Unit 2

Interpolation

Suppose we are given the following values of $y = f(x)$ for a set of values of x

x : x_0 x_1 x_2 ... x_n

y : y_0 y_1 y_2 ... y_n

Then, the process of finding the value of y corresponding to any value of $x = x_i$, between x_0 and x_n is called interpolation. Thus, interpolation is the method of estimating the value of a function for any intermediate value of the independent variable while the process of computing the value of the function outside the range is called extrapolation.

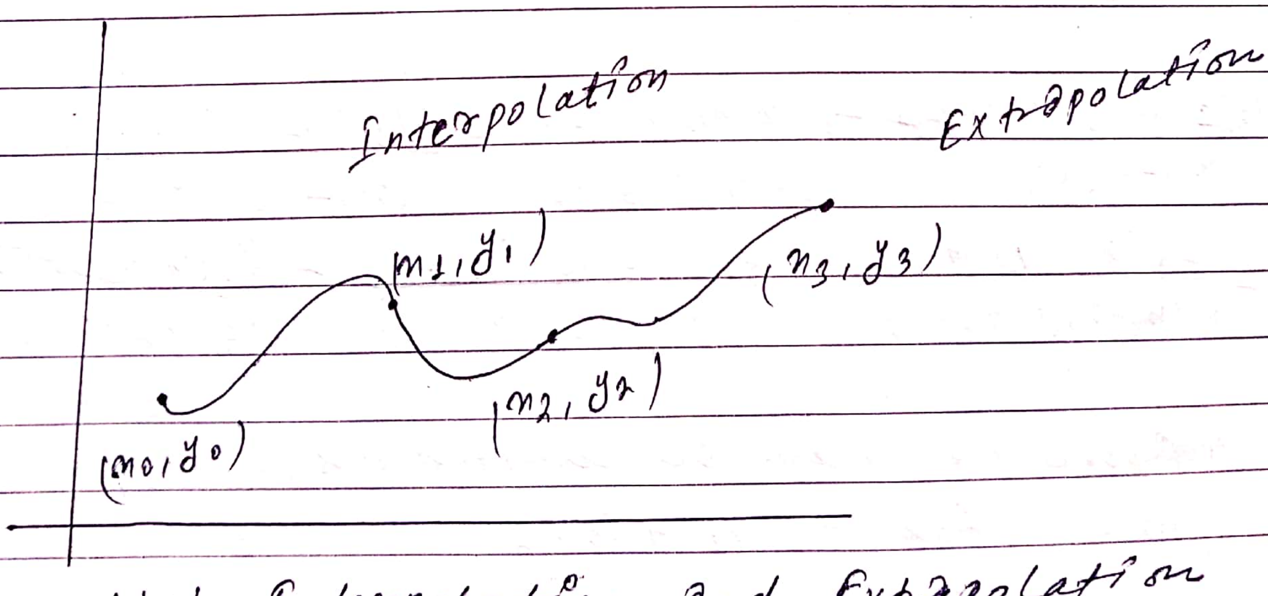

fig:- Interpolation and Extrapolation

classmate

Date

Page

II Lagrange Interpolation: A second order polynomial can be written as: (i)

$$P_2(x) = b_1(x-x_0)(x-x_1) + b_2(x-x_1)(x-x_2) + b_3(x-x_0)(x-x_2)$$

$P_2(x)$ Let $(x_0, f_0), (x_1, f_1)$ and (x_2, f_2) are three interpolation points. Substituting these points in eq^n (i), we get

(x_0, f_0) $P_2(x_0) = f_0 = b_2(x_0-x_1)(x_0-x_2)$

(x_1, f_1) $P_2(x_1) = f_1 = b_3(x_1-x_0)(x_1-x_2)$

(x_2, f_2) $P_2(x_2) = f_2 = b_1(x_2-x_0)(x_2-x_1)$

$$b_2 = \frac{f_0}{(x_0-x_1)(x_0-x_2)}, \quad b_3 = \frac{f_1}{(x_1-x_0)(x_1-x_2)}$$

$$b_1 = \frac{f_2}{(x_2-x_0)(x_2-x_1)}$$

Also, substituting these values of b_1, b_2 & b_3 in eq^n (i) we get

$$P_2(x) = \frac{(x-x_0)(x-x_1)f_2}{(x_2-x_0)(x_2-x_1)} + \frac{f_0(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + \frac{f_1(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}$$

$$= \frac{f_0(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + f_1\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + \frac{f_2(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

Above eq^n can be written as,

$$P_2(x) = f_0 l_0 + f_1 l_1 + f_2 l_2$$

$$= \sum_{i=0}^{2} f_i l_i(x) \quad \ldots \text{(ii)}$$

where,

$$l_i(x) = \prod_{\substack{i=0,\\ j=i}}^{2} \frac{(x-x_j)}{(x_i-x_j)}$$

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Generalizing eq.^{n} (ii) for $(n+1)$ points, we get

$$P_n(x) = \sum_{i=0}^{n} f_i l_i(x) \quad \text{--- (iii)}$$

where,

$$l_i(x) = \prod_{\substack{j=0, \\ j \neq i}}^{n} \frac{(x - x_j)}{(x_i - x_j)}$$

eq^n (iii) is called Lagrange interpolation formula.

Example: The velocity of a rocket is given as a function of time in table below:

x	Time (t)	0	10	15	20	22.5	30
$f = y$	velocity (v)	0	227.04	362.78	517.35	602.97	901.67

Determine the value of velocity at $t = 16$ seconds ~~seconds~~ using first order Language polynomial

Sol^n:

⇒ For first order polynomial interpolation the velocity is given by,

$$P_1(x) = \sum_{i=0}^{1} f_i l_i(x) = f_0 l_0(x) + f_1 l_1(x)$$

Since, we want to find the velocity at $t = 16$ and we are using a first order polynomial, we need to choose the two data points that are closer to $t=16$, that also bracket $t=16$ to evaluate it.

The two points are $x_0 = 15$ and $x_1 = 20$, this gives,

$$l_i(x) = \prod_{\substack{j=0 \\ j \neq 0}}^{1} \frac{x - x_j}{x_i - x_j}$$

$i = 0, j \le 1$
if $(i \neq j)$
$\frac{x - x_j}{x_i - x_j}$

$$l_0(x) = \prod_{\substack{j=0 \\ j\neq 0}}^{1} \frac{x-20}{15-20} \quad \left[\because \frac{x - x_1}{x_0 - x_1}\right]$$

$i=0$
$j=1$

$$= -\frac{x-20}{5}$$

$$l_1(x) = \prod_{\substack{i=1 \\ j\neq 1}}^{1} \frac{x-15}{20-15} \quad \left[\because \frac{x - x_0}{x_1 - x_0}\right]$$

$i=0$
$j \neq 1$

$i=1$
$j\neq 1$

$$= \frac{x-15}{5}$$

when $x = 16$

$$l_0(x) = \prod_{\substack{j=0 \\ j\neq 0}}^{1} \frac{x - x_1}{x_0 - x_1} = \frac{16-20}{15-20} = 4/5$$

$$l_1(x) = \frac{16-15}{5} = 1/5$$

$$p_1(x) = f_0 l_0(x) + f_1 l_1(x)$$

$$= 362.78 \times 4/5 + 517.35 \times 1/5$$

$$= 393.694$$

Q. The value of e^x is given in table below:

x	0	1	2	3
e^x	1	2.7183	7.3891	20.0855

Determine the value at $e^{1.2}$ with second order polynomial interpolation using Lagrangian polynomial interpolation.

→ Sol^n:

For second order polynomial interpolation ~~the velocity is given by~~, value at $e^{1.2}$,

$$P_2(x) = \sum_{i=0}^{2} f_i l_i(x)$$

$$= f_0 l_0(x) + f_1 l_1(x) + f_2 l_2(x)$$

Since, we want to find the value at $e^{1.2}$ and we are using second order polynomial, we need to choose three data points that are closer to $e^{1.2}$, that also bracket $e^{1.2}$ to evaluate it.

The three points are $x_0 = 0$, $x_1 = 1$, $x_2 = 2$

$$l_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{2} \frac{x - x_j}{x_i - x_j}$$

$$l_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)}$$

when $x = 1.2$

$$l_0(x) = \frac{(1.2 - 1)(1.2 - 2)}{(0 - 1)(0 - 2)}$$

$$= -0.08$$

$$l_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)}$$

$$= \frac{(x - 0)(x - 2)}{(1 - 0)(1 - 2)}$$

when $x = 1.2$ $= \frac{(1.2 - 0)(1.2 - 2)}{-1}$

$$= 0.96$$

$$l_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}$$

$$= \frac{(x - 0)(x - 1)}{(2 - 0)(2 - 1)}$$

when $x = 1.2$ $= \frac{(1.2 - 0)(1.2 - 1)}{2}$

$$= 0.12$$

$$P_2(x) = f_0 l_0(x) + f_1 l_1(x) + f_2 l_2(x)$$

$$= 1 \times (-0.08) + 2.7083 \times 0.96 + 7.3891 \times 0.12$$

$$= 3.32666$$ //

Q. Consider the table below, gives square root for Integers:

i	0	1	2	3	4
x	1	2	3	4	5
$f(x)$	1	1.414			

Find the square root of 2.5 using the second order Lagrange Interpolation method.

i	0	1	2	3	4
x	1	2	3	4	5
$f(x)$	1	1.4142	1.7321	2	2.2361

→ Sol^n:

For second order Interpolation, the square root ~~of 2.5~~ is given by,

$$P_2(x) = \sum_{i=0}^{2} f_i l_i(x)$$

$$= f_0 l_0(x) + f_1 l_1(x) + f_2 l_2(x)$$

Since we want to find the square root of 2.5 and we are using second order polynomial, we need to choose three data points that are closer to 2.5, that also bracket 2.5 to evaluate it.

The three points are $x_0 = 1$, $x_1 = 2$, $x_2 = 3$, $x = 2.5$

$$l_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{2} \frac{x - x_j}{x_i - x_j}$$

$$l_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} = \frac{(x-2)(x-3)}{(1-2)(1-3)}$$

when $x = 2.5$

$$l_0(x) = \frac{(2.5-2)(2.5-3)}{(1-2)(1-3)} = -0.125$$

$$l_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)}$$

$$= \frac{(2.5-1)(2.5-3)}{(2-1)(2-3)}$$

$$= 0.75$$

$$l_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

$$= \frac{(2.5-1)(2.5-2)}{(3-1)(2-1)}$$

$$= 0.375$$

$$P_2(x) = 1(-0.125) + 1.4142 \times 0.75 + 1.7321 \times 0.375$$

$$= 1.535$$

Algorithm:-

1. Read number of data points say n.
2. Read the value at which the value is needed, say x.
3. Read n data points.
4. for $i=0$ to $n-1$
   for $j=0$ to $n-1$
   if(j != i)
   $ln[i] = ln[i] * ((x - x_j) / (x[i] - x[j]))$
5. for $i=0$ to $n-1$
   value = value + $fx[i]$ * $ln[i]$
6. print the interpolated value at x

Newton's Divided Difference Interpolation :-

Let us consider a polynomial of degree n of the form $P_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \ldots + a_n(x-x_0)(x-x_1)\ldots(x-x_{n-1})$ — (i)

To construct the polynomial, we need to find the coefficient $a_0, a_1, a_2, \ldots a_n$.

Let us suppose that $(x_0, f(x_0))$, $(x_1, f(x_1))$, ... $(x_n, f(x_n))$ are given interpolating points. Now at $x = x_0$ eq^n (i) becomes,

$$P(x_0) = f(x_0) = a_0 \Rightarrow a_0 = f(x_0) \quad \text{— (ii)}$$

Similarly, at $x = x_1$, eq^n (i) becomes

$$p(x_1) = f(x_1) = a_0 + a_1(x_1 - x_0)$$

$$\Rightarrow a_1 = \frac{f(x_1) - a_0}{(x_1 - x_0)} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

at $x = x_2$ eq^n (i) becomes

$$p(x_2) = f(x_2) = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$$

$$\rightarrow f(x_2) = f(x_0) + \frac{f(x_1) - f(x_0)}{(x_1 - x_0)}(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$$

Rearranging the terms, we get

$$a_2 = \frac{\dfrac{f(x_2) - f(x_1)}{(x_2 - x_1)} - \dfrac{f(x_1) - f(x_0)}{x_1 - x_0}}{(x_2 - x_0)}$$

Now a_0, a_1, a_2 are finite divided differences a_0, a_1, a_2 are first, second and third finite divided differences respectively. We denote the first dividend

difference by
$f[x_0] = f(x_0)$ [given from table]
Second divided difference by,
$$f[x_1, x_0] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$
and third divided difference by,
$$f[x_2, x_1, x_0] = \frac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0}$$
$$= \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

This leads us to writing the general form of the Newton's divided difference polynomial for $n+1$ data points $(x_0, y_0), (x_1, y_1) \ldots (x_n, y_n)$ are

$$P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \ldots + a_n(x - x_0)(x - x_1)\ldots(x - x_{n-1})$$
where,
$a_0 = f[x_0]$
$a_1 = f[x_1, x_0]$
$a_2 = f[x_2, x_1, x_0]$
$\vdots$
$a_{n-1} = f[x_{n-1}, x_{n-2}, \ldots, x_0]$
$a_n = f[x_n, x_{n-1}, \ldots x_0]$
where, m^{th} divided difference is,
$a_m = f[x_m, \ldots, x_0]$

$$= \frac{f[x_m, \ldots, x_1] - f[x_{m-1}, \ldots x_0]}{x_m - x_0}$$

Now, eqⁿ ① can be written as,

$$P_n(x) = f[x_0] + \sum_{i=1}^{n} f[x_0, x_1, \ldots x_n] \prod_{j=0}^{i-1} (x - x_j) \quad \text{— (iii)}$$

eqⁿ (iii) is called Newtons divided difference interpolation polynomial.

Q. Given the following data table:

x	0	10	15	20	22.5	30
$f(x)$	0	227.04	362.78	517.35	602.97	901.67

Determine the value at $x=16$ with third order polynomial interpolation using Newton's divided difference polynomial method.

⇒ Solⁿ:

For, third order polynomial, we need to choose four data points as:

$x_0 = 10$, $x_1 = 15$, $x_2 = 20$, $x_3 = 22.5$

$f(x_0) = 227.04$, $f(x_1) = 362.78$, $f(x_2) = 517.35$, $f(x_3) = 602.97$

For the third order polynomial, Newton's divided difference polynomial is given by

$$P_3(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2) \quad \text{— (i)}$$

Now,

$$a_0 = f[x_0] = f(x_0) = 227.04$$

$$a_1 = f[x_1, x_0] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{362.78 - 227.04}{15 - 10} = 27.148$$

8. Given below is a table of data for $\log x$. Estimate $\log(2.5)$, using second order Newton Interpolation polynomial.

x_i	1	2	3	4
$\log x_i$	0	0.3010	0.4771	0.6021

⇒ Sol^n://

For second order polynomial, we need to choose three data points as:

$x_0 = 1$, $\log x_0 = 0$

$x_1 = 2$, $\log x_1 = 0.3010$

$x_2 = 3$, $\log x_2 = 0.4771$

For second ~~third~~ order polynomial, Newton's divided polynomial is given as

$$P_2(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) \quad \text{--- (1)}$$

Now,

$a_0 = f(x_0) = 0$

$$a_1 = f[x_1, x_0] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$= \frac{0.3010 - 0}{2 - 1}$$

$$= 0.3010$$

$a_2 = f[x_0, x_1, x_2]$

$$= \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

$$= \frac{\frac{0.4771 - 0.3010}{3 - 2} - 0.3010}{3 - 1} = -0.06$$

Now, substituting the value of a_0, a_1 & a_2 in eqn ①

$P_2(x) = 0 + 0.3010(x-1) + (-0.06245)(x-1)(x-2)$

At $x = 2.5$

$= 0.3010(2.5-1) - 0.06245(2.5-1)(2.5-2)$

$= 0.4046$

8. H.W. Given the following data table, evaluate $f(2.4)$ using third order Newton's Divided Difference Interpolation polynomial

i	0	1	2	3	4
x_i	0	1	2	3	4
$f(x_i)$	1	2.25	3.75	4.25	5.81

⇒ Soln:

For third order Newton's Divided polynomial, we need to choose four data points as

$x_0 = 0 \qquad f(x_0) = 1$

$x_1 = 1 \qquad f(x_1) = 2.25$

$x_2 = 2 \qquad f(x_2) = 3.75$

$x_3 = 3 \qquad f(x_3) = 4.25$

For third order polynomial, Newton's divided difference polynomial is given by

$$P_3(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + a_3(x-x_0)(x-x_1)(x-x_2) \quad \text{—} ①$$

Now,

$a_0 = f[x_0] = f(x_0) = 1$

$a_1 = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0} = \dfrac{2.25 - 1}{1} = 1.25$

$$a_2 = f[x_2, x_1, x_0]$$

$$= \frac{\frac{f(x_2)-f(x_1)}{x_2-x_1} - \frac{f(x_1)-f(x_0)}{x_1-x_0}}{x_2-x_0}$$

$$= \frac{\frac{3.75-2.25}{2-1} - 1.25}{2-0}$$

$$= \frac{1.5-1.25}{2}$$

$$= 0.125$$

$$a_3 = f[x_0, x_1, x_2, x_3]$$

$$= \frac{f[x_3, x_2, x_1] - f[x_2, x_1, x_0]}{x_3-x_0}$$

then,

$$f[x_3, x_2, x_1] = \frac{\frac{f(x_3)-f(x_2)}{x_3-x_2} - \frac{f(x_2)-f(x_1)}{x_2-x_1}}{x_3-x_1}$$

$$= \frac{\frac{4.25-3.75}{3-2} - 1.5}{3-1}$$

$$= \frac{0.5-1.5}{2}$$

$$= -0.5$$

Now,

$$a_3 = \frac{-0.5-0.125}{3}$$

$$= -0.2083$$

Substituting the value of a_0, a_1, a_2 and a_3 in eqn (1) we get

at $x = 2.4$

$$P_3(x) = 1 + 1.25(2.4-0) + 0.125(2.4-0)(2.4-1) + (-0.2083)(2.4-0)(2.4-1)(2.4-2)$$

$$= 4.1400$$

Algorithm :

1. Start
2. Read the number of points say n.
3. Read the value at which interpolated value say x.
4. For i=0 to n-1
   dd[i] = fx[i]
   end for
5. for i=0 to n-1
   for (j=n-1 to i+1)
   $$dd[j] = \frac{dd[j] - dd[j-1]}{x[j] - x[j-1-i]}$$
   End for
   End for
6. Initialize v=0, p=1
7. for i=0 to n-1
   for (j=0 to i-1)
   $p = p * (x - x_j)$
   end for
   v = v + dd[i] * p
   Reset p=1,
   End for
8. print the interpolated value v.
9. Stop

II Divided Difference Table:

We know that coefficients of Newton Divided Difference interpolation polynomial are evaluated using divided differences at the interpolating points. The first order divided difference are calculated as,

$$f[x_1, x_0] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

and,

$$f[x_2, x_1] = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Second order divided difference are calculated from first order divided difference as,

$$f[x_2, x_1, x_0] = \frac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0}$$

Thus we can say that higher order divided difference can be calculated from lower order divided differences recursively as:-

$$f[x_m, \ldots, x_0] = \frac{f[x_m, \ldots, x_1] - f[x_{m-1}, \ldots, x_0]}{x_m - x_0}$$

eg:

A divided difference table for five points is shown in following table:

Q: A divided difference table for five data points is shown in following table

i	x_i	$f(x_i)$	f.d 1st	s.d 2nd	T.d 3rd	f.d 4th
		a_0				
0	x_0	$f(x_0)$	$a_1 =$	a_2		
			$f[x_1, x_0]$		a_3	
1	x_1	$f(x_1)$		$f[x_2, x_1, x_0]$		a_4
			$f[x_2, x_1]$		$f[x_3, x_2, x_1, x_0]$	
2	x_2	$f(x_2)$		$f[x_3, x_2, x_1]$		$f[x_4, x_3, x_2, x_1, x_0]$
			$f[x_3, x_2]$		$f[x_4, x_3, x_2, x_1]$	
3	x_3	$f(x_3)$		$f[x_4, x_3, x_2]$		
			$f[x_4, x_3]$			
4	x_4	$f(x_4)$				

Q. Given the following set of data points, Obtain the table of divided differences. Use the table to estimate the value of $f(1.5)$

i	0	1	2	3	4
x_i	1	2	3	4	5
$f(x_i)$	0	7	26	63	124

⇒ Soln//

We have, fourth order divided difference is given as,

$$P_4(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2) + a_4(x - x_0)(x - x_1)(x - x_2)(x - x_3)$$

Interpolation with evenly spaced data:
Newton forward difference interpolation.

Newton forward formula is used mainly to interpolate the value of $f(x)$ near the begining of a set of tabular values and also for extrapolating the value of $f(x)$ at short distance ahead of $f(x)$. That is, it is used to find the unknown values of y for some x which lies at the begining of a set of tabular values.

We know that Newton's divided difference polynomial is given as,

$$P_n(x) = f[x_0] + f[x_1, x_0](x - x_0) + \dots + f[x_n, x_{n-1} \dots x_1, x_0](x - x_0)(x - x_1)\dots(x - x_{n-1})$$

Let us introduce a notation $h = x_{i+1} - x_i$ for each $i = 0, 1, 2, \dots n-1$,

Suppose,

$$x = x_0 + sh$$

$$x_k = x_0 + kh \quad \text{// value at non tabular point}$$

$$\Rightarrow x - x_k = (s - k)h,$$

This gives,

$$\left.\begin{aligned} x - x_0 &= sh \\ x - x_1 &= (s-1)h \\ &\vdots \\ x - x_{n-1} &= (s - n + 1)h \end{aligned}\right\}$$

Now,

Newton's Interpolation divided difference formula becomes,

$$P_n(x) = P_n(x_0 + sh) = f[x_0] + f[x_1, x_0]sh + f[x_2, x_1, x_0] s(s-1)h^2 + \dots + f[x_n, x_{n-1}, \dots x_1, x_0] s(s-1)(s-2) \dots (s-n+1)h^n$$

$$= \sum_{k=0}^{n} f[x_0, x_1, \dots, x_k] s(s-1) \dots (s-k+1) h^k$$

Using binomial coefficient notation:

$$\binom{s}{k} = \frac{s(s-1) \dots (s-k+1)}{k!}$$

We can express $P_n(x)$ compactly as,

$$P_n(x) = P_n(x_0 + sh) = f[x_0] + \sum_{k=1}^{n} \binom{s}{k} k!\, h^k f[x_0, x_1, \dots x_k] \quad \text{(1)}$$

Newton's forward difference formula is constructed by making use of the forward difference notation 'Δ'. With this notation,

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{1}{h} \Delta f(x_0)$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$$

$$= \frac{1}{2h} \frac{1}{h} [\Delta f(x_1) - \Delta f(x_0)]$$

$$= \frac{1}{2h^2} \Delta^2 f(x_0)$$

And in general,

$$f[x_0, x_1, \dots x_k] = \frac{1}{k!\, h^k} \Delta^k f(x_0)$$

Now, eqn (1) can be written as:

$(x - x_0) = sh$
$(x - x_1) = (s-1)h$ | $h = x_1 - x_0$

$$P_n(x) = P_n(x_0 + sh) = f[x_0] + \sum_{k=1}^{n} \binom{s}{k} \Delta^k f(x_0) \quad \text{--- (2)}$$

eq^n (2) is called Newton - Gregory forward difference formula.

$$P_n(x) = f[x_0] + s\Delta f(x_0) + \frac{s(s-1)}{2!}\Delta^2 f(x_0) + \frac{s(s-1)(s-2)\Delta^3 f(x_0)}{3!} + \dots + \frac{s(s-1)\dots(s-k+1)\Delta^k f(x_0)}{k!}$$

Example :
Table of forward difference for a cubic polynomial.

x	f	Δf_i	$\Delta^2 f_i$	$\Delta^3 f_i$
x_0	$\boxed{f(x_0)}$			
		$\boxed{\Delta f(x_0)}$		
x_1	$f(x_1)$		$\boxed{\Delta^2 f(x_0)}$	
		$\Delta f(x_1)$		$\boxed{\Delta^3 f(x_0)}$
x_2	$f(x_2)$		$\Delta^2 f(x_1)$	
		$\Delta f(x_2)$		
x_3	$f(x_3)$			

8. Estimate the value of $\sin\theta$ at $\theta = 25°$, using Newton-Gregory forward difference formula with the help of the following data

θ	10	20	30	40	50
$\sin\theta$	0.1736	0.3420	0.5000	0.6428	0.7660

⇒ Soln:

Here, $h = 20 - 10 = 10$, $x = 25$

Since, $x = x_0 + sh$

$$\Rightarrow s = \frac{x - x_0}{h} = \frac{25 - 10}{10} = 1.5$$

Now, forward difference table is calculated as:

θ	$\sin\theta$	$\Delta f(x_0)$	$\Delta^2 f(x_0)$	$\Delta^3 f(x_0)$	$\Delta^4 f(x_0)$
10	0.1736				
		0.1684			
20	0.3420		-0.0104		
		0.158		-0.0048	
30	0.5000		-0.0152		0.0004
		0.1428		-0.0044	
40	0.6428		-0.0196		
		0.1232			
50	0.7660				

$$P_4(x) = f(x_0) + s\Delta f(x_0) + \frac{s(s-1)\Delta^2 f(x_0)}{2!} + \frac{s(s-1)(s-2)\Delta^3 f(x_0)}{3!} + \frac{s(s-1)(s-2)(s-3)\Delta^4 f(x_0)}{4!}$$

$$= 0.1736 + 1.5 \times 0.1684 + \frac{1.5(1.5-1)(-0.0104)}{2!} + \frac{1.5(1.5-1)(1.5-2)(-0.0048)}{3!} + \frac{1.5(1.5-1)(1.5-2)(1.5-3)\,0.000}{4!}$$

$$= 0.4226$$ //

8. Construct Newton's forward difference table for data points given in table below and then approximate the value of $f(1.1)$ by using Newton's forward difference formula.

x	1	1.3	1.6	1.9	2.2
$f(x)$	0.7651977	0.6200860	0.4554022	0.2818186	0.1103623

⇒ Soln //

$h = 1.3 - 1 = 0.3$, $x = 1.1$, $x_0 = 1$

Since, $x = x_0 + sh$

$$s = \frac{x - x_0}{h} = \frac{1.1 - 1}{0.3} = 0.3333333$$

Now, forward difference table is calculated as

x	$f(x)$	$\Delta f(x_0)$	$\Delta^2 f(x_0)$	$\Delta^3 f(x_0)$	$\Delta^4 f(x_0)$
1	0.7651977				
		-0.1451117			
1.3	0.6200860		-0.0195721		
		-0.1646838		0.0284719	
1.6	0.4554022		0.0088998		-0.0352444
		-0.1735836		-0.0067725	
1.9	0.2818186		0.0021273		
		-0.1714563			
2.2	0.1103623				

0.7168271
0.0021746
0.0017575
0.0014503

$$P_4(x) = f(x_0) + s\Delta f(x_0) + \frac{s(s-1)\Delta^2 f(x_0)}{2!} + \frac{s(s-1)(s-2)\Delta^3 f(x_0)}{3!} + \frac{s(s-1)(s-2)(s-3)\Delta^4 f(x_0)}{4!}$$

$$= 0.7651977 + 0.3333333 \times (-0.1451117) + \frac{0.3333333(0.3333333-1)(-0.0195721)}{2!} + \frac{0.3333333(0.3333333-1)(0.3333333-2)(0.0284719)}{3!} + \frac{0.3703592 \times 0.0939862}{4!}$$

$= \cancel{0.7222741} = \cancel{0.7222141}$
$= 0.7222095$

Algorithm:

1. Read the number of data points say n.
2. Read the value at which interpolated value is needed say x_p.
3. Read n data points say $x[i]$ and $fx[i]$.
4. Compute $h = x[1] - x[0]$ and $s = (x_p - x[0])/h$
5. for $i = 0$ to $n-1$
   $fd[i] = fx[i]$
   end for
6. for $i = 0$ to $n-1$
   for $j = n-1$ to $i+1$
   $fd[j] = fd[j] - fd[j-1]$
   end for
   end for
7. $v = fd[0]$
8. for $i = 1$ to $n-1$
   for $k = 1$ to i
   $p = p * (s - k + 1)$
   end for

$\frac{x - x_0}{h}$

$$v = v + \frac{fd[i] * p}{i!}$$

Reset $p = 1$

end for

9. print the value (interpolated value) v ~~set~~ up

Newton Backward Difference Interpolation:

If the table is too large and if the required point is close to the end of the table, we can use another formula known as Newton-Gregory backward difference formula.

In the case of backward difference, the reference point is x_n, instead of x_0. Therefore, we have

$$x = x_n + sh$$

$$x_k = x_n - kh$$

$$\boxed{x - x_k = (s+k)h}$$

$$\frac{x - x_n}{h} = s$$

Then, the newton-Gregory backward difference formula is given by:

$$P_n(x) = f(x_n) + s\nabla f(x_n) + \frac{s(s+1)}{2!}\nabla^2 f(x_n) + \dots$$

$$+ \frac{s(s+1)(s+2) + \dots + (s+n-1)}{n!}\nabla^n f(x_n)$$

for a given table of data, the backward difference table will be identical to the forward difference table. However, the reference point will be x_n.

Table of backward differences for five data points is given as:

x	f	∇f	$\nabla^2 f$	$\nabla^3 f$	$\nabla^4 f$
x_0	f_0				
		∇f_1			
x_1	f_1		$\nabla^2 f_2$		
		∇f_2		$\nabla^3 f_3$	
x_2	f_2		$\nabla^2 f_3$		$\boxed{\nabla^4 f_4}$
		∇f_3		$\boxed{\nabla^3 f_4}$	
x_3	f_3		$\boxed{\nabla^2 f_4}$		
		$\boxed{\nabla f_4}$			
x_4	$\boxed{f_4}$				

Algorithm (Backward difference)

1. Start
2. Read number of data points say n,
3. Enter the value at which interpolated value is required by xp
4. Read the data points
5. h = x[1] - x[0] and s = (xp - x[n-1]) / h
6. for i = 0 to n-1
   bd[i] = fx[i]
   end for
7. for i = n-1 to 1
   for j = 0 to i-1
   bd[j] = bd[j+1] - bd[j]
   end for
   end for
8. v = bd[n-1]
9. for i = 1 to n-1
   for k = 1 to i
   initialize p=1, p = p * (s + k - 1)
   end for
   $v = v + \frac{bd[n-i-1] * p}{i!}$
   reset p = 1
10. print the interpolated value (v)
11. Stop

Q. The table gives the distance in nautical miles of the visible horizon for the given heights in feet above the earth's surface:

x = height	100	150	200	250	300	350	400
y = distance	10.63	13.03	15.04	16.81	18.42	19.90	21.27

find the values of y when

1) $x = 218$ ft

2. $x = 410$ ft

⇒ Soln:

1. $h = 150 - 100 = 50$, $x = 218$ (forward)

$$s = \frac{x - x_0}{h} = \frac{218 - 100}{50} = 2.36$$

when $x = 410$

$$s = (x - x_n)/h = (410 - 400)/50 = 0.2$$ (Backward)

x	y	$\Delta f(x_0)$ / $\nabla f(x_6)$	$\Delta^2 f(x_0)$ / $\nabla^2 f(x_6)$	$\Delta^3 f(x_0)$ / $\nabla^3 f(x_6)$	$\Delta^4 f(x_0)$ / $\nabla^4 f(x_0)$	$\Delta^5 f(x_0)$ / $\nabla^5 f(x_6)$	$\Delta^6 f(x_0)$ / $\nabla^6 f(x_6)$
100	10.63						
		2.4					
150	13.03		-0.39				
		2.01		0.15			
200	15.04		-0.24		-0.07		
		1.77		0.08		0.02	
250	16.81		-0.16		-0.05		0.02
		1.61		0.03		0.04	$\nabla^6 f(x_6)$
300	18.42		-0.13		-0.01	$\nabla^5 f(x_6)$	
		1.48		0.02	$\nabla^4 f(x_6)$		
350	19.90		-0.11	$\nabla^3 f(x_6)$			
		1.37	$\nabla^2 f(x_6)$				
400	21.27	$\nabla f(x_6)$					
	$f(x_6)$						

$$P_6(218) = f(x_0) + s\Delta f(x_0) + \frac{s(s-1)\Delta^2 f(x_0)}{2!} + \frac{s(s-1)(s-2)\Delta^3 f(x_0)}{3!} +$$

$$\frac{s(s-1)(s-2)(s-3)\Delta^4 f(x_0)}{4!} + \frac{s(s-1)(s-2)(s-3)(s-4)\Delta^5 f(x_0)}{5!} +$$

$$\frac{s(s-1)(s-2)(s-3)(s-4)(s-5)\Delta^6 f(x_0)}{6!}$$

$$= 10.63 + 2.36(2.4) + \frac{2.36(2.36-1)(-0.39)}{2!} + \frac{2.36(1.36) * 0.36 * 0.15}{3!}$$

$$\frac{2.36 * 1.36 * 0.36 * (-0.64) \times -0.07}{4!} + \frac{2.36 * (-0.313) \times -1.64 * 0.02}{5!} +$$

$$\frac{1.211 * -2.64 * 0.02}{6!}$$

$$= 15.668 + 0.028 + 0.002 + 0.0002 + 0.00002$$

$$= 15.69822 \text{ nautical miles}$$

$$P_6(410) = f(x_6) + s\nabla f(x_6) + \frac{s(s+1)\nabla^2 f(x_6)}{2!} + \frac{s(s+1)(s+2)\nabla^3 f(x_6)}{3!} +$$

$$\frac{s(s+1)(s+2)(s+3)\nabla^4 f(x_6)}{4!} + \frac{s(s+1)(s+2)(s+3)(s+4)\nabla^5 f(x_6)}{5!} +$$

$$\frac{s(s+1)(s+2)(s+3)(s+4)(s+5)\nabla^6 f(x_6)}{6!}$$

$$= 21.5322 \text{ nautical miles}$$

Curve fitting: Regression

objective: less the error i.e. more effective

Least Square Approximations:

This is the most popular method of parameter estimation for coefficients of regression model. Ideally, if all the residuals (error) e_i are zero, one may have found an equation in which all the points lie on a model.

Thus, minimization of the residual is an objective of obtaining coefficients. In the least squares method, estimates of the constants of the models are choosen such that minimization of the sum of the squared residuals is achieved i.e. minimized.

$$\sum_{i=1}^{n} e_i^2$$

Linear Regression (fitting linear equation):

Fitting a straight line is simplest approach of regression analysis.

Let us consider the mathematical equation of straight line, $y = f(x) = a + bx$ where a and b are regression coefficients to be determined.

The technique of minimizing the sum of squares of errors is known as least square regression.

Let, the sum of squares of individual errors can be expressed as:

$$E = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_i - f(x_i))^2 = \sum_{i=1}^{n} (y_i - a - bx_i)^2$$

We should choose regression coefficient a and b such that E is minimum.
Necessary conditions for E to be minimum is $\frac{\partial E}{\partial a} = 0$ and $\frac{\partial E}{\partial b} = 0$

$$\Rightarrow \frac{\partial E}{\partial a} = 2 \sum_{i=1}^{n} (y_i - a - bx_i)(-1) = 0$$

$$\Rightarrow \frac{\partial E}{\partial b} = 2 \sum_{i=1}^{n} (y_i - a - bx_i)(-x_i) = 0$$

Giving,

$$-\sum_{i=1}^{n} y_i + \sum_{i=1}^{n} a + \sum_{i=1}^{n} bx_i = 0$$ and

$$-\sum_{i=1}^{n} x_i y_i + \sum_{i=1}^{n} a x_i + \sum_{i=1}^{n} b x_i^2 = 0$$

$$\sum_{i=1}^{n} y_i = na + b \sum_{i=1}^{n} x_i \quad \text{—— (1)}$$

And

$$\sum_{i=1}^{n} x_i y_i = a \sum_{i=1}^{n} x_i + b \sum_{i=1}^{n} x_i^2 \quad \text{—— (2)}$$

Now, solving eqn (1) & (2)

$$b = \frac{n \sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \cdot \sum_{i=1}^{n} y_i}{n \sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}$$

$$a = \frac{\sum_{i=1}^{n} y_i}{n} - b \frac{\sum_{i=1}^{n} x_i}{n} = \bar{y} - b\bar{x}$$

where,

$$\bar{y} = \frac{\sum_{i=1}^{n} y_i}{n} \quad \text{and} \quad \bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}.$$

Example:

Fit a straight line to the following data points.

x	1	2	3	4	5
$y = f(x)$	3	5	7	10	12

Soln:

x	$y=f(x)$	x^2	xy
1	3	1	3
2	5	4	10
3	7	9	21
4	10	16	40
5	12	25	60
$\Sigma x = 15$	$\Sigma y = 37$	$\Sigma x^2 = 55$	$\Sigma xy = 134$

$n = 5$

$$b = \frac{n\,\Sigma xy - \Sigma x\,\Sigma y}{n\,\Sigma x^2 - (\Sigma x)^2}$$

$$= \frac{5 \times 134 - 15 \times 37}{5 \times 55 - (15)^2}$$

$$= 2.3$$

$$a = \frac{\Sigma y}{n} - b\,\frac{\Sigma x}{n}$$

$$= \frac{37}{5} - 2.3 \times \frac{15}{5}$$

$$= 0.5$$

$\therefore y = a + bx$

$f(x) = y = 0.5 + 2.3x$ is the required eqn //

Algorithm:

1. Read the number of data points say n.
2. for $i = 0$ to $n-1$
   Read $x[i]$ and $y[i]$
3. for $i = 0$ to $n-1$
   $S_x = S_x + x[i]$
   $S_y = S_y + y[i]$
   $S_{xy} = S_{xy} + x[i]\, y[i]$
   $S_{x2} = S_{x2} + x[i] * x[i]$
4. Calculate
   $b = ((n * S_{xy}) - (S_x * S_y)) / ((n * S_{x2}) - (S_x * S_x))$
   $a = (S_y / n) - ((b * S_x) / (n))$
5. print the value of a and b.
6. fits in eqn $y = a + bx$ and display it.

II Polynomial Regression (fitting a polynomial function)

When a given set of data does not appear to satisfy a linear eqn, we can try a suitable polynomial as a regression curve to fit the data.

Given n data points as $(x_1, y_1), (x_2, y_2) \ldots (x_n, y_n)$

Use least square method to regress the data to an m^{th} order polynomial.

Consider a polynomial eqn as

$$y = a_0 + a_1 x + a_2 x^2 + \ldots + a_m x^m \quad \text{--- (1)} \quad (m < n)$$

The residual at each data point is given by,

$$e_i = y_i - a_0 - a_1 x_i - a_2 x_i^2 \cdots - a_M x_i^M \quad \text{—— (2)}$$

The sum of the square of residual is given by,

$$E = \sum_{i=1}^{n} e_i^2$$

$$= \sum_{i=1}^{n} (y_i - a_0 - a_1 x_i - \ldots - a_M x_i^M)^2 \quad \text{—— (3)}$$

To find the constant of the polynomial regression model, we put the derivative w.r.t a_i to zero i.e,

$$\frac{\partial E}{\partial a_0} = \sum_{i=1}^{n} 2(y_i - a_0 - a_1 x_i - \ldots - a_M x_i^M)(-1) = 0$$

$$\frac{\partial E}{\partial a_1} = \sum_{i=1}^{n} 2(y_i - a_0 - a_1 x_i - \ldots - a_M x_i^M)(-x_i) = 0$$

$$\vdots \qquad \vdots$$

$$\frac{\partial E}{\partial a_M} = \sum_{i=1}^{n} 2(y_i - a_0 - a_1 x_i - \ldots - a_M x_i^M)(-x_i^M) = 0$$

This gives,

$$n a_0 + a_1 \sum_{i=1}^{n} x_i + a_2 \sum_{i=1}^{n} x_i^2 + \ldots + a_M \sum_{i=1}^{n} x_i^M = \sum_{i=1}^{n} y_i$$

$$a_0 \sum_{i=1}^{n} x_i + a_1 \sum_{i=1}^{n} x_i^2 + a_2 \sum_{i=1}^{n} x_i^3 + \ldots + a_M \sum_{i=1}^{n} x^{M+1} = \sum_{i=1}^{n} x_i y_i$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$a_0 \sum_{i=1}^{n} x_i^M + a_1 \sum_{i=1}^{n} x_i^{M+1} + a_2 \sum_{i=1}^{n} x_i^{M+2} + \ldots + a_M \sum_{i=1}^{n} x_i^{2M} = \sum_{i=1}^{n} y_i x_i^M$$

Now, setting these equation in matrix form

$$
\begin{array}{ccc}
C & A & B \\
\begin{bmatrix}
n & \left(\sum_{i=1}^{n} x_i\right) & \cdots & \left(\sum_{i=1}^{n} x_i^m\right) \\
\left(\sum_{i=1}^{n} x_i\right) & \left(\sum_{i=1}^{n} x_i^2\right) & \cdots & \left(\sum_{i=1}^{n} x_i^{m+1}\right) \\
\vdots & \vdots & & \\
\left(\sum_{i=1}^{n} x_i^m\right) & \left(\sum_{i=1}^{n} x_i^{m+1}\right) & \cdots & \left(\sum_{i=1}^{n} x_i^{2m}\right)
\end{bmatrix}
&
\begin{bmatrix}
a_0 \\ a_1 \\ \vdots \\ a_m
\end{bmatrix}
=
&
\begin{bmatrix}
\left(\sum_{i=1}^{n} y_i\right) \\
\left(\sum_{i=1}^{n} x_i y_i\right) \\
\vdots \\
\left(\sum_{i=1}^{n} y_i x_i^m\right)
\end{bmatrix}
\end{array}
$$

i.e. $CA = B$

The above system can be solved for $a_0, a_1, \ldots a_m$.

~~II~~ Example :-

Fit a second order polynomial to the data points in the table below:

x	1	2	3	4
y	6	11	18	27

⇒ Soln://

The order of polynomial is 2 & therefore we will have 3 simultaneous eqn are :-

$$n a_0 + a_1 \Sigma x_i + a_2 \Sigma x_i^2 = \Sigma y_i$$

$$a_0 \Sigma x_i + a_1 \Sigma x_i^2 + a_2 \Sigma x_i^3 = \Sigma x_i y_i$$

$$a_0 \Sigma x_i^2 + a_1 \Sigma x_i^3 + a_2 \Sigma x_i^4 = \Sigma x_i^2 y_i$$

Here, $n = 4$

x	y	x^2	x^3	x^4	xy	x^2y
1	6	1	1	1	6	6
2	11	4	8	16	22	44
3	18	9	27	81	54	162
4	27	16	64	256	108	432
$\Sigma x = 10$	$\Sigma y = 62$	$\Sigma x^2 = 30$	$\Sigma x^3 = 100$	$\Sigma x^4 = 354$	$\Sigma xy = 190$	$\Sigma x^2y = 64$

$a_0 + 1.25 a_1 + 2.29 a_2 = 2.29$

or, $a_0 + 0 + 2.29 = 2.29$

$\therefore a_0 = 0$

The required eqn is $y = a_0 + a_1 x + a_2 x^2$

$= x^2$ //

Algorithm:

1. Start
2. Read numbers of data points n and order of polynomial Mp.
3. If $n < Mp$,
   "Regression is not possible" and stop
   else
   continue;
4. set $M = Mp + 1$
5. Compute coefficient of C matrix.
6. Compute coefficient of B matrix.
7. Solve for the coefficients $a_0, a_1, \ldots, a_m$
8. Write the coefficients.
9. Display the polynomial
10. Stop

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Exponential Regression:-

Let us consider the exponential function $y = ae^{bx}$ — ①

Taking log on both sides we get

$$\log y = \log a + bx \quad — ②$$

This equation is similar in form, to the linear equation $y = a + bx$. Thus we can evaluate parameters a and b by using the equation of linear regression model as:

$$b = \frac{n\sum_{i=1}^{n} x_i \log y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} \log y_i}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}$$

$$\log a = \frac{\sum_{i=1}^{n} \log y_i}{n} - b \cdot \frac{\sum_{i=1}^{n} x_i}{n}$$

$$\text{Let } R = \frac{\sum_{i=1}^{n} \log y_i}{n} - b \cdot \frac{\sum_{i=1}^{n} x_i}{n}$$

$$\Rightarrow a = e^R$$

Algorithm:

1. Input number of n data points.
2. for i=1 to n,
   input x[i], y[i]
3. for i=0 to n-1
   Sx = Sx + x[i];
   Slgy = Slgy + lg y[i]
   Sxlgy = Sxlgy + x[i] * lg y[i]
   Sx_2 = Sx_2 + x[i] * x[i]
4. Compute $b = \frac{(n \times Sxlgy) - (Sx * Slgy)}{(n * Sx_2) - (Sx_2 * Sx_2)}$
5. compute loga = (Slgy)/n - (b * Sx/n)
6. Compute $a = e^{a}$
7. print the equation $y = ae^{bx}$

8. Fit an exponential model to the following data set.

x	0.4	0.8	1.2	1.6	2	2.3
y	750	1000	1400	2000	2700	3750

⇒ Soln: n = 6

x	y	x^2	log y	x log y
0.4	750	0.16	2.87	1.148
0.8	1000	0.64	3	2.4
1.2	1400	1.44	3.14	3.768
1.6	2000	2.56	3.30	5.28
2	2700	4	3.43	6.86
2.3	3750	5.29	3.57	8.211
Σx = 8.3	Σy = 11600	Σx^2 = 14.09	Σ logy = 19.31	Σ x logy = 27.667

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$$b = \frac{6 \times 27.667 - 8.3 \times 19.31}{6 \times 14.09 - (8.3)^2}$$

$$= +0.365$$

$$R = \frac{19.31}{6} + \frac{0.365 \times 8.3}{6}$$

$$= 2.71625$$

$$a = e^R = e^{2.71623} = 15.12348$$

$\therefore$ The required eqn is $y = 15.12348\, e^{(0.365)x}$

8. The latent heat of vaporisation of steam r is given in the following table at different temperatures t:

t :	40	50	60	70	80	90	100	110
r :	1069.1	1063.6	1058.2	1052.7	1049.3	1041.8	1036.3	1030.8

For this range of temperature, a relation of the form $r = a + bt$ is known as to fit the data. Find the values of a and b.

$\Rightarrow$ soln:

t (x)	r (y)	t^2 (x^2)	rt (xy)
40	1069.1	1600	42764
50	1063.6	2500	53180
60	1058.2	3600	63492
70	1052.7	4900	73689
80	1049.3	6400	83944
90	1041.8	8100	93762
100	1036.3	10000	103630
110	1030.8	12100	113388
$\Sigma t = 600$	$\Sigma r =$ 8401.8	$\Sigma t^2 =$ 49200	$\Sigma rt =$ 627849