The LNM Institute of Information Technology
Jaipur, Rajasthan
MATH-II
Assignment 7

---

1. Find the eigenvalues and eigen functions of the following Strum-Liouville problems:
   (i) $y'' + \lambda y = 0, \quad y(0) = y'(1) + y(1) = 0$
   (ii) $(xy')' + \frac{\lambda}{x} y = 0, \quad y(1) = y'(e) = 0.$

2. If $p(x)$, $q(x)$, $r(x)$ are all greater than zero on (a, b), then prove that the eigenvalues of the Strum-Liouville problem, $(p(x)y')' - q(x)y + \lambda r(x)y = 0$, are positive with any of the boundary conditions: (i) $p(a) = 0$, $p(b) = 0$, (ii) $p(a) = p(b)$ with $y(b) = y(a)$, $y'(b) = y'(a)$ (iii) $y(a) - ky'(a) = 0$, $y(b) + my'(b) = 0$, $k, m > 0$.

3. Consider the Strum-Liouville problem
$$(p(x)y')' + [q(x) + \lambda r(x)]y = 0$$
   with $p(x) > 0$ on $[a, b]$ and $y(a) \neq y(b)$, $y'(a) \neq y'(b)$. Show that every eigen function is unique except for a constant factor.

4. Let $F(s)$ be the Laplace transform of $f(t)$. Find the Laplace transform of $f(at)$ $(a > 0)$.

5. Find the Laplace transforms:
   (a) $[t]$, (greatest integer function) (b) $t^m \cosh bt$ ($m \in$ non-negative integers), (c) $e^t \sin at$,
   (d) $\dfrac{e^t \sin at}{t}$, (e) $\dfrac{\sin t \cosh t}{t}$, (f) $f(t) = \begin{cases} \sin 3t, & \text{if} \quad 0 < t < \pi, \\ 0, & \text{if} \quad t > \pi. \end{cases}$

6. Find the Laplace transforms (Hint: second shifting theorem):
$$(a)\ f(t) = \begin{cases} 1, & 0 < t < \pi, \\ 0, & \pi < t > 2\pi, \\ \cos t, & t > 2\pi. \end{cases} \qquad (b)\ f(t) = \begin{cases} 0, & 0 < t < 1, \\ \cos \pi t, & 1 < t > 2, \\ 0, & t > 2. \end{cases}$$

7. Find the inverse Laplace transforms of
$$(a)\ \tan^{-1}(a/s), \quad (b)\ \ln \frac{s^2+1}{(s+1)^2}, \quad (c)\ \frac{s+2}{(s2+4s-5)^2}, \quad (d)\ \frac{se^{-\pi s}}{s^2+4}, \quad (e)\ \frac{(1-e^{-2s})(1-3e^{-2s})}{s^2}.$$

8. Using Convolution, find the inverse Laplace transforms:
$$(a)\ \frac{1}{(s^2-5s+6)}, \quad (b)\ \frac{-6}{(s^2-1)}, \quad (c)\ \frac{1}{s^2(s^2+4)}, \quad (d)\ \frac{1}{(s-1)^2}.$$

9. Use Laplace transform to solve the initial value problems:
   (a) $y'' + 4y = \cos 2t,\ y(0) = 0,\ y'(0) = 1.$
$$(b)\ y'' + 3y' + 2y = \begin{cases} 4t, & \text{if} \ \ 0 < t < 1, \\ 8, & \text{if} \ \ t > 1, \end{cases} \qquad y(0) = y'(0) = 0.$$
$$(c)\ y'' + 9y = \begin{cases} 8 \sin t, & \text{if} \ \ 0 < t < \pi, \\ 0, & \text{if} \ \ t > \pi, \end{cases} \qquad y(0) = 0,\ y'(0) = 4.$$
$$(d)\ y_1' + 2y_1 + 6\int_0^t y_2(\tau)d\tau = 2u(t),\ y_1' + y_2' = -y_2,\ y_1(0) = -5,\ y_2(0) = 6.$$