The LNM Institute of Information Technology
Jaipur, Rajasthan

MATHEMATICS-I November 18, 2015

End Term Exam (Part-2) Time: 3 Hours, Maximum Marks: 50

Note: You should attempt all questions. Your writing should be legible and neat. Make an index showing the question number and page number on the front page of your answer sheet in the following format, otherwise you may be penalized by the deduction of **2 marks**.

Question No.				
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1. (a) Let S be a nonempty subset of $\mathbb{R}$ and $\beta \in \mathbb{R}$. If $\beta = \inf S$, then show that for every $\epsilon > 0$, there is some $x \in S$ such that $\beta + \epsilon > x$. **[2 marks]**

   (b) Let S be a nonempty subset of $\mathbb{R}$ and $\beta \in \mathbb{R}$ be a lower bound. If for every $\epsilon > 0$, there is some $x \in S$ such that $\beta + \epsilon > x$ then show that $\beta = \inf S$ **[3 marks]**

   (c) Let (a_k) be a real sequence such that (a_{2k}) and (a_{2k-1}) diverges to $+\infty$. Then Show that (a_k) diverges to $+\infty$. **[5 marks]**

2. (a) For $k \in \mathbb{N}$, let $a_{2k-1} := \dfrac{1}{4^k}$ and $a_{2k} := \dfrac{1}{9^k}$. Discuss the convergence/divergence of the series $\displaystyle\sum_{k=1}^{\infty} a_k$. **[3 marks]**

   (b) Let D be a non-empty subset of $\mathbb{R}$ and $f : D \to \mathbb{R}$ be uniformly continuous. Prove that f is continuous on D. **[2 marks]**

   (c) Let r be a nonnegative rational number and consider $a_n := \displaystyle\sum_{k=1}^{n} \frac{k^r}{n^{r+1}}$ for $n \in \mathbb{N}$. Determine the limit of the sequence (a_n) by expressing the nth term as a Riemann sum for a suitable function. **[5 marks]**

3. (a) *Viviani's curve* is the intersection of the cylinder $(x-\alpha)^2 + y^2 = \alpha^2$ and the sphere $x^2 + y^2 + z^2 = 4\alpha^2$ and has parametric equation: **[5 Marks]**

$$\alpha : [0, 4\pi] \longrightarrow \mathbb{R}^3 : t \mapsto \alpha\left(1 + \cos t, \sin t, 2\sin\frac{t}{2}\right).$$

   Show that the curvature and torsion of this curve are given by

$$\kappa(t) = \frac{\sqrt{13 + 3\cos t}}{\alpha(3 + \cos t)^{3/2}}, \quad \tau(t) = \frac{6\cos\frac{t}{2}}{\alpha(13 + 3\cos t)}.$$

   (b) Let $f(x, y, z) = x^2 + y^2 + z^2$. Prove f is differentiable at $(1, 1, 1)$ with linear transformation $T(x, y, z) = 2x + 2y + 2z$ as its derivative. **[3 Marks]**

   (c) A mosquito is flying around a room in which the temperature is given by $T(x, y, z) = x^2 + y^4 + 2z^2$. The mosquito is at the point $(1, 1, 1)$ and realizes that he's cold. In what direction should he fly to warm up most quickly? **[2 Marks]**

4. (*a*) Many airlines require that carry-on luggage have a linear distance (sum of length, width, height) of no more than 45 inches with an additional requirement of being able to slide under the seat in front of you. Assuming that the carry-on is to have the shape of a rectangular box and one dimension is half of one of the other dimensions (to insure "slide under seat" is possible), what dimensions of the carry-on will lead to maximum storage (i.e., maximum volume)? **[6 Marks]**
[Hint: *Apply Lagrange's multiplier method with more than one constraints*]

(*b*) Suppose S is a "*light-bulb-shaped region*" shown in Figure 1. Imagine a light-bulb cut

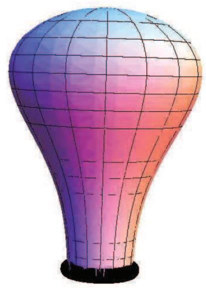

Figure 1: Bulb-shaped region S

off at the base so that its boundary is the unit circle $x^2 + y^2 = 1$, oriented with the outward-pointing normal. Suppose $F = (xe^{z^2-2z}, 1+y+\sin(xyz), e^{z^2}\sin(z^2))$. Compute the flux integral $\int\int_S curl\, F \cdot d\sigma$ using Stokes' theorem. **[4 Marks]**

5. (*a*) Compute the integral

$$\int_C \left(5y + \sqrt{1+x^5}\right) dx + \left(5x - e^{y^2}\right) dy,$$

where C is a circle of radius 5 centered at the origin. **[3 Marks]**

(*b*) In this problem S is the surface given by the quarter of the right-circular cylinder centered on the $z-$axis, of radius 2 and height 4, which lies in the first octant. A vector field $F(x,y,z) = y\hat{j}$ is defined on S.
(*i*) Compute the flux integral $\int\int_S F \cdot \hat{n}\, d\sigma$.
(*Hint: Use the normal which points outward from S, i.e. on the side away from the $z-$axis.*)
(*ii*) D be the solid in the first octant given by the interior of the quarter cylinder defined above. Use the divergence theorem to compute the flux of the field $F = y\hat{j}$ out of the solid D. **[4+3 Marks]**

✠ **End of paper** ✠