The LNM Institute of Information Technology
Jaipur, Rajasthan
MATH-II
Assignment #5

---

1. Let $u(x)$ be any nontrivial solution of $u'' + q(x)u = 0$ on a closed interval $[a, b]$. Show that $u(x)$ has at most a finite number of zeros in $[a, b]$.

2. Show that any nontrivial solution of $u'' + q(x)u = 0$, $q(x) < 0$ has at most one zero.

3. Let $u(x)$ be any nontrivial solution of $u'' + [1 + q(x)]u = 0$, where $q(x) > 0$. Show that $u(x)$ has infinitely many zeros.

4. The equation $y'' + y' - xy = 0$ has a power series solution of the form $\sum a_n x^n$.
   (i) Find the power series solutions $y_1(x)$ and $y_2(x)$ such that $y_1(0) = 1$, $y_1'(0) = 0$ and $y_2(0) = 0$, $y_2'(0) = 1$.
   (ii) Find the radius of convergence for $y_1(x)$ and $y_2(x)$.

5. Consider the differential equation $(1 + x^2)y'' - 4xy' + 6y = 0$.
   (i) Find its general solution in the form $y = a_0 y_1(x) + a_1 y_2(x)$, where y_1 and y_2 are power series.
   (ii) Find the radius of convergence for $y_1(x)$ and $y_2(x)$.

6. (a) Show that the fundamental system of solutions of Legendre equation

$$(1 - x^2)y'' - 2xy' + p(p + 1)y = 0$$

consists of $y_1(x) = \sum_{n=0}^{\infty} a_{2n} x^{2n}$ and $y_2(x) = \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1}$, where $a_0 = a_1 = 1$ and

$$a_{2n+2} = -\frac{(p - 2n)(p + 2n + 1)}{(2n + 1)(2n + 2)} a_{2n} \quad n = 0, 1, 2 \ldots$$
$$a_{2n+1} = -\frac{(p - 2n + 1)(p + 2n)}{2n(2n + 1)} a_{2n-1} \quad n = 1, 2, 3, \ldots$$

   (b) Verify that

$$y_1(x) = P_0(x) = 1, \quad y_2(x) = \frac{1}{2}\log\left\{\frac{1 + x}{1 - x}\right\} \quad \text{for } p = 0$$
$$y_2(x) = P_1(x) = x, \quad y_1(x) = 1 - \frac{x}{2}\log\left\{\frac{1 + x}{1 - x}\right\} \quad \text{for } p = 1.$$

   (c) The expression, $P_n(x) = \dfrac{1}{2^n n!}\dfrac{d^n}{dx^n}[(x^2 - 1)^n]$, is called the Rodrigues' formula for Legendre polynomial P_n of degree n. Assuming this, find P_1, P_2, P_3, P_4.

7. Using Rodrigues' formula for $P_n(x)$, show that
   (i) $P_n(-x) = (-1)^n P_n(x)$ *(ii)* $P_n'(-x) = (-1)^{n+1} P_n'(x)$
   (iii) $\int_{-1}^{1} P_n(x)P_m(x)dx = \dfrac{2}{2n + 1}\delta_{mn}$ *(iv)* $\int_{-1}^{1} x^m P_n(x)dx = 0$ if $m < n$.

8. Suppose $m > n$. Show that $\int_{-1}^{1} x^m P_n(x)dx = 0$ if $m - n$ is odd. What happens if $m - n$ is even?

9. The function on the left side of $\dfrac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n$ is called the generating function of the Legendre polynomial P_n. Using this relation, show that

$(i)\ (n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0$ $\quad (ii)\ nP_n(x) = xP_n'(x) - P_{n-1}'(x)$

$(iii)\ P_{n+1}'(x) - xP_n'(x) = (n+1)P_n(x)$ $\quad (iv)\ P_n(1) = 1,\ P_n(-1) = (-1)^n$

$(v)\ P_{2n+1}(0) = 0,\ P_{2n}(0) = (-1)^n \dfrac{1.3.5.\ldots(2n-1)}{2^n n!}$.