Work & Energy.

Most of the material you have studied earlier. We need to be clear about work Energy theorem and PE function. We need to have clear feeling about conservative force and non conservative force. Most of the time we define it mathematically but we don't feel about it. We have to learn how to do the line integral in straight line and in curve.

Aneog.
28 Aug, 2016.

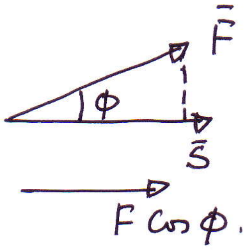
Work & Energy.

Work Done – / Work done by a Const Force.
\ Work done by a Variable force.

• Work Done by a Constant Force.

$$\boxed{W = \bar{F} \cdot \bar{S} = F S \cos \phi}$$

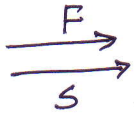
Force   displacement.



Work done in Physics is the dot product (Scalar Product) of force & displacement. The component of the force along the displacement vector (or otherwise) times the displacement gives the work done.

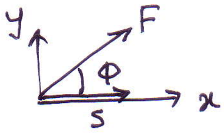
Positive & Negative Work.

(a) If $\phi = 0$



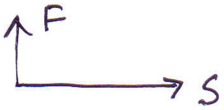
W = positive.

(b) If $0 < \phi < 90$



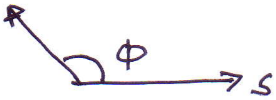
W = positive.

(c) If $\phi = 90^\circ, 270^\circ$



W = zero.

(d) If $90 < \phi < 270^\circ$



W = negative.

• When Force / Component of force is along the displacement W is +ve.

• When Force / Comp of force is opp to displacement W is −ve.

Work done by a Variable force.

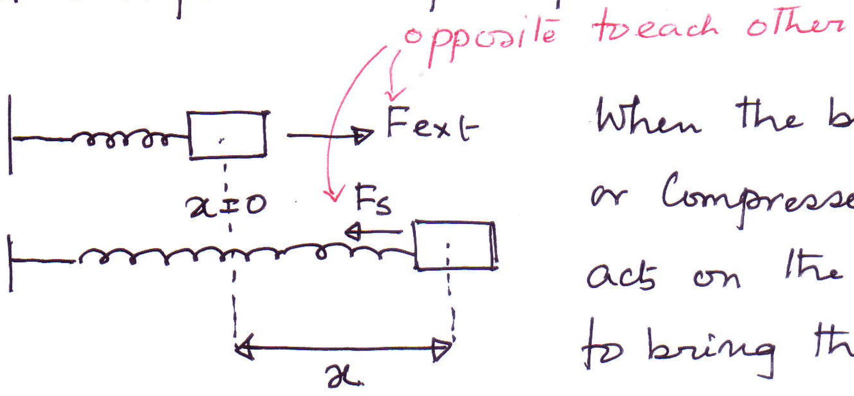
$$dW = \bar{F} \cdot d\bar{r}$$

$$W = \int dW = \int_{r_1}^{r_2} \bar{F} \cdot d\bar{r} = \int_{x_1}^{x_2} F_x \cdot dx$$ if the displacement is along x axis.

It is important to write proper limits of integration.

Simple Example.

A Simple example of a Variable force is Spring Force.



When the block is either Stretched or Compressed a restoring force (Fs) acts on the block which tries to bring the block back to its equilibrium position.

The stretching Force (Fext) is always opposite to the restoring force (Fs).

$$\bar{F}_{ext} = -\bar{F}_s.$$

Here $F_s \propto x$.

$$F_s = -kx$$

The work done by Restoring force

$$W_s = \int dw = \int_0^x F_s\, dx = \int_0^x -kx\, dx = -\frac{1}{2}kx^2$$

Here work done by restoring force is negative because force is in opposite direction to the displacement of spring.

Work Energy Theorem.

This is also known as Work kinetic energy Theorem. This Theorem is very fundamental which relats work to KE. It stats.

(U.V. Imp)

The net Work done by all The forces acting on a body is equal to change in KE of the body.

Here the most important point to understand is work done by all the forces must be taken into Consideration otherwise the Theorem will not hold good. All the forces include Conservative, non conservative, external & internal force.

$$\boxed{W_{net} = \bar{F}_{net} \cdot \bar{s} \Rightarrow \int_a^b \bar{F}_{net} \cdot d\bar{r} = \Delta k = \frac{1}{2} m V_b^2 - \frac{1}{2} m V_a^2}$$

$\Downarrow$ If Const Force $\Downarrow$ If Variable force.

We can apply this law everywhere.

V. Imp

Q: Suppose I hold an object in my hand & I raise it to a height H. The work done by me will be mgH. But the body is still at rest & there is no change in KE. Why? Where have I gone Wrong?

General Derivation of the Work Energy Theorem.

Case 1 : The general case of rectilinear motion, when the net Force F is not Constant in magnitude but is Constant in direction & parallel to the Velocity of the particle.

$$W = \int_a^b \bar{F}\cdot d\bar{r} = \int_{t_1}^{t_2} \bar{F}\cdot\bar{v}\,dt \qquad \left[\frac{d\bar{r}}{dt} = \bar{v}\right]$$

a → limit changes.

$$\text{So } W = \int_{t_1}^{t_2} \bar{F}\cdot\bar{v}\,dt = \int_{t_1}^{t_2} F v\,dt = \int_{t_1}^{t_2} m a v\,dt = m\int_{t_1}^{t_2} a v\,dt$$

$$= m\int_{t_1}^{t_2} v\frac{dv}{dt}\,dt = m\int_{v_1}^{v_2} v\,dv = \frac{1}{2}m\left(v_2^2 - v_1^2\right)$$

$$= \Delta E_K.$$

Case 2 : It is a more general Case of a force acting on a particle moving along any Curvilinear path

$$W = \int_{t_1}^{t_2} \bar{F}\cdot\bar{v}\,dt = m\int_{t_1}^{t_2} (\bar{a}\cdot\bar{v})\,dt \Rightarrow$$

What is $\bar{a}\cdot\bar{v}$? This Can be Written as

identity $\quad \bar{a}\cdot\bar{v} = \frac{1}{2}\frac{dv^2}{dt}$ (How) See below.

$$\frac{dv^2}{dt} = \frac{d}{dt}(\bar{v}\cdot\bar{v}) = \frac{d\bar{v}}{dt}\cdot v + v\cdot\frac{d\bar{v}}{dt} = 2\frac{d\bar{v}}{dt}\cdot\bar{v} = 2\bar{a}\cdot\bar{v}$$

So the above expression Can be Written as.

$$W = m\int_{t_1}^{t_2} \bar{a}\cdot\bar{v}\,dt = \frac{m}{2}\int_{t_1}^{t_2}\frac{dv^2}{dt}\,dt = \frac{m}{2}\int_{v_1^2}^{v_2^2} dv^2 = \frac{m v_2^2}{2} - \frac{m v_1^2}{2}$$

→ limit changes

$$= \Delta E_k.$$

Conservative & Non Conservative Forces.

Lot has been written about Conservative & Non-Conservative forces but still there is confusion about this topic. For a Conservative force work done is path independent or work done around a closed path is zero. But this does not give a feel about the two forces.

There are different kinds of forces. One way to classify them is contact forces & ~~forces~~ action at a distance force. Contact forces are those forces which arise when two surfaces are in contact where as there are many forces like gravitational forces which act when two bodies are at a distance.

There is another way to classify different forces in nature depending on whether they conserve the total mechanical energy or not. They forces can be ~~are~~ commonly ~~named~~ divided as internal and external forces. Internal forces like gravitational force, spring force, electrical force between two charges are between two bodies (a System). When work is done agaist these forces energy is stored in the force field/system as potential energy. These forces are named as Conservative forces. Work done by these forces (+ve or -ve) conserves the Total Mechanical Energy of the system.

There are some other forces which can be grouped under external forces which either increase or decrease the total mechanical energy of the body or the system. These forces are grouped under Non Conservative forces. Examples of this kind of forces are ~~&~~ applied forces (not all), frictional forces, ~~air dr~~ viscous forces, tension forces etc.

Now this is a general discussion to get a feel ~~of~~ ~~what~~ about conservative & non conservative force which we see in nature.

Now suppose there is an externally applied force. (say)

$F_1 = 2\hat{i} + 3\hat{j} + 4\hat{k}$

$F_2 = 3x^2\hat{i} + 2y\hat{j}$

$F_3 = A(xy\hat{i} + y^2\hat{j})$

Now are these forces conservative or non conservative. ?

How to find out ?

They may or may not.

Now we have to go to Mathematical definition of conservative and non conservative force.

But the above discussion is important because it gives us a feel about conservative / non conservative forces which we don't get from mathematical formula.

To Summarize:

Forces in Nature
- Conservative force (Gravity, Spring force)
  - +ve work
  - −ve work
  - But Total ME conserved
- Non Conservative force (friction.)
  - +ve work
  - −ve work
  - Total ME not conserved

External Forces by humans/machine
- Conservative force
- Non Conservative force.
- Same as above case respectively.

Before we go into the mathematical discussion there are two more important qustions.

(1) So all the forces can be subdivided into two major categoris – conservative & non conservative.
Is it true?

Yes that is true but some authors like H. C. Verma, D. C. Pandey prefers to divide into three categoris: F_{cons}, $F_{non\ cons}$, F_{ext}. They have not cleared (Pl correct me if I am wrong) what they mean about F_{ext}. I feel that by F_{ext} they mean all those forces which are man made but are not conservative. Man made forces are not spread through out the whole region as a vector field or force field & hence we can find some paths where $\oint \vec{F}.d\vec{l}$ is around a closed ~~loom~~ path is not zero & hence non conservative. But they have not included them as non conservative. Because they want to preserve the word non conservative for fiction like forces which are naturally occuring but not ~~mak~~ made.

But Kleppner & Kolenkow have divided all forces into two category only conservative and non-conservative. There man made external forces which are usually non conservative are taken in nonconservative category. (Pl. give your comments)

(2) What is the significance of this study – dividing / Grouping the Conservative & non Conservative forces.

It is because the work done by Conservative forces (whether +ve or –ve) can be expressed in the form of potential energy function. We must remember that when a weight is lifted from height h_1 to h_2, the work done by the person lifting the weight is +ve but work done by gravity is negative. So when Conservative force does negative work pot energy of the system increases.

$$-\int_{r_1}^{r_2} F.dr = U_{r_2} - U_{r_1} = \Delta U.$$

Why this negative Sign.

This is Conservative force which is force of gravity & not the force applied by the man in lifting the weight & that is why the –ve sign in beginning.

Now when the above function is combined with work KE Theorem for a Conservative force we get the famous equation of Law of Conservation of Mechanical energy which is only valid for Conservative forces.

Mathematical Formulation.

(1) WORK done by a Conservative force in moving a particle between two points is independent of the path taken and depend only on the end points

(2) Equivalently, if a particle travels a closed loop, the net work done (the sum of the force acting along the path multiplied by the distance travelled) by a Conservative force is zero.

(3) If a force is Conservative, it is possible to assign a numerical value for the potential at any point.

(4) If the force is non Conservative above 3 Conditions are not fulfilled.

A force field F, defined everywhere in space (or within a Simply Connected volume of Space) is called a Conservative force or Conservative vector field if it meets any of these Three equivalent Conditions:

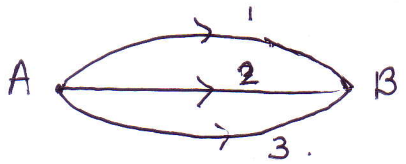
1. The Curl of F is the zero Vector.

$$\vec{\nabla} \times \bar{F} = \vec{0} \quad , \quad \left[\bar{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right]$$

2. There is zero net work done by the force when moving a particle through a trajectory that starts and ends in Same space

$$W = \oint_C \bar{F} \cdot d\bar{r} = 0$$

OR



The work done in going from A → B depends only on end points i.e $W_1 = W_2 = W_3$.

3. The force Can be Written as negative Gradient of a potential u

$$F = -\nabla u.$$

Few Questions. to Think upon.

1) Is the electric field (or electric force) produced by static change Conservative.

2) Is the electric field produced by time Varying magnetic field conservative.

3) Is the magnetic force Conservative?

4) Is the normal force Conservative.

5) If the Curl of a vector field is zero, is it always a Conservative field? (when not)

To find an answer of this question see Wikipedia
Conservative vector field / Irrotational vector fields.

6) If $F(x,y) = -ky\,\hat{i} - kx\,\hat{j}$, Is it a Conservative force.? Find Curl F.

Potential Energy.

In the Work energy Theorem we have seen

$$\int_a^b \bar{F}\cdot d\bar{r} = \frac{1}{2}mV_b^2 - \frac{1}{2}mV_a^2 = \text{change in KE.} \qquad (1)$$

This equation can be Written in many ways - sometims in one dimension, sometims for a single particle, Sometims for the Center of mass.

Now we write almost the same equation here, that is the first part. for the expression of PE.

$$-\int_{r_i}^{r_f} \bar{F}\cdot d\bar{r} = U_f - U_i = \Delta U = \text{Change in PE.} \qquad (2)$$

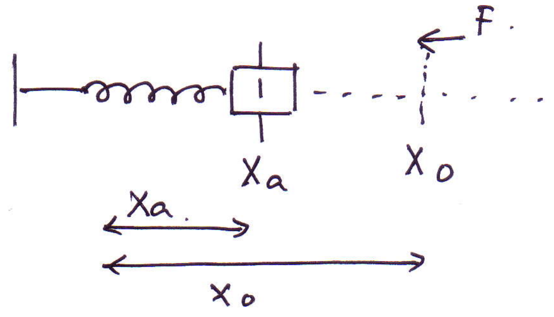
We have done lot of problems with PE but the above two equations are little bit confusing.

Few Important Points.

(1) Is the '$\bar{F}$' in the first equation & second equation same ?

No it is not. In the first equation F is the total force acting on a body that is work done by all the forces. In the second equation it is work done by conservative force only.

In the Second equation we talk about potential energy of the system rather than a particular object. like a ball earth system, a block spring system. These cases are there in some special situations & not everywhere. Here PE changes because of change of position/Configuration. The examples are ball earth system, block spring system, two / more electric charges as a system. Suppose there is a block spring System.



Suppose I give a force F which compress a block by $(X_o - X_a)$. Now while Calculating PE I only Consider the ~~force of~~ Conservative force of the Spring & the work done by it. The negative of the work done by the Spring is the pot energy U. Here if we go according to the first equation the total work done that is work done by me as well as work done by spring will be 0.

Now.

$$\Delta U = U_f - U_i = -\int_{r_i}^{r_f} F.\,dr.$$

We generally choose the reference point at infinity & assume potential energy to be zero There i.e if we take $r_i = \infty$ (infinite) and $U_i = 0$ We can Write.

$$U = -\int_{\infty}^{r} \bar{F}.\,d\bar{r} = -W.$$

Potential Energy of a body or system is the negative of Work done by the Conservative forces in bringing from infinity to the present position

- One can find out Conservative force F from PE function.

$$F = -\frac{dU}{dr} \quad \text{or} \quad -\frac{dU}{dx}.$$

Conservative forces always act in a direction where PE of the Systems is decreased.

PE of Mass Spring system. (Elastic PE).

When the spring is stretched or Compressed by an amount x from its unstretched position the work done by spring force $= -\frac{1}{2}kx^2$

$$U = -W = -\left(-\frac{1}{2}kx^2\right)$$

$$\boxed{U = \frac{1}{2}kx^2}$$

Gravitational PE.

- The Gravitational PE of two particles of masses m_1 and m_2 separated by a distance r given by

$$\boxed{U = -G\frac{m_1 m_2}{r}}$$

$G = 6.67 \times 10^{-11} \frac{N\text{-}m^2}{kg^2}$

$\downarrow$

Universal Gravitation Const.

- If a body of mass m is raised to a height h from the surface of the earth, the change in PE of the system (earth + body) comes out to be

$$\boxed{\Delta U = \frac{mgh}{\left(1+\frac{h}{R}\right)}}$$

$\Delta U \simeq mgh$ if $h \ll R$.

Electric PE.

The electric potential energy of two point charges q_1 & q_2 separated by a dist r in vacuum is given by.

$$\boxed{U = \frac{1}{4\pi\epsilon_0} \cdot \frac{q_1 q_2}{r}}$$

$$\frac{1}{4\pi\epsilon_0} = 9 \times 10^9 \frac{N\text{-}m^2}{C^2} = \text{const.}$$

Conservation of Mechanical Energy.

$W = -(U_f - U_i)$ for Conservative force.

$W = K_f - K_i$ Work energy Theorem

So $K_f - K_i = -(U_f - U_i)$

$$\boxed{K_f + U_f = K_i + U_i}$$

The Sum of potential Energy and KE is called the total mechanical energy. We see that total mechanical energy of a system remains Constant ⇒ Conservation of Mechanical Energy.

Please NOTE: The above expression is true only if Conservative forces are acting on a system of particles and the work done by all other forces is zero.

Non Conservative Forces.

Suppose non Conservative force such as Friction is acting on the system. Work Energy Theorem is still valid.

$$W_c + W_{nc} + W_{ext} = K_f - K_i$$

$$W_c = -(U_f - U_i)$$

Subtracting Second from the first

$$W_{nc} + W_{ext} = (K_f + U_f) - (K_i + U_i)$$

$$W_{nc} + W_{ext} = E_f - E_i = \Delta E$$

Where $E = K + U$ is total Mechanical Energy.

If $W_{ext} = 0$ $\boxed{W_{nc} = \Delta E}$

To Summarize.

(1) Work done by Conservative forces is equal to change in Potential energy

$$W_c = -\Delta u = -(u_f - u_i) = u_i - u_f$$

(2) Work done by all the forces is equal to change in KE

$$W_{all} = \Delta k = k_f - k_i$$

(3) Work done by the forces other than Conservative forces (non-Conservative + external forces) is equal to change in Mechanical energy.

$$W_{nc} + W_{ext} = \Delta E = E_f - E_i = (K_f + U_f) - (k_i + U_i)$$

\# A body is displaced from position A to position B. Kinetic and potential energies of the body at positions A & B are: $k_A = 50\,J$, $U_A = -30\,J$ $k_B = 10\,J$ $U_B = 20\,J$.

Find work done by (a) Conservative forces (b) all forces (c) forces other than Conservative forces.

⇒ (a) $W_c = -(U_f - U_i) = U_i - U_f = -30 - 20 = -50\,J$

WORK done by Conservative Force is −ve so Pot. Energy is rising.

(b) $W_{all} = \Delta k = k_f - k_i = k_B - k_A = 10 - 50 = -40\,J$

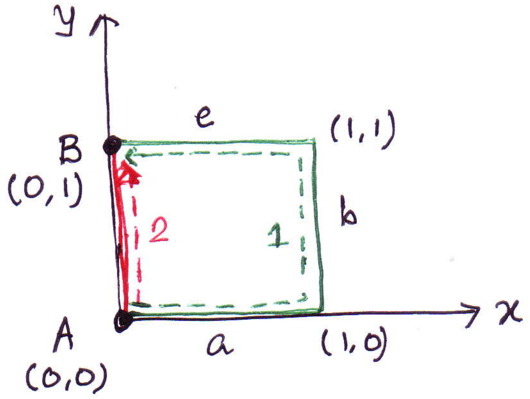
(c) ~~For~~ work done by the forces other than Conservative

$$= \Delta E = E_f - E_i = (k_B + U_B) - (k_A + U_A)$$

$$= (10 + 20) - (50 - 30) = 10\,J$$

Example of a Non Conservative force – A path dependent line Integral.

Consider a force $\bar{F} = A(xy\,\hat{i} + y^2\,\hat{j})$. Calculate $\int \bar{F}\cdot d\bar{l}$ along two paths in going from initial point A (0,0) to B (0,1)



A (0,0) & B (0,1) are two end points. We will go from A to B via two different paths & calculate $\int \bar{F}\cdot d\bar{l}$ — along path 1 (green) — along path 2 (red)

Integral along path 1 $\Rightarrow$ $\bar{F}\cdot d\bar{r} = (F_x\hat{i} + F_y\hat{j} + F_z\hat{k})\cdot(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$

General expression.

$$\oint_1 \bar{F}\cdot d\bar{r} = \int_a \bar{F}\cdot d\bar{r} + \int_b \bar{F}\cdot d\bar{r} + \int_c \bar{F}\cdot d\bar{r}$$

$$\int_a F\cdot dr = \int F_x\,dx = A\,xy\,dx, \quad y=0 \text{ along } a.$$

$$= 0. \quad (\text{Because } y = 0)$$

$$\int_b F\cdot dr = \int F_y\,dy = A\int y^2\,dy = \frac{A}{3}\Big[\,y^3\,\Big]_{y=0}^{y=1} = \frac{A}{3}.$$

$$\int_c F\cdot dr = A\int F_x\,dx = A\int xy\,dx = \frac{A}{2}\,x^2\Big|_1^0 = -\frac{A}{2}.$$

So $\oint_1 \bar{F}\cdot d\bar{r} = \frac{A}{3} - \frac{A}{2} = -A/6.$

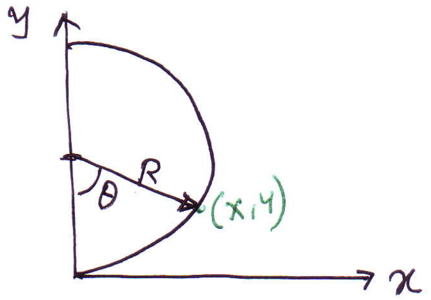
Along path 2.

$$\oint_2 F\cdot dr = \int F_y\,dy = A\int y^2\,dy = \frac{A}{3}\,\big|\,y^3\,\big|_0^1 = \frac{A}{3}.$$

$$\neq \oint_1 F\cdot dr.$$

Parametric Evaluation of a Line Integral.

Evaluate the line integral of $F = A(x^3\hat{i} + xy^2\hat{j})$ from $(x=0, y=0)$ to $(x=0, y=2R)$ along the semicircle shown.



⇒ The natural parameter to use here is θ and it varies from 0 to π.

$x = R\,\overset{\sin\theta}{\cancel{\cos\theta}}$, $dx = R\cos\theta\, d\theta$

$y = R(1-\cos\theta)$ $\quad dy = R\sin\theta\, d\theta$.

$F_x = A x^3 = A R^3 \sin^3\theta$, $\quad F_y = A R^3 \sin\theta\,(1-\cos\theta)^2$

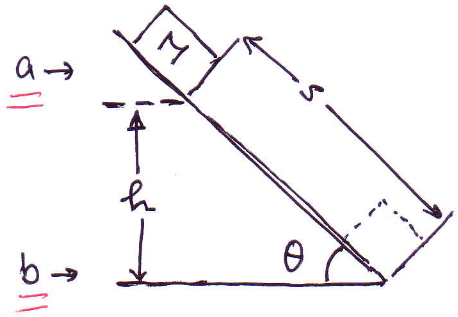
$$\oint \bar{F}\cdot d\bar{r} = A\int_0^{\pi} \left[[R\sin\theta]^3 R\cos\theta + R^3 \sin\theta\,(1-\cos\theta)^2 R\sin\theta \right] d\theta$$

$$= R^4 A \int_0^{\pi} \left[\sin^3\theta\cos\theta + \sin^2\theta\,(1-\cos\theta)^2 \right] d\theta.$$

Try substituting $u = \cos\theta$ & evaluate the integral.

Problem related to Non Conservative Force.

\# A block of mass M slides down a plane of angle θ. The problem is to find the speed of the block after it has descended through height h, assuming that it start from rest and coeff of friction μ is const.



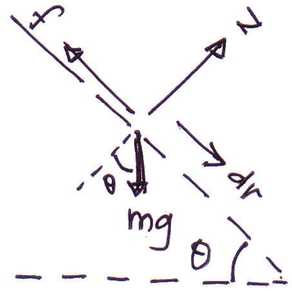
Initially block is at height h.
Finally block is moving at speed v, $h=0$.

(a)	(b)
$U_a = Mgh$.	$U_b = 0$.
$K_a = 0$	$K_b = \frac{1}{2}Mv^2$
$E_a = Mgh$.	$E_b = \frac{1}{2}Mv^2$

Non Conservative Force

Friction is non conservative force f

$f = \mu N = \mu Mg\cos\theta$



Non Conservative Work will be

$$W_{ba}^{nc} = \int_a^b \bar{f}\cdot d\bar{r} = -fs$$

(-ve sign arises because the direction of f is always opp to disp)

Geometry $\frac{h}{s} = \sin\theta$.

$$\Rightarrow s = h/\sin\theta.$$

$$W_{ba}^{nc} = -\mu Mg\cos\theta \frac{h}{\sin\theta}$$ [$f = \mu N$]

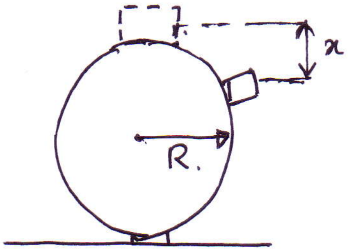
$$= -\mu\cot\theta\, Mgh.$$

Energy Eq is $\boxed{E_b - E_a = W_{ba}^{nc}}$

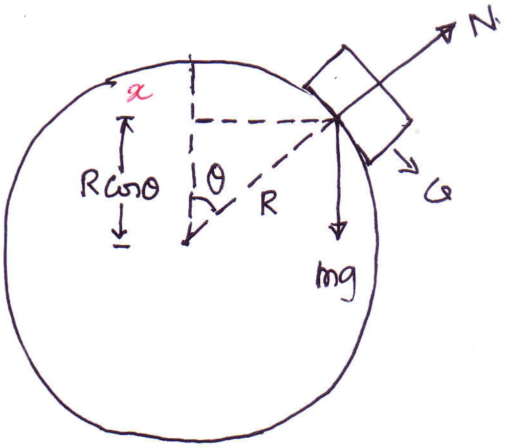
$$\frac{1}{2}Mv^2 - Mgh = -\mu\cot\theta\, Mgh.$$

$$\Rightarrow v = [2(1-\mu\cot\theta)gh]^{1/2}$$

A small block slides from rest from the top of a frictionless sphere of radius R. How far below the top x does it lose contact with the sphere? The sphere does not move.



This is a very Common problem & a very important problem. It involves. Three important concepts.



(i) Energy conservation.

$$mg\,(R - R\cos\theta) = \frac{1}{2} m v^2 \qquad (1).$$

(ii) In a circular motion the net force acting towards the Centre givs rise to Centripetal acc.

$$mg\cos\theta - N = m\frac{v^2}{R}. \qquad (11)\ [F = ma]$$

It is better to use the form of radial acc. derived in chapter 2.

(iii) When block leaves the surface at that point $N = 0$ (iii).

Solving them we get $\cos\theta = 2/3$. that is $R\cos\theta = \frac{2R}{3}$.

So $x = \frac{R}{3}$.