The LNM Institute of Information Technology
Jaipur, Rajsthan

MATH-I ■ Assignment #1

(Real Number System, Sequences)

Q1. Let x be a real number such that x^2 is irrational. Show that x is also irrational. Deduce that $\sqrt{2}+\sqrt{3}$ is irrational.

Q2. Using the result "Let $m \in \mathbb{Z}, n \in \mathbb{N}$ and p be a prime. If $p|m^n$, then $p|m$." show that

1. $\sqrt{p}$ is irrational for any prime p.
2. $\sqrt{15}, \sqrt[3]{2}, \sqrt[5]{16}$ are irrational.

Q3. Find the infimum and supremum (if exists) of the sets $S_1 = \left\{\frac{m}{m+n} : m, n \in \mathbb{N}\right\}, S_2 = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$.

Q4. Using Archimedean property of real numbers show that for any $a \in \mathbb{R}$, there is some $m, n \in \mathbb{N}$ such that $-m < a < n$.

Q5. Use the Archimedean property of real numbers to show that $\displaystyle\bigcap_{n\in\mathbb{N}} \left(0, \frac{1}{n}\right] = \Phi$.

Q6. Let $a_n \to a$ and $a \neq 0$. Then show that there is $m \in \mathbb{N}$ such that $a_n \neq 0$ for all $n \geq m$.

Q7. Investigate the convergence/divergence of the following sequences:
(a) $x_n = \frac{1}{n^2+1} + \frac{2}{n^2+2} + \cdots + \frac{n}{n^2+n}$
(b) $x_n = \frac{n^2}{n^3+n+1} + \frac{n^2}{n^3+n+2} + \cdots + \frac{n^2}{n^3+2n}$
(c) $x_n = (n+1)^\alpha - n^\alpha$ for some $\alpha \in (0,1)$
(d) $x_n = \left(\sqrt{2} - 2^{\frac{1}{3}}\right)\left(\sqrt{2} - 2^{\frac{1}{5}}\right)\cdots\left(\sqrt{2} - 2^{\frac{1}{2n+1}}\right)$
(e) $x_n = \frac{n!}{(2n+1)!!}$.

Q8. Let $a > 0$ and $x_1 > 0$. Define $x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$ for all $n \in \mathbb{N}$. Prove that the sequence (x_n) converges to $\sqrt{a}$. *These sequences are used in the numerical calculation of* $\sqrt{a}$.