The LNM Institute of Information Technology
Jaipur, Rajsthan

MATH-I ■ Assignment #5

(Rolle's Theorem, Mean Value Theorem, Taylor's Theorem)

Q1. Prove that the ploynomial $f(x) = x^3 - 3x + c$ has at most one root in $[0, 1]$, no matter what c may be.

Q2. Suppose f is continuous on $[a, b]$, differentiable on (a, b), and satisfies $f^2(a) - f^2(b) = a^2 - b^2$. Then show that the equation $f'(x)f(x) = x$ has at least one root in (a, b).

Q3. Verify that $x^3 + 2x + 1$ satisfies the hypotheses of the Mean Value Theorem on $[0, 1]$. Then find all numbers that satisfy the conclusion of the Mean Value Theorem.

Q4. Using Mean Value Theorems (MVT or CMVT) show that
(a) $\log(1 + x) > \frac{x}{1+x}$, for all $x > 0$
(b) $e^x \geq 1 + x$ for $x \in \mathbb{R}$
(c) $1 - \frac{x^2}{2!} < \cos x$ for $x \neq 0$
(d) $x - \frac{x^3}{3!} < \sin x < x - \frac{x^3}{3!} + \frac{x^5}{5!}$ for $x > 0$
(e) $1 - \frac{x^2}{2!} < \cos x < 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$ for $x \neq 0$.

Q5. Find $\lim\limits_{x \to 5} (6 - x)^{\frac{1}{x-5}}$ and $\lim\limits_{x \to 0^+} \left(1 + \dfrac{1}{x}\right)^x$.

Q6. Suppose f is a three times differentiable function on $[-1, 1]$ such that $f(-1) = 0, f(1) = 1$ and $f'(0) = 0$. Using Taylor's theorem prove that $f'''(c) \geq 3$ for some $c \in (-1, 1)$.

Q7. For $x > -1$, $x \neq 0$ prove that
(a) $(1 + x)^\alpha > 1 + \alpha x$ whenever $\alpha < 0$, or $\alpha > 1$
(b) $(1 + x)^\alpha < 1 + \alpha x$ whenever $0 < \alpha < 1$.

Q8. Using Taylor's theorem, for any $k \in \mathbb{N}$ and for all $x > 0$, show that

$$x - \frac{1}{2}x^2 + \ldots + \frac{1}{2k}x^{2k} < \log(1 + x) < x - \frac{1}{2}x^2 + \ldots + \frac{1}{2k+1}x^{2k+1}.$$