The LNM Institute of Information Technology
Jaipur, Rajasthan
MATH-II
Assignment #3

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1. Solve the following linear first order differential equations using the method of variation of parameters:
$$(i)\ xy' - 2y = x^4 \qquad (ii)\ y' + (\cos x)y = \sin x \cos x$$
{Method: To solve $y' + P(x)y = R(x)$, first find the general solution $y_h(x) = cy_1(x)$ of the homogeneous equation $y' + P(x)y = 0$. Assume that $y_p(x) = u_1(x)y_1(x)$ is a (particular) solution of the original equation. Find u_1 and the general solution of the original equation is given by $y(x) = y_h(x) + y_p(x)$.}

2. Solve $y' + y = x - 1$, $y' + y = \cos 2x$. Hence solve $y' + y = \cos^2 x - x/2$.

3. **(Ricatti equation)** The equation $y' = p(x) + q(x)y + r(x)y^2$, is called *Ricatti equation*, which is non-linear and in general cannot be solved by elementary methods. However, if a particular solution $y_p(x)$ is known, then the general solution has the form $y(x) = y_p(x) + z(x)$. Show that $z(x)$ satisfies the Bernoulli type equation $z' - (q + 2ry_p)z = rz^2$. Hence solve $y' - x^2y + xy^2 = 1$. (Observe that $y = x$ is a particular solution)

4. Show that the initial value problem
$$y' = y^{2/3},\ y(0) = 0,$$
has more than one solutions. Does this contradict the existence and uniqueness theorem.

5. Let $f(x, y)$ be continuous on the closed rectangle $R : |x - x_0| \le a, |y - y_0| \le b$.
   (i) Show that y is a solution of the initial value problem $y' = f(x, y)$, $y(x_0) = y_0$ if and only if
$$y(x) = y_0 + \int_{x_0}^{x} f[t, y(t)]dt.$$
   (ii) Let $|f(x, y)| \le M$ and $y_n(x) = y_0 + \int_{x_0}^{x} f[t, y_{n-1}(t)]dt$, with $y_0(x) = y_0$. Show by the method of induction that $|y_n(x) - y_0| \le b$ for $|x - x_0| \le h$, where $h = \min\{a, b/M\}$.

6. Use Picard's method of successive approximation to solve the following initial value problems and compare these results with the exact solutions:
$$(i)\ y' = 2\sqrt{x},\ y(0) = 1 \qquad (ii)\ y' + xy = x,\ y(0) = 0 \qquad (iii)\ y' = 2\sqrt{y}/3,\ y(0) = 0.$$

7. Apply (i) Euler method and (ii) improved Euler method to compute $y(x)$ at $x = 0.2,\ 0.4,\ 0.6,\ 0.8,\ 1.0$ for the initial value problem: $y' = xy + xy^2$, $y(0) = 1$. Compare the errors at each point with the exact solution.

8. Solve $y' = (y - x)^{2/3} + 1$. Show that $y = x$ is also a solution. What can be said about the uniqueness of the initial value problem consisting of the above equation with $y(x_0) = y_0$, where (x_0, y_0) lies on the line $y = x$.

9. Find all initial conditions so that the initial value problem $(x^2 - 2x)y' = 2(x - 1)y$, with $y(x_0) = y_0$ has (i) no solution, (ii) more than one solution, and (iii) only one solution. Discuss with reference to the existence and uniqueness theorem.

Supplementary problems from "Advanced Engg. Maths. by E. Kreyszig (8^{th} Edn.)
(i) Page 58 – 59, Q.1,2,5,14,18,19
(ii) Page 951, Q.1,2,6,7