Classification of 2nd order PDE and canonical forms

Let $D \subseteq \mathcal{R}^2$ be a smooth domain. Let $R, S, T : D \longrightarrow \mathcal{R}$ be smooth functions. Let $u_x = \partial u/\partial x$ and $u_y = \partial u/\partial y$ and

$$L := R\frac{\partial^2}{\partial x^2} + 2S\frac{\partial^2}{\partial x \partial y} + T\frac{\partial^2}{\partial y^2}, \tag{1}$$

where $R^2 + S^2 + T^2 \neq 0$. Consider, an equation

$$Lu + g(x, y, u, u_x, u_y) = 0, \tag{2}$$

where g is a smooth function. The quantity $\nabla = S^2 - RT$ is called the discriminant of (2).
Classification: Equation (2) is called

i. hyperbolic in D if $\nabla > 0$ in D

ii. parabolic in D if $\nabla = 0$ in D

iii. elliptic in D if $\nabla < 0$ in D.

Let us consider a coordinate transformation

$$\xi = \xi(x, y),\ \eta = \eta(x, y),$$

which is an invertible transformation locally i.e.

$$\xi_x \eta_y - \xi_y \eta_x \neq 0. \tag{3}$$

Simple calculation shows

$$u_x = u_\xi \xi_x + u_\eta \eta_x$$

$$u_y = u_\xi \xi_y + u_\eta \eta_y$$

$$u_{xx} = u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + u_\xi \xi_{xx} + u_\eta \eta_{xx}$$

$$u_{yy} = u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + u_\xi \xi_{yy} + u_\eta \eta_{yy}$$

$$u_{xy} = u_{\xi\xi}\xi_x\xi_y + (\xi_x\eta_y + \xi_y\eta_x)u_{\xi\eta} + u_{\eta\eta}\eta_x\eta_y + u_\xi \xi_{xy} + u_\eta \eta_{xy}$$

Substituting these in (2), we have

$$(R\xi_x^2 + 2S\xi_x\xi_y + T\xi_y^2)u_{\xi\xi} + (R\eta_x^2 + 2S\eta_x\eta_y + T\eta_y^2)u_{\eta\eta} +$$

$$2\left[R\xi_x\eta_x + S(\xi_x\eta_y + \xi_y\eta_x) + T\xi_y\eta_y\right]u_{\xi\eta} = G(\xi, \eta, u, u_\xi, u_\eta). \tag{4}$$

Let

$$A = R\xi_x^2 + 2S\xi_x\xi_y + T\xi_y^2$$

$$B = R\xi_x\eta_x + S(\xi_x\eta_y + \xi_y\eta_x) + T\xi_y\eta_y$$

$$C = R\eta_x^2 + 2S\eta_x\eta_y + T\eta_y^2$$

Then (4) can be written as

$$Au_{\xi\xi} + 2Bu_{\xi\eta} + Cu_{\eta\eta} = G(\xi, \eta, u, u_\xi, u_\eta). \tag{5}$$

A simple calculation leads to (Homework!)

$$B^2 - AC = (S^2 - RT)(\xi_x \eta_y - \xi_y \eta_x)^2 \tag{6}$$

Thus if we choose a smooth invertible transformation of coordinate, the sign of the discriminant does not change.

Since ξ and η are arbitrary, we choose it so that (4) takes a simpler form.

Case 1: $S^2 - RT > 0$: Let λ_1, λ_2 be distinct real roots of $R\alpha^2 + 2S\alpha + T = 0$. We choose ξ and η so that

$$\frac{\partial \xi}{\partial x} = \lambda_1 \frac{\partial \xi}{\partial y}; \quad \frac{\partial \eta}{\partial x} = \lambda_2 \frac{\partial \eta}{\partial y} \tag{7}$$

With such a choice, we note that, $A = C = 0$ and (5) reduces to

$$u_{\xi\eta} = \chi(\xi, \eta, u, u_\xi, u_\eta) \tag{8}$$

Coming back to (7), note that λ_1 and λ_2 are functions of x and y. Let $f_1(x, y) = c_1$ and $f_2(x, y) = c_2$ be solutions of

$$\frac{dy}{dx} + \lambda_1(x, y) = 0; \quad \frac{dy}{dx} + \lambda_2(x, y) = 0$$

respectively. Then we know (!)

$$\xi = f_1(x, y), \quad \eta = f_2(x, y)$$

are solution of (7). The curves ξ = constant and η = constant are called the characteristic curves of (2) and (8) is the canonical form of (2).

Case 2: $S^2 - RT = 0$. In this case the quadratic $R\alpha^2 + 2S\alpha + T = 0$ has a repeated real root, say $\lambda(x, y)$. Now $A = 0$ if we choose $\xi_x = \lambda \xi_y$. Let $f(x, y) = c_1$ be a solution of

$$\frac{dy}{dx} + \lambda(x, y) = 0$$

Then $\xi = f(x, y)$. Now choose η so that $\xi_x \eta_y - \eta_x \xi_y \neq 0$. Then by (6), $B = 0$ and (5) reduces to

$$u_{\eta\eta} = \zeta(\xi, \eta, u, u_\xi, u_\eta), \tag{9}$$

which is the canonical form of (2). ξ = constant and η =constant are called characteristic equations of (2).

Case 3: $S^2 - RT < 0$: Let λ_1, λ_2 be complex roots of $R\alpha^2 + 2S\alpha + T = 0$. As in the case 1, we again find (2) reduces to (8) but ξ and η are complex conjugate. In this case there are no real characteristic curves. Let

$$\alpha = \frac{\xi + \eta}{2}; \quad \beta = \frac{\xi - \eta}{2i}$$

Now $\alpha(x, y)$ and $\beta(x, y)$ are real and they form a smooth invertible cordinate transformation by which (2) reduces to the following cannonical form

$$u_{\alpha\alpha} + u_{\beta\beta} = \phi(\alpha, \beta, u, u_\alpha, u_\beta)$$

Examples:

1. $u_{tt} = c^2 u_{xx}, \quad c > 0$, is hyperbolic on every domain D of $\mathcal{R}^2$. This equation is called the wave equation.

2. The heat equation $u_t = \kappa u_{xx}$ is parabolic type

3. The Laplace's equation (also called potential equation) $u_{xx} + u_{yy} = 0$ is elliptic type

Example: Let us find the canonical form of

$$y^2 u_{xx} - 2xy u_{xy} + x^2 u_{yy} = \frac{y^2}{x} u_x + \frac{x^2}{y} u_y$$

Here $R = y^2, S = -xy, T = x^2$, thus $S^2 - RT = 0$ and so it is parabolic. Let us find the roots of

$$R\alpha^2 + 2S\alpha + T = 0$$

i.e.

$$y^2\alpha^2 - 2xy\alpha + x^2 = 0 \quad \text{or} \quad (y\alpha - x)^2 = 0$$

i.e. $\lambda = x/y$ is the repeated real root. Solution of

$$\frac{dy}{dx} + \frac{x}{y} = 0$$

is $x^2 + y^2 = c_1$. So $\xi = x^2 + y^2$. We choose $\eta = x^2 - y^2$ (!!) and we note that $\partial(\xi, \eta)/\partial(x, y) \neq 0$. Transforming from x, y to ξ, η, we finally get

$$u_{\eta\eta} = 0$$

which is the canonical form of the given equation.
Remark: $u_{\eta\eta} = 0 \implies u_n = f(\xi) \implies u = f(\xi)\eta + g(\xi)$, where f and g are arbitrary functions. In terms of x, y, the solution is

$$u(x, y) = f(x^2 + y^2)(x^2 - y^2) + g(x^2 + y^2)$$