**The LNM Institute of Information Technology**
**Jaipur, Rajsthan**

**MATH-I** ■ Solutions Assignment #6

(Riemann Integral, Improper Integrals)

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Q1. If $f$ is a bounded function such that $f(x) = 0$ except at a point $c \in [a, b]$. then show that $f$ is integrable on $[a, b]$ and that $\int_a^b f = 0$.

Ans. *Using the definition:* Let $f(c) > 0$ and $P$ be any partition. Suppose $c \in [x_i, x_{i+1}]$. Then $L(P, f) = 0$ and $U(P, f) = f(c)\Delta x_i$. Since $P$ is arbitrary, $\inf_P U(P, f) = 0$ and $\sup_P L(P, f) = 0$. Hence $f$ is integrable and $\int_a^b f(x)dx = 0$.
*Using the "$\epsilon - P$ argument (essentially the same)":* Let $\epsilon > 0$. Note that if $P$ is a partition such that $\max_i \Delta x_i < \delta$ then $L(P, f) = 0$ and $U(P, f) \leq 2f(c)\delta$. Choose $\delta < \frac{\epsilon}{2f(c)}$. Then $U(P, f) - L(P, f) < \epsilon$ and hence $f$ is integrable by the Riemann criterion. Since the lower integral is 0 and the function is integrable, $\int_a^b f(x)dx = 0$.

Q2. Let $f : [0, 1] \to \mathbb{R}$ such that $f(x) = \begin{cases} \frac{1}{n}, & \text{if } x = \frac{1}{n} \\ 0, & \text{otherwise.} \end{cases}$ Show that $f$ is integrable on $[0, 1]$ and $\int_0^1 f(x)dx = 0$.

Ans. We will use the Riemann criterion to show that $f$ is integrable on $[0, 1]$. Let $\epsilon > 0$ be given. We need to find a partition $P$ such that $U(P, f) - L(P, f) < \epsilon$. Since $\frac{1}{n} \to 0$, there exists $N$ such that $\frac{1}{n} \in [0, \epsilon]$ for all $n > N$.
So only finite number of $\frac{1}{n}'$s lie in the interval $[\epsilon, 1]$. Cover these finite number of $\frac{1}{n}'$s by the intervals $[x_1, x_2], [x_3, x_4], \ldots [x_{m-1}, x_m]$ such that $x_i \in [\epsilon, 1]$ for all $i = 1, 2, \ldots m$ and the sum of the length of these $m$ intervals is less than $\epsilon$. Consider the partition $P = \{0, \epsilon, x_1, x_2, \ldots, x_m\}$. It is clear that $U(P, f) - L(P, f) < 2\epsilon$.
Hence by the Reimann criterion the function is integrable. Since the lower integral is 0 and the function is integrable, $\int_0^1 f(x)dx = 0$.

Q3. Define $f : [-1, 1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} -1, & -1 \leq x \leq 0 \\ 1, & 0 < x \leq 1. \end{cases}$$

Is the function continuous on $[-1, 1]$? Is the function Riemann integrable?

Ans. Clearly $f$ is not continuous at $x = 0$. Rest part is similar to exercise (1).

Q4. Does there exist a continuous function $f$ on $[0,1]$ such that

$$\int_0^1 x^n f(x)dx = \frac{1}{\sqrt{n}} \quad \text{for all} \quad n \in \mathbb{N}.$$

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Ans. Suppose there is such a function $f$. Then, by the previous problem, for every $n$ there exist $c_n \in [0,1]$ such that $f(c_n)\int_0^1 x^n dx = \frac{1}{\sqrt{n}}$. This implies that $f(c_n) = \frac{n+1}{n!} \to \infty$. That is, $f$ is not bounded on $[0,1]$ which is a contradiction.

**Aliter:** This problem can also done without using the previous problem. Suppose there is such a function $f$ and $\sup f = M$. Then

$$\frac{1}{\sqrt{n}} = \left|\int_0^1 f(x)x^n dx\right| \leq M\left|\int_0^1 x^n dx\right| = \frac{M}{n+1}.$$

This implies that $1 \leq \frac{M\sqrt{n}}{n+1} \to 0$ which is a contradiction.

Q5. Let $f : [0,1] \to \mathbb{R}$ such that $g_n(y) = \begin{cases} \frac{ny^{n-1}}{1+y}, & \text{if } 0 \leq y < 1 \\ 0, & y = 1. \end{cases}$ Then prove that $\lim_{n\to\infty}\int_0^1 g_n(y)dy = \frac{1}{2}$ whereas $\int_0^1 \lim_{n\to\infty} g_n(y)dy = 0$.

Ans. From the ratio test for the sequence we can show that $\lim_{n\to\infty}\frac{ny^{n-1}}{1+y} = 0$, for each $0 < y < 1$. Therefore $\int_0^1 \lim_{n\to\infty} g_n(y)dy = 0$.

For the other part, use integration by parts to see that $\int_0^1 \frac{ny^{n-1}}{1+y}dy = \frac{1}{2} + \int_0^1 \frac{y^n}{(1+y)^2}dy$.

Note that $\int_0^1 \frac{y^n}{(1+y)^2}dy \leq \int_0^1 y^n = \frac{1}{n+1} \to 0$ as $n \to \infty$. Therefore,

$$\lim_{n\to\infty}\int_0^1 g_n(y)dy = \frac{1}{2}.$$

Q6. Test the convergence/divergence of the following improper integrals:

$(a)\ \int_0^1 \frac{dx}{\log(1+\sqrt{x})}$ $\quad (b)\ \int_0^1 \frac{dx}{x - \log(1+x)}$ $\quad (c)\ \int_0^1 \frac{\log x}{\sqrt{x}}dx$ $\quad (d)\ \int_0^1 \sin\left(\frac{1}{x}\right)dx$

$(e)\ \int_1^\infty \frac{\sin\left(\frac{1}{x}\right)}{x}dx$ $\quad (f)\ \int_0^\infty e^{-x^2}dx$ $\quad (g)\ \int_0^{\pi/2} \frac{dx}{x - \sin x}$ $\quad (h)\ \int_0^{\pi/2} \text{cosec} x dx.$

Ans. $(a)$ Converges by limit comparison test (LCT) with $\frac{1}{\sqrt{x}}$.

$(b)$ Diverges by LCT with $\frac{1}{x^2}$.

$(c)$ The integral $-\displaystyle\int_0^1 \frac{\log x}{\sqrt{x}}$ converges by LCT with $\frac{1}{x^p}$, where $\frac{1}{2} < p < 1$.

$(d)$ Since $|\sin\frac{1}{x}| \leq 1$, the integral converges. Note that in this case the integral is a proper integral.

$(e)$ Converges by LCT with $\frac{1}{x^2}$.

$(f)$ Converges by LCT with $\frac{1}{x^p}$, where $p \geq 2$.

$(g)$ Apply LCT with $\frac{1}{x^3}$. The integral diverges.

$(h)$ $\displaystyle\int_0^{\pi/2} \text{cosec} x\, dx = \int_0^{\pi/2} \frac{1}{\sin x}\, dx$. Apply LCT with $\frac{1}{x}$. The integral is divergent.

Q7. In each case, determine the values of $p$ for which the following improper integrals converge

$(a)$ $\displaystyle\int_0^\infty \frac{1-e^{-x}}{x^p}$ $\quad(b)$ $\displaystyle\int_0^\infty \frac{t^{p-1}}{1+t}dt.$

Ans. $(a)$

$$\int_0^\infty \frac{1-e^{-x}}{x^p} = \int_0^1 \frac{1-e^{-x}}{x^p} + \int_1^\infty \frac{1-e^{-x}}{x^p} = I_1 + I_2.$$

Now one has to see how the function $\frac{1-e^{-x}}{x^p}$ behaves in the respective intervals and apply the LCT.

Since $\displaystyle\lim_{x\to 0} \frac{1-e^{-x}}{x} = 1$, by LCT with $\frac{1}{x^{p-1}}$, we see that $I_1$ is convergent iff $p-1<1$, i.e. $p<2$. Similarly, $I_2$ is convergent (by applying LCT with $\frac{1}{x^p}$ ) iff $p>1$. Therefore $\displaystyle\int_0^\infty \frac{1-e^{-x}}{x^p}$ converges iff $1<p<2$.

$(b)$

$$\int_0^\infty \frac{t^{p-1}}{1+t}dt = \int_0^1 \frac{t^{p-1}}{1+t}dt + \int_1^\infty \frac{t^{p-1}}{1+t}dt = I_1 + I_2.$$

For $I_1$, use LCT with $t^{p-1}$. We see that the integral converges iff $p>0$. Similarly, for $I_2$, Use LCT with $t^{p-2}$. The integral converges iff $p<1$. Therefore, $\displaystyle\int_0^\infty \frac{t^{p-1}}{1+t}dt$ converges iff $0<p<1$.

Q8. Show that the integrals $\displaystyle\int_0^\infty \frac{\sin x^2}{x^2}dx$ and $\displaystyle\int_0^\infty \frac{\sin x}{x}dx$ converge. Further, prove that

$$\int_0^\infty \frac{\sin x^2}{x^2}dx = \int_0^\infty \frac{\sin x}{x}dx.$$

Ans.
$$\int_0^\infty \frac{\sin x^2}{x^2}dx = \int_0^1 \frac{\sin x^2}{x^2}dx + \int_1^\infty \frac{\sin x^2}{x^2}dx = I_1 + I_2.$$
$I_1$ is a proper integral and $I_2$ converges by a comparison with $\frac{1}{x^2}$.
Similarly $\displaystyle\int_0^\infty \frac{\sin x}{x}dx.$ converges by Dirichlet test.
Using integration by parts we see that
$$\int_0^\infty \frac{\sin x^2}{x^2}dx = -\frac{\sin x^2}{x}\bigg|_0^\infty + \int_0^\infty \frac{2\sin x\cos x}{x}dx = \int_0^\infty \frac{\sin 2x}{2x}d(2x) = \int_0^\infty \frac{\sin x}{x}dx.$$

Q9. Show that $\displaystyle\int_0^\infty \frac{x\log x}{(1+x^2)^2}dx = 0.$

Ans.
$$\int_0^\infty \frac{x\log x}{(1+x^2)^2}dx = \int_0^1 \frac{x\log x}{(1+x^2)^2}dx + \int_1^\infty \frac{x\log x}{(1+x^2)^2}dx = I_1 + I_2.$$
Since, $\lim\limits_{x\to 0} x\log x = 0$, $I_1$ is a proper integral.
For large $x$, $\log x \le x$. Hence $\frac{x\log x}{(1+x^2)^2} \le \frac{x^2}{(1+x^2)^2} \le \frac{1}{1+x^2}$ and $I_2$ converges. Use the substitution $x = \frac{1}{t}$ in $I_1$ to get $I_1 = -I_2$.

Q10. Prove that $\displaystyle\int_1^\infty \frac{\sin x}{x^p}dx$ converges conditionally for $0 < p \le 1$ and and absolutely for $p > 1$.

Ans. By Dirichlets Test, $\displaystyle\int_1^\infty \frac{\sin x}{x^p}dx$ converges for all $p > 0$.
$\displaystyle\int_1^\infty \frac{|\sin x|}{x^p}dx \le \int_1^\infty \frac{1}{x^p}dx.$ Therefore, the function converges absolutely for $p > 1$.
Now, let $0 < p \le 1$. Since, $|\sin x| \ge \sin^2 x$, we see that $\left|\frac{\sin x}{x^p}\right| \ge \frac{\sin^2 x}{x^p} = \frac{1-\cos 2x}{2x^p}$. By Dirichlets Test, $\displaystyle\int_1^\infty \frac{\cos 2x}{x^p}$ converges $\forall p > 0$. But $\displaystyle\int_1^\infty \frac{1}{2x^p}$ diverges for $p \le 1$.
Hence, $\displaystyle\int_1^\infty \frac{\sin x}{x^p}dx$ converges conditionally for $0 < p \le 1$ and absolutely for $p > 1$.

Q11. Show that $\displaystyle\int_0^s \frac{1+x}{1+x^2}dx$ and $\displaystyle\int_{-s}^0 \frac{1+x}{1+x^2}dx$ do not approach a limit as $s \to \infty$. However $\displaystyle\lim_{s\to\infty}\int_{-s}^s \frac{1+x}{1+x^2}dx$ exists.

Ans. $\displaystyle\int_0^s \frac{1+x}{1+x^2}dx$ diverges by limit comparison with $\frac{1}{x}$.

$$\begin{aligned}\int_{-s}^s \frac{1+x}{1+x^2}dx &= \int_{-s}^0 \frac{1+x}{1+x^2}dx + \int_0^s \frac{1+x}{1+x^2}dx \\ &= \int_0^s \frac{1-u}{1+u^2}du + \int_0^s \frac{1+x}{1+x^2}dx \\ &= \int_0^s \frac{2du}{1+u^2}du,\end{aligned}$$

which converges.

Q12. Investigate the convergence of the improper integral

$$I = \int_0^1 \frac{dx}{\sqrt{1-x^3}}.$$

Ans. Note that $1-x^3 = (1-x)(1+x+x^2)$. Let us compare the given function with $\frac{1}{\sqrt{1-x}}$.

$$\lim_{x\to 1}\frac{1/\sqrt{1-x^3}}{1/\sqrt{1-x}} = \lim_{x\to 1}\frac{\sqrt{1-x}}{\sqrt{1-x^3}} = \lim_{x\to 1}\frac{1}{\sqrt{1+x+x^2}} = \frac{1}{\sqrt{3}}.$$

Now

$$\int_0^1 \frac{dx}{\sqrt{1-x}} = \left[-\frac{2}{\sqrt{1-x}}\right]_0^1 = 2.$$

and so by LCT the integral $I$ converges.