The LNM Institute of Information Technology
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MATH-I ■ Solutions Assignment #3

(Infinite Series)

Q1. Let $a_n \geq 0$. Then show that both the series $\sum_{n\geq 1} a_n$ and $\sum_{n\geq 1} \frac{a_n}{a_n+1}$ converge or diverge together.

Ans. Suppose $\sum_{n\geq 1} a_n$ converge. Since $0 \leq \frac{a_n}{a_n+1} \leq a_n$ by comparison test $\sum_{n\geq 1} \frac{a_n}{a_n+1}$ converges. Suppose $\sum_{n\geq 1} \frac{a_n}{a_n+1}$ converges. By the necessary condition $\frac{a_n}{a_n+1} \to 0$. Hence $a_n \to 0$ and therefore $1 \leq 1 + a_n < 2$ eventually. Hence $0 \leq \frac{1}{2} a_n \leq \frac{a_n}{a_n+1}$. Apply the comparison test.

Q2. In each of the following cases, discuss the convergence/divergence of the series $\sum_{n\geq 1} a_n$, where a_n equals:
(a) $1 - n \sin \frac{1}{n}$, (b) $\frac{1}{n} \log(1 + \frac{1}{n})$, (c) $1 - \cos \frac{1}{n}$,
(d) $2^{-n-(-1)^n}$, (e) $\left(1 + \frac{1}{n}\right)^{n(n+1)}$, (f) $\frac{n \log n}{2^n}$.

Ans. (a) Use Limit Comparison Test (LCT) with $\frac{1}{n^2}$. Since $1 - n \sin \frac{1}{n} \leq \frac{1}{3! n^2} < \frac{1}{n^2}$, one can also use comparison test.
(b) Use LCT or comparison test with $\frac{1}{n^2}$.
(c) Use LCT with $\frac{1}{n^2}$ or comparison test because $1 - \cos \frac{1}{n} \leq \frac{1}{2! n^2} < \frac{1}{n^2}$ or $1 - \cos \frac{1}{n} = 2 \sin^2 \frac{1}{2n} < \frac{1}{2n^2}$.
(d) Use root test to show that $a_n^{\frac{1}{n}}$ converges to $\frac{1}{2}$ and therefore the series converges.
(e) Use root test to show that $a_n^{\frac{1}{n}}$ converges to $e > 1$ and hence the series is divergent.
(f) By ratio test, we get $\frac{a_{n+1}}{a_n} \to \frac{1}{2}$ and therefore the series converges.

Q3. Test the series $\sum_{n\geq 1} \tan^{-1}(e^{-n})$ and the series $\sum_{n\geq 1} \left(1 - \frac{1}{n}\right)^{n^2}$ for convergence.

Ans. Applying ratio test we get

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{\tan^{-1}(e^{-(n+1)})}{\tan^{-1}(e^{-n})} = \lim_{n\to\infty} \frac{\frac{-e^{-(n+1)}}{1+e^{-2(n+1)}}}{\frac{-e^{-n}}{1+e^{-2n}}}$$
$$= \frac{1}{e} \lim_{n\to\infty} \frac{1 + e^{-2n}}{1 + e^{-2(n+1)}} = \frac{1}{e}.$$

Since $\frac{1}{e} < 1$, therefore by Ratio test the series converges.

Applying root test, we get $|a_n|^{\frac{1}{n}} = \left(1 - \frac{1}{n}\right)^n$. Also,

$$\lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^n = \frac{1}{e} < 1$$

$\Big($To see this, Let $y = \left(1 - \frac{1}{n}\right)^n$. Then $\ln y = n \ln\left(1 - \frac{1}{n}\right) = \frac{\ln\left(1-\frac{1}{n}\right)}{\frac{1}{n}}$. Therefore, using L'Hopital rule we get $\lim\limits_{n\to\infty} \ln y = -1 \implies \lim\limits_{n\to\infty} y = \dfrac{1}{e}.\Big)$

Hence, the series converges by root test.

Q4. Let $\{a_n\}$ be a decreasing sequence, $a_n \geq 0$ and $\lim\limits_{n\to\infty} a_n = 0$. For each $n \in \mathbb{N}$, let $b_n = \frac{a_1+a_2+\ldots+a_n}{n}$. Show that $\sum\limits_{n\geq 1}(-1)^n b_n$ converges.

Ans. $b_{n+1} - b_n = \frac{a_1+a_2+\ldots+a_{n+1}}{n+1} - \frac{a_1+a_2+\ldots+a_n}{n} = \frac{a_{n+1}}{n+1} - \frac{a_1+a_2+\ldots+a_n}{n(n+1)}$. Since $\{a_n\}$ is decreasing, $a_1 + a_2 + \ldots + a_n \geq n a_n$. Therefore, $b_{n+1} - b_n \leq \frac{a_{n+1}-a_n}{n+1} \leq 0$. Therefore, $\{b_n\}$ is decreasing.

We now need to show that $b_n \to 0$. For a given $\epsilon > 0$, since $a_n \to 0$, there exists n_0 such that $a_n < \epsilon/2$, $\forall n \geq n_0$.

Therefore, $\left|\frac{a_1+a_2+\ldots+a_n}{n}\right| = \left|\frac{a_1+a_2+\ldots+a_{n_0}}{n} + \frac{a_{n_0+1}+a_2+\ldots+a_n}{n}\right| \leq \left|\frac{a_1+a_2+\ldots+a_{n_0}}{n}\right| + \frac{n-n_0}{n}\frac{\epsilon}{2}$.

Choose $N \geq n_0$ large enough so that $\frac{a_1+a_2+\ldots+a_{n_0}}{N} < \frac{\epsilon}{2}$. Then, for all $n \geq N$, $\frac{a_1+a_2+\ldots+a_n}{n} < \epsilon$. Hence, $b_n \to 0$. Use the Leibnitz test for convergence.

Q5. Show that if $\sum\limits_{n=1}^{\infty}|a_n|$ converges then $\sum\limits_{n=1}^{\infty}a_n$ converges. Give an example to show that the converse need not be true.

Ans. Let $\sum\limits_{n=1}^{\infty}|a_n|$ converges. Then the sequence of partial sums of $\sum\limits_{n=1}^{\infty}|a_n|$ satisfies the Cauchy criterion. Therefore, the sequence of partial sums of $\sum\limits_{n=1}^{\infty}a_n$ also satisfies the Cauchy criterion (why?). This shows that the series $\sum\limits_{n=1}^{\infty}a_n$ converges.

For the converse part, consider the series $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^{n-1}}{n}$. This series converges by Leibnitz test, however the series $\sum\limits_{n=1}^{\infty}\dfrac{1}{n}$ obtained on taking the absolute values of the terms of the original series diverges.

Q6. Determine the values of x for which the following series converges:

$$(a) \sum_{n\geq 1} \frac{(x-1)^{2n}}{n^2 3^n}, \qquad (b) \sum_{n\geq 1} \frac{n^3}{3^n} x^n, \qquad (c) \sum_{n\geq 1} \frac{(2n)!}{(2^n n!)^2} \frac{x^{2n+1}}{2n+1}.$$

Ans. (a) By root test the series converges for $|x-1| < \sqrt{3}$. If $x - 1 = \pm\sqrt{3}$ the series converges. Therefore the series converges for $|x-1| \leq \sqrt{3}$.
(b) Use the ratio test to see that the series converges for $|x| < 3$. For $x = \pm 3$, the series diverges.
(c) Use ratio test to see that the series converges for $|x| < 1$. For $x = \pm 1$, the corresponding series will converge.

Q7. Let (a_n) be a constant sequence. If $\sum_n a_n$ converges then show that $a_n = 0$ for all n.

Ans. Let $a_n = c$ for some $c \in \mathbb{R}$ such that $c \neq 0$. Then $S_n = nc$. Given that S_n converges, $\frac{1}{c} S_n = n$ converges, which is a contradiction.

Q8. Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series of positive terms satisfying $\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n}$ for $n \geq N$. Show that if $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ also converges. Test the series $\sum_{n\geq 1} \frac{n^{n-2}}{e^n n!}$ for convergence.

Ans. Clearly, $c_{n+1} = \frac{a_{n+1}}{b_{n+1}} \leq \frac{a_n}{b_n} = c_n \; \forall n \geq \mathbb{N}$. Thus $0 < c_n = \frac{a_n}{b_n} < \frac{a_N}{b_N} \; \forall n > \mathbb{N}$. Use the comparison test.
For the other part, note that $\frac{a_{n+1}}{a_n} = \frac{\left(1+\frac{1}{n}\right)^{n-2}}{e} = \frac{\left(1+\frac{1}{n}\right)^{n}}{\left(1+\frac{1}{n}\right)^{2} e} < \frac{e}{\left(1+\frac{1}{n}\right)^{2} e} = \frac{\frac{1}{(1+n)^2}}{\frac{1}{n^2}}$.