Differential Equation

* $a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_n y = Q(x)$

$D \equiv \frac{d}{dx}$

$(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_n) y = 0$

Auxiliary Equⁿ is

$a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + \cdots + a_n = 0$

$m = m_1, m_2, m_3, \cdots, m_n$

* Direct method for obtaining particular integral:-

→ This methods depend on the nature of $Q(x)$.
Particular Integral by this method can be obtained where $Q(x)$ has the following form.

(i) $Q(x) = e^{ax+b}$

(ii) $Q(x) = \sin(ax+b)$ or $\cos(ax+b)$

(iii) $Q(x) = x^m$ or polynomial in x.

(iv) $Q(x) = e^{ax} \cdot V(x)$

(v) $Q(x) = x \cdot V(x)$

(i) $\boxed{Q(x) = e^{ax+b}}$

$P.I. = \frac{1}{f(D)} e^{ax+b}$

$= \frac{1}{f(a)} e^{ax+b}$, (provided $f(a) \neq 0$)

If $f(a) = 0$

Then, $P.I = \frac{1}{f(D)} e^{ax+b} = x \cdot \frac{1}{f'(a)} \cdot e^{ax+b}$ provided $f'(a) \neq 0$

if $f'(a) = 0$

Then, $P.I. = \frac{1}{f(D)} e^{ax+b} = x^2 \cdot \frac{1}{f''(a)} e^{ax+b}$, Provided $f''(a) \neq 0$

Q) Solve → $(D^2 - 3D + 2)y = e^{3x}$

Given $(D^2 - 3D + 2)y = e^{3x}$ —(1)

Auxiliary eqn is

$m^2 - 3m + 2 = 0$

$m^2 - 2m - m + 2 = 0$

$m(m-2) - 1(m-2) = 0$

$(m-1)(m-2) = 0 \Rightarrow m = (1, 2)$

C.F = $\boxed{c_1 e^{x} + c_2 e^{2x}}$
(Complementary function)

(Particular Integral) P.F. $= \frac{1}{f(D)} e^{3x} = \frac{1}{D^2 - 3D + 2} e^{3x}$

$\boxed{\text{Put } D = a = 3}$

$= \frac{1}{2} e^{3x}$

∴ General Solution ⇒ $y = c_1 e^{x} + c_2 e^{2x} + \frac{1}{2} e^{3x}$
where c_1 & c_2 are the arbitrary constant.

Q) $(D^2 + 6D + 9)y = 5^{x} - \log 2$

$m^2 + 6m + 9$

$m^2 + 3m + 3m + 9$

$(m+3)(m+3) = 0$

$m = (-3, -3)$

C.F = $(c_1 + c_2 x) e^{-3x}$

P.F. $= \frac{1}{f(D)}(5^{x} - \log 2) = \frac{1}{D^2 + 6D + 9}(5^{x} - \log 2)$

$$= \frac{1}{D^2+6D+9} e^{x\log 5} - \frac{1}{D^2+6D+9} \log 2 e^{0.x}$$

$$= \frac{1}{(\log 5)^2 - 6\log 5 + 9} \cdot e^{x\log 5} - \frac{1}{(0)^2+6 0 + 9} \log 2 e^{0.x}$$

$$P.I = \frac{e^{x.\log 5}}{(\log 5 \cancel{} + 3)^2} - \frac{\log 2}{9}$$

General soln $\Rightarrow y = (C_1 + C_2 x) e^{-3x} + \frac{e^{x\log 5}}{(\log 5 + 3)^2} - \frac{\log 2}{9}$

Q) ~~Solve~~ $(D^6 - 64) y = e^x \cosh 2x$ $\quad \left(\cosh = \frac{e^x + e^{-x}}{2} \right)$

$m^6 - 64 = 0$

$(m^3 - 8)(m^3 + 8) = 0$

$\cancel{m = \sqrt[3]{8} = 2, 2, 2}$

$\cancel{m = \sqrt[3]{-8} = 2\times2\times2}$

$=$

$(m-2)(m^2 + 4 + 2m)(m+2)(m^2 + 4 - 2m) = 0$

$(m-2)(m+2)(m^2 + 2m + 4)(m^2 - 2m + 4) = 0$

$m = 2$ or $m = \frac{-2 \pm \sqrt{4-16}}{2}$ or $m = \frac{2 \pm \sqrt{4-16}}{2}$
$m = -2$

$m = 2$ or $m = -1 \pm \sqrt{3} i$ or $m = 1 \pm \sqrt{3} i$
$m = -2$

$$C.F. = C_1 e^{2x} + C_2 e^{-2x} + C_3 e^{-1 + i\sqrt{3} x} + C_4 e^{-1 - i\sqrt{3} x} + C_5 e^{1 + i\sqrt{3} x} + C_6 e^{1 - i\sqrt{3} x}$$

$\therefore$ $C.F. = c_1 e^{2x} + c_2 e^{-2x} + e^{1.x}(c_3 \cos\sqrt{3}x + c_4 \sin\sqrt{3}x)$
$+ e^{-x}(c_5 \cos\sqrt{3}x + c_6 \sin\sqrt{3}x)$

$P.I. = \frac{1}{D^6-64} \cdot e^x \left[\frac{e^{2x}+e^{-2x}}{2}\right]$

$= \frac{1}{2} \cdot \frac{1}{D^6-64}(e^{3x}+e^{-x})$

$= \frac{1}{2} \cdot \frac{1}{D^6-64} \cdot e^{3x} + \frac{1}{2} \cdot \frac{1}{D^6-64} e^{-x}$

$= \frac{1}{2} \cdot \frac{1}{729-64} \cdot e^{3x} + \frac{1}{2 \cdot (1-64)} e^{-x}$

$= \frac{1}{2}\left(\frac{e^{3x}}{625} - \frac{e^{-x}}{63}\right)$

(ii) $\boxed{Qx = \sin(ax+b) \text{ or } \cos(ax+b)}$

$P.I. = \frac{1}{\phi(D)} \sin(ax+b)$

$= \frac{1}{\phi(D^2)} \sin(ax+b)$

$= \frac{1}{\phi(-a^2)} \sin(ax+b)$ provided $\phi(-a^2) \neq 0$

if $\phi(-a^2) = 0$

$P.I. = x \cdot \frac{1}{\phi'(-a^2)} \sin(ax+b)$, provided $\phi'(a^2) \neq 0$

if $\phi(-a^2) = 0$

$P.I. = x^2 \cdot \frac{1}{\phi''(-a^2)} \sin(ax+b)$, provided $\phi''(-a^2) \neq 0$

Q) $(D^2+9)y = \sin 4x$

A.E is $m^2+9=0$

$m = \pm 3i$

$C.F = e^{0\cdot x}(c_1 \cos 3x + c_2 \sin 3x)$

$P.I. = \frac{1}{D^2+9} \cdot \sin 4x$

$= \frac{1}{-16+9} \cdot \sin 4x \quad (D=a=4)$

$= \frac{-1}{7} \sin 4x.$

General soln $\Rightarrow y = c_1 \cos 3x + c_2 \cos 3x - \frac{1}{7} \sin 4x$

Q) $(D^2+3D+2)y = \sin 2x$

$m^2+3m+2=0$

$m^2+2m+m+2=0$

$(m+2)(m+1)=0$

$m=-2,-1$

$C.F. = c_1 e^{-2x} + c_2 e^{-x}$

$P.I. = \frac{1}{D^2+3D+2} \cdot \sin 2x$

$= \frac{1}{-(2)^2+3D+2} \cdot \sin 2x$

$= \frac{1}{-2+3D} \cdot \sin 2x$

$= \frac{3D+2}{9D^2-4} \sin 2x$

$= \frac{(3D+2)\cdot \sin 2x}{9\times-(2)^2-4} = -\frac{(3D+2)\cdot \sin 2x}{40}$

$3D \sin 2x + 2\sin 2x$

$6\cos 2x + 2\sin 2x$

Q) $(D^4 + 2a^2D^2 + a^4)y = 8\cos ax$

Solve> $m^4 + 2a^2m^2 + a^4 = 0$

$$m^4 + a^2m^2 + a^2m^2 + a^4 = 0$$

$$m^2(m^2 + a^2) + a^2(m^2 + a^2) = 0$$

$$(m^2 + a^2)(m^2 + a^2) = 0$$

$$m^2 = -a^2$$

$$m = \pm ia, \pm ia$$

$$C.F. = \{(c_1 + c_2x)\cos ax + (c_3 + c_4x)\sin ax\} \times e^{0x}$$

$$P.I = \frac{1}{D^4 + 2a^2D^2 + a^4} \cdot 8\cos ax$$

$$= \frac{1}{D^2 \times D^2 + 2a^2D^2 + a^4} \cdot 8\cos ax$$

$$= \frac{1}{a^4 + a^4 - 2a^4} \cdot 8\cos ax$$

(First Derivative)

$$P.I = x \cdot \frac{1}{4D^3 + 4a^2D} \cdot 8\cos ax$$

$$= \frac{1}{4D(D^2 + a^2)} \cdot 8\cos ax$$

$$\cancel{= \frac{D}{4D^2 \times (-a^2+1)} \cdot 8\cos ax = \frac{D}{4(a^4 - a^2)} \cdot 8\cos ax}$$

(Second Derivative)

$$P.I = x^2 \cdot \frac{1}{12D^2 + 4a^2} \cdot 8\cos ax$$

$$= -\frac{1}{8a^2} \cdot 8\cos ax = \frac{-x^2\cos ax}{a^2}$$

Q) $(D^3 - 3D^2 + 4D - 2)y = \cos x$

$m^3 - 3m^2 + 4m - 2 = 0$

$m = 1$

$$P.I = \frac{1}{D^3 - 3D^2 + 4D - 2} \cdot \cos x$$

$$= \frac{1}{-D + 3 + 4D - 2} \cdot \cos x$$

$$= \frac{1}{3D + 1} \cdot \cos x$$

$$= \frac{3D - 1}{(3D+1)(3D-1)} \cdot \cos x = \frac{3D - 1}{9D^2 - 1} \cos x$$

$$= -\frac{(3D - 1)\cos x}{10} = (-)\frac{(-3\sin x - \cos x)}{10}$$

$\boxed{3 D.\cos x - \cos x)}$

$\boxed{3(-\sin x) - \cos x}$

$$= \frac{3\sin x + \cos x}{10}$$

(iii) $Q(x) = x^m$ or polynomial in x.

$$P.I. = \frac{1}{f(D)} Q(x)$$

$$= \frac{1}{1 \pm \phi(D)} Q(x)$$

$$= [1 \pm \phi(D)]^{-1} Q(x)$$

Q). $(D^2+2D+1)y = x.$

$m^2+2m+1=0$

$m = \frac{-2 \pm \sqrt{4-4}}{2}$

$m = -1$

$C.F. = (c_1+c_2x)e^{-x}$

$P.I = \frac{1}{D^2+2D+1} \cdot x$

$= \frac{1}{(D+1)^2} \cdot x$

$= (1+D)^{-2} \cdot x = (1-2D+3D^2-4D^3+\ldots) x$

$= (x - 2Dx + 3D^2x + \ldots)$

$= (x-2)$

General soln $\Rightarrow$ $y = (c_1+c_2x)e^{-x} + (x-2)$

Q) $(D^4-2D^3+D^2)y = x^3$

$P.I. = \frac{1}{D^4-2D^3+D^2} \times x^3$

$= \frac{1}{D^2[1+(D^2-2D)]} \times x^3$

$= \frac{1}{D^2}[1+(D^2-2D)]^{-1} x^3$

$= \frac{1}{D^2}[1-(D^2-2D)+(D^2-2D)^2 - \ldots] x^3$

$= \frac{1}{D^2}[x^3 - x^3(D^2-2D) + \ldots]$

$= \frac{1}{D^2}[x^3 - 6x + 6x^2 + 24x + 24]$

Date → 23/1/23

Q) $(D^3 - D^2 - 6D)y = 1 + x^2$

$m^3 - m^2 - 6m = 0$

$m(m^2 - m - 6) = 0$

$m\{m^2 - 3m + 2m - 6\} = 0$

$m(m-3)(m+2) = 0$

$m = 0 \mid m = 3 \mid m = -2$

$C.F = c_1 e^{0x} + c_2 e^{3x} + c_3 e^{-2x}$

$= c_1 + c_2 e^{3x} + c_3 e^{-2x}$

$P.I = \frac{1}{D^3 - D^2 - 6D}(1+x^2)$

$= \frac{-1}{6D\left\{1 - \frac{(D^3 - D^2)}{6D}\right\}}(1+x^2)$

$= \frac{-1}{6D}\left[1 - \frac{1}{6}(D^2 - D)\right]^{-1}(1+x^2)$

$= -\frac{1}{6D}\left[1 + \frac{1}{6}(D^2 - D) + \frac{1}{36}(D^2 - D)^2 + \frac{1}{216}(D^2 - D)^3 + \dots\right](1+x^2)$

$= \frac{-1}{6D}\left[(1+x^2) + \frac{1}{6}(D^2 - D)(1+x^2) + \frac{1}{36}(D^4 - 2D^3 + D^2)(1+x^2) + \dots\right]$

$= -\frac{1}{6D}\left[(1+x^2) + \frac{1}{3}(1-x) + \frac{1}{36}\cdot 2\right]$

$= \frac{-1}{6D}\left[1 + x^2 + \frac{1}{3}(1-x) + \frac{1}{18}\right]$

$= \frac{-1}{6}\left[x + \frac{x^3}{3} + \frac{1}{3}\left(x - \frac{x^2}{2}\right) + \frac{1}{18}x\right]$

$$= -\frac{1}{6}\left[\frac{x^3}{3} - \frac{x^2}{6} + \frac{25}{18}x\right]$$

General soln is, $y = C.F. + P.I.$

(iv) $\boxed{Q(x) = e^{ax}.\ V(x)}$, where v is a function of x.

$$P.I. = \frac{1}{f(D)} e^{ax}.V = e^{ax}.\frac{1}{f(D+a)}.V$$

~~Solve~~

Q) $(D^2 - 2D - 1)y = e^x.\cos x$

$m^2 - 2m - 1 = 0$

$\cancel{m^2 - m - m - 1 = 0}$

$\cancel{m(m-1) - 1(}$

$$m = \frac{2 \pm \sqrt{4+4}}{2}$$

$$= \frac{2 \pm \sqrt{8}}{2}$$

$$= 1 \pm \sqrt{2} \text{ (real \& distinct)}$$

$$C.F = C_1 e^{(1+\sqrt{2})x} + C_2 e^{(1-\sqrt{2})x}$$

$$P.I. = \frac{1}{D^2 - 2D - 1} e^x.\cos x$$

$$= e^x.\frac{1}{(D+1)^2 - 2(D+1) - 1} \cos x$$

$$= \frac{e^x.\cos x}{D^2 + \cancel{2D} + \cancel{1} - \cancel{2D} - 2 - \cancel{1}}$$

$$= e^x.\frac{1}{D^2 - 2}.\cos x$$

(Here apply 2nd Rule)

$$= e^x.\frac{1}{-1^2 - 2} \cos x = e^x \frac{-1}{3}.\cos x = -\frac{1}{3}e^x\cos x$$

Q) $(D^3+3D^2-4D+12)y = 12x e^{-2x}$

Sol^n $\to$ $m^3+3m^2-4m+12=0$

$m^2(m+3)-4(m+3)=0$

$(m^2-4)(m+3)=0$

$(m-2)(m+2)(m+3)=0$

$m=(2,-2,-3)$

$C.F. = c_1 e^{2x} + c_2 e^{-2x} + c_3 e^{-3x}$

$P.I. = \frac{1}{D^3+3D^2-4D+2} e^{-2x} \cdot 12x$

$= \frac{1}{(D+3)(D+2)(D-2)} e^{-2x} \cdot 12x$

$= 12 e^{-2x} \cdot \frac{1}{(D+3-2)(D+2-2)(D-2-2)} \cdot x$

$= 12 e^{-2x} \cdot \frac{1}{(D+1)D(D-4)} \cdot x$

$= 12 e^{-2x} \cdot \frac{1}{D^3-4D^2+D-4D} \cdot x$

$= 12 e^{-2x} \cdot \frac{1}{[-4D]\left\{1-\frac{(D^3-3D^2)}{4D}\right\}} \cdot x$

$= 12 e^{-2x} \times \frac{-1}{4D} \cdot \left[1-\left(\frac{D^3-3D^2}{4D}\right)\right]^{-1} \cdot x$

$$= 12e^{-2x} x - \frac{1}{4D}\left[1 + \frac{1}{4}(D^2 - 3D) + \frac{1}{16}(D^2 - 3D)^2 + \cdots\right\} x$$

$$= 12e^{-2x} x - \frac{1}{4D}\left[x + \frac{1}{4}(-3)\right\}$$

$$= -3e^{-2x}\left[\frac{x^2}{2} - \frac{3}{4}x\right]$$

Q) $(D^2 - 2D + 10)y = 16e^x \cos 3x + 24e^x \sin 3x$.

$m^2 - 2m + 10 = 0$

$$m = \frac{2 \pm \sqrt{4 - 40}}{2} = \frac{2 \pm 6i}{2} = 1 \pm 3i$$

$C.F. = e^x(\cos 3x + \sin 3x)$.

$$P.I. = \frac{1}{D^2 - 2D + 10}\{16e^x \cos 3x + 24e^x \sin 3x\}$$

$$= \frac{1}{D^2 - 2D + 10} 16e^x \cos 3x + \frac{1}{D^2 - 2D + 10} 24e^x \sin 3x.$$

$$= 16e^x \cdot \frac{1}{(D+1)^2 - 2(D+1) + 10} \cdot \cos 3x + 24e^x \cdot \frac{1}{(D+1)^2 - 2(D+1) + 10} \cdot \sin 3x$$

$$= 16e^x \cdot \frac{1}{D^2 + 1 + \cancel{2D} - \cancel{2D} - 2 + 10} \cdot \cos 3x + 24e^x \cdot \frac{1}{D^2 + 9} \cdot \sin 3x$$

$$= 16e^x \cdot \frac{1}{D^2 + 9} \cdot \cos 3x + 24e^x \cdot \frac{1}{D^2 + 9} \cdot \sin 3x$$

$$= \frac{16 e^{x} \cdot x \cos 3x}{2D} + \frac{24 e^{x} \cdot x \sin 3x}{2D}$$

$$= \frac{\cancel{16}^{8} e^{x} \cdot 1 \cdot \cos 3x}{-\cancel{18}\,9} + \frac{\cancel{16}^{8} e^{x} \cdot 1 \cdot \sin 3x}{-\cancel{18}\,9}$$

$$= -\frac{8 e^{x} \cos 3x}{9} - \frac{8 e^{x} \sin 3x}{9}$$

$$= -\frac{8 e^{x}}{9}(\cos 3x + \sin 3x)$$

$$= \frac{\cancel{16}^{8} e^{x} \cdot x \cdot \cos 3x}{\cancel{2}D} + \frac{\cancel{24}^{12} e^{x} \cdot x \cdot \sin 3x}{\cancel{2}D}$$

$$= 8 e^{x} x \left\{ \frac{\sin 3x}{3} - \frac{\cancel{12}^{4} \cos 3x}{\cancel{3}} \right\}$$

$$= \frac{4x e^{x}}{3}(2 \sin 3x - 3 \cos 3x)$$

(V) $\boxed{Q(x) = x \cdot V(x)}$, where v is function of x.

$$P.I = \frac{1}{f(D)} \cdot Q = \frac{1}{f(D)} x \cdot V$$

$$= x \frac{1}{f(D)} \cdot V - \frac{f'(D)}{[f(D)]^{2}} \cdot V$$

Q) $(D^{2} - 5D + 6) y = x \cdot \cos 2x$

$m^{2} - 5m + 6 = 0$

$m^{2} - 3m - 2m + 6 = 0$

$m(m-3) - 2(m-3) = 0$

$$(m-3)(m-2)=0$$

$$m=3,2$$

$$\boxed{C.F. = C_1 e^{3x} + C_2 e^{2x}}$$

C.F = complementary function

$$P.I. = \frac{1}{D^2-5D+6}\, x \cdot \cos 2x$$

$$= x \cdot \frac{1}{D^2-5D+6} \cdot \cos 2x - \frac{2D-5}{(D^2-5D+6)^2} \cdot \cos 2x$$

$$= x \cdot \frac{1}{[-2^2-5D+6]} \cdot \cos 2x - \frac{(2D-5)}{(-2^2-5D+6)^2} \cdot \cos 2x$$

$$= x \cdot \frac{1}{2-5D} \cos 2x - \frac{(2D-5)}{(2-5D)^2} \cdot \cos 2x$$

$$= x \frac{2+5D}{(2-5D)(2+5D)} \cos 2x - \frac{(2D-5)}{4-20D+25D^2} \cos 2x$$

$$= x \frac{(2+5D) \cos 2x}{4-25D^2} - \frac{(2D-5)}{4-20D+25(-2^2)} \cos 2x$$

$$(\because D^2=-2^2)$$

$$= \frac{x(2+5D) \cos 2x}{104} - \frac{(2D-5)}{4-20D-100} \cos 2x$$

$$= \frac{x \cdot (2\cos 2x + 5D\cos 2x)}{104} + \frac{(2D-5)}{20D+96} \cos 2x$$

$$= \frac{x}{104}\left(2\cos 2x - 5 \cdot 2 \sin 2x\right) + \frac{2D-5}{20D+96} \cos 2x$$

$$\frac{x}{104}\left(2\cos 2x - 10\sin 2x\right) + \frac{1}{4}\frac{(2D-5)}{(5D+24)}\cos 2x.$$

$$\frac{x}{104}\left(2\cos 2x - 10\sin 2x\right) + \frac{1}{4}\frac{(2D-5)(5D-24)}{25D^2-576}\cos 2x$$

$$\frac{x}{104}\left(2\cos 2x - 10\sin 2x\right) + \frac{1}{4}\frac{(10D^2-48D-25D+120)}{-676}\cos 2x$$

$$\cancel{\frac{x}{104}\left(2\cos 2x - 10\sin 2x\right) + \frac{1}{4}\frac{(-40+120-73D)}{-676}\cos 2x}$$

$$\cancel{\frac{x}{104}\left(2\cos 2x - 10\sin 2x\right) - \frac{1}{2704}(80-73D)\cos 2x.}$$

$$\cancel{\frac{x}{104}\left(2\cos 2x - 10\sin 2x\right) - \frac{80\cos 2x}{2704}}$$

$$\frac{x}{104}\left(2\cos 2x - 10\sin 2x\right) - \frac{1}{2704}\left[40\cos 2x - 73(-2\sin 2x) + 120\cos 2x\right]$$

$$\frac{x}{104}\left(2\cos 2x - 10\sin 2x\right) - \frac{1}{2704}\left[-40\cos 2x + 146\sin 2x + 120\cos 2x\right]$$

$$\frac{x}{104}\left(2\cos 2x - 10\sin 2x\right) + \frac{80}{2704}\cos 2x + \frac{146}{2704}\sin 2x$$

$$\frac{x}{104}\left(2\cos 2x - 10\sin 2x\right) + \frac{5\cos 2x}{676} + \frac{73}{1352}\sin 2x$$

Q) $(D^2+3D+2)y = x e^x . \sin x$

$m^2+3m+2=0$

$m^2+2m+m+2=0$

$m(m+2)+1(m+2)=0 \qquad m=-1,-2$

$$P.I = \frac{1}{D^2+3D+2} e^x (x.\sin x)$$

$$= e^x . \frac{1}{(D+1)^2+3(D+1)+2} (x.\sin x)$$

$$= e^x . \frac{1}{D^2+1+2D+3D+3+2} (x.\sin x)$$

$$= e^x . \frac{1}{D^2+5D+6} (x.\sin x)$$

$$= e^x \left[x \frac{1}{D^2+5D+6} \sin x - \frac{2D+5}{(D^2+5D+6)^2} . \sin x \right]$$

$$= e^x \left[\frac{x}{-1^2+5D+6} \sin x - \frac{(2D+5)\sin x}{(-1^2+5D+6)^2} \right]$$

$$= e^x \left[\frac{x}{5+5D} \sin x - \frac{2D+5}{5+5D} . \sin x \right]$$

$$= e^x \left[\frac{x(1-D)}{5(1+D)(1-D)} \sin x - \frac{(2D+5)(1-D)}{5(1+D)(1-D)} \sin x \right]$$

$$= e^x \left[\frac{x}{5} \frac{(1-D)\sin x}{(1-D^2)} - \frac{(2D+5)(1-D)\sin x}{5(1-D^2)} \right.$$

$$= e^x\left[\frac{x}{10}(1-D)\sin x - \frac{(2D+5)(1-D)\sin x}{10}\right]$$

$$= e^x\left[\frac{x}{10}(\sin x - D\sin x) - \left(\frac{2D - 2D^2 + 5 - 5D}{10}\right)\sin x\right]$$

$$= e^x\left[\frac{x}{10}(\sin x - \cos x) - \left(\frac{-3\cos x - 2(-\sin x) + 5\cos x}{10}\right)\right]$$

$$= e^x\left[\frac{x}{10}(\sin x - \cos x) + \left(\frac{-3\cos x + 5\cos x + 2\sin x}{10}\right)\right]$$

$$= \frac{e^x}{10}\left[(x\sin x - x\cos x) - (2\sin x + 2\cos x)\right]$$

$$= -\frac{e^x}{5}\left[\frac{x}{2}(\cos x - \sin x) + \frac{1}{5}\left(\sin x - \frac{5}{2}\cos x\right)\right]$$

* **General method for obtaining particular integral :-**

$$P.I. = \frac{1}{f(D)}Q(x)$$

$$= \frac{1}{(D-m_1)(D-m_2)\cdots(D-m_n)} \times Q(x)$$

$$= \left[\frac{A_1}{(D-m_1)} + \frac{A_2}{D-m_2} + \cdots + \frac{A_n}{D-m_n}\right]Q(x)$$

$$= \frac{A_1}{(D-m_1)}Q(x) + \frac{A_2}{(D-m_2)}Q(x) + \cdots + \frac{A_n}{D-m_n}(Qx)$$

$$= A_1 e^{m_1 x}\int Q(x)e^{-m_1 x}dx + A_2 e^{m_2 x}\int Q(x)e^{-m_2 x}dx + \cdots + A_n e^{m_n x}\int Q(x)e^{-m_n x}dx$$

Q) Solve $\rightarrow (D^2+3D+2)y = e^{e^x}$

Auxiliary Eqn

$$m^2+3m+2=0$$

$$m^2+2m+m+2=0$$

$$m=-2,-1$$

$$C.F. = C_1 e^{-2x} + C_2 e^{-x}$$

$$P.I. = \frac{1}{D^2+3D+2} e^{e^x}$$

$$= \frac{1}{(D+2)(D+1)} e^{e^x}$$

$$= \frac{1}{(D+2)}\left[\frac{1}{(D+1)} e^{e^x}\right]$$

$$= \frac{1}{(D+2)}\left[\frac{1}{D-(-1)} e^{e^x}\right]$$

$$= \frac{1}{(D+2)}\left[e^{-x}\int e^{e^x}\cdot e^x\,dx\right]$$

Put $e^x = p$

$e^x dx = dp$

$$= \frac{1}{(D+2)}\left[e^{-x}\int e^p\cdot dp\right]$$

$$= \frac{1}{(D+2)}\left[e^{-x}\, e^p\right] \Rightarrow \frac{1}{(D+2)}\left(e^{-x}e^{e^x}\right)$$

$$= \frac{1}{D-(-2)}\left[e^{-x}e^{e^x}\right]$$

$$= e^{-2x}\int e^{-x} e^{e^x} e^{2x}\,dx$$

$$= e^{-2x}\int e^{x}\cdot e^{e^x}\,dx$$

* Geometric mean $\Rightarrow \log G = \int_a^b \log x \, f(x) \, dx$.

* Harmonic mean $\frac{1}{H} = \int_a^b \frac{f(x)}{x} dx$

$$\frac{1}{4}\left(3m^2 - m^3\right) = \frac{3}{2}$$

Date → 30/01/23

Q) $(D^2 + a^2)y = \sec ax$

Auxilary Eq^n $m^2 + a^2 = 0$

$m = \pm ai$

$C.F. = (c_1 \cos ax + c_2 \sin ax)$

$$P.I. = \frac{1}{D^2+a^2} \sec ax$$

$$= \frac{1}{(D+ai)(D-ai)} \sec ax$$

$$= \frac{1}{2ia}\left[\frac{1}{D-ia} - \frac{1}{D+ia}\right] \sec ax$$

$$= \frac{1}{2ia}\left[\frac{1}{D-ia} \sec ax - \frac{1}{D+ia} \sec ax\right]$$

$$\frac{1}{D-ia} \sec ax = e^{iax} \int \sec ax \, e^{-iax} dx$$

$$\cos ax = \frac{e^{iax} + e^{-iax}}{2}$$

$$\sec ax = \frac{2}{e^{iax} + e^{-iax}}$$

$$e^{iax} \int \frac{2}{e^{iax} + e^{-iax}} \times e^{-iax} dx$$

$$e^{iax} \int \frac{2 \cdot e^{-2iax}}{1 + e^{-2iax}} dx.$$

$$\left.\frac{e^{iax}}{-ia}\right] \frac{dt}{t}$$

$1+e^{-2iax} = t$

$-2ia\, e^{-2iax}\, dx = dt$

$$\Rightarrow -\frac{e^{iax}}{ia} \log t$$

$$\Rightarrow -\frac{e^{iax}}{ia} \log(1+e^{-2iax})$$

$$\Rightarrow -\frac{e^{iax}}{ia} \log(1+\cos 2ax - i\sin 2ax)$$

$$\Rightarrow -\frac{e^{iax}}{ia} \log(2\cos^2 ax - 2i \sin ax \cos ax)$$

$$\Rightarrow -\frac{e^{iax}}{ia} \log(2\cos ax \cdot e^{-iax})$$

$$\Rightarrow -\frac{e^{iax}}{ia} \left[\log(2\cos ax) - iax\right]$$

Replacing i by $(-i)$ we get

$$\frac{1}{D+ia} \sec ax$$

$$= \frac{1}{ia} e^{-iax}\left[\log(2\cos ax + iax\right]$$

$$P.I = \frac{1}{2ai}\left[-\frac{e^{iax}}{ai}\left[\log(2\cos ax) - iax\right] - \frac{e^{-iax}}{ia}\left[\log(2\cos ax\right.\right.$$

Q)

Cauchy's linear Eq^n → An equation of

form: $a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}}$

$+ \dots + a_n y = Q(x)$

where, $a_0, a_1, a_2, \dots, a_{n-1}, a_n$ are constant is called Cauchy's linear Eq^n.

Let, $x = e^z$

$1 = e^z \cdot \frac{dz}{dx}$

$\frac{dz}{dx} = \frac{1}{e^z} = \frac{1}{x}$

$\Rightarrow \boxed{x \cdot \frac{dz}{dx} = 1}$

$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{x} \times \frac{dy}{dz}$

$\Rightarrow \boxed{x \cdot \frac{dy}{dx} = \frac{dy}{dz}}$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

$= \frac{d}{dx}\left(\frac{1}{x} \cdot \frac{dy}{dz}\right)$

$= -\frac{1}{x^2} \frac{dy}{dz} + \frac{1}{x} \cdot \frac{d^2y}{dz^2} \cdot \frac{dz}{dx}$

$= -\frac{1}{x^2} \frac{dy}{dz} + \frac{1}{x} \frac{d^2y}{dz^2} \times \frac{1}{x}$

$= \frac{1}{x^2}\left(-\frac{dy}{dz} + \frac{d^2y}{dz^2}\right)$

$= \frac{1}{x^2}\left(-D' + D'^2\right)$.

$\frac{d^2y}{dx^2} = \frac{1}{x^2}(D'^2 - D')y$

$$\Rightarrow \boxed{x^2 \frac{d^2y}{dx^2} = D'(D'-1)y} \quad \because \boxed{D' = \frac{d}{dz}}$$

Similarly,

$$x^3 \frac{d^3y}{dx^3} = D'(D'-1)(D'-2)\,y$$

$$\vdots$$

$$x^n \frac{d^ny}{dx^n} = [D'(D'-1)(D'-2)\cdots(D'-(n-1))]$$

Q.) $(4x^2D^2 + 16xD + 9)y = 0$

~~$4x^2m^2 + 16xm + 9 = 0$~~

Let. $x = e^z$

~~$4x^2m^2$~~ $\therefore$ The given differential eqn can be rewritten as.

$$[4D'(D'-1) + 16D' + 9]y = 0$$

$$[4D'^2 - 4D' + 16D' + 9]y = 0$$

$$[4D'^2 + 12D' + 9]y = 0$$

A.E is $\Rightarrow$ $4m^2 + 12m + 9 = 0$

$$\boxed{m = -\frac{3}{2}, -\frac{3}{2}}$$

$$C.F = (c_1 + c_2 z)e^{-\frac{3}{2}z}$$

$$= (c_1 + c_2 \log x)\, e^{-\frac{3}{2}\log x}$$

$$= (c_1 + c_2 \log x)\, x^{-3/2}$$

$\therefore$ General Sqn $\Rightarrow$ $\boxed{y = (c_1 + c_2 \log x)x^{-3/2}}$

$(4x^2D^2+1)y = 19\cos(\log x) + 22\sin(\log x)$

$\cancel{(4D'(D'-1)+1)y = 19\cos\log x}$

Let $x = e^z$

$\therefore$ the given eqn reduces.

$[4D'(D'-1)+1]y = 19\cos(\overset{z}{\cancel{\log x}}) + 22\sin(\overset{z}{\cancel{\log x}})$

$[4D'^2 - 4D' + 1]y = 19\cos z + 22\sin z$

$(2D'-1)^2 y = 19\cos z + 22\sin z$

$A.E \Rightarrow (2m-1)^2 = 0$

$m = \frac{1}{2}, \frac{1}{2}$

$C.F = (c_1 + c_2 z)e^{\frac{1}{2}z}$

$= (c_1 + c_2\log x)e^{\frac{1}{2}\log x}$

$= (c_1 + c_2\log x)x^{1/2}$

$P.I = \frac{1}{(2D'-1)^2}(19\cos z + 22\sin z)$

$= \frac{1}{4D'^2 - 4D' + 1}19\cos z + \frac{1}{4D'^2 - 4D' + 1}22\sin z$

$= \frac{1}{4\times -1^2 - 4D' + 1}19\cos z + \frac{1}{4\times -1^2 - 4D' + 1}22\sin z$

$= \frac{1}{-3-4D'}19\cos z + \frac{1}{-3-4D'}22\sin z$

$= \frac{-1}{4D'+3}19\cos z + \frac{-1}{4D'+3}22\sin z$

$$\frac{-1(4D'-3)\,19\cos z}{16D'^2-9} + \frac{-1(4D'-3)\,22\sin z}{16D'^2-9}$$

$$\frac{1}{25}\left[(4D'-3)\,19\cos z + (4D'-3)\,22\sin z\right]$$

$$\frac{1}{25}\left[-76\sin z - 57\cos z + 88\cos z - 66\sin z\right]$$

$$\Rightarrow \frac{1}{25}\left[31\cos z - 142\sin z\right]$$

$$\Rightarrow \frac{1}{25}\left[31\cos(\log x) - 142\sin(\log x)\right]$$

General Eq^n $\rightarrow$ $\boxed{C.F + P.I}$

Q) $(x^3D^3 + x^2D^2 - 2)y = x + \frac{1}{x^3}$

Let $x = e^z$

$\therefore$ The given eq^n reduces.

$$[D'(D'-2)(D'-1) + D'(D-1) - 2]y = e^z + \frac{1}{e^{3z}}$$

$$[(D'^2 - 2D')(D'-1) + D'^2 - D' - 2]y$$

$$[D'^3 - \cancel{D'^2} - 2D'^2 + 2D' + \cancel{D'^2} - D' - 2]y$$

$$[D'^3 - 2D'^2 + D' - 2]y$$

A.E $=$ $m^3 - 2m^2 + m - 2 = 0$

$m^2(m-2) + 1(m-2) = 0$ $\quad |(m^2+1)(m-2) = 0$

$m = 2 \quad | \quad m = \sqrt{-1}$
$= i, -i$

$C.F. = c_1 e^{2z} + c_2 e^{iz} + c_3 e^{-iz}$

$= c_1 e^{2\log x} + c_2 e^{i \log x} + c_3 e^{-i \log x}$

$= c_1 x^2 + \cancel{c_2 x^i + c_3 x^{-i}}$

$c_2 \cos z + c_3 \sin z$

$\Rightarrow \boxed{c_1 x^2 + c_2 \cos \log x + c_3 \sin \log x}$

$P.I. = \dfrac{1}{(D'^2+1)(D'-2)} (e^z + e^{-3z})$

$\Rightarrow \dfrac{1}{(D'^2+1)(D'-2)} e^z + \dfrac{1}{(D'^2+1)(D'-2)} e^{-3z}$

$= \dfrac{1}{(1^2+1)(1-2)} e^z + \dfrac{1}{((-3)^2+1)(-3-2)} e^{-3z}$

$= \dfrac{1}{-2} \times e^z + \dfrac{e^{-3z}}{10 \times -5}$

$= \dfrac{-e^z}{2} - \dfrac{e^{-3z}}{50}$

$= -\dfrac{e^{\log x}}{2} - \dfrac{e^{-3\log x}}{50}$

$\boxed{P.I. = -\dfrac{x}{2} - \dfrac{1}{50x^3}}$

* Legendre's Linear Equation

An equation in the form:

$$a_0(a+bx)^n \cdot \frac{d^n y}{dx^n} + a_1(a+bx)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots$$

$$a_{n-1}(a+bx)\frac{dy}{dx} + a_n y = Q(x)$$

$$Q\cdot\left(\frac{e^z-a}{b}\right)$$

Where, $a_0, a_1, a_2, \ldots a_n$ are constant is called Legendre's Linear eq^n.

Let $(a+bx) = e^z$

$(a+bx)\frac{dy}{dx} = b\frac{dy}{dz} = bD'y$ where, $D' = \frac{d}{dz}$

$(a+bx)^2\frac{d^2y}{dx^2} = b^2\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right) = b^2 D'(D'-1)y$

$\vdots$

$(a+bx)^n\frac{d^ny}{dx^n} = b^n D'(D'-1)(D'-2)\cdots[D'-(n-1)]$

Q) $\left[(2+3x)^2\frac{d^2y}{dx^2} + 3(2+3x)\frac{dy}{dx} - 36\right]y = 3x^2+4x+1$

Let, $(2+3x) = e^z$

Reduce Eq^n is

$= [(3)^2 D'(D'-1) + 3\cdot 3D' - 36]$

$= 3\left(\frac{e^z-2}{3}\right)^2 + 4\left(\frac{e^z-2}{3}\right) + 1$

$\Rightarrow [9D'^2 - 36]y = \frac{(e^z-2)^2}{3} + \frac{4}{3}(e^z-2) + 1$

$\Rightarrow [9D'^2 - 36]y = \frac{e^{2z} + 4 - \cancel{4e^z} + \cancel{4e^z} - 8 + 3}{3} = \frac{e^{2z}-1}{}$

$(D'^2 - 4)y = \frac{1}{27}(e^{2z} - 1)$

A.E. is $m^2 - 4 = 0$

$m = \pm 2$

C.F. $= c_1 e^{2z} + c_2 e^{-2z}$

$= c_1 e^{2\log(2+3x)} + c_2 e^{-2\log(2+3x)}$

$= c_1 (2+3x)^2 + c_2 (2+3x)^{-2}$

P.I. $= \frac{1}{D'^2 - 4} \frac{1}{27}(e^{2z} - 1)$

$= \frac{z}{2D'} \frac{1}{27}(e^{2z} - 1)$

$= \frac{z}{54} \frac{1}{D'}(e^{2z} - 1)$

$= \frac{z}{54}\left(\frac{e^{2z}}{2} - z\right)$

$= \frac{z}{2D'} \frac{1}{27}(e^{2z}) + \frac{1}{4} \cdot \frac{1}{27} \cdot 1$

$= \frac{\log(2+3x)}{54} \cdot \frac{(2+3x)^2}{2} + \frac{1}{108}$

$= \frac{\log(2+3x)}{54}\left[\frac{(2+3x)^2}{2} - \log(2+3x)\right]$

$\frac{\log(2+3x) \cdot (2+3x)^2 + 1}{108}$

Q) $[(x+1)^2 D^2 + (x+1)D]y = (2x+3)(2x+4)$

Q) $\left[(x-1)^3 \frac{D^3 y}{dx^3} + 2(x-1)^2 \frac{d^2 y}{dx^2} - 4(x-1)\frac{dy}{dx} + 4y\right]$

$= 4\log(x-1)$