

# Posterior Sampling with MCMC

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LANL: CCS-6

## Tasks

Say you have  $\mathbf{y}_1, \dots, \mathbf{y}_n \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$  where each  $\mathbf{y}_i$  is a vector of length  $p$ . Use  $n = 100$ ,  $p = 3$ ,  $\boldsymbol{\mu} = (1, 2, 3)$  and

$$\boldsymbol{\Sigma} = \begin{pmatrix} 1.0 & 1.4 & 2.1 \\ 1.4 & 4.0 & 4.2 \\ 2.1 & 4.2 & 9.0 \end{pmatrix}$$

to generate some data

**Task 1:** Use  $\mathbf{y}_1, \dots, \mathbf{y}_n \sim N(\boldsymbol{\mu}, \text{diag}(\sigma_1^2, \dots, \sigma_p^2))$  as your likelihood, with  $\boldsymbol{\mu} \sim N(\mathbf{m}, \mathbf{S})$  as your prior for  $\boldsymbol{\mu}$  and  $\sigma_i^2 \sim \text{InvGamma}(a, b)$  as your prior for  $\sigma_i^2$ . Use Gibbs sampling to sample the resulting posterior.

**Task 2:** Now use  $\mathbf{y}_1, \dots, \mathbf{y}_n \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$  as your likelihood, with  $\boldsymbol{\mu} \sim N(\mathbf{m}, \mathbf{S})$  as your prior for  $\boldsymbol{\mu}$  and the diagonal elements of  $\boldsymbol{\Sigma}$  as  $\sigma_i^2 \sim \text{InvGamma}(a, b)$  and each correlation parameter as  $\rho_{ij} \sim \text{Beta}(a, b)$ . Use Gibbs sampling and Metropolis-Hastings to sample the resulting posterior.

## Data Generation

```
set.seed(12)
library(mvtnorm)
library(invgamma)
mu <- c(1, 2, 3); mu

## [1] 1 2 3

sig <- cbind(c(1, 1.4, 2.1), c(1.4, 4.0, 4.2), c(2.1, 4.2, 9.0)); sig

##      [,1] [,2] [,3]
## [1,]  1.0  1.4  2.1
## [2,]  1.4  4.0  4.2
## [3,]  2.1  4.2  9.0

data <- rmvnorm(1000, mu, sig)
n <- nrow(data)
str(data)

##  num [1:1000, 1:3] -0.0251 -0.6187 0.4604 0.4057 0.3188 ...
```

## Task 1

Our likelihood follows a multivariate normal distribution with mean  $\boldsymbol{\mu}$  and variance  $\text{diag}(\sigma_1^2, \dots, \sigma_p^2)$ .

We multiply the likelihood by both priors to get our posterior distribution,  $P(\boldsymbol{\mu}, \tilde{\boldsymbol{\Sigma}} \mid \alpha, \beta, \mathbf{m}, \mathbf{S})$ , where  $\tilde{\boldsymbol{\Sigma}} = \text{diag}(1, 4, 9)$ .

We get the following result:

$$P(\boldsymbol{\mu}, \tilde{\boldsymbol{\Sigma}} \mid \alpha, \beta, \mathbf{m}, \mathbf{S}) \propto \exp((\boldsymbol{\mu} - \mathbf{m})' \mathbf{S}^{-1} (\boldsymbol{\mu} - \mathbf{m})) \prod_{i=1}^{P=3} \{(\sigma_i^2)^{-\alpha-1} \exp(-\frac{\beta}{\sigma_i^2})\} \prod_{i=1}^{100} \{ \prod_{j=1}^3 [\sigma_j^2]^{-1/2} \exp\{-\frac{1}{2}(\mathbf{y}_i - \boldsymbol{\mu})' \tilde{\boldsymbol{\Sigma}}^{-1} (\mathbf{y}_i - \boldsymbol{\mu})\} \}$$

Our  $\sigma_i^2$  values follow an inverse gamma distribution with parameters  $\alpha + 50$  and  $\beta + \frac{1}{2} \sum_{j=1}^{100} (y_{ji} - \mu_i)^2$

$$\sigma_i^2 \sim IG(\alpha + 50, \beta + \frac{1}{2} \sum_{j=1}^{100} (y_{ji} - \mu_i)^2)$$

Our  $\mu_i$  values follow a univariate normal distribution:

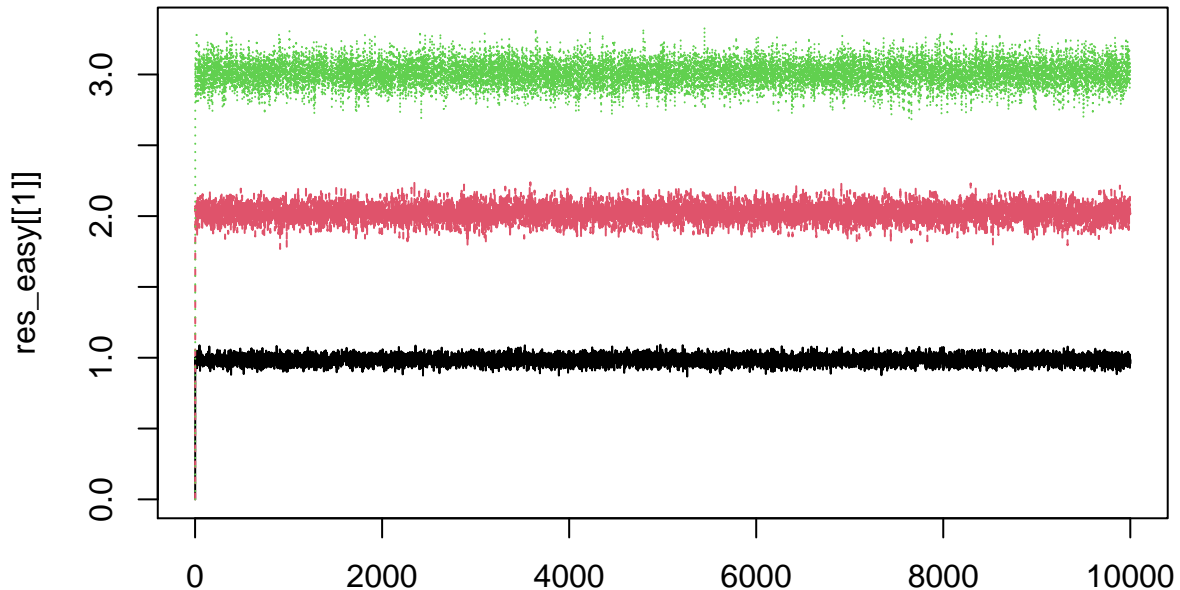
$$\mu_i \sim N(\frac{\sigma_i^2}{\sigma_i^2 + 100s_i^2} m_i + \frac{s_i^2 \sum_{j=1}^{100} y_{ji}}{\sigma_i^2 + 100s_i^2}, \frac{\sigma_i^2 s_i^2}{\sigma_i^2 + 100s_i^2})$$

## Gibbs Sampling

Please see the `mcmc_gibbs_script.R` file for the code used to generate these results.

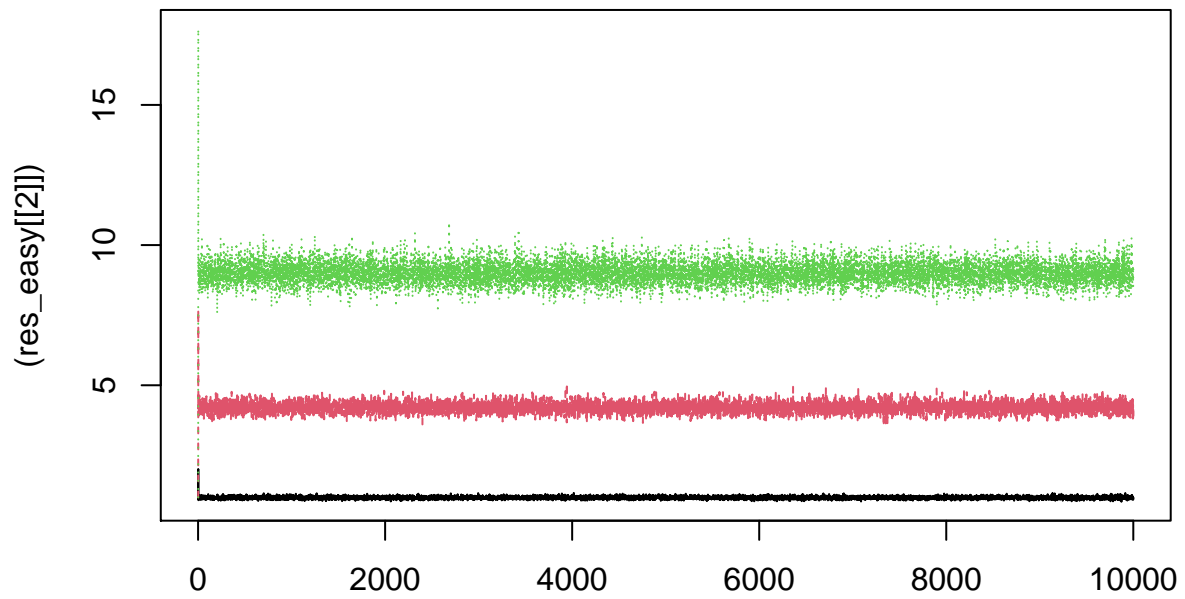
```
res_easy <- gibbs_easy()
matplot(res_easy[[1]], type = "l", main = "Plot of mu values vs. iterations")
```

**Plot of mu values vs. iterations**



```
matplot((res_easy[[2]]), type = "l", main = "Plot of sigma^2 values vs. iterations")
```

**Plot of sigma<sup>2</sup> values vs. iterations**



## Task 2: The Hard Task

Our likelihood follows a multivariate normal distribution with mean  $\boldsymbol{\mu}$  and variance  $\text{diag}(\sigma_1^2, \dots, \sigma_p^2)$ .

We multiply the likelihood by both priors to get our posterior distribution,  $P(\boldsymbol{\mu}, \boldsymbol{\Sigma} \mid \alpha, \beta, \mathbf{m}, \mathbf{S})$ , where

$$\boldsymbol{\mu} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

and

$$\boldsymbol{\Sigma} = \begin{pmatrix} 1.0 & 1.4 & 2.1 \\ 1.4 & 4.0 & 4.2 \\ 2.1 & 4.2 & 9.0 \end{pmatrix}$$

Our values of  $\mathbf{S}$  and  $\mathbf{m}$  used in the prior of  $\boldsymbol{\mu}$ .

```
S <- diag(3)
Sinv <- solve(S)
m <- c(0,0,0)
```

It follows that

$$\boldsymbol{\mu} \mid \cdot \sim \mathcal{N}[(\mathbf{S}^{-1} + n\boldsymbol{\Sigma}^{-1})^{-1}(\mathbf{S}^{-1}\mathbf{m} + n\boldsymbol{\Sigma}^{-1}\tilde{\mathbf{y}}), (\mathbf{S}^{-1} + n\boldsymbol{\Sigma}^{-1})^{-1}]$$

which is the full conditional distribution for  $\boldsymbol{\mu}$ .

Recall that for our correlation coefficient  $\rho_i$ , we have that

$$\rho_{ij} = \rho_{ji} = \frac{\text{cov}(i,j)}{\sigma_i \sigma_j}$$

which we simplify to  $\rho_i$  and work with the upper and lower triangular portion of the matrices.

## Metropolis-Hastings and Gibbs

Please see the `mcmc_gibbs_script.R` file for the code used to generate these results.

```
its <- 10000
a <- met_gibbs(its = its)
```

## Comparison of Results

### Mean Vector

Our acceptance rates of  $\sigma_i^2$  and  $\rho_i^2$ , respectively.

```
a[[4]] / its #sigma
```

```
## [1] 0.0673 0.2393 0.4541
```

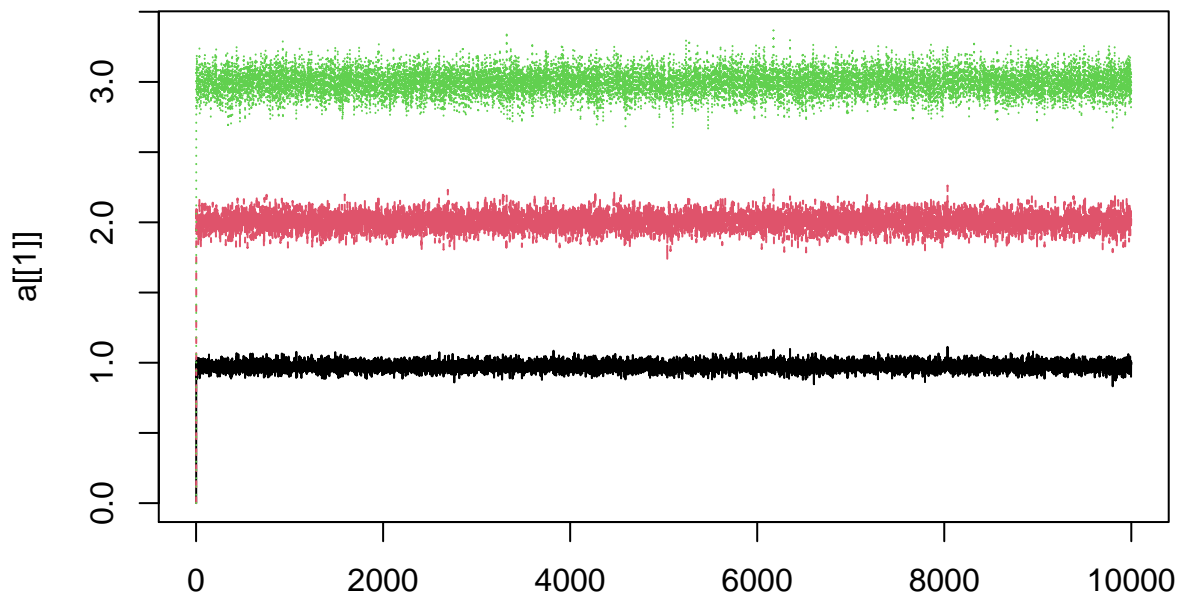
```
a[[5]] / its #rho
```

```
## [1] 0.1650 0.1750 0.1597
```

A plot of our sampled values of  $\mu$  is shown below, along with the column means:

```
matplot(a[[1]], type = "l", main = "Plot of mu values vs. iterations")
```

**Plot of mu values vs. iterations**



Column means of our sampled values:

```
colMeans(a[[1]])
```

```
## [1] 0.9759861 2.0070181 2.9962056
```

The true mean from the generated data:

```
colMeans(data)
```

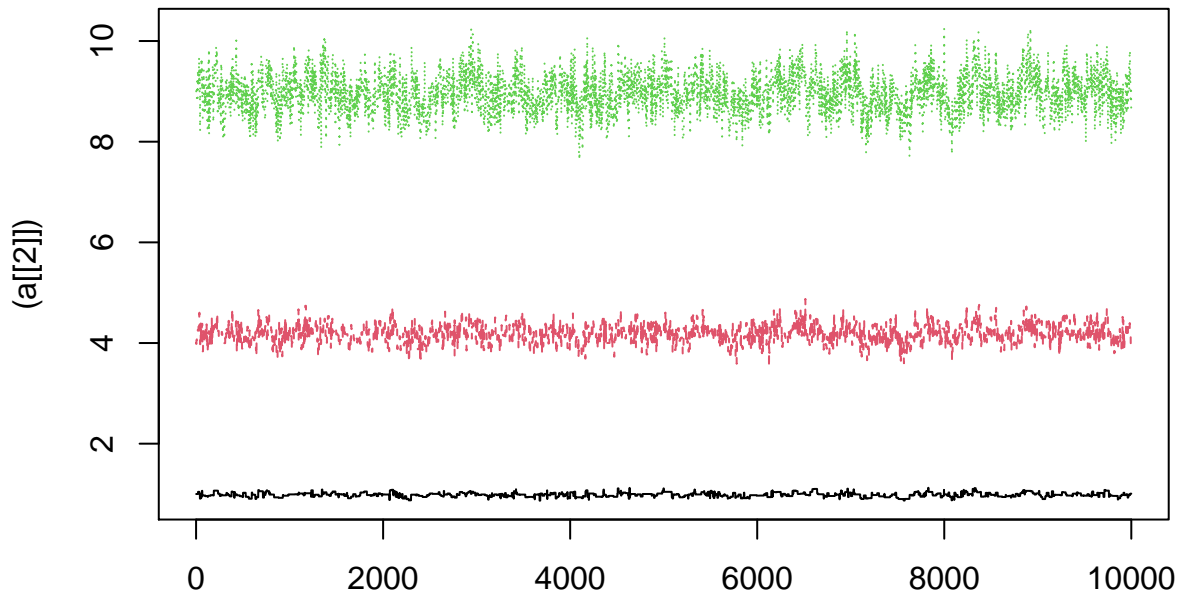
```
## [1] 0.9864302 2.0314982 3.0356240
```

## Covariance Matrix

For  $\sigma_i^2$ , the plot of the sampled data and their column means is shown below.

```
matplot((a[[2]]), type = "l", main = "Plot of sigma^2 values vs. iterations")
```

**Plot of sigma^2 values vs. iterations**



Column means of sampled values:

```
colMeans(a[[2]]) #average sigma^2 values
```

```
## [1] 0.985311 4.186508 8.956232
```

Final covariance matrix  $\hat{\Sigma}$

```
.cov <- a[[6]]; .cov #sampled
```

```
##          [,1]      [,2]      [,3]
## [1,] 1.006545 1.415843 2.022820
## [2,] 1.415843 3.966981 4.206504
## [3,] 2.022820 4.206504 8.627688
```

Compare this to the true values:

```
cov_data <- cov(data); cov_data #true values
```

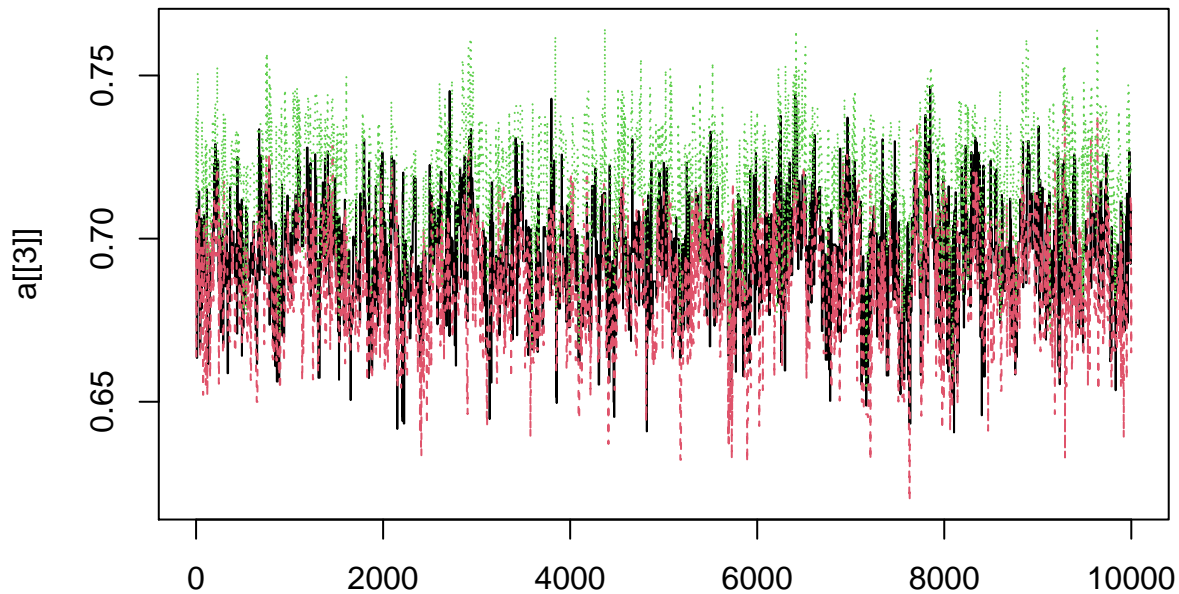
```
##          [,1]      [,2]      [,3]
## [1,] 0.9897208 1.429226 2.065696
## [2,] 1.4292258 4.210307 4.454647
## [3,] 2.0656956 4.454647 9.012204
```

## Correlation Coefficients

Finally, for  $\rho_i^2$ , our results are as follows:

```
matplot(a[[3]], type = "l", main = "Plot of rho values vs. iterations")
```

**Plot of rho values vs. iterations**



Column means of sampled values:

```
colMeans(a[[3]]) #average rho values
```

```
## [1] 0.6948384 0.6858425 0.7185440
```

We convert our covariance matrices to correlation matrices of our true and sampled data.

Final values of rho:

```
cor_sampled <- cov2cor(.cov); cor_sampled #sampled values
```

```
##           [,1]      [,2]      [,3]
## [1,] 1.0000000 0.7085466 0.6864256
## [2,] 0.7085466 1.0000000 0.7190251
## [3,] 0.6864256 0.7190251 1.0000000
```

```
cor_sampled[upper.tri(cor_sampled)]
```

```
## [1] 0.7085466 0.6864256 0.7190251
```

Compare this to the true correlation matrix:

```
cor_true <- cov2cor(cov_data); cor_true #true values
```

```
##           [,1]      [,2]      [,3]
```



```
## [1,] 1.0000000 0.7001444 0.6916629
## [2,] 0.7001444 1.0000000 0.7231708
## [3,] 0.6916629 0.7231708 1.0000000
cor_true[upper.tri(cor_true)]

## [1] 0.7001444 0.6916629 0.7231708
```

As seen here, our sampled values of  $\mu$ ,  $\sigma_i^2$  and  $\rho_i$  closely match the true value from the generated data.