Collisions in LAB Frame & Center of Mass Frame.

Here we learn how to visualize a collision that is taking place in Lab frame to a ~~syste~~ center of mass frame. Then a joint vector diagram is made where all the velocity vectors ~~of~~ as seen in lab frame as well as in COM frame are plotted. It is very useful. It is sometimes very easy to visualize the collision problem in COM frame because total momentum is always zero. In case of elastic collision the initial velocity is equal to final velocity in COM frame. So it is much simpler to calculate the velocities as well as angles of deflection in COM frame Through velocity diagrams / vector diagram of velocity vectors.

- If you find any mistake in the note please inform me.
- You are welcome to add more numerical problems to this note.

Amit Negi
(AMIT NEOGI)
30 Aug, 2016.

Collisions in Center of Mass Frame & Laboratory Frame.

Consider two particle of mass m_1 and m_2.

The Center of Mass, say R_{cm} is given by

$$\bar{R}_{cm} = \frac{m_1 \bar{r}_1 + m_2 \bar{r}_2}{m_1 + m_2} = \frac{\sum m_i \bar{r}_i}{\sum m_i} \quad \text{(for many particles)}$$

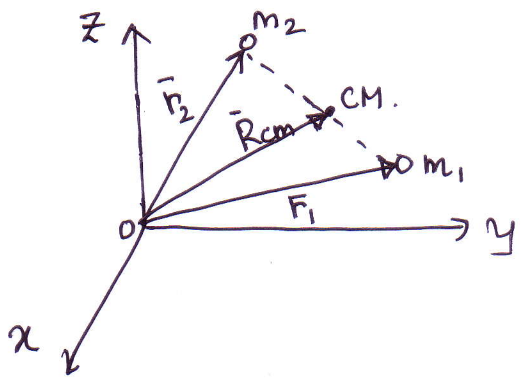
Differentiating the above wrt ~~to~~ t gives Center of mass velocity given by

$$\bar{V}_{cm} = \frac{m_1 \bar{v}_1 + m_2 \bar{v}_2}{m_1 + m_2}$$

Further differentiation with t gives Center of Mass acceleration

$$\bar{a}_{cm} = \frac{m_1 \bar{a}_1 + m_2 \bar{a}_2}{m_1 + m_2}$$

Now all These expressions are known to us. What We need to be careful That all the three quantities $\bar{R}_{cm}$, $\bar{V}_{cm}$ and $\bar{a}_{cm}$ are measured from a Laboratory frame fixed to earth.



xyz is the laboratory frame which is fixed to earth. The position vector of m_1 and m_2 are $\bar{r}_1$ & $\bar{r}_2$. The Center of mass is $\bar{R}_{cm}$.

Center of Mass Frame.

Suppose I fix a frame $x'y'z'$ with the Center of mass. Then in that frame what will be the position of Center of mass, velocity of CM, and acceleration of CM ?

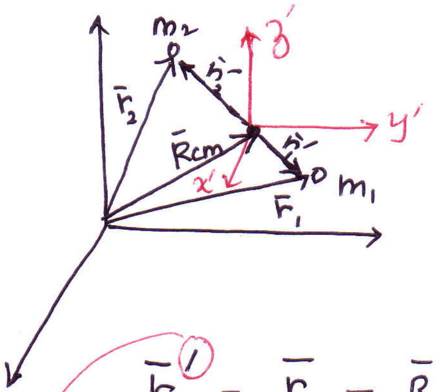
They will all be equal to 0. Because if I am sitting in the frame I will always find its velocity and acceleration to be 0. So total momentum is always 0 in this frame. That is why this frame is also known as zero momentum reference frame.

Few Important Points.

- One must be clear about Center of mass and Center of mass frame. When we talk about R_{cm}, V_{cm} or a_{cm} we are measuring these quantities from Laboratory frame of reference (xyz) fixed to the earth. But when one talks of Center of mass frame then it means that we have fixed a frame $(x'y'z')$ in the Center of mass and it is moving with velocity V_{cm} or accelerating with a_{cm}. So in short there are two frames.

Laboratory frame (xyz)	CM frame $(x'y'z')$
R_{cm} = Some fixed number	$R'_{cm} = 0$
V_{cm} = "	$V'_{cm} = 0$
a_{cm} = "	$a'_{cm} = 0$
No Primed.	Primed Coordinates

Pictorially.



$x'y'z'$ is the dash coordinate is the Center of mass frame. If net ext force $= 0$ Then it moves with const velocity.

$$\bar{r}_1' = \bar{r}_1 - \bar{R}_{cm},$$

$$\bar{r}_2' = \bar{r}_2 - \bar{R}_{cm},$$

dash means it is in CM frame $x'y'z'$

No dash, measured in Lab frame xyz.

$\bar{r}_1'$ is the position vector of the mass m_1 which is expressed in $\bar{r}_1$ and R_{cm} which are unprimed & measured in Lab frame $(x\,y\,z)$. Similarly $\bar{r}_2'$.

To avoid confusion we should be very clear about the notations. For two body collision.

Quantities		Lab frame (No dash)		CM frame. (dash)
Initial Vel. of mass 1	→	u_1	→	u_1'
" " mass 2	→	u_2	→	u_2'
Final Velocity of mass 1	→	v_1	→	v_1'
" " mass 2	→	v_2	→	v_2'

Velocity Diagrams of CM frame & Lab frame.

Consider two bodies m_1 and m_2 with initial Velocity $\bar{u}_1$ and $\bar{u}_2$ in Lab frame.

$m_1 \bullet \longrightarrow \bar{u}_1$

$m_2 \bullet \longrightarrow \bar{u}_2$

In Lab frame.

$$\Rightarrow \quad V_{cm} = \frac{m_1 \bar{u}_1 + m_2 \bar{u}_2}{m_1 + m_2}$$

Center of mass velocity in Lab frame.

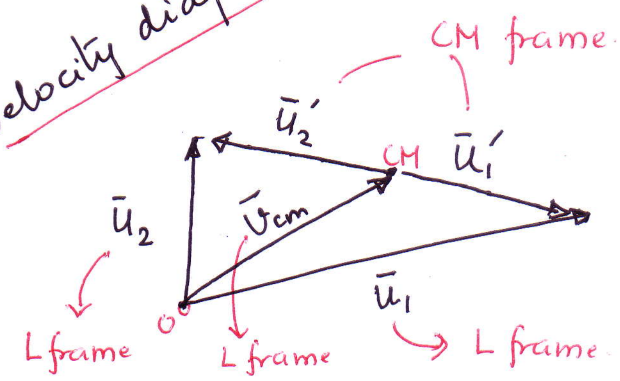
Center of Mass frame.

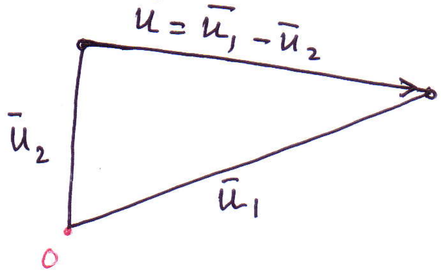
$$u_1' = u_1 - v_{cm} = \bar{u}_1 - \frac{m_1\bar{u}_1 + m_2\bar{u}_2}{m_1+m_2} = \frac{m_2(\bar{u}_1 - \bar{u}_2)}{m_1+m_2}$$

(Note the dash notation for CM frame)

$$u_2' = u_2 - v_{cm} = \bar{u}_2 - \frac{m_1\bar{u}_1 + m_2\bar{u}_2}{m_1+m_2} = \frac{-m_1(\bar{u}_1 - \bar{u}_2)}{m_1+m_2}$$

Velocity diagram





$\bar{u}_1'$ and $\bar{u}_2'$ lie back to back along the relative Velocity Vector $\bar{u} = \bar{u}_1 - \bar{u}_2$.

Momentum in Center of Mass Frame.

Momentum of the system of Particles can be Calculated in two ways.

(i) $\bar{P} = \sum M_i V_i$

(ii) $\bar{P} = M V_{cm}$.

As we have seen that V_{cm} in CM frame that is V'_{cm} is 0. Because that is how it is defined. So the total momentum is zero from second method. Let us Calculate from method (i) For a two body system.

$$\bar{P}_1' = m_1 \bar{u}_1' = \frac{m_1 m_2}{m_1 + m_2}(\bar{u}_1 - \bar{u}_2) = \mu \bar{u}$$

$$\bar{P}_2' = m_2 \bar{u}_2' = -\frac{m_1 m_2}{m_1 + m_2}(\bar{u}_1 - \bar{u}_2) = -\mu \bar{v}$$

Here $\mu = m_1 m_2 / (m_1 + m_2)$ is the reduced mass of the system.

The total momentum $\bar{P}' = \bar{P}_1' + \bar{P}_2' = 0$.

So

- $\boxed{\text{Total Momentum in CM frame is always } 0.}$
- Total Momentum in Lab frame is

$$m_1 \bar{u}_1 + m_2 \bar{u}_2 = (m_1 + m_2) V_{cm}$$

Another way to prove that total momentum in a CM is 0.

$$\bar{P}' = \sum m_i u_i' = \sum m_i (u_i - v_{cm}) = \sum m_i u_i - \sum m_i v_{cm}$$

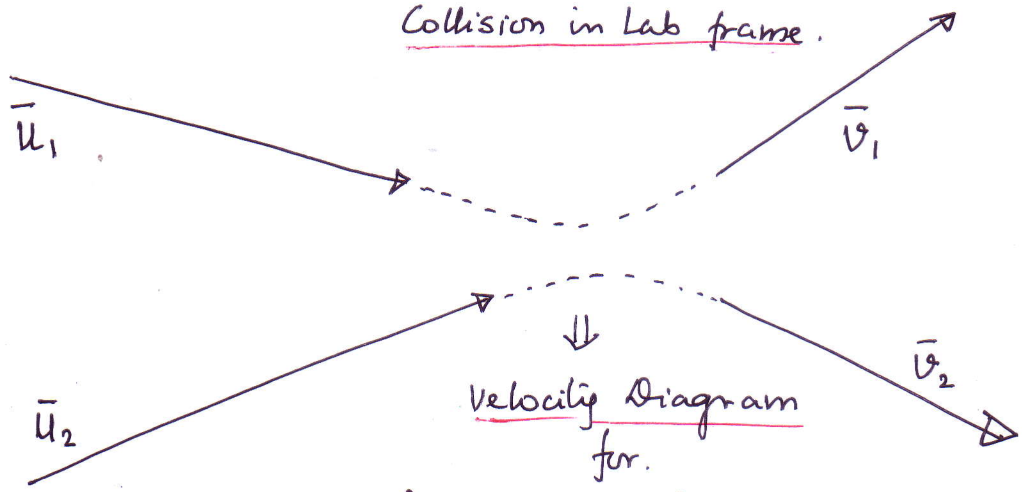
$$= M v_{cm} - v_{cm} \sum m_i = \cancel{Mcn}\, M v_{cm} - M v_{cm} = 0.$$

Please note how u_i' is the velocity in CM frame. Then it is converted to Lab frame and total momentum is Calculated.

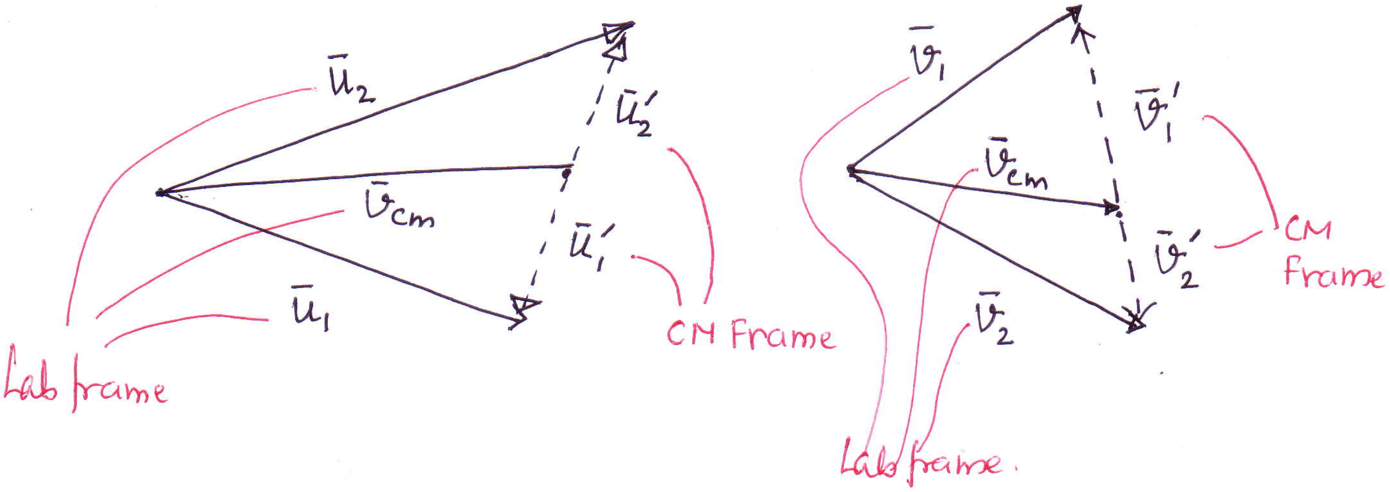
- **Velocity Diagram in Case of a two dimensional Collision.**

Velocity diagrams gives a vectorial representation of initial & final velocities in lab & Centre of Mass frame.

Collision in Lab frame.



Velocity Diagram for. Initial & final Velocity in LAB & CM Frame



Special Cases.

(a) If the collision is elastic

Applying Energy conservation in CM frame

$$\frac{1}{2} m_1 u_1'^2 + \frac{1}{2} m_2 u_2'^2 = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2$$

Since momentum $= 0$ in CM frame

$$m_1 u_1' - m_2 u_2' = 0$$

$$m_1 v_1' - m_2 v_2' = 0$$

Eliminating u_2' and v_2'

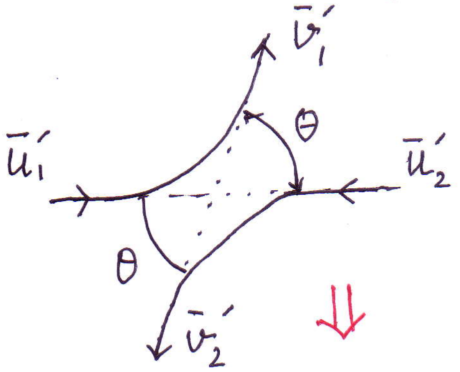
$$\frac{1}{2}\left(m_1 + \frac{m_1^2}{m_2}\right) u_1'^2 = \frac{1}{2}\left(m_1 + \frac{m_1^2}{m_2}\right) v_1'^2$$

$\Rightarrow$ $\boxed{u_1' = v_1'}$

Similarly $\boxed{u_2' = v_2'}$

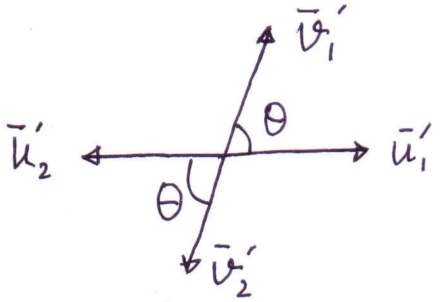
In Center of Mass Frame Speed remains Const in elastic Collision but direction gets changed

In an elastic collision the speed of each particle in the CM frame is same before and after the collision. Velocity Vector simply rotate in the scattering plane. The Schematic diagram & the corresponding vector diagram in the CM frame is given below.



$\Rightarrow$ scattering of two particles in CM frame. pictorial diagram.

$\Downarrow$



$\Rightarrow$ The Velocity diagram along with the scattering angle.

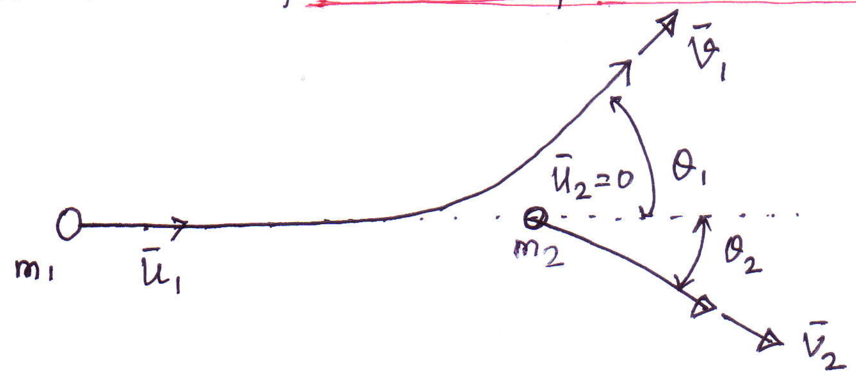
(b) One of the particle say m_2 is at rest initially

$m_1 \xrightarrow{u_1}$ $\quad\quad$ $\circ\, m_2$

Now most of the problems that you get in this chapter is a combination of part (a) & (b) that is mass m_2 is at rest and the collision is elastic.

In the problems you will be asked to relate the quantities of Lab frame with the Centre of Mass frame.

Let us write the pictorial representation in Lab frame.



Now.

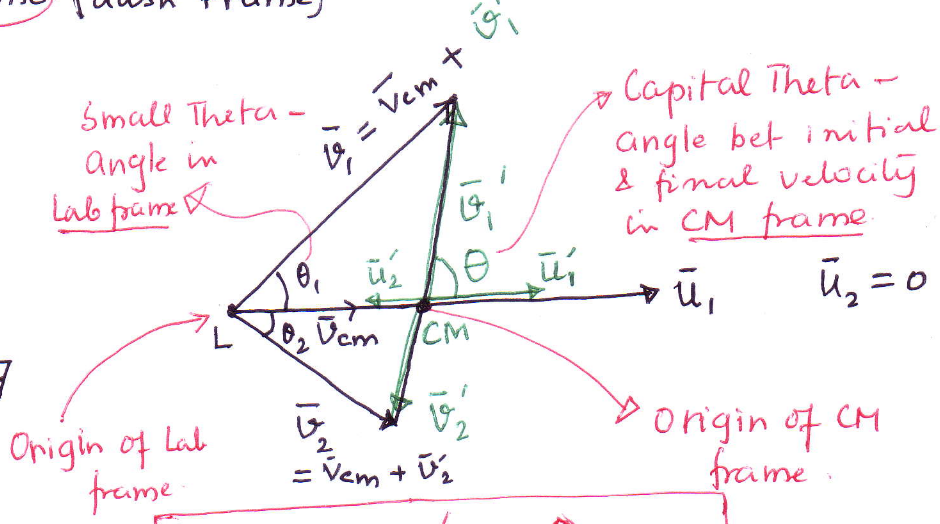
$\boxed{\text{Draw Velocity Diagram}}$ V. Imp.

HOW

We need to represent all these quantities in velocity diagram

Lab frame	CM Frame (dash frame)
$\bar{u}_1$	$\bar{u}_1'$
$\bar{u}_2$	$\bar{u}_2'$
$\bar{v}_1$	$\bar{v}_1'$
$\bar{v}_2$	$\bar{v}_2'$
$\bar{v}_{cm}$	

$\boxed{\text{How ?}}$



$$\boxed{\tan\theta_1 = \frac{v_1' \sin\Theta}{v_{cm} + v_1'\cos\Theta}}$$ V. Imp exp

Steps.

1) Draw the Origin of Lab frame, a point.
2) Draw $\bar{u}_1, \bar{u}_2, \bar{v}_1$, $\bar{v}_2$, v_{cm}.
3) Where $\bar{v}_{cm}$ ends it is origin of CM frame.
(4) Draw u_1', u_2', v_1', v_2' from CM origin. (Green color above diagram)

Continued from previous page.

Some Mathematical expressions.

$$v_{cm} = \frac{m_1 \bar{u}_1}{m_1 + m_2} \quad -(1)$$

$$\bar{u}_1' = \bar{u}_1 - \bar{v}_{cm} = \frac{m_2}{m_1 + m_2} \bar{u}_1 \quad -(2)$$

$$\bar{u}_2' = \bar{u}_2 - \bar{v}_{cm} = -\bar{v}_{cm} = -\frac{m_1}{m_1 + m_2} \bar{u}_1 \quad -(3)$$

Now. From (1) & (2)

$$\left(\frac{V_{cm}}{u_1'}\right) = \frac{m_1}{m_2} \quad ; \quad \frac{u_1'}{u_2'} = -\frac{m_2}{m_1}$$

$\Downarrow$

Above expressions are useful.

Now. Writing the last expression of previous page

$$\boxed{\tan\theta_1 = \frac{v_1' \sin\theta}{v_{cm} + v_1' \cos\theta}}$$

Since the scattering is elastic $v_1' = u_1'$

Large Theta in CM frame.

Hence
$$\tan\theta_1 = \frac{u_1' \sin\theta}{v_{cm} + u_1' \cos\theta} = \frac{\sin\theta}{\left(\frac{v_{cm}}{u_1'}\right) + \cos\theta}$$

Small Theta in Lab frame.

But $\frac{v_{cm}}{u_1'} = \frac{m_1}{m_2}$ as derived at the top., So

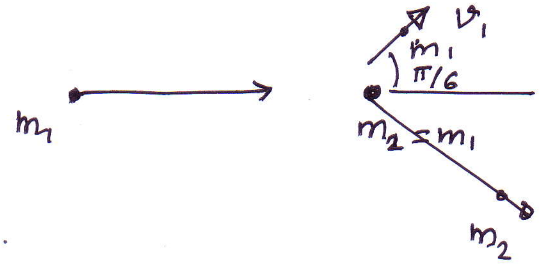
$$\tan\theta_1 = \frac{\sin\theta}{\left(\frac{m_1}{m_2}\right) + \cos\theta}$$

This is an important expression. But if you get a problem in exam you can't write this formula directly but have to derive it.

Depending on whether $m_1 > m_2$, $m_1 = m_2$, $m_1 < m_2$ The vector $\bar{v}_1'$ will rotate & θ will change. The above expression relates the angle between Lab frame & CM frame.

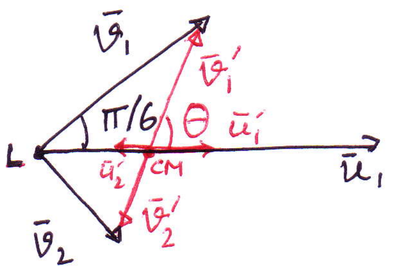
#1 A particle collides elastically with another identical particle at rest. If the first particle deflects through an angle $\pi/6$ as seen from laboratory, what is the angle of deflection in the CM frame.?

In Lab frame



Velocity Diagram.

Quantities in Lab frame: $u_1, u_2, v_1, v_2, v_{cm}$. (Black color)
CM frame: $\bar{u}_1', \bar{u}_2', \bar{v}_1', \bar{v}_2'$ (Red color).



$$\tan \pi/6 = \frac{v_1' \sin\theta}{V_{cm} + v_1' \cos\theta} \quad — (1)$$

Elastic Collision $\bar{v}_1' = \bar{u}_1'$

So
$$\tan \pi/6 = \frac{u_1' \sin\theta}{V_{cm} + u_1' \cos\theta} = \frac{\sin\theta}{\left(\frac{V_{cm}}{u_1'}\right) + \cos\theta} \quad — (2)$$

Now $\bar{V}_{cm} = \frac{m_1 u_1}{m_1 + m_2}$; $\bar{u}_1' = \frac{m_2}{m_1 + m_2} \bar{u}_1$ $\left[\bar{u}_1' = \bar{u}_1 - \bar{v}_{cm}\right]$

$$\Rightarrow \frac{\bar{V}_{cm}}{\bar{u}_1'} = \frac{m_1}{m_2} \quad — (3).$$

Identical particles $\frac{m_1}{m_2} = 1.$

Substitute (3) in (2)
$$\tan \pi/6 = \frac{\sin\theta}{1 + \cos\theta} = \frac{2 \sin\theta/2 \cos\theta/2}{1 + 2\cos^2\frac{\theta}{2} - 1}$$

$$\Rightarrow \tan \pi/6 = \tan\frac{\theta}{2}$$

$$\Rightarrow \frac{\theta}{2} = \frac{\pi}{6} \Rightarrow \theta = \frac{\pi}{3}$$

Angle between initial & final velocity of mass 1 is 60°.

#2 A proton with a Speed of 500 m/s collides elastically with another proton at rest. The Original proton is Scattered Through 60° from its initial direction.

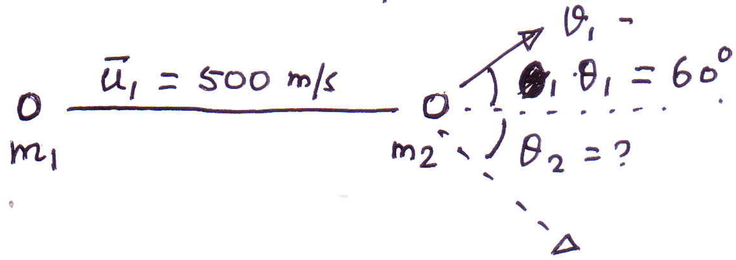
(a) Find the scattering angle of this proton in the CM frame.

(b) What is the direction of the Velocity of the target proton in the lab frame after the collision?

(c) What are the speeds of the two protons in the lab frame after collision.

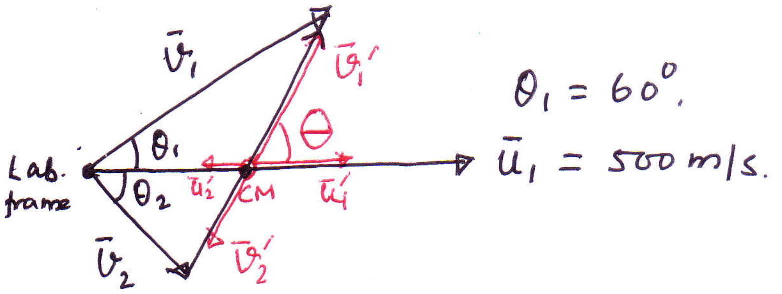
⇒

Step 1 : First draw the picture in lab frame



$\boxed{m_1 = m_2}$ $= m$.

Step 2 : Let us draw a rough diagram of Velocity Vectors.



$\theta_1 = 60°$.

$\bar{u}_1 = 500$ m/s.

$\theta_1 =$ angle bet $\bar{u}_1$ & $\bar{v}_1$ in Lab frame.

$\Theta =$ angle bet $\bar{u}_1'$ & $\bar{v}_1'$ in Center of Mass frame

Now earlier we have derived: $\tan\theta_1 = \dfrac{\sin\Theta}{\frac{m_1}{m_2} + \cos\Theta}$

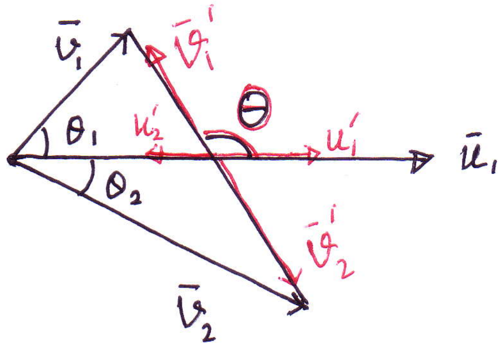
Since $m_1 = m_2$

$\tan\theta_1 = \dfrac{2\sin\Theta/2\cos\Theta/2}{1 + 2\cos^2\frac{\Theta}{2} - 1}$ $\Rightarrow$ $\tan\theta_1 = \tan\Theta/2$

$\Rightarrow$ $\theta_1 = \Theta/2$

$\Theta = 60 \times 2 = 120°$.

Step 3. Since $\theta = 120°$, our diagram was not properly drawn. Lets draw it properly.



$\theta = 120°$

$\bar{u}_1 = 500\,m/s$.

(b) $\theta_2 = ?$ θ_2 is asked now.

u_1 is given.

$$\bar{u}_1' = \bar{u}_1 - \bar{V}_{cm}$$

In Scalar form. $u_1' = u_1 - V_{cm}$.

$$= u_1 - \frac{m_1 u_1}{m_1+m_2} = u_1 - \frac{u_1}{2} = \frac{u_1}{2}$$

(Because $m_1 = m_2 = m$) (1)

So

$$u_1' = \frac{u_1}{2} \quad -(2)$$

Since collision is elastic

$$u_1' = v_1' = \frac{u_1}{2} \quad \text{from (2).} \quad -(3)$$

Total Momentum in CM frame is always 0.

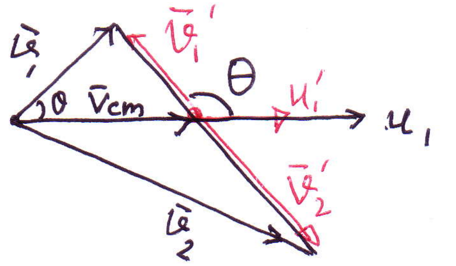
$$m_1 v_1' = m_2 v_2' \qquad [m_1 = m_2]$$

$$v_1' = v_2' = \frac{u_1}{2} \quad \text{from (3).}$$

Now $$\tan\theta_2 = \frac{v_2' \sin(\pi - \theta)}{V_{cm} + v_2' \cos(\pi-\theta)} = \frac{u_1/2 \; \sin 60}{\frac{u_1}{2} + \frac{u_1}{2}\cos 60}$$

$$\tan\theta_2 = \tan 30 \Rightarrow \theta_2 = 30°$$

(c) $|\bar{v}_1|$ and $|\bar{v}_2|$ is asked. Refer to the figure.



$\theta = 120°$.

$\bar{v}_2' = \frac{u_1}{2}$

$\bar{v}_1' = \frac{u_1}{2}$

$\bar{v}_{cm} = \frac{u_1}{2}$

Cosine Law.

$$v_2^2 = v_{cm}^2 + v_2'^2 - 2 v_{cm} v_2' \cos 120.$$

$$v_2^2 = \left(\frac{u_1}{2}\right)^2 + \left(\frac{u_1}{2}\right)^2 - 2\left(\frac{u_1}{2}\right)\left(\frac{u_1}{2}\right)\left(-\frac{1}{2}\right)$$

$$= 2\frac{u_1^2}{4} + \frac{u_1^2}{4} = \frac{3}{4}u_1^2$$

$$v_2 = \frac{\sqrt{3}}{2} u_1$$

$$v_1^2 = v_{cm}^2 + v_1'^2 - 2 v_{cm} v_1' \cos 60$$

$$= \left(\frac{u_1}{2}\right)^2 + \left(\frac{u_1}{2}\right)^2 - 2\left(\frac{u_1}{2}\right)\left(\frac{u_1}{2}\right)\frac{1}{2}$$

$$= \frac{u_1^2}{4}$$

$$v_1 = \frac{u_1}{2}.$$

(Please verify the answers yourself.)