Momentum.

Center of Mass.

This Chapter Mostly deals with Centre of Mass Calculation. When it is Calculated for a Solid extended object you need to use integration. It is important. The body may be Circular, spherical, rectangular or triangular. You need to learn how to Calculate dm for different geometries. Try to practise diff exampls given here.

- Mostly taken from Kleppner
- Please let me know if there are/is any mistake.
- Add to more exampls & problems.

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18 Aug, 2016.

Momentum.

1.) Particle.

As you know momentum $\bar{P}$ for a particle is defined as

$$\bar{P} = m\bar{v}$$

Using Newton's Second Law of motion, for Const Mass.

$$\frac{d\bar{P}}{dt} = \frac{d}{dt}(m\bar{v}) = m\frac{d\bar{v}}{dt} = m\bar{a}$$

OR.

$$F_{\cancel{ext}\,net} = \frac{d\bar{P}}{dt} = m\bar{a} \quad (\text{if mass is Const})$$

A. NOTE: Momentum is more fundamental than mass or Velocity

2.) System of Particles or extended body.

For a system of particles the total momenta is the vector sum of the momenta of individual particls

$$\vec{P}_{sys} = \sum_i m_i\bar{v}_i = \sum_i \bar{P}_i$$

Now this can be written in terms of Centre of Mass.

$$\bar{P}_{sys} = \sum m_i\bar{v}_i = M\bar{V}_{cm}.$$

$\bar{V}_{cm}$ is the Velocity of Centre of Mass.

Then $$\frac{d\bar{P}_{sys}}{dt} = M\frac{d\bar{V}_{cm}}{dt} = M\bar{a}_{cm}.$$

According to Newton's Second Law of Motion.

$$\sum \vec{F}_{ext} = \bar{F}_{net-ext} = \frac{d\vec{P}_{sys}}{dt}$$

external because internal forces cancel each other according to third's Law of motion.

If $F_{ext} = 0$ Then $\bar{P}_{sys} = \sum_i m_i \bar{v}_i = M V_{cm} = \text{Const}$

This is known as Conservation of Momentum.

Center of Mass

What is Centre of Mass.

There are few things that We need to remember

(1) We Speak of Center of Mass when we dont talk about a particle but we talk about an extended solid object or a system of particles.

(2) Centre of Mass is a point, a coordinate.

(3) If we have to consider an extended solid object or a system of particles as a point where the whole mass is concentrated then that point is the Centre of mass for that solid body or system of particles.

(4) Finally Center of Mass is defined as a mass-weighted average position of particles.

What is the difference between Center of Mass & Center of Gravity?

Center of Gravity.

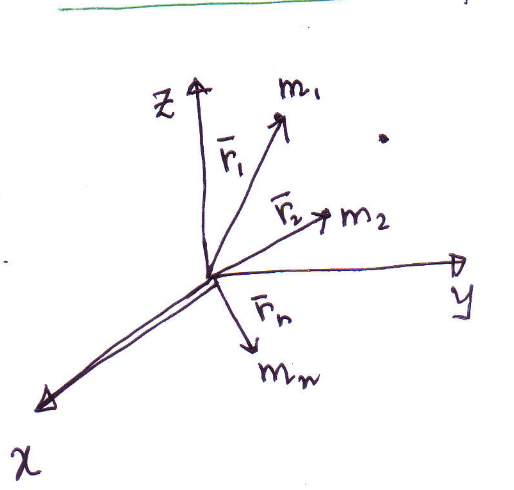
(1) Center of Gravity is the point in a body around which the resultant torque due to gravity force Vanishes.

(2) Center of mass is a fixed property for a given rigid body where as Center of Gravity may, in addition depend upon its Orientation in a non Uniform gravitational field. Centre of Gravity may change with Orientation.

(3) When Gravity gradient effects are negligible C.g and C.m are Same and are used inter changeably In solid objects it is the Centroid.

(4) Center of mass may be located outside the physical body as the Case for hollow or open shaped objects such as horse shoe.

Mathematical Expressions



$$\bar{R}_{\text{[redacted]}} = \frac{\sum_{i=1}^{n} m_i \bar{r}_i}{\sum_i m_i}$$

CM.

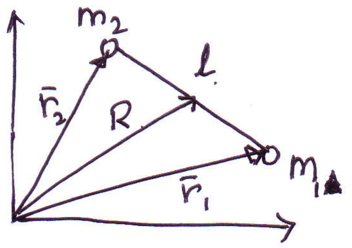
Similarly Velocity of Centre of Mass

$$\bar{v}_{cm} = \frac{m_1 \bar{v}_1 + m_2 \bar{v}_2 + \cdots}{m_1 + m_2 + \cdots} = \frac{\sum m_i v_i}{\sum m_i}$$

OR $M \bar{V}_{cm} = \bar{P}$

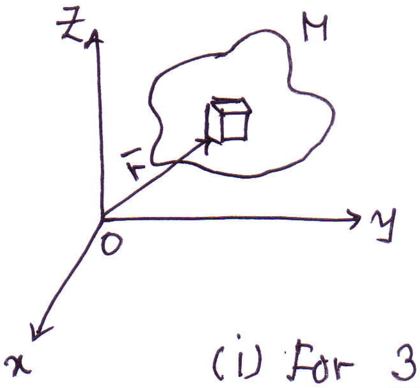
Similarly $a_{cm} = \cdots$

Ex 3.2 Find the Centre of mass of two bodies connected by a massless rod.



$$\bar{R} = \frac{m_1 \bar{r}_1 + m_2 \bar{r}_2}{m_1 + m_2}$$

- A solid extended object can be considered as large number of bodies with $N \to \infty$



$$\bar{R} = \frac{\lim\limits_{N\to\infty} \sum\limits_{i=1}^{N} m_i \bar{r}_i}{\lim\limits_{N\to\infty} \sum\limits_{i=1}^{N} m_i} \Rightarrow \boxed{\bar{R} = \frac{1}{M}\int \bar{r}\, dm}$$

(i) For 3d object $\Rightarrow$ $dm = \rho\, dv$.

ρ = ~~vol~~ mass per unit vol. known as vol mass density

$$\boxed{\bar{R} = \frac{1}{M}\int \bar{r}\rho\, dv.}$$

(ii) For 2d Object $\Rightarrow$ $\boxed{\bar{R} = \frac{1}{M}\int \bar{r}\,\sigma\, dA}$

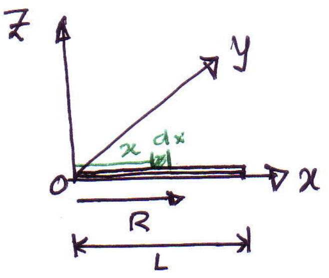
Applicable for thin plats & not for plats whose thickness is finite

σ = mass per unit area
Area mass density

(iii) For 1d object $\Rightarrow$ $\boxed{R = \frac{1}{M}\int \bar{r}\,\lambda\, dl}$

λ = mass per unit length
length mass density

A rod of length L has a non uniform density. λ, the mass per unit length of the rod, varies as $\lambda = \lambda_0 (s/L)$, where λ_0 is a constant and s is the distance from the end marked 0. Find the Center of mass.



$$\bar{R} = \frac{1}{M}\int \bar{r}\,\underbrace{\lambda\, dl}_{dm}$$

$$M = \int dm = \int_0^L \lambda\, dx = \int_0^L \frac{\lambda_0 x\, dx}{L}$$

Here s is taken along x axis.

$$= \frac{\lambda_0}{L}\int_0^L x\, dx = \frac{\lambda_0}{L}\left.\frac{x^2}{2}\right|_0^L = \frac{\lambda_0 L}{2}$$

The Centre of Mass is

$$R = \frac{1}{M}\int \bar{r}\,\underbrace{\lambda\, dl}_{dm} = \frac{2}{\lambda_0 L}\int_0^L (x\hat{i} + 0\hat{j} + 0\hat{k})\frac{\lambda_0 x\, dx}{L}$$

$$= \frac{2}{L^2}\frac{\hat{i}}{3}\left. x^3\right|_0^L = \frac{2}{3}L\hat{i}$$

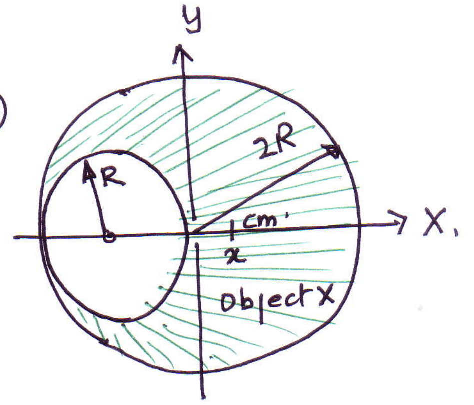
The density of a Thin rod of length l varies with the distance x from one end as $\rho = \rho_0 x^2/l^2$. Find the position of the Center of mass.

[Ans. $X = 3l/4$]

The Value of Center of Mass does not depend upon the choice of Origin. You can try shift your Origin k dist away & Recheck the answer.

The Figure below shows a circular metal plate of radius 2R which a disk of radius R has been removed. Let us call it object X. Its centre of mass is shown as a dot on the x axis. Find the location of this point.

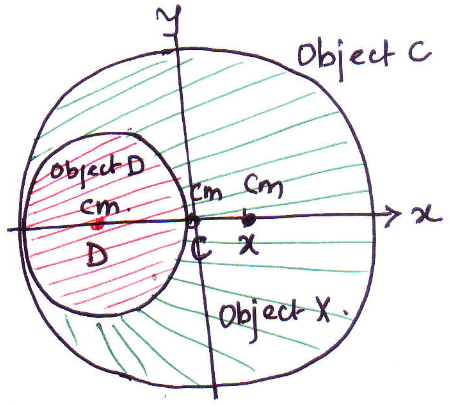
(a)



⇒ By symmetry object X should have its CM along x axis. Let us solve the following way.

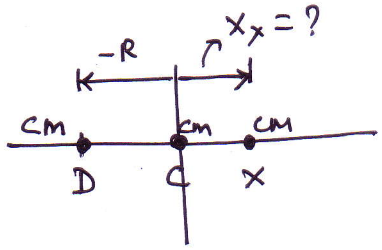
(a) The first fig (a) is object X is a metal disk of radius 2R with a hole of radius R cut in it.

(b)



(b) Let us fill the hole with another disk of the same material of radius R which we call object D.

(c) Let us label as object C the large uniform composite disk so formed. By symmetry its CM is at the center – origin.



(d) Let us represent the object X and object D as pt masses. The object C can be treated as equivalent to two particles.

The position of centre of mass of object C is given by

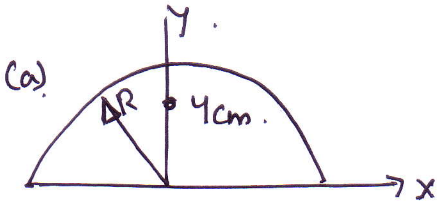
$$x_c = \frac{m_D x_D + m_x x_x}{m_D + m_x}$$

Now $x_c = 0$ at the centre

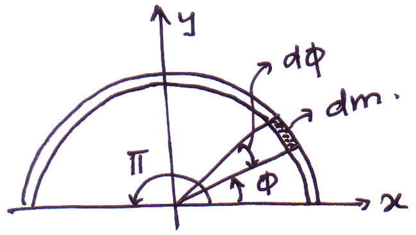
So $x_x = -\left(\frac{m_D}{m_x}\right) x_D$ | $\Rightarrow \frac{m_D}{m_x} = \frac{\text{area of } D}{\text{area of } X} = \frac{\text{area of } D}{\text{area } C - \text{area } D}$

with $x_D = -R$ $\quad x_x = R/3$ | $= \pi R^2 / (\pi (2R)^2 - \pi R^2) = 1/3$

A thin strip of material is bent into the shape of a Semicircle of Radius R. Find its Centre of mass.



$\Rightarrow$



By Symmetry one can say That $x_{cm} = 0$. The centre of mass must lie on The Y axis.

Consider the small element of mass dm which subtends an angle $d\phi$ at the Center. Now.

$$Y_{cm} = \frac{1}{M}\int y\,dm.$$

Now it Can be Solved two ways –

(1). dm can be calculated from the ratio of angles because the strip is of Uniform mass density.

$$\frac{dM}{M} = \frac{d\phi}{\pi}$$ [ratio of dm by M should be equal to the angles subtended by them]

$$\Rightarrow dm = M\frac{d\phi}{\pi}$$

The element dm is located at coord. $y = R\sin\phi$.

So
$$Y_{cm} = \frac{1}{M}\int y\,dm = \frac{1}{M}\int_0^{\pi}(R\sin\phi)\,\frac{M}{\pi}\,d\phi$$

$$= \frac{R}{\pi}\int_0^{\pi}\sin\phi\,d\phi = \frac{2R}{\pi} = 0.637R \text{ (Ans)}$$

(2) Second way – Assume mass per Unit length as λ_0

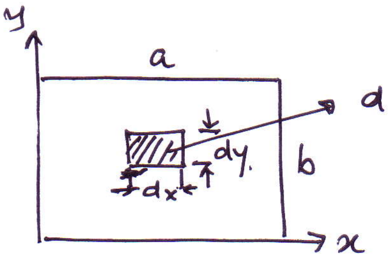
$dm = \lambda_0 R\,d\phi$, $M = \pi R\lambda_0$

You get back the same equation as above.

Find the Center of mass of a thin rectangular plate with sides of length a and b, whose mass per Unit area σ Varies in the following fashion $\sigma = \sigma_0 (xy/ab)$

V. Imp word.

$\Rightarrow$



$dm = \sigma \, dx \, dy.$

$$\bar{R} = \frac{1}{M} \iint (x\hat{i} + y\hat{j}) \sigma \, dx \, dy.$$

First let us find out the mass of the plate

$$M = \int_0^b \int_0^a \sigma \, dx \, dy. = \int_0^b \int_0^a \sigma_0 \frac{xy}{ab} \, dx \, dy.$$

This is a double integral. First integrate over x, treating y as a constant. Then integrate y.

$$M = \int_0^b \left(\int_0^a \sigma_0 \frac{x}{a} \frac{y}{b} . dx \right) dy = \int_0^b \left(\sigma_0 \frac{y}{b} \frac{x^2}{2a} \Big|_{x=0}^{x=a} \right) dy.$$

$$= \int_0^b \sigma_0 \frac{y}{b} . \frac{a}{2} \, dy = \frac{\sigma_0 a}{2} \frac{y^2}{2b} \Big|_{y=0}^{y=b} = \frac{1}{4} \sigma_0 ab.$$

The x Component of R is

$$X = \frac{1}{M} \iint x \sigma \, dx \, dy = \frac{1}{M} \int_0^b \left(\int_0^a x \sigma_0 \frac{xy}{ab} \, dx \right) dy.$$

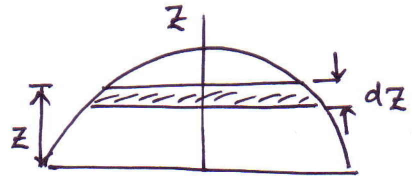
$$= \frac{1}{M} \int_0^b \left(\frac{\sigma_0 y}{ab} \right) \frac{x^3}{3} \Big|_0^a \Big) dy = \frac{1}{M} \frac{\sigma_0}{ab} \int_0^b \frac{y a^3}{3} \, dy.$$

$$= \frac{1}{M} \frac{\sigma_0}{ab} \frac{a^3}{3} \frac{b^2}{2} = \frac{4}{\sigma_0 ab} \frac{\sigma_0 a^2 b}{6} = \frac{2}{3} a.$$

Similarly $Y = \frac{2}{3} b$

Differentiate this with earlier triangular sheet which was of finite thickness. In finite Thickness σ Cannot be used.

Find the Center of mass of a Uniform Solid hemisphere of radius R and mass M.



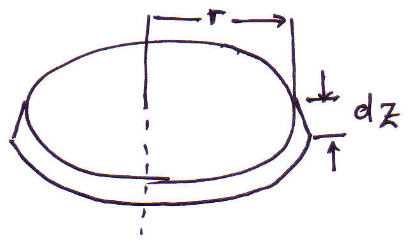
From Symmetry it is apparent that the Center of mass lies on the z axis. Its height above the equatorial plane is

$$Z = \frac{1}{M}\int z\, dM$$

CM. dist from dm

The problem is a 3d problem but its symmetry helps to become a one dimensional problem.

Let us divide the hemisphere into a pile of thin disks.



Consider the Circular disk of rad = r, Thickness = dZ

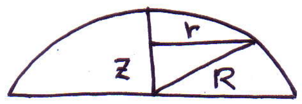
$$dV = \pi r^2 dZ$$

$$dM = \rho\, dv = \left(\frac{M}{V}\right)(dv)$$

$$Z = \frac{1}{M}\int z\, dM = \frac{1}{M}\int \left(\frac{M}{V}\right) z\, dV = \frac{1}{V}\int_{Z=0}^{R} \pi r^2 z\, dz$$

CM.

We need to a find a relation between r & Z.



$$r^2 = R^2 - z^2$$

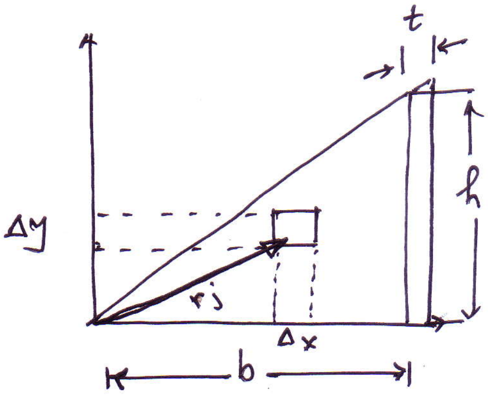
$$Z = \frac{\pi}{V}\int_0^R z\,(R^2 - z^2)\, dZ = \frac{\pi}{V}\left(\frac{1}{2} z^2R^2 - \frac{z^4}{4}\right)\Big|_0^R$$

$$= \frac{\pi}{V}\left(\frac{1}{2}R^4 - \frac{1}{4}R^4\right) = \frac{\frac{1}{4}\pi R^4}{\frac{2}{3}\pi R^3} = \underline{\underline{\frac{3}{8}R.}}$$

[V = vol of hemisphere is $= \frac{2}{3}\pi R^3$]

Find out the Center of mass of a right triangular sheet of mass M, base b, height h and small thickness t. (V. Imp, This is not a thin plate)

⇒ Now here it becomes little tedious problem because it has a finite thickness. It is a 3d problem but it can easily be changed into 2d problem because the 't' can be cancelled from both numerator and denominator. But if it is done within ∫ then it will be 3d. So it is better to take small discrete mass objects, change the problem to 2d and then apply integration.



$$R \approx \frac{\sum m_j r_j}{M}$$

$$= \frac{\sum \rho_j t \, \Delta x \, \Delta y \, r_j}{M}$$

Sheet is Uniform $\rho_j = \text{Const} = \frac{M}{V} = \frac{M}{At}$

We can carry out the sum by summing first over the Δx and then over the Δy.

$$\bar{R} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \left(\frac{M}{At}\right)\left(\frac{t}{M}\right) \sum \sum \bar{r}_j \, \Delta x \, \Delta y = \frac{1}{A} \iint \bar{r} \, dx \, dy.$$

$$\bar{R} = (X\hat{i} + Y\hat{j}) = \frac{1}{A} \iint (x\hat{i} + y\hat{j}) \, dx \, dy.$$

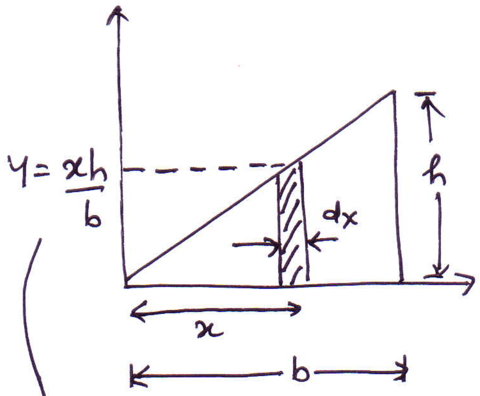
$$= \underbrace{\frac{1}{A}\left(\iint x \, dx \, dy\right)}_{X} \hat{i} + \underbrace{\frac{1}{A}\left(\iint y \, dx \, dy\right)}_{Y} \hat{j}$$

Now to find the x coordinate of CM.

$$X = \frac{1}{A}\iint x\,dx\,dy.$$

Now this integral means, take each element, multiply its area by its x coordinate and sum the results.

How to do it – Since it is a triangle.



$$\frac{b}{h} = \frac{x}{y}$$

First consider the element in a strip parallel to y axis. The strip runs from $y=0$ to $y = xh/b$. Let us integrate y first keeping x const. & Then integrate x.

$$X = \frac{1}{A}\iint x\,dx\,dy = \frac{1}{A}\,x\,dx\int_0^{xh/b} dy$$

$$= \frac{h}{bA}\,x^2\,dx.$$

Finally sum the contributions of all such strips $x=0$ to $x=b$.

$$X = \frac{h}{bA}\int_0^b x^2\,dx = \frac{h}{bA}\cdot\frac{b^3}{3} = \frac{hb^2}{3A}$$

Since $A = \frac{1}{2}bh \Rightarrow X = \frac{2}{3}b.$

Similarly.

$$Y = \frac{1}{A}\int_0^b\left(\int_0^{xh/b} y\,dy\right)dx = \frac{h^2}{2Ab^2}\int_0^b x^2\,dx = \frac{h^2 b}{6A} = \frac{h}{3}.$$

Hence

$$R = \frac{2}{3}b\,\hat{i} + \frac{1}{3}h\,\hat{j}.$$