National Institute Of Technology, Agartala

Mathematical Foundation

Assignment-1.

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1: Two dice are rolled. Let X denote the random variable which counts the total number of points on upturned faces. Construct a table giving the non-zero values of pmf. Also find the probability distribution function of X.

1: Sample Space:

11	12	13	14	15	16
21	22	23	24	25	26
31	32	33	34	35	36
41	42	43	44	45	46
51	52	53	54	55	56
61	62	63	64	65	66

The possible values of X are number 2 through 12 $X=2$ is event (11), so $P(2)=1/36$, $X=3$ is event (12, 21).

So $P(3)=2/36$

So the table is

x	2	3	4	5	6	7	8	9	10	11	12
$P(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

2: The following is the distribution function of a discrete random variable X :

x	-3	-1	0	1	2	3	5	8
$f(x)$	0.10	0.30	0.45	0.50	0.75	0.90	0.95	1.00

(a) find P(X is even)

$P(X=2 \,\&\, X=8 \,\&\, x=0) \Rightarrow 0.45 + 0.75 + 1.00$

$\Rightarrow 2.20$

(b) $P(1 \leq X \leq 8)$

$\Rightarrow P(X=1) + P(X=2) + P(X=3) + P(X=5) + P(X=8)$

$\Rightarrow 0.50 + 0.75 + 0.90 + 0.95 + 1.00$

$\Rightarrow 4.10$

(c) $P(X=-3 \mid x<0)$

$\Rightarrow \dfrac{P(X=-3 \cap X<0)}{P(X<0)}$

$\Rightarrow \dfrac{0.10}{0.10+0.30} \Rightarrow \dfrac{1}{4}$

(d) $P(X \geq 3 \mid X > 0)$

$\Rightarrow \dfrac{P(X \geq 3 \cap X>0)}{P(X>0)}$

$\Rightarrow \dfrac{0.90 + 0.95 + 1.00}{0.95 + 0.50 + 0.75 + 0.90 + 1.00}$

$\Rightarrow \dfrac{2.85}{4.10}$

3. A random variable X is distributed at random between the values 0 & 1 so that its probability density function is $f(x) = kx^2(1-x^3)$ where k is a constant. Find k. Using k, find mean & variance.

3. $\Rightarrow \int_{-\infty}^{\infty} kx^2(1-x^3) = 1$

$\Rightarrow \int_0^1 kx^2(1-x^3) = 1$

$\Rightarrow \left[\frac{kx^3}{3} - \frac{kx^6}{6}\right]_0^1 = 1$

$\Rightarrow \frac{k}{3} - \frac{k}{6} = 1$

$\Rightarrow 2k - k = 6$

$\Rightarrow \boxed{k = 6}$

mean, $\mu_1' = \int_{-\infty}^{\infty} x f(x) dx = 6\int_0^1 (x^3 - x^6) dx$

$= 6\left[\frac{x^4}{4} - \frac{x^7}{7}\right]_0^1$

$= 6\left[\frac{1}{4} - \frac{1}{7}\right] = 6 \times \frac{3}{28}$

$\boxed{\text{Mean} = \frac{9}{14}}$

variance $\mu_2' = \int_{-\infty}^{\infty} x^2 f(x) dx = 6\int_0^1 (x^4 - x^7) dx$

$\Rightarrow 6\left[\frac{x^5}{5} - \frac{x^8}{8}\right]_0^1$

$\Rightarrow 6\left[\frac{1}{5} - \frac{1}{8}\right] = 6 \times \frac{3}{40}$

$\boxed{\text{variance} = \frac{9}{20}}$

4. Suppose that the life in hours of a certain part of radio tube is a continuous r.v. x with p.d.f given by:

$$f(x) = \begin{cases} \frac{100}{x^2} & \text{when } x \geq 100 \\ 0 & \text{elsewhere} \end{cases}$$

(i) What is the probability that all of three such tubes in a given radio set will have to be replaced during first 150 hours of operation?

(i) A tube in radio set will have to be replaced during first 150 hours, if its life is less than 150 hours.

So req. probability

$$P(X \leq 150) \Rightarrow \int_{100}^{150} f(x)\,dx = \int_{100}^{150} \frac{100}{x^2}$$

$$\Rightarrow \left[\frac{-100}{x}\right]_{100}^{150}$$

$$\Rightarrow -\frac{100}{150} + 1 = \frac{50}{150}$$

$$\Rightarrow \frac{1}{3}$$

Probability that all 3 will be replaced during 1st 150 hours is $= \left(\frac{1}{3}\right)^3 = \frac{1}{27}$

(ii) What is the probability that none of three of the original tubes will have to be replaced during first 150 hours of operation.

(ii) Probability that a tube is not replaced is:

$\Rightarrow P(X > 150)$

$\Rightarrow 1 - P(X \leq 150) \Rightarrow 1 - \frac{1}{3} \Rightarrow \frac{2}{3}$

Probability that none of 3 tubes will be replaced is $= \left(\frac{2}{3}\right)^3 = \frac{8}{27}$

(iii) What is the probability that a tube will last less than 200 hours if it is known that tube is ~~still~~ functioning after 150 hours of service?

(iii) $P(X<200 \mid x \geq 150) = \dfrac{P(150<X<200)}{P(X>150)}$

$$= \frac{\int_{150}^{200} \frac{100}{x^2}\,dx}{\int_{150}^{\infty} \frac{100}{x^2}\,dx}$$

$$= \frac{\left[-\frac{100}{x}\right]_{150}^{200}}{\frac{2}{3}} \Rightarrow \frac{\left[-\frac{100}{200}+\frac{100}{150}\right]}{\frac{2}{3}}$$

$$= \frac{\frac{1}{6}}{\frac{2}{3}} = \frac{1}{4}$$

5. A petrol pump is supplied with petrol once a day. If its daily volume of sales (X) in thousands of litre is distributed by

$$f(x) = 5(1-x)^4 \quad 0 \leq x \leq 1,$$

What must be the capacity of its tank in order that the probability that its supply will be exhausted in a given day shall be 0.01?

5. Probability density function $= 5(1-x)^4$

Given $P(X>V) = 0.01$, x = sales

V = tank volume

$$F(t) = 5\int_0^t (1-x)^4\,dx$$

$$P(X>V) = 1-F(V)$$

$$F(V) = 5\int_0^V (1-x)^4\,dx$$

$$1-x = t$$

$-dx = -dt$

$x = 0 \; ; \; t = 1$

$x = v \; ; \; t = 1-v$

$$f(v) \Rightarrow 5 \int_{1}^{1-v} (t)^4 (-dt)$$

$$\Rightarrow -5 \left[\frac{t^5}{5} \right]_1^{1-v}$$

$$\Rightarrow -5 \left[\frac{(1-v)^5}{5} - \frac{1}{5} \right]$$

$$f(v) = -(1-v)^5 + 1$$

$$P(X > v) = 0.01$$

$$1 - F(v) = 0.01$$

$$1 - [-(1-v)^5 + 1] = 0.01$$

$$+(1-v)^5 = 0.01$$

$$v = 0.602$$

$v = 0.602 *$ thousand litres

$\boxed{v = 602 \text{ litres}}$

6. An experiment consist of three independent tosses of fair coin. Let X = number of heads, Y = number of head runs, Z = length of head runs, a head run is defined as consecutive occurence of at least two heads, its length then being the number of heads occuring together in three tosses of coin.

Find probability function of:

(a) X

possible outcomes of experiment is

$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

x = number of heads

It takes values {0, 1, 2, 3}

P(0 head) = 1/8 P(2 head) = 3/8

P(1 head) = 3/8 P(3 head) = 1/8

pdf :

X	0	1	2	3
P(X)	1/8	3/8	3/8	1/8

(b) Y = number of head runs.

It takes up values 0 & 1

Y	0	1
P(Y)	5/8	3/8

$P(Y=0) = P(0) = 5/8$

$P(Y=1) = P(1) = 3/8$

(c) Z = length of head runs

S.No.	Elementary Event	Random Var. X	Y	Z	X+Y	XY
1	HHH	3	1	3	4	3
2	HHT	2	1	2	3	2
3	HTH	2	0	0	2	0
4	HTT	1	0	0	1	0
5	THH	2	1	2	3	2
6	THT	1	0	0	1	0
7	TTH	1	0	0	1	0
8	TTT	0	0	0	0	0

Z takes values 0, 2 & 3

Z	0	2	3
P(Z)	5/8	2/8	1/8

(d) $P(x+Y)$ takes values $0, 1, 2, 3, 4$

$(X+Y)$	0	1	2	3	4
$P(X+Y)$	1/8	3/8	1/8	2/8	1/8

(e) $P(XY)$ takes values $0, 2, 3$

XY	0	2	3
$P(XY)$	5/8	2/8	1/8

7. The kms X in thousands of kms which car owners get with a certain kind of tyre is a random variable having p.d.f :

$$f(x) = \begin{cases} \frac{1}{20} e^{-x/20} & , x > 0 \\ 0 & , x \leq 0 \end{cases}$$

Find probability that one of these tyre will last

(i) atmost 10,000 kms

(i) Let r.v. X denote kms (in 000 kms) with a certain kind of tyre for 10,000 kms

$$\Rightarrow P(x \leq 10) = \int_0^{10} f(x)\,dx = \frac{1}{20}\int_0^{10} e^{-x/20}\,dx$$

$$= \frac{1}{20}\left[\frac{e^{-x/20}}{-1/20}\right]_0^{10}$$

$$= -[e^{-1/2} - 1]$$

$$= 1 - e^{-1/2}$$

$$= 1 - 0.6065$$

$$= 0.3935$$

(ii) anywhere b/w 16000 to 24000 kms.

(ii) $P(16 \leq x \leq 24) = \frac{1}{50} \int_{16}^{24} e^{-x/20} \, dx$

$= \frac{1}{20} \left[\frac{e^{-x/20}}{-1/20} \right]_{16}^{24}$

$= -\left[e^{-6/5} - e^{-4/5} \right]$

$= e^{-4/5} - e^{-6/5}$

$= 0.4493 - 0.3012$

$= 0.1481$

(iii) at least 30,000 kms

$P(X \geq 30) = \int_{30}^{\infty} e^{-x/20} \, dx$

$= \frac{1}{20} \left[\frac{e^{-x/20}}{-1/20} \right]_{30}^{\infty}$

$= -\left[e^{\infty/20} - e^{-3/2} \right]$

$= -\left[0 - e^{-1.5} \right]$

$= e^{1.5}$

$= 0.2231$

8. Suppose that time in minutes that a person has to wait at a certain bus stop is found to be a random phenomenon with a probability function specified by distribution function:

$$f(x) = \begin{cases} 0 & x < 0 \\ x/8 & 0 \leq x < 2 \\ x^2/10 & 2 \leq x < 4 \\ 1, & x \geq 4 \end{cases}$$

(i) Is the distribution function continuous? If so give formule for its p.d.f.

(i) Yes it is continuous. It is defined for all values of $-\infty < x < \infty$

Now $\frac{d}{dx} F(x) = f(x)$

So differentiating distribution function gives the density function

for $x<0$, $F(x) = 0$ $\quad \frac{d}{dx} F(x) = 0$

for $0 \le x < 2$, $F(x) = \frac{x}{8}$ $\quad \frac{d}{dx}\left(\frac{x}{8}\right) = \frac{1}{8}$

for $2 \le x < 4$, $F(x) = \frac{x^2}{16}$ $\quad \frac{d}{dx}\left(\frac{x^2}{16}\right) = \frac{x}{8}$

for $x \ge 4$ $\quad F(x) = 1$ $\quad \frac{d}{dx}(1) = 0$

So pdf is given by

$$f(x) = \begin{cases} 0 & x<0 \\ 1/8 & 0 \le x < 2 \\ x/8 & 2 \le x < 4 \\ 0 & x \ge 4 \end{cases}$$

(ii) What is the probability that a person will have to wait

(a) more than 2 minutes

(a) $P(X>2) \Rightarrow 1 - P(X \le 2)$

$\Rightarrow 1 - \frac{1}{8} \cdot 2$

$\Rightarrow \frac{3}{4}$

(b) less than two minutes

(b) $P(X<2) = \frac{1}{8} \cdot 2 = \frac{1}{4}$

(c) between 1 & 2 minutes

(c) $P(1 \le X \le 2) = \int_1^2 f(x)\,dx$

$= \int_{-\infty}^{2} f(x)\,dx - \int_{-\infty}^{1} f(x)\,dx$

$\Rightarrow P(X \leq 2) - P(X \leq 1)$

$\Rightarrow \frac{1}{8} \cdot 2 - \frac{1}{8} \cdot 1$

$\Rightarrow \frac{2}{8} - \frac{1}{8} \Rightarrow \frac{1}{8}$

(iii) What is the conditional probability that person will have to wait for a bus for

(a) more than two minutes, given it is more than one minutes

(a) $P(X \geq 2 | x \geq 1) = \frac{P(X > 2 \cap X > 1)}{P(X > 1)}$

$= \frac{P(X > 2)}{P(X > 1)}$

$= \frac{3/4}{1 - P(X < 1)}$

$= \frac{3}{4} \cdot \frac{8}{7}$

$= \frac{6}{7}$

(b) less than 2 minutes given that it is more than one minutes

(b) $P(X < 2 | x > 1) = \frac{P(X 1 < X < 2)}{P(X > 1)}$

$= \frac{1/8}{7/8}$

$= 1/7$

9. In a continuous distribution whose relative frequency distribution is given by

$f(x) = y_0 . x(2-x) \quad 0 \leq x \leq 2$

find mean, variance, median, mode of distribution & also show that for the distribution $\mu_{2n+1} = 0$

9: Now

Total probability = unity

$$\int_0^2 f(x)\,dx = 1$$

$$y_0 \int_0^2 x(2-x)\,dx = 1$$

$$y_0 \left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_0^2 = 1$$

$$y_0 \left[4 - \frac{8}{3}\right] = 1$$

$$y_0 \left[\frac{4}{3}\right] = 1$$

$$y_0 = \frac{3}{4}$$

$$f(x) = \frac{3}{4}x(2-x)$$

mean, $\mu_1' = \int_0^2 x f(x)\,dx = \frac{3}{4}\int_0^2 x^2(2-x)$

$$= \frac{3}{4}\left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^2 = \frac{3}{4}\cdot\left[\frac{16}{3} - 4\right]$$

$$= \frac{3}{4}\cdot\frac{4}{3}$$

$$= 1$$

variance, $\mu_2 = \mu_2' - \mu_1'$

$$= \frac{3}{4}\int_0^2 x^2 f(x)\,dx - 1$$

$$= \frac{3}{4}\int_0^2 x^2(x(2-x))\,dx - 1$$

$$= \frac{3}{4}\int_0 (2x^3 - x^4)\,dx - 1$$

$$= \frac{3}{4}\left[\frac{2x^4}{4} - \frac{x^5}{5}\right]_0^2 - 1$$

$$= \frac{3}{4}\left[8 - \frac{32}{5}\right] - 1$$

$$= \frac{1}{5}$$

Median :

$$\int_0^m f(x)dx = \frac{1}{2}$$

$$\frac{3}{4}\int_0^m x(2-x)dx = \frac{1}{2}$$

$$\frac{3}{4}\left|\frac{2x^2}{2} - \frac{x^3}{3}\right|_0^m = \frac{1}{2}$$

$$\left[x^2 - \frac{x^3}{3}\right]_0^m = \frac{2}{3}$$

$$\frac{3m^2 - m^3}{3} = \frac{2}{3}$$

$$3m^2 - m^3 = 2$$

$$m^3 - 3m^2 + 2 = 0$$

$$(m-1)(m^2 - 2m - 2) = 0$$

$$m = 1,\ 1 \pm \sqrt{3}$$

The only value lying b/w 0 & 2 is 1.

$$\boxed{\text{Median} = 1}$$

Mode :

$$f(x) = \frac{3}{4}(2x - x^2)$$

$$f'(x) = \frac{3}{4}(2 - 2x) = 0$$

$$\Rightarrow x = 1$$

$$f''(x) = \frac{3}{4}(-2) = -\frac{3}{2} < 0$$

Hence

$$\boxed{\text{Mode} = 1}$$

$$\mu_{2n+1} = \int_0^2 (x - \text{mean})^{2n+1} f(x)dx$$

$$= \frac{3}{4}\int_0^2 (x-1)^{2n+1} x(2-x)\,dx$$

$$= \frac{3}{4}\int_{-1}^1 (t)^{2n+1}(t+1)(1-t)\,dt \quad [x-1=t]$$

$$= \frac{3}{4}\int_{-1}^1 t^{2n+1}(1-t^2)\,dt$$

Now t^{2n+1} = odd function of t & $(1-t^2)$ is an even function of t, the integrand $t^{2n+1}(1-t^2)$ is an odd function of t.

Hence $\int_{-1}^{1}$ odd $= 0$

So $\boxed{\mu_{2n+1} = 0}$

10. The diameter say X of an electric cable, is assumed to be continuous r.v. with pdf $f(x) = 6x(1-x) \quad 0 \le x \le 1$

(i) Check that $f(x)$ is a p.d.f

(i) $f(x)$ is pdf in range [a to b] if

$$\int_a^b f(x) = 1$$

Then. $\int_0^1 6x(1-x)\,dx = \left[\frac{6x^2}{2} - \frac{6x^3}{3}\right]_0^1$

$= \left[3x^2 - 2x^3\right]_0^1$

$= 3-2$

$= 1$

So $f(x)$ is pdf

(ii) Obtain an expression for distribution function of x.

(ii) distribution function $= \int 6x(1-x)$

$= \frac{6x^2}{2} - \frac{6x^3}{3}$

$= 3x^2 - 2x^3$

distribution function:

$$F(x) = \begin{cases} 3x^2 - 2x^3 & x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

(iii) Compute the number k, such that $P(X<k) = P(x>k)$

$$\int_0^k [6x(1-x)]\,dx = \int_k^1 [6x(1-x)]\,dx$$

$$[3x^2-2x^3]_0^k = [3x^2-2x^3]_k^1$$

$$3k^2-2k^3-0 = 3-2-3k^2+2k^3$$

$$6k^2-4k^3 = 1$$

$$2(3k^2-2k^3) = 1$$

$$4k^3-6k^2+1 = 0$$

$$(2k-1)(2k^2-2k-1) = 0$$

$$k = \frac{1}{2} \quad \text{or} \quad k = \frac{2 \pm \sqrt{(2^2+4\times2\times1)}}{4}$$

$$k = \frac{1 \pm \sqrt{3}}{2} > 0 \quad \text{(not acceptable)}$$

So $\boxed{k = \frac{1}{2}}$

11. In four tosses of a coin, let x be number of heads. Tabulate all possible outcomes with corresponding value of X. By simple counting derive the probability distribution of X & hence calculate expected value of X.

Possible Outcomes	X
HHHH	4
HHHT	3
HHTH	3
HHTT	2
HTHH	3
HTHT	2
HTTH	2
HTTT	1
THHH	3
THHT	2
THTH	2
THTT	1
TTHH	2
TTHT	1
TTTH	1
TTTT	0

Probability distribution

x takes values 0, 1, 2, 3, 4

X	0	1	2	3	4
P(X)	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

Expected value

$$E(X) = \sum_{x=0}^{4} x\, p(x)$$

$$= 0 \cdot \frac{1}{10} + 1 \cdot \frac{1}{4} + 2 \cdot \frac{3}{8} + 3 \cdot \frac{1}{4} + 4 \cdot \frac{1}{6}$$

$$= \frac{1}{4} + \frac{3}{4} + \frac{3}{4} + \frac{1}{4}$$

$$= 2$$

12. A coin is tossed until a head appears. What is the expectation of number of tosses required.

12. Let x denote number of tosses req. to get first head.

$$E(X) = \sum_{x=1}^{\infty} x\, p(x)$$

Event	x	Prob. $p(x)$
H	1	1/2
TH	2	1/2 * 1/2 = 1/4
TTH	3	1/2 * 1/2 * 1/2 = 1/8
⋮	⋮	⋮

$$E(X) = 1 \times \frac{1}{2} + 2 \times \frac{1}{4} + 3 \times \frac{1}{8} + 4 \times \frac{1}{16} \cdots$$

This is an AGP with $r = 1/2$

$$S = 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{8} + \cdots$$

$$\frac{1}{2} S = \cancel{\frac{1}{4}} + \cancel{2} \frac{1}{4} + 2 \cdot \frac{1}{8} + 3 \cdot \frac{1}{16} + \cdots$$

$$\frac{1}{2} S = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

$$\frac{1}{2}S = \frac{1/2}{1-1/2} = 1$$

$$S = 2$$

So, $\boxed{E(X) = 2}$

13. What is the expectation of number of failures preceding the first success in an infinite series of independent trials with constant probability p of success in each trial ?

13. Let X denote number of failures

So it can take values $0, 1, 2 \ldots \infty$

$p(x) = P(X = x) = P$ (x failures precede first success)

$= q^x p.$

where, $q = (1-p)$ probability of failure in trial

Then by def :

$$E(X) = \sum_{x=0}^{q} x\, p(x)$$

$$= \sum_{x=0}^{\infty} x \cdot q^x p$$

$$= pq \sum_{x=1}^{\infty} x \cdot q^{x-1}$$

$$= pq\,[1 + 2q + 3q^2 + \ldots]$$

Now,

[$1 + 2q + 3q^2 + \ldots$ is an infinite AGP]

$$S = 1 + 2q + 3q^2 + 4q^3 + \ldots$$

$$qS = q + 2q^2 + 3q^3 + \ldots$$

$$(1-q)S = 1 + q + q^2 + q^3 + \ldots = \frac{1}{1-q}$$

$$S = \frac{1}{(1-q)^2}$$

So $E(x) = \frac{pq}{(1-q)^2} \Rightarrow \frac{pq}{p^2} \Rightarrow \boxed{E(x) = \frac{q}{p}}$

14. Starting from origin, unit steps are taken to right ------ . Find mean, variance of distance moved from origin after n steps

Let us associate a variable x_i with ith step

14. defined as follows

$$X_i = +1, \text{ ith step towards right.}$$
$$= -1, \text{ ith step towards left.}$$

Then, $S = X_1 + X_2 + \cdots + X_n = \sum x_i$ (random distance moved from origin after n steps)

$$E(x_i) = 1 \times p + (-1) \times q = p - q$$
$$E(x_i^2) = 1^2 \times p + (-1)^2 \times q = p + q = 1$$
$$\mathrm{var}(x_i) = E(x_i)^2 - [E(x_i)]^2 = (p+q)^2 - (p-q)^2$$
$$= 4pq$$
$$E(S_n) = \sum_{i=1}^{n} E(x_i) = n(p-q)$$
$$V(S_n) = \sum_{i=1}^{n} V(x_i) = 4npq$$

15. A man with n keys wants to open his door & tries the keys independently & at random. Find mean & variance of the number of trials required to open the door.

(i) if unsuccessful keys are not eliminated from further selection,

Suppose man gets first success in x trials, i.e. he is unable to open the door in first $(x-1)$ trials.

If unsuccessful keys are not eliminated then X is a random variable which can take values $1, 2, 3 \cdots \infty$.

$$P(\text{Success at first trial}) = 1/n$$
$$P(\text{Failure at first trial}) = 1 - 1/n$$

If unsuccessful keys are not eliminated then probability of success of consequently of failure in each trial.

$P(x)$ = Prob. of first success at x trial

$= (1-1/n)^{x-1} \cdot 1/n$

$\therefore E(x) = \sum_{x=1}^{\infty} x \cdot p(x)$

$= \sum_{x=1}^{\infty} x \cdot \left(1-\frac{1}{n}\right)^{x-1} \cdot \frac{1}{n}$

$= \frac{1}{n} \sum_{x=1}^{\infty} x \cdot A^{x-1}, \quad A = 1-\frac{1}{n}$

$E(x) = \frac{1}{n} [1+2A+3A^2+4A^3+\cdots]$

$= \frac{1}{n} (1-A)^{-2}$

$= \frac{1}{n} \left[1-\left(1-\frac{1}{n}\right)\right]^{-2}$

$= n$

$E(x^2) = \sum_{x=1}^{\infty} x^2 \cdot px$

$= \sum_{x=1}^{\infty} x^2 \left(1-\frac{1}{n}\right)^{x-1} \cdot \frac{1}{n}$

$= \frac{1}{n} \sum_{x=1}^{\infty} x^2 \cdot A^{x-1}$

$= \frac{1}{n} [1+2^2 \cdot A + 3^2 \cdot A^2 + \cdots]$

$= \frac{1}{n} (1+A)(1-A)^{-3}$

$= \frac{1}{n} \left[1+\left(1-\frac{1}{n}\right)\right]\left[1-\left(1-\frac{1}{n}\right)\right]^{-3}$

$= (2n-1)n$

$V(x) = E(x^2) - [E(x)]^2 = (2n-1)n - n^2$

$= n^2 - n = n(n-1)$

(ii) if they are :

P of success at first trial $= 1/n$

P of success at 2nd trial $= \frac{1}{n-1}$

P of success at 3rd trial $= \frac{1}{n-2}$

Hence, Probability of 1st success at 2nd trial

$$= \left(1-\frac{1}{n}\right)\frac{1}{n-1} \quad \frac{1}{n-2}$$

$$= \frac{1}{n} \quad \text{\& so on}$$

In general

$P(x) =$ Prob. of 1st success at x trial $= \frac{1}{n}$

$$E(x) = \sum_{x=1}^{\infty} x \cdot P(x) = \frac{1}{n}\sum_{x=1}^{n} x = \frac{n+1}{2}$$

$$E(x^2) = \sum_{x=1}^{\infty} x^2 \cdot p(x) = \frac{1}{n} \cdot \sum_{x=1}^{n} x^2$$

$$= \frac{n(n+1)(2n+1)}{6n}$$

$$V(x) = E(x^2) - [E(x)]^2$$

$$= \frac{(n+1)(2n+1)}{6} - \left(\frac{n(n+1)}{2}\right)^2$$

$$= \frac{n+1}{12}\left[2(2n+1) - 3(n+1)\right]$$

$$= \frac{n^2-1}{12}$$

Binomial distribution

16. Ten coins are thrown simultaneously. Find probability of getting atleast 7 heads.

16. $P(\text{getting head}) = 1/2$
$q = 1/2$

$P(\text{at least } 7) = P(7 \text{ head or } 8 \text{ head or } 9 \text{ head or } 10 \text{ head})$
$= P(X=7) + P(X=8) + P(X=9) + P(X=10)$

$P(X=r) = {}^{10}C_r\, p^r q^{10-r} = {}^{10}C_r (1/2)^{10}$
$= {}^{10}C_7 (1/2)^{10} + {}^{10}C_8 (1/2)^{10}$
$+ {}^{10}C_9 (1/2)^{10} + {}^{10}C_{10} (1/2)^{10}$
$= \left(\frac{1}{2}\right)^{10} [120 + 45 + 10 + 1]$
$= \frac{176}{1024}$

17. A & B plays a game in which their chances of winning are in the ratio 3:2. Find A's chance of winning atleast three games out of five games played.

17. $P(\text{win a game}) = 3/5$
$P(\text{winning at least } 3) = 1 - P(\text{winning } 0, 1, 2)$
$\Rightarrow 1 - \left[{}^5C_0 (3/5)^0 (2/5)^5 + {}^5C_1 (3/5)^1 (2/5)^4 + {}^5C_2 (3/5)^2 (2/3)^3\right]$
$\Rightarrow 1 - 0.31744$
$\Rightarrow 0.68250$

18. A coffee connoisseur ------. Find chance of having claim of

(i) accepted

$$P(a) = \frac{75}{100} = \frac{3}{4}$$

$$P(\text{at least } 5) = P(5) + P(6)$$

$$= 6C_5 (3/4)^5 (1/4) + 6C_6 (3/4)^6$$

$$= \frac{2186}{4096}$$

$$= 0.5336$$

(ii) rejected

$$P(\text{rejected}) = 1 - P(\text{accepted})$$

$$= 1 - 0.5336$$

$$= 0.4660645$$

19. A multiple choice test ---- the student secures a distinction?

19.

$$P(C) = \frac{2}{6} = \frac{1}{3}$$

$$P(I) = 1 - \frac{1}{3} = \frac{2}{3}$$

$$n = 8$$

$$P(x) = nC_x p^x q^{n-x}$$

atleast 75% correct answer

$$\Rightarrow (75/100) \times 8 = 6$$

Correct answers = 6, 7, 8

Probability that student secures distinctions =

$$P(6) + P(7) + P(8)$$

$$= 8C_6 (1/3)^6 (2/3)^2 + 8C_7 (1/3)^7 (2/3)^1$$

$$+ 8C_8 (1/3)^8 (2/3)^6$$

$$= 28 \times \frac{4}{3^8} + 8 \times \frac{2}{3^8} + \frac{1}{3^8}$$

$$= 0.01966$$

20. An irregular six-faced --- not an even numbers?

20. Let r.v. x denote number of even numbers

$P(\text{getting } x \text{ even numbers}) = P(X=x) = {}^{n}C_x\, p^x q^{10-x}$

$P(\text{getting 5 even numbers}) = 2P(\text{getting 4 even numbers})$

$P(X=5) = 2P(X=4)$

${}^{10}C_5\, p^5 q^5 = 2\; {}^{10}C_4\, p^4 q^6$

$\frac{p}{5} = \frac{q}{3} \Rightarrow 3p = 5q \Rightarrow 5(1-p)$

$\boxed{p = \frac{5}{8}}$

$q = 1-p \Rightarrow 1-\frac{5}{8} \Rightarrow \frac{3}{8}$

$P(x \text{ even numbers}) = {}^{10}C_x \left(\frac{5}{8}\right)^x \left(\frac{3}{8}\right)^{10-x}$

$x = 0, 1, 2 \ldots n$

required numbers of times that in 1000 sets of 10 throws each we get no even number

$= 1000 \times P(X=0)$

$= 1000 \times {}^{10}C_0 \left(5/8\right)^0 \left(3/8\right)^{10}$

$= 1$ (approx.)

21. In a binomial ...... find 'p' of distribution.

21. $n = 5$

probability of 1 & 2 successes are 0.4096 & 0.2048 respectively

${}^{5}C_1\, p^1 (1-p)^{5-1} = 0.4096$ — (1)

${}^{5}C_2\, p^2 (1-p)^3 = 0.2048$ — (2)

$\frac{(1-p)}{2p} = 2$

$5p = 1$

$\boxed{p = \frac{1}{5}}$

22. With the usual notations, find 'p' for the binomial variate x, if $n=6$ & $9P(X=4)=P(X=2)$

22. $9P(X=4) = P(X=2)$

$$9 \cdot {}^6C_4 \, p^4 (1-p)^2 = {}^6C_2 \, p^2 (1-p)^4$$

$$9p^2 = (1-p)^2$$

$$3p = 1-p$$

$$\boxed{p = \frac{1}{4}}$$