The LNM Institute of Information Technology
Jaipur, Rajsthan

MATH-I ■ Solutions Assignment #4

(Continuity, Intermediate Value Property, Derivatives)

Q1. Determine the points of continuity for the function $f : [0,1] \to [0,1]$ defined by

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational.} \end{cases}$$

Ans. f is discontinuous everywhere.
For, if x is a rational point, then we can find a sequence of irrationals (x_n) converging to x. However, $(f(x_n)) \to 1 \neq f(x) = 0$. Similarly, f is not continuous at any irrational point.

Q2. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and let $c \in \mathbb{R}$. Show that if $x_0 \in \mathbb{R}$ is such that $f(x_0) > c$, then there exists a $\delta > 0$ such that $f(x) > c$ for all $x \in (x_0 - \delta, x_0 + \delta)$.

Ans. Since $f(x_0) - c > 0$, choose ϵ such that $0 < \epsilon < f(x_0) - c$. Since, f is continuous at x_0, for this choice of ϵ, there exists a $\delta > 0$, such that $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$. Hence, for all $x \in (x_0 - \delta, x_0 + \delta)$ $f(x) > f(x_0) - \epsilon > c$.

Q3. Prove that if a continuous function $f : \mathbb{R} \to \mathbb{R}$ is also one-one then either f is a strictly increasing function or f is a strictly decreasing function.

Ans. Since f is $1-1$, if $x \neq y$, $f(x) \neq f(y)$. Assume $x < y$ and $f(x) < f(y)$. Let $c \in (x, y)$. We claim that $f(c) \in (f(x), f(y))$.
Clearly, $f(c) \neq f(x), f(y)$. If possible, assume that $f(x) > f(c)$. Then, $\frac{f(x)+f(c)}{2}$ lies in both the intervals $(f(c), f(x))$ and $(f(c), f(y))$. By the intermediate value theorem, we can find, $x_1 \in (x, c)$ and $x_2 \in (c, y)$ such that $f(x_1) = \frac{f(x)+f(c)}{2}$ and $f(x_2) = \frac{f(x)+f(c)}{2}$. Here, $x_1 \neq x_2$, but $f(x_1) = f(x_2)$, a contradiction. Thus $f(x) < f(c)$.

Q4. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function which takes only rational values. Show that f is a constant function.

Ans. Suppose $f(x) \neq f(y)$ for some $x, y \in \mathbb{R}$. Let us choose an irrational number α between $f(x)$ and $f(y)$. Since f is continuous by IVP, there exists $z \in (x, y)$ such that $f(z) = \alpha$ which is a contradiction (as f takes only rational values).

Q5. Show that the polynomial $x^4 + 2x^3 - 9$ has at least two real roots.

Ans. Let $p(x) = x^4 + 2x^3 - 9$. Then $p(0) = -9$ and $p(x) \to \infty$ as $x \to \pm\infty$. Therefore, by intermediate value theorem there exist two real roots say $a > 0, b < 0$ of $p(x)$. Note that $p'(x) = 4x^3 + 6x^2 = 2x^2(2x+3)$ and $p\left(-\frac{3}{2}\right) =\neq 0$. a, b are simple roots of p. Since complex roots occur in pair, if p has three real roots, it will have all four as real roots. Also none of them is a repeated root. Therefore, p' must vanish at three distinct points, which is not true. Hence p has exactly two real roots.

Q6. Let $f : [1,3] \to \mathbb{R}$ be a continuous function. Prove that there exist real numbers $x_1, x_2 \in [1,3]$ such that

$$x_2 - x_1 = 1 \quad \text{and} \quad f(x_2) - f(x_1) = \frac{1}{2}(f(3) - f(1)).$$

Ans. Define $g(x) = f(x+1) - f(x) - \frac{1}{2}(f(3) - f(1))$. Then $g(1) = -g(2)$. By the intermediate value property, there exists $c \in (1,2)$ such that $g(c) = 0$ i.e. $f(c+1) - f(c) = \frac{1}{2}(f(3) - f(1))$. Here $x_1 = c$ and $x_2 = c+1$.

Q7. Show that the function $f(x) = x|x|$ is differentiable at 0. More generally, if f is continuous at 0, then $g(x) = xf(x)$ is differentiable at 0.

Ans. Easy. Use definition of the derivative of a function.

Q8. Check the function $f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$ for differentiability.

Ans. $f'(0)$ does not exist as $\displaystyle\lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \sin \frac{1}{h}$, which doesn't exist.

Q9. Show that the function $f(x) = \begin{cases} x^2 \sin \frac{1}{x^2}, & x \neq 0 \\ 0, & x = 0. \end{cases}$ is differentiable at all $x \in \mathbb{R}$. Also show that the function $f'(x)$ is not bounded on the interval $[-1,1]$. From this deduce that $f'(x)$ is not continuous at $x = 0$. Thus, a function that is differentiable at every point of $\mathbb{R}$ need not have a continuous derivative $f'(x)$.

Ans. Note that $f'(0) = \displaystyle\lim_{h \to 0} h \sin \frac{1}{h^2} = 0$. Therefore, f is differentiable at 0. At any other point f is differentiable being the product of two differentiable functions. Hence f is differentiable for all real x.

We have $f'(x) = \begin{cases} 2x \sin \frac{1}{x^2} - \frac{2}{x} \cos \frac{1}{x}, & x \neq 0 \\ 0, & x = 0. \end{cases}$

Here $2x \sin \frac{1}{x^2}$ is bounded in $[-1,1]$ but $\frac{2}{x} \cos \frac{1}{x^2}$ is not bounded in any interval containing 0. Hence $f'(x)$ is not bounded on $[-1,1]$ and so it can not be continuous at $[-1,1]$.

Q10. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is an even function $[(f(-x) = f(x)$ for all $x \in \mathbb{R}]$ and has a derivative at every point, then the derivative f' is an odd function $[(f(-x) = -f(x)$ for all $x \in \mathbb{R}]$.

Ans. $f'(-x) = \lim\limits_{h\to 0} \dfrac{f(-x+h) - f(-x)}{h} = \lim\limits_{x\to 0} \dfrac{f(x-h) - f(x)}{h} = -\lim\limits_{k\to 0} \dfrac{f(x+k) - f(x)}{k} = -f'(x)$.

i.e. the derivative of f is an odd function.