II Doolittle LU Decomposition

The coefficient matrix A of a system of linear equation can be decomposed into two triangular matrix L and U such that,

$$A = \begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} \cdots & a_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} L_{11} & 0 \cdots & 0 \\ L_{21} & L_{22} \cdots & 0 \\ \vdots & \vdots & \vdots \\ L_{n1} & L_{n2} \cdots & L_{nn} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} \cdots & U_{1n} \\ 0 & u_{22} \cdots & u_{2n} \\ \vdots & \vdots & \vdots \\ 0 & 0 \cdots & u_{nn} \end{bmatrix}$$

If L has 1's on its diagonal, then it is called Doolittle factorization, Thus, Doolittle algorithms assumes that $L_{11} = 1$, $L_{22} = 1$, $\ldots$ $L_{nn} = 1$

Now, from above matrices,

$a_{11} = L_{11} U_{11} \Rightarrow u_{11} = a_{11}$ $\quad [\because L_{11} = 1]$

$a_{12} = L_{11} U_{12} \Rightarrow u_{12} = a_{12}$ $\quad [\because L_{11} = 1]$

$a_{1n} = L_{11} U_{1n} \Rightarrow U_{1n} = a_{1n}$ $\quad [\because L_{11} = 1]$

$a_{21} = L_{21} U_{11} \Rightarrow L_{21} = \dfrac{a_{21}}{U_{11}} = \dfrac{a_{21}}{a_{11}}$

$a_{22} = L_{21} U_{12} + L_{22} U_{22} \Rightarrow U_{22} = \dfrac{a_{22} - L_{21} U_{12}}{L_{22}}$

$= a_{22} - L_{21} U_{12}$

$\vdots$

$a_{2n} = L_{21} U_{1n} + L_{22} U_{2n} \Rightarrow U_{2n} = \dfrac{a_{2n} - L_{21} U_{1n}}{L_{22}} = a_{2n} - L_{21} U_{1n}$

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and

$a_{n1} = l_{n1} u_{11} \Rightarrow u_{n1} = \dfrac{a_{n1}}{a_{11}}$ $[\because a_{11} = u_{11}]$

$a_{n2} = l_{n1} u_{12} + l_{n2} u_{22} \Rightarrow l_{n2} = \dfrac{a_{n2} - l_{n1} u_{12}}{u_{22}}$

$= \dfrac{1}{u_{22}} (a_{n2} - l_{n1} u_{12})$

$a_{n3} = l_{n1} u_{13} + l_{n2} u_{23} + l_{n3} u_{33} \Rightarrow l_{n3}$

$= \dfrac{a_{n3} - l_{n1} u_{13} - l_{n2} u_{23}}{u_{33}}$

$= \dfrac{1}{u_{33}} (a_{n3} - l_{n1} u_{13} - l_{n2} u_{23})$

$\vdots$

$a_{nn} = l_{n1} u_{1n} + l_{n2} u_{2n} + \dots + l_{nn} u_{nn}$

$\Rightarrow u_{nn} = \dfrac{a_{nn} - l_{n1} u_{1n} - l_{n2} u_{2n} - \dots - l_{n\,n-1} \quad u_{n-1\,n}}{l_{nn}}$

$\therefore u_{nn} = a_{nn} - l_{n1} u_{1n} - l_{n2} u_{12} - \dots - l_{n\,n-1} \quad u_{n-1\,n}$

Generalizing this, we get

$u_{ij} = a_{ij} - \sum_{k=1}^{i-1} l_{ik} u_{kj}, \quad j = 1, 2, \dots n)$

$[\text{if } i \le j]$

where,

$u_{11} = a_{11}, \; u_{12} = a_{12}, \dots$

and if $i > j$,

$l_{ij} = \dfrac{1}{u_{jj}} \left[a_{ij} - \sum_{k=1}^{j-1} l_{ik} u_{kj} \right], \quad j = 1, 2, 3, \dots n$

Example:

Find the Doolittle LU decomposition of the following matrix.

$$A = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

$\Rightarrow$ Soln:

We know that, $A = LU$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

where,

$$A = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

from Doolittle LU decomposition gives

$L_{11} = L_{22} = L_{33} = 1$

Now,

$a_{11} = L_{11} u_{11} \Rightarrow u_{11} = a_{11}$ $[\because L_{11} = 1]$

$\Rightarrow u_{11} = 25$

$a_{12} = L_{11} u_{12} \Rightarrow u_{12} = a_{12} = 5$ $[\because L_{11} = 1]$

$a_{13} = L_{11} u_{13} \Rightarrow u_{13} = a_{13} = 1$ $[\because L_{11} = 1]$

$a_{21} = L_{21} u_{11} \Rightarrow L_{21} = \dfrac{a_{21}}{u_{11}} = \dfrac{64}{25} = 2.56$

$$a_{22} = L_{21} U_{12} + L_{22} U_{22} \Rightarrow U_{22} = \frac{a_{22} - L_{21} U_{12}}{L_{22}}$$

$$= 8 - \frac{64}{25} \times 5$$

$$= -24/5$$

$$a_{23} = L_{21} U_{13} + L_{22} U_{23} \Rightarrow U_{23} = \frac{a_{23} - L_{21} U_{13}}{L_{22}}$$

$$= 1 - \frac{64}{25} \times 1$$

$$= -1.56$$

$$a_{31} = L_{31} U_{11} \Rightarrow \underline{L_{31}} = \frac{a_{31}}{U_{11}} = \frac{144}{25} = 5.76$$

$$a_{32} = L_{31} U_{12} + L_{32} U_{22} \Rightarrow \underline{L_{32}} = \frac{a_{32} - L_{31} U_{12}}{U_{22}}$$

$$= \frac{12 - 5.76 \times 5}{-4.8}$$

$$= 3.5$$

$$a_{33} = L_{31} U_{13} + L_{32} U_{23} + L_{33} U_{33}$$

$$U_{33} = \frac{a_{33} - L_{31} U_{13} - L_{32} U_{23}}{L_{33}}$$

$$= 1 - 5.76 \times 1 - 3.5 \times -1.56$$

$$= 0.7$$

$$L = \begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix}, \quad U = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{pmatrix}$$

$$= \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$

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Q. Solve the following system of equations by using Doolitte LU Decomposition Method.

$3x_1 + 2x_2 + x_3 = 10$

$2x_1 + 3x_2 + 2x_3 = 14$

$x_1 + 2x_2 + 3x_3 = 14$

⇒ Soln:

Given equations can be written in matrix form as:

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 10 \\ 14 \\ 14 \end{bmatrix}$$

Again,

we have from Doolittle decomposition,

$A = LU$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

And, from Doolittle decomposition, we have

$L_{11} = L_{22} = L_{33} = 1$

Now,

$a_{11} = L_{11} u_{11} \Rightarrow u_{11} = a_{11} = 3 \quad [\because L_{11} = 1]$

$a_{12} = L_{11} u_{12} \Rightarrow u_{12} = a_{12} = 2$

$a_{13} = L_{11} u_{13} \Rightarrow u_{13} = a_{13} = 1$

$a_{21} = L_{21} u_{11} \Rightarrow L_{21} = \dfrac{a_{21}}{u_{11}} = \dfrac{2}{3}$

$a_{22} = L_{21} u_{12} + L_{22} u_{22} \Rightarrow u_{22} = a_{22} - L_{21} u_{12}$

$= 3 - \frac{2}{3} \times 2$

$= 5/3$

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$a_{23} = L_{21}U_{13} + L_{22}U_{23} \Rightarrow U_{23} = a_{23} - L_{21}U_{13}$

$= 2 - \frac{2}{3} \times 1$

$= 4/3$

$= 1.333$

$a_{31} = L_{31}U_{11} \Rightarrow L_{31} = \frac{a_{31}}{u_{11}} = \frac{1}{3} = 0.333$

$a_{32} = L_{31}U_{12} + L_{32}U_{22} \Rightarrow L_{32} = \frac{a_{32} - L_{31}U_{12}}{U_{22}}$

$= \frac{2 - 0.333 \times 2}{1.667}$

$= 0.800$

$a_{33} = L_{31}U_{13} + L_{32}U_{23} + L_{33}U_{33}$

$\Rightarrow U_{33} = \frac{a_{33} - L_{31}U_{13} - L_{32}U_{23}}{L_{33}}$

$= \frac{3 - 0.333 \times 1 - 0.8 \times 1.333}{1}$

$= 1.6006$

$L = \begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1/3 & 0.8 & 1 \end{bmatrix}$ $U = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 5/3 & 4/3 \\ 0 & 0 & 1.6006 \end{bmatrix}$

Now,

$LZ = C$ using forward substitution,

$$\begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1/3 & 0.8 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \\ 14 \end{bmatrix}$$

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Now, $z_1 = 10$

$\frac{2}{3} z_1 + z_2 = 14$

or, $\frac{2}{3} \times 10 + z_2 = 14$

$\therefore z_2 = 14 - 20/3 = 22/3$

$\frac{1}{3} z_1 + 0.8 z_2 + z_3 = 14$

or, $\frac{1}{3} \times 10 + 0.8 \times \frac{22}{3} + z_3 = 14$

or, $z_3 = 14 - \frac{46}{5} = \frac{24}{5}$

Now, we solve for x as:

$Ux = z$

$$\begin{bmatrix} 3 & 2 & 1 \\ 0 & 5/3 & 4/3 \\ 0 & 0 & 1.6006 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 22/3 \\ 24/5 \end{bmatrix}$$

Now,

$1.6006 x_3 = 24/5$

$\therefore x_3 = 2.99 \approx 3$

$5/3 \, x_2 + 4/3 \times 3 = 22/3$

$x_2 = \frac{22/3 - 4}{5/3}$

$\therefore x_2 = 2$

$3x_1 + 2x_2 + x_3 = 10$

$x_1 = \frac{10 - 4 - 3}{3} = 1$

Hence, the required solution is

$x_1 = 1, \; x_2 = 2, \; x_3 = 3$

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Cholecky Method / factorization:-

In case, the coefficient matrix A is symmetric. Cholecky method can be used to factorize the coefficient matrix and hence to solve system of linear equation. If coefficient matrix is symmetric, the upper factor is transpose of lower factor or vice-versa.

i.e. $A = [L][L^T]$ or $[U^T][U]$

Thus,

$$[A] = \begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} \cdots & a_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} u_{11} & 0 \cdots & 0 \\ u_{12} & u_{22} \cdots & 0 \\ & & \vdots \\ u_{1n} & u_{2n} & u_{nn} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} \cdots & u_{1n} \\ 0 & u_{22} & u_{2n} \\ \vdots & & \\ 0 \cdots 0 & & u_{nn} \end{bmatrix}$$

Like Doolittle ~~description~~ decomposition multiply the matrix of right hand side and comparing it with coefficient. At matrix of LHS, we get

$$u_{ii} = \sqrt{a_{ii} - \sum_{k=1}^{i-1} u_{ki}^2} \quad i = 1, 2, \ldots n$$

$$u_{ij} = \frac{1}{u_{ii}} \left(a_{ij} - \sum_{k=1}^{i-1} u_{ki} u_{kj} \right) \quad j > i$$

Example: find the Cholesky decomposition of the matrix:

$$[A] = \begin{bmatrix} 1 & 4 & 7 \\ 4 & 80 & 44 \\ 7 & 44 & 89 \end{bmatrix}$$

⇒ Sol^n:

$$u_{11} = \sqrt{a_{11}} = \sqrt{1} = 1$$

$$u_{12} = \frac{1}{u_{11}}(a_{12}) = \frac{1}{1} \times 4 = 4$$

$$u_{13} = \frac{1}{u_{11}}(a_{13}) = \frac{1}{1} \times 7 = 7$$

$$u_{22} = \sqrt{a_{22} - \sum_{k=1}^{1} u_{12}^2} = \sqrt{80 - 16} = 8$$

$$u_{23} = \frac{1}{u_{22}}\left(a_{23} - \sum_{k=1}^{1} u_{12}\, u_{13}\right)$$

$$= \frac{1}{8}(44 - 4 \times 7)$$

$$= \frac{1}{8} \times 16$$

$$= 2$$

$$u_{33} = \sqrt{a_{33} - \sum_{k=1}^{2} u_{13}^2}$$

$$= \sqrt{89 - (u_{13}^2 + u_{23}^2)}$$

$$= \sqrt{89 - (49 + 4)}$$

$$= \sqrt{36}$$

$$= 6$$

$$U = \begin{bmatrix} 2 & 4 & 7 \\ 0 & 8 & 2 \\ 0 & 0 & 6 \end{bmatrix}, \quad U^T = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 8 & 0 \\ 7 & 2 & 6 \end{bmatrix}$$

Hence, $A = [U^T][U]$

Iterative methods for solving system of linear equation.

Iterative methods for solving general, Large sparse linear system. Because of round-off error direct method, became less efficient than iterative method when they applied to large system. Iterative methods are more efficient than direct methods.

Jacobi iterative method:

The jacobi method is a method of solving system of linear equation that has the non zeroes along its diagonal. Each diagonal elements is solved, and an approximate value plugged in. The process is then iterated until it converge.

Let A be $n \times n$ non-singular matrix and $AX = B$ is the system to be solved ie we have to solve the system of equations.

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$$

This can be written as:

$$x_1 = \frac{b_1 - (a_{12} x_2 + \cdots + a_{1n} x_n)}{a_{11}}$$

$$x_2 = \frac{b_2 - (a_{21} x_1 + \cdots + a_{2n} x_n)}{a_{22}}$$

$\vdots$

$$x_n = \frac{b_n - (a_{n1} x_1 + a_{n2} x_2 + \cdots + a_{n,n-1} x_{n-1})}{a_{nn}}$$

Now, we can compute values of $x_1, x_2, \ldots x_n$ by using initial guess and Jacobian's method is used to find the new values of unknown.

In general, it can be represented as:

$$x_i^{k+1} = \frac{b_i - \left(\sum_{j=1, j \neq i}^{n} a_{ij} \cdot x_j^k \right)}{a_{ii}}, \quad k = 0, 1, 2 \ldots, \quad i = 1, 2, \ldots n$$

At the end of each iteration, the absolute relative approximate error for each x_i is calculated as:

$$|\epsilon_a|_i = \left| \frac{x_i^{new} - x_i^{old}}{x_i^{new}} \right| \times 100$$

where, x_i^{new} is the recently obtained value of x_i & x_i^{old} is the previous value of x_i.

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When the absolute relative approximate error for each x_i is less than the pre-specified tolerance, iteration are stop.

Example:- Use Jacobian Iteration method, to obtain the solution of the following eqn.

$$6x_1 - 2x_2 + x_3 = 11$$
$$-2x_1 + 7x_2 + 2x_3 = 5$$
$$x_1 + 2x_2 - 5x_3 = -1$$

⇒ Soln:

step 1: First, solve the equation for unknown on the diagonal; ie

$$x_1 = \frac{11 + 2x_2 - x_3}{6}$$

$$x_2 = \frac{2x_1 - 2x_3 + 5}{7}$$

$$x_3 = \frac{x_1 + 2x_2 + 1}{5}$$

Guess isn't given so assume "0"

Step 2: Assume the Initial guess $(x_1)^0 = (x_2)^0 = (x_3)^0 = 0$ then calculate $(x_1)^1, (x_2)^1, (x_3)^1$

$$(x_1)^1 = \frac{11 + 2\times 0 - 0}{6} = \frac{11}{6}$$

$$(x_2)^1 = \frac{2\times 0 - 2\times 0 + 5}{7} = 5/7$$

$$(x_3)^1 = \frac{0 + 0 + 1}{5} = 1/5$$

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Step3: Calculate $(x_1)^2, (x_2)^2, (x_3)^2$ using the value of $(x_1)', (x_2)', (x_3)'$

$$(x_1)^2 = \frac{11 + 2 \times 5/7 - 1/5}{6} = 2.0380$$

$$(x_2)^2 = \frac{2 \times 11/6 - 2 \times 1/5 + 5}{7} = 1.1809$$

$$(x_3)^2 = \frac{11/6 + 2 \times 5/7 + 1}{5} = 0.8523$$

Step 4: Calculate $(x_1)^3, (x_2)^3, (x_3)^3$ using $(x_1)^2, (x_2)^2, (x_3)^2$

$$(x_1)^3 = \frac{11 + 2 \times 1.1809 - 0.8523}{6} = 2.0849$$

$$(x_2)^3 = \frac{2 \times 2.0380 - 2 \times 0.8523 + 5}{7} = 1.0530$$

$$(x_3)^3 = \frac{2.0380 + 2 \times 1.1809 + 1}{5} = 1.0799$$

Step 5:

$$(x_1)^4 = \frac{11 + 2 \times 1.0530 - 1.0799}{6} = 2.0043$$

$$(x_2)^4 = \frac{2 \times 2.0849 - 2 \times 1.0799 + 5}{7} = 1.0014$$

$$(x_3)^4 = \frac{2.0849 + 2 \times 1.0530 + 1}{5} = 1.0381$$

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Step 6:

$$(x_1)^5 = \frac{11 + 2 \times 1.0014 - 1.0381}{6} = 1.9941$$

$$(x_2)^5 = \frac{2 \times 2.0043 - 2 \times 1.0381 + 5}{7} = 0.9903$$

$$(x_3)^5 = \frac{2.0043 + 2 \times 1.0014 + 1}{5} = 1.0014$$

Step 7:

$$(x_1)^6 = \frac{11 + 2 \times 0.9903 - 1.0014}{6} = 1.9965$$

$$(x_2)^6 = \frac{2 \times 1.9941 - 2 \times 1.0014 + 5}{7} = 0.9979$$

$$(x_3^6) = \frac{1.9941 + 2 \times 0.9903 + 1}{5} = 0.9949$$

Step 8:

$$(x_1)^7 = \frac{11 + 2 \times 0.9979 - 0.9949}{6} = 2.0001$$

$$(x_2)^7 = \frac{2 \times 1.9965 - 2 \times 0.9949 + 5}{7} = 1.0004$$

$$(x_3)^7 = \frac{1.9965 + 2 \times 0.9979 + 1}{5} = 0.9984$$

Step 9:

$$(x_1)^8 = \frac{11 + 2 \times 1.0004 - 0.9984}{6} = 2.0004$$

$$(x_2)^8 = \frac{2 \times 2.0001 - 2 \times 0.9984 + 5}{7} = 1.0004$$

$(x_3)^8 = \frac{2.0501 + 2\times 1.0004 + 1}{5} = 1.0001$

Step 10:

$(x_1)^9 = \frac{11 + 2\times 1.0504 - 1.0001}{6} = 2.0001$

$(x_2)^9 = \frac{2\times 2.0004 - 2\times 1.0001 + 5}{7} = 1.0000$

$(x_3)^9 = \frac{2.0004 + 2\times 1.0004 + 1}{5} = 1.0002$

Step 11:

$(x_1)^{10} = \frac{11 + 2\times 1.0000 - 1.0002}{6} = 1.9999$

$(x_2)^{10} = \frac{2\times 2.0001 - 2\times 1.0002 + 5}{7} = 0.9999$

$(x_3)^{10} = \frac{2.0001 + 2\times 1 + 1}{5} = 1.0000$

Step 12:

$(x_1)^{11} = \frac{11 + 2\times 0.9999 - 1}{6} = 1.9999$

$(x_2)^{11} = \frac{2\times 1.9999 - 2\times 1.0000 + 5}{7} = 0.9999$

$(x_3)^{11} = \frac{1.9999 + 2\times 0.9999 + 1}{5} = 0.9999$

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Step 13:

$$(x_1)^{12} = \frac{11 + 2\times 0.9999 - 0.9999}{6} = 1.9999$$

$$(x_2)^{12} = \frac{2\times 1.9999 - 2\times 0.9999 + 5}{7} = 1$$

$$(x_3)^{12} = \frac{1.9999 + 2\times 0.9999 + 1}{5} = 0.9994$$

Step 14:

$$(x_1)^{13} = \frac{11 + 2\times 1 - 0.9994}{6} = 2.0001$$

$$(x_2)^{13} = \frac{2\times 1.9999 - 2\times 0.9994 + 5}{7} = 1.0001$$

$$(x_3)^{13} = \frac{1.9999 + 2\times 1 + 1}{5} = 0.9999$$

Step 15:

$$(x_1)^{14} = \frac{11 + 2\times 1.0001 - 0.9999}{6} = 2.0000$$

$$(x_2)^{14} = \frac{2\times 2.0001 - 2\times 0.9999 + 5}{7} = 1.0000$$

$$(x_3)^{14} = \frac{2.0001 + 2\times 1.0001 + 1}{5} = 1.0000$$

Step 16:

$$(x_1)^{15} = \frac{11 + 2\times 1 - 1}{6} = 2$$

$$(x_2)^{15} = \frac{2\times 2 - 2\times 1 + 5}{7} = 1$$

$$(x_3)^{15} = \frac{2 + 2\times 1 + 1}{5} = 1$$

$\therefore x_1 = 2$

$x_2 = 1$

$x_3 = 1$

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Q. Use Jacobian Iteration method, to obtain the solution of the following equations:

$5x_1 - 2x_2 + 3x_3 = -1$
$-3x_1 + 9x_2 + x_3 = 2$
$2x_1 - x_2 - 7x_3 = 3$

⇒ Sol^n:

Step1: Solve the equation for unknown on the diagonal;

$$x_1 = \frac{2x_2 - 3x_3 - 1}{5}$$

$$x_2 = \frac{3x_1 - x_3 + 2}{9}$$

$$x_3 = \frac{2x_1 - x_2 - 3}{7}$$

Step2: Assume the initial guess $(x_1)^0 = (x_2)^0 = (x_3)^0 = 0$ then calculate $(x_1)^1, (x_2)^1, (x_3)^1$

$$(x_1)^1 = \frac{2\times0 - 3\times0 - 1}{5} = -0.2$$

$$(x_2)^1 = \frac{3\times0 - 0 + 2}{9} = 0.2222$$

$$(x_3)^1 = \frac{2\times0 - 0 - 3}{7} = -0.4285$$

Step3: Calculate $(x_1)^2, (x_2)^2, (x_3)^2$ using the value of $(x_1)^1, (x_2)^1, (x_3)^1$

$$(x_1)^2 = \frac{2\times0.2222 - 3\times-0.4285 - 1}{5} = 0.1459$$

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$$(x_2)^2 = \frac{3 \times -0.2 + 0.4285 + 2}{9} = 0.2031$$

$$(x_3)^2 = \frac{2 \times -0.2 - 0.2222 - 3}{7} = -0.5174$$

Step 4:

$$(x_1)^3 = \frac{2 \times 0.2031 + 3 \times 0.5174 - 1}{5} = 0.1916$$

$$(x_2)^3 = \frac{3 \times 0.1459 + 0.5174 + 2}{9} = 0.3283$$

$$(x_3)^3 = \frac{2 \times 0.1459 - 0.2031 - 3}{7} = -0.4159$$

Step 5:

$$(x_1)^4 = \frac{2 \times 0.3283 + 3 \times 0.4159 - 1}{5} = 0.1808$$

$$(x_2)^4 = \frac{3 \times 0.1916 + 0.4159 + 2}{9} = 0.3323$$

$$(x_3)^4 = \frac{2 \times 0.1916 - 0.3283 - 3}{7} = -0.4207$$

Step 6:

$$(x_1)^5 = \frac{2 \times 0.3323 + 3 \times 0.4207 - 1}{5} = 0.1853$$

$$(x_2)^5 = \frac{3 \times 0.1808 + 0.4207 + 2}{9} = 0.3292$$

$$(x_3)^5 = \frac{2 \times 0.1808 - 0.3323 - 3}{7} = -0.4243$$

Step 7:

$$(x_1)^6 = \frac{2\times 0.3292 + 3\times 0.4243 - 1}{5} = 0.1862$$

$$(x_2)^6 = \frac{3\times 0.1853 + 0.4243 + 2}{9} = 0.3311$$

$$(x_3)^6 = \frac{2\times 0.1853 - 0.3292 - 3}{7} = -0.4226$$

Step 8:

$$(x_1)^7 = \frac{2\times 0.3311 + 3\times 0.4226 - 1}{5} = 0.186$$

$$(x_2)^7 = \frac{3\times 0.1862 + 0.4226 + 2}{9} = 0.3312$$

$$(x_3)^7 = \frac{2\times 0.1862 - 0.3311 - 3}{7} = -0.4226$$

Step 9:

$$(x_1)^8 = \frac{2\times 0.3312 + 3\times 0.4226 - 1}{5} = 0.1860$$

$$(x_2)^8 = \frac{3\times 0.186 + 0.4226 + 2}{9} = 0.3311$$

$$(x_3)^8 = \frac{2\times 0.186 - 0.3312 - 3}{7} = -0.4227$$

Step 10:

$$(x_1)^9 = \frac{2\times 0.3311 + 3\times 0.4227 - 1}{5} = 0.1860$$

$$(x_2)^9 = \frac{3\times 0.1860 + 0.4227 + 2}{9} = 0.3311$$

$$(x_3)^9 = \frac{2\times 0.1860 - 0.3311 - 3}{7} = -0.4227$$

Hence, $x_1 = 0.1860$, $x_2 = 0.3311$, $x_3 = -0.4227$

Algorithm:

1. Read dimension of system of equations say n.
2. Read coefficients Row-wise.
3. Read elements of RHS vector.
4. Read accuracy limit, say error
5. for i = 1 to n

   old_x[i] = 0

   end for
6. for i = 1 to n

   sum = b[i]

   for j = 1 to n

   if (i ≠ j)

   sum = sum - a[i][j] * old_x[j]

   new_x[i] = sum / a[i][i]

   $$E[i] = \left| \frac{new\_x[i] - old\_x[i]}{new\_x[i]} \right|$$
7. for i = 1 to n

   if (E[i] > error)

   for j = 1 to n

   old_x[j] = new_x[j]

   goto step 6
8. Display results in "new_x" vector.

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Gauss-Seidal Methods:-

Given a general set of n equations and n unknowns, we have

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$$
$$\vdots \qquad\qquad\qquad \vdots \qquad\qquad \vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \dots + a_{nn}x_n = b_n$$

If the diagonal elements are non-zero, each equation is rewritten for corresponding unknown ie the first eq is rewritten with x_1 on the LHS, and second equation is rewritten with x_2 on the LHS and so on as follows

$$x_1 = \frac{b_1 - a_{12}x_2 - a_{13}x_3 - \dots - a_{1n}x_n}{a_{11}}$$

$$x_2 = \frac{b_2 - a_{21}x_1 - a_{23}x_3 - \dots - a_{2n}x_n}{a_{22}}$$

$$\vdots$$

$$x_n = \frac{b_n - a_{n1}x_1 - a_{n2}x_2 - \dots - a_{n\,n-1}x_{n-1}}{a_{nn}}$$

Now, to find x_i's, one assumes an initial guess for x_i's and then uses the rewritten equations to calculate the new estimats. Note that, unlike Jacobi iteration method Gaus-Seidal method always uses the most recent estimates to calculate the next estimates for x_i.

This means Gauss-Seidal's method is used to find the new values of unknown as,

$$x_i^{(k+1)} = \frac{b_i - \left(a_{i1} x_1^{k+1} + \dots a_{i,i-1} x_{i-1}^{(k+1)} + a_{i,i+1} x_{i+1}^{k} + \dots + a_{in} x_n^{k}\right)}{a_{ii}}$$

recent previous

$k = 0, 1, 2 \dots$

The absolute error for each x_i is calculated as

$$|E_a|_i = \left| \frac{x_i^{new} - x_i^{old}}{x_i^{new}} \right| \times 100$$

Example:

Use the Gauss-Seidal method to obtain solution of the following equations.

$6x_1 - 2x_2 + x_3 = 11$

$-2x_1 + 7x_2 + 2x_3 = 5$

$x_1 + 2x_2 - 5x_3 = -1$

⇒ Soln:

Assume the initial guesses,

$(x_1)^0 = (x_2)^0 = (x_3)^0 = 0$

Rewrite the eqn as:

$$x_1 = \frac{2x_2 - x_3 + 11}{6}$$

$$x_2 = \frac{2x_1 - 2x_3 + 5}{7}$$

$$x_3 = \frac{x_1 + 2x_2 + 1}{5}$$

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Iteration 1: Calculate $(x_1)^1, (x_2)^1, (x_3)^1$

$$(x_1)^1 = \frac{2\times 0 - 0 + 11}{6} = 1.833$$

$$(x_2)^1 = \frac{2\times 1.833 - 2\times 0 + 5}{7} = 1.238$$

$$(x_3)^1 = \frac{1.833 + 2\times 1.238 + 1}{5} = 1.061$$

Iteration 2:

$$(x_1)^2 = \frac{2\times 1.238 - 1.061 + 11}{6} = 2.069$$

$$(x_2)^2 = \frac{2\times 2.069 - 2\times 1.061 + 5}{7} = 1.002$$

$$(x_3)^2 = \frac{2.069 + 2\times 1.002 + 1}{5} = 1.014$$

Iteration 3:

$$(x_1)^3 = \frac{2\times 1.002 - 1.014 + 11}{6} = 1.998$$

$$(x_2)^3 = \frac{2\times 1.998 - 2\times 1.014 + 5}{7} = 0.995$$

$$(x_3)^3 = \frac{1.998 + 2\times 0.995 + 1}{5} = 0.997$$

Iteration 4:

$$(x_1)^4 = \frac{2\times 0.995 - 0.997 + 11}{6} = 1.998$$

$$(x_2)^4 = \frac{2\times 1.998 - 2\times 0.997 + 5}{7} = 1$$

$$(x_3)^4 = \frac{1.998 + 2 \times 1 + 1}{5} = 0.999$$

Iteration 5:

$$(x_1)^5 = \frac{2 \times 1 - 0.999 + 11}{6} = 2$$

$$(x_2)^5 = \frac{2 \times 2 - 2 \times 0.999 + 5}{7} = 1$$

$$(x_3)^5 = \frac{2 + 2 \times 1 + 1}{5} = 1$$

Iteration 6:

$$(x_1)^6 = \frac{2 \times 1 - 1 + 11}{6} = 2$$

$$(x_2)^6 = \frac{2 \times 2 - 2 \times 1 + 5}{7} = 1$$

$$(x_3)^6 = \frac{2 + 2 \times 1 + 1}{5} = 1$$

$\therefore$ Hence, The value of $x_1 = 2$, $x_2 = 1$, $x_3 = 1$

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Algorithm:

1. Read dimension of system of equation say n.
2. Read coefficient matrix row-wise.
3. Read elements of RHS vector as b.
4. Read accuracy limit say error.
5. for i = 1 to n
   new_x[i] = 0;
   end for
6. for i = 1 to n
   sum = b[i]
   for j = 1 to n
   if (i ≠ j)
   sum = sum - a[i][j] * new_x[j]
   end for
   old_x[i] = new_x[i]
   new_x[i] = sum / a[i][i]

$$E[i] = \left| \frac{new\_x[i] - old\_x[i]}{new\_x[i]} \right|$$

   end for
7. for i = 1 to n
   if (E[i] > error)
   goto step 6
   end for
8. Display result in "new_x" vector.

Q. Use Gauss-Seidal method to obtain the solution of the following equations.

$$5x_1 - 2x_2 + 3x_3 = -1$$
$$-3x_1 + 9x_2 + x_3 = 2$$
$$2x_1 - x_2 - 7x_3 = 3$$

⇒ Solⁿ:

Assume the initial guess as,

$(x_1)^0 = (x_2)^0 = (x_3)^0 = 0$

Rewrite the eqⁿ as:

$$x_1 = \frac{2x_2 - 3x_3 - 1}{5}$$

$$x_2 = \frac{3x_1 - x_3 + 2}{9}$$

$$x_3 = \frac{2x_1 - x_2 - 3}{7}$$

Iteration 1: Calculate $(x_1)^1, (x_2)^1, (x_3)^1$

$$(x_1)^1 = \frac{2\times 0 - 3\times 0 - 1}{5} = -0.2$$

$$(x_2)^1 = \frac{2\times -0.2 - 0 + 2}{9} = 0.177$$

$$(x_3)^1 = \frac{2\times -0.2 - 0.177 - 3}{7} = -0.511$$

Iteration 2:

$$(x_1)^2 = \frac{2\times 0.177 + 3\times 0.511 - 1}{5} = 0.177$$

$$(x_2)^2 = \frac{2\times 0.177 + 0.511 + 2}{9} = 0.318$$

$$(x_3)^2 = \frac{2\times 0.177 - 0.318 - 3}{7} = -0.423$$

Iteration 3:

$$(x_1)^3 = \frac{2\times 0.318 + 3\times 0.423 - 1}{5} = 0.181$$

$$(x_2)^3 = \frac{3\times 0.181 + 0.423 + 2}{9} = 0.329$$

$$(x_3)^3 = \frac{2\times 0.181 - 0.329 - 3}{7} = -0.423$$

Iteration 4:

$$(x_1)^4 = \frac{2\times 0.329 + 3\times 0.423 - 1}{5} = 0.185$$

$$(x_2)^4 = \frac{3\times 0.185 + 0.423 + 2}{9} = 0.330$$

$$(x_3)^4 = \frac{2\times 0.185 - 0.330 - 3}{7} = -0.422$$

Iteration 5:

$$(x_1)^5 = \frac{2\times 0.330 + 3\times 0.422 - 1}{5} = 0.185$$

$$(x_2)^5 = \frac{3\times 0.185 + 0.422 + 2}{9} = 0.330$$

$$(x_3)^5 = \frac{2\times 0.185 - 0.330 - 3}{7} = -0.422$$

$\therefore$ Hence, the required value are $x_1 = 0.185$, $x_2 = 0.330$, $x_3 = -0.422$

H.W

Application of Eigen vector:

Many applications of matrices in both engineering and science utilize eigenvalues and eigenvectors. control theory, vibration analysis, electric circuits, advanced dynamics and quantum machanics are just a few application area.

Eigen vectors gives the direction of spread of data, while eigen value is the intensity of spread in a particular direction or of that respective eigen vector.

→ Dimensionality Reduction.
→ Low rank factorization for collaborative predictions.
→ The google page range algorithm.
→ Using singular value decomposition for image compression.
→ Deriving special relatively is more natural in the language of linear Algebra.

{Application / Physical meaning of eigen value}

Eigen Vector and Eigen value of matrix:

The eigen vectors of a square matrix are the non-zero vectors which after being multiplied by the matrix, remain proportional to the original vector. ie any vector x that satisfies the equation
$Ax = \lambda X$,
where 'A' is the given square matrix, X is the eigen vector and λ is the associated eigen value.

Inorder to find the eigen vectors of a matrix we must start by finding the eigen values. To do this we consider,

$$AX - \lambda X = 0$$

then,

$$(A - \lambda I)X = 0$$

If $A - \lambda I$ does have an inverse we find,
$X = (A - \lambda I)^{-1} = 0$ ie only solution is the zero vector.

Then in only way this can be solved. If $A - \lambda I$ does not have an inverse, therefore, we find values of λ such that the determinant of $A - \lambda I$ is zero.
ie $|A - \lambda I| = 0$

Once we have set of eigen values we can substitute them back, into the original equation to find the eigen vectors.

$x_1 = x_2$

Example: find the eigen values and eigen vectors of the matrix.

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

⇒ Soln: let $(A - \lambda I)x = 0$

$$A - \lambda I = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}$$

Now, we have

$|A - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$(2-\lambda)(2-\lambda) - 1 = 0$

or, $4 - 2\lambda - 2\lambda + \lambda^2 - 1 = 0$

or, $\lambda^2 - 4\lambda + 3 = 0$, $\lambda^2 - 3\lambda - \lambda + 3 = 0$

or, $\lambda(\lambda - 3) - 1(\lambda - 3) = 0$

or, $(\lambda - 1)(\lambda - 3) = 0$

$\therefore \lambda = 1, 3$

For $\lambda = 3$

$(A - \lambda I)x = 0$

or, $\left(\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \right) x = 0$

or, $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} x = 0$

~~$R_2 \to R_2 + R_1$~~

for $\lambda = 1$

$$A - \lambda I = \begin{bmatrix} 2-1 & 1 \\ 1 & 2-1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Augmented matrix form:

$$\left[\begin{array}{cc:c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right]$$

$R_2 \to R_2 - R_1$

$$\left[\begin{array}{cc:c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_1 + x_2 = 0$

$x_1 = -x_2$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} x_2$$

↳ eigen vector

for $\lambda = 3$

$$A - \lambda I = \begin{bmatrix} 2-3 & 1 \\ 1 & 2-3 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

Augmented matrix form,

$$\left[\begin{array}{cc:c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right]$$

$R_2 \to R_2 + R_1$

$$\left[\begin{array}{cc:c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$-x_1 + x_2 = 0$

$x_1 = x_2$

$$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} x_2$$

↳ eigen vector

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Power Method:

Eigen values can be ordered in terms of their absolute values to find the largest eigen value of matrix.

One of the simplest methods for finding the largest eigen values and eigen vectors of a matrix is the power method. It uses iterative approach and starts with initial guess for vector x. Power method can be implemented as:

$$Y = AX \quad \text{—— (1)}$$

$$X = \frac{1}{k} Y \quad \text{—— (2)}$$

Now, value of x can be obtained from eq^n (2) and then value of Y is calculated from eq^n (1)

This process is repeated until the desired level of accuracy is obtained. The parameter k is known as scaling factor and it is the element of Y with the largest magnitude.

Q. Find the eigen value and corresponding eigen vector of the matrix given below using power method with $X = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

[If initial guess is not given then keep $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ ....]

3×3 3×1

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⇒ Soln:

1st iteration:

We have initial guess for eigen vector.

$$X = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$Y = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

⇒ $k = 2$ (since 2 is largest element of resultant matrix ie Y)

Now, New value of x can be calculated as,

$$X = \frac{1}{K} Y = \frac{1}{2} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix}$$

2nd iteration:

$$Y = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1 \\ 2+0.5 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 2.5 \\ 0 \end{bmatrix}$$

⇒ $K = 2.5$ (since 2.5 is largest element of resultant matrix ie Y)

$$x = \frac{1}{2.5}\begin{bmatrix} 2 \\ 2.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 4/5 \\ 1 \\ 0 \end{bmatrix}$$

3rd iteration :

$$Y = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 4/5 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 14/5. \\ 13/5 \\ 0 \end{bmatrix}$$

$\Rightarrow k = 14/5$ (since $14/5$ is largest element of resultant matrix ie Y)

$$x = \frac{5}{14}\begin{bmatrix} 14/5 \\ 13/5 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 13/14 \\ 0 \end{bmatrix}$$

4th iteration:

$$Y = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 13/14 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 20/7 \\ 41/14 \\ 0 \end{bmatrix}$$

$\Rightarrow k = 41/14$ (since it is largest)

$$X = \frac{14}{41}\begin{bmatrix} 20/7 \\ 41/14 \\ 0. \end{bmatrix} = \begin{bmatrix} 40/41 \\ 1 \\ 0 \end{bmatrix}$$

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5^{th} iteration:

$$Y = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 40/41 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 122/41 \\ 121/41 \\ 0 \end{bmatrix}$$

$\Rightarrow k = 122/41$ (since it is largest)

$$X = \frac{41}{122} \begin{bmatrix} 122/41 \\ 121/41 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 121/122 \\ 0 \end{bmatrix}$$

6^{th} iteration:

$$Y = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 121/122 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 182/61 \\ 365/122 \\ 0 \end{bmatrix}$$

$\Rightarrow k = 365/123$

$$X = \frac{122}{365} \begin{bmatrix} 182/61 \\ 365/123 \\ 0 \end{bmatrix} = \begin{bmatrix} 364/365 \\ 1 \\ 0 \end{bmatrix}$$

7^{th} iteration:

$$Y = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 364/365 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1094/365 \\ 1093/365 \\ 0 \end{bmatrix}$$

$\Rightarrow k = 1094/365$ (since it is largest

$$x = \frac{365}{1094}\begin{bmatrix} 1094/365 \\ 1093/365 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.99 \\ 0 \end{bmatrix} \approx \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

↳ eigen vector

Eigen value $= k = 1094/365 = 2.99$

Unit 5

Solutions of Ordinary Differential Equations:

Mathematical models are based on empirical observations that describes the changes in the states of systems are often expressed in terms of not only certain system parameters but also their derivatives, such mathematical models, which use differential calculus to express relationship between variables are known as differential equations.

Example:

1. Newton's law of cooling,

$$\frac{dT(t)}{dt} = k(T_s - T(t))$$

where, T_s is the temperature of surrounding, $T(t)$ is the temperature of liquid at