Sturm Comparison Theorem

We shall now discuss some qualitative result about Linearly homogeneous equation of 2nd order. One of the important discussion is a bout zeros of a solution of differential equation.

We have already seen

(i) Any nontrivial solution of $y''Q(x)y' + R*x)y = 0$ [$Q(x)$ & $R(x)$ continuous on I = [a, b]has simple zeros only on I.

(ii) Two linear independent solution of above equation have no common zeros.

This implies that if $\varphi(x)$ & $\psi(x)$ are two linear independent function on an interval I having common zero at some point $x_0 \in I$, then there we can not find continuous functions $Q(x)$ and $R(x)$ such that $\varphi(x)$ & $\psi(x)$ are solutions of the equation :

$$y''+Q(x)y' + R(x)y = 0 \qquad (1)$$

(iii) Zeros of two linear independent solutions of above equation alternate.

Before going for further discussion, note that equation (1) can be transformed to the *standard form*

$$(p(x)y')' + r(x)y = 0$$

By multiplying eqn. (1) by $\exp(\int Q(x)dx)$

Where, $p(x) = e^{\int Q(x)dx}$ & $r(x) = e^{\int Q(x)dx}\ R(x)$.

Further: $y''+Q(x)y' + R(x)y = 0$ (2)

Can also be transformed to <u>normal</u> <u>form</u> by change of dependent variable $y = u(x)z(x)$ to normal form:

$$z''+r(x)z = 0 \qquad (3)$$

where u is so chosen that coefficient of z' becomes zero, which gives:

$$u(x) = \exp(-\frac{1}{2}\int Q(x)dx);\ r(x) = R(x) - \frac{1}{2}Q'(x) - \frac{1}{4}Q^2(x)$$

Note that if z (x) is a solution of (3) then $y(x) = u(x)z(x)$ is solution of (2).

Further, if Q(x) is continuous then $u(x) \neq 0$ at any point on I. Therefore, the zeros of y & z will coincide.

e.g. if we consider $x^2y'' + xy' + (x^2 - p^2)y = 0 \quad x > 0$ then we have:

$$Q(x) = \frac{1}{x},\ R(x) = 1 - \frac{p^2}{x^2} \text{ which gives}$$

$$u(x) = \exp(-\frac{p}{2}\int\frac{dx}{x}) = \frac{1}{\sqrt{x}}$$

$$\&\ r(x) = 1 + \frac{1-4p^2}{4x^2}$$

Thus, $x^2 y'' + xy' + (x^2 - p^2)y = 0 \equiv z'' + (1 + \frac{1-4p^2}{4x^2})z = 0$ where $z(x) = \left(\sqrt{x}\right) y(x)$

Now, we shell see the main result:

Sturm's comparison Theorem: "Let $p(x) > 0$ & $p'(x), r_1(x), r_2(x)$ continuous functions on $I = [a, b]$ and $r_2(x) \geq r_1(x)$ on I.

Further, if $\varphi(x)$ is a nontrivial solution of

$$(p(x)y')' + r_1(x)y = 0$$

and $\psi(x)$ is nontrivial solution of

$$(p(x)y')' + r_2(x)y = 0,$$

then between any two consecutive zeros of $\varphi(x)$ there exists a zero of $\psi(x)$".

(the case of $r_2(x) \equiv r_1(x)$ will be considered later)

$$e.g.\ consider\ \ y'' + y = 0 \quad ....(i) \quad p = 1; r_1 \equiv 1$$

$$\&\ y'' + 4y = 0 \quad ....(ii)\ \ p \equiv 1, r_2 = 4 > r_1$$

$\varphi = \sin x$ is a solution of (i) having zeros at $x = n\pi$ and $y = \sin 2x + \cos 2x$ is a solution of (ii).

So Sturm Comparison Theorem says that between 0 and $\pi, \psi = \sin 2x + \cos 2x$ will have at least one zero.

Note: It just indicates that there will be at least are zero of ψ, but it can have more than one zero -- $\psi(x) = \cos 2x$ is a solution of (ii) having zeros at $2x = \pi/2, 3\pi/2$ i.e. at $x = \pi/4\ \&\ 3\pi/4,$ both of then lie between 0 and π.

Proof. Given φ & ψ be solution of (i) & (ii) respectively.

Thus,

$$(p\varphi')' + r_1\varphi = 0 \qquad \text{(iii)}$$

and

$$(p\psi')' + r_2\psi = 0 \qquad \text{(iv)}$$

Let x_1 & x_2 be two consecutive zeros of $\varphi \Rightarrow \varphi(x_1) = 0$ & $\varphi(x_2) = 0$ & $\varphi(x) \neq 0$ for $x \in (x_1, x_2)$. then due to continuity of φ we can assume without loss of generality (*wlog*) that $\varphi(x) > 0\ on\ (x_1, x_2) \Rightarrow \varphi'(x_1) > 0$ & $\varphi'(x_2) < 0.$

If possible, let $\psi(x)$ have no zero on $(x_1, x_2),$ then its continuity implies that it does not change its sign on (x_1, x_2), hence, *wlog*, $\psi > 0\ on\ (x_1, x_2).$

Now multiply (iii) with ψ & (iv) with φ.

Integrate between x_1 & x_2 after subtracting the resultant equations:

$$\int_{x_1}^{x_2} \left\{[\psi(p\varphi')' - \varphi(p\psi')'] + (r_1 - r_2)\varphi\psi\right\} dx = 0$$

$$\Rightarrow \int_{x_1}^{x_2} (r_2 - r_1)\varphi\psi dx = -\int_{x_1}^{x_2} \frac{d}{dx}[pW(\varphi, \psi)]dx$$

Integrand on LHS is positive, so this integral is positive.

However, RHS $= -p(x_2)W(\varphi,\psi)\Big|_{x_2} + p(x_1)W(\varphi,\psi)\Big|_{x_1}$

$$= \underbrace{-p(x_2)[-\psi(x_2)\varphi'(x_2)+\psi(x_1)\varphi'(x_1)]}_{(-)ve\ quantity}$$

Thus LHS > 0 and RHS < 0 which is a contradiction.
Thus, ψ must have a zero between x_1 & x_2 .

Cor. If $r_1 \equiv r_2$ & φ,ψ are linear independent solution of $(py')' + r_1(x)y = 0,$ then zero of φ & ψ separate each other, *i.e.*, between any two consecutive zeros of φ, there is a zero of ψ and *vice versa*.

In case $r_1 \equiv r_2$, then LHS equals zero. And the above proof will still work as one side is zero and other side is negative. Further the role of ψ and φ can be interchanged., so the result.