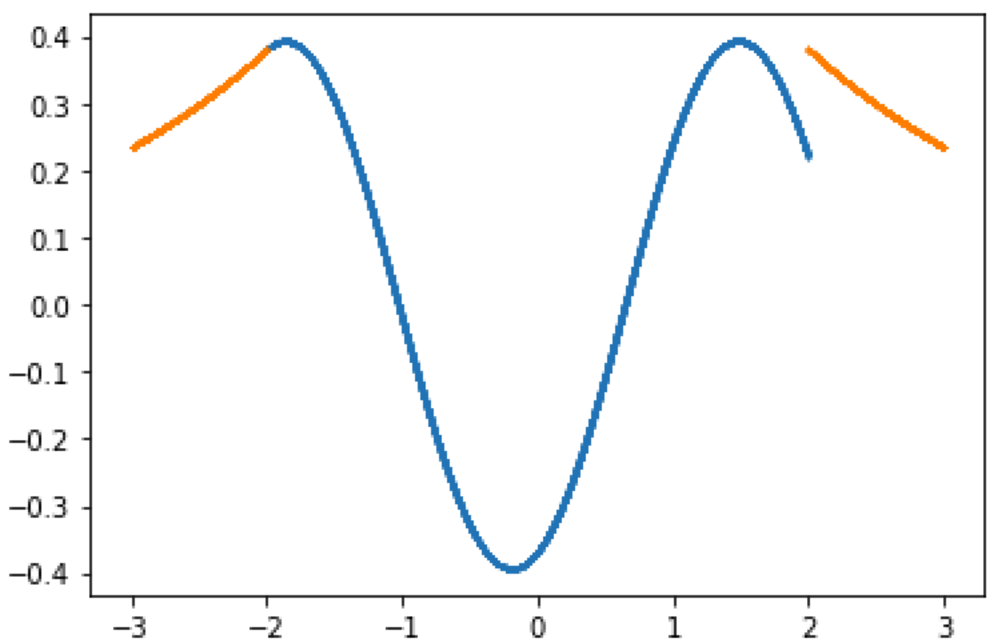
1. **Solving an ODE like it's the Transcontinental Railroad**

   When studying the problem of a particle in a finite potential well, we solved for the wavefunction, ψ(x), by integrating a second-order linear differential equation from left-to-right (in position, x, space) and from right-to-left, iterating over the value of the particle's energy E until the two integrations match at some prescribed location, x_{match}. What was different about this particular problem as compared to other ODE problems? Why was it important that we integrate from both directions? Discuss the advantages (and disadvantages) of solving the problem in this way. Can you envision an- other way to solve the problem while still satisfying all of the requirements that originally motivated the two-directional integration solver?

The major differences between this problem and other ODE problems we have solved is that it was (1) a piecewise function, and (2) we had to find a value for the energy, "E" in order to make the solution continuous across the well boundary. These two factors meant we had to use a Eulerian, or RK2 scheme inside the well to determine what the wave function was inside the well, and then solve for the eigenvalue "E", at the well boundary. In order to solve for this, we had to find the value of the function from the left of the well boundary and the from right of the well boundary for a guessed "E", we will call E_1.



There will be some difference between the value of the function inside the well versus outside of the well. We can guess another value for "E", E_2 and see if that guess gets us closer to having the value of the wave function inside the well match the value for the wave function outside of the well. Using E_1 and E_2, we can use a bisection-esque method to make a logical guess for our next guess, E_3. This method will walk us towards an E value where the wave-function inside the well will match the wave function outside of the well to some set tolerance. One advantage to using a bisection method to solve for "E" is that the bisection method will *always* converge to a root. However, depending on the initial guesses, and how far they are from the solution, the bisection method will converge slowly.

Perhaps instead of using the bisection method and two-directional integration to find the eigenvalue, we could try fitting a function to the numerically determined wave-function inside the well. With the fit function, we could use root finding find the where the fit function is equal to the wave function at the boundary of the well, defined by the analytic solution. Them, we could adjust E so the wave function is equal to the analytic solution at the boundary of the well at the x_{match} point.

(2) **van der Pol's Equation & the Electric Field of a Laser**

$$\ddot{E} = -\omega_0^2 E - \frac{1}{\tau}\dot{E} + (g - \tilde{g}E^2)\dot{E}$$

(a) I'm choosing to do the RK2 method because it is more accurate than the Euler method, but less computationally expensive than the RK4 method, which is preferred if I want to see the long

$$y^{(0)} = E$$

$$y^{(1)} = \frac{dy^{(0)}}{dt} = \frac{dE}{dt}$$

$$= \frac{\Delta y}{\Delta t} = \frac{\Delta y}{h} = \frac{y^{(0)}(t_{n+1}) - y^{(0)}(t_n)}{h} = f(E,t)$$

$$\therefore y^{(0)}(t_{n+1}) = y^{(0)}(t_n) + h \cdot f(E,t)$$

$$= y^{(0)}(t_n) + h \cdot \frac{dE}{dt}(t_n)$$

$$\frac{dy^{(1)}}{dt} = -\omega_0^2 y^{(0)} - \frac{1}{\tau} y^{(1)} + (g - \tilde{g} y^{(0)2}) y^{(1)}$$

$$\therefore \frac{\Delta y^{(1)}}{h} = -\omega_0^2 y^{(0)} - \frac{1}{\tau} y^{(1)} + (g - \tilde{g} y^{(0)2}) y^{(1)}$$

$$\therefore \frac{y^{(1)}(t_{n+1}) - y^{(1)}(t_n)}{h} = -\omega_0^2 y^{(0)} - \frac{1}{\tau} y^{(1)} + (g - \tilde{g} y^{(0)2}) y^{(1)}$$

$$\therefore y^{(1)}(t_{n+1}) = h\left(-\omega_0^2 y^{(0)} - \frac{1}{\tau} y^{(1)} + (g - \tilde{g} y^{(0)2}) y^{(1)}\right) + y^{(1)}(t_n)$$

Final equations:

$$\boxed{\begin{aligned} E(t_{n+1}) &= E(t_n) + h \cdot \dot{E}(t_n) \\ \dot{E}(t_{n+1}) &= \dot{E}(t_n) + h\left(-\omega_0^2 E(t_n) - \frac{1}{\tau}\dot{E}(t_n) + (g - \tilde{g}E^2(t_n))\dot{E}(t_n)\right) \end{aligned}}$$

term behavior of the system with a small time step. In order to get to the RK2 equations, I have to start with the Euler equations shown below.

$$E(t_n + \tfrac{h}{2}) = E + \tfrac{h}{2}\dot{E}(t_n)$$
$$\dot{E}(t_n + \tfrac{h}{2}) = \dot{E}(t_n) + \tfrac{h}{2}\left(-\omega_0^2 E(t_n) - \tfrac{1}{\tau}\dot{E}(t_n) + (g - \tilde{g}E^2(t_n))\dot{E}(t_n)\right)$$
$$E(t_{n+1}) = E(t_n) + h\dot{E}(t_n + \tfrac{h}{2})$$
$$\dot{E}(t_{n+1}) = \dot{E}(t_n) + h\left(-\omega_0^2 E(t_n + \tfrac{h}{2}) - \tfrac{1}{\tau}\dot{E}(t + \tfrac{h}{2}) + (g - \tilde{g}E^2(t_n + \tfrac{h}{2}))\dot{E}(t_n + \tfrac{h}{2})\right)$$

With the Eulerian scheme, I will calculate the relevant quantities at the midpoint, and use the midpoint values to calculate the next time step. These equations are shown below:

(c) With the constants defined as:

$\omega_0 = 1\ rad\ s^{-1}$

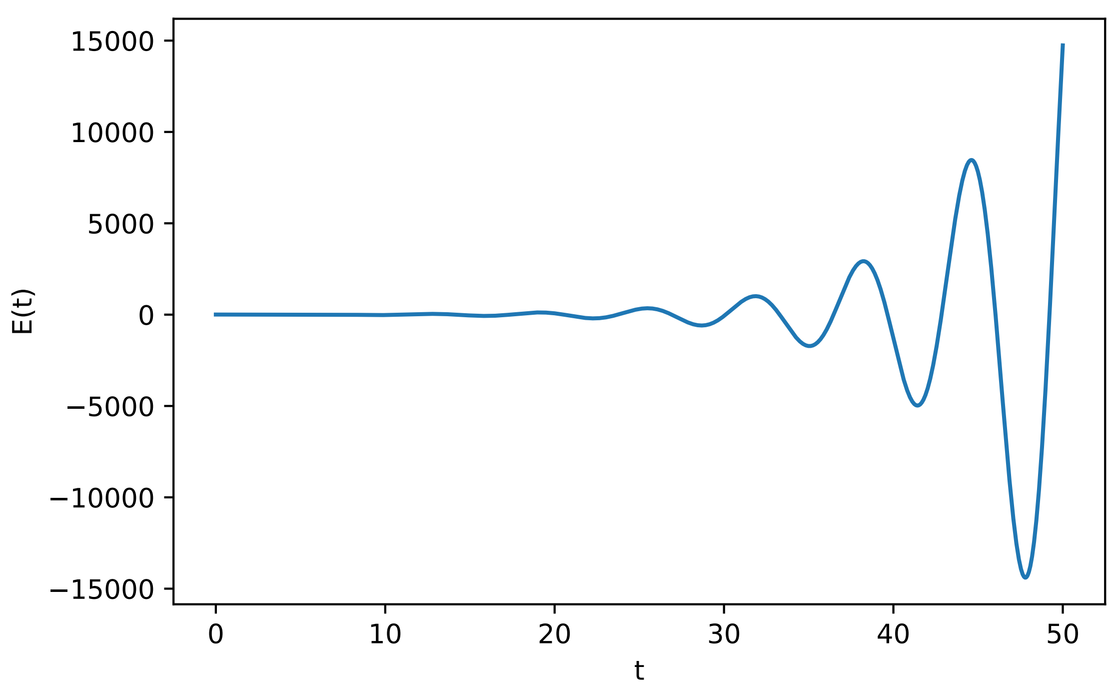
$\tau = 1.5$

$g = 1$

$\tilde{g} = 0$

$E_0 = 5$

$\dot{E} = 0$



We see that the amplitude of the energy oscillations continually increases with time.

(d) With constants of:

$\omega_0 = 0.5$

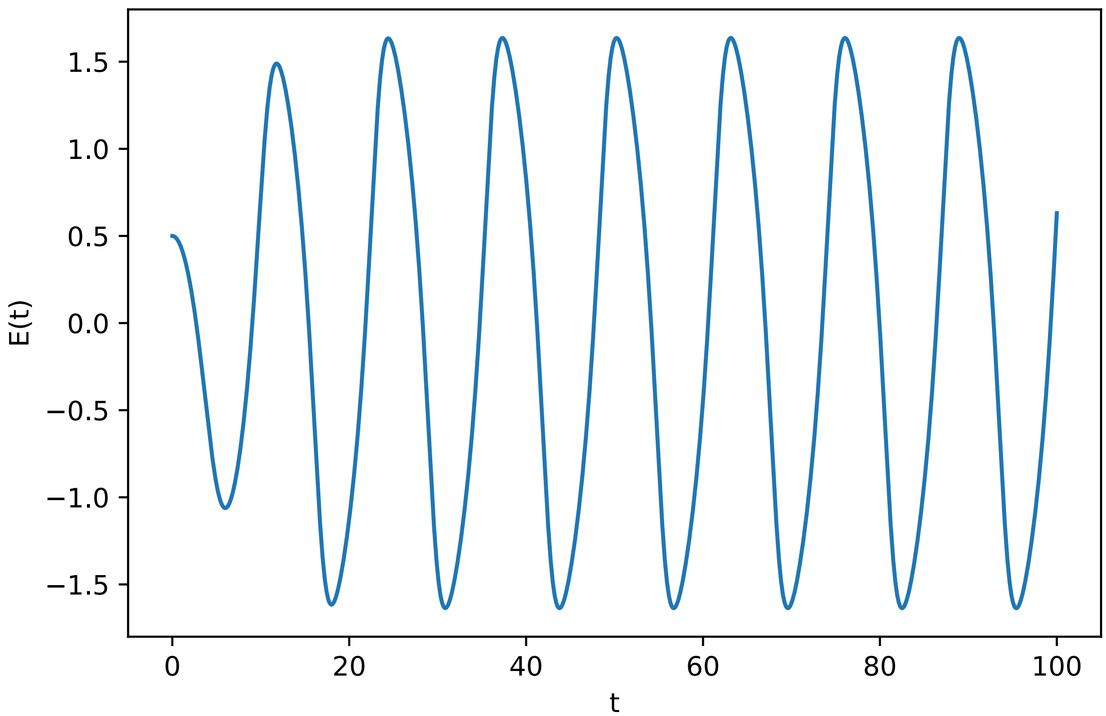
$\tau = 1.5$

$g = 1$

$\tilde{g} = 0.5$

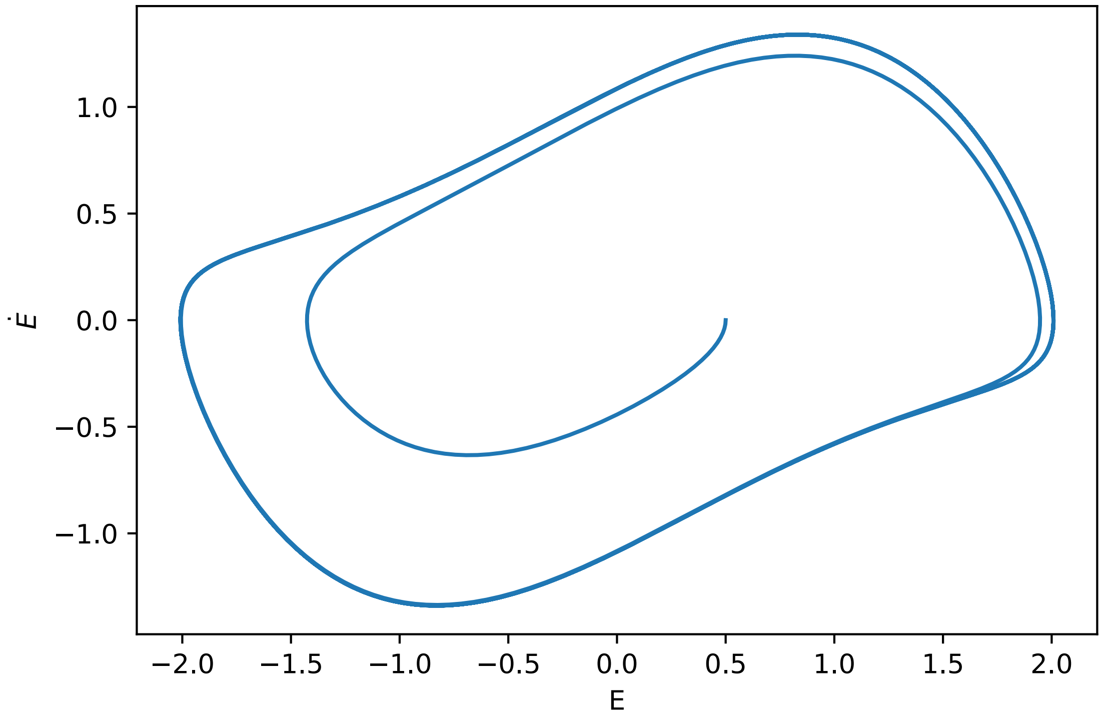
$E_0 = 0.5$

$\dot{E}_0 = 0$



With $\tilde{g} > 0$, the amplitude of oscillations first increases with each oscillation until the amplitude of oscillations reaches a certain amplitude, in this case when E(t) = 1.5. By changing the variables, I found the maximum amplitude the system reaches is equal to τ. After reaching the maximum amplitude, the system oscillates with a constant amplitude. Something else to note is the oscillations are not exactly sinusoidal. They bow inward so that the left side of the oscillation is curved inward and the right side of the oscillation is curved outward.

(e) Looking at the phase space figure for when $\tilde{g} > 0$, we see the initial path differs from the long term path.



This initial path is showing the relaxation time, or the time it takes the laser to reach a steady state of oscillation. We see at first, the amplitude of oscillations and the time derivative of the amplitude of oscillations has not reached its maximum value. We see this in the energy amplitude over time, as it takes a few oscillations for the amplitude to reach its maximum value. We can also see based on the shape of the phase-space diagram that the energy amplitude and the time derivative are out of phase with one another. If they were completely in phase, we would see a straight line in the phase-space diagram, and if if they were 180 degrees out of phase we would see a circle. This indicates the energy and time derivative of energy are out of phase somewhere in between 0 degrees and 180 degrees. Also, because of the odd, concave shape of the phase-space diagram, we know that the motion is not perfectly sinusoidal because out of phase sinusoidal functions would look like an ellipse. While the motion is not sinusoidal, we see the functions are periodic because after reaching a solid state, the functions trace the same path in phase-space.