Closures of Relations

Chapter 10
22c:19
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Relational closures

- Three types we will study
  - Reflexive
    - Easy
  - Symmetric
    - Easy
  - Transitive
    - Hard

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Reflexive closure

- Consider a relation R. We want to add edges to make the relation reflexive.
- $R' = R \cup \{ (x, y) \}$
- With matrices, we set the diagonal to all 1's
- By adding those edges, we have made a non-reflexive relation R into a reflexive relation
- This new relation R' is called the **reflexive closure** of R

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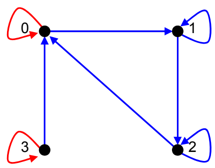
Reflexive closure example

- Let R be a relation on the set { 0, 1, 2, 3 } containing the ordered pairs (0,1), (1,1), (1,2), (2,0), (2,2), and (3,0)
- What is the reflexive closure of R?
- We add all pairs of edges (a,a) that do not already exist



We add edges:
(0,0), (3,3)

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Symmetric closure

- In order to find the symmetric closure of a relation R, we add an edge from a to b, where there is already an edge from b to a
- The symmetric closure of R is $R \cup R^{-1}$
  - If $R = \{ (a,b) \mid \ldots \}$
  - Then $R^{-1} = \{ (b,a) \mid \ldots \}$

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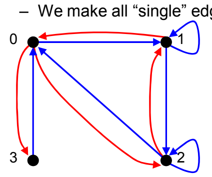
Symmetric closure example

- Let R be a relation on the set { 0, 1, 2, 3 } containing the ordered pairs (0,1), (1,1), (1,2), (2,0), (2,2), and (3,0)
- What is the symmetric closure of R?
- We add all pairs of edges (a,b) where (b,a) exists
  - We make all "single" edges into anti-parallel pairs



We add edges:
(0,2), (0,3)
(1,0), (2,1)

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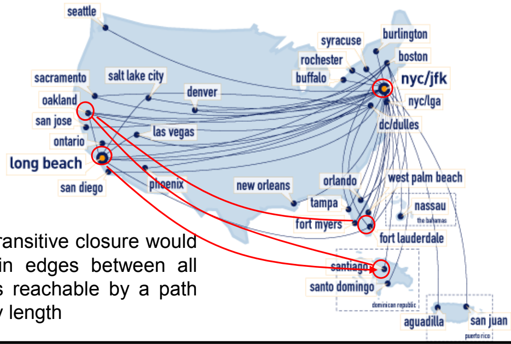
Transitive closure

Given the flight relation between cities, what's the relation "reachable by air" between cities?

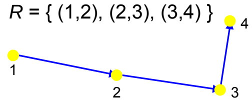


The transitive closure would contain edges between all nodes reachable by a path of any length

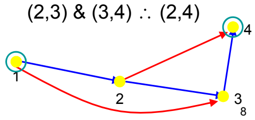
Transitive closure

- Informal definition: If there is a path from *a* to *b*, then there should be an edge from *a* to *b* in the transitive closure
- First take of a definition:
  - In order to find the transitive closure of a relation *R*, we add an edge from *a* to *c*, when there are edges from a to *b* and *b* to *c*

$R = \{ (1,2), (2,3), (3,4) \}$



(1,2) & (2,3) ∴ (1,3)
(2,3) & (3,4) ∴ (2,4)



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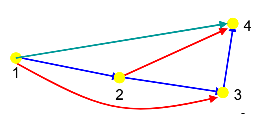
Transitive closure

- Informal definition: If there is a path from *a* to *b*, then there should be an edge from *a* to *b* in the transitive closure
- Second take of a definition:
  - In order to find the transitive closure of a relation *R*, we add an edge from *a* to *c*, when there are edges from a to *b* and *b* to *c*
  - Repeat this step until no new edges are added to the relation
- We will study different algorithms for determining the transitive closure
- red means added on the first repeat
- teal means added on the second repeat



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6 degrees of separation

- The idea that everybody in the world is connected by six degrees of separation
  - Where 1 degree of separation means you know (or have met) somebody else
- Let R be a relation on the set of all people in the world
  - $(a,b) \in R$ if person a has met person b
- So six degrees of separation for *any* two people a and g means:
  - (a,b), (b,c), (c,d), (d,e), (e,f), (f,g) are all in R
- Or, for any a and g, $(a,g) \in R^6$

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Connectivity relation

- R contains edges between all the nodes reachable via 1 edge
- $R \circ R = R^2$ contains edges between nodes that are reachable via 2 edges or less in R (a path of length 2 or less)
- $R^2 \circ R = R^3$ contains edges between nodes that are reachable via 3 edges or less in R (a path of length 3 or less)
- R^n = contains edges between nodes that are reachable via n edges or less in R (a path of length n or less)

- R^* contains edges between nodes that are reachable via any number of edges (i.e. via any path) in R
  - Rephrased: R^* contains all the edges between nodes a and b when is a path of length at least 1 between a and b in R

- R^* is the transitive closure of R
  - The definition of a transitive closure is that there are edges between any nodes (a,b) whenever there is a path from a to b.

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How long are the paths in a transitive closure?

- Let R be a relation on set A, where $|A| = n$
  - Rephrased: consider a graph G with n nodes and some number of edges
- Lemma 1: If there is a path from a to b in R, then there is a path between a and b of length $< n$.

- Proof preparation:
  - Suppose there is a path from a to b in R
  - Let the length of that path be m
  - Let the path be edges (x_0, x_1), (x_1, x_2), …, (x_{m-1}, x_m)
  - That's m+1 nodes $x_0, x_1, x_2, \ldots, x_{m-1}, x_m$
  - If a node exists twice in our path, then it's not a shortest path
    - As we made no progress in our path between the two occurrences of the repeated node
  - Thus, each node may exist at most once in the path, m+1 <= n.

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How long are the paths in a transitive closure?

- Proof by contradiction:
  - Assume that the shortest path from a to b is at least n, i.e., $m >= n$
  - There are m+1 >= n+1 nodes in the path.
  - By the pigeonhole principle, there must be at least one node in the graph that has two occurrences in the path
    - Not possible, as the path would not be the shortest path
  - Thus, it cannot be the case that $m >= n$

- If there exists a path from a to b, then there is a path from a to b of at most length n -1.

Finding the transitive closure

- Let $\mathbf{M}_R$ be the zero-one matrix of the relation R on a set with n elements. Then the zero-one matrix of the transitive closure R^* is:

$$\mathbf{M}_{R^*} = \mathbf{M}_R \vee \mathbf{M}_R^{[2]} \vee \mathbf{M}_R^{[3]} \vee \cdots \vee \mathbf{M}_R^{[n-1]}$$

Nodes reachable with one application of the relation

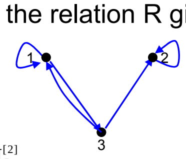
Nodes reachable with two applications of the relation

Nodes reachable with n-1 applications of the relation

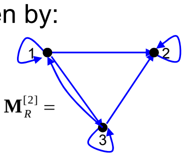
Sample questions

- Find the zero-one matrix of the transitive closure of the relation R given by:

$$\mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$



$$\mathbf{M}_R^{[2]} =$$



$$\mathbf{M}_{R^*} = \mathbf{M}_R \vee \mathbf{M}_R^{[2]}$$

$$\mathbf{M}_R^{[2]} = \mathbf{M}_R \odot \mathbf{M}_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \odot \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Transitive closure algorithm

- What we did (or rather, could have done):
  - Compute the next matrix $\mathbf{M}_R^{[i]}$, where $1 \le i < n$
  - Do a Boolean join with the previously computed matrix
- For our example:
  - Compute $\mathbf{M}_R^{[2]} = \mathbf{M}_R \circ \mathbf{M}_R$
  - Join that with $\mathbf{M}_R$ to yield $\mathbf{M}_R \vee \mathbf{M}_R^{[2]}$
  - Compute $\mathbf{M}_R^{[3]} = \mathbf{M}_R^{[2]} \circ \mathbf{M}_R$
  - Join that with $\mathbf{M}_R \vee \mathbf{M}_R^{[2]}$ from above

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Transitive closure algorithm

procedure *transitive_closure* ($\mathbf{M}_R$: zero-one $n \times n$ matrix)
  A := $\mathbf{M}_R$
  B := **A**
  for i := 2 **to** n - 1
  begin
    A := **A** $\odot$ $\mathbf{M}_R$
    B := **B** $\vee$ **A**
  end { **B** is the zero-one matrix for R^* }

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Transitive closure algorithms

- More efficient algorithms exist, such as Warshall's algorithm
  - We won't be studying it in this class

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Equivalence vs Partial Order

- Certain combinations of relation properties are very useful
- Equivalence relations
  - A relation that is reflexive, symmetric and transitive
- Partial orderings
  - A relation that is reflexive, antisymmetric, and transitive
- The difference is whether the relation is symmetric or antisymmetric

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Equivalence relations

- A relation on a set A is called an *equivalence relation* if it is reflexive, symmetric, and transitive
- Consider relation $R = \{ (a,b) \mid len(a) = len(b) \}$
  - Where $len(a)$ means the length of string a
  - It is reflexive: $len(a) = len(a)$
  - It is symmetric: if $len(a) = len(b)$, then $len(b) = len(a)$
  - It is transitive: if $len(a) = len(b)$ and $len(b) = len(c)$, then $len(a) = len(c)$
  - Thus, R is an equivalence relation

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Equivalence relation example

- Consider the relation $R = \{ (a,b) \mid m \mid a-b \}$
  - Called "congruence modulo m"
- Is it reflexive: $(a,a) \in R$ means that $m \mid a-a$
  - $a-a = 0$, which is divisible by m
- Is it symmetric: if $(a,b) \in R$ then $(b,a) \in R$
  - (a,b) means that $m \mid a-b$
  - Or that $km = a-b$. Negating that, we get $b-a = -km$
  - Thus, $m \mid b-a$, so $(b,a) \in R$
- Is it transitive: if $(a,b) \in R$ and $(b,c) \in R$ then $(a,c) \in R$
  - (a,b) means that $m \mid a-b$, or that $km = a-b$
  - (b,c) means that $m \mid b-c$, or that $lm = b-c$
  - (a,c) means that $m \mid a-c$, or that $nm = a-c$
  - Adding these two, we get $km+lm = (a-b) + (b-c)$
  - Or $(k+l)m = a-c$
  - Thus, m divides $a-c$, where $n = k+l$
- Thus, congruence modulo m is an equivalence relation

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Sample questions

- Which of these relations on {0, 1, 2, 3} are equivalence relations? Determine the properties of an equivalence relation that the others lack

a) { (0,0), (1,1), (2,2), (3,3) }
- Has all the properties, thus, is an equivalence relation

b) { (0,0), (0,2), (2,0), (2,2), (2,3), (3,2), (3,3) }
- Not reflexive: (1,1) is missing
- Not transitive: (0,2) and (2,3) are in the relation, but not (0,3)

c) { (0,0), (1,1), (1,2), (2,1), (2,2), (3,3) }
- Has all the properties, thus, is an equivalence relation

d) { (0,0), (1,1), (1,3), (2,2), (2,3), (3,1), (3,2) (3,3) }
- Not transitive: (1,3) and (3,2) are in the relation, but not (1,2)

e) { (0,0), (0,1) (0,2), (1,0), (1,1), (1,2), (2,0), (2,2), (3,3) }
- Not symmetric: (1,2) is present, but not (2,1)
- Not transitive: (2,0) and (0,1) are in the relation, but not (2,1)

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Sample questions

- Suppose that A is a non-empty set, and f is a function that has A as its domain. Let R be the relation on A consisting of all ordered pairs (x,y) where $f(x) = f(y)$
  - Meaning that x and y are related if and only if x and y have the same image under *f*.
- Show that R is an equivalence relation on A

- Reflexivity: $f(x) = f(x)$
  - True, as given the same input, a function always produces the same output
- Symmetry: if $f(x) = f(y)$ then $f(y) = f(x)$
  - True, by the definition of equality
- Transitivity: if $f(x) = f(y)$ and $f(y) = f(z)$ then $f(x) = f(z)$
  - True, by the definition of equality

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Sample questions

- Show that the relation R, consisting of all pairs (x,y) where x and y are bit strings of length three or more that agree except perhaps in their first three bits, is an equivalence relation on the set of all bit strings

- Let $f(x)$ = the bit string formed by deleting the first 3 bits of x, i.e., the last n-3 bits of the bit string x where n is the length of the string.

- Thus, we want to show: let R be the relation on A consisting of all ordered pairs (x,y) where $f(x) = f(y)$

- This has been shown in question 5 on the previous slide

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Equivalence classes

- Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the *equivalence class* of a.
- The equivalence class of a with respect to R is denoted by $[a]_R$
- When only one relation is under consideration, the subscript is often deleted, and $[a]$ is used to denote the equivalence class
- Note that these classes are disjoint!
  - As the equivalence relation is symmetric

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More on equivalence classes

- Consider the relation $R = \{ (a,b) \mid a \bmod 2 = b \bmod 2 \}$
  - Thus, all the even numbers are related to each other
  - As are the odd numbers
- The even numbers form an equivalence class
  - As do the odd numbers
- The equivalence class for the even numbers is denoted by [2] (or [4], or [784], etc.)
  - $[2] = \{ \ldots, -4, -2, 0, 2, 4, \ldots \}$
  - 2 is a *representative* of it's equivalence class
- There are only 2 equivalence classes formed by this equivalence relation

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More on equivalence classes

- Consider the relation $R = \{ (a,b) \mid a = b \text{ or } a = -b \}$
  - Thus, every number is related to additive inverse
- The equivalence class for an integer a:
  - $[7] = \{ 7, -7 \}$
  - $[0] = \{ 0 \}$
  - $[a] = \{ a, -a \}$
- There are an infinite number of equivalence classes formed by this equivalence relation

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Partitions

- Consider the relation R = { (a,b) | a mod 2 = b mod 2 }
- This splits the integers into two equivalence classes: even numbers and odd numbers
- Those two sets together form a partition of the integers
- Formally, a partition of a set S is a collection of non-empty mutually-disjoint subsets of S whose union is S
- In this example, the partition is { [0], [1] }
  - Or { {…, -3, -1, 1, 3, …}, {…, -4, -2, 0, 2, 4, …} }

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Sample questions

- Which of the following are partitions of the set of integers?

a) The set of even integers and the set of odd integers
   - Yes, it's a valid partition

b) The set of positive integers and the set of negative integers
   - No: 0 is in neither set

c) The set of integers divisible by 3, the set of integers leaving a remainder of 1 when divided by 3, and the set of integers leaving a remaineder of 2 when divided by 3
   - Yes, it's a valid partition

d) The set of integers less than -100, the set of integers with absolute value not exceeding 100, and the set of integers greater than 100
   - Yes, it's a valid partition

e) The set of integers not divisible by 3, the set of even integers, and the set of integers that leave a remainder of 3 when divided by 6
   - The first two sets are not disjoint (2 is in both), so it's not a valid partition

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Partial Ordering

- An equivalence relation is a relation that is reflexive, symmetric, and transitive
- A partial ordering (or partial order) is a relation that is reflexive, *anti*symmetric, and transitive
  - Recall that antisymmetric means that if (a,b) $\in$ R, then (b,a) $\notin$ R unless b = a
  - Thus, (a,a) is allowed to be in R
  - But since it's reflexive, all possible (a,a) must be in R
- A set S with a partial ordering R is called a *partially ordered set*, or *poset*
  - Denoted by (S,R)

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Partial ordering examples

- Show that ≥ is a partial order on the set of integers
  - It is reflexive: $a \geq a$ for all $a \in \mathbf{Z}$
  - It is antisymmetric: if $a \geq b$ then the only way that $b \geq a$ is when $b = a$
  - It is transitive: if $a \geq b$ and $b \geq c$, then $a \geq c$
- Note that ≥ is the partial ordering on the set of integers
- ($\mathbf{Z}$, ≥) is the partially ordered set, or poset

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Symbol usage

- The symbol $\lhd$ is used to represent *any* relation when discussing partial orders
  - Not just the less than or equals to relation
  - Can represent ≤, ≥, ⊆, etc
  - Thus, $a \lhd b$ denotes that $(a,b) \in R$
  - The poset is (S, $\lhd$)
- The symbol $\prec$ is used to denote $a \lhd b$ but $a \neq b$
  - If $\lhd$ represents ≥, then $\prec$ represents >

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Comparability

- The elements a and b of a poset (S, $\lhd$) are called *comparable* if either $a \lhd b$ or $b \lhd a$.
  - Meaning if $(a,b) \in R$ or $(b,a) \in R$
  - It can't be both because $\lhd$ is antisymmetric
    - Unless $a = b$, of course
  - If neither $a \lhd b$ nor $b \lhd a$, then a and b are *incomparable*
    - Meaning they are not related to each other
- If all elements in S are comparable, the relation is a *total ordering*

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Comparability examples

- Let $\lhd$ be the "divides" operator |
- In the poset ($\mathbf{Z}^+$,|), are the integers 3 and 9 comparable?
  - Yes, as 3 | 9
- Are 7 and 5 comparable?
  - No, as $7 \nmid 5$ and $5 \nmid 7$
- Thus, as there are pairs of elements in $\mathbf{Z}^+$ that are not comparable, the poset ($\mathbf{Z}^+$,|) is a partial order

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Comparability examples

- Let $\lhd$ be the relation $\leq$
- In the poset ($\mathbf{Z}^+$,$\leq$), are the integers 3 and 9 comparable?
  - Yes, as $3 \leq 9$
- Are 7 and 5 comparable?
  - Yes, as $5 \leq 7$
- As all pairs of elements in $\mathbf{Z}^+$ are comparable, the poset ($\mathbf{Z}^+$,$\leq$) is a total order
  - a.k.a. totally ordered poset, linear order, chain, etc.

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Well-ordered sets

- (S, $\lhd$) is a well-ordered set if:
  - (S, $\lhd$) is a totally ordered poset
  - Every non-empty subset of S has at least element
- Example: ($\mathbf{Z}$, $\leq$)
  - It is a total ordered poset (every element is comparable to every other element)
  - It has no least element
  - Thus, it is not a well-ordered set
- Example: (S, $\leq$) where S = { 1, 2, 3, 4, 5 }
  - It is a total ordered poset (every element is comparable to every other element)
  - Has a least element (1)
  - Thus, it is a well-ordered set
- Example: ($\mathbf{Z}^+$, $\leq$) is a well-ordered set.

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Lexicographic ordering

- Consider two posets: $(S, \lhd_1)$ and $(T, \lhd_2)$
- We can order Cartesian products of these two posets via lexicographic ordering
  - Let $s_1 \in S$ and $s_2 \in S$
  - Let $t_1 \in T$ and $t_2 \in T$
  - $(s_1,t_1) \lhd (s_2,t_2)$ if either:
    - $s_1 \prec_1 s_2$
    - $s_1 = s_2$ and $t_1 \lhd_2 t_2$
- Lexicographic ordering is used to order dictionaries.

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Lexicographic ordering

- Let S be the set of word strings (i.e. no spaces)
- Let T bet the set of strings with spaces
- Both the relations are alphabetic sorting
  - We will formalize alphabetic sorting later
- Thus, our posets are: $(S, \lhd)$ and $(T, \lhd)$
- Order (“run”, “noun: to…”) and (“set”, “verb: to…”)
  - As “run” $\prec$ “set”, the first Cartesian product comes before the “set” one
- Order (“run”, “noun: to…”) and (“run”, “verb: to…”)
  - Both the first part of the Cartesian products are equal
  - “noun” is first (alphabetically) than “verb”, so it is ordered first

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Lexicographic ordering

- We can do this on more than 2-tuples
- $(1,2,3,5) \lhd (1,2,4,3)$
  - When $\lhd$ is $\leq$

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Lexicographic ordering

- Consider the two strings $a_1a_2a_3\ldots a_m$, and $b_1b_2b_3\ldots b_n$
- Here follows the formal definition for lexicographic ordering of strings
- If $m = n$ (i.e. the strings are equal in length)
  - $(a_1, a_2, a_3, \ldots, a_m) \lhd (b_1, b_2, b_3, \ldots, b_n)$ using the comparisons just discussed
  - Example: "run" $\lhd$ "set"
- If $m \neq n$, then let t be the minimum of m and n
  - Then $a_1a_2a_3\ldots a_m$, is less than $b_1b_2b_3\ldots b_n$ if and only if either of the following are true:
  - $(a_1, a_2, a_3, \ldots, a_t) \prec (b_1, b_2, b_3, \ldots, b_t)$
    - Example: "run" $\lhd$ "sets" ($t = 3$)
  - $(a_1, a_2, a_3, \ldots, a_t) = (b_1, b_2, b_3, \ldots, b_t)$ and $m < n$
    - Example: "run" $\lhd$ "running"

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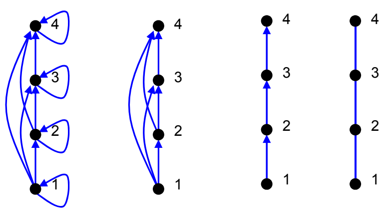
Hasse Diagrams

- Consider the graph for a finite poset ({1,2,3,4},≤)
- When we KNOW it's a poset, we can simplify the graph



Called the Hasse diagram

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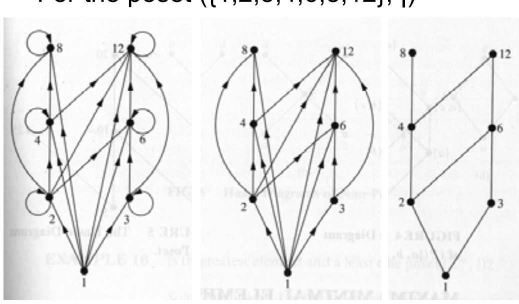
Hasse Diagram

- For the poset ({1,2,3,4,6,8,12}, |)



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