Lecture 8: More Partiality & Differentiability

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Let $f : B_r(x_0, y_0) \to \mathbb{R}$ be a function such that such that $f_x(x_0, y_0)$ exists at every $(x_0, y_0) \in B_r(x_0, y_0)$, then we obtain a function from $B_r(x_0, y_0)$ to $\mathbb{R}$ given by $(x, y) \longmapsto f_x(x, y)$. It is denoted by f_x and called the partial derivative of f with respect to x on $B_r(x_0, y_0)$. In case f_x is defined on $B_r(x_0, y_0)$, we can consider its partial derivatives at any point of $B_r(x_0, y_0)$. The partial derivative of $f_x : \mathbb{R}^2 \to \mathbb{R}$ with respect to x at (x_0, y_0), if it exists, is denoted by $f_{xx}(x_0, y_0)$. Also, the partial derivative of f_x with respect to y at (x_0, y_0), if it exists, is denoted by $f_{xy}(x_0, y_0)$. We can similarly define $f_{yx}(x_0, y_0)$ and $f_{yy}(x_0, y_0)$. Collectively, these are referred to as the second-order partial derivatives or simply the second partials of f at (x_0, y_0). Among these, $f_{xy}(x_0, y_0)$ and $f_{yx}(x_0, y_0)$ are called the mixed (second-order) partial derivatives of f , or simply the mixed partials of f.

Example 8.1 *Consider $f : \mathbb{R}^2 \to \mathbb{R}$ given by*

$$f(x, y) := \begin{cases} xy\dfrac{x^2 - y^2}{x^2 + y^2} & \text{if} \quad (x, y) \neq (0, 0) \\ 0 & \text{if} \quad (x, y) = (0, 0) \end{cases}$$

Show that $f_{xy}(0, 0) \neq f_{yx}(0, 0)$.

Solution: Note that

$$f_{xy}(0, 0) = \lim_{k\to 0} \frac{f_x(0, 0 + k) - f_x(0, 0)}{k}, \quad f_{yx}(0, 0) = \lim_{h\to 0} \frac{f_y(h, 0) - f_y(0, 0)}{h}$$

Hence it suffices to calculate f_x and f_y only along y-axis and x-axis, respectively. For any $y_0 \in \mathbb{R}$ we have

$$f_x(0, y_0) = \lim_{h\to 0} \frac{f(0 + h, y_0) - f(0, y_0)}{h} = \lim_{h\to 0} \frac{hy_0\frac{h^2-y_0^2}{h^2+y_0^2} - 0}{h} = \lim_{h\to 0} -y_0 = -y_0$$

$$\text{Hence} \quad f_{xy}(0, 0) = \lim_{k\to 0} \frac{-k - 0}{k} = -1$$

Similarly for any $x_0 \in \mathbb{R}$ we have

$$f_y(x_0, 0) = \lim_{k\to 0} \frac{f(x_0, 0 + k) - f(x_0, 0)}{k} = \lim_{k\to 0} \frac{x_0k\frac{x_0^2-k^2}{x_0^2+k^2} - 0}{k} = \lim_{k\to 0} x_0 = x_0$$

$$\text{Hence} \quad f_{yx}(0, 0) = \lim_{h\to 0} \frac{h - 0}{h} = 1$$

Theorem 8.2 (Mixed Partials Theorem) *Let f and its partial derivatives f_x, f_y, f_{xy} and f_{yx} are defined in some open disk with center (x_0, y_0). If either f_{xy} or f_{yx} are continuous at (x_0, y_0), then $f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)$.*

Using above theorem, we can say that for function f in Example 8.1, none of f_{yx} and f_{xy} is continuous at $(0, 0)$.

Exercise 8.3 *For function f in Example 8.1, Compute f_{xx}, f_{yy} on $\mathbb{R}^2$ if it exists and also compute f_{xy}, f_{yx} at all non-zero points. Discuss continuity of each of the second order partial derivative.*

Differentiability of functions of two variable

If look at the definition of continuity and limit of real-valued function of two variables, it's natural extension of the notion of continuity and limit of real-valued function of one variable. So it is natural to ask that, can we extend the notion of derivative in single variable calculus, to higher dimension, i.e., the differentiability of a real-valued function of two variables?

First recall the differentiability of a single variable function.

Definition 8.4 *Let $f : (a, b) \to \mathbb{R}$, and $c \in (a, b)$. We say that f is differentiable at point c if the limit*

$$\lim_{h \to 0} \frac{f(c + h) - f(c)}{h}$$

exists.

Let us try extend this definition to define differentiability of a real-valued function of two variables.

Let $f : B_r(x_0, y_0) \to \mathbb{R}$ be any function. It might seem natural to consider a limit such as

$$\lim_{(h,k) \to (0,0)} \frac{f(x_0 + h, y_0 + k) - f(x_0, y_0)}{(h, k)}$$

But this doesn't make sense for the simple reason that division of a real number by a point in $\mathbb{R}^2$ has not been defined. Now how to overcome this difficulty ?

Exercise 8.5 *Let $f : (a, b) \to \mathbb{R}$, and $c \in (a, b)$. Show that f is differentiable at point c if and only if there is $\alpha \in \mathbb{R}$ such that*

$$\lim_{h \to 0} \frac{|f(c + h) - f(c) - \alpha h|}{|h|} = 0.$$

Solution: First we assume that f is differentiable at c. That is

$$\lim_{h\to 0} F(h) \text{ exists}, \quad \text{where } F(h) := \frac{f(c+h)-f(c)}{h}.$$

We denote this limit by $f'(c)$, i.e., $\lim_{h\to 0} F(h) = f'(c)$. Now we consider a constant function $G(h) \equiv f'(c)$. Trivially $\lim_{h\to 0} G(h) = f'(c)$. Therefore, by properties of limits of functions of one variable

$$\begin{aligned}
\lim_{h\to 0}[F(h) - G(h)] &= 0\\
\text{i.e., } \lim_{h\to 0}\left[\frac{f(c+h)-f(c)}{h} - f'(c)\right] &= 0\\
\text{i.e., } \lim_{h\to 0}\left[\frac{f(c+h)-f(c)-f'(c)h}{h}\right] &= 0\\
\iff \lim_{h\to 0}\left|\frac{f(c+h)-f(c)-f'(c)h}{h}\right| &= 0,
\end{aligned}$$

where the last equivalence is follows from the fact that " $\lim_{x\to c} g(x) = 0 \iff \lim_{x\to c} |g(x)| = 0$"

Hence we choose $\alpha = f'(c)$.

Now we assume that there is $\alpha \in \mathbb{R}$ such that

$$\lim_{h\to 0} \frac{|f(c+h)-f(c)-\alpha h|}{|h|} = 0.$$

Then by previous discussion, we have

$$\lim_{h\to 0}\left[\frac{f(c+h)-f(c)}{h} - \alpha\right] = 0$$

Hence

$$\lim_{h\to 0}\left(\left[\frac{f(c+h)-f(c)}{h} - \alpha\right] + G(h)\right) = 0 + \alpha$$

Above statement is same as saying that

$$\lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = \alpha$$

That is f' at c exists and is equal to α. ∎

Now the last exercise and a realization that the derivative of a real-valued function of two variables may not be a single number but possibly a pair of real numbers suggests the way to define the differentiability of function of two variables. Good time to recall that $|(h,k)| = \sqrt{h^2+k^2}$.

Definition 8.6 *Let $D \subseteq \mathbb{R}^2$ and Let $(x_0, y_0) \in D$ be an interior point of D. A function $f : D \to \mathbb{R}$ is said to be differentiable at (x_0, y_0) if there is $(\alpha_1, \alpha_2) \in \mathbb{R}^2$ such that*

$$\lim_{(h,k)\to(0,0)} \frac{|f(x_0+h, y_0+k) - f(x_0, y_0) - \alpha_1 h - \alpha_2 k|}{\sqrt{h^2+k^2}} = 0$$

In this case, we call the pair (α_1, α_2) the derivative of f at (x_0, y_0).

Let us note that if f is differentiable at (x_0, y_0) and if (α_1, α_2) is the derivative of f at (x_0, y_0), then letting (h, k) approach $(0, 0)$ along the x-axis we see that

$$\lim_{h\to 0} \frac{|f(x_0+h, y_0) - f(x_0, y_0) - \alpha_1 h|}{\sqrt{h^2}} = \lim_{h\to 0} \frac{|f(x_0+h, y_0) - f(x_0, y_0) - \alpha_1 h|}{|h|} = 0$$

that is $\alpha_1 = f_x(x_0, y_0)$. Similarly, letting (h, k) approach $(0, 0)$ along the y-axis we see that

$$\lim_{k\to 0} \frac{|f(x_0, y_0+k) - f(x_0, y_0) - \alpha_2 k|}{\sqrt{k^2}} = \lim_{k\to 0} \frac{|f(x_0, y_0+k) - f(x_0, y_0) - \alpha_2 k|}{|k|} = 0$$

that is $\alpha_2 = f_y(x_0, y_0)$. Hence if f is differentiable, then the gradient of f at (x_0, y_0) exists and the derivative of f at $(x_0, y_0) = \nabla f(x_0, y_0)$. Thus in checking the differentiability of f at (x_0, y_0), First check wether partial derivatives exists and then check whether the corresponding two-variable limit exists and is equal to zero. Also, if either of the partial derivatives does not exist at a point, then we can be sure that f is not differentiable at that point.

Example 8.7 *Let $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = |xy|$. Since $f(x, 0) = 0$ for all $x \in \mathbb{R}$ and $f(0, y) = 0$ for all $y \in \mathbb{R}$ hence $f_x(0, 0) = 0 = f_y(0, 0)$. Now to check wether*

$$\lim_{(h,k)\to(0,0)} \frac{|f(0+h, 0+k) - f(0, 0) - 0.h - 0k|}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{|hk|}{\sqrt{h^2+k^2}} = 0$$

Let $((h_n, k_n))$ be sequence in $\mathbb{R}^2 \setminus \{(0, 0)\}$ such that $(h_n, k_n) \to (0, 0)$.

Note that $|h_n| \leq \sqrt{h_n^2 + k_n^2}$ Hence

$$0 \leq \frac{|h_n k_n|}{\sqrt{h_n^2 + k_n^2}} \leq |k_n|$$

Since $k_n \to 0$ hence $|k_n| \to 0$. Hence $\dfrac{|h_n k_n|}{\sqrt{h_n^2+k_n^2}} \to 0$. Hence, f is differentiable at $(0, 0)$ and $\nabla f(0, 0) = (0, 0)$.

Exercise 8.8 *For function f in Example 8.7, check the differentiability at all non-zero points.*