Maths

Date: 16/01/2023

Differential Equation

$*\ a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_n y = Q(x)$

$D \equiv \frac{d}{dx}$

$(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_n) y = 0$

Auxiliary Equⁿ is

$a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + \cdots + a_n = 0$

$m = m_1, m_2, m_3, \ldots, m_n$

*Direct method for obtaining particular integral:-

→ This methods depend on the nature of $Q(x)$. Particular Integral by this method can be obtained where $Q(x)$ has the following form.

(i) $Q(x) = e^{ax+b}$

(ii) $Q(x) = \sin(ax+b)$ or $\cos(ax+b)$

(iii) $Q(x) = x^m$ or polynomial in x.

(iv) $Q(x) = e^{ax} \cdot V(x)$

(v) $Q(x) = x \cdot V(x)$

(1) $\boxed{Q(x) = e^{ax+b}}$

$P.I. = \frac{1}{f(D)} e^{ax+b}$

$= \frac{1}{f(a)} e^{ax+b}$, (provided $f(a) \neq 0$)

If $f(a) = 0$

Then, $P.I = \frac{1}{f(D)} e^{ax+b} = x \cdot \frac{1}{f'(a)} \cdot e^{ax+b}$, provided $f'(a) \neq 0$

if $f'(a) = 0$

Then, P.I. $= \frac{1}{f(D)} e^{ax+b} = \frac{x^2 \cdot 1 \cdot e^{ax+b}}{f''(a)}$, Provided $f''(a) \neq 0$

Q) $(D^2 - 3D + 2)y = e^{3x}$

Solve→ Given $(D^2 - 3D + 2)y = e^{3x}$ —①.

Auxiliary eq^n is

$m^2 - 3m + 2 = 0$

$m^2 - 2m - m + 2 = 0$

$m(m-2) - 1(m-2) = 0$

$(m-1)(m-2) = 0 \Rightarrow m = (1, 2)$

C.F = $\boxed{c_1 e^{x} + c_2 e^{2x}}$
(Complementary function)

(Particular Integral) P.I. $= \frac{1}{f(D)} e^{3x} = \frac{1}{D^2 - 3D + 2} e^{3x}$

$\boxed{\text{Put } D = a = 3}$ $= \frac{1}{2} e^{3x}$

∴ General Solution ⇒ $y = c_1 e^{x} + c_2 e^{2x} + \frac{1}{2} e^{3x}$

Where c_1 & c_2 are the arbitrary constant.

Q) $(D^2 + 6D + 9)y = 5^x - \log 2$

$m^2 + 6m + 9$

$m^2 + 3m + 3m + 9$

$(m+3)(m+3) = 0$

$m = (-3, -3)$

C.F $= (c_1 + c_2 x) e^{-3x}$

P.F. $= \frac{1}{f(D)} (5^x - \log 2) = \frac{1}{D^2 + 6D + 9} (5^x - \log 2)$

$$= \frac{1}{D^2+6D+9} e^{x\log 5} - \frac{1}{D^2+6D+9} \log 2 e^{0\cdot x}$$

$$= \frac{1}{(\log 5)^2 - 6\log 5 + 9} \cdot e^{x\log 5} - \frac{1}{0+0+9} \log 2 e^{0\cdot x}$$

$$P.I = \frac{e^{x\log 5}}{(\log 5 + 3)^2} - \frac{\log 2}{9}$$

General Sol^n $y = (C_1 + C_2 x)e^{-3x} + \frac{e^{x\log 5}}{(\log 5 + 3)^2} - \frac{\log 2}{9}$

Q) ~~Solve~~ $(D^6 - 64)y = e^x \cosh 2x$ $\qquad \left(\cosh = \frac{e^x + e^{-x}}{2} \right)$

$$m^6 - 64 = 0$$

$$(m^3 - 8)(m^3 + 8) = 0$$

$$\cancel{m = \sqrt[3]{8} = 2, 2, 2}$$

$$\cancel{m = \sqrt[3]{-8} = 2 \times 2, \ldots}$$

$$\cancel{\sqrt[3]{2^3} \quad 2 \times 2 \times 2}$$

$$(m-2)(m^2 + 4 + 2m)(m+2)(m^2 + 4 - 2m) = 0$$

$$(m-2)(m+2)(m^2 + 2m + 4)(m^2 - 2m + 4) = 0$$

$m = 2$, $m = -2$ or $m = \frac{-2 \pm \sqrt{4-16}}{2}$ or $m = \frac{2 \pm \sqrt{4-16}}{2}$

$m = 2$, $m = -2$ or $m = -1 \pm \sqrt{3}i$ or $m = 1 \pm \sqrt{3}i$

$$C.F. = C_1 e^{2x} + C_2 e^{-2x} + C_3 e^{-1 + i\sqrt{3}x} + C_4 e^{-1 - i\sqrt{3}x} + C_5 e^{1 + i\sqrt{3}x} + C_6 e^{1 - i\sqrt{3}x}$$

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$\therefore$ $C.F. = c_1 e^{2x} + c_2 e^{-2x} + e^{x}(c_3 \cos\sqrt{3}x + c_4 \sin\sqrt{3}x)$

$+ e^{-x}(c_5 \cos\sqrt{3}x + c_6 \sin\sqrt{3}x)$

$$P.I. = \frac{1}{D^6-64} \cdot e^{x}\left[\frac{e^{2x}+e^{-2x}}{2}\right]$$

$$= \frac{1}{2} \cdot \frac{1}{D^6-64}(e^{3x} + e^{-x})$$

$$= \frac{1}{2} \cdot \frac{1}{D^6-64} \cdot e^{3x} + \frac{1}{2} \cdot \frac{1}{D^6-64} \cdot e^{-x}$$

$$= \frac{1}{2} \cdot \frac{1}{729-64} \cdot e^{3x} + \frac{1}{2 \cdot (1-64)} e^{-x}$$

$$= \frac{1}{2}\left(\frac{e^{3x}}{625} - \frac{e^{-x}}{63}\right)$$

(ii) $\boxed{Q x = \sin(ax+b) \text{ or } \cos(ax+b)}$

$$P.I. = \frac{1}{Q(D)} \sin(ax+b)$$

$$= \frac{1}{\phi(D^2)} \sin(ax+b)$$

$$= \frac{1}{\phi(-a^2)} \sin(ax+b) \text{ provided } \phi(-a^2) \neq 0$$

if $\phi(-a^2) = 0$,

$$P.I. = x \cdot \frac{1}{\phi'(-a^2)} \sin(ax+b) \text{ , provided } \phi'(a^2) \neq 0$$

If $\phi(-a^2) = 0$

$$P.I. = x^2 \cdot \frac{1}{\phi''(-a^2)} \sin(ax+b) \text{ , provided } \phi''(-a^2) \neq 0$$

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Q) $(D^2+9)y = \sin 4x$

A.E is $m^2+9=0$

$m = \pm 3i$

$C.F = e^{0\cdot x}(c_1 \cos 3x + c_2 \sin 3x)$

$P.I. = \frac{1}{D^2+9} \cdot \sin 4x$

$= \frac{1}{-16+9} \cdot \sin 4x \quad (D = a = 4)$

$= \frac{-1}{7} \sin 4x.$

General soln $\Rightarrow y = c_1 \cos 3x + c_2 \cos 3x - \frac{1}{7} \sin 4x$

Q) $(D^2+3D+2)y = \sin 2x$

$m^2+3m+2=0$

$m^2+2m+m+2=0$

$(m+2)(m+1)=0$

$m = -2, -1$

$C.F. = c_1 e^{-2x} + c_2 e^{-x}$

$P.I. = \frac{1}{D^2+3D+2} \cdot \sin 2x$

$= \frac{1}{-(2)^2+3D+2} \cdot \sin 2x$

$= \frac{1}{-2+3D} \cdot \sin 2x$

$= \frac{3D+2}{9D^2-4} \sin 2x$

$= \frac{(3D+2)\cdot \sin 2x}{9x-(2)^2-4} = -\frac{(3D+2)\cdot \sin 2x}{40}$

$3D \sin 2x + 2\sin 2x$

$6 \cos 2x + 2\sin 2x$

$= -\frac{1}{40}(6 \cos 2x + 2\sin 2x)$

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Q) $(D^4 + 2a^2D^2 + a^4)y = 8\cos ax$

Solve) $m^4 + 2a^2m^2 + a^4 = 0$

$m^4 + a^2m^2 + a^2m^2 + a^4 = 0$

$m^2(m^2 + a^2) + a^2(m^2 + a^2) = 0$

$(m^2 + a^2)(m^2 + a^2) = 0$

$m^2 = -a^2$

$m = \pm ia, \pm ia$

$C.F. = \{(c_1 + c_2x)\cos ax + (c_3 + c_4x)\sin ax\} \times e^{0x}$

$P.I = \frac{1}{D^4 + 2a^2D^2 + a^4} \cdot 8\cos ax$

$= \frac{1}{D^2 \times D^2 + 2a^2D^2 + a^4} \cdot 8\cos ax$

$= \frac{1}{a^4 + a^4 - 2a^4} \cdot 8\cos ax$

First Derivative)

$P.I = x \cdot \frac{1}{4D^3 + 4a^2D} \cdot 8\cos ax$

$= \frac{1}{4D(D^2 + a^2)} \cdot 8\cos ax$

$= \cancel{\frac{D}{4D \times (-a^2+1)}} \cdot \cancel{8\cos ax} = \cancel{\frac{D}{4(a^4 - a^2)}} \cdot \cancel{8\cos ax}$

Second Derivative)

$P.I = x^2 \cdot \frac{1}{12D^2 + 4a^2} \cdot 8\cos ax$

$= -\frac{1}{8a^2} \cdot 8\cos ax = \frac{-x^2\cos ax}{a^2}$

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Q) $(D^3-3D^2+4D-2)y = \cos x$

$m^3 - 3m^2 + 4m - 2 = 0$

$m = 1$

$$P.I = \frac{1}{D^3 - 3D^2 + 4D - 2} \cdot \cos x$$

$$= \frac{1}{-D + 3 + 4D - 2} \cdot \cos x$$

$$= \frac{1}{3D + 1} \cdot \cos x$$

$$= \frac{3D-1}{(3D+1)(3D-1)} \cdot \cos x = \frac{3D-1}{9D^2-1} \cos x$$

$$= -\frac{(3D-1)\cos x}{10} = (-)\frac{(-3\sin x - \cos x)}{10}$$

$$= \frac{3\sin x + \cos x}{10}$$

$\boxed{3D.\cos x - \cos x)}$

$\boxed{3(-\sin x) - \cos x}$

(iii) $Q(x) = x^m$ or polynomial in x.

$$P.I. = \frac{1}{f(D)} Q(x)$$

$$= \frac{1}{1 \pm \phi(D)} Q(x)$$

$$= [1 \pm \phi(D)]^{-1} Q(x)$$

(Q). $(D^2+2D+1)y = x$.

$m^2+2m+1=0$

$m = \dfrac{-2 \pm \sqrt{4-4}}{2}$

$m = -1$.

$C.F. = (c_1+c_2x)e^{-x}$

$P.I = \dfrac{1}{D^2+2D+1} \cdot x$

$= \dfrac{1}{(D+1)^2} \cdot x$

$= (1+D)^{-2} \cdot x \quad = (1-2D+3D^2-4D^3+\ldots)x$

$= (x - 2Dx + 3D^2x + \ldots$

$= (x-2)$

General soln $\Rightarrow$ $y = (c_1+c_2x)e^{-x} + (x-2)$

(Q) $(D^4 - 2D^3 + D^2)y = x^3$.

$P.I. = \dfrac{1}{D^4-2D^3+D^2} \times x^3$

$= \dfrac{1}{D^2[1+(D^2-2D)]} \times x^3$

$= \dfrac{1}{D^2} [1+(D^2-2D)]^{-1} x^3$

$= \dfrac{1}{D^2} [1 - (D^2-2D) + (D^2-2D)^2 - \ldots] x^3$

$= \dfrac{1}{D^2} [x^3 - x^3 (D^2-2D) + \ldots]$

$= \dfrac{1}{D^2} [x^3 - 6x + 6x^2 + 24x + 24]$

$64 + 4b^2 x^3 + 4b^2 x^3$

$= \frac{1}{D^2}[x^3 + 18x + 6x^2 + 24]$

$= \boxed{\frac{x^5}{20} + \frac{x^4}{2} + 3x^3 + 12x}$

Date → 18/01/23

Probability

* Random variable (R.V.) → 1) Discrete 2) Continuous.

$X : S \to R$

X : no. of heads

$X(HH) = 2$

$X(HT) = 1$

$X(TT) = 0$

→ A random variable is function X with domain (S) and range (R).

→ If a random variable take finite no. of value then it is called random variable.

→ If a random variable take all the possible value of the space and infinite value.

* Probability function :- 1) Prob. mass function (pmf) (discrete)

$P_X(x) = \begin{cases} P(X = x_i) = p_i & \text{if } x = x_i \\ 0, & x \neq x_i \end{cases}$

i) $0 \leq P_X(x) \leq 1$

ii) $\sum_x P_X(x) = 1$

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2) Prob. density function (pdf) (continuous R.v.)

$$f_X(x) = \lim_{\delta x \to 0} \frac{P(x \le X \le x+\delta x)}{\delta x}$$

i) $0 \le f_X(x) \le 1$

ii) $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

$P(a \le X \le b) = \int_a^b f_X(x)\,dx$

3) Commulative distribution system (cdf)

$$F_X(x) = \begin{cases} P(X \le x_1), & \text{discrete r.v.} \\ \int_{-\infty}^{x_1} f(x)\,dx, & \text{Continous r.v.} \end{cases}$$

Q) A R.v. X has the following probability function. (discrete r.v.)

x: 0, 1, 2, 3, 4, 5, 6, 7

$P(x)$: 0, k, $2k$, $2k$, $3k$, k^2, $2k^2$, $(7k^2+k)$

(i) find k (ii) find $P(X \le 5)$

(iii) $P(0 < X < 5)$

(iv) If $P(X \le a) > \frac{1}{2}$, find min. value of a.

(v) Complete distribution function of X.

(i) $0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$

$10k^2 + 9k - 1 = 0$

$10k^2 + 10k - k - 1 = 0$

$10k(k+1) - 1(k+1) = 0$

$k = -1$ or $k = \frac{1}{10}$

5

(ii) $\sum_{i=0} P(x_i) = 0 + \cancel{8}k + 2k + \cancel{8}k + 3k + k^2$

$= 0 + \frac{1}{10} + \frac{2}{10} + \frac{2}{10} + \frac{3}{10} + \frac{1}{100}$

$= \frac{81}{100} = 0.81$

(iii) $\sum_{i=1}^{4} (Px_i) =$

(iv) $0 \quad \frac{1}{10} \quad \frac{2}{10} \quad \frac{2}{10} \quad \frac{3}{10} \quad \frac{1}{100}, \quad \frac{2}{100} \quad \left(\frac{7}{100} + \frac{1}{10}\right)$

$\frac{170}{100}$

~~$0 \quad \frac{1}{10} \quad \frac{3}{10} \quad \frac{3}{10}$~~

min. value = 4.

(v) $F(x) = P(X \leq x_i)$

x_i	$P(x)$	$F(x)$
0	0	0
1	0.1	0.1
2	0.2	0.3
3	0.2	0.5
4	0.3	0.8
5	0.01	0.81
6	0.02	0.83
7	0.17	1

Q) The diameter of a table is a continous r.v. with pdf.

$f(x) = kx(1-x)$, $0 \le x \le 1$

1. find k
2. find a no. c such that $(P(x \le c) = P(x > c)$
3. Compute $P(x \le \frac{1}{2} \mid \frac{1}{3} \le x \le \frac{2}{3})$
4. find C.d.f of x

(1) $\int_0^1 f(x)\, dx = 1$

$\int_0^1 kx - kx^2 = 1$

$\left[\frac{kx^2}{2} - \frac{kx^3}{3}\right]_0^1 = 1$

$k\left[\frac{1}{2} - \frac{1}{3}\right] = 1$

$k = 6$

(2) $\int_0^c kx - x^2 = \int_c^1 kx - x^2$

$k\left[\frac{x^2}{2} - \frac{x^3}{3}\right] = k\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_c^1$

$\left[\frac{c^2}{2} - \frac{c^3}{3}\right] = \frac{1}{2} - \frac{1}{3} - \frac{c^2}{2} + \frac{c^3}{3}$

$\frac{3c^2 - 2c^3}{6} = \frac{3 - 2 - 3c^2 + 2c^3}{6}$

$C = 1$

$$6c^2 - 4c^3 = 1$$

$$4c^3 - 6c^2 - 1 = 0$$

$$c = \frac{1}{2}$$

$9 \,|\, 18, 81$; $2, 9$

(3)

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$2 \,|\, 12, 164$; $3 \,|\, 6, 81$; $2, 27$

$3 \,|\, 18, 81$; $3 \,|\, 6, 27$; $2, 9$

$$= \frac{P\left(\frac{1}{3} \le x \le \frac{1}{2}\right)}{P\left(\frac{1}{3} \le x \le \frac{2}{3}\right)}$$

$$\Rightarrow \int_{1/3}^{1/2} kx - kx^2$$

Numerator

$$6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{1/3}^{1/2} = \frac{1}{8} - \frac{1}{24} - \frac{1}{18} + \frac{1}{81}$$

$$\frac{3-1}{24} - \frac{9+2}{164}$$

$$\frac{\cancel{2}}{\cancel{24}_{12}} - \frac{11}{164}$$

$$\frac{27 - 22}{2 \times 3 \times 2 \times 27}$$

$$= \frac{5}{2 \times 3 \times 2 \times 27}$$

Denominator

$$\frac{4}{18} - \frac{8}{81} - \frac{1}{18} + \frac{1}{81}$$

$$\frac{3}{18} - \frac{7}{81}$$

$$\frac{27 - 6}{3 \times 3 \times 2 \times 9} = \frac{19}{3 \times 3 \times 2 \times 9}$$

$$\frac{\frac{5}{2 \times 3 \times 3 \times 2 \times 27}}{\frac{19}{3 \times 3 \times 2 \times 9}} = \boxed{\frac{5}{38}}$$

(4) $0 \le x \le 1$ $\quad F(x) = \int_{-\infty}^{x} f(x)\,dx$

$-\infty < x \le 0 \rightarrow Fx = \int_{\infty}^{0} f(x)dx = 0$

$0 < x < 1 \rightarrow f(x) = \int_{-\infty}^{0} \text{limit} \int_{0}^{x=1} f(x)dx$

$1 \le x < \infty$

$\downarrow$

$= 0 + \int_{0}^{x} f(x)\,dx$

$\int_{-\infty}^{0} + \int_{0}^{1} + \int_{1}^{x=\infty} f(x)dx$

$= 3x^2 - 2x^3$

$= 1 \quad (\text{Put } x=1)$

$0 + 1 + 0$

$= 1$

(5) $f(x) = ax, \quad 0 \le x \le 1$

$a, \quad 1 \le x \le 2$

$-ax + 3a, \quad 2 \le x \le 3$

$0, \quad \text{else.}$

$f(x) = \int_{0}^{1} ax \; = a\left[\frac{x^2}{2}\right]_0^1 = \frac{a}{2}$

$f(x) = \int_{0}^{1} a + \int_{1}^{2} a$

$= a + a[x]_1^2$

$= a + a = 2a$

$f(x) = \int_{0}^{1} -ax+3a + \int_{1}^{2} -ax+3a + \int_{2}^{3} -ax+3a$

$$\Rightarrow\left[-a\frac{x^2}{2}\right]_0^1 + 3a[x]_0^1 + \left[-a\frac{x^2}{2}\right]_1^2 + 3a[x]_1^2$$

$$+\left[-\frac{ax^2}{2}\right]_2^3 + 3a[x]_2^3$$

$$\Rightarrow -\frac{a}{2} + 3a + \left(-\frac{a\cancel{4}2}{\cancel{2}} + \frac{a}{2}\right) + 3a + 3a + \left[-\frac{9a}{2} + \frac{4a}{2}\right]$$

$$\Rightarrow \underline{a + 3a + 3a} - \frac{9a}{2} + 2a$$

$$9a - \frac{9a}{2} = \frac{9a}{2}$$

~~$\frac{a}{2} + 2a + \frac{9a}{2} = \frac{14a}{2}$~~

(i) find cdf

(ii) find $P(X \leq 1.5)$

(iii) $\frac{d}{dx} F(x) = f_x(x)$

Date → 23/1/23

Q) $(D^3 - D^2 - 6D)y = 1+x^2$

$m^3 - m^2 - 6m = 0$

$m(m^2 - m - 6) = 0$

$m\{m^2 - 3m + 2m - 6\} = 0$

$m(m-3)(m+2) = 0$

$m = 0 \mid m = 3 \mid m = -2$

$C.F = C_1 e^{0x} + c_2 e^{3x} + c_3 e^{-2x}$

$= C_1 + c_2 e^{3x} + c_3 e^{-2x}$

$P.I = \frac{1}{D^3 - D^2 - 6D}(1+x^2)$

$= \frac{-1}{6D\left\{1 - \frac{(D^3 - D^2)}{6D}\right\}}(1+x^2)$

$= \frac{-1}{6D}\left[1 - \frac{1}{6}(D^2 - D)\right]^{-1}(1+x^2)$

$= \frac{-1}{6D}\left[1 + \frac{1}{6}(D^2 - D) + \frac{1}{36}(D^2 - D)^2 + \frac{1}{216}(D^2 - D)^3 + \dots\right](1+x^2)$

$= \frac{-1}{6D}\left[(1+x^2) + \frac{1}{6}(D^2 - D)(1+x^2) + \frac{1}{36}(D^4 - 2D^3 + D^2)(1+x^2) + \dots\right]$

$= \frac{-1}{6D}\left[(1+x^2) + \frac{1}{3}(1-x) + \frac{1}{36}\cdot 2\right]$

$= \frac{-1}{6D}\left[1+x^2 + \frac{1}{3}(1-x) + \frac{1}{18}\right]$

$= \frac{-1}{6}\left[x + \frac{x^3}{3} + \frac{1}{3}\left(x - \frac{x^2}{2}\right) + \frac{1}{18}x\right]$

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$$= -\frac{1}{6}\left[\frac{x^3}{3} - \frac{x^2}{6} + 2\right.$$

General sol^n is, $y =$ C.F. + P.I.

(iv) $\boxed{Q(x) = e^{ax} \cdot V(x)}$, where v is a function of x.

$$P.I. = \frac{1}{f(D)} e^{ax} \cdot V = e^{ax} \cdot \frac{1}{f(D+a)} \cdot V$$

~~Solve~~

Q) $(D^2 - 2D - 1)y = e^x \cdot \cos x$

$m^2 - 2m - 1 = 0$

$\cancel{m^2 - m - m - 1 = 0}$

$\cancel{m(m-1) - 1(}$

$$m = \frac{2 \pm \sqrt{4+4}}{2}$$

$$= \frac{2 \pm \sqrt{8}}{2}$$

$= 1 \pm \sqrt{2}$ (real & distinct)

$$C.F. = c_1 e^{(1+\sqrt{2})x} + c_2 e^{(1-\sqrt{2})x}.$$

$$P.I. = \frac{1}{D^2 - 2D - 1} e^x \cdot \cos x$$

$$= e^x \cdot \frac{1}{(D+1)^2 - 2(D+1) - 1} \cos x$$

$$= \frac{e^x \cdot \cos x}{D^2 + \cancel{2D} + \cancel{1} - \cancel{2D} - 2 - \cancel{1}}$$

$$= e^x \cdot \frac{1}{D^2 - 2} \cdot \cos x$$

(Here apply 2nd Rule

$$= e^x \cdot \frac{1}{-1^2 - 2} \cos x = e^x \frac{-1}{3} \cdot \cos x = -\frac{1}{3} e^x$$

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Q) $(D^3 + 3D^2 - 4D - 12)y = 12x e^{-2x}$

Sol^n → $m^3 + 3m^2 - 4m - 12 = 0$

$$m^2(m+3) - 4(m+3) = 0$$

$$(m^2 - 4)(m+3) = 0$$

$$(m-2)(m+2)(m+3) = 0$$

$$m = (2, -2, -3)$$

$$C.F. = c_1 e^{2x} + c_2 e^{-2x} + c_3 e^{-3x}$$

$$P.I. = \frac{1}{D^3 + 3D^2 - 4D + 2} e^{-2x} \cdot 12x$$

$$= \frac{1}{(D+3)(D+2)(D-2)} e^{-2x} \cdot 12x$$

$$= 12 e^{-2x} \cdot \frac{1}{(D+3-2)(D+2-2)(D-2-2)} \cdot x$$

$$= 12 e^{-2x} \cdot \frac{1}{(D+1) D (D-4)} \cdot x$$

$$= 12 e^{-2x} \cdot \frac{1}{D^3 - 4D^2 + D - 4D} \cdot x$$

$$= 12 e^{-2x} \cdot \frac{1}{[-4D]\left\{1 - \frac{(D^3 - 3D^2)}{4D}\right\}} \cdot x$$

$$= 12 e^{-2x} \times \frac{-1}{4D} \cdot \left[1 - \left(\frac{D^3 - 3D^2}{4D}\right)\right]^{-1} \cdot x$$

$$= 12e^{-2x} x - \frac{1}{4D}\left[1 + \frac{1}{4}(D^2-3D) + \frac{1}{16}(D^2-3D)^2 + \cdots\right]x$$

$$= 12e^{-2x} x - \frac{1}{4D}\left[x + \frac{1}{4}(-3)\right]$$

$$= -3e^{-2x}\left[\frac{x^2}{2} - \frac{3}{4}x\right].$$

Q) $(D^2 - 2D + 10)y = 16e^x \cos 3x + 24e^x \sin 3x.$

$m^2 - 2m + 10 = 0$

$$m = \frac{2 \pm \sqrt{4-40}}{2} = \frac{2 \pm 6i}{2} = 1 \pm 3i$$

$$C.F. = e^x(c_1\cos 3x + c_2\sin 3x).$$

$$P.I. = \frac{1}{D^2 - 2D + 10}\{16e^x \cos 3x + 24e^x \sin 3x\}$$

$$= \frac{1}{D^2-2D+10} 16e^x \cos 3x + \frac{1}{D^2-2D+10} 24e^x \sin 3x.$$

$$= 16e^x \cdot \frac{1}{(D+1)^2 - 2(D+1) + 10} \cdot \cos 3x + 24e^x \cdot \frac{1}{(D+1)^2 - 2(D+1) + 10} \cdot \sin 3x$$

$$= 16e^x \cdot \frac{1}{D^2 + 1 + 2D - 2D - 2 + 10} \cdot \cos 3x + 24e^x \cdot \frac{1}{D^2+9} \cdot \sin 3x$$

$$= 16e^x \cdot \frac{1}{D^2+9} \cdot \cos 3x + 24e^x \cdot \frac{1}{D^2+9} \cdot \sin 3x$$

$$= \frac{16 e^x \cdot x \cos 3x}{2D} + \frac{\cancel{16} e^x \cdot x \sin 3x}{2D}$$

$$= \frac{\overset{8}{\cancel{16}} e^x \cdot 1 \cdot \cos 3x}{-\cancel{18}\, 9} + \frac{\overset{8}{\cancel{16}} e^x \cdot 1 \cdot \sin 3x}{-\cancel{18}\, 9}$$

$$= \frac{-8e^x \cos 3x}{9} - \frac{8e^x \sin 3x}{9}$$

$$= -\frac{8e^x}{9}(\cos 3x \cdot \sin 3x)$$

$$= \frac{\overset{8}{\cancel{16}} e^x \cdot x \cdot \cos 3x}{\cancel{2}D} + \frac{\overset{8\,12}{\cancel{24}} e^x \cdot x \cdot \sin 3x}{\cancel{2}D}$$

$$= 8 e^x x \left\{ \frac{8(\sin 3x)}{3} - \frac{\overset{4}{\cancel{12}} \cos 3x}{\cancel{3}} \right\}$$

$$= \frac{4x e^x}{3}(2 \sin 3x - 3 \cos 3x)$$

(8) (v) $\boxed{Q(x) = x \cdot V(x)}$, where v is function of x.

$$P.I = \frac{1}{f(D)} \cdot Q = \frac{1}{f(D)} x.V$$

$$= x \frac{1}{f(D)} \cdot V - \frac{f'(D)}{[f(D)]^2} V$$

Q) $(D^2 - 5D + 6) y = x \cdot \cos 2x$

$m^2 - 5m + 6 = 0$

$m^2 - 3m - 2m + 6 = 0$

$m(m-3) - 2(m-3) = 0$

$$(m-3)(m-2)=0$$

$$m = 3, 2$$

$$\boxed{C.F. = c_1 e^{3x} + c_2 e^{2x}}$$

C.F = complementary function

$$P.I. = \frac{1}{D^2-5D+6} x \cdot \cos 2x$$

$$= x \cdot \frac{1}{D^2-5D+6} \cdot \cos 2x - \frac{2D-5}{(D^2-5D+6)^2} \cdot \cos 2x$$

$$= x \cdot \frac{1}{[-2^2-5D+6]} \cdot \cos 2x - \frac{(2D-5)}{(-2^2-5D+6)^2} \cdot \cos 2x$$

$$= x \cdot \frac{1}{2-5D} \cos 2x - \frac{(2D-5)}{(2-5D)^2} \cdot \cos 2x$$

$$= x \frac{2+5D}{(2-5D)(2+5D)} \cos 2x - \frac{(2D-5)}{4-20D+25D^2} \cos 2x$$

$$= x \frac{(2+5D) \cos 2x}{4-25D^2} - \frac{(2D-5)}{4-20D+25(-2^2)} \cos 2x$$

$$(\because D^2 = -2^2)$$

$$= \frac{x(2+5D)\cos 2x}{104} - \frac{(2D-5)}{4-20D-100} \cos 2x$$

$$= \frac{x}{104} \cdot (2\cos 2x + 5D\cos 2x) + \frac{(2D-5)}{20D+96} \cos 2x$$

$$= \frac{x}{104}\left(2\cos 2x - 5.2\sin 2x\right) + \frac{2D-5}{20D+96} \cos 2x$$

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$$\frac{x}{104}\left(2\cos 2x - 10\sin 2x\right) + \frac{1}{4}\frac{(2D-5)}{(5D+24)}\cos 2x.$$

$$\frac{x}{104}\left(2\cos 2x - 10\sin 2x\right) + \frac{1}{4}\frac{(2D-5)(5D-24)}{25D^2-576}\cos 2x$$

$$\frac{x}{104}\left(2\cos 2x - 10\sin 2x\right) + \frac{1}{4}\frac{(10D^2-48D-25D+120)}{-676}\cos 2x$$

$$\cancel{\frac{x}{104}\left(2\cos 2x - 10\sin 2x\right) + \frac{1}{4}\frac{(-40+120-73D)}{-676}\cos 2x}$$

$$\cancel{\frac{x}{104}\left(2\cos 2x - 10\sin 2x\right) - \frac{1}{2704}(80-73D)\cos 2x.}$$

$$\cancel{\frac{x}{104}\left(2\cos 2x - 10\sin 2x\right) - \frac{80\cos 2x}{2704}}$$

$$\frac{x}{104}\left(2\cos 2x - 10\sin 2x\right) - \frac{1}{2704}\left[40\cos 2x - 73(-2\sin 2x) + 120\cos 2x\right]$$

$$\frac{x}{104}\left(2\cos 2x - 10\sin 2x\right) - \frac{1}{2704}\left[-40\cos 2x + 146\sin 2x + 120\cos 2x\right]$$

$$\frac{x}{104}\left(2\cos 2x - 10\sin 2x\right) + \frac{80}{2704}\cos 2x + \frac{146}{2704}\sin 2x$$

$$\frac{x}{104}\left(2\cos 2x - 10\sin 2x\right) + \frac{5\cos 2x}{676} + \frac{73}{1352}\sin 2x$$

Q) $(D^2+3D+2)y = xe^x \cdot \sin x$

$m^2+3m+2=0$

$m^2+2m+m+2=0$

$m(m+2)+1(m+2)=0 \qquad m=-1,-2$

$$P.I = \frac{1}{D^2+3D+2} e^x (x.\sin x)$$

$$= e^x . \frac{1}{(D+1)^2+3(D+1)+2} (x.\sin x)$$

$$= e^x . \frac{1}{D^2+1+2D+3D+3+2} (x.\sin x)$$

$$= e^x . \frac{1}{D^2+5D+6} (x.\sin x)$$

$$= e^x \left[x \frac{1}{D^2+5D+6} \sin x - \frac{2D+5}{(D^2+5D+6)^2} . \sin x \right]$$

$$= e^x \left[\frac{x}{-1^2+5D+6} \sin x - \frac{(2D+5)\sin x}{(-1^2+5D+6)^2} \right]$$

$$= e^x \left[\frac{x}{5+5D} \sin x - \frac{2D+5}{5+5D} . \sin x \right]$$

$$= e^x \left[\frac{x(1-D)}{5(1+D)(1-D)} \sin x - \frac{(2D+5)(1-D)}{5(1+D)(1-D)} \sin x \right]$$

$$= e^x \left[\frac{x}{5} \frac{(1-D)\sin x}{(1-D^2)} - \frac{(2D+5)(1-D)\sin x}{5(1-D^2)} \right]$$

$$= e^x\left[\frac{x}{10}(1-D)\sin x - \frac{(2D+5)(1-D)\sin x}{10}\right]$$

$$= e^x\left[\frac{x}{10}(\sin x - D\sin x) - \left(\frac{2D-2D^2+5-5D}{10}\right)\sin x\right]$$

$$= e^x\left[\frac{x}{10}(\sin x - \cos x) - \left(\frac{-3\cos x - 2(-\sin x) + 5\cos x}{10}\right)\right]$$

$$= e^x\left[\frac{x}{10}(\sin x - \cos x) + \left(\frac{-3\cos x + 5\cos x + 2\sin x}{10}\right)\right]$$

$$= \frac{e^x}{10}\left[(x\sin x - x\cos x) - (3\sin x - 2\cos x)\right]$$

$$= -\frac{e^x}{5}\left[\frac{x}{2}(\cos x - \sin x) + \frac{1}{2}\left(\sin x - \frac{5}{2}\cos x\right)\right]$$

* General method for obtaining particular integral :-

$$P.I. = \frac{1}{f(D)}Q(x)$$

$$= \frac{1}{(D-m_1)(D-m_2)\cdots(D-m_n)} \times Q(x)$$

$$= \left[\frac{A_1}{(D-m_1)} + \frac{A_2}{D-m_2} + \cdots + \frac{A_n}{D-m_n}\right]Q(x)$$

$$= \frac{A_1}{(D-m_1)}Q(x) + \frac{A_2}{(D-m_2)}Q(x) + \cdots + \frac{A_n}{D-m_n}Q(x)$$

$$= A_1 e^{m_1 x}\int Q(x)e^{-m_1 x}dx + A_2 e^{m_2 x}\int Q(x)e^{-m_2 x}dx + \cdots + A_n e^{m_n x}\int Q(x)e^{-m_n x}dx$$

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Q) Solve → $(D^2+3D+2)y = e^{e^x}$

Auxiliary Eqn $m^2+3m+2=0$

$m^2+2m+m+2=0$

$m=-2,-1$

$C.F. = C_1e^{-2x}+C_2e^{-x}$

$P.I. = \frac{1}{D^2+3D+2} e^{e^x}$

$= \frac{1}{(D+2)(D+1)} e^{e^x}$

$= \frac{1}{(D+2)}\left[\frac{1}{(D+1)} e^{e^x}\right]$

$= \frac{1}{(D+2)}\left[\frac{1}{D-(-1)} e^{e^x}\right]$

$= \frac{1}{(D+2)}\left[e^{-x}\int e^{e^x}\cdot e^x\,dx\right]$ Put $e^x=p$

$e^x dx = dp$

$= \frac{1}{(D+2)}\left[e^{-x}\int e^p\cdot dp\right]$

$= \frac{1}{(D+2)}\left[e^{-x}e^p\right] \Rightarrow \frac{1}{(D+2)}(e^{-x}e^{e^x})$

$= \frac{1}{D-(-2)}\left[e^{-x}e^{e^x}\right]$

$= e^{-2x}\int e^{-x}e^{e^x}e^{2x}\,dx$

$= e^{-2x}\int e^{x}\cdot e^{e^x}\,dx$

$e^{-2x}\int e^{t}\,dt$ Let $e^{x}=t$.
$e^{x}dx=dt$.

$=e^{-2x}e^{t} \Rightarrow e^{-2x}e^{e^{x}}$ Ans

25/01/23

$\frac{a+2a-5a+6a}{2}$

(1) $f(x)=\begin{cases} ax & 0\le x\le 1 \\ a & 1\le x\le 2 \\ -ax+3a & 2\le x\le 3 \\ 0 & \text{else} \end{cases}$

$$\int_{-\infty}^{\infty} f(x)\,dx=1$$

$$\int_{-\infty}^{0} f(x)\,dx+\int_{0}^{1} f(x)\,dx+\int_{1}^{2} f(x)\,dx+\int_{2}^{3} f(x)\,dx+\int_{3}^{\infty} f(x)\,dx=1$$

(first term $=0$, last term $=0$)

$$\int_{0}^{1} a(x)\,dx+\int_{1}^{2} a\,dx+\int_{2}^{3}(-ax+3a)\,dx=1$$

$$a\left[\frac{x^2}{2}\right]_0^1+a[x]_1^2+a\left[-\frac{x^2}{2}+3x\right]_2^3=1$$

$$\frac{a}{2}+a+3a-\frac{5a}{2}=1$$

$$2a=1$$

$$a=\frac{1}{2}$$

(2) c.d.f = Case 1 $x<0$

$$\int_{-\infty}^{x} f(x)\,dx=0$$

Case 2 $0\le x\le 1$

$$\int_{-\infty}^{x} f(x)\,dx=\int_{-\infty}^{0} f(x)\,dx+\int_{0}^{x} f(x)\,dx$$

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$$0 + \int_0^x \frac{1}{2}x\,dx = \frac{1}{4}x^2$$ $\boxed{= \frac{1}{4} \quad (\because x=1)}$

Case 3 :- $1 \le x \le 2$

$$\int_{-\infty}^{0} b(x)\,dx + \int_0^1 b(x)\,dx + \int_1^x b(x)\,dx$$

$$= 0 + \int_0^1 \frac{1}{2}x + \int_1^x \frac{1}{2}$$

$$= 0 + \frac{1}{4} + \frac{x}{2} - \frac{1}{2}$$

$$= \frac{x}{2} - \frac{1}{4}$$ $\boxed{= 1 - \frac{1}{4} = \frac{3}{4} \quad (\because x=2)}$

Case 4 :- $2 \le x \le 3$

$$\int_{-\infty}^{0} b(x)\,dx + \int_0^1 b(x)\,dx + \int_1^2 b(x)\,dx + \int_2^x b(x)\,dx$$

$$0 + \frac{1}{4} + \frac{1}{2}[x]_1^2 + \left[\frac{3}{2}x - \frac{1}{4}x^2\right]_2^x$$

$$0 + \frac{1}{4} + \frac{1}{2} + \left[\frac{3x}{2} - \frac{1}{4}x^2 - 3 + 1\right]$$

$$\frac{3}{4} - 2 + \frac{3x}{2} - \frac{1}{4}x^2$$

$$= \frac{3x}{2} - \frac{1}{4}x^2 - \frac{5}{4}$$

$$\boxed{= -\frac{5}{4} + \frac{9}{2} - \frac{9}{4} \quad [\because x=3] \quad = \frac{-5+18-9}{4} = 1}$$

Cases $x > 3$

$$\int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx + \int_{2}^{3} f(x)\,dx + \int_{3}^{x} f(x)\,dx$$

$$0 + \frac{1}{4} + \frac{1}{2} + \left[\frac{3x}{2} - \frac{1}{4}x^2\right]_2^3 + 0$$

$$\frac{3}{4} + \left[\frac{9}{2} - \frac{9}{4} - \frac{6}{2} + 1\right]$$

$$\frac{3}{4} + \frac{9}{4} - \frac{6}{2} + 1$$

$$= 1$$

Q) The time (in min.) that a person has to wait at a certain bus stop for a bus is found to be a random phenomena. with a function specified by,

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{x}{8}, & 0 \le x < 2 \\ \frac{x^2}{16}, & 2 \le x < 4 \\ 1, & x \ge 4 \end{cases}$$

① find Pdf of x.

② find the prob. that a person has to wait

a) more than 2 min.

b) betn 1 & 2 min.

③ find the prob. that a person has to wait more than 2 min. given it in more than 1 min.

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① $\frac{d}{dx}F(x) = b(x)$

$$b(x) = \begin{cases} 0, & x<0 \\ \frac{1}{8}, & 0 \le x < 2 \\ \frac{x}{8}, & 2 \le x \le 4 \\ 0, & x \ge 4 \end{cases}$$

② a) $P(x \ge 2) = 1 - P(x \le 2)$

$1 - P(x \le 2)$ or $= 1 - \int_0^2 b(x)\,dx$

$1 - F(2)$ or

$1 - \frac{1}{4}$ or $= 1 - \left[\int_0^2 b(x)dx + \cancel{\int^2 b(x)dx}\right]$

$\Rightarrow \frac{3}{4}$

$= 1 - \int_0^2 \frac{x}{8}$

$= 1 - \frac{1}{8}\left[\frac{x^2}{2}\right]_0^2$

$= 1 - \frac{1}{8}\left(\frac{2}{4}\right) = \frac{3}{4}$

b) $P(1 \le x \le 2)$ $\boxed{P(a \le x \le b) = F(b) - F(a)}$

$F(2) - F(1)$

$\frac{(2)^2}{16} - \frac{1}{8}$

$\frac{4}{16} - \frac{1}{8} = \frac{1}{4} - \frac{1}{8} = \frac{2-1}{8} = \frac{1}{8}$

S.C. Gupta mathematical Statistical

(3) $P(x>2 \mid x>1) = \frac{P(A\cap B)}{P(B)}$

$= \frac{P(x>2)}{P(x>1)}$

$= \frac{\frac{3}{4}}{\frac{7}{8}} = \frac{3}{\cancel{4}} \times \frac{\cancel{8}2}{7} = \frac{6}{7}$

* Let $f(x)$ be a pdf defined on $[a,b]$ Then, rth moment about any point A is

$$\mu_r^A = \int_a^b (x-A)^r f(x)\,dx$$

Put $A=0 \Rightarrow \mu_r' = \int_a^b x^r f(x)\,dx \rightarrow$ rth moment about origin.

$A=\bar{x} \Rightarrow \mu_r = \int_a^b (x-\bar{x})^r f(x)\,dx \rightarrow$ " " " " mean

* → Arithmatic mean (AM)
→ Geometric mean (GM)
→ Harmonic mean (HM)

Mean

$\mu_1' = \int_a^b x f(x)\,dx$ (Put $r=1$)

$= \text{mean} = \bar{x}$

$\mu_1 = \int_a^b (x-\bar{x}) f(x)\,dx$ (Put $r=1$)

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$3ab(a-b)$

$$\mu_1 = \int_a^b x\, b(x)\, dx - \int_a^b \bar{x}\, b(x)\, dx$$

$$= \bar{x} - \bar{x}\int_a^b b(x)\, dx$$

$$\Rightarrow \bar{x} - \bar{x} = 0$$

$\boxed{\text{Put } r=2}$

$$\mu_2 = \int_a^b (x-\bar{x})^2\, b(x)\, dx$$

$$= \int_a^b (x-\bar{x})(x+\bar{x})\, b(x)\, dx$$

$$x^2 + (\bar{x})^2 - 2x\bar{x}\, b(x$$

$$= \int_a^b x^2 + x\bar{x} - x\bar{x} \quad x^2 + (\bar{x})^2$$

$$2(\bar{x})^2 + 2(\bar{x})^2$$

$$4(\bar{x})^2$$

variance $\boxed{\mu_2 = \mu_2' - (\mu_1')^2}$

$$\mu_3' = \int_a^b x^3\, b(x)\, dx$$

$$= x^3 \int_a^b b(x)\, dx.$$

$$\mu_3 = \int_a^b (x-\bar{x})^3\, b(x)\, dx.$$

$$= \int_a^b (x^3 - (\bar{x})^3 - 3x^2\bar{x} + 3x(\bar{x})^2)\, b(x)\, dx$$

$$= \cancel{\mu_3'}$$

$$= \cancel{-3x^2\bar{x} + 3x(\bar{x})^2}$$

$$= \cancel{3x\bar{x}(\bar{x}-x)} \Rightarrow$$

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$$\mu_3 = \mu_3' - 3\mu_1' \cdot \mu_2' + 2(\mu_1')^3$$

$$\mu_4 = \mu_4' - 4\mu_3' \mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4$$

<u>meadium</u> : $\int_a^m f(m)\,dx = \int_m^b f(x)\,dx = \frac{1}{2}$

<u>mode</u> ! $f'(x) = 0$ & $f''(x) < 0$

Q) In a continuous distribution with relative density function $f(x) = Kx(2-x)$; $0 \leq x \leq 2$

(i) find K

(ii) find mean, variance etc.

Solve → (i) $\int_0^2 2Kx - Kx^2 = 1$

$$\frac{2K}{2}\left[x^2\right]_0^2 - K\left[\frac{x^3}{3}\right]_0^2 = 1$$

$$K \times 4 - K \times \frac{8}{3} = 1$$

$$4K - \frac{8K}{3} = 1$$

$$\frac{4K}{3} = 1$$

$$K = \frac{3}{4}$$

(ii) mean $= \bar{x} = \int_a^b Kx(2-x)\,dx$

$$= \frac{3}{4}\int_0^2 2x - x^2\,dx$$

$$= \frac{3}{4}\left[2\frac{x^2}{2} - \frac{x^3}{3}\right]_0^2$$

$$= \frac{3}{4}\left[4 - \frac{8}{3}\right] = 1$$

(iii) variance $\Rightarrow \mu_2 = \mu_2' - (\mu_1')^2$

$$\int_0^2 x^2 f(x)\,dx - 1$$

$$= \cancel{\int_0^2 x^2 \times x^2(2-x)} - 1$$

$$= \cancel{\frac{3}{4}\int_0^2 2x^2 - x^3} - 1$$

$$\frac{3}{4}\int_0^2 x^3(2-x)\,dx - 1$$

$$= \cancel{\frac{3}{4}\left(\left[2\frac{x^3}{3}\right]_0^2 - \left[\frac{x^4}{4}\right]_0^2\right)} - 1$$

$$\frac{3}{4}\left[2\frac{x^4}{4_2} - \frac{x^5}{5}\right]_0^2 - 1$$

$$= \cancel{\frac{3}{4}\left(\frac{16}{3} - 4\right)} - 1$$

$$\frac{3}{4}\left[8 - \frac{32}{5}\right] - 1$$

$$= \cancel{\frac{4}{3}}$$

$$\frac{3}{4} \times \frac{8}{5}^2 - 1$$

$$\frac{6}{5} - 1 = \frac{1}{5}$$

(iv) Meadian $= \int_0^m \frac{3}{4}(2x - x^2)\,dx$

$$= \frac{3}{4}\int_0^m 2x - x^2$$

$$= \frac{3}{4}\left[x^2 - \frac{x^3}{3}\right]_0^m$$

$$\frac{3}{4}\left[m^2 - \frac{m^3}{3}\right] = \frac{1}{2}$$

$$\frac{3}{4}\int_m^2 2x - x^2\,dx$$

$$\frac{3}{4}\left[x^2 - \frac{x^3}{3}\right]_m^2$$

$$\frac{3}{4}\left[m^2 - \frac{m^3}{3} - 4 \quad \frac{8}{3}\right.$$

$$\frac{3}{4}\left[m^2 - \frac{m^3}{3}\right] + \frac{3}{4}($$

$$\frac{1}{2} + \cancel{\frac{x}{9}} = \cancel{\frac{4}{3}}$$

$$= \frac{1}{2} - 1 = -\frac{1}{2}$$

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* Geometric mean $\Rightarrow \log G = \int_a^b \log x \, f(x) \, dx$

* Harmonic mean $\Rightarrow \frac{1}{H} = \int_a^b \frac{f(x)}{x} \, dx$

$3m^2 - m^3 = 2$

$\frac{1}{y_2}(3m^2 - m^3) = \frac{3}{x}$

Date → 30/01/23

Q1) $(D^2 + a^2)y = \sec ax$

Auxilary Eqn

$$m^2 + a^2 = 0$$

$$m = \pm ai$$

$$C.F = (c_1 \cos ax + c_2 \sin ax)$$

$$P.I. = \frac{1}{D^2 + a^2} \sec ax$$

$$= \frac{1}{(D + ai)(D - ai)} \sec ax$$

$$= \frac{1}{2ia}\left[\frac{1}{D - ia} - \frac{1}{D + ia}\right] \sec ax$$

$$= \frac{1}{2ia}\left[\frac{1}{D - ia} \sec ax - \frac{1}{D + ia} \sec ax\right]$$

$4 iax]$

$$\frac{1}{D - ia} \sec ax = e^{iax} \int \sec ax \, e^{-iax} \, dx$$

$$e^{iax} \int \frac{2}{e^{iax} + e^{-iax}} \times e^{-iax} \, dx$$

$\cos ax = \frac{e^{iax} + e^{-iax}}{2}$

$\sec ax = \frac{2}{e^{iax} + e^{-iax}}$

$$e^{iax} \int \frac{2 \cdot e^{-2iax}}{1 + e^{-2iax}} \, dx$$

$$\left.\frac{e^{iax}}{-ia}\right] \frac{dt}{t} \qquad 1+e^{-2iax} = t$$

$$-2ia\, e^{-2iax}\, dx = dt.$$

$$\Rightarrow -\frac{e^{iax}}{ia} \log t$$

$$\Rightarrow -\frac{e^{iax}}{ia} \log(1+e^{-2iax})$$

$$\Rightarrow -\frac{e^{iax}}{ia} \log(1+\cos 2ax - i\sin 2ax)$$

$$\Rightarrow -\frac{e^{iax}}{ia} \log(2\cos^2 ax - 2i \sin ax \cos ax)$$

$$\Rightarrow -\frac{e^{iax}}{ia} \log(2\cos ax \cdot e^{-iax})$$

$$\Rightarrow -\frac{e^{iax}}{ia} \left[\log(2\cos ax) - iax\right]$$

Replacing i by $(-i)$, we get

$$\frac{1}{D+ia} \sec ax$$

$$= \frac{1}{ia} e^{-iax}\left[\log(2\cos ax) + iax\right]$$

$$P.I = \frac{1}{2ai}\left[-\frac{e^{iax}}{ai}\left[\log(2\cos ax) - iax\right] - \frac{e^{-iax}}{ia}\left[\log(2\cos\right.\right.$$

Cauchy's Linear Eq^n

$\to$ An equation of the form: $a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2}y}{dx^{n-2}}$

$$+ \cdots + a_n y = Q(x)$$

Where, $a_0, a_1, a_2, \cdots, a_{n-1}, a_n$ are constant is called Cauchy's Linear Eq^n.

Let, $x = e^z$

$$1 = e^z \cdot \frac{dz}{dx}$$

$$\frac{dz}{dx} = \frac{1}{e^z} = \frac{1}{x}$$

$$\Rightarrow \boxed{x \cdot \frac{dz}{dx} = 1}$$

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{x} \cdot \frac{dy}{dz}$$

$$\Rightarrow \boxed{x \cdot \frac{dy}{dx} = \frac{dy}{dz}}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right)$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d^2y}{dz^2} \cdot \frac{dz}{dx}$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d^2y}{dz^2} \times \frac{1}{x}$$

$$= \frac{1}{x^2}\left(-\frac{dy}{dz} + \frac{d^2y}{dz^2}\right)$$

$$= \frac{1}{x^2}\left(-D' + D'^2\right).$$

$$\frac{d^2y}{dx^2} = \frac{1}{x^2}(D'^2 - D')y$$

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$$\Rightarrow \boxed{x^2 \frac{d^2y}{dx^2} = D'(D'-1)y} \quad \because \boxed{D' = \frac{d}{dz}}$$

Similarly,

$$x^3 \frac{d^3y}{dx^3} = D'(D'-1)(D'-2)\,y$$

$$x^n \frac{d^ny}{dx^n} = [D'(D'-1)(D'-2)\cdots(D'-(n-1))]$$

Q.) $(4x^2D^2 + 16xD + 9)y = 0$

~~$4x^2m^2 + 16xm + 9 = 0$~~

Let $x = e^z$

$\therefore$ The given differential eqn can be rewritten as,

$$[4D'(D'-1) + 16D' + 9]y = 0$$

$$[4D'^2 - 4D' + 16D' + 9]y = 0$$

$$[4D'^2 + 12D' + 9]y = 0$$

A.E is $\Rightarrow$ $4m^2 + 12m + 9 = 0$

$$\boxed{m = -\frac{3}{2}, -\frac{3}{2}}$$

$$C.F = (c_1 + c_2 z)e^{-\frac{3}{2}z}$$

$$= (c_1 + c_2 \log x)\, e^{-\frac{3}{2}\log x}$$

$$= (c_1 + c_2 \log x)\, x^{-3/2}$$

$\therefore$ General Eqn $\rightarrow$ $\boxed{y = (c_1 + c_2 \log x)x^{-3/2}}$

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Q) $(4x^2D^2+1)y = 19\cos(\log x)+22\sin(\log x)$

~~$(4D'(D'-1)+1)y = 19\cos\log x$~~

Let $x = e^z$

$\therefore$ the given eqn reduces.

$[4D'(D'-1)+1]y = 19\cos(\overset{z}{\log x})+22\sin(\overset{z}{\log x})$

$[4D'^2-4D'+1]y = 19\cos z+22\sin z$

$(2D'-1)^2 y = 19\cos z+22\sin z$

A.E $\Rightarrow (2m-1)^2=0$

$m=\frac{1}{2}, \frac{1}{2}$

$C.F = (c_1+c_2z)e^{\frac{1}{2}z}$

$= (c_1+c_2\log x)e^{\frac{1}{2}\log x}$

$= (c_1+c_2\log x)x^{1/2}$

$P.I = \frac{1}{(2D'-1)^2}(19\cos z+22\sin z)$

$= \frac{1}{4D'^2-4D'+1}19\cos z+\frac{1}{4D'^2-4D'+1}22\sin z$

$= \frac{1}{4\times-1^2-4D'+1}19\cos z+\frac{1}{4\times-1^2-4D'+1}22\sin z$

$= \frac{1}{-3-4D'}19\cos z+\frac{1}{-3-4D'}22\sin z$

$= \frac{-1}{4D'+3}19\cos z+\frac{-1}{4D'+3}22\sin z$

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$$-\frac{1(4D'-3)}{16D'^2-9}19\cos z + -\frac{1(4D'-3)}{16D'^2-9}22\sin z$$

$$\frac{1}{25}\left[(4D'-3)19\cos z + (4D'-3)22\sin z\right]$$

$$\frac{1}{25}\left[-76\sin z - 57\cos z + 88\cos z - 66\sin z\right]$$

$$\Rightarrow \frac{1}{25}\left[31\cos z - 142\sin z\right]$$

$$\Rightarrow \frac{1}{25}\left[31\cos(\log x) - 142\sin(\log x)\right]$$

General Eq^n $\Rightarrow$ $\boxed{C.F + P.I}$

Q) $(x^3D^3 + x^2D^2 - 2)y = x + \frac{1}{x^3}$

Let $x = e^z$

$\therefore$ The given eq^n reduces.

$$[D'(D'-2)(D'-1) + D'(D'-1) - 2]y = e^z + \frac{1}{e^{3z}}$$

$$[(D'^2 - 2D')(D'-1) + D'^2 - D' - 2]y$$

$$[D'^3 - \cancel{D'^2} - 2D'^2 + 2D' + \cancel{D'^2} - D' - 2]y$$

$$[D'^3 - 2D'^2 + D' - 2]y$$

A.E $= m^3 - 2m^2 + m - 2 = 0$

$m^2(m-2) + 1(m-2) = 0 \quad | \quad (m^2+1)(m-2) = 0$

$m = 2 \quad | \quad m = \sqrt{-1}$

$= \cancel{\alpha i \beta, \alpha i \beta} + i, -i$

$C.F. = c_1 e^{2z} + c_2 e^{iz} + c_3 e^{-iz}$

$= c_1 e^{2\log x} + c_2 e^{i \log x} + c_3 e^{-i \log x}$

$= c_1 . x^2 + \cancel{c_2 x^i + c_3 x^{-i}}$

$c_2 \cos z + c_3 \sin z$

$\Rightarrow \boxed{c_1 x^2 + c_2 \cos \log x + c_3 \sin \log x}$

$P.I. = \frac{1}{(D'^2+1)(D'-2)} (e^z + e^{-3z})$

$\Rightarrow \frac{1}{(D'^2+1)(D'-2)} e^z + \frac{1}{(D'^2+1)(D'-2)} e^{-3z}$

$= \frac{1}{(1^2+1)(1-2)} e^z + \frac{1}{((-3)^2+1)(-3-2)} e^{-3z}$

$= \frac{1}{-2} e^z + \frac{e^{-3z}}{10 \times -5}$

$= \frac{-e^z}{2} - \frac{e^{-3z}}{50}$

$= -\frac{e^{\log x}}{2} + \frac{e^{-3\log x}}{50}$

$$\boxed{P.I. = -\frac{x}{2} - \frac{1}{50 x^3}}$$

* Legendre's Linear Equation

An equation in the form:

$$a_0(a+bx)^n \cdot \frac{d^n y}{dx^n} + a_1(a+bx)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots$$

$$a_{n-1}(a+bx)\frac{dy}{dx} + a_n y = Q(x)$$

$$Q\cdot\left(\frac{e^z-a}{b}\right)$$

Where, $a_0, a_1, a_2, \ldots a_n$ are constant is called Legendre's Linear eq^n.

Let $(a+bx) = e^z$

$(a+bx)\frac{dy}{dx} = b\frac{dy}{dz} = bD'y$ where, $D' = \frac{d}{dz}$

$(a+bx)^2\frac{d^2y}{dx^2} = b^2\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right) = b^2D'(D'-1)y$

$\vdots$

$(a+bx)^n\frac{d^ny}{dx^n} = b^n D'(D'-1)(D'-2)\cdots[D'-(n-1)]$

Q) $\left[(2+3x)^2\frac{d^2y}{dx^2} + 3(2+3x)\frac{dy}{dx} - 36\right]y = 3x^2+4x+1$

Let, $(2+3x) = e^z$

Reduce Eq^n is $= [(3)^2D'(D'-1) + 3\cdot3D' - 36]$

$$= 3\left(\frac{e^z-2}{3}\right)^2 + 4\left(\frac{e^z-2}{3}\right) + 1$$

$\Rightarrow [9D'^2 - 36]y = \frac{(e^z-2)^2}{3} + \frac{4}{3}(e^z-2) + 1$

$\Rightarrow [9D'^2 - 36]y = \frac{e^{2z} + 4 - \cancel{4e^z} + \cancel{4e^z} - 8 + 3}{3} = \frac{e^{2z}-1}{3}$

$(D'^2 - 4)y = \frac{1}{27}(e^{2z} - 1)$

A.E.y $m^2 - 4 = 0$

$m = \pm 2$

$C.F = C_1 e^{2z} + C_2 e^{-2z}$

$= C_1 e^{2\log(2+3x)} + C_2 e^{-2\log(2+3x)}$

$= C_1 (2+3x)^2 + C_2 (2+3x)^{-2}$

$P.I. = \frac{1}{D'^2 - 4} \frac{1}{27}(e^{2z} - 1)$

$= \frac{z}{2D'} \frac{1}{27}(e^{2z} - 1)$ $= \frac{z}{2D'} \frac{1}{27}(e^{2z})$

$+ \frac{1}{4} \cdot \frac{1}{27} \cdot 1$

$= \frac{z}{54} \cdot \frac{1}{D'}(e^{2z} - 1)$

$= \frac{\log(2+3x)}{54} \cdot \frac{(2+3x)^2}{2}$

$= \frac{z}{54}\left(\frac{e^{2z}}{2} - z\right)$ $+ \frac{1}{108}$

$= \frac{\log(2+3x)}{54}\left[\frac{(2+3x)^2}{2} - \cancel{\log(2+3x)}\right]$

$\frac{\log(2+3x) \cdot (2+3x)^2 + 1}{108}$

Q) $[(x+1)^2 D^2 + (x+1)D + 1]y = (2x+3)(2x+4)$

Q) $\left[(x-1)^3 \frac{D^3y}{dx^3} + 2(x-1)^2 \frac{d^2y}{dx^2} - 4(x-1)\frac{dy}{dx} + 4y\right]$

$= 4\log(x-1)$

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* Mathematical expectation :- (expected value)

$$E(x^r) = \begin{cases} \sum_x x^r p(x) , & \text{discrete} \\ \int_{-\infty}^{\infty} x^r f(x)\, dx , & \text{Continuous} \end{cases}$$

$$\mu'_r = E(x^r)$$

$$\mu_r = E[x - \bar{x}]^r$$

$$\text{Mean} = E(x) = \begin{cases} \sum x\, p(x) \\ \int x\, f(x)\, dx \end{cases}$$

$$\text{Var.} = E[x - \bar{x}]^2 = \int_{-\infty}^{\infty} (x - \bar{x})^2 f(x)\, dx$$

$$= \mu'_2 - (\mu'_1)^2$$

$$= E(x^2) - (E(x))^2$$

Property,

① $E(X+Y) = E(x) + E(Y)$

② $E(xy) = E(x) E(y)$ if x, y are independent.

③ $E(Kx) = K E(x)$

④ $E(k) = k$.

⑤ $\text{Var}(ax+b) = a^2 V(x)$

* Covariance.

$$\text{cov}(x, y) = E(x - \bar{x})(y - \bar{y})$$

$$= E(xy) - 2E(x)E(y) + E(x)E(y)$$

$$= E(xy) - E(x)E(y)$$

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$$E(xy - x\bar{y} - \bar{x}y + \bar{x}\bar{y})$$

$$E(xy) + E(\bar{x}\bar{y}) - [E(x\bar{y}) + E(\bar{x}y)]$$

$$E(x)E(y) + E(\bar{x})E(\bar{y}) - [E(x)E(\bar{y}) + E(\bar{x})E(y)]$$

$$E(x)(E(y) - E(\bar{y})) + E(\bar{x})(E\bar{y} - E(y))$$

$$(E(x) - E(\bar{x}))(E(y) - E(\bar{y})) \times$$

H/W

1. $Cov(ax, by) = ab\,Cov(x,y)$
2. $Cov(x+a, by+b) = Cov(x,y)$

Q) Two unbiased dice are thrown. Find the expected value of the sum of nos. of points on the face.

⇒

$x \rightarrow$	2	3	4	5	6	7	8	9	10	11	12
	$\frac{1}{1}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{1}$

7
6 — 7 — 8
5 — 4   5 — 9
4 — 3   3 — 10
3 — 2   2 — 11
2 — 1   1 — 12

$E(x) = \sum x p(x)$

$E(x) = 2 + 6 + 12 + 20 + 30 + 42 + 40 + 36 + 30 + 22 + 12$

$= \frac{252}{36} = 7$ Ans

112
40
152
36
188
30
218
240

x1 : no. of points on ith dice

$E(x_1 + x_2) = E(x_1) + E(x_2)$

$= \frac{7}{2} + \frac{7}{2}$

$= 7$

$\boxed{\text{For } n \text{ dice} = \frac{7n}{2}}$

(H/W) In four tosses of coin find the expected values of no. of heads.

Q) A coin is tossed until the head is appear. Find the expected of toss required.

1 2 3

$\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}$

$\frac{1}{2}$ $\frac{1}{4}$ $\frac{1}{8}$

$= \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$

$X = 1 \quad 2 \quad 3 \quad 4 \quad 5 \ldots$

$P(x) = \frac{1}{2} \quad \frac{1}{4} \quad \frac{1}{8} \quad \frac{1}{16} \quad \frac{1}{32} \ldots$

$E(x) = 1\cdot\frac{1}{2} + 2\times\frac{1}{4} + 3\times\frac{1}{8} + 4\times\frac{1}{16} + \ldots$

$\frac{1}{2}\left(\frac{1}{1} + \frac{2}{2} + \frac{3}{4} + \frac{4}{8} + \ldots\right)$

$\frac{1}{1-2}\cdot\frac{1\times\frac{1}{2}}{(1-r)^2}$

$\frac{1}{1-\frac{1}{2}} + \frac{1\times\frac{1}{2}}{(1-\frac{1}{2})^2} = 2$

$2 - 1 = 1$

$S_n = \frac{1}{2} + 2\cdot\frac{1}{4} + 3\cdot\frac{1}{8} \ldots$

$\frac{1}{2}S_n = \frac{1}{4} + \ldots$

$S_n - \frac{1}{2}S_n =$

* Chebychev's inequality

$$P(|x-\mu| \geq k\sigma) \leq \frac{1}{k^2}$$

or $P(|x-\mu| < k\sigma) \geq 1 - \frac{1}{k^2}$

let $k\sigma = c$

$$P(|x-\mu| \geq c) \leq \frac{\sigma^2}{c^2}$$

$$\Rightarrow P(|x-\mu| \geq c) \leq \frac{Var(x)}{c^2}$$

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Q A symmetric dice is thrown 600 times, find the lower bound of probability of getting 80 to 120 sixes.

$\Rightarrow$ $\boxed{\mu = n \times p \quad \sigma^2 = npq \; ; \; q = 1-p}$

solve $\Rightarrow$ $n \times p = 100$

$= \frac{500}{6}$

$k = 80 - 120$

$\Rightarrow P(|x - 100| < k \cdot \sqrt{\frac{500}{6}}) \geq 1 - \frac{1}{k^2}$

$\Rightarrow P\left(100 - k\sqrt{\frac{500}{6}} < x < 100 + k\sqrt{\frac{500}{6}}\right) \geq 1 - \frac{1}{k^2}$

$\Rightarrow P(80 < x < 120) \geq \frac{19}{24}$

7.10.

Q) Two unbiased dice if x is the sum of the no. showing up then find the upper band of the probability of $P(|x-7| \geq 3)$.