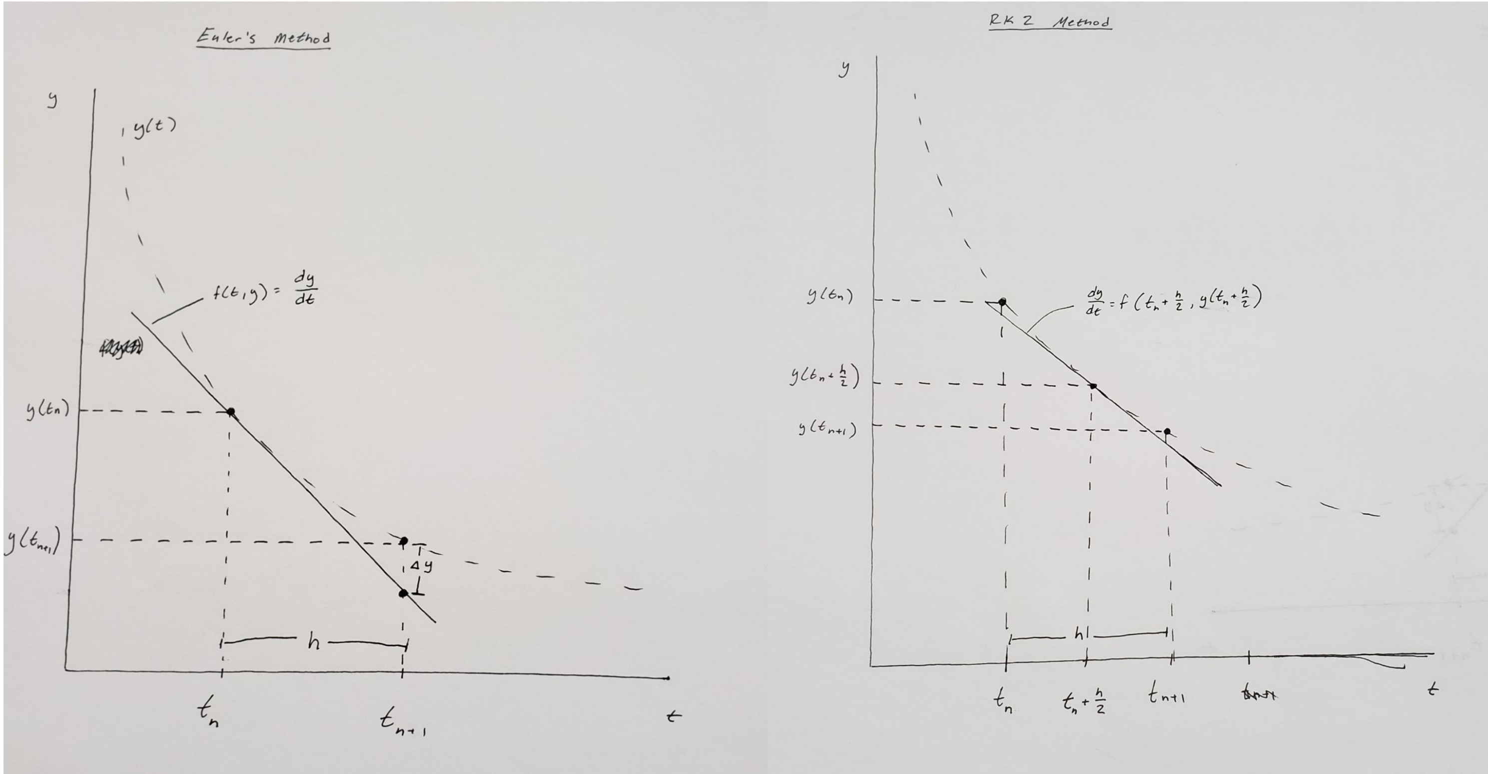
$\frac{d^4 y}{dt^4} = \alpha y^2$ where α is a constant

(e) Two coupled, first-order linear differential equations

$\frac{dy}{dt} = \alpha z + \gamma y$ where α and γ are a constants

$\frac{dz}{dt} = \alpha z + \zeta y$ where α and ζ are a constants

(2) ODE Algorithms: Explain why (and how) a 2nd-order Runge-Kutta (RK2) method is more precise (i.e., it has a smaller numerical error) than Euler's method for solving a 2nd-order non-linear differential equation, assuming equal step sizes. Use figures to aid your explanation, if necessary. What is different between an RK2 and an RK4 method and why is RK4 more precise?



For a given step size, the RK2 method is more precise because the slope or derivative it uses to calculate the function value at the next time step is closer to the value of the derivative at the next time step. In the Euler method, the value of the function at the next time step is calculated using the derivative of the function at the current function value. The RK2 method on the other hand uses the value of the derivative at the *midpoint* between the current value, $y(t_n)$ and the function value at the next time step $y(t_{n+1})$. The value of the slope at the midpoint has to be calculated using another method, i.e. the Euler method. Given the same time step, the RK2 method uses a slope which is a closer approximation to the value of the slope at the next time step, assuming the function is continuous and differentiable. This leads to a smaller numerical difference between the function's actual value at t_{n+1} and the approximated value at t_{n+1}.

The RK2 method differs from the RK4 method in that the RK4 method uses 4 "k" terms to approximate the next time step whereas the RK2 method only uses 2. The "k" terms are the time step multiplied by the derivative at intervals between t_n and t_{n+1}. For similar reasoning as the precision between the Euler method and the RK2 method, the RK4 method is a better approximation to the actual function because the slope it uses to calculate the next time step is a closer approximation to the slope at the next time step.

3. Physical Pendulum

Imagine a mass m suspended from a massless rod of length l. You raise the mass so that the rod makes an angle θ with the vertical (i.e., with the mass's resting position). Write a code to solve for the mass's motion over time once it is released.

(a) Setup the problem by first sketching the system and writing down the relevant equations.

See attached

(b) Select an algorithm (or algorithms) to solve the problem. Explain and justify your choice of algorithm.

I will use the Euler method and the RK2 method to solve the problem as they use the derivative to calculate the value of the function at the next time step. Given I have a second derivative, acceleration, I can calculate the first derivative at the next time step using the second derivative, and then the function value at the next time step using the first derivative.I'll use both methods because I will need the Euler method in the RK2 method because I need the Euler method to calculate the value of the function and the derivative of the function at the mid-point between the current step and the next step.

(c) Write pseudocode outlining how you plan to solve for the mass's motion.

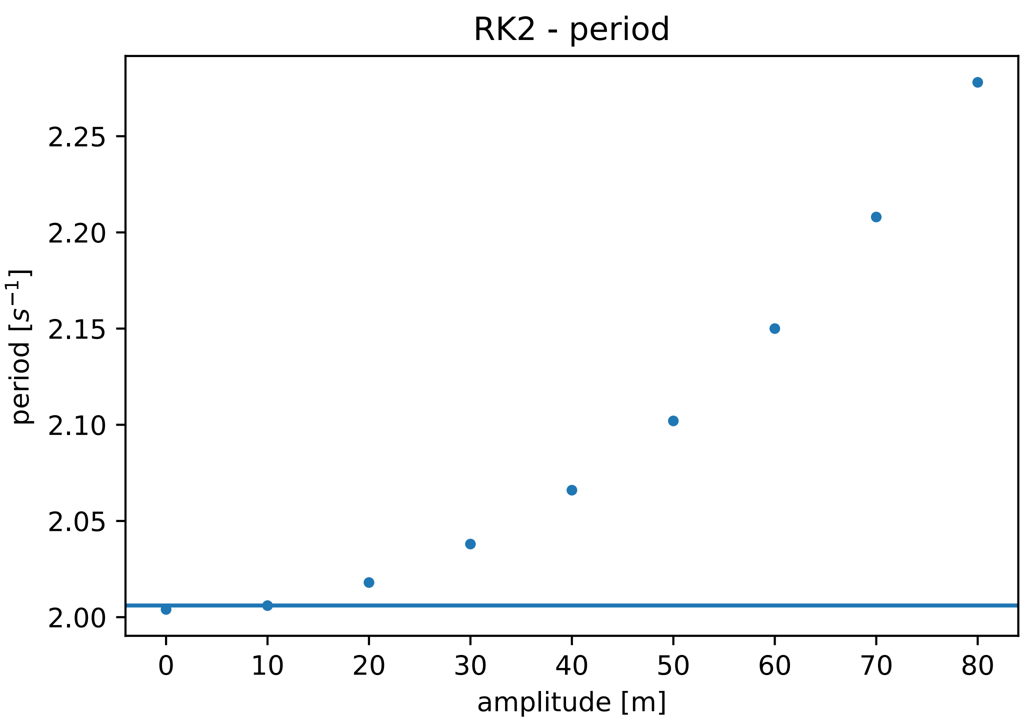
See attached.

(d) Make a plot of the mass's position in Cartesian space and also plot the y-velocity against the x-velocity. Describe your figures and explain why they make sense.

See attached figures. In Cartesian space, we see both the position and the velocity travel in an arc. In position space, this makes sense because the pendulum should swing in an arc. In velocity space this also makes sense because the velocity of the mass not eh pendulum will begin at zero at the top of its arc, or when the pendulum is at its maximum angle, and will swing through, increasing in velocity until it reaches a maximum at the bottom of its arc. The velocity will then decrease and become negative as it travels away from the origin towards the top of its arc on the opposite side. In three dimensions with time as the third dimension, we can see the position and velocity curves are out of phase with each other. This makes sense given the position will be at a maximum when the velocity is at zero. That is the velocity will be zero when the bob is at the top of its arc. Then the velocity is a maximum when the bob passes through the (t, 0, 0) in position space, this also makes sense given the bob will have maximum kinetic energy at this point the change in its gravitational potential energy is a maximum at this point.

(e) Find the period of your pendulum for a few different initial angles and rod lengths. How do your results compare with the theoretical expectation for a simple pendulum:

At small angles, the predicted period matches well with the theoretical predictions for period, as we would expect given the small angle approximation.



When do your results begin to differ significantly from the theoretical expectation?
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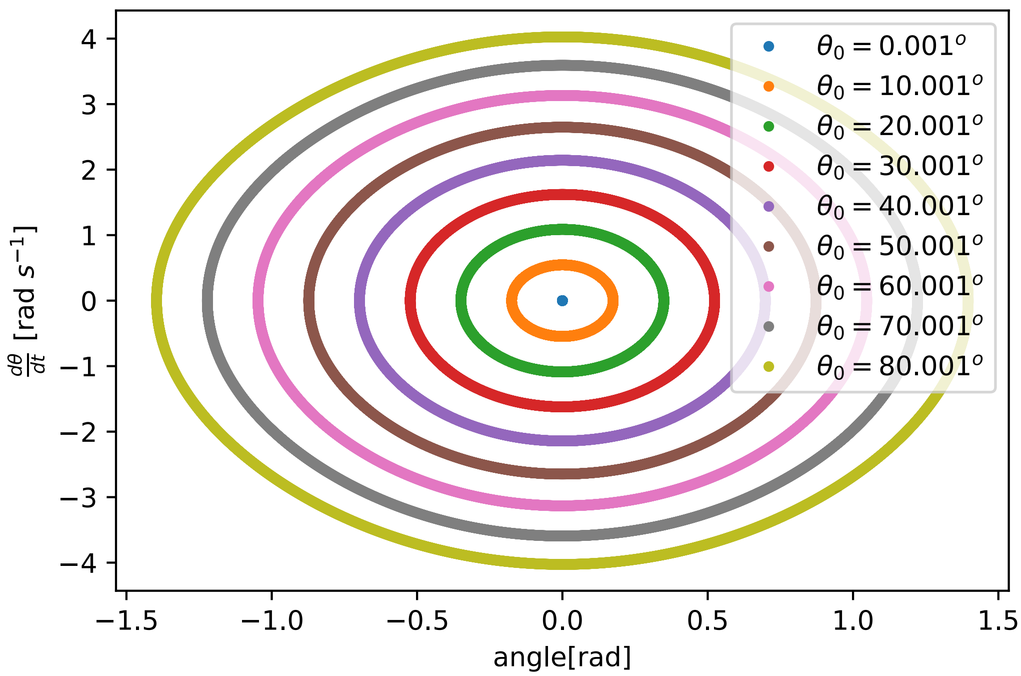
However at $\theta_0 \approx 15°$, the predicted period begins to deviate from the theoretical value, and becomes greater. This is the point where the small angle approximation will break down.

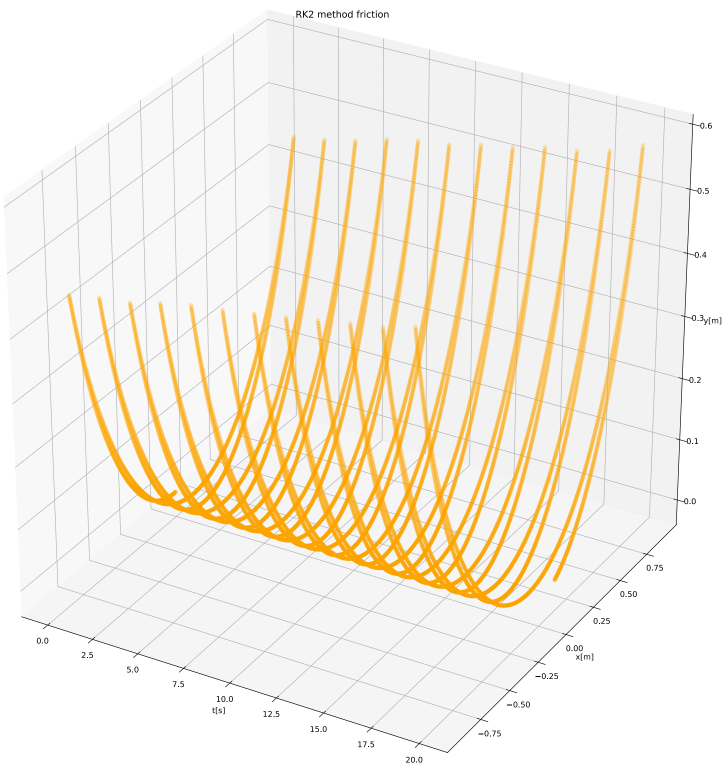
(f) Plot the system in phase-space (i.e., θ vs θ) for a series of different initial angles. Explain the resulting figure and its significance.

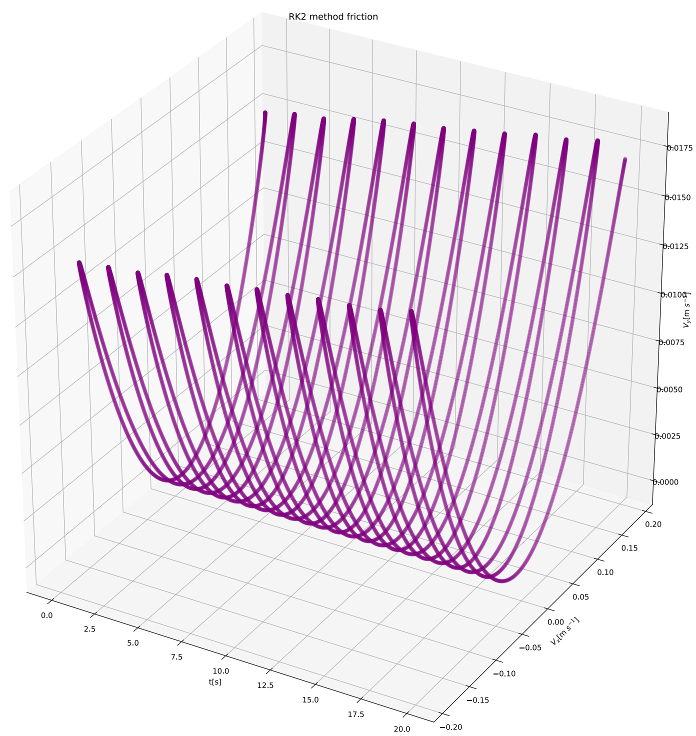


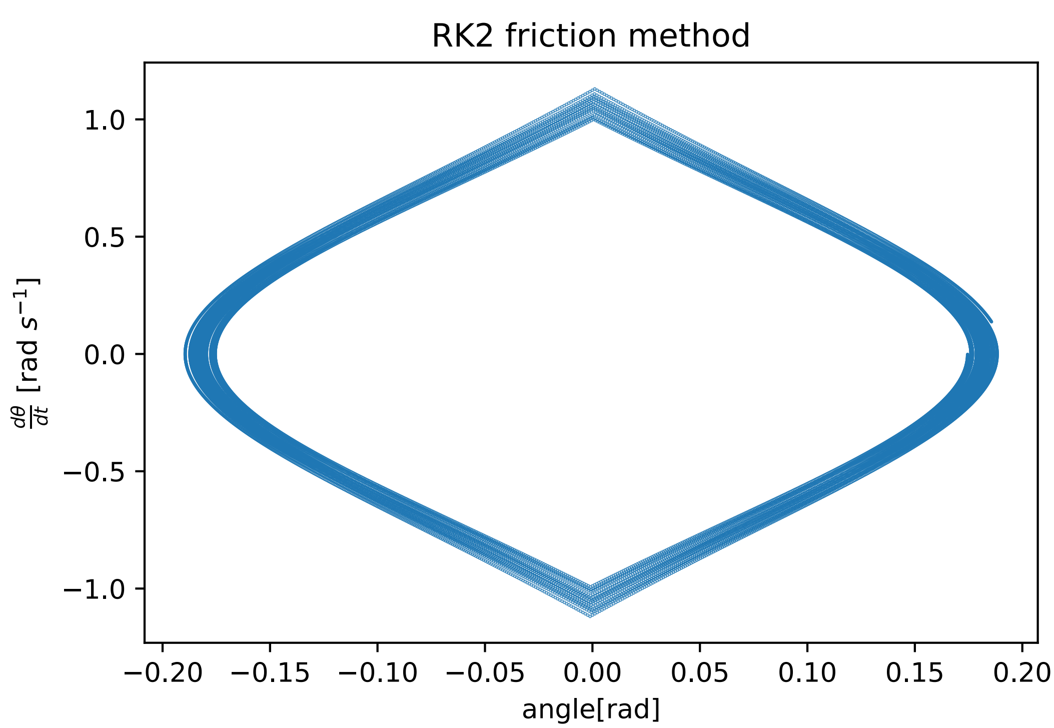
The phase space curve looks like an ellipse, indicating the angle amplitude and the angular velocity are out of phase with one another. We see the angular velocity has a maximum value when the angle is at zero. This makes sense given the bob is passing through the lowest point in it's arc where all of it has the greatest change in gravitational potential energy and therefore the greatest kinetic energy. When the angle is at a maximum, the bob has no angular velocity, which makes sense since it will only have gravitational potential energy at this point. As you decrease the initial amplitude, the absolute values of the maximum and minimum values in both velocity and angle amplitude decrease. This makes sense since the angular velocity is determined by the change in potential energy when the bob is at the top of its arc to the bottom of its arc. This value decreases as the initial amplitude decreases.

(g) Challenge: Add friction in the form of air drag. How do your figures change?

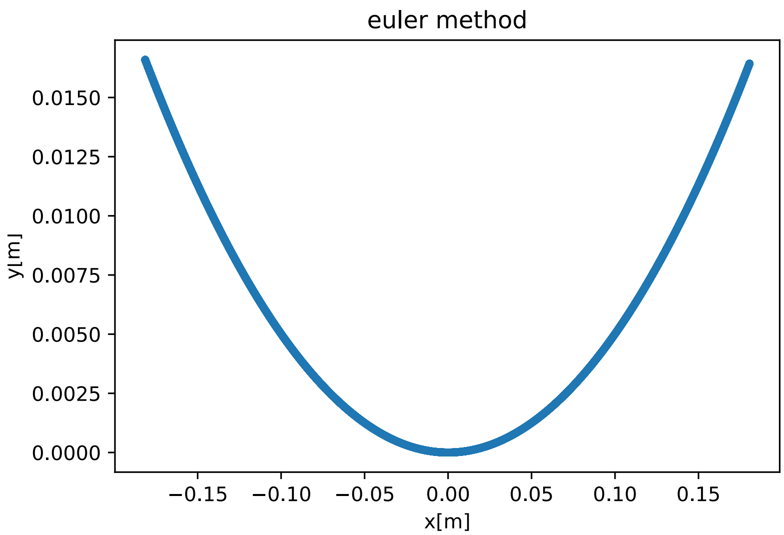
First looking at the figures for x and y as a function of time and velocity in the x and y as a function of time, we see the maximum amplitude of the bob decreases over time, as you would expect given friction will oppose the acceleration of the bob. Also, as a note, the coefficient of friction has been set to b = 50, and the mass to m = 0.1kg. In phase space, we see thee figure is no longer an ellipse, however it still follows the characteristic that when the position is at a maximum or minimum, the velocity is zero, and when velocity is at a maximum or minimum, the position is zero.

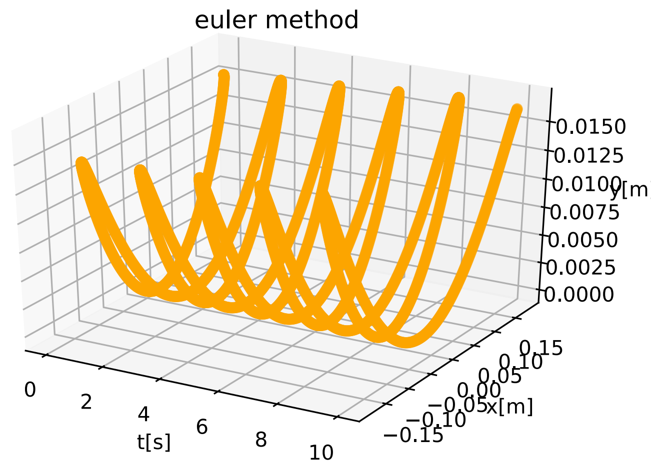


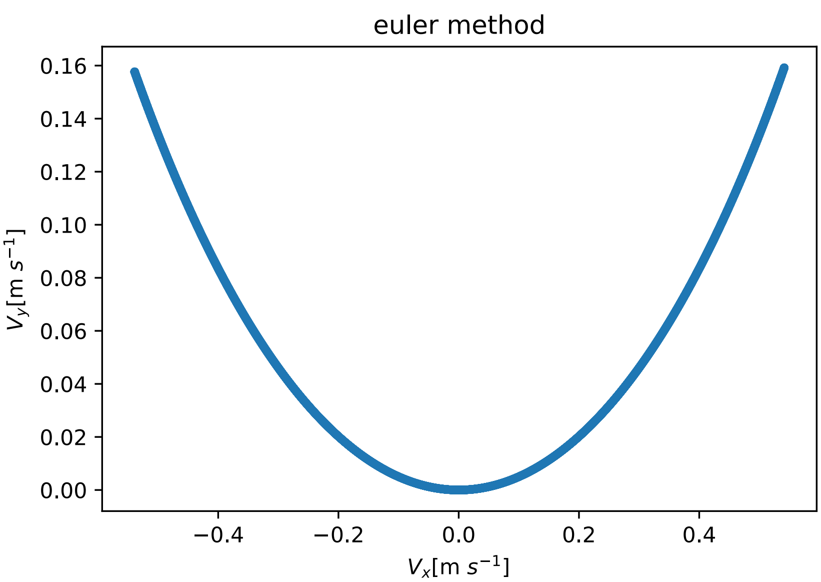


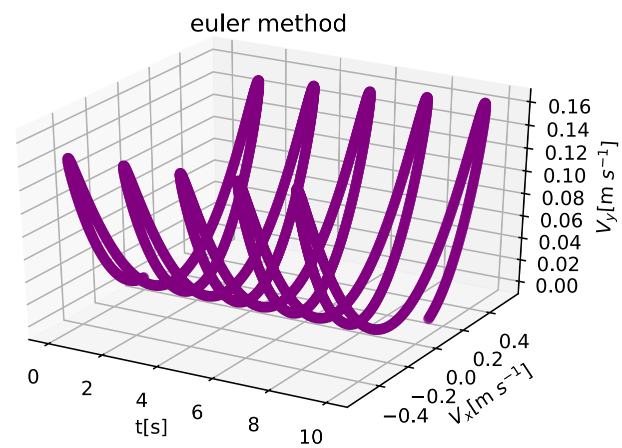


Euler method figures:

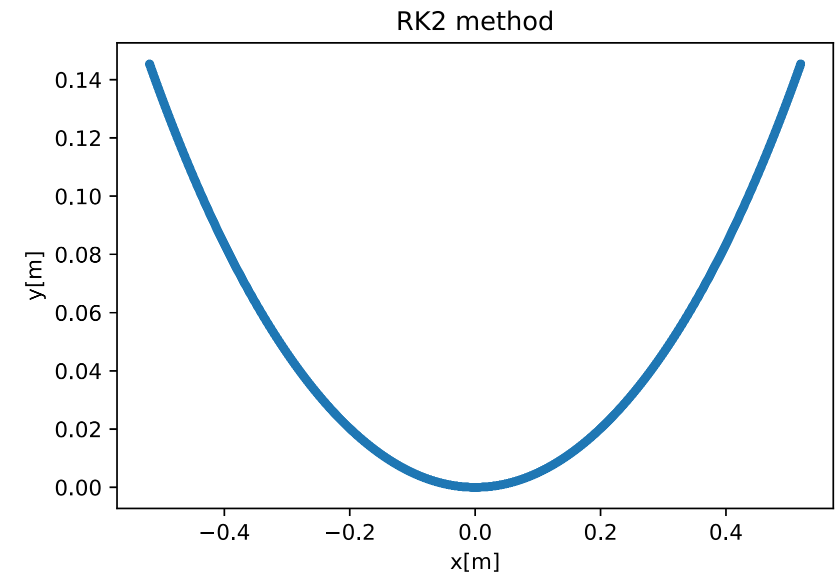


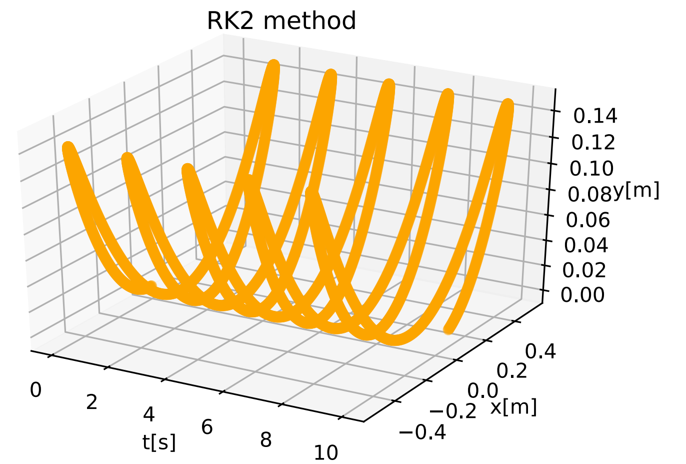


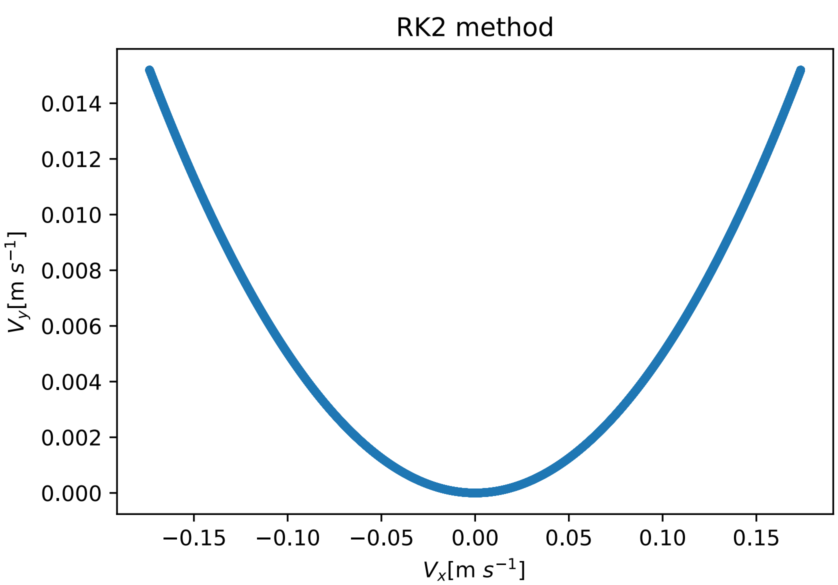


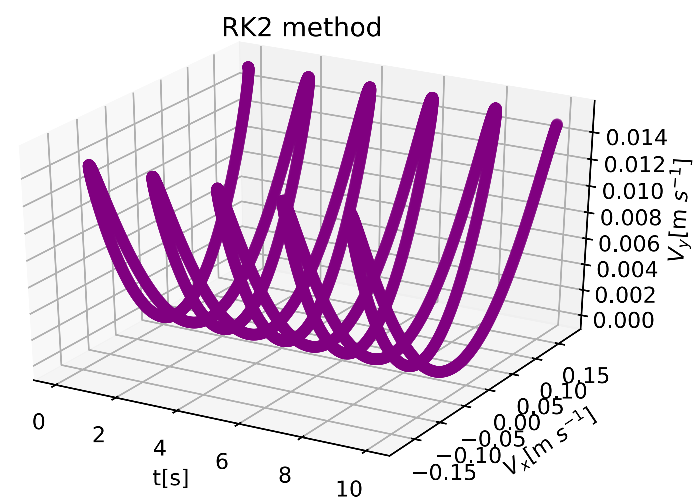


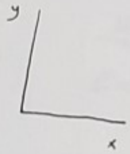
RK2 Method Figures:

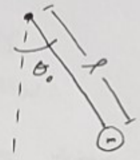


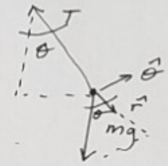












$x = l \sin\theta$
$y = l\cos\theta$

$\vec{r} = r\hat{r} = l\hat{r}$

$\dot{\vec{r}} = l\frac{d\hat{r}}{dt} = l\dot{\theta}\hat{\theta}$

$\ddot{\vec{r}} = l\ddot{\theta}\hat{\theta} + l\dot{\theta}\frac{d\hat{\theta}}{dt}$

$= l\ddot{\theta}\hat{\theta} + l\dot{\theta}(-\dot{\theta}\hat{r})$

$= l\ddot{\theta}\hat{\theta} - l\dot{\theta}^2\hat{r}$

$\Rightarrow$

Forces in $\hat{r}$ + $\hat{\theta}$:

$\hat{r}: -T + mg\cos\theta = -ml\dot{\theta}^2$

$\boxed{\hat{\theta} = -mg\sin\theta = ml\ddot{\theta}}$

$\Downarrow$

want to find $\theta(t)$

$\hat{\theta}: -mg\sin\theta = ml\ddot{\theta}$

$\therefore ml\frac{d^2\theta}{dt^2} = -mg\sin\theta$

$\therefore \frac{d^2\theta}{dt^2} = -\frac{g}{\ell}\sin\theta$

Euler method

$y^{(0)} = \theta$

$y^{(1)} = \frac{dy^{(0)}}{dt} = \frac{d\theta}{dt} \Rightarrow \frac{\Delta\theta}{h} = \frac{\theta(t_{n+1}) - \theta(t_n)}{h} = \dot{\theta}(t_n) \therefore \boxed{\theta(t_{n+1}) = \theta(t_n) + h\dot{\theta}(t_n)}$

$\frac{dy^{(1)}}{dt} = -\frac{g}{\ell}\sin\theta \Rightarrow \frac{\Delta y^{(1)}}{h} = \frac{y^{(1)}(t_{n+1}) - y^{(1)}(t_n)}{h} = -\frac{g}{l}\sin\theta$

$\therefore y^{(1)}(t_{n+1}) = y^{(1)}(t_n) - h\cdot\frac{g}{l}\sin\theta$

$\therefore \boxed{\dot{\theta}(t_{n+1}) = \dot{\theta}(t_n) - h\frac{g}{\ell}\sin\theta}$

RK2 method

$\theta(t_n + \frac{h}{2}) = \theta(t_n) + \frac{h}{2}\dot{\theta}$

$\dot{\theta}(t_n + \frac{h}{2}) = \dot{\theta}(t_n) - \frac{h}{2}\frac{g}{\ell}\sin(\theta(t_n))$

$\theta(t_{n+1}) = \theta(t_n) + h\cdot\dot{\theta}(t_n + \frac{h}{2})$

$\dot{\theta}(t_{n+1}) = \dot{\theta}(t_n) - h\cdot\frac{g}{\ell}\sin(\theta(t_n + \frac{h}{2}))$

Pseudocode

Euler method:

define time array

$\theta = \theta_0$ $\longrightarrow$ set initial conditions

$\dot{\theta} = \dot{\theta}_0$

for value in time array:

$\dot{\theta}_{n+1} = \dot{\theta} - \left(h \cdot \frac{g}{\ell} \cdot \sin(\theta)\right)$ $\longrightarrow$ calculate $\dot{\theta}_{n+1}$

$\theta_{n+1} = \theta + h \cdot \dot{\theta}$ $\longrightarrow$ calculate θ_{n+1}

- calculate x + y coordinates
- Calculate velocity in x + y directions
- Save the new $\dot{\theta}, \theta, x, y, V_x, V_y$ values to an array
- update the θ + $\dot{\theta}$ values to the new $\dot{\theta}$ + θ values:

$\theta = \theta_{n+1}$

$\dot{\theta} = \dot{\theta}_{n+1}$

- rinse and repeat over the time array

RK2 method

- define time array
- set initial θ_0 + $\dot{\theta}_0$ conditions:
  $\theta = \theta_0$
  $\dot{\theta} = \dot{\theta}_0$

for t in time array:

- calculate θ + $\dot{\theta}$ at the midpoint:

$$\theta_{1/2} = \theta + \frac{h}{2}\dot{\theta}$$

$$\dot{\theta}_{1/2} = \dot{\theta} - \left(\frac{h}{2} \cdot \frac{g}{\ell} \cdot \sin(\theta)\right)$$

- calculate θ + $\dot{\theta}$ at next time step using $\theta_{1/2}$ + $\dot{\theta}_{1/2}$:

$$\dot{\theta}_{n+1} = \dot{\theta} - h \cdot \frac{g}{\ell} \cdot \sin(\theta_{1/2})$$

$$\theta_{n+1} = \theta + h \cdot \dot{\theta}_{1/2}$$

- ~~save~~
- calculate x, y, V_x, V_y at θ_{n+1}
- Save θ_{n+1}, $\dot{\theta}_{n+1}$, x, y, V_x, V_y to array
- update θ + $\dot{\theta}$

$$\theta = \theta_{n+1}$$

$$\dot{\theta} = \dot{\theta}_{n+1}$$

rinse + repeat over the time array