Unit 4

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Solution of Linear Algebric Equations

⇒ System of linear equation:

A general set of m linear equations and n unknowns,

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = c_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = c_2$$
$$\vdots \qquad \vdots \qquad \dots \qquad \vdots \qquad \vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = c_m$$

Which can be written in the matrix form as:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix}$$

$$[A][X] = [C]$$

Denoting the matrix by $[A]$, $[X]$ and $[C]$ the system of equation is $[A][X] = [C]$, where $[A]$ is called the coefficient matrix, $[X]$ is called the solution vector and $[C]$ is called RHS vector.

Existence of Solution:

$$[A][X] = [C]$$

Consistent System

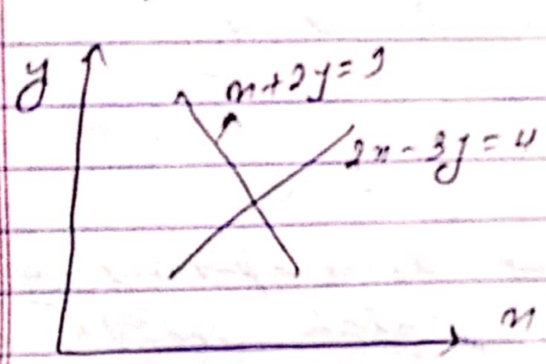
Unique solution

Infinite solution

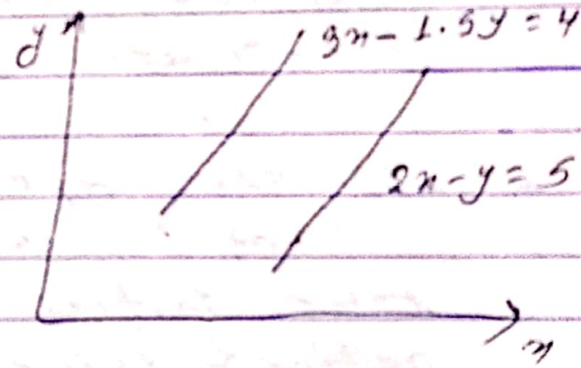
Inconsistent System

→ no solution

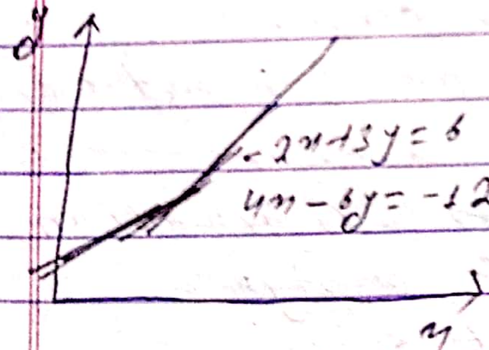
① Unique Solution



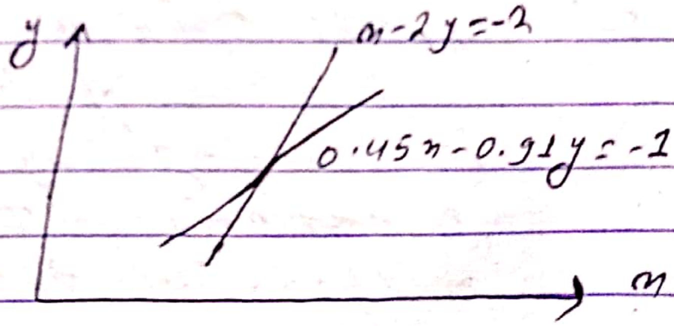
② No Solution



3. Infinite Solution



4. Ill conditioned System



II Solving System of linear equations:-

There are two techniques for solving system of linear algebric equations.

1) Elimination technique
2) Iterative technique

1. Elimination Approach:

It is also known as direct method, reduces the given system of equations to a form from which the solution can be obtained by substitution.

2. Iterative Approach:

As usual, involves assumptions of some initial values which are then refined repeatedly till they reach at some accepted level of accuracy.

II Gauss Elimination Method:

One of the most popular method for solving system of linear equations is the Gauss elimination method. The approach is designed to solve a general set of n equations and n unknowns.

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

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$R_2 \rightarrow a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n = b_2$

$\vdots \quad \vdots \quad \cdots \quad \vdots \quad \vdots$

$a_{n1} x_n + a_{n2} x_2 + \cdots + a_{nn} x_n = b_n$

Gaussian elimination method consists of two steps:

1. Forward elimination of unknowns:
In this step the unknown are eliminated in each equation starting with first equations. The equations are reduced to one equation and one unknown in each equation.

2. Back Substitution:
In this step, starting from the last equation each of the unknown is found.

Forward Elimination of unknowns:

In the first step of forward elimination, the first unknown x_1 is eliminated from all rows below the first row. The first equation is selected as the pivot equation to eliminate x_1. So to eliminate x_1 in the second equation, first divides the first equation by a_{11} and then multiply it by a_{21}. Now first equation becomes,

$R_1 \quad a_{21} x_1 + \frac{a_{21}}{a_{11}} a_{12} x_2 + \cdots + \frac{a_{21}}{a_{11}} a_{1n} x_n = \frac{a_{21}}{a_{11}} b_1$

$R_2 \leftarrow R_2 - R_1$

Now, the equation can be subtracted from the second equation to give,

$$0 + \left(a_{22} - \frac{a_{21}}{a_{11}} a_{12}\right) x_2 + \dots + \left(a_{2n} - \frac{a_{21}}{a_{11}} a_{1n}\right) x_n$$

$$= b_2 - \frac{a_{21}}{a_{11}} b_1$$

or, $a_{22}' x_2 + \dots + a_{2n}' x_n = b_2'$

where,

$$a_{22}' = a_{22} - \frac{a_{21}}{a_{11}} a_{12}$$

$$\vdots \qquad \vdots$$

$$a_{2n}' = a_{2n} - \frac{a_{21}}{a_{11}} a_{1n}$$

$$b_2' = b_2 - \frac{a_{21}}{a_{11}} b_1$$

The procedure of eliminating x_1, is now repeated for the third equation of the n^{th} equation to reduce the set of equation as,

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$0 + a_{22}' x_2 + a_{23}' x_3 + \dots + a_{2n}' x_n = b_2'$$

$$0 + 0 + a_{32}' x_2 + a_{33}' x_3 + \dots + a_{3n}' x_n = b_3'$$

$$\vdots \qquad \vdots$$

$$a_{n2}' x_2 + a_{n3}' x_3 + \dots + a_{nn}' x_n = b_n'$$

This is the end of the first step of forward elimination. Now for the second step of forward elimination, we start with the second equation as the pivot equation and a'_{22} as the pivot element. So, to eliminate x_2 in the third equation, divides the second equation by a'_{22} and then multiply it by a'_{32}. This is the same as multiplying the second eqn by a'_{32}/a'_{22} and subtracting it from this equation.

This makes the coefficient of x_2 zero in the third equation. The same procedure is now repeated for the fourth eqn till the n^{th} equation to give,

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\
0 + a'_{22}x_2 + a'_{23}x_3 + \cdots + a'_{2n}x_n &= b'_2 \\
0 + 0 + a''_{33}x_3 + \cdots + a''_{3n}x_n &= b''_3 \\
&\vdots \\
a''_{n3}x_3 + \cdots + a''_{nn}x_n &= b''_n
\end{aligned}
$$

The same procedure is repeated i.e. there will be a total $n-1$ steps of forward elimination. At the end of $n-1$ steps of forward elimination, we get set of equations as,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$
$$a'_{22}x_2 + a'_{23}x_3 + \dots + a'_{2n}x_n = b'_2$$
$$a''_{33}x_3 + \dots + a''_{3n}x_n = b''_3$$
$$\vdots$$
$$a_{nn}^{(n-1)}x_n = b_n^{(n-1)}$$

II Back Substitution:

Now the equations are solved starting from the last equation, it has only one unknown,

$$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

Then the second last equation, ie $(n-1)^{th}$ equation has two unknown x_n and x_{n-1}, but x_n is already known. This reduces the $(n-1)^{th}$ equation also to one unknown. Back substitution hence can be represented for all equation by the formula,

$$x_i = \frac{b_i^{(i-1)} - \sum_{j=i+1}^{n} a_{ij}^{(i-1)} x_j}{a_{ii}^{(i-1)}} \quad \text{for } i = n-1, n-2 \dots, 1$$

and $$x_n = \frac{b_n^{(n-1)}}{a_{nn}^{(n-1)}}$$

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8. Solve the following system of linear equation using Gaussian elimination method:

$$20x_1 + 15x_2 + 10x_3 = 45$$
$$-3x_1 - 2.249x_2 + 7x_3 = 1.751$$
$$5x_1 + x_2 + 5x_3 = 9$$

⇒ Sol^n:

The matrix form is:

$$\begin{bmatrix} 20 & 15 & 10 \\ -3 & -2.249 & 7 \\ 5 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 45 \\ 1.751 \\ 9 \end{bmatrix}$$

Forward elimination:

Step 1: ~~Multiply R_2 by~~ $R_2 \to R_2 + \frac{3}{20} R_1$

$$\cancel{\begin{bmatrix} 20 & 15 & 10 \\ 0 & 0.001 & 8.5 \\ 5 & 1 & 5 \end{bmatrix}} = \cancel{\begin{bmatrix} 45 \\ 1.751 \\ 9 \end{bmatrix}}$$

Step 1: Multiply Row 1 by $-3/20$ and subtract it from Row 2 $= R_2 - (-3/20 R_1)$.

$$R_2 \to R_2 - \left(\frac{-3}{20} R_1\right) \begin{bmatrix} 20 & 15 & 10 \\ 0 & 0.001 & 8.5 \\ 5 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 45 \\ 8.501 \\ 9 \end{bmatrix}$$

Step 2: Multiply Row 1 by 5/20 and subtract it from Row 3 = $R_3 - 5/20\, R_1$

$$\begin{bmatrix} 20 & 15 & 10 \\ 0 & 0.001 & 8.5 \\ 0 & -2.75 & 2.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 45 \\ 8.501 \\ -2.25 \end{bmatrix}$$

Step 3: Reduce x_2 from third equation:

~~Multiply Row 2 by 15/0.001 and subtract it from Row 1 = $R_1 - 15/0.001\, R_2$~~

~~20 0~~

Multiply Row 2 by $\frac{-2.75}{0.001}$ and Subtract it from

$$R_3 = R_3 - \left(\frac{-2.75}{0.001}\right) R_2$$

$$\begin{bmatrix} 20 & 15 & 10 \\ 0 & 0.001 & 8.5 \\ 0 & 0 & 23377.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 45 \\ 8.501 \\ 23375.5 \end{bmatrix}$$

$$23377.5\, x_3 = 23375.5$$

$$x_3 = 0.999144477$$

$$0.001 x_2 + 8.5 x_3 = 8.501$$

$$0.001 x_2 + 8.5 \times 0.999 = 8.501$$

$$x_2 = \cancel{8.5}\ \cancel{9.5}\ 1.72719455$$

$$20x_1 + 15x_2 + 10x_3 = 45$$

$$x_1 = 0.454646863 7$$

Algorithm:

1. Read dimension of system of equation say n.
2. Read coefficients of matrix row-wise.
3. Read RHS vector say b
4. for k = 1 to n-1

    pivot = a[k][k]

    if (pivot < 0.0000001)

    print "Method fails";

    else

    for (i = k+1 to n)

    $$\text{value} = \frac{a[i][k]}{\text{pivot}}$$

    - Multiply row k of coefficient matrix by "value" and subtract it from row i,
    - Multiply row k of b matrix by "value" and subtract it from row i.

    end for

5. x[n] = b[n] / a[n][n]
6. for i = n-1 to 1

    sum = 0

    for j = i+1 to n

    sum = sum + a[i][j] * x[j]

    x[i] = (b[i] - sum) / a[i][i]

7. Display the solution vector

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Gaussian Elimination with Pivoting:

Gauss elimination method and Gauss elimination with pivoting are same, ~~expe~~ except in the begining of each step of forward elimination a row-switching is done based on the following citeria. If there are n equations, then there are $n-1$ forward elimination steps. At the begining of the k^{th} step of forward elimination, find the maximum of $|a_{kk}|, |a_{k+1\,k}|, \ldots, |a_{nk}|$. Then if the maximum of these values is $|a_{pk}|$ in the p^{th} row, $k \le p \le n$, then switch rows p and k. The other steps of forward elimination are the same as the Gauss elimination method.

Q. Solve:

$$20x_1 + 15x_2 + 10x_3 = 45$$
$$-3x_1 - 2.249x_2 + 7x_3 = 1.751$$
$$5x_1 + x_2 + 5x_3 = 9$$

⇒ Soln:

The matrix form of given system is

$$\underbrace{\begin{bmatrix} 20 & 15 & 10 \\ -3 & -2.249 & 7 \\ 5 & 1 & 5 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{B} = \underbrace{\begin{bmatrix} 45 \\ 1.751 \\ 9 \end{bmatrix}}_{C}$$

So, the largest absolute value is in the row 1. So, as per Gaussian elimination with pivoting to switch in between Row 1 with itself to give

$$\begin{bmatrix} 20 & 15 & 10 \\ -3 & -2.249 & 7 \\ 5 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 45 \\ 1.751 \\ 9 \end{bmatrix}$$

Step 1 Multiply Row 1 by $-3/20$ = .. and Subtract it from Row 2 = $R_2 - (-3/20) R_1$,

$$\begin{bmatrix} 20 & 15 & 10 \\ 0 & 0.001 & 8.5 \\ 5 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 45 \\ 8.501 \\ 9 \end{bmatrix}$$

Step 2: Multiply Row 1 by $5/20$ and subtract it from Row 3 = $(R_3 - (5/20) R_1)$ we get

$$\begin{bmatrix} 20 & 15 & 10 \\ 0 & 0.001 & 8.5 \\ 0 & -2.75 & 2.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 45 \\ 8.501 \\ -9.14 \end{bmatrix}$$

Now, for the second step of forward elimination, the absolute value of the second column elements in Row 2 and Row 3 is 0.001 and 2.75 respectively. So, the largest absolute value is in Row 3 so Row 2 is switched with Row 3 to give,

$$\begin{bmatrix} 20 & 15 & 10 \\ 0 & -2.75 & 2.5 \\ 0 & 0.001 & 8.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 45 \\ \cancel{8.501}\ -2.25 \\ \cancel{-2.25}\ 8.501 \end{bmatrix}$$

Multiply Row 2 by $\frac{0.001}{2.75}$ and subtract from

Row 3 = $R_3 - \left(\frac{0.001}{2.75}\right) R_2$, we get

$$\begin{bmatrix} 20 & 15 & 10 \\ 0 & -2.75 & 2.5 \\ 0 & 0 & 8.5009 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 45 \\ -2.25 \\ 8.5001 \end{bmatrix}$$

Now,

$$8.5009\, x_3 = 8.5001$$

$$x_3 = \frac{8.5001}{8.5009} = 0.9999$$

$$-2.75x_2 + 2.5x_3 = -2.25$$

or, $-2.75x_2 + 2.5 \times 0.9999 = -2.25$

or, $x_2 = \cancel{0.09}\ 1.7271$

$$20x_1 + 15x_2 + 10x_3 = 45$$

or, $20x_1 + 15 \times 1.7271 + 10 \times 0.9999 = 45$

$\therefore x_1 = 0.4546$

Algorithm:-

1. Read dimensions of system of equation say 'n'.
2. Read coefficients of matrix row-wise.
3. Read RHS vector by 'b'.
4. Begin from first equation, for checking the pivoting if it is the largest below it, obtain the equation otherwise, identify the largest and make pivot. Interchange, the original pivot equation with one having largest element.

5. for k = 1 to n-1
pivot = a[k][k]
for (i = k+1 to n)
value = a[k][k] / pivot

- Multiply row k of coefficient matrix by "value" and substract it from row i.
- multiply row k of b matrix by "value" and subtract it from row i.

6. x[n] = b[n] / a[n][n]
7. for i = n-1 to 1
sum = 0;
for j = i+1 to n
sum = sum + a[i][j] * x[j]
end for
x[i] = (b[i] - sum) / a[i][i]
end for
8. Display solution vector.

Gauss Jordon Method :-

It is the variation of Gauss elimination process. The difference is that when an unknown is eliminated from an equation, it is also eliminated from all other equations. All rows are normalized by dividing them by their pivot element. Hence the elimination steps result as identity matrix rather than a triangular matrix. Back substitution is, therefore, not necessary. All the techniques developed for Gaussian elimination are still valid for Gauss-Jordan method. However Gauss-Jordan requires more computational work than Gauss-elimination (approximately 50% more operations).

The goal of Gauss Jordan elimination is to use row operations to change the augmented matrix to row reduced echelon form as :-

$$\left[\begin{array}{ccc:c} 1 & 0 & 0 & b_1 \\ 0 & 1 & 0 & b_2 \\ 0 & 0 & 1 & b_3 \end{array}\right]$$

To get matrix in row reduced echelon form we can perform the following method.

First :-

$$\left[\begin{array}{ccc:c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$$

Use row operation to make $a_{11} = 1$
Then use row operations to make the other elements in column 1 zero (0).

Second:

$$\left[\begin{array}{ccc:c} 1 & a_{12}' & a_{13}' & b_1' \\ 0 & a_{22}' & a_{23}' & b_2' \\ 0 & a_{32}' & a_{33}' & b_3' \end{array}\right]$$

Use row operations to make $a_{22} = 1$. Then use row operation to make other elements in column 2 zero.

Third:

$$\left[\begin{array}{ccc:c} 1 & 0 & a_{13}'' & b_1'' \\ 0 & 1 & a_{23}'' & b_2'' \\ 0 & 0 & a_{33}'' & b_3'' \end{array}\right]$$

Use row operation to make $a_{33} = 1$.
Then use row operations to make other elements in column 3 zero.

$$\left[\begin{array}{ccc:c} 1 & 0 & 0 & b_1''' \\ 0 & 1 & 0 & b_2''' \\ 0 & 0 & 1 & b_3''' \end{array}\right] \rightarrow \text{Normalization process}$$

Now, we can obtain values of unknowns directly from RHS of augmented matrix without doing back substitution.
Thus the solution vector is

$$\begin{bmatrix} b_1''' \\ b_2''' \\ b_3''' \end{bmatrix}$$

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Example: Solve the system of linear equations by Gauss-Jordan elimination method.

$$2x_1 - x_2 + 4x_3 = 15$$
$$2x_1 + 3x_2 - 2x_3 = 1$$
$$3x_1 + 2x_2 - 4x_3 = -4$$

⇒ Sol^n: Augmented form of given system is:

$$\left[\begin{array}{ccc:c} 2 & -1 & 4 & 15 \\ 2 & 3 & -2 & 1 \\ 3 & 2 & -4 & -4 \end{array}\right]$$

Step 1: $R_1 \to R_1/2$

$$\sim \left[\begin{array}{ccc:c} 1 & -1/2 & 2 & 15/2 \\ 2 & 3 & -2 & 1 \\ 3 & 2 & -4 & -4 \end{array}\right]$$

~~Step~~ $R_2 : R_2 - 2R_1$

$$\sim \left[\begin{array}{ccc:c} 1 & -1/2 & 2 & 15/2 \\ 0 & 4 & -6 & -14 \\ 3 & 2 & -4 & -4 \end{array}\right]$$

~~Step~~: $R_3 \to R_3 - 3R_1$

$$\sim \left[\begin{array}{ccc:c} 1 & -1/2 & 2 & 15/2 \\ 0 & 4 & -6 & -14 \\ 0 & 7/2 & -10 & -53/2 \end{array}\right]$$

Step 2: $R_2 \to R_2/4$

$$\sim \left[\begin{array}{ccc:c} 1 & -1/2 & 2 & 15/2 \\ 0 & 1 & -1.5 & -3.5 \\ 0 & 7/2 & -10 & -53/2 \end{array}\right]$$

~~Step 5~~: $R_1 \to R_1 + 1/2\, R_2$

$$\sim \begin{bmatrix} 1 & 0 & 5/4 & : & 23/4 \\ 0 & 1 & -1.5 & : & -3.5 \\ 0 & 7/2 & -10 & : & -53/2 \end{bmatrix}$$

~~Step 6~~: $R_3 \to R_3 - 7/2\, R_2$

$$\sim \begin{bmatrix} 1 & 0 & 5/4 & : & 23/4 \\ 0 & 1 & -1.5 & : & -3.5 \\ 0 & 0 & -19/4 & : & -57/4 \end{bmatrix}.$$

Step 7 : $R_3 \to 4/19\, R_3$

$$\sim \begin{bmatrix} 1 & 0 & 5/4 & : & 23/4 \\ 0 & 1 & -1.5 & : & -3.5 \\ 0 & 0 & 1 & : & 3 \end{bmatrix}$$

~~Step 8~~ $R_2 \to R_2 + 3/2\, R_3$

$$\sim \begin{bmatrix} 1 & 0 & 5/4 & : & 23/4 \\ 0 & 1 & 0 & : & 1 \\ 0 & 0 & 1 & : & 3 \end{bmatrix}$$

~~Step 9~~: $R_1 \to R_1 - 5/4\, R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & : & 2 \\ 0 & 1 & 0 & : & 1 \\ 0 & 0 & 1 & : & 3 \end{bmatrix}$$

$\therefore$ The solution vector is $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$

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Algorithm:

1. Read dimensions of system of equation say n.
2. Read coefficient of augmented matrix.
3. for K=1 to n
   pivot = a[k][k]
   for p=1 to n+1
   a[k][p]= a[k][p] / pivot
   for i=1 to n
   term = a[i][k]
   if (i ≠ k)
   Multiply row k by term and subtract it from row i.
4. Display solution vector which is last column of augmented matrix.

Matrix Inversion

A matrix X is said to be inverse of A if AX=I, where I is the identity matrix of same order as that of matrix A. Matrix inverse can be computed by using Gauss Jordan method as:

Step1: Augmented the coefficient matrix with identity matrix as:

$$\left[\begin{array}{ccc:ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{array}\right]$$

Step2: Apply Gauss-Jordan method to the augmented matrix to reduce coefficient matrix to identity matrix.

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & a'_{11} & a'_{12} & a'_{13} \\ 0 & 1 & 0 & a'_{21} & a'_{22} & a'_{23} \\ 0 & 0 & 1 & a'_{31} & a'_{32} & a'_{33} \end{array}\right]$$

Now, right hand side of augmented matrix is the inverse of the coefficient matrix.

Example:
Using Gauss-Jordan method find the inverse of the matrix.

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$

⇒ Solⁿ:

The given matrix is

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$

Step1: Augmented the matrix with identity matrix:

$$\left[\begin{array}{ccc:ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 1 & 3 & -3 & 0 & 1 & 0 \\ -2 & -4 & -4 & 0 & 0 & 1 \end{array}\right]$$

$-4+2$

$-4+6$

Step 2: Applying Gauss-Jordan method to the augmented matrix to reduce coefficient matrix to identity matrix

$R_2 \to R_2 - R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & -6 & -1 & 1 & 0 \\ -2 & -4 & -4 & 0 & 0 & 1 \end{array}\right]$$

$R_3 \to R_3 + 2R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & -6 & -1 & 1 & 0 \\ 0 & -2 & 2 & 2 & 0 & 1 \end{array}\right]$$

$R_2 \to R_2 / 2$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -1/2 & 1/2 & 0 \\ 0 & -2 & 2 & 2 & 0 & 1 \end{array}\right]$$

$2+2x-3$
$2-6$

$R_1 \to R_1 - R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 6 & 3/2 & -1/2 & 0 \\ 0 & 1 & -3 & -1/2 & 1/2 & 0 \\ 0 & -2 & 2 & 2 & 0 & 1 \end{array}\right]$$

$R_3 \to R_3 + 2R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 6 & 3/2 & -1/2 & 0 \\ 0 & 1 & -3 & -1/2 & 1/2 & 0 \\ 0 & 0 & -4 & 1 & 1 & 1 \end{array}\right]$$

$R_3 \to R_3/{-4}$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 6 & 3/2 & -1/2 & 0 \\ 0 & 1 & -3 & -1/2 & 1/2 & 0 \\ 0 & 0 & 1 & -1/4 & -1/4 & -1/4 \end{array}\right]$$

$R_1 \to R_1 - 6R_3$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & \overset{3}{\cancel{2.25}} & \overset{1}{\cancel{0.25}} & 3/2 \\ 0 & 1 & -3 & -1/2 & 1/2 & 0 \\ 0 & 0 & 1 & -1/4 & -1/4 & -1/4 \end{array}\right]$$

$R_2 \to R_2 + 3R_3$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 3 & 1 & 3/2 \\ 0 & 1 & 0 & -5/4 & -1/4 & -3/4 \\ 0 & 0 & 1 & -1/4 & -1/4 & -1/4 \end{array}\right]$$

$$\therefore A^{-1} = \begin{bmatrix} 3 & 1 & 3/2 \\ -5/4 & -1/4 & -3/4 \\ -1/4 & -1/4 & -1/4 \end{bmatrix}$$

Q. Solve the system of linear equation by Gaun - Jordan method

$$2x_1 + 2x_2 + x_3 = 6$$
$$4x_1 + 2x_2 + 4x_3 = 4$$
$$x_1 + x_2 + x_3 = 0$$

Q. Find the inverse of matrix, given below using Gauss - Jordan method.

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

⇒ Soln:

The given coefficient matrix is:

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

Step 1: Augmented the coefficient matrix with identity matrix:-

$$\left[\begin{array}{ccc:ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 0 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right]$$

Step 2: Applying Gauss-Jordan method to the augmented matrix to reduce coefficient matrix to identity matrix

$R_2 \to R_2 - 2R_1$

$$\left[\begin{array}{ccc:ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 5 & -2 & -2 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \to R_2/5$

$$\left[\begin{array}{ccc:ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2/5 & -2/5 & 1/5 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$R_1 \to R_1 + R_2$, $R_3 \to R_3 + 2R_2$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0.6 & 0.6 & 1/5 & 0 \\ 0 & 1 & -2/5 & -2/5 & 1/5 & 0 \\ 0 & 0 & 0.2 & -0.8 & 0.4 & 1 \end{array}\right]$$

$R_3 \to R_3/0.2$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0.6 & 0.6 & 1/5 & 0 \\ 0 & 1 & -2/5 & -2/5 & 1/5 & 0 \\ 0 & 0 & 1 & -4 & 2 & 5 \end{array}\right]$$

$R_1 \to R_1 - 0.6R_3$, $R_2 \to R_2 + 2/5\, R_3$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 3 & -1 & -3 \\ 0 & 1 & 0 & -2 & 1 & 2 \\ 0 & 0 & 1 & -4 & 2 & 5 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 3 & -1 & -3 \\ -2 & 1 & 2 \\ -4 & 2 & 5 \end{bmatrix}$$

Q. Solve the system of linear equation by Gauss-Jordan method

$$2x_1 + 2x_2 + x_3 = 6$$
$$4x_1 + 2x_2 + 4x_3 = 4$$
$$x_1 + x_2 + x_3 = 0$$

$\Rightarrow$ Augmented form of given system is:

$$\left[\begin{array}{ccc:c} 2 & 2 & 1 & 6 \\ 4 & 2 & 4 & 4 \\ 1 & 1 & 1 & 0 \end{array}\right]$$

Step1 : $R_1 \to R_1/2$.

$$\left[\begin{array}{ccc:c} 1 & 1 & 1/2 & 3 \\ 4 & 2 & 4 & 4 \\ 1 & 1 & 1 & 0 \end{array}\right]$$

$R_2 \to R_2 - 4R_1$, $R_3 \to R_3 - R_1$

$$\left[\begin{array}{ccc:c} 1 & 1 & 1/2 & 3 \\ 0 & -2 & 2 & -8 \\ 0 & 0 & 1/2 & -3 \end{array}\right]$$

Step 2 :

$R_2 \to R_2/-2$

$$\left[\begin{array}{ccc:c} 1 & 1 & 1/2 & 3 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 1/2 & -3 \end{array}\right]$$

$R_1 \to R_1 - R_2$, R

$$\left[\begin{array}{ccc:c} 1 & 0 & -3/2 & 1 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 1/2 & -3 \end{array}\right]$$

Step 3 : $2R_3 \to 2R_3$

$$\left[\begin{array}{ccc:c} 1 & 0 & -3/2 & 1 \\ 0 & 1 & -1 & 4 \\ 0 & 0 & 1 & -6 \end{array}\right]$$

$R_1 \to R_1 + 3/2 R_3$, $R_2 \to R_2 + R_3$

$$\left[\begin{array}{ccc:c} 1 & 0 & 0 & -8 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -6 \end{array}\right]$$

$\therefore$ Solⁿ vector $= \begin{bmatrix} -8 \\ -2 \\ -6 \end{bmatrix}$

$4 - 4 \times 3 = 4-$

$-1 - \frac{1}{2}$

$\frac{-2-1}{2}$ $-3/2$

$4 - 3$

$-1 - \frac{1}{2}$

$2 \times 1/2$

-3×2

$1 + \frac{3}{2} \times -6$

$1 - 9$

$4 - 6$

I Matrix Inversion:

A matrix X is said to be inverse of A if AX = I, where I is the identity matrix of same order as that of matrix. Matrix inverse can be computed by using Gauss Jordan Method as:

Step1: Augment the coefficient matrix with identity matrix as:

$$\left[\begin{array}{ccc:ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{array}\right]$$

Step2: Apply Gauss-Jordan method to the augmented matrix to reduce coefficient matrix to identity matrix:

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & a'_{11} & a'_{12} & a'_{13} \\ 0 & 1 & 0 & a'_{21} & a'_{22} & a'_{23} \\ 0 & 0 & 1 & a'_{31} & a'_{32} & a'_{33} \end{array}\right]$$

Now, right hand side of augmented matrix is the inverse of the coefficient matrix.

II Matrix Factorization:-

In this method, we factorize the given matrix, as A = LU, where L is a lower triangular matrix and U is upper triangular matrix.

ie.

$$L = \begin{bmatrix} L_{11} & 0 & \cdots & 0 \\ L_{21} & L_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ L_{n1} & L_{n2} & \cdots & L_{nn} \end{bmatrix} \text{ \& } U = \begin{bmatrix} U_{11} & U_{12} & \cdots & U_{1n} \\ 0 & U_{22} & \cdots & U_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & U_{nn} \end{bmatrix}$$

Now,

system of linear equations $AX = C$ can be written as,

$LUX = C$, since $A = LU$.

Multiplying both sides by L^{-1}, we get

$$L^{-1}LUX = L^{-1}C$$

$$IUX = L^{-1}C \quad [\because L^{-1}L = I]$$

$$UX = L^{-1}C$$

Let $L^{-1}C = Z$, where Z is unknown vector.

then, lower coefficient

$LZ = C \ldots$ (i)

& $UX = Z \ldots$ (ii) upper

for linear eqn

Now,

we can solve system of equations in two steps:

1) Solve equation (i) first to Z by using forward substitution

2) Solve eqn (ii) to calculate the solution vector X by back substitution.

II Doolittle LU Decomposition

The coefficient matrix A of a system of linear equation can be decomposed into two triangular matrix L and U such that,

$$A = \begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} \cdots & a_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} L_{11} & 0 \cdots & 0 \\ L_{21} & L_{22} \cdots & 0 \\ \vdots & \vdots & \vdots \\ L_{n1} & L_{n2} \cdots & L_{nn} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} \cdots & U_{1n} \\ 0 & u_{22} \cdots & u_{2n} \\ \vdots & \vdots & \vdots \\ 0 & 0 \cdots & u_{nn} \end{bmatrix}$$

If L has 1's on its diagonal, then it is called Doolittle factorization, Thus, Doolittle algorithms assumes that $L_{11} = 1,\ L_{22} = 1, \ldots L_{nn} = 1$

Now, from above matrices;

$a_{11} = L_{11} U_{11} \Rightarrow u_{11} = a_{11}$ $[\because L_{11} = 1]$

$a_{12} = L_{11} U_{12} \Rightarrow U_{12} = a_{12}$ $[\because L_{11} = 1]$

$a_{1n} = L_{11} U_{1n} \Rightarrow U_{1n} = a_{1n}$ $[\because L_{11} = 1]$

$a_{21} = L_{21} U_{11} \Rightarrow L_{21} = \frac{a_{21}}{U_{11}} = \frac{a_{21}}{a_{11}}$

$a_{22} = L_{21} U_{12} + L_{22} U_{22} \Rightarrow U_{22} = \frac{a_{22} - L_{21} U_{12}}{L_{22}}$

$= a_{22} - L_{21} U_{12}$

$\vdots$

$a_{2n} = L_{21} U_{1n} + L_{22} U_{2n} \Rightarrow U_{2n} = \frac{a_{2n} - L_{21} U_{1n}}{L_{22}} = a_{2n} - L_{21} U_{1n}$

and

$a_{n1} = l_{n1} u_{11} \Rightarrow l_{n1} = \frac{a_{n1}}{a_{11}}$ $[\because a_{11} = u_{11}]$

$a_{n2} = l_{n1} u_{12} + l_{n2} u_{22} \Rightarrow l_{n2} = \frac{a_{n2} - l_{n1} u_{12}}{u_{22}}$

$= \frac{1}{u_{22}} (a_{n2} - l_{n1} u_{12})$

$a_{n3} = l_{n1} u_{13} + l_{n2} u_{23} + l_{n3} u_{33} \Rightarrow l_{n3}$

$= \frac{a_{n3} - l_{n1} u_{13} - l_{n2} u_{23}}{u_{33}}$

$= \frac{1}{u_{33}} (a_{n3} - l_{n1} u_{13} - l_{n2} u_{23})$

$\vdots$

$a_{nn} = l_{n1} u_{1n} + l_{n2} u_{2n} + \dots + l_{nn} u_{nn}$

$\Rightarrow u_{nn} = \frac{a_{nn} - l_{n1} u_{1n} - l_{n2} u_{2n} - \dots - l_{n\,n-1} u_{n-1\,n}}{l_{nn}}$

$\therefore u_{nn} = a_{nn} - l_{n1} u_{1n} - l_{n2} u_{12} - \dots - l_{n\,n-1} u_{n-1\,n}$

Generalizing this, we get

$u_{ij} = a_{ij} - \sum_{k=1}^{i-1} l_{ik} u_{kj}, \quad j = 1, 2, \dots n)$

$[\text{if } i \leq j]$

where,

$u_{11} = a_{11}, \ u_{12} = a_{12}, \dots$

and if $i > j$,

$l_{ij} = \frac{1}{u_{jj}} \left[a_{ij} - \sum_{k=1}^{i-1} l_{ik} u_{kj} \right], \quad j = 1, 2, 3, \dots n$

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Example:

Find the Doolittle LU decomposition of the following matrix.

$$A = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

⇒ Soln:

We know that, $A = LU$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

where,

$$A = \begin{bmatrix} 25 & 5 & 1 \\ 64 & 8 & 1 \\ 144 & 12 & 1 \end{bmatrix}$$

from Doolittle LU decomposition gives

$L_{11} = L_{22} = L_{33} = 1$

Now,

$a_{11} = L_{11} U_{11} \Rightarrow u_{11} = a_{11}$ $[\because L_{11} = 1]$

$\Rightarrow u_{11} = 25$

$a_{12} = L_{11} u_{12} \Rightarrow u_{12} = a_{12} = 5$ $[\because L_{11} = 1]$

$a_{13} = L_{11} U_{13} \Rightarrow U_{13} = a_{13} = 1$ $[\because L_{11} = 1]$

$a_{21} = L_{21} U_{11} \Rightarrow L_{21} = \frac{a_{21}}{U_{11}} = \frac{64}{25} = 2.56$

$$a_{22} = L_{21}U_{12} + L_{22}U_{22} \Rightarrow U_{22} = \frac{a_{22} - L_{21}U_{12}}{L_{22}}$$

$$= 8 - \frac{64}{25} \times 5$$

$$= -24/5$$

$$a_{23} = L_{21}U_{13} + L_{22}U_{23} \Rightarrow U_{23} = \frac{a_{23} - L_{21}U_{13}}{L_{22}}$$

$$= 1 - \frac{64}{25} \times 1$$

$$= -1.56$$

$$a_{31} = L_{31}U_{11} \Rightarrow L_{31} = \frac{a_{31}}{U_{11}} = \frac{144}{25} = 5.76$$

$$a_{32} = L_{31}U_{12} + L_{32}U_{22} \Rightarrow L_{32} = \frac{a_{32} - L_{31}U_{12}}{U_{22}}$$

$$= \frac{12 - 5.76 \times 5}{-4.8}$$

$$= 3.5$$

$$a_{33} = L_{31}U_{13} + L_{32}U_{23} + L_{33}U_{33}$$

$$U_{33} = \frac{a_{33} - L_{31}U_{13} - L_{32}U_{23}}{L_{33}}$$

$$= 1 - 5.76 \times 1 - 3.5 \times -1.56$$

$$= 0.7$$

$$L = \begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2.56 & 1 & 0 \\ 5.76 & 3.5 & 1 \end{bmatrix}, \quad U = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{pmatrix}$$

$$= \begin{bmatrix} 25 & 5 & 1 \\ 0 & -4.8 & -1.56 \\ 0 & 0 & 0.7 \end{bmatrix}$$