Rotational Dynamics - 2.

This chapter deals with mostly ω, L, τ and their applications. Most of this material you have studied before. This is just a revision. I have consulted many books and videos for making this note — Kleppner, Resnick Halliday Walker, Young & Freedman, D C. Pandey, wikipedia, video of Walter Lewin of MIT and Kalyan Dutt. MR. Kalyan Dutt has uploaded quite a good number of videos on Youtube on this topic which are really good & worth seeing. There are many ways to contribute to the society. And this is one way to do that. I want each of you to contribute something worth in the internet knowledge in these four years of LNMIIT

I wrote in a hurry at many places. Please let me know if you find any mistake anywhere. This lesson has become long - 43 pages but it won't take much time for you to go through because you have read all these before.

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(AMIT NEGI)
11 Sept. 2016.

Rotational Dynamics - 2.

Theory.

1) ω — Particle / Rigid body } axis

2) L — Particle / Rigid body } Point / Axis

3) τ — Particle / Rigid body } Point / axis

Applications

- Translation.
- Fixed Axis Rotation
  ↓
  Small oscillations
- Plane Motion —
  Translation +
  Fixed Axis Rotation.
  Theory / Examples.
- Rigid body Dynamics.
  ↓
  Axis is changing direction
  Top / Gyroscope.

(In the next Write up)

Kinematics.

Here we don't talk about Force or torque. But we derive equations based on Constant acceleration or Constant angular acceleration.

Linear Kinematics.	Rotational Kinematics.
$a = \text{Const}$	$\alpha = \frac{d\omega}{dt} = \ddot{\theta} = \text{Const}.$
$\bar{v} = \bar{u} + \bar{a}t$	$\omega = \omega_0 + \alpha t$
$\bar{s} = \bar{s}_0 + \bar{u}t + \frac{1}{2}\bar{a}t^2$	$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$
$v^2 = u^2 + 2\bar{a}\cdot\bar{s}$	$\omega^2 = \omega_0^2 + 2\alpha\theta$
$\bar{s} = \left(\frac{\bar{u}+\bar{v}}{2}\right) t$	$\theta = \left(\frac{\omega+\omega_0}{2}\right) t.$
$\Downarrow$	$\Downarrow$
Vector Equations	Scalar Equations.

Here ω is angular speed That is rate at which angular displacement is changing. (Please note we dont call ω as angular velocity. but angular speed here — a scalar quantity) All the equations of Linear kinematics are vector ~~quantities~~ equations. Where as the equations on the right side are all scalar equations. We treat θ as Scalar — later we will see why. Later we will see where ω will be Vector.

Now books like Resnick Halliday krane have written the Scalar equations little differently. That is

$$\phi = \phi_0 + \omega_{0z} t + \frac{1}{2} \alpha_z t^2$$

They have taken ω as a vector – angular Velocity and written it as ω_{0z} that is z component of ω. When we write one component of a vector it is a scalar. When it is a particle the ω if taken as a Vector, its direction will be always along z direction.

⇒ I think it is better for a particle – When we write the Rotational kinematics equations we treat ω as ω_0 ~~z~~ & treat ω as angular Speed rather than treating it as a vector & writing its z component. Your comments are welcome.

Rotational Quantities as Vectors

Vectorial nature of linear displacement, Velocity & acceleration comes naturally but in angular displacement things are different & Vectorial nature does not come naturally.

Now a Vector must obey law of vector addition.

$$\bar{A} + \bar{B} = \bar{B} + A.$$

θ = angular displacement in case of rigid body does not follow this rule.

$$\boxed{\Delta \phi_1 + \Delta \phi_2 \neq \Delta \phi_2 + \Delta \phi_1}$$

So finite angular displacement cannot be represented as vectors

4

Larger angle Rotation

$\theta_1 + \theta_2 \neq \theta_2 + \theta_1$

Smaller angle Rotation

$\theta_1 + \theta_2 = \theta_2 + \theta_1$

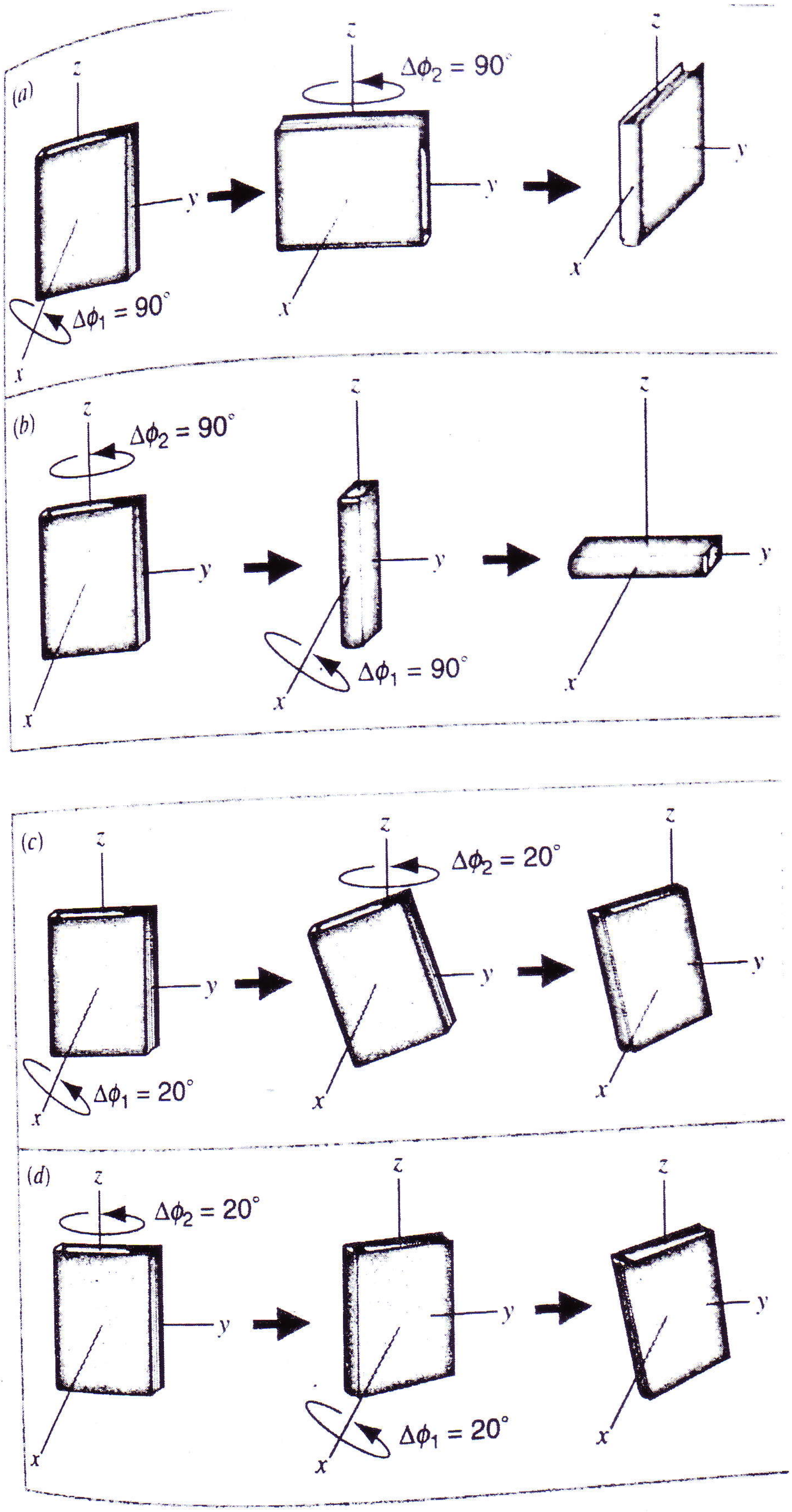


FIGURE 8-5. (*a*) The book is given two successive rotations: $\Delta\phi_1 = 90°$ about the x axis and $\Delta\phi_2 = 90°$ about the z axis. (*b*) If the order of the rotations is reversed, the final position of the book is different. (*c*) Now the book is rotated as in (*a*) but by two smaller angles: $\Delta\phi_1 = 20°$ about the x axis and $\Delta\phi_2 = 20°$ about the z axis. (*d*) If the order of the rotations in (*c*) is reversed, the final position more closely resembles that of (*c*).

This is what we observe in the first figure. Where as in second figure you will observe that commutative addition law holds good if the displacement is small.

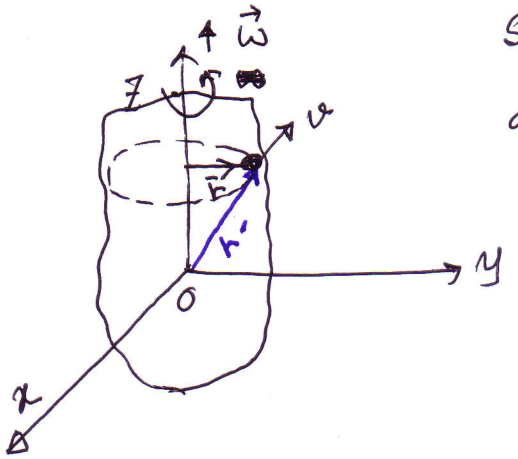
$$\boxed{d\phi_1 + d\phi_2 = d\phi_2 + d\phi_1}$$

Hence infinitesimal angular rotations can be represented as vectors.

Now $\vec{d\phi}$ is a vector dt is a scalar. So

$\bar{\omega} = \frac{\vec{d\phi}}{dt}$ is a vector

Angular velocity can therefore represented as vector.
Now what will be the direction. The direction does not come naturally here like linear velocity.
We have to assign a direction such that it is in accordance with the laws of Physics.
It is given by right-hand rule. If the fingers of the right hand curl around the axis in the direction of rotation of the body, the extended thumb points along the direction of angular velocity vector.



Suppose a rigid body is rotating about any axis OZ then

$$\boxed{\bar{v} = \bar{\omega} \times \bar{r}}$$

$\bar{\omega}$ is along z axis, then $\bar{\omega} \times \bar{r}$ is in the direction of v, so things are consistent.

Now it is interesting to note that

$$\boxed{\bar{v} = \bar{\omega} \times \bar{r} = \bar{\omega} \times \bar{r}'}$$

Both gives the correct direction & magnitude of velocity

$\bar{r}$ and $\bar{r}'$ – Both gives correct magnitude & direction of $\bar{v}$.

What does it mean?

Any point on the axis will give the correct answer.

$$\bar{v} = \bar{\omega} \times \bar{r}'$$

$$\frac{d\bar{v}}{dt} = \bar{a} = \frac{d\omega}{dt} \times \bar{r}' + \bar{\omega} \times \frac{d\bar{r}'}{dt}$$

- - - - - - - -

Finally $= \bar{\alpha} r - \omega^2 \bar{r}$

tangential Component

Radial Component.

So every thing is consistent, so above equation is correct.

Angular Momentum.

Why study of Angular Momentum is important -

We know at many places ($\tau_{ext} = 0$) Angular mom. is conserved. Any quantity that is conserved in Physics is quite important.

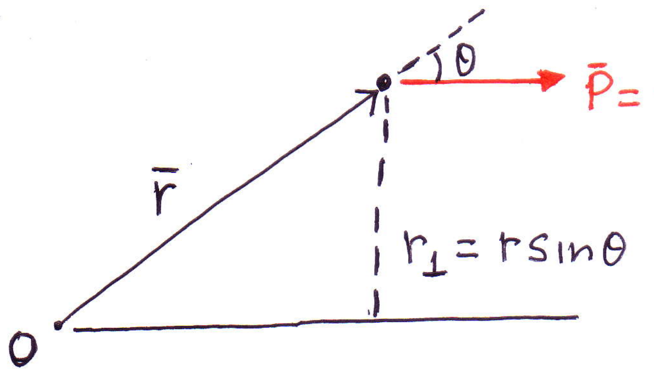
Angular Momentum can be defined

(a) about a point

(b) about an axis.

Generally angular momentum of a particle / body is calculated about a point. If we take a component of this vector about an axis, that is known as about an axis.

Ang. Momentum of a Particle about a fixed pt.



$$\boxed{\bar{L}_0 = \bar{r}_0 \times \bar{P}}$$

about pt O.

It is a Common Practice.

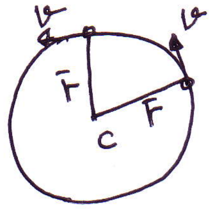
Let a particle of mass m moving with linear momentum $\bar{P} = m\bar{v}$. Its angular momentum about O is

$$L_0 = \bar{r}_0 \times \bar{P} = \bar{r}_0 \times (m\bar{v}) = m(\bar{r}_0 \times \bar{v}) = mvr_0 \sin\theta$$
$$= mvr_\perp$$

Subscript O is written to stress this point that angular

momentum is Calculated about a fixed pt.
It is not the Case of linear momentum. So.
Angular Momentum is not an intrinsic property of a moving particle because it will change from pt to pt.

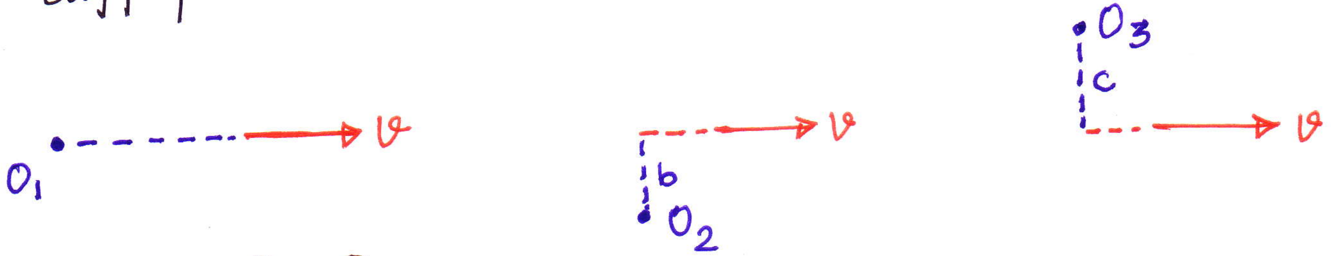
But linear momentum of a moving particle is its intrinsic property given by $\bar{p} = m\bar{v}$.



- If a particle rotates in a circle then radius vector is always perpendicular to the tangential Velocity So $L = mvR$.

Though velocity is changing at every point still L is Constant. The torque is always zero here because $\bar{r}$ and Centripetal force responsible for Circular motion are always along the same direction.

- If a particle is moving with constant vel (Speed & direction) Then angular momentum about any point always remains Const but different about diff points.



$L_{O1} = 0$ $\quad r_{\perp} = 0$

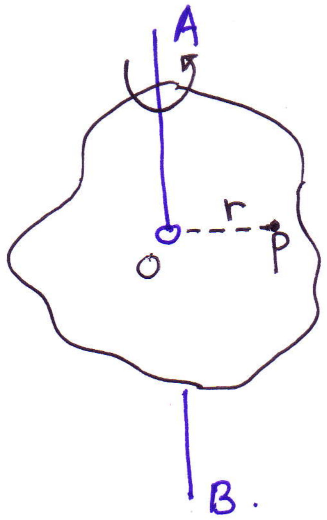
$L_{O2} = mvb$ $\otimes$

Inward

$L_{O3} = mvc$ $\odot$

upward.

Angular Momentum of a rigid body Rotating about a fixed axis.



Suppose a rigid body made up of large number of particls rotating about an axis of rotation AB. Here O can be chosen at any point of the axis. The Component of $\bar{r} \times \bar{p}$ along the axis is mvr. The angular momentum of all the particles about pt O

$$L = \Sigma m_i r_i v_i \qquad v_i = r_i \omega$$

$$L = \Sigma m_i r_i^2 \omega = \omega \Sigma m_i r_i^2 \Rightarrow \boxed{L = I\omega.}$$

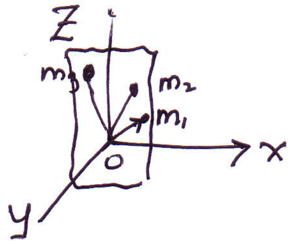
(not a Vector eqn)

Here I is the moment of Inertia of the body about AB. Here L is upward or along the axis of rotation from B to A.

- Here $\bar{L} = I\bar{\omega}$ is not correct because $\bar{L}$ and $\bar{\omega}$ may not point in the same direction but we could write $L_{AB} = I\omega$. If the body is symmetric about the axis of rotation $\bar{L}$ and $\bar{\omega}$ are $\parallel$ & we can write $\bar{L} = I\bar{\omega}$

<u>Confusion</u>: Now the above proof of calculating L for a rigid body is very confusing. The r_i is defined from a fixed origin. How that r_i becoms distance from axis & become I.

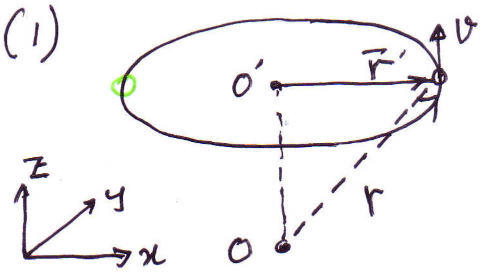
I hope you are clear about the confusion. The confusion is if I have to calculate the angular momentum of a solid body (ie $\bar{r} \times m\bar{v}$), the r is the distance from a common origin and not from the axis of rotation.



To solve this confusion / derivation – Resnick Halliday, Young & Freedman & other books have explained. But I found the best explanation by a video in Youtube by Kalyan Dutt – Lecture 66, Rotational dynamics. I suggest you to see that video. – if you want to be clear about this confusion.

Let me summarize:

(1)



Suppose a particle rotates in a circle in XY Plane. as shown. Its angular momentum about O'

$$\bar{L} = mvr' = m(r'\omega)r' = \underbrace{mr'^2}_{I}\omega$$
$$= I\omega\hat{k}$$

(2) Now if you calculate its angular momentum about any other point O (say origin) then its angular momentum will not be equal to $I\omega$ but its Z component will be equal to $I\omega$.

(3) Now you take another particle diametrically opposite to the first particle, & calculate the $\bar{L}$ for the system about both these points.

$\bar{L}_{O'} = I\omega\hat{k}$ (Here I is I of the system of two particles)

$L_O = I\omega\hat{k}$.

$\Downarrow$

When it is Calculated about any other point O, then $\bar{L}$ will have Component along $\bar{r}$ and along $\hat{k}$. The r component of $\bar{L}$ will Cancel each other but $\hat{k}$ Component will be added up. Now it will be $I\omega$ for both these points.

(4) From the above result we conclude something about Symmetric objects rotating about a symmetric axis. $\bar{L}$ about any point in the axis is $I\omega\hat{k}$. So this is how we arrive at angular momentum of a solid object which is symmetric rotating ~~as~~ a about a symmetric axis is $I\omega$. So the earlier proof that we saw was too simplistic.

Some important Points to Remember.

(a) Symmetric objects Rotating about an axis of Symmetry - Example is a disc that is rotating about a symmetric axis passing through its Center of mass.

$\bar{L} \parallel \bar{\omega}$ $\Rightarrow$ L is parallel to ω about any point in axis.

$\boxed{\bar{L}_{axis} = I_{axis}\,\bar{\omega}}$ $\Rightarrow$ True for any point in the axis.

(b) Symmetric objects rotating about an axis that is not an axis of symmetry - Example: A disc which is symmetric but rotating about an axis passing through any point between Center and circumference.

$\bar{L}$ is not $\parallel$ to $\bar{\omega}$ about any point in the axis.

The Component of the angular momentum about the axis of rotation is (This means that z comp of angular momentum = $I\omega$)

$\boxed{L_{axis} = I_{axis}\,\omega.}$ $\qquad L_z = I\omega$

$\Downarrow$

No Vector.

Kleppner Notation.

Notation L_{axis} $\Rightarrow$ It means that any point in the axis will follow that.

Conservation of Angular Momentum.

$\bar{L} = \bar{r} \times \bar{p}$ (Angular Momentum of particle)

$$\frac{dL}{dt} = \bar{r} \times \frac{dp}{dt} + \frac{d\bar{r}}{dt} \times \bar{p} = \underbrace{\bar{r} \times \bar{F}}_{\tau} + \underbrace{\bar{v} \times \bar{p}}_{0}$$

So $\frac{dL}{dt} = \bar{r} \times \bar{F} = \bar{\tau}$ So $\boxed{\bar{\tau} = \frac{d\bar{L}}{dt}}$.

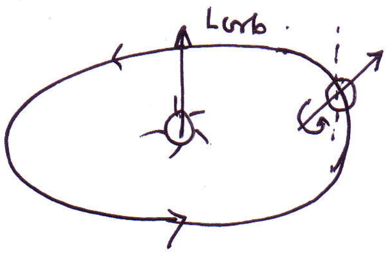
Both L & τ should be calculated about the same point.

If $\bar{\tau}_{ext} = 0 \Rightarrow \frac{d\bar{L}}{dt} = 0$ $L = \text{Const}$

$I\omega = \text{Const.}$

Or $\boxed{I_1 \omega_1 = I_2 \omega_2}$

SPIN & Orbital Angular Momentum.



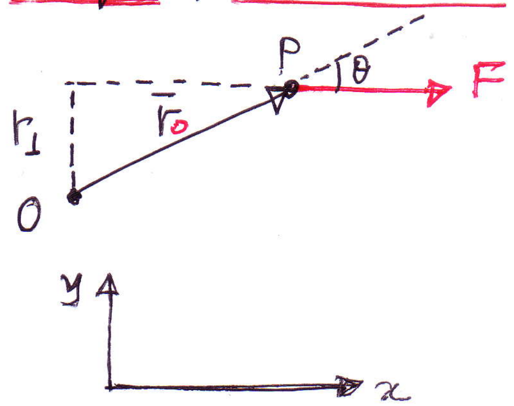
The earth rotats about its axis known as spin angular momentum. The earth also also rotats in an orbit around the Sun which is known as orbital angular momentum (mvR).. They are not parallel to each other.

We will refer to these terms later again

Torque

A force causes acceleration in translational motion. But if a body is free to rotate around a particular axis, force causes angular acceleration: But here it is also important to know where the force is applied & depending on that angular acceleration changes.

Torque for a Particle.



A force F is acting on a particle P. The position vector of this particle from a reference point is $\bar{r}$. The torque about O.

$$\bar{\tau}_0 = \bar{r}_0 \times \bar{F}$$

V. Imp notation.

It is very very important to remember that torque like angular momentum depends on the Origin we choose but force does not. Since Origin is V. Imp so to stress this point a certain custom is followed by many Physicists as well as Kleppner & that is to Write that point as Subscript. In the above we have Written in red.

$|\tau_0| = r_0 F \sin\theta$ ⇒ Direction is defined by Right hand rule.

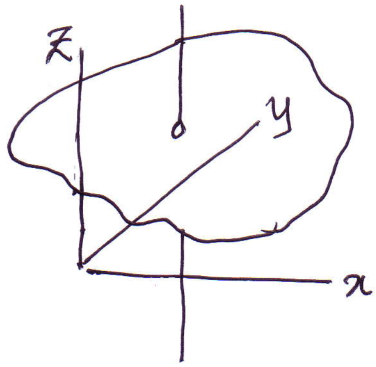
$|\tau_0| = |r_\perp| \, |F|$ $r_\perp$ = often known as Moment arm

$|\tau_0| = |r| \, |F_\perp|$ The direction along which Force acts is known as line of action

$$\bar{\tau} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}$$

Torque For a Rigid body.

Please see some standard book like Resnick Halliday or Young & Freedman for the proof of this.



Suppose a body rotats ~~for~~ by the application of a force at certain pt. The force gets distributed to the whole body — to every particle. Then every particle experience a torque. When we add up all such Torque we get an important expression.

Say for a particle which is rotating along with the rigid body about an axis.

$$F_1 = m_1 a_1$$

Now this is tangential comp of force which gives tangential acceleration.

$$F_{1,tan} = m_1 a_{1,tan}. \qquad \Rightarrow \quad a_{tan} = r\alpha$$

angular acc

For Simplicity lets remove tan. $\alpha = \alpha_z$ if the ~~body~~ particle rotats about z axis.

$$F_1 = m_1 r_1 \alpha$$

Multiply by r_1 on both sides.

$$F_1 r_1 = m_1 r_1^2 \alpha$$

$$\tau_1 = I_1 \alpha \quad \Rightarrow \text{ Adding for all particles.}$$

$$\Sigma \tau_i = \left(\Sigma m_i r_i^2\right)\alpha \quad \text{we get} \quad \boxed{\Sigma \tau = I\alpha}$$

I is moment of Inertia about rotating axis.

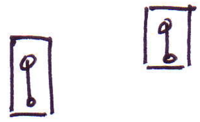
In rigid body we talk more in terms of axis and Concept of moment of Inertia I Comes into picture.

If you see the right handed Coordinate system, if the body is rotating about z axis then Counter clockwise torque will give rise to α in the direction of $+z$ axis. So

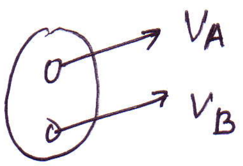
- Counter clockwise torque is taken as +ve.
- Clockwise torque is taken as −ve.

Applications.

1) <u>Translation</u>. : line joining the particles does not change direction.



When an object is undergoing a translatory motion all particles have same velocity at any time.



$$\bar{V}_A = \bar{V}_B$$

$$\bar{a}_A = \bar{a}_B .$$

2) <u>Fixed Axis Rotation</u>.

All particles are moving in a circle about a fixed axis. All particles have same angular velocity at any time t.

3) <u>Plane Motion</u> –

It is a Combination of Translation + Fixed axis Rotation. – Ex – A cylinder rolling down an incline plane or along a plane surface.

Here the axis of rotation moves along a straight line but direction is Constant.

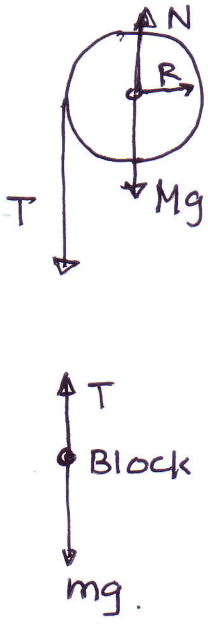
4) In <u>Rigid body Motion</u> – <u>General Motion</u> like <u>Top</u>, <u>Gyroscope</u> the ~~ax~~ direction of axis changes.

Simple Problems on Fixed Axis Rotation.

#1. A Uniform disk of mass $M = 2.5$ kg and radius $R = 20$ cm, mounted on a fixed (frictionless) horizontal axle. A block of mass $m = 1.2$ kg hangs from a light cord that is wrapped around the rim of the disk. Find the acceleration of the falling block, the tension in the cord and the angular acceleration of the disk.

This is one of the simplest problem on fixed axis Rotation but teaches us the fundamentals.

$\Rightarrow$



Rotational Motion.

$$\boxed{\Sigma \tau = I\alpha}$$

$$TR = I\alpha$$

$$I = MR^2/2$$

No slip Condition

Combining linear acc with angular acc. $\Rightarrow$ $a_y = R\alpha$

Translational Motion.

$$\boxed{\Sigma F_y = m a_y}$$

$$mg - T = ma_y$$

Solving $a = g \dfrac{2m}{M+2m} = 4.8 \text{ m/s}^2$

$$T = mg \frac{M}{M+2m} = 6.0 \text{ N}.$$

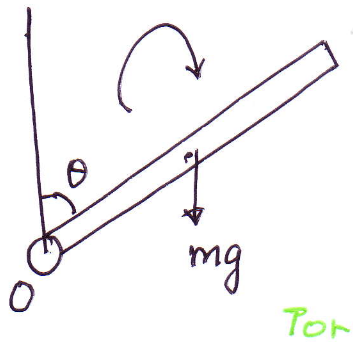
[Sample Problem: 9-10 from Resnick Halliday Krane]

#2 A uniform rod of length L and mass M is pivoted freely at one end

(a) What is the angular acceleration of the rod when it is at angle θ to the vertical ?

(b) What is the tangential linear acceleration of the free end when the rod is horizontal. ?

The moment of inertia of a rod about one end is $\frac{1}{3}ML^2$.



Torque about O.

Only Rotational Motion.

$$\sum \tau = I\alpha$$

$$\frac{mgL}{2}\sin\theta = \frac{ML^2}{3}\alpha$$

$$\alpha = \frac{3g\sin\theta}{2L}$$

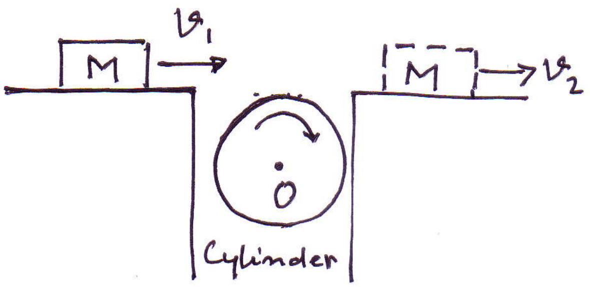
When $\theta = \pi/2 \rightarrow \alpha = 3g/2L$

The tangential linear acceleration of the free end is

$$a_t = \alpha L = \frac{3g}{2}$$

#3 The cylinder shown in fig has a fixed axis and is initially at rest. The block of mass M is initially moving to the right without friction with speed v_1. It passes over the cylinder to the dotted position. When it first makes contact with cylinder, it slips on the cylinder but the friction is large enough so that slipping cease before M loses contact with cylinder.

Calculate the final speed v_2 in terms of v_1, M, I and Radius.



How to do this problem?

We will face similar problem later.

This kind of problem is collision type of problem where small torque is created because of internal forces — causing internal torques which cancel each other.

We can then apply conservation of angular momentum similar to conservation of linear momentum problem in case of collisions.

Lets take angular momentum about O.

$$M v_1 R = M v_2 R + I\omega$$

(Lets assume ω be angular velocity of cylinder)

$$v_2 = R\omega \Rightarrow \omega = \frac{v_2}{R}$$

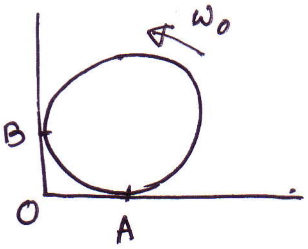
$$v_2 = \frac{M v_1 R}{MR + \frac{I}{R}} = \frac{v_1}{1 + \frac{I}{MR^2}}$$

[From Resnick Halliday]

#4 A uniform Cylinder of radius R is spinned about its axis to the angular speed ω_0 and Then placed into a Corner as in fig. The coeff of friction between Corner walls & cylinder is μ_k. How many turns will the cylinder accomplish before it stops.?

<u>Step 1</u>: What is that asked?

turns $\Rightarrow$ total angle θ $\Rightarrow \frac{\theta}{2\pi}$ = turns.

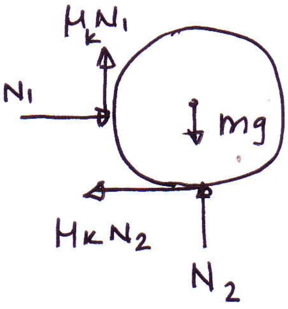


θ can be Calculated from $\omega^2 = \omega_0^2 + 2\alpha\theta$

If we know α

α Can be Calculated from Torque eqn.

<u>Step 2.</u> F.B.D.



Apply $\Sigma F = 0$.

$N_1 = \mu N_2$ — (1)

$N_2 + \mu N_1 = mg$ — (2)

Solving.

$N_1 = \mu mg/(1+\mu^2)$

$N_2 = \frac{mg}{1+\mu^2}$

<u>Step 3</u>

Now taking τ at CM $\Rightarrow$ $\tau_{cm} = I_{cm}\alpha$.

$$\mu N_2 R + \mu N_1 R = I\alpha = \frac{mR^2}{2}\alpha.$$

$$\mu \frac{\cancel{m}g}{1+\mu^2}\cancel{R} + \mu \frac{\mu \cancel{m}g}{1+\mu^2}\cancel{R} = \frac{\cancel{m}R^{\cancel{2}}}{2}\alpha$$

$$\alpha = \frac{2}{R}\left[\frac{\mu}{1+\mu^2} + \frac{\mu^2 g}{1+\mu^2}\right] = \frac{2}{R}\,\frac{\mu + \mu^2 g}{1+\mu^2} = \frac{2\mu}{R}\,\frac{(1+\mu g)}{1+\mu^2}$$

$$\omega_f^2 = \omega_0^2 - 2\alpha\theta \Rightarrow 0 + \omega_0^2 = +2\left(\frac{2\mu}{R}\,\frac{(1+\mu g)}{1+\mu^2}\right).\theta.$$

Calculate θ $\qquad n = \theta/2\pi$

[IRODOV - 1.248]

- There is another way to Calculate / solve this problem. Calculate τ, L about any point which is stationary / Inertial and then apply $\tau = \frac{dL}{dt}$.

We will see in later section how to do that.

One can Calculate τ, L about O, A, B. & solve it.

Problems.

On Small Oscillations.

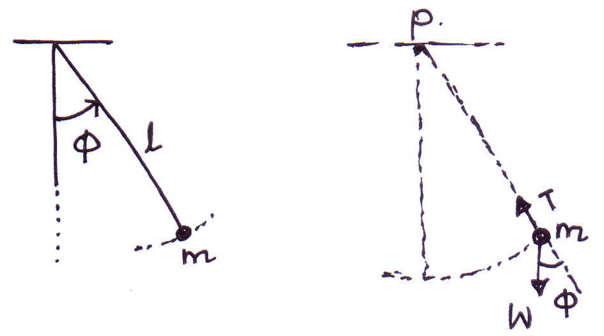
Here we will see three examples on small oscillations & find out the time period.

The method is:

1) Write down the FBD - Draw all the forces, Write Force eqn. $\sum F_i = ma$.

2) $\sum \tau_i = I\alpha$ put $\alpha = \ddot{\theta}$, $\sin\theta \approx \theta$ for small θ & then solve the ODE. (ordinary Differential eq).

#1 Time period of Simple Pendulum.



Force along the tangential direction $= -W\sin\phi$.

Force Eqn: $-W\sin\phi = m\,l\ddot{\phi}$ $[F = ma = mr\alpha]$

Torque Eqn: $-Wl\sin\phi = ml^2\ddot{\phi}$ [Torque about P]

$\Rightarrow -W\sin\phi = ml\ddot{\phi}$

Both these equations give the same equation - Simplify

$l\ddot{\phi} + g\sin\phi = 0$ for small θ, $\sin\phi \to \phi$.

$l\ddot{\phi} + g\phi = 0$ This is eqn of SHM. Solution is given by:

Sol $\phi = A\sin\omega t + B\cos\omega t$ $\omega = \sqrt{g/l}$.

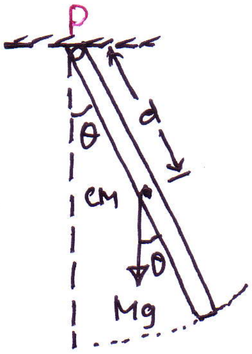
at $t = 0$ $\phi = \phi_0$ $\Rightarrow$ $\phi_0 = B\cos 0 = B$.

Solution is $\Rightarrow$ $\phi = \phi_0\cos\omega t$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}$$

[Kleppner]

#2. Time period of a bar Pendulum.



$$|\tau_p| = -Mgd\sin\theta = I_p\alpha$$

about pt P (under $|\tau_p|$)   about pt P. (under $I_p\alpha$)

For small θ, $\sin\theta \approx \theta$.

$$Mgd\theta + I_p\ddot{\theta} = 0$$

or $$\ddot{\theta} + \left(\frac{Mgd}{I_p}\right)\theta = 0$$

It can be written as $\ddot{\theta} + \omega^2\theta = 0$ $\omega^2 = \frac{Mgd}{I_p}$ (angular freq.)

solution of this eqn. $\theta = \theta_0 \cos(\omega t + \phi_0)$.

Now $$T = 2\pi\sqrt{\frac{\cancel{Mg}\, I_p}{Mgd}} = 2\pi\sqrt{\frac{I_0 + Md^2}{Mgd}}$$ (parallel axis Theorem).

- If the rod is thin & uniform (homogeneous)

$$I_0 = \frac{Ml^2}{12} = \frac{M(2d)^2}{12} = \frac{Md^2}{3}.$$

- If the rod is not uniform & thin one can express in terms of radius of Gyration.

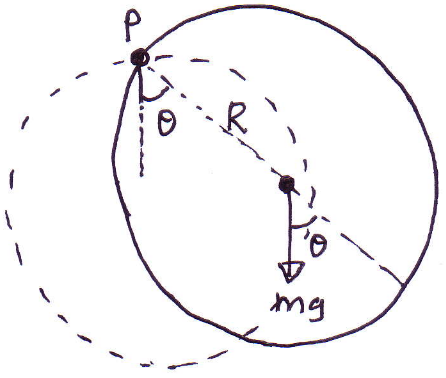
$I_0 = MK^2$ where k is radius of Gyration.

Let us take a more general case, the latter one.

$$\underline{T} = 2\pi\sqrt{\frac{MK^2 + Md^2}{Mgd}} = \underline{2\pi\sqrt{\frac{K^2+d^2}{gd}}}$$

Exactly the same way we can calculate T for a Physical pendulum & we get the same answer — only the initial figure is diff otherwise all steps are same.

#3. Time period of a Circular hoop / ring.



The problem is almost similar to the previous example of bar pendulum. Only the diagram is different. — Concept wise same.

$$|\tau_p| = -MgR\sin\theta = I_p\ddot{\theta}$$

For small angle $\sin\theta \approx \theta$

$$I_p\ddot{\theta} + MgR\theta = 0$$

$$\ddot{\theta} + \left(\frac{MgR}{I_p}\right)\theta = 0. \qquad \omega^2 = \frac{MgR}{I_p}.$$

Here $I_p = MR^2 + I_{cm} = MR^2 + MR^2 = 2MR^2$

$$\omega^2 = \frac{\cancel{M}gR}{2\cancel{M}R^{\cancel{2}}} = \frac{g}{2R} \quad \Rightarrow \quad T = 2\pi\sqrt{\frac{2R}{g}}$$

Now if there is a pendulum of length 2R then it will have same answer. Only geometry is different here.

Prof. Walter Lewin has demonstrated this experiment in the class & showed that a simple pendulum of length 2R and a circular hoop of ~~be~~ diameter 2R has almost the same time period.

[From Kleppner]

Motion Involving both Translation & Rotation.

This is little complicated topic. I would like to divide this topic into different subsections.

- Choosing frame of reference
  - Non Inertial frame — fixed to rotating body.
  - Inertial frame fixed to Lab
- Uniform Pure Rolling
  - forward slipping
  - Backward slipping
- Instantaneous axis of Rotation.
- Accelerated Pure Rolling
  - forward slip.
  - Backward slip.
- Direction of friction for different Cases.

You are requested to revise these topics from any book which you have studied. I am giving a brief Summary of the same.

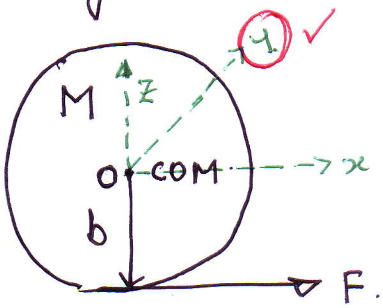
(a) Choosing a reference frame.

Let me choose example 6.15 of Kleppner to explain this concept.

6.15 (K&K). A disk of mass M and radius b is pulled with constant force F by a thin tape wound around its circumference. The disc slids on ice without friction. What is its motion?

Now We can solve this problem in two different ways.

$\boxed{\text{Method 1 : Non inertial/or inertial}}$



(a) The first way (may be easier) is to fix a frame whose one axis is passing through the axis of rotating body & its centre of mass. In this axis the body will be seen as pure rolling. But if the body accelerats (i.e COM accelerats) it will become an noninertial frame. In order to solve problem in noninertial frame we have to add pseudo forcs (−ma) & psedo torque. It is seen that total torque of all these pseudo forcs is 0 about this axis. So if a symmetric axis is chosen which passes through COM & does not change direction, one can apply all the equations (Newton's Second Law)

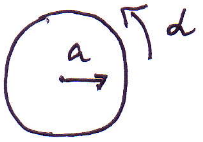
Refer to H.C. Verma (page 180, Section 10.16)
Refer to Mechanics by Young & Freedman.

Let us solve this problem using this method. Looking at the motion about Center of Mass.

$$\tau_0 = bF = I_0 \alpha \quad \Rightarrow \quad \alpha = \frac{bF}{I_0}.$$

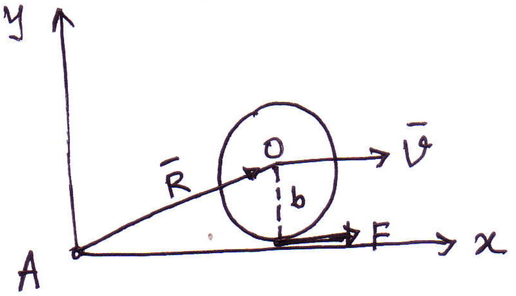
The acceleration of COM.

$$a = \frac{F}{m}.$$



Please note that here rotation is anticlockwise in contrary to usual case of rotation.

$$\boxed{\text{Method 2 : Purely Inertial Frame}}.$$



We choose a Coordinate system fixed to the lab whose origin A is along the line of F.

The torque about A is

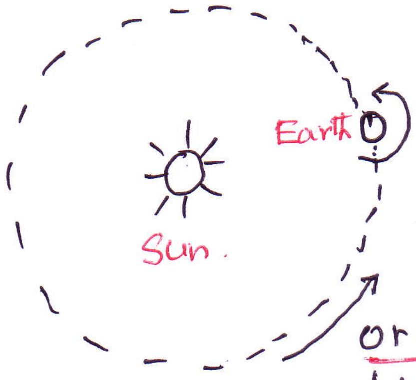
$$\boxed{\tau_{A(z)} = \tau_0 + (\bar{R} \times \bar{F})_{(z)}} = bF - bF = 0$$

Add this formula in your formula list

$\Downarrow$

You may wonder from where this formula has come. The proof of this formula is given in page 263 of K&K

This means that Torque/angular momentum of a rigid body is the sum of Torque/angular momentum about its Center of mass and the Torque/angular momentum of the Center of mass about the Origin. These two terms are often referred to as the spin & orbital terms respectively. Earth illustrate them nicely.



Spin motion ⇒ gives rise to spin angular momentum.

Orbital Motion ⇒ gives rise to orbital angular momentum.

An important feature of the spin angular momentum is that it is independent of the coordinate system. In this sense it is intrinsic to the body: no change in coordinate system can eliminate spin where as orbital angular momentum disappears if the origin is along the line of rotation.

(Section 6.7 in K & K)

Now referring back to previous problem, The torque = 0 so $\frac{dL}{dt} = 0$

velocity of center of mass.

$$\boxed{L_{A(z)} = I_0 w + (\bar{R} \times M\bar{V})_z} = I_0 w - bMV = 0$$

(direction is opp)

$dL/dt = 0$, differentiate — Add this in your formula list.

$0 = \cancel{bM}\, I_0 \alpha - bMa$

OR $\quad \alpha = \frac{bMa}{I_0} = \frac{bF}{I_0}$ as before.

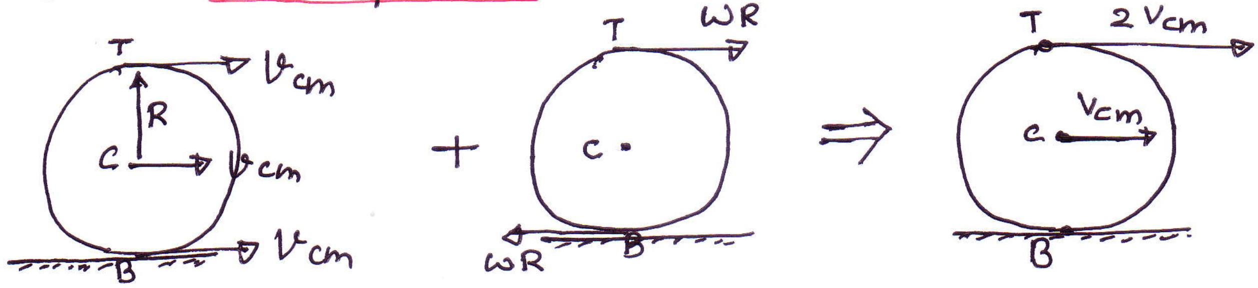
[By mistake I have named axes differently in Method 1 & Method 2. We should keep them same]

Uniform Pure Rolling.

Uniform → $a_{cm} = 0$ Uniform Velocity

Pure Rolling → No Sliding

In this case Center of mass is moving with Constant velocity and there is no sliding. This can be viewed as a superposition of pure translation and rotation about the center of mass.



Translational	Rotational	Superposition.
• All points move with same linear velocity	• All points move with same ω	Pure Rolling

✡ $v = R\omega$. Condition for pure rolling

✡ $v_B = 0$ ⇒ The bottom most point touching the ground is having zero velocity, that is at rest.

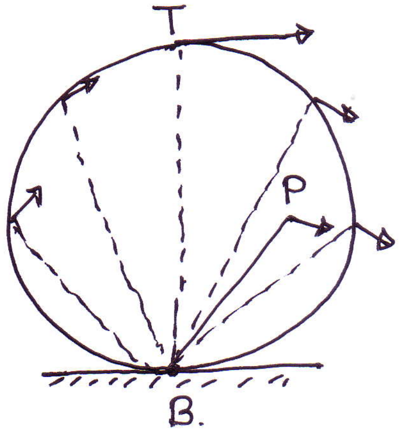
- Forward Slipping ⇒ $v > R\omega$.
- Backward slipping ⇒ $v < R\omega$.

} Not a case of pure Rolling

- Uniform Pure Rolling ⇒ Both v and ω are Const.
- In pure Rolling distance moved by Center of mass of the rigid body in one full rotation $= 2\pi R$
- In pure Rolling the path of a point on circumference is a Cycloid

Instantaneous Axis of Rotation.

The combined rotational and translational motion of a rigid body can be thought of as a pure rotation with same ω about an axis called Instantaneous axis of rotation, That is the ~~bottomost~~ Bottom most point in case of pure rolling.



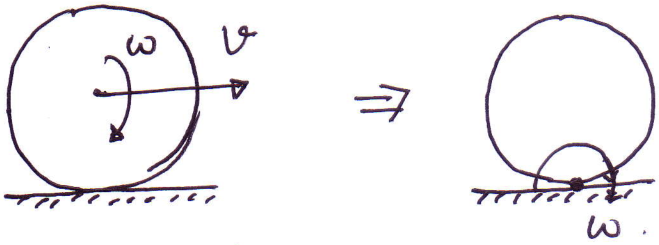
- Velocity of any point P can be obtained by a single term

$$v = r\omega.$$

$$v\,(\text{at pt } T) = (2R)\,\omega = 2R\omega.$$

- Total KE $= \frac{1}{2} I \omega^2$

Where I is the ~~total~~ moment of Inertia about instantaneous axis of rotation.



Combined motion of pure rolling about COM & translation.

pure rotation about bottom most point which is at rest.

You might have solved many problems to find out instantaneous axis of rotation. Revise them.

Accelerated Pure Rolling.

In uniform pure rolling v and ω are constants. Now if an external force is applied to the rigid body the motion will no longer remain Uniform.

The Condition for pure rolling is

$$\boxed{v = R\omega} \quad - (1)$$

Differentiating wrt to time.

$$\frac{dv}{dt} = R\frac{d\omega}{dt}$$

$$\boxed{a = R\alpha} \quad - (2)$$

So now we have two conditions to maintain pure rolling.

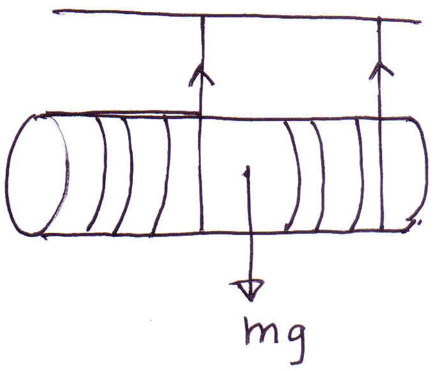
Role & Direction of Friction.

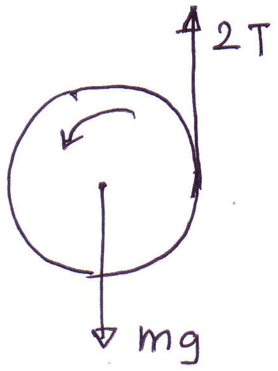
Here friction plays an important role in maintaining pure rolling. It may act in forward direction, backward direction or zero. If

- If $a > R\alpha$, Then to Support the angular motion the force of friction will act in backward direction
- If $a < R\alpha$ Then to Support the linear motion the force of friction f will act in forward direction
- If $a = R\alpha$ Either force of friction $= 0$ or in the backward direction.

Sample Problems on Plane Motion.

\# A solid Cylinder of mass = 4 kg and radius R = 10 cm, has two ropes wrapped around it one near each end. The cylinder is held horizontally by fixing the two free ends of the cords to the hooks on the Ceiling such that both the cords are exactly vertical. The Cylinder is released to fall under gravity. Find the tension in the cords when they unwind and the linear acceleration of the cylinder.





First Draw Free Body Diagram.

$\boxed{\text{Translational Motion}}$

$$\Sigma F = ma.$$

$$mg - 2T = m a_y.$$

$\boxed{\text{Rotational Motion}}$

$$\Sigma \tau = I\alpha.$$

$$a_y = R\alpha$$

$$2TR = I\alpha$$

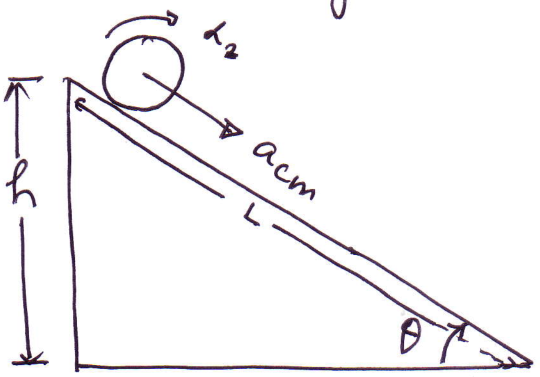
$$I = \frac{mR^2}{2}$$

Combining all Three equations

$$mg = ma + \frac{ma}{2} \Rightarrow a = \frac{2g}{3}.$$

[IRODOV, 1.253 Resnick Halliday Krane]

A solid cylinder of mass M and radius R starts from rest and rolls without slipping down an inclined plane of length L and height h. Find the speed / (acceleration) of its center of mass when the cylinder reaches the bottom.



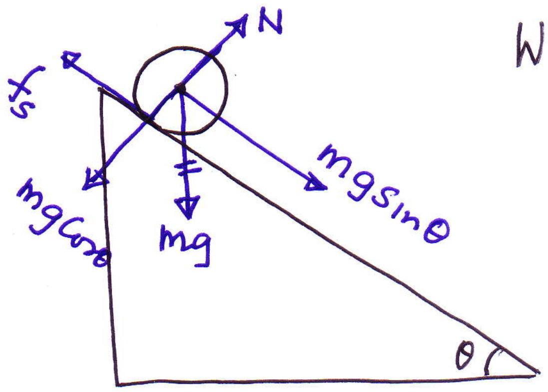
This is a very important problem because it clears the concept - many concepts.

Moreover this problem can be solved by many ways.

Method 1 : Energy Method. (1)

In the first step one has to see what is being asked. If it is speed then it is easier to calculate speed by Energy Method. Even in energy method there are two ways.

First Way.



Applying Work Energy Theorem.

$$W = \oint \bar{F} \cdot d\bar{r} = \Delta KE.$$

Here we have to calculate workdone by all the forces which should be equal to change in KE.

[Many Books Kleppner]

$$\underset{\substack{\Downarrow \\ mgh}}{W_{mg}} + \underset{\substack{\Downarrow \\ 0}}{W_N} + \underset{\substack{\Downarrow \\ 0}}{W_{fs}} = K_2 - K_1$$

(displacement of the pt is 0, the pt touching the ground is always at rest.

Work done by friction force is zero here. This is a very important Concept which will be used in many other problems.

KE = Translation of CM + Rotation around CM.

$$mgh = \frac{1}{2} m V_{cm}^2 + \frac{1}{2} I \omega^2 = \frac{1}{2} m V_{cm}^2 + \frac{1}{2}\left(\frac{MR^2}{2}\right)\frac{V_{cm}^2}{R^2}$$

For pure Rotation $V_{cm} = R\omega$. (another Imp Concept)

Solving it gives $V_{cm} = \sqrt{\frac{4}{3} gh}$

Second Way. – Energy Method. (2)

The Second way of applying energy method – You Visualize it as translation of CM and rotation about CM.

Translation Motion:

If we consider the whole mass Concentrated in COM then the net force $= mg\sin\theta - f_s$.

Increase in KE due to translational motion only

$$= \frac{1}{2} M V^2$$

So $(mg\sin\theta - f_s)L = \frac{1}{2} M V_{cm}^2$ — (1)

Rotational Motion.

$$\int_{\theta_i}^{\theta_f} \tau \, d\theta = \frac{1}{2} I_0 \omega_f^2 - \frac{1}{2} I_0 \omega_i^2 = \frac{1}{2} I_0 \omega^2$$

$$f b \theta = \frac{1}{2} I_0 \omega^2$$

$b\theta = L$ so $fL = \frac{1}{2} I_o \omega^2$ (2).

Substitute $\omega = v/L$.

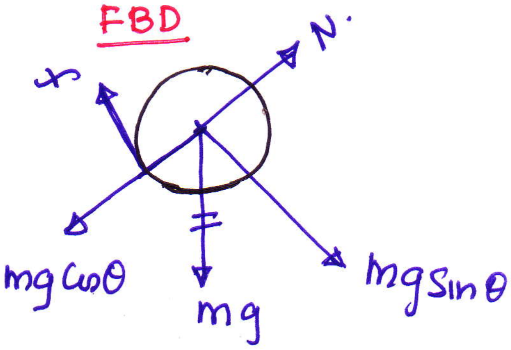
Now here we observe That if we add (1) & (2) We get the first equation of Previous Energy Method So it boils down to the same equation.

In this method we are treating translational motion & Rotational Separately. ⇒ Translation of Center of Mass + Rotation about CM ⇒ & solve the problem.

In the previous method we do both the steps together & it was simpler.

Third Way : Force Eqns & Torque Eqns.

FBD



Force Eqn. $mg\sin\theta - f_s = ma_{cm}$. —(1)

Torque Eqn. $f_s R = \frac{mR^2}{2}\alpha$. —(2)

Pure Rolling (no slipping) $a_{cm} = \alpha R$. — (3)

Solving we get $a_{cm} = \frac{2}{3} g\sin\theta$

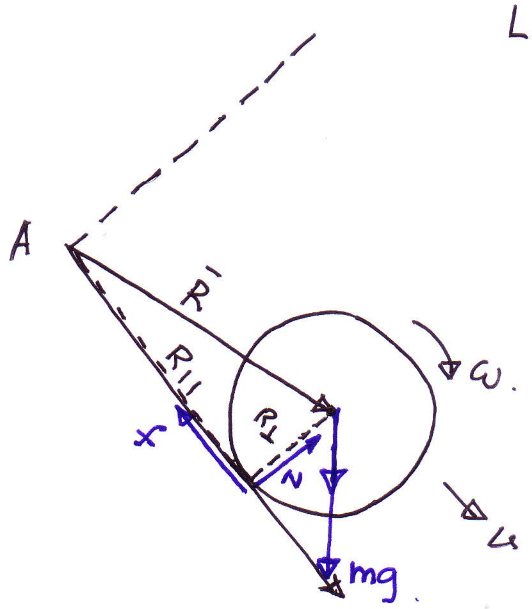
a_{cm} = Given $u = 0$ $v = ?$ $L\theta = \frac{h}{\sin\theta}$

$v^2 = u^2 + 2as$.

$v^2 = 0 + 2 \times \frac{2}{3} g\sin\theta \times \frac{h}{\sin\theta}$

$v = \sqrt{\frac{4}{3} gh}$

Fourth Method: Choose a fixed inertial pt. Then apply $\tau_{ext} = \frac{dL}{dt}$. – Calculate τ & L about that fixed pt.

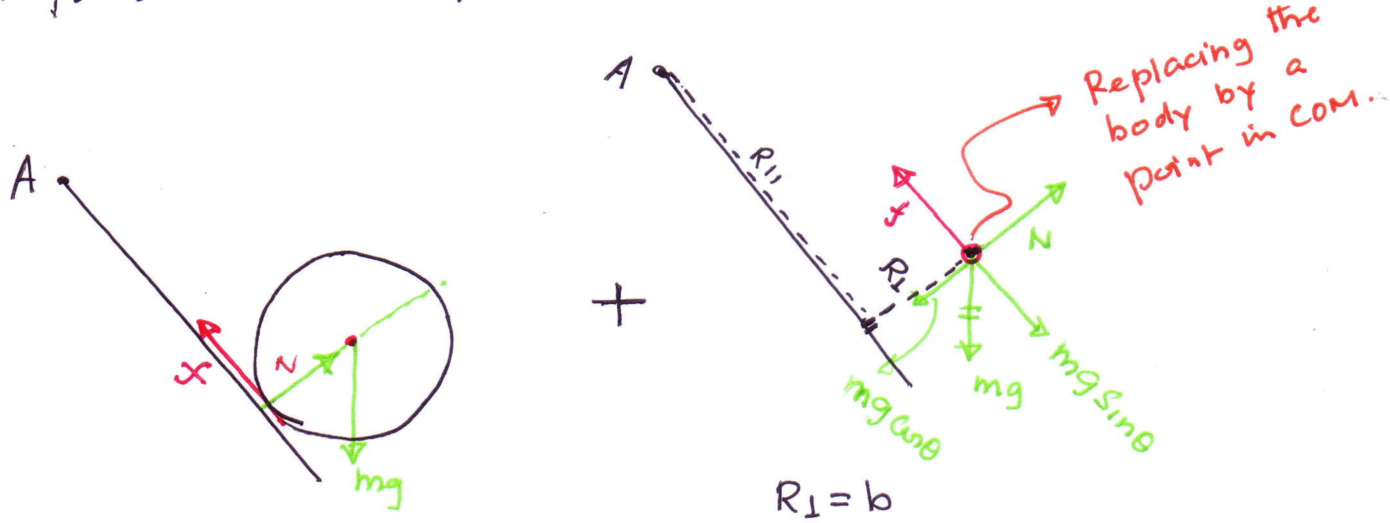


Choose a coordinate system fixed to the lab whose origin is A. The torque about A is

$$\tau_A = \tau_{CM} + \bar{r}_{cm} \times \bar{F}$$

$$L_A = L_{cm} + \bar{r}_{cm} \times MV_{cm}.$$

So what we do here We calculate both τ and L about some fixed pt (say A here). Then it is equal to torque / angular momentum about center of mass and then we replace COM by a point & see all the forces on that point. So it can be visualized as.



$R_\perp = b$

$$\tau_A = \tau_{cm} + (\bar{R} \times \bar{F}) = -R_\perp f + R_\perp(f - mg\sin\theta) + R_{\parallel}(N - Mg\cos\theta)$$

$$= -bmg\sin\theta$$

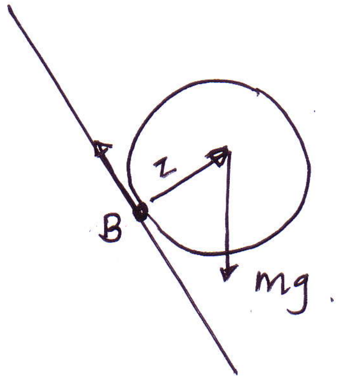
$$L_A = L_{cm} + \bar{r}_{cm} \times MV_{cm} = -I_0\omega - Mb^2\omega = -\frac{3}{2}Mb^2\omega.$$

$$\tau = \frac{dL}{dt} \Rightarrow bmg\sin\theta = \frac{3}{2}Mb^2\alpha \Rightarrow \alpha = \frac{2}{3}\frac{g\sin\theta}{b}$$

Since $a = b\alpha$

$a = \frac{2}{3} g \sin\theta$. $\Rightarrow$ Getting back our previous ans.

Choosing Origin differently.



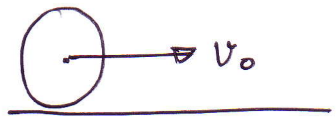
Analysis would have been simpler if we would have chosen B as the Origin. You may try.

Now Some questions are asked related to this question.

- If there are two Solid Cylinders both same mass but Radius of one is larger than the second which one will reach the ground first.
- Two solid cylinders $R_1 = R_2$ but $M_1 > M_2$ which one will reach faster
- Two ~~Solid~~ cylinders – one solid & other hollow of same mass & radius. – which one will reach earlier.

If you see your result & the steps as long as it is a Solid Cylinder the mass & radius is getting Cancelled. But if the expression of I changes then it will change. So solid & hollow Cylinder will matter – Find out from the expression.

A ball is thrown down an alley in such a way that it slides with a speed V_0 initially without rolling. Find its speed when it starts pure rolling. Also what is workdone by friction in this transition.

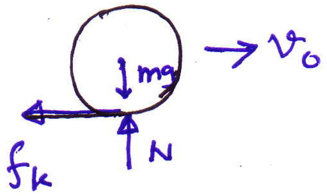


Now this is typically a slipping problem. And there is a way to do.

Direction of friction is very important here. Here it acts in opposite direction to that of V_0 & helps the body roll — the torque produced by friction. So two things are happening simultaneously.

$V_0 \downarrow$ $\omega \uparrow$ finally $V = R\omega$.

V_0 is decreasing because of opposite force of friction.
ω is increasing because of Torque produced by "



$$a_{cm} = -\frac{f_k}{m}.$$

$$v = V_0 - \frac{f_k t}{m} \quad - (1)$$

$$f_k R = I\alpha \Rightarrow \cancel{\alpha}$$

$$f_k R = \frac{2}{5}MR^2\alpha \Rightarrow \alpha = \frac{5 f_k}{2MR}.$$

$$\omega = \omega_0 + \alpha t \Rightarrow \omega = 0 + \frac{5}{2}\frac{f_k t}{mR} \quad - (2)$$

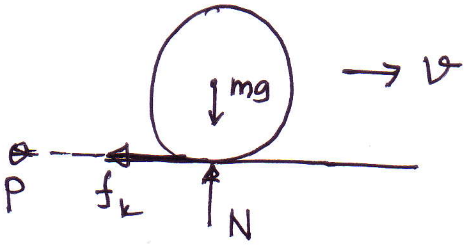
Solving (1) (2) (3)

$$V = R\omega \quad - (3)$$

$$V = \frac{5V_0}{7}$$

[Resnick Halliday]

Method 2 : Calculate τ about a stationary point.



Let us take a point P.

$\tau_p = 0$ [Just calculate τ by all three forces] otherwise you can apply the formula.

If $\tau_p = 0$ Then L_p = Conserved.

Initial Angular Mom = Final Ang. Mom.

$$m V_0 R = I_{cm} \omega + m v R.$$

$$= \frac{2}{5} m R^2 \omega + m v R.$$

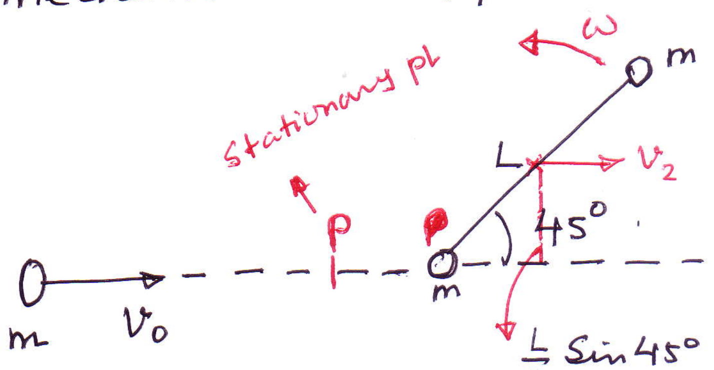
$$= \frac{2}{5} m V R + m v R$$

$$V = \frac{5 V_0}{7}.$$

[Use the formula $L_p = L_{cm} + R \times mv.$]

This is far simpler method.

A rigid massless rod of length L joins two particles each of mass m. The rod lies on a frictionless table and is struck by a particle of mass m and velocity V_0, moving as shown. After the collision, the projectile moves straight back. Find the angular velocity of the rod about its center of mass after the collision, assuming that mechanical energy is conserved.



$$\frac{L}{2}\sin 45^\circ = \frac{L}{2}\frac{1}{\sqrt{2}}$$

$\boxed{\text{Mom Cons}}$ $\quad mV_0 = -mV_1 + 2mV_2$

$\boxed{\text{Ang Mom Cons}}$ $\quad 0 = 0 - \left(\frac{L}{2}\frac{1}{\sqrt{2}}(2m)V_2\right) + \left(I\omega\right)$

ang mom of CM. ($\bar{r} \times m\bar{v}$)   about CM

$$0 = -\frac{L}{2}\frac{1}{\sqrt{2}}(2m)V_2 + 2m\left(\frac{L}{2}\right)^2\omega.$$

$(2 \times mr^2)\omega$

$\boxed{\text{Energy Cons}}$

$$\frac{1}{2}mV_0^2 = \frac{1}{2}mV_1^2 + \underbrace{\frac{1}{2}(2m)V_2^2}_{\text{Translational Energy}} + \underbrace{\frac{1}{2}I\omega^2}_{\text{Rotational Energy}}$$

Three Unknowns V_1, V_2, ω & Three equations

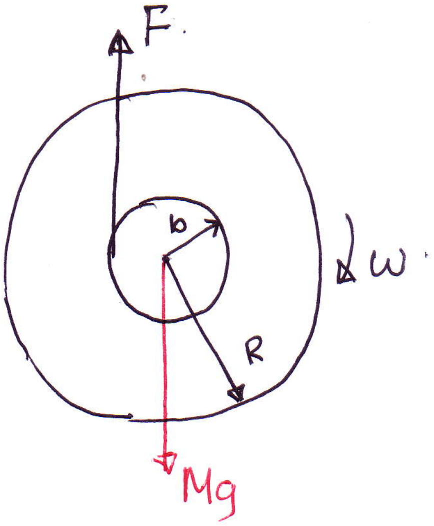
Here $V_2 \neq L/2\,\omega$

[Kleppner – 6.38]

A Yo-Yo of mass M has an axle of radius b and a spool of radius R. Its moment of inertia can be taken to be $MR^2/2$ and the thickness of the string can be neglected. The Yo-Yo is released from rest.

a) What is the tension in the cord as the Yo-Yo descends and as it ascends.

b) The center of the Yo-Yo descends distance h before the string is fully unwound. Assuming that it reverses direction with uniform spin velocity, find the average force on the string while the Yo-Yo turns around. [Kleppner – 6.29]



$$\left.\begin{aligned} Mg - F &= Ma \\ Fb &= I\alpha \\ a &= b\alpha \end{aligned}\right\} \Rightarrow \text{Solving} \quad F = \frac{Mg}{1+\frac{2b^2}{R^2}}$$

ascends same.

(b) Descends dist $= h$. → Reverse direction with uniform spin velocity
Average Force = ?

$F_{av}(\Delta t) = 2Mv$

Energy: $Mgh = \underbrace{\frac{1}{2}Mv^2}_{CM} + \underbrace{\frac{1}{2}\left(\frac{1}{2}MR^2\right)\omega^2}_{\frac{1}{2}I\omega^2}$ $\quad \omega \to v/b$

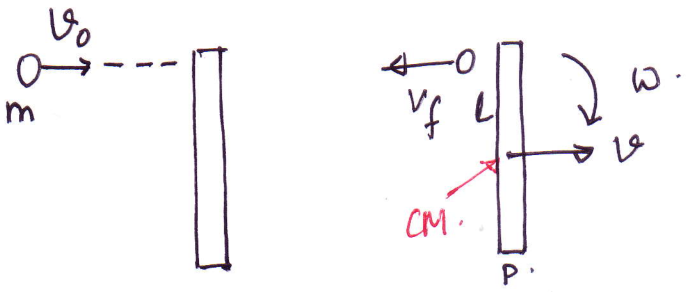
$$\Rightarrow v^2 = \frac{2gh}{1+R^2/2b^2}$$

$\Delta t = \frac{\pi}{\omega}$

$$F_{av} = 4Mgh/(\pi b)(1+R^2/2b^2)$$

A planck of mass $2l$ and mass M lies on a frictionless Plane. A ball of mass m and speed v_0 strikes its end as shown. Find the final velocity of the ball v_f, assuming that mechanical energy is conserved and that v_f is along the original line of motion.

[Kleppner – 6.37]



In this kind of problems appy momentum conservation, Energy Conservation & angular momentum conservation.

mom Cons. $\quad m v_0 = -m v_f + M v$

Energy Cons. $\quad \frac{1}{2} m v_0^2 = \frac{1}{2} m v_f^2 + \frac{1}{2} M v^2 + \frac{1}{2} I \omega^2$

Ang Mom Cons ⇓ About CM.

$$m v_0 l = - m v_f l + I_0 \omega .$$

$I_0 = \frac{1}{12} M (2l)^2 = \frac{1}{3} M l^2$

$m v_0 l$: Incoming particle; $m v_f l$: reflected particle; $I_0 \omega$: rod.

Solving $\quad v_f = \left(\dfrac{1 - \frac{4m}{M}}{1 + \frac{4m}{M}} \right) v_0 .$

(b) Find v_f assuming that stick is pivoted at the lower end.
Forces act at pivot so Momentum cons will not hold good.
other two eqns about I_p will give the result. $I_p = \frac{1}{3} M (2l)^2$