Variable Mass

This is an important topic in Mechanics. Problems are sometime tricky. It is necessary to understand the equation & each & every term.

$$\bar{F}_{ext} + \bar{u}_{rel} \frac{dM}{dt} = M \frac{d\bar{v}}{dt}$$

Sign of these vectors are very important. Please go through all different examples to see how each & every term changes with the situation. This is a very fundamental equation but sometimes it is easier not to use this equation & use $\bar{F}\Delta t = \Delta \bar{P}$

- If you find any mistake please let me know. I will correct it & reload the file.
- This is for internal circulation only.
- The main book which I followed to make my notes is Resnick Halliday Krane. I also used internet sites & videos. Few questions from Irodov is being solved here.
- If you find any interesting problem please let me know so that I can include that as well.

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Freshers Night - 2016.

Rubaroo - 2016.

Variable Mass Problem.

We cannot directly apply Newtons Law in a system which continuously changs mass like Rocket. Like.

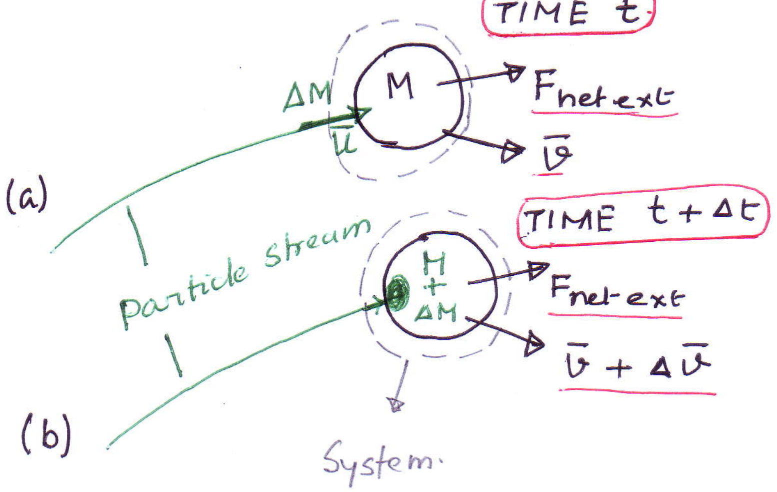
$$F_{net} = \frac{d}{dt}(\bar{P}) = \frac{d}{dt}(m(t)\,V(t)) = m(t)\frac{dv}{dt} + v(t)\frac{dm}{dt} \text{ is } \times$$

The Falsehood of this formula can be seen by noting that it does not respect Galilean invariance, a Variable mass object with $F=0$ in one frame will be seen to have $F \neq 0$ in another frame. The correct equation of motion for a body whose mass m varies with time by either ejecting or accreting mass is obtained by applying second Law to the entire Constant mass System consisting of the body and its ejected/accreted mass; The result is

$$\boxed{\bar{F}_{ext} + \bar{V}_{rel}\frac{dM}{dt} = M\frac{d\bar{v}}{dt}}$$

where V_{rel} is the velocity of the escaping/incoming mass relative to the body.

Derivation.



Suppose

(a) At Time t a Continuous stream of matter moving at Velocity $\bar{u}$ is Impacting an object of mass M that is moving with Velocity $\bar{v}$

(b) At Time $t+\Delta t$, These impacting particls stick to The object Increasing its mass by ΔM. In addition v changs by Δv.

Now we have to consider the whole system & apply Newton's second Law or impulse momentum Th.

$$\bar{F}_{net\ ext}\,\Delta t = \Delta \bar{P} = \bar{P}_f - \bar{P}_i = [(M+\Delta M)(\bar{v}+\Delta\bar{v}) - (M\bar{v} + \Delta M\bar{u})]$$

OR.

$$\bar{F}_{net\text{-}ext}\,\Delta t = M\Delta\bar{v} + \Delta M(\bar{v}-\bar{u}) + \Delta M\Delta\bar{v}$$

Dividing by Δt through out.

$$\bar{F}_{net\,ext} = M\frac{\Delta\bar{v}}{\Delta t} + \frac{\Delta M}{\Delta t}(\bar{v}-\bar{u}) + \frac{\Delta M}{\Delta t}\Delta\bar{v}$$

Taking the limit $\Delta t \to 0$, that means $\Delta M \to 0$, $\Delta\bar{v} \to 0$

$$\bar{F}_{net\,ext} = M\frac{d\bar{v}}{dt} - \frac{dM}{dt}(\bar{u}-\bar{v}) + \cancel{\frac{dM}{dt}}\,\cancel{(0)}\,\frac{dv}{dt}$$

as $\Delta t \to 0$
$dM \to 0$.
So neglected.

$$\bar{F}_{net\,ext} + \bar{u}_{rel}\frac{dM}{dt} = M\frac{d\bar{v}}{dt}.$$

$$\boxed{M\frac{d\bar{v}}{dt} = \bar{F}_{net\text{-}ext} + \bar{u}_{rel}\frac{dM}{dt}}$$

This term is known as thrust of the rocket.

$\bar{u}_{rel} = \bar{u} - \bar{v}$ = Velocity of the impacting material relative to the object. Velocity of the Smaller mass w.r.t. to bigger mass.

Note that other than $\bar{u}_{rel}\frac{dM}{dt}$ in the above equation is identical to the equation for Newton's Second Law. $\bar{F} = M\bar{a}$

$\bar{u}_{rel}\frac{dM}{dt}$ term = Thrust of the rocket = Force exerted on the rocket by ejected mass.

So Thrust can be increased either by increase (a) Speed of ejected gas or (b) Rate at which it is ejected.

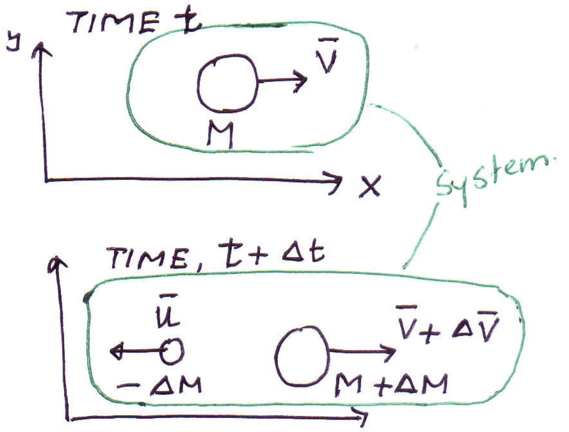
The last equation that we derived is true for all cases of variable mass problem.

Question: ?

If the mass is ejected away like the case of a rocket shall we get the same equation as before?

The answer is Yes !!

How ? Please derive and show.



Question: When mass of rocket is decreasing why it is written $M+\Delta M$ and other smaller mass $-\Delta M$, why not $M-\Delta M$ & ΔM. In this case when we write $M+\Delta M$, it means ΔM is −ve.

But it is written as $+\Delta M$ so that later we will get a term $\frac{dM}{dt}$ which will be >0 for mass increasing and will be <0 when mass is decreasing like that of a rocket. Here

This is a vector Equation.

$$F_{net\ ext}\,\Delta t = (M+\Delta M)(\bar{V}+\Delta\bar{V}) + (-\Delta M)\bar{u} - M\bar{V}$$

We get back the old equation after following earlier steps

$$\boxed{M\frac{d\bar{V}}{dt} = \bar{F}_{net\ ext} + \bar{u}_{rel}\frac{dM}{dt}}$$

One can also get this using mass $(M-\Delta M)$ & ΔM but then it will not be a general equation as before i.e $\frac{dM}{dt}<0$ for mass ejecting & $\frac{dM}{dt}>0$ for mass gaining.

Further Confusion:

Now in the given eq.

$$M \cdot \frac{d\bar{v}}{dt} = \bar{F}_{net\ ext} + \bar{u}_{rel} \frac{dM}{dt}$$

In case of rocket

(i) Should we put $\bar{u}_{rel}$ as $-u_e$ because the Velocity of ejected mass is in negative direction compared to the direction of rocket ?

Ans: Yes, when you change the vector equator to a scalar equation.

(ii) Should we write $\frac{dM}{dt}$ as $-\frac{dM}{dt}$ because $\frac{dM}{dt} < 0$ in case of Rocket type problem.

Ans: At the beginning NO | sometimes yes. At the end if it is necessary we must. Sometimes integration takes care of negative sign when we put the limits.

Rocket Equation.

(a) Rocket in Free Space.

The general equation for variable Mass system

$$\boxed{M \cdot \frac{d\bar{v}}{dt} = \bar{F}_{net\,ext} + \bar{u}_{rel} \frac{dM}{dt}}$$

In Free space no external force on a Rocket.

$$\boxed{F_{net\,ext} = 0.}$$

$$\bar{u}_{rel} \frac{dM}{dt} = M \cdot \frac{d\bar{v}}{dt}.$$

Now changing the vector equation to scalar equation.
(You may not do it. One may keep vector eq also)

$\bar{u}_{rel} \Rightarrow$ can be written as $-u_{rel}$

$\frac{d\bar{v}}{dt} \Rightarrow$ can be written as $\frac{dv}{dt}$.

$$-u_{rel} \frac{dM}{dt} = M \frac{dv}{dt}$$

$$\Rightarrow \quad -u_{rel}\, dM = M\, dv.$$

$$-u_{rel} \int_{m_0}^{m_1} \frac{dM}{M} = \int_{v_0}^{v_1} dv$$

$$\Rightarrow \quad u_{rel} \ln \frac{m_0}{m_1} = v_1 - v_0 = \Delta v$$

Now this number is +ve.

$$\Rightarrow \quad \boxed{\Delta v = u_{rel} \ln \frac{m_0}{m_1}}$$

One can get this eq in vector form as well.

m_0 = Initial total mass including propellant
m_1 = total final mass
u_{rel} = effective exhaust velocity, Δv = change of speed of vehicle.

(b) Rocket in Gravitational field.

Let us Write down the general equation.

$$\boxed{M \frac{d\bar{v}}{dt} = \bar{F}_{net\ ext} + \bar{u}_{rel} \frac{dM}{dt}.}$$

In Gravitational Field $F_{net\ ext} = M\bar{g}$

$$M \frac{d\bar{v}}{dt} = M\bar{g} + \bar{u}_{rel} \frac{dM}{dt}$$

Let us Convert in Scalar Form.

$$M \frac{dv}{dt} = -Mg - u_{rel} \frac{dM}{dt}$$

Rocket is moving straight up so $V = V_y$.

$$\frac{dV_y}{dt} = -g - \frac{u_{rel}}{M} \frac{dM}{dt}$$

Integrating wrt to time. For a Rocket at $t=0$, $V_y = 0$.

$$\boxed{V_y = -gt + u_{rel} \ln \frac{m_0}{m_f}}$$

It is a positive number.

One better way to Convert a Vector equation to Scalar is to Write the Vector equation in terms of Unit Vector $\hat{i}, \hat{j}$ and then to Cancel them from both sids - like

$$M \frac{d\bar{v}}{dt} = M\bar{g} + \bar{u}_{rel} \frac{dM}{dt}$$

$$M \frac{dv}{dt}\hat{j} = -Mg\hat{j} - u_{rel}\hat{j} \frac{dM}{dt}$$

Canceling $\hat{j}$ through out

$$M \frac{dv}{dt} = -Mg - u_{rel} \frac{dM}{dt}$$

A space ship with a total mass of 13,600 kg is moving relative to a certain inertial frame with a speed of 960 m/s in a region of space of negligible gravity. Its fires its rocket engines to give an acceleration parallel to the initial velocity. The rockets ejects gas at a constant rate of 146 kg/s with a constant speed (relative to the space ship) of 1520 m/s and they are fired until 9100 kg of fuel has been burned & ejected

(a) What is the thrust produced by rocket

(b) What is the velocity of the space ship after rockets have fired?

Ans ⇒ (a) Thrust $= \left| \bar{u}_{rel} \frac{dM}{dt} \right| = (1520 \text{ m/s})(146 \frac{kg}{s}) = 2.2 \times 10^5 N$

(b) Choosing positive x direction to be space ship initial velocity. $\Sigma \bar{F}_{ext} = 0$

$$\boxed{M \frac{d\bar{v}}{dt} = \cancel{\bar{F}_{net\ ext}}^{\,0} + \bar{u}_{rel} \frac{dM}{dt}}$$ (Vector equation)

$$\Rightarrow \quad M \frac{d\bar{v}_x}{dt} = \bar{u}_{rel,x} \frac{dM}{dt}$$ (Scalar equation)

[Please note both u_{rel} & $\frac{dM}{dt}$ is < 0 so RHS > 0 so v_x is ↑]

Rewriting the equation ⇒ $dV_x = u_{rel,x} \left(\frac{dM}{M} \right)$.

$$\int_{V_{ix}}^{V_{fx}} d\bar{V}_x = \bar{u}_{rel,x} \int_{M_i}^{M_f} \left(\frac{dM}{M} \right)$$

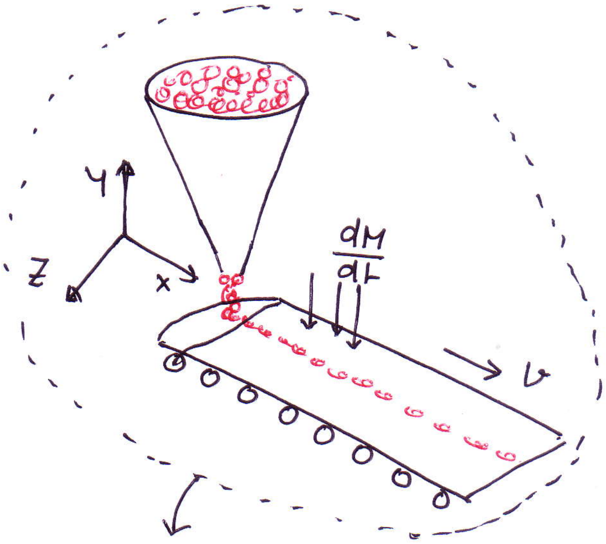
$$\left[\bar{V}_{fx} - \bar{V}_{ix} \right] = \bar{u}_{rel,x} \ln \frac{M_f}{M_i}$$ ⇒ Here we keep the vector form unlike the scalar form derived earlier. So −ve sign below.

$$\Rightarrow \quad V_{fx} = 960 \text{ m/s} + (-1520) \ln \frac{4500 \text{ kg}}{13,600 \text{ kg}} = 2640 \text{ m/s}$$

negative

* Sample Problem 7-9 ⇒ Resnick Halliday Krane.

Sand drops from a stationary hopper at a rate of 0.134 kg/s onto a conveyor belt moving with a speed of 0.96 m/s as shown below. What net force must be applied to the conveyor belt to keep it moving at constant speed?



System boundary.

- Direction of the motion of the belt is taken as positive x direction.
- Fix the coordinate system in the laboratory and locate our inertial reference frame in which the hopper is at rest
- Let us write the eqn.

$$M \frac{d\bar{v}}{dt} = \bar{F}_{net\,ext} + \bar{u}_{rel} \frac{dM}{dt}$$

In scalar form.

$$M \frac{dv_x}{dt} = F_{ext} + (-u_{rel,x}) \cdot \frac{dM}{dt}$$

Now $\frac{dv_x}{dt} = 0$, $u_{rel,x} = -u_x$, $\frac{dM}{dt} > 0$.

V. Imp The observer travelling with the belt would see the sand leaving the hopper moving in the negative x direction.

Solving

$$0 = F_{ext} + (-0.96\,m/s)(0.134\,kg/s)$$

$$\Rightarrow \quad F_{ext} = 0.129\,N.$$

The force has a positive x component. It must be applied in the direction of the motion of the belt to increase the x component of the velocity of each grain of sand that drops onto the belt from 0 to 0.96 m/s.

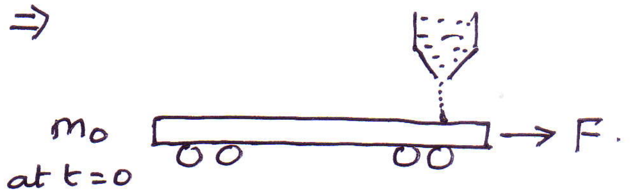
[* Sample Problem 7-10 ⇒ Resnick Halliday Krane.]

• Mass Flowing in • Vertically.

\# A Flatcar of mass m_0 starts moving to the right due to a constant horizontal force F. Sand spills on the flatcar from a stationary hopper. The velocity of loading is constant and equal to μ kg/s. Find the time dependence of the velocity and acceleration of the flatcar in the process of loading. The friction is negligibly small.

(IRODOV problem - 1.183)

⇒



Step 1 : Equation for Variable Mass. System.

$$\bar{F}_{ext} + \bar{u}_{rel}\frac{dm}{dt} = m\frac{d\bar{v}}{dt}$$

Step 2

$$F\hat{i} \qquad -v\hat{i} \qquad \mu \qquad \text{mass at any instant } t.$$

$$m = m_0 + \mu t$$

Why $-v$

$\bar{u}_{rel} = \bar{u} - \bar{v}$ ⇒ Here u is the velocity of the smaller mass compared to bigger mass. That is the velocity of smaller mass wrt to bigger mass. It means what a person sitting in the cart will observe. The ~~drop~~ sand which is falling on the cart has 0 horizontal velocity. So

$u_{rel} = \bar{u} - \bar{v} = 0 - \bar{v}\hat{i} = -v\hat{i}$ [Important step]

Now the above equation reads (in scalar form)

$$F - v\mu = (m_0 + \mu t)\frac{dv}{dt}$$

which can be solved using separation of variable

$$\int_0^{v_f} \frac{dv}{F - vb} = \int_0^t \frac{dt}{m_0 + \mu t}$$

After solving you get $v = f(t)$

By applying a small trick one can avoid integration.

How ?

Don't write $\mu = \frac{dm}{dt}$ in the beginning. So the eqn is scalar form.

$$m \frac{d\vec{v}}{dt} = \vec{F} - v \frac{dm}{dt}$$

$$\Rightarrow \quad m \frac{dv}{dt} + v \frac{dm}{dt} = F.$$

$$\Rightarrow \quad \frac{d}{dt}(mv) = F. \quad \Rightarrow \int d(mv) = \int F\, dt$$

$$\Rightarrow \quad mv = Ft \quad \Rightarrow \quad v = \frac{Ft}{m} = \frac{Ft}{m_0 + \mu t}$$

Acceleration $\frac{dv}{dt} = \frac{F}{m_0 \left(1 + \frac{\mu t}{m_0}\right)^2}$

Here direction of both v and a is along $\hat{i}$ axis.

A Shorter Method.

Another shorter method is available which works only for this problem. But best way is to use the variable mass ~~problem system~~ equation so that you don't make mistake.

Here as the sand is falling into the cart, it does not have any velocity in the horizontal direction. So it is not adding any momentum to the system in the x direction. So momentum in x direction is conserved.

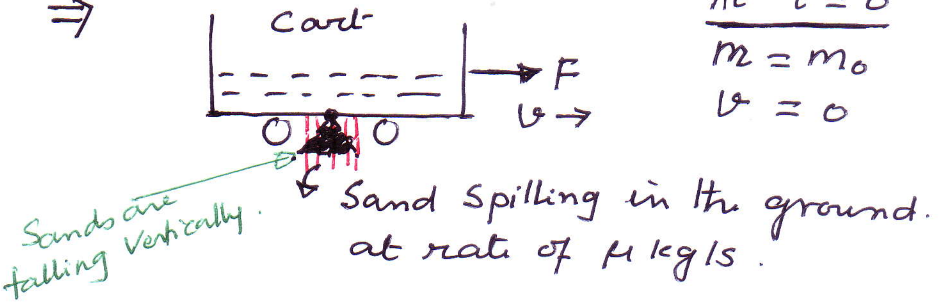
$$\bar{F} = \frac{d\bar{P}}{dt} \quad \Rightarrow \quad \int_0^t \bar{F}\, dt = \int_{P_f}^{P_i} d\bar{P} \quad \Rightarrow \quad \bar{F}t = \bar{P}_f - \bar{P}_i \quad (\text{if } \bar{F} = \text{const})$$

Using that you get answer in one line. Please remember that it is not a general method & it works only for this situation as explained above.

• Mass Flowing out • Vertically

\# A cart loaded with sand moves along a horizontal plane due to a constant force, F coinciding in direction with constant velocity vector. In the process sand spills through a hole in the bottom with a constant rate of μ kg/s. Find the acceleration and velocity of cart at moment t, if at the initial moment t = 0, the cart with loaded sand has mass m_0 and velocity 0. (Neglect friction) (IRODOV- 1.182)

$\Rightarrow$



At $t=0$	At $t=t_f$
$m = m_0$	$v = ?$
$v = 0$	$a = ?$

$$\bar{F}_{ext} + \bar{u}_{rel}\frac{dM}{dt} = M\frac{d\bar{v}}{dt}$$

$\bar{F}_{ext} = F\hat{i}$ (given)

$\bar{u}_{rel} = 0$ (why) The sand that is coming out has a horizontal velocity v. Remember when a man jumps from running train why does he fall – he is having same velocity as that of train.

$M = (m_0 - \mu t)$

M is mass at any instant t. so.

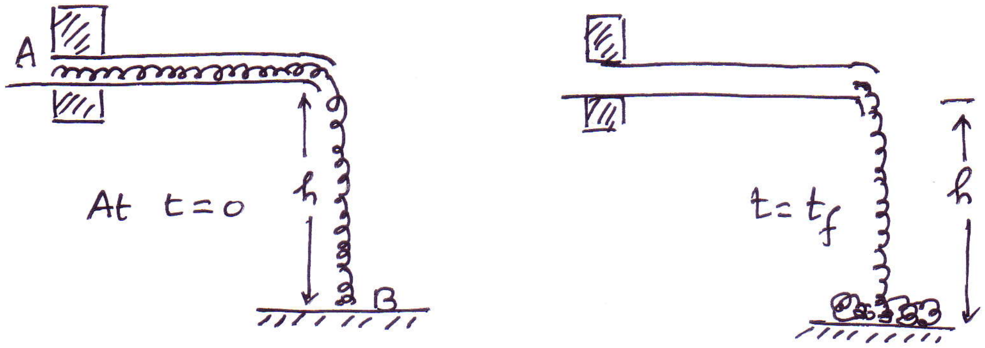
$$F\hat{i} + 0 = M\frac{dv}{dt}\hat{i}. \Rightarrow \frac{dv}{dt} = a = \frac{F}{m_0 - \mu t}.$$

or. $$\int_0^v dv = \int_0^t \frac{F\,dt}{m_0 - \mu t} \Rightarrow v = \frac{F}{\mu}\ln\frac{m_0}{m_0 - \mu t}.$$

Chain Problem.

\# A chain AB of length l is located in a smooth horizontal tube so that its fraction of length h hangs freely and touches the surface of the table with its end B. At a certain moment the end A of the chain is set free. With what velocity will this end of the chain slip out of the tube?

(Irodov - 1.184)



This problem can be solved in many ways. Two principal ways are — (a) Using Variable Mass equation (b) Without using that equation.

⇒ Let's assume that mass of the chain $= m_0$.

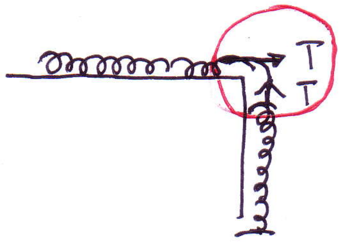
Mass of the hanging portion $= \left(\frac{m_0}{l}\right)h$.

Mass of the portion in tube $= \left(\frac{m_0}{l}\right)x$.

x is that part which is in tube.

$$\frac{dm}{dt} = \frac{d}{dt}\left(m_0\frac{x}{l}\right) = \frac{m_0}{l}\frac{dx}{dt}$$

Now the force of Tension which pulls the chain.



writing $F = ma$.

Tube Part $\quad T = \left(\frac{m_0}{l}\right)x\,a \quad$ — (I)

Hanging Part $\quad \left(\frac{m_0 h}{l}\right)g - T = \left(\frac{m_0 h}{l}\right)a$ — (II)

Now eliminating T from (I) & (II) you get the Value of a and from there one can get the Velocity

$$a = \frac{hg}{x+h}$$

$$\frac{dv}{dt} = \frac{hg}{x+h}$$

Then Writing $\frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = \frac{dv}{dx} \cdot v$. So

Is it Correct?

(It should be $-v$)

$$-v \cdot \frac{dv}{dx} = \frac{hg}{x+h}$$

$$-\int v \cdot dv = hg \int \frac{1}{x+h} dx$$

limit of v from $0 \rightarrow v_f$

limit of x from $(l-h) \rightarrow 0$

This was the length which was there in tube initially.

Finally by doing so you get

$$v = \sqrt{2gh \ln\left(\frac{l}{h}\right)}$$ (Ans) Without using Variable Mass equation.

So we have got the answer without using our Variable mass problem.

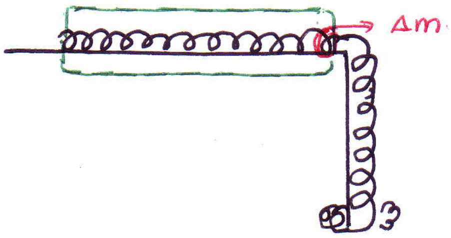
Using Variable Mass Equation

But If we want to use that equation we have to eliminate a from eq (I) & (II) & get for T.

$$\frac{\frac{m_0 h g}{l} - T}{T} = \frac{h}{x} \Rightarrow T = \frac{m_0 h g x}{l(x+h)}$$

What is our System?

At $t = t$



$$\bar{F}_{ext} + \bar{u}_{rel}\frac{dm}{dt} = m\frac{d\bar{v}}{dt}$$

$\bar{F}_{ext}$ = Tension which we have Calculated.

$\bar{u}_{rel} = 0$, Δm that is going out is having same velocity as the rest part which is inside the tube. So. the eqn in Scalar form is

$$\frac{m_0 h g x}{l(x+h)} = m\frac{dv}{dt}$$

$$\frac{m_0 h g x}{l(x+h)} = m \cdot \frac{dv}{dx}\frac{dx}{dt} = m \cdot \frac{dv}{dx}(-v).$$

Why?

$m = m_0\left(\frac{x}{l}\right).$

You get back the old equation which gives rise to

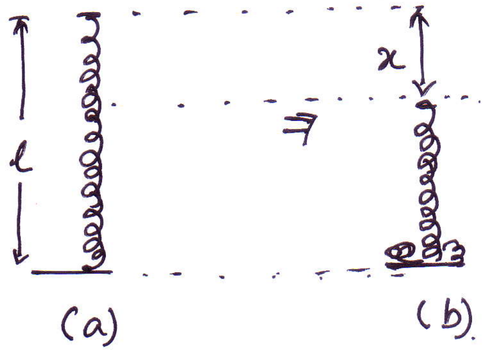
$$\frac{\cancel{m_0} g \cancel{x} \cdot h}{\cancel{l}(x+h)} = -\cancel{m_0}\left(\frac{\cancel{x}}{\cancel{l}}\right)\frac{dv}{dx}v$$

$$\int_0^V v\,dv = -\int_{l-h}^{0}\frac{hg\,dx}{x+h}$$

$$\underline{v = \sqrt{2gh\ln\left(\frac{l}{h}\right)}}$$

Using Variable Mass Equation.

A chain of Mass M and Length L is falling Vertically on a table. Find the reaction by the table when the chain has fallen by a distance x from top. (Assume that each link of the chain is falling freely)



A rocket of total mass 1.11×10^5 kg, of which 8.70×10^4 kg is fuel, is to be launched vertically. The fuel will be burned at a constant rate of 820 m/s. Relative to the rocket what is the minimum exhaust speed that allows lift-off at launch.? (Ans: 1.33 Km/s)

[Hint: There may be some extra information in the question. Don't get confused by that]

(a)
A rocket set for vertical firing weights 50 kg and contains 450 kg of fuel. It can have a maximum exhaust velocity of 2 km/s. What should be its minimum rate of fuel consumption

(i) to just lift it off the launching pad

(ii) to give it an initial acceleration of $20 m/s^2$

(b) What will be the speed of the rocket when the rate of consumption of fuel is 10 kg/s after whole of the fuel is consumed. ($g = 9.8 m/s^2$)

Ans (a) (i) 2.45 kg/s (ii) 7.45 kg/s

(b) 4.164 km/s.

⇒ Always write down the vector equation — Then change that to scalar equation with proper sign. Then you will never make mistake. In case of rocket when $\frac{dM}{dt} < 0$ one can write it as $\left(-\frac{dm}{dt}\right)$ & keep it like this through out to avoid confusion

Summary.

1) We derived the equation for variable mass by Considering a system where mass is not changing at two different times t & $t + \Delta t$.

2) The equation that we used to derive ~~this~~ is $\bar{F} dt = \Delta \bar{P}$. Many problems Can be Solved using this.

3) We must try to understand each & every term of Variable Mass equation.

$$\bar{F}_{ext} + \bar{u}_{rel} \frac{dM}{dt} = M \frac{d\bar{v}}{dt}.$$

(a) This is a Vector equation.

(b) If we work with a scalar equation it is better to write this eqn in terms of $\hat{i}, \hat{j}, \hat{k}$ -- & cancel the vectors from both sids. We will make less mistake

(c) $\bar{u}_{rel}$ is the relative velocity of the smaller mass wrt to bigger mass. This means what is the velocity of the smaller body as seen from the bigger body.

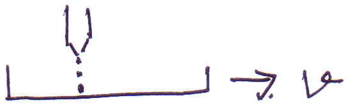
(d) When mass is const we write $F_{ext} = M \frac{dv}{dt}$.
The Second term in Variable Mass Eqn is Coming because of changing mass & this gets added up with F_{ext}. That means that not only force can change the acceleration but changing mass has the Capability to change acceleration & hence acts like a force when multiplied by u_{rel}.

(e) M here means mass at any general time t.
($M = M_0 + \mu t$, $M = M_0 - \mu t$).

(f) See how this equation is different for different cases.

1) 

gas coming out from rocket

2) 

Mass getting added up from top.

3) 

Mass leaking out from the bottom.

(g) Sometimes you need not apply the Variable Mass equation. It is easier to solve without using the equation.