The LNM Institute of Information Technology
Jaipur, Rajsthan

MATH-I ■ Drill Assignment

(Calculus of Functions of Several Variables, Directional Derivatives, Double, Triple Integrals)

Note:- *The assignment is devoted to the practice problems on limits, continuity and differentiability for functions of more than one variables and their applications to directional derivatives, gradients, problems on maxima/minima and double/triple integrals.*

P1. *This problems is concerned with the Sandwich theorem for functions of two variables:* Does knowing that $2|xy| - \frac{x^2y^2}{6} < 4(1 - \cos\sqrt{|xy|}) < 2|xy|$ tell you anything about $\lim_{(x,y)\to(0,0)} \frac{4(1-\cos\sqrt{|xy|})}{|xy|}$? Give reasons for your answer.

P2. *The following problem illustrates the fact that the partial derivatives of a function may exist at a point but the function need not be differentiable at that point and may not be even continuous.*
Show that the function

$$f(x,y) = \begin{cases} \frac{2xy}{x^2+y^2}, & \text{if } \ (x,y) \neq (0,0) \\ 0, & \text{if } \ (x,y) = (0,0) \end{cases}$$

has both the partial derivatives at $(0,0)$. However it is not continuous at (0,0).

P3. *Change along a helix:* Find the derivative of $f(x,y,z) = x^2 + y^2 + z^2$ in the direction of the unit tangent vector to the helix

$$R(t) = a\cos t\,\overrightarrow{i} + a\sin t\,\overrightarrow{j} + bt\,\overrightarrow{k}$$

at the points where $t = -\frac{\pi}{4}, 0$ and $\frac{\pi}{4}$.
(**Note:** *The function f in this problem gives the square of the distance of a point on the helix from the origin and the derivative will give the rate at which this distance is changing with respect to t as one moves through the given points on the helix.*).

P4. (*Temperature finding problem*) A flat circular plate has the shape of the region $x^2 + y^2 \leq 1$. The plate, including the boundary where $x^2 + y^2 = 1$, is heated so that the temperature (in $°C$) at the point (x,y) is given by $T(x,y) = x^2 + 2y^2 - x$. Find the temperature at the hottest and the coldest points on the plate.
(*Hint: This problem is asking you to find the extreme temperatures on the plate and we hope you can now open up yourself to solve the problem!*).

P5. *This problem illustrates application of the method of Lagrange multipliers to a problem from industry.*
L&T produces steel boxes at three different plants in amounts x, y and z, respectively, producing an annual revenue of $R(x,y,z) = 8xyz^2 - 200(x+y+z)$. The company is to produce 100 units annually. How should production be distributed to maximize revenue?
(*Hint: Use the method of lagrange multipliers, with* $\nabla R = \lambda \nabla F$. *Here,* $F(x,y,z) = x + y + z = 100$).

P6. By changing to polar co-ordinates, show that

$$\int_0^{a\sin\beta} \int_{y\cot\beta}^{\sqrt{a^2-y^2}} \ln(x^2+y^2)\,dx\,dy = a^2\beta\left(\ln a - \frac{1}{2}\right),$$

where $a > 0$ and $0 < \beta < \frac{\pi}{2}$. Rewrite the Cartesian integral with the order of integration reversed.

P7. (*Green's theorem and Laplace's equation*) Assuming that all the necessary derivatives exist and are continuous, show that if $f(x,y)$ satisfies the Laplace equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0,$$

then

$$\oint \frac{\partial f}{\partial y}dx - \frac{\partial f}{\partial x}dy = 0$$

for all closed curves C to which Green's theorem applies.
(**Note:** *The converse of this result is also true.*)