The LNM Institute of Information Technology
Jaipur, Rajsthan

MATH-I ■ Solutions Assignment #7

(Functions of several variables: Continuity, Differentiability, Directional derivatives, Maxima, Minima and Lagrange Multipliers)

Q1. Examine the following functions for continuity at the point $(0,0)$ where $f(0,0)=0$ and $f(x,y)$ for $(x,y)\neq(0,0)$ is given by
(a) $|x|+|y|$, (b) $\frac{-x}{\sqrt{x^2+y^2}}$, (c) $\frac{2x}{x^2+x+y^2}$, (d) $\frac{x^4-y^2}{x^4+y^2}$, (e) $\frac{x^4}{x^4+y^2}$.

Ans. (a) Given that $f(0,0)=0$. Let $\epsilon>0$ be given then $\big|(|x|+|y|)-0\big|=\big||x|+|y|\big|\leq |x|+|y|<\epsilon$, whenever $|x|<\delta=\epsilon/2$ and $|y|<\delta=\epsilon/2$. Therefore the function is continuous at (0,0).
Alternatively, the given function is continuous being the sum of two continuous functions.
(b) Let $y=mx$. Then $\displaystyle\lim_{(x,y)\to(0,0)}\frac{-x}{\sqrt{x^2+y^2}}=\frac{-1}{\sqrt{1+m^2}}$. Thus we get different limits for different values of m. Therefore, f is discontinuous at $(0,0)$.
(c) Let $x=r\cos\theta$, $y=r\sin\theta$. Then

$$\frac{2x}{x^2+x+y^2}=\frac{2r\cos\theta}{r^2\cos^2\theta+r\cos\theta+r^2\sin^2\theta}=\frac{2\cos\theta}{r+\cos\theta}.$$

Now, $\displaystyle\lim_{(x,y)\to(0,0)}\frac{2x}{x^2+x+y^2}=\lim_{r\to0}\frac{2\cos\theta}{r+\cos\theta}=2$. Therefore the function is continuous at (0,0).
(d) Let $y=mx^2$. Then $\displaystyle\lim_{(x,y)\to(0,0)}\frac{x^4-y^2}{x^4+y^2}=\frac{1-m^2}{1+m^2}$. Thus we get different limits for different values of m. Therefore, f is discontinuous at $(0,0)$.
(e) Let $y=mx^2$. Then $\displaystyle\lim_{(x,y)\to(0,0)}\frac{x^2y}{x^4+y^2}=\frac{m}{1+m^2}$. Thus we get different limits for different values of m. Therefore, f is discontinuous at $(0,0)$.

Q2. Consider the function $f:\mathbb{R}^2\to\mathbb{R}$ defined by

$$f(x,y)=\begin{cases}1, & \text{if } x=0 \text{ or if } y=0\\ 0, & \text{otherwise.}\end{cases}$$

Show that the function satisfy the following:
(a) The iterated limits $\displaystyle\lim_{x\to0}\left[\lim_{y\to0}f(x,y)\right]$ and $\displaystyle\lim_{y\to0}\left[\lim_{x\to0}f(x,y)\right]$ exist and equals 0,
(b) $\displaystyle\lim_{(x,y)\to(0,0)}f(x,y)$ does not exist,
(c) $f(x,y)$ is not continuous at $(0,0)$,
(d) the partial derivatives exist at $(0,0)$.

Ans. (a) Let $x \neq 0$, then $\lim_{y \to 0} f(x,y) = 0$.

Similarly, if $y \neq 0$, then $\lim_{x \to 0} f(x,y) = 0$.

Therefore, $\lim_{x \to 0}\left[\lim_{y \to 0} f(x,y)\right] = \lim_{y \to 0}\left[\lim_{x \to 0} f(x,y)\right] = 0$.

(b) Along the line $x = 0$, we have $\lim_{(x,y) \to (0,0)} f(x,y) = 1$.

Along the line $y = x$, we have $\lim_{(x,y) \to (0,0)} f(x,y) = 0$.

Hence, the limit does not exist.

(c) From above, the function is not continuous.

(d) Easy. Leave it to the students.

Q3. Let

$$f(x,y) = \begin{cases} xy\left(\frac{x^2-y^2}{x^2+y^2}\right), & \text{if} \quad (x,y) \neq (0,0) \\ 0, & \text{if} \quad (x,y) = (0,0). \end{cases}$$

Prove that

(a) $f_x(0,y) = -y$ and $f_y(x,0) = x$ for all x and y,

(b) $f_{xy}(0,0) = -1$ and $f_{yx}(0,0) = 1$ and

(c) $f(x,y)$ is differentiable at $(0,0)$.

Ans. ***Discussed in the class.***

(a) Note that

$$f_x(0,k) = \lim_{h \to 0} \frac{f(0+h,k) - f(0,k)}{h} = \lim_{h \to 0} \frac{f(h,k) - f(0,k)}{h} = \lim_{h \to 0} k\left(\frac{h^2-k^2}{h^2+k^2}\right) = -k.$$

Thus $f_x(0,y) = -y$.

Similarly, $f_y(x,0) = x$.

(b) Note that $f_{xy}(0,0) = \lim_{h \to 0} \dfrac{f(h,0) - f(0,0)}{h} = -1$ and $f_{xy}(0,0) = 1$.

(c) We need to show that

$$f(\Delta x, \Delta y) - f(0,0) = f_x(0,0)\Delta x + f_y(0,0)\Delta y + \epsilon_1(\Delta x, \Delta y)\Delta x + \epsilon_2(\Delta x, \Delta y)\Delta y,$$

where $\epsilon_1, \epsilon_2 \to 0$ as Δx and $\Delta y \to 0$. Since $f_x(0,0) = 0$ and $f_y(0,0) = 0$ we will show that

$$f(\Delta x, \Delta y) - f(0,0) = \epsilon_1(\Delta x, \Delta y)\Delta x + \epsilon_2(\Delta x, \Delta y)\Delta y.$$

Now $f(\Delta x, \Delta y) - f(0,0) = f(\Delta x, \Delta y) = \Delta x \Delta y\left(\frac{(\Delta x)^2-(\Delta y)^2}{(\Delta x)^2+(\Delta y)^2}\right) = \frac{(\Delta x)^2\Delta y}{(\Delta x)^2+(\Delta y)^2}\Delta x - \frac{\Delta x(\Delta y)^2}{(\Delta x)^2+(\Delta y)^2}\Delta y = \epsilon_1 \Delta x + \epsilon_2 \Delta y$.

Here $\epsilon_1(\Delta x, \Delta y) = \frac{(\Delta x)^2\Delta y}{(\Delta x)^2+(\Delta y)^2}$, $\epsilon_2(\Delta x, \Delta y) = \frac{\Delta x(\Delta y)^2}{(\Delta x)^2+(\Delta y)^2} \to 0$ as Δx and $\Delta y \to 0$.

So f is differentiable at $(0,0)$.

Q4. Suppose f is a function with $f_x(x,y) = f_y(x,y) = 0$ for all (x,y). Then show that f is constant.

Ans. This follows immediately from the MVT for functions of several variables.

Q5. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \frac{y^3}{x^2+y^2}, & \text{if } (x,y) \neq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Show that f is continuous at $(0,0)$, it has all directional derivatives at $(0,0)$ but not differentiable at $(0,0)$.

Ans. Using polar coordinates, we see that

$$\frac{y^3}{x^2+y^2} = \frac{r^3 \sin^3\theta}{r^2} = r\sin^3\theta.$$

Therefore,

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{y^3}{x^2+y^2} = \lim_{r\to 0} r\sin^3\theta = 0.$$

$\implies f$ is continuous at $(0,0)$.
Let $U = (u_1, u_2)$ be a unit vector. Now $D_{(0,0)}f(U) = \lim_{t\to 0} \frac{f((0,0)+t(u_1,u_2)) - f(0,0)}{t} = \lim_{t\to 0} \frac{f(tu_1, tu_2)}{t} = 0$. Therefore directional derivatives in all directions exist.
Note that $f_x(0,0) = 0$ and $f_y(0,0) = 1$. If f is differentiable at $(0,0)$ then $f'(0,0) = (0,1)$. Now

$$\begin{aligned} f(\Delta x, \Delta y) - f(0,0) = f(\Delta x, \Delta y) &= \Delta x + \frac{(\Delta y)^3}{(\Delta x)^2 + (\Delta y)^2} - \Delta x \\ &= \Delta x + \frac{(\Delta y)^2}{(\Delta x)^2 + (\Delta y)^2}\Delta y - \Delta x. \end{aligned}$$

Here $\epsilon_1 = -1$, $\epsilon_2 = \frac{(\Delta y)^2}{(\Delta x)^2+(\Delta y)^2} \nrightarrow 0$ as $\Delta x, \Delta y \to 0$.
Therefore the function is not differentiable at $(0,0)$.

Q6. Examine the following functions for local maxima, local minima and saddle points: $(i)\ 4xy - x^4 - y^4$, $\quad (ii)\ x^3 - 3xy$, $\quad (iii)\ (x^2+y^2)\exp^{-(x^2+y^2)}$.

Ans. (i) For $f(x,y) = 4xy - x^4 - y^4$, $f_x(x_0,y_0) = f_y(x_0,y_0) = 0$ for $(x_0,y_0) = (0,0), (1,1)$ or $(-1,-1)$. These are the critical points. By second derivative test, $(0,0)$ ia a saddle point and $(-1,1)$ and $(1,1)$ are local maxima.

(*ii*) $f(x, y) = x^3 - 3xy^2$, $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$ for $(x_0, y_0) = (0, 0)$. So $(0, 0)$ is the only critical point. Second derivative fails here. Along $y = 0$, $f(x, y) = x^3$, hence $(0, 0)$ is a saddle point.
(*iii*) Similar, leave it to the students as an exercise.

Q7. Let $f(x, y) = 3x^4 - 4x^2y + y^2$. Show that f has a local minimum at $(0, 0)$ along every line through $(0, 0)$. Does f have a minimum at $(0, 0)$? Is $(0, 0)$ a saddle point for f?

Ans. Let $f(x, y) = 3x^4 - 4x^2y + y^2$. Along, the $x-$axis, the local minimum of the function is at $(0, 0)$. Let $x = r\cos\theta$ and $y = r\sin\theta$, for a fixed $\theta \neq 0, \pi$ (or let $y = mx$). Then, $f(r\cos\theta, r\sin\theta) = 3r^4\sin^4\theta - 4r^3\cos 2\theta\sin\theta + r^2\sin^2\theta$ which is function of one variable. By the second derivative test (for functions of one variable), we see that $(0, 0)$ is a local minima.
Since $f(x, y) = (3x^2 - y)(x^2 - y)$, we see that in the region between the parabolas $3x^2 = y$ and $y = x^2$, the function takes negative values and is positive everywhere else. Thus, $(0, 0)$ is a saddle point for f.

Q8. Find the absolute maxima of $f(x, y) = xy$ on the unit disc $\{f(x, y) : x^2 + y^2 \leq 1\}$.

Ans. Given that $f(x, y) = xy$. Clearly, f is differentiable so f can assume extreme values at the points where $f_x = 0$, $f_y = 0$ and boundary points on the disk.
$f_x = 0$, $f_y = 0 \Longrightarrow (x, y) = (0, 0)$. The value of f at $(0, 0)$ is $f = 0$.
On the boundary of the disk we have $f(x, y) = g(x) = x\sqrt{1 - x^2}$, $-1 \leq x \leq 1$. For maxima/minima we have $g'(x) = 0$. This gives $x = \pm\frac{1}{\sqrt{2}}$ and for this value of x, we have $y = \pm\frac{1}{\sqrt{2}}$. Moreover, $g''(x) = -\frac{1}{2} < 0$ at $x = \frac{1}{\sqrt{2}}$ and $g''(x) = \frac{1}{2}$ at $x = -\frac{1}{\sqrt{2}}$. Therefore, the function function $f(x, y)$ takes values $-\frac{1}{2}$ and $\frac{1}{2}$ at the points $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ and $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ respectively.
Thus all maxima/minima for f are $-\frac{1}{2}, 0, \frac{1}{2}$. Hence, the maximum of $f(x, y)$ is $\frac{1}{2}$ which occur at $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ and the minimum is $-\frac{1}{2}$ which occur at $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.

Q9. Find the equation of the surface generated by the normals to the surface $x + 2yz + xyz^2 = 0$ at all points on the $z-$axis.

Ans. $f(x, y, z) = x + 2yz + xyz^2 = 0$. Any point P_0 on the $z-$axis is of the form $(0, 0, z_0)$. The gradient is

$$\nabla f|_{P_0} = ((1 + yz^2)\overrightarrow{i} + (2z + xz^2)\overrightarrow{j} + 2(y + xyz)\overrightarrow{k})_{(0,0,z_0)} = \overrightarrow{i} + 2z_0\overrightarrow{j}.$$

Equation of the normal lines is given by

$$\frac{x - 0}{1} = \frac{y - 0}{2z_0} = \frac{z - z_0}{0}$$

Solving, we get

$$y = 2xz_0, z = z_0.$$

Eliminating z_0, we get equation of the surface as

$$2xz - y = 0.$$

Q10. Given n positive numbers $a_1, a_2, \ldots, a_n$, find the maximum value of the expression the function $a_1x_1 + a_2x_2 + \ldots + a_nx_n$ where $x_1^2 + x_2^2 + \ldots + x_n^2 = 1$.

Ans. Note that here $f(x_1, x_2, \ldots, x_n) = a_1x_1 + a_2x_2 + \ldots + a_nx_n$ and $g(x_1, x_2, \ldots, x_n) = x_1^2 + x_2^2 + \ldots + x_n^2 - 1$.
Using the method of lagrange multipliers let λ be such that $\nabla f = \lambda \nabla g$. This gives,

$$a_1 = \lambda x_1, a_2 = \lambda x_2, \ldots, a_n = \lambda x_n \qquad \text{and} \qquad x_1^2 + x_2^2 + \ldots + x_n^2 - 1 = 0.$$

Therefore, $a_1^2 + a_2^2 + \ldots + a_n^2 = 4\lambda^2$. This gives $\lambda = \pm\frac{\sqrt{a_1^2+a_2^2+\ldots+a_n^2}}{2}$. Since the continuous function f achieves its minimum and maximum on the closed and bounded set $x_1^2 + x_2^2 + \ldots + x_n^2 = 1$, $\lambda = \pm\frac{\sqrt{a_1^2+a_2^2+\ldots+a_n^2}}{2}$ leads to the maximum value $f\left(\frac{a_1}{2\lambda}, \frac{a_2}{2\lambda}, \ldots, \frac{a_n}{2\lambda}\right) = \sqrt{a_1^2 + a_2^2 + \ldots + a_n^2}$ and $\lambda = \pm\frac{\sqrt{a_1^2+a_2^2+\ldots+a_n^2}}{2}$ leads to the minimum value of f.

Q11. Assume that among all rectangular boxes with fixed surface area of 20 square meters, there is a box of largest possible volume. Find its dimensions.

Ans. Let the box have sides of length $x, y, z > 0$. Then $V(x, y, z) = xyz$ and $xy+yz+xz = 10$. Using the method of lagrange multipliers, we see that $yz = \lambda(y+z), xz = \lambda(x+z)$ and $xy = \lambda(x+y)$. It is easy to see that $x, y, z > 0$. Now, we can see that $x = y = z$ and therefore, $x = y = z = \sqrt{\frac{10}{3}}$.

Q12. Minimize the function $x^2 + y^2 + z^2$ subject to the constraints $x + 2y + 3z = 6$ and $x + 3y + 9z = 9$.

Ans. Let $F(x, y, z) = x^2 + y^2 + z^2, g(x, y, z) = x + 2y + 3z$ and $h(x, y, z) = x + 3y + 9z$, where $x + 2y + 3z = 6$ and $x + 3y + 9z = 9$.
Using the method of lagrange multipliers let λ and μ be such that $\nabla F = \lambda\nabla h + \mu\nabla g$. We get

$$\lambda + \mu = 2x, 2\lambda + 3\mu = 2y \quad \text{and} \quad 3\lambda + 9\mu = 2z. \tag{1}$$

From here, using $x + 2y + 3z = 6$ and $x + 3y + 9z = 9$, we get $7\lambda + 17\mu = 6$ and $34\lambda + 91\mu = 18$.
Hence, $\mu = -\frac{78}{59}$ and $\lambda = \frac{240}{59}$.
From equation (1), we get $2(x^2 + y^2 + z^2) = 6\lambda + 9\mu$, hence the minimum value of of f is $\frac{369}{59}$.