The LNM Institute of Information Technology
Jaipur, Rajasthan
MATH-II
Assignment #6

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1. Expand the following functions in terms of Legendre polynomials over $[-1, 1]$:

$$(i)\, f(x) = x^3 + x + 1 \quad (ii)\, f(x) = \begin{cases} 0 & \text{if } \quad -1 \le x < 0, \\ x & \text{if } \quad 0 \le x \le 1 \end{cases} \quad \text{(first three non-zero terms)}$$

2. Locate and classify the singular points in the following:

$$(i)\, x^3(x-1)y'' - 2(x-1)y' + 3xy = 0 \quad (ii)\, (3x+1)xy'' - xy' + 2y = 0.$$

3. For each of the following, verify that the origin is a regular singular point and find two linearly independent solutions:
   (a) $9x^2y'' + (9x^2 + 2)y = 0$ $\qquad$ (b) $x^2(x^2-1)y'' - x(1+x^2)y' + (1+x^2)y = 0,$
   (c) $xy'' + (1-2x)y' + (x-1)y = 0,$ $\qquad$ (d) $x(x-1)y'' + 2(2x-1)y' + 2y = 0.$

4. Show that the equation $2x^3y'' + (\cos 2x - 1)y' + 2xy = 0$ has only one Frobenius series solution.

5. Reduce $x^2y'' + xy' + (x^2 - 1/4)y = 0$ to normal form and hence find its general solution. (Infer that $J_{1/2}(x) = A\dfrac{\sin x}{\sqrt{x}}$).

6. Find a solution bounded near $x = 0$ of the following ODE:

$$x^2y'' + xy' + (\lambda^2x^2 - 1)y = 0.$$

7. Using recurrence relations, show that

$$(i)\; J_0''(x) = -J_0(x) + J_1(x)/x \qquad (ii)\; xJ_{n+1}'(x) + (n+1)J_{n+1}(x) = xJ_n(x).$$

8. Show that

$$(i)\; \int x^4 J_1(x)dx = (4x^3 - 16x)J_1(x) - (x^4 - 8x^2)J_0(x) + C,$$

$$(ii)\; \int J_5(x)dx = -2J_4(x) - 2J_2(x) - J_0(x) + C$$

9. Express
   (i) $J_3(x)$ in terms of $J_1(x)$ and $J_0(x)$ $\qquad$ (ii) $J_2'(x)$ in terms of $J_1(x)$ and $J_0(x)$
   (iii) $J_4(ax)$ in terms of $J_1(ax)$ and $J_0(ax)$.

10. Prove that between each pair of consecutive positive zeros of $J_\nu(x)$, there is exactly one zero of $J_{\nu+1}(x)$ and vice versa.

11. Let $y_\nu(x)$ be a nontrivial solution of Bessel's equation of order ν on the positive $x-$axis. Show that (i) If $0 \le \nu < 1/2$, then every interval of length π contains at least one zero of $y_\nu(x)$; (ii) If $\nu = 1/2$, then the distance between successive zeros of $y_\nu(x)$ is exactly π; and (iii) if $\nu > 1/2$, then every interval of length π contains at most one zero of $y_\nu(x)$.

12. Show that the Bessel's function J_ν, $(\nu \ge 0)$ satisfy

$$\int_0^1 xJ_\nu(\lambda_m x)J_\nu(\lambda_n x)dx = \frac{1}{2}J_{\nu+1}^2(\lambda_n)\delta_{mn},$$

where, λ_i are the positive zeros of J_ν.