Electrostatic Fields in Matter

The basic questions we want to answer are:

- what happens to a dielectric when it is placed in an electric field? and
- what effect does the dielectric have on the field?

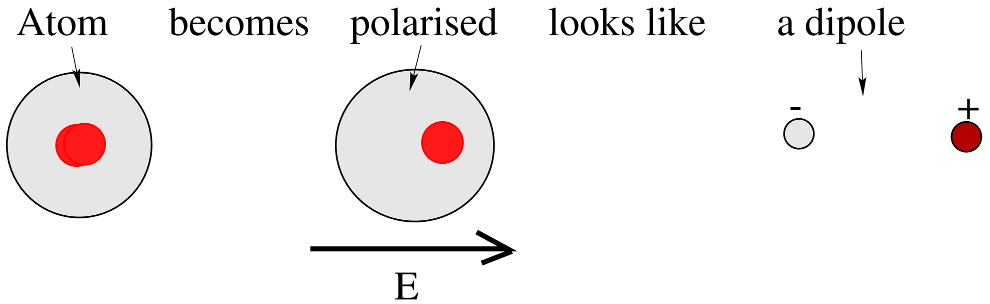
1 Revision

We need two ideas:

1. Charges generate electric fields.
2. Electric dipoles can be thought of as two point charges, $+q$ and $-q$, separated by a distance d. The *dipole moment* of the dipole is: $\vec{p} = q\vec{d}$, where $\vec{d}$ is directed from $-q$ to $+q$.

2 Atomic Dipoles and Polarisation

When an atom is placed in an electric field, the positive and negative charges are pulled in opposite directions.



The atom is said to be *polarised* by the field.

The induced dipole moment is usually proportional to the field. The constant of proportionality is called the *polarisability*

$$\vec{p} = \alpha \vec{E} \tag{1}$$

The total dipole moment of all the atoms in a material is described by the *polarisation*, which is defined to be the net dipole moment per unit volume:

$$\vec{P} = \frac{\Sigma_i \vec{p_i}}{V} \tag{2}$$

If all the atoms have the same dipole moment, equation (2) becomes:

$$\vec{P} = \frac{N\vec{p}}{V} = n\vec{p} \tag{3}$$

where N is the total number of atomic dipoles in the dielectric and n is the number per unit volume.

Since $\vec{p}$ is proportional to $\vec{E}$, then so also is $\vec{P}$.
The constant of proportionality is written:

$$\vec{P} = \epsilon_0 \chi_e \vec{E} \tag{4}$$

.

The quantity χ_e is called the *susceptibility* of the dielectric. Some books do not include the ϵ_0 in the definition, although most modern books do. From equations (1), (2) and (4) we have $n\alpha = \epsilon_0 \chi_e$.

3 Polarisation Mechanisms

There are 3 mechanisms involved in the polarisation of atoms and molecules:

1. Electronic polarization - displacement of electrons from their normal position in molecules.
2. Ionic polarization (or atomic polarization) - the displacement of ions from their normal positions within molecules.
3. Dipolar polarization - alignment of permanent molecular dipoles by the field.

The size of the effect is largest for 3, then 2 then 1. However effects 3 and 2 are not always present.

4 Frequency Dependence (non-examinable)

The polarisation generated by an alternating field varies with frequency and temperature.

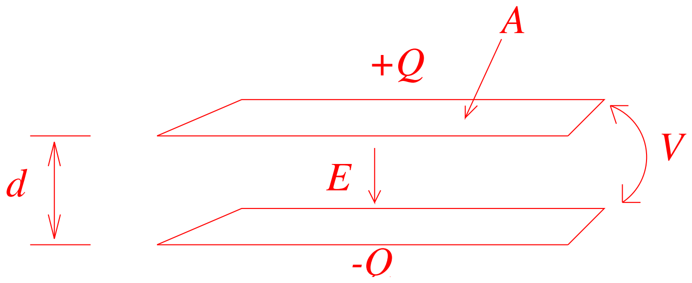
- Dipole polarisation occurs only at relatively low frequencies (up to high radio frequencies) and disappears at low temperatures.
- Atomic polarisation extends to the infrared.
- Electronic polarisation occurs for virtually all frequencies.

The refractive index is due to electronic polarisation.

5 Parallel Plate Capacitor.

A parallel plate capacitor is a useful model for demonstrating some of the basic ideas about dielectrics.

The figure shows two parallel conducting plates. A charge Q has been transferred from the bottom plate to the top and the plates are now isolated.



$$\text{From Gauss's law } \vec{E} = \frac{\sigma}{\epsilon_0} = \frac{Q/A}{\epsilon_0} \tag{5}$$

$$V = \int \vec{E} \cdot \vec{ds} = Ed \tag{6}$$

$$C = \frac{Q}{V} = \frac{\epsilon_0 A}{d} \tag{7}$$

These equations are based on the assumption that the field between the plates is constant, which is reasonable if the plates are close together.

What happens when a dielectric is placed between the plates?

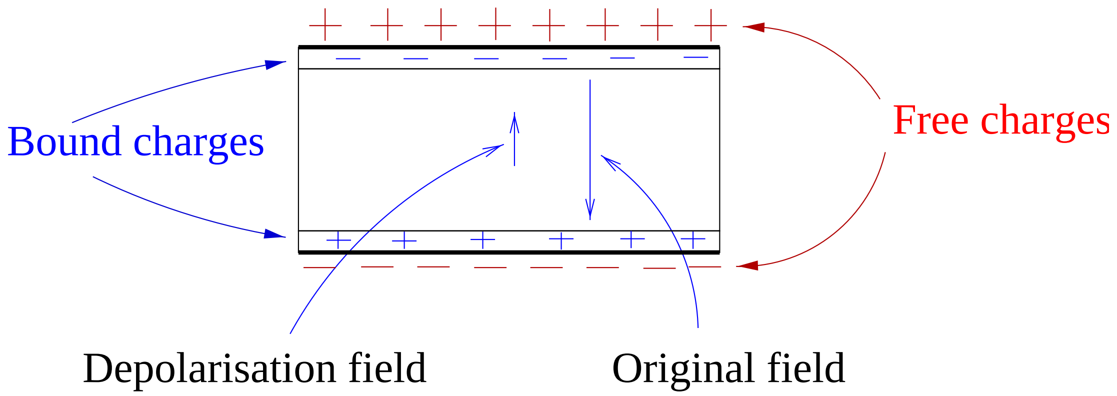
The original field, produced by the charges on the plates, pushes the +ve charges in the dielectric down and -ve charges up. The dielectric becomes *polarised*.

This leaves a layer of negative charge on the top surface of the dielectric and a positive layer at the bottom.

The original charge Q, which was placed on the capacitor plates, is referred to as *free charge*.

The charge induced on the dielectric surface is called *bound charge*.

The bound charge generates a new field, in the opposite direction to the original field. This new field is called the *depolarisation field*. The total field in the dielectric is the sum of the original field (from the free charges) and the depolarisation field - it will be smaller than the original field.



Looking at equations (6) and (7), we see that when a dielectric is placed between charged plates which are not connected, then Q remains constant while V decreases; hence C increases. The factor by which C increases is called the *dielectric constant* K (or the *relative permittivity*, ϵ_r).

6 Bound Charges

- Detailed derivation: Griffiths section 4.2.1, page 166 (non-examinable)
- Simpler derivation: Griffiths section 4.2.2, page 170.

The following is similar to Griffith's simpler derivation.

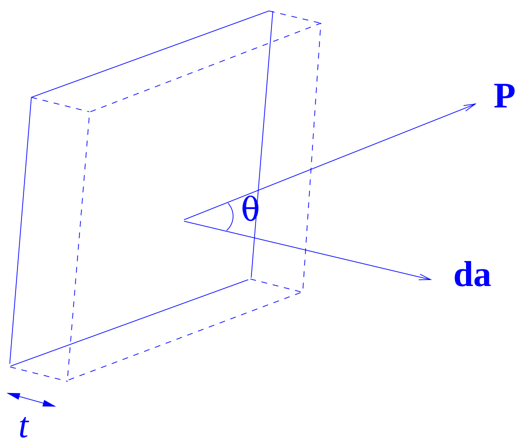
When a dielectric is polarised, the movement of charge creates regions in the dielectric which can have a net charge. We will derive expressions for the densities of these bound charges.

The electric field produced by a polarised dielectric is calculated simply by calculating the field due to the bound charges.

6.1 Surface Bound Charge Density

The diagram shows a small area da at the surface of a polarised dielectric. This area is represented by a vector $\vec{da}$, normal to the surface. The polarisation at this point is at some angle θ to the surface normal.

In each atom the positive charges have been displaced by $\vec{d}$ relative to the negative, so $\vec{p} = Q\vec{d}$ (and remember $\vec{P} = n\vec{p}$).



The charges move in the direction of $\vec{P}$. At the surface this leaves a layer of uncompensated charge, with a thickness $t = d\cos\theta$.

The amount of charge in the element in the diagram is the product of the element's volume and the charge per unit volume:

i.e. the charge is: $(da \times t) \times (nQ) = nQd\cos\theta da = n\vec{p} \cdot \vec{da} = \vec{P} \cdot \vec{da}$ (8)

The surface charge density (i.e. the charge per unit area) is thus:

$$\sigma_b = \vec{P} \cdot \hat{n} \tag{9}$$

where $\hat{n}$ is a unit vector, perpendicular to the surface and pointing out of the surface.

6.2 Volume Bound Charge Density

We saw above, that the net charge flowing across a surface when a dielectric is polarised is $\vec{P} \cdot \vec{da}$. If the volume is enclosed by a surface, then the net charge which flows out of that volume is given by :

$$Q_{out} = \oint_{area} \vec{P} \cdot \vec{da} \tag{10}$$

The charge left behind in the volume is just $-Q_{out}$.

$$-Q_{out} = -\oint_{area} \vec{P} \cdot \vec{da} = -\oint_{vol} \nabla \cdot \vec{P} d\tau \tag{11}$$

The last step uses Gauss' theorem.

What this tells us is that the net charge density inside the volume element is :

$$\rho_b = -\nabla \cdot \vec{P} \tag{12}$$

Note that since the volume charge density depends on spatial derivatives of the polarisation, it will be zero inside a uniformly polarised dielectric.

7 Permittivity and Susceptibility

The permittivity (and dielectric constant) are related to the susceptibility.

For simplicity we consider the case of a constant field, as found between parallel plates. We can then omit the vector nature of the field.

From Gauss's law

$$E = \frac{\sigma_f - \sigma_b}{\epsilon_0} \tag{13}$$

Using (9) and (4):

$$E = \frac{\sigma_f - P}{\epsilon_0} = \frac{\sigma_f - \epsilon_0 \chi_e E}{\epsilon_0}$$

$$\text{giving } E = \frac{\sigma_f}{\epsilon_0} \frac{1}{1 + \chi_e} \tag{14}$$

The field is reduced by the factor $\chi_e + 1$. The capacitance is increased by the same factor. Keeping in mind the definition of the dielectric constant, we have:

$$K = \epsilon_r = \chi_e + 1 \tag{15}$$

The permittivity of a dielectric is : $\epsilon = \epsilon_r \epsilon_0$

Equation (14) could be written in a variety of ways:

$$E = \frac{\sigma_f}{\epsilon_0 (1 + \chi_e)} = \frac{\sigma_f}{\epsilon_r \epsilon_0} = \frac{\sigma_f}{\epsilon} \tag{16}$$

Equation (16) illustrates an important point. In dielectrics it is usually possible to use the normal equations for calulating electric fields due to free charges by replacing ϵ_0 by ϵ.

This could all get confusing!
Remember

- ϵ will be a very small number (since $\epsilon_0 = 8.84 \times 10^{-12}$).
- Relative permittivity and dielectric constant are the same thing, with values usually between 1 and 100 (although 50,000 is possible).
- χ_e is one less than ϵ_r. (So what is its value for free space?).

8 Electric Displacement

The differential form of Gauss's law is:

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \tag{17}$$

We now think of the charge as two parts: $\rho = \rho_f + \rho_b$.

$$\text{This means that } \nabla \cdot \vec{E} = \frac{\rho_f + \rho_b}{\epsilon_0} = \frac{\rho_f - \nabla \cdot \vec{P}}{\epsilon_0}$$

$$\text{and this gives us } \nabla \cdot (\epsilon_0 \vec{E} + \vec{P}) = \rho_f \tag{18}$$

We now define a new quantity, the *electric displacement vector* $\vec{D}$, defined as

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} \tag{19}$$

$$\text{(This also means that } \vec{D} = \epsilon \vec{E}) \tag{20}$$

$$\text{We then have } \nabla \cdot \vec{D} = \rho_f \tag{21}$$

We can think of equation (21) as Gauss's law in a dielectric.

$\vec{D}$ is convenient because it depends only on the free charge. You can calculate $\vec{D}$ in the same way as you calculate $\vec{E}$, except that you only need include the free charge and there will be an ϵ_0 missing somewhere.

The integral form of equation (21) would be:

$$\oint \vec{D} \cdot \vec{da} = Q_{fenc} \tag{22}$$

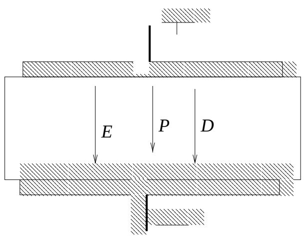
In a dielectric then, we have two equations:

$$\nabla \cdot \vec{D} = \rho_f \text{ and } \nabla \times \vec{E} = 0 \tag{23}$$

Example

A parallel plate capacitor with plates 5 mm apart is filled with polythene, which has $\epsilon_r = 2.26$. The capacitor is connected to a 9 V battery.

1. Calculate $\vec{E}$, $\vec{P}$, σ_b, $\vec{D}$ and σ_f in the dielectric.
2. Calculate the field due to the free charge and the depolarising field.
3. What extra information would you need if you wished to calculate $\vec{p}$, the atomic dipole moment?



Solution

Since in this simple geometry, all the vector quantities are in the same direction, we will ignore their vector nature.

1.
$$E = \frac{V}{d} = \frac{9\ \mathrm{V}}{5\ \mathrm{mm}} = 1.8 \times 10^3\ \mathrm{Vm}^{-1}$$

$$\begin{aligned} P &= \epsilon_0(K-1)E \\ &= 8.85 \times 10^{-12}(2.26-1)1.8 \times 10^3 \\ &= 2.01 \times 10^{-8}\ \mathrm{Cm}^{-2} \end{aligned}$$

$$\sigma_b = P = 2.01 \times 10^{-8}\ \mathrm{Cm}^{-2}$$

$$\begin{aligned} D &= \epsilon_0 E + P \\ &= 8.85 \times 10^{-12} \times 1.8 \times 10^3 + 2.01 \times 10^{-8} \\ &= 3.60 \times 10^{-8}\ \mathrm{Cm}^{-2} \end{aligned}$$

$$\sigma_f = D = 3.60 \times 10^{-8}\ \mathrm{Cm}^{-2}$$

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2. For a sheet of charge, with charge per unit area of σ, Gauss's law gives the field as σ/ϵ_0.

$$\text{So } E_{free} = \frac{\sigma_f}{\epsilon_0} = 4.07 \times 10^3 \text{ Vm}^{-1}$$

$$\text{and } E_{bound} = \frac{\sigma_b}{\epsilon_0} = 2.27 \times 10^3 \text{ Vm}^{-1}$$

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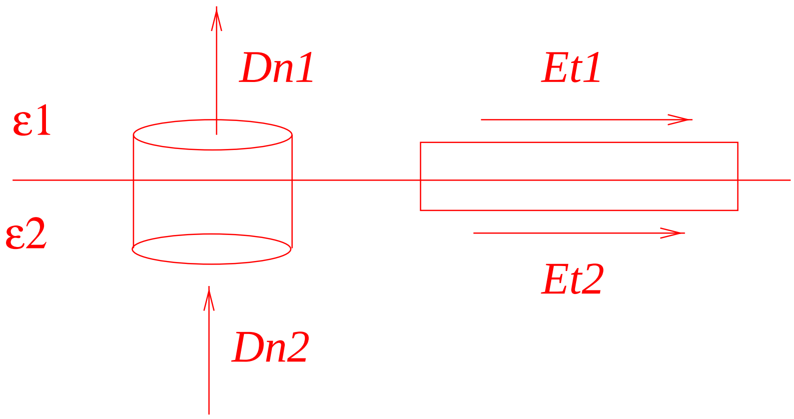
3. The polarisation $\vec{P}$ is defined as the total dipole moment per unit volume. In order to calculate $\vec{p}$, the moment of an individual dipole, we would need to know the number of dipoles per unit volume.

9 Boundary Conditions.

What happens at the junction of two dielectrics?

In the figure, the dielectric above the horizontal line has permittivity ϵ_1, while below the line it is ϵ_2.



Take the Gaussian pill-box on the left in the figure and imagine its height goes to zero. If there is no free charge at the boundary, there is no net flux of $\vec{D}$ out of the pill-box. Hence the boundary condition for $\vec{D}$ is:

$$D_{n1} = D_{n2} \tag{24}$$

We also know that the line integral of the electric field around the rectangle on the right must be zero (since $\nabla \times \vec{E} = 0$). As we imagine the rectangle shrinking to an infinitesimal size, we see that:

$$E_{t1} = E_{t2} \tag{25}$$

i.e. across the boundary of 2 dielectrics, the normal component of $\vec{D}$ is continuous and the tangential component of $\vec{E}$ is continuous.

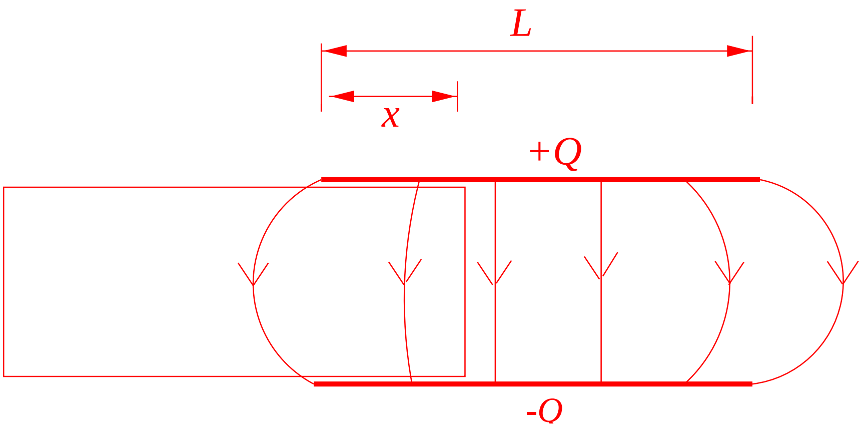
10 Force on a Dielectric

In general, an electric field will exert a torque on a dipole which is not parallel to the field and a force on a dipole if the two centres of charge are at points where the field is different.

We will not look at this in detail, although it is covered in Griffiths.

However there is an interesting example, which shows how lateral thinking can sometimes get answers out of apparently difficult situations.

Consider the situation shown below.



A parallel plate capacitor with a slab of dielectric partly inserted between the plates. The plates are isolated, so Q, the free charge on the plates, is constant.

Our task is to calculate the force on the dielectric slab. The situation is complex. The field is not constant and hence the polarisation will also vary with position.

The method used is to consider the energy. If the slab moves a distance dx into the capacitor, the energy changes since the capacitance changes. From conservation of energy we know that the work done by the force moving the dielectric must be equal to the change in the stored energy.

$$\text{i.e. } F dx = \Delta\left(\frac{1}{2}\frac{Q^2}{C}\right) = \left(-\frac{1}{2}\frac{Q^2}{C^2}\right) dC = \left(-\frac{1}{2}V^2\right) dC \qquad (26)$$

Take the length of the plates to be L and the distance the dielectric intrudes into the space between the plates to be x. The total capacitance is thus:

$$C = \frac{\epsilon_0 A}{d} \frac{L - x}{L} + \frac{K \epsilon_0 A}{d} \frac{x}{L} = \frac{\epsilon_0 A}{d} + \frac{\epsilon_0 A}{d} \frac{(K - 1)x}{L} \tag{27}$$

If x changes by dx, then the change in C will be:

$$dC = \frac{\epsilon_0 A}{d} \frac{(K - 1)}{L} dx \tag{28}$$

Combining equations (26), (27) and (28) gives finally:

$$F = -\frac{1}{2} V^2 \frac{\epsilon_0 A}{d} \frac{(K - 1)}{L} \tag{29}$$

The expression itself is not particularly important. The interesting point is that it is possible to calculate the force in a complex situation in such a simple way.

11 Energy in a Dielectric.

The following represents a simple derivation, although the result obtained is quite general.

We know that the energy stored in a charged capacitor is

$$W = \frac{1}{2}CV^2 \tag{30}$$

For a parallel plate capacitor $C = K\epsilon_o A/d$ and $V = Ed$.

$$\text{Thus } W = \frac{1}{2}K\epsilon_o AdE^2 \tag{31}$$

The energy per unit volume stored in the capacitor is thus:

$$U = \frac{1}{2}K\epsilon_0 E^2 = \frac{1}{2}\epsilon E^2 = \frac{1}{2}\vec{D}\cdot\vec{E} \tag{32}$$

The energy stored in a dielectric, for a given field, is K times larger then the energy stored in a vacuum. The extra energy comes from the polarization of the dielectric.

12 Summary

The state of a dielectric is characterised by three vector quantities:

The electric field $\vec{E}$

The polarisation $\vec{P}$ (= total dipole moment per unit volume).

The displacement $\vec{D}$ defined by $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$.

There are relations between these quantities.

$$\vec{D} = \epsilon \vec{E} = \epsilon_r \epsilon_0 \vec{E} \text{ and } \vec{P} = \epsilon_0 \chi_e \vec{E}$$

The polarisation gives rise to bound charges:

$$\sigma_b = \vec{P} \cdot \hat{n} \text{ and } \rho_b = -\nabla \cdot \vec{P}$$

The displacement $\vec{D}$ is determined by the free charges.

$$\nabla \cdot \vec{D} = \rho_f \text{ (in integral form this is } \oint \vec{D} \cdot \vec{da} = Q_{fenc})$$

The field $\vec{E}$ is determined by ALL the charges (i.e. free plus bound)

At a dielectric boundary the normal component of $\vec{D}$ and the tangential component of $\vec{E}$ are continuous.

The energy density in a polarised dielectric is $\frac{1}{2}\vec{D} \cdot \vec{E}$.

Example

A hollow dielectric sphere, with inner radius R_1 and outer radius R_2 has susceptibility χ_e. A charge Q is spread uniformly over the inner surface of the sphere'

1. Find $\vec{D}$, $\vec{E}$ and $\vec{P}$ in the three regions $r < R_1$, $R_1 \leq r \leq R_2$ and $r > R_2$.

2. Calculate the bound charges.