Assignment-4

1. Determine whether each of the following relation is a function or not. If, it is, then determine its range and type, i.e, one-to-one, or onto, or both, or none.

a) $f: \mathbb{Z} \to \mathbb{Z}$; $f(x) = \dfrac{x^3}{2x+1}$, $\forall x \in \mathbb{Z}$

b) $f: \mathbb{R} \to \mathbb{R}$; $f(x) = \dfrac{e^x}{1+\log|x|}$, $\forall x \in \mathbb{R}$

c) $f: \mathbb{Z} \to \mathbb{Z}$; $f(x) = x - 1$, $\forall x \in \mathbb{Z}$

d) $f: \mathbb{R} \to \mathbb{R}^+ \cup \{0\}$; $f(x) = |x|$, $\forall x \in \mathbb{R}$

e) $f: \mathbb{Z} \to \{0, 1\}$, such that $f(x) = \begin{cases} 0, & \text{if } x \text{ is even} \\ 1, & \text{if } x \text{ is odd} \end{cases}$

f) $f: \mathbb{R} \to \{0, 1\}$, such that $f(x) = \begin{cases} 0, & \text{if } x \notin \mathbb{Z} \\ 1, & \text{if } x \in \mathbb{Z} \end{cases}$

g) $f: \mathbb{R} \to \mathbb{Z}$, such that $f(x) = \lceil x \rceil$, where $\lceil\ \rceil$ is the ceiling function.

2. Show that each of the following function is a bijection and find its inverse.

i) $f: \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$; $f((x, y)) = (x+y, x-y)$, $\forall (x, y) \in \mathbb{R} \times \mathbb{R}$

ii) Let, $A = \{x \mid x \in \mathbb{R} \text{ and } x \geq 0\}$, $B = \{y \mid y \in \mathbb{R} \text{ and } y \geq -1\}$. Define, $f: A \to B$, such that $f(x) = x^2 - 1$, $\forall x \in A$.

iii) $f: \mathbb{R} \to \mathbb{R}$; $f(x) = x^3 + 1$, $\forall x \in \mathbb{R}$.

iv) $A = \{1, 2, 3, 4, 5\}$ and $f: A \to A$, such that
$$f = \{(1,3), (2,2), (3,4), (4,5), (5,1)\}$$

3. Let, $f: A \to B$ and $g: B \to C$ be two functions. Show that

a) $g \circ f$ is one-to-one implies that f is one-to-one

b) $g \circ f$ is onto implies that g is onto

4. Show that:

a) Every logarithmic function $f(n) = \log n$ is of order $O(\lg n)$, where $\lg n$ signifies logarithm of n to the base 2.

b) $f(n) = n^2(7n-2)$ is $O(n^3)$.

c) $f(n) = n\lg n$, $g(n) = n^2$. Then f is $O(g)$, but g is not $O(f)$.

d) $f(n) = n^{100}$, $g(n) = 2^n$. Then, f is $O(g)$, but g is not $O(f)$.

e) $f(n) = 5n^2+4n+3$ and $g(n) = n^2+100n$ are of same order.

5. Consider the following functions:

$f_1(n) = 5n\log n$, $f_2(n) = 6n^2 - 3n + 7$, $f_3(n) = \sqrt{n} + \lg(\sqrt[3]{n})$,

$f_4(n) = \lg(n^4)$, $f_5(n) = 15000$, $f_6(n) = -15n$,

$f_7(n) = n + \lg n$, $f_8(n) = \sqrt{n} + 12n$, $f_9(n) = \lg(n!)$.

Now, determine the Θ class of each of these functions from the list: $\Theta(1)$, $\Theta(n)$, $\Theta(n\lg n)$, $\Theta(\lg n)$, $\Theta(\sqrt{n})$, $\Theta(n^2)$, $\Theta(2^n)$.

6. For each of the following relations R, find $Dom(R)$, $Ran(R)$, $\overline{R}$, R^{-1} and M_R. Also, whenever the domain and co-domain of R are identical sets, find its diagraph.

a) $A = \{a, b, c, d\}$, $B = \{1, 2, 3\}$

$R = \{(a,1), (a,2), (b,1), (c,2), (d,1)\}$

b) $A = \{1, 2, 3, 4\}$, $B = \{1, 4, 6, 8, 9\}$; aRb, iff $a^2 = b$.

c) $A = B = \{1, 2, 3, 4, 8\}$; aRb, iff $a = b$.

d) $A = B = \{1, 2, 3, 4, 8\}$; aRb, iff a divides b.

e) $A = B = \{1, 2, 3, 5, 7, 8\}$; aRb, iff either $a+b \le 9$ or $a^2 - b^2 \ge 10$.

7. Let, R be a relation on a set A, whose Boolean matrix is given. Find the relation R.

i) $M_R = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$, ii) $M_R = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$

8. i) Let, $A = \{1, 2, 3, 4, 5, 6\}$ and R be a relation on A, defined as: $R = \{(1,2), (1,3), (2,2), (2,4), (2,5), (3,4), (4,5), (5,6)\}$. Draw the diagraph of R and from this diagraph, find R^2 and R^3. Verify that $M_{R^2} = M_R \odot M_R$ and $M_{R^3} = M_R \odot M_R \odot M_R$. Also, draw the diagraphs of R^2 and R^3.

ii) Let, $A = \{1, 2, 3, 4, 5\}$ and R be a relation on R, defined as aRb iff $a < b$. Then, determine R, R^2, and R^3. Find M_R, M_{R^2}, and M_{R^3}. Draw the diagraphs of R, R^2, and R^3.

9. Determine whether each of the following relations R on the sets A is reflexive, irreflexive, symmetric, asymmetric, antisymmetric, or transitive.

i) $A = \mathbb{Z}$; aRb, iff $a \leq b+1$.

ii) $A = \mathbb{Z}^+$; aRb, iff $|a-b| \leq 2$.

iii) A = set of all ordered pair of real numbers; $(a,b) R (c,d)$, iff $a = c$.

iv) A = set of all straight lines in the XY plane. For two lines L_1, and L_2, $L_1 R L_2$, iff L_1 intersects L_2.

v) A = set of all circles in the XY plane. For two circles C_1 and C_2, $C_1 R C_2$, iff C_1 and C_2 are concentric.

10. Determine, whether each of the following relation R is an equivalence relation on the set A or not with proper justification. In case, R is an equivalence relation, then find A/R (the equivalence class of A by R).

i) $A = \{1, 2, 3\}$, $M_R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

ii) $A = \{1, 2, 3\}$, $M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

iii) $A = \{1, 2, 3, 4, 5\}$, $R = \{(1,1), (1,2), (1,3), (2,1), (2,2), (3,1), (2,3), (3,3), (4,4), (3,2), (5,5)\}$

iv) $A = \{a, b, c, d\}$, $R = \{(a,a), (b,a), (b,b), (c,c), (d,d), (d,c)\}$

v) $A = \{a, b, c, d, e\}$, $M_R = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix}$.

vi) $A = \mathbb{R} \times \mathbb{R}$; $(a,b) R (c,d)$, iff $a^2 + b^2 = c^2 + d^2$, $\forall$ $(a,b), (c,d) \in A$.

11. i) Let, $A = \mathbb{Z}^+$ and R be a relation on A, defined as aRb, iff $b = a + 1$. Give the transitive closure of A.

ii) Let, A = set of all people. Define the relation R on A, such that aRb iff "b is the mother of a". Give the transitive closure of A.

12. Prove that if R is reflexive as well as transitive, then $R^n = R$, $\forall n \in \mathbb{N}$.

13. Let, $A = \{a_1, a_2, a_3, a_4, a_5\}$ and R be a relation on A, having the following Boolean matrix:

$$M_R = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

. Find the transitive closure of R, by using Warshall's algorithm.

14. Let, $A = \{1, 2, 3, 4\}$. Then, find the equivalence closure of each of the following relations, by using Warshall's algorithm.

i) $R = \{(1,1), (1,4), (2,1), (2,2), (3,4), (4,4)\}$

ii) $R = \{(1,4), (2,1), (2,4), (3,2), (3,4), (4,3)\}$

15. Let, $A = \{a, b, c, d, e\}$ and let R and S be two relations on A, defined as:

$R = \{(a,a), (a,c), (a,e), (b,d), (c,a), (d,c), (d,d), (e,a), (e,c)\}$

$S = \{(a,b), (a,d), (b,a), (b,b), (b,e), (c,a), (c,b), (c,c), (d,b), (e,b), (e,d)\}$.

Use Warshall's algorithm to find the smallest equivalence relation containing R and S.