$\Rightarrow k = 1094/365$ (since it is largest

$$X = \frac{365}{1094}\begin{bmatrix} 1094/365 \\ 1093/365 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.99 \\ 0 \end{bmatrix} \approx \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

↳ eigen vector

Eigen value $= k = 1094/365 = 2.99$

Unit 5

Solutions of Ordinary Differential Equations:

Mathematical models are based on empirical observations that describes the changes in the states of systems are often expressed in terms of not only certain system parameters but also their derivatives, such mathematical models, which use differential calculus to express relationship between variables are known as differential equations.

Example:

1. Newton's law of cooling,

$$\frac{dT(t)}{dt} = k(T_s - T(t))$$

where, T_s is the temperature of surrounding $T(t)$ is the temperature of liquid at time t and k is the constant of proportionality.

2) Law of motion:
Law governing the velocity $v(t)$ of a moving body is $m\frac{d\,v(t)}{dt} = F$, where m is the mass of the body and F is the force acting on it.

Types of variable (Dependent & Independent)
The quantity being differentiated is called dependent variable and the quality with respect to which the dependent variable is differentiated is called independent variables.

Types of differential Equations:
If there is only one independent variable, then equation is called an Ordinary Differential equation. If it contains two or more independent variables, then the derivative will be partial and equation is called partial differential equation.

Example:
$\frac{dT(t)}{dt} = k(T_s - T_{t})$ belongs to the class of ODE.

$$\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial y^2} = \cancel{f(x)}\, f(x, y)$$ is the class of PDE

Order of Differential Equations:

The order of differential equation is the highest derivative that appears in the equation. When the equation contains only a first derivative it is called first order differential equation. On the other hand if the ~~equation is called~~ highest derivative is ~~cat~~ second derivative, the equation is called second order differential equation.

A first order differential eq^n can be expressed in the form

$$\frac{dy}{dx} = f(x, y)$$

A second order equation can be expressed in the form $y'' = f(x, y, y')$ and so on.

Degree of Differential Equations:

The degree of differential equation is the power of the highest-order derivative.

eg:- $xy'' + y^2 y' = 2x + 3$ is first degree, second order equation.

While,

H.W (Boundary value problems)

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$(y''')^2 + 5y' = 0$ is second degree, third order equation.

Initial Value problem (IVP)

In order to obtain the values of the constants we need additional information, eg: the solution $y = ae^{x}$ to the equation $y' = y$. If we are given a value of y for some x, the constant 'a' can be determined.

Suppose $y=1$ at $x=0$ then

$y(0) = ae^{0} = 1$

$\therefore a = 1$, $y = e^{x}$

If the order of equation is n, we will have to obtain n constants and therefore, we need n conditions in order to obtain a solution when all the conditions are specified at a particular value of independent variable x, then the problem is called initial value problem.

Boundary Value Problem (BVP):

It is not always necessary to specify the conditions at one point of the independent variable. They can be specified at different points in the interval (a, b) and therefore, such problems are called the boundary

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value problems.

Here, we seek solutions at specified points within the domain of given boundaries, for instance, given

$$\frac{d^2y}{dx^2} = f(x, y, y')$$

$$y(a) = y_a, \quad y(b) = y_b$$

we are interested in finding the values of y in the range $a \leq x \leq b$.

$y' = 1 + xy$
$y'' = y + xy'$
$y''' = 2y' + xy''$

0.1 0.1

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Taylor Series Method:-

Taylor series expansion of function $y(x)$ about a point $x = x_0$ is given by the relation,

$$y(x) = y(x_0) + (x - x_0) y'(x_0) + (x - x_0)^2 \frac{y''(x_0)}{2!} + (x - x_0)^3 \frac{y'''(x_0)}{3!} + \dots + (x - x_0)^n \frac{y^n(x_0)}{n!} \quad \text{--- (1)}$$

The value of the function $y(x)$ can be calculated if we know the values of its derivatives at some point. Thus if we are given the equation $y' = f(x, y)$ we can differentiate it repeatedly and evaluate them at $x = x_0$. Finally these values can be substituted in eqn (1) to obtain $y(x)$

ie if $y' = f(x, y)$ then

$$y'' = \frac{d}{dx}(dy/dx)$$

$$= \frac{d(f(x, y))}{dx}$$

$$= \frac{\partial [f(x, y)]}{\partial x} + \frac{\partial [f(x, y)]}{\partial y} \frac{dy}{dx}$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f$$

$$= f_x + f_y \cdot f$$

where, f denotes the function $f(x,y)$, f_x denotes partial derivative of $f(x,y)$ wrt x and f_y denotes partial derivatives of $f(x,y)$ wrt y.
Similarly, we can calculate y''' as

$$y''' = f_{xx} + 2f f_{xy} + f^2 f_{yy} + f_x f_y + f f_y^2$$

$$\frac{d(f_x + f_y \cdot f)}{dx}$$

$$= \frac{d f_x}{dx} + \frac{d(f_y \cdot f)}{dx}$$

$$= f_{xx} + f \cdot \frac{d f_y}{dx} + f_y \frac{df}{dx}$$

Example:
Given y'

$$= f_{xx} + f \cdot \frac{df_y}{dy} \frac{dy}{dx} + f_y \cdot \frac{df}{dx}$$

$$= f_{xx} + f \cdot f_y^2 + f_y f_x$$

$$\frac{df_x}{df} \frac{df}{dyx} + f \frac{df_y}{dx} + f_y \frac{df}{dx}$$

$$= f_{xx} \cdot f_{yx} + f \cdot \frac{df_y}{df} \cdot \frac{df}{dx} + f_y \cdot \frac{df}{dx}$$

$$= f_{xx} \cdot f_{yx} + f \cdot f_{yy} \cdot f_x + f_y \cdot f_x$$

Example:

Given $y' = x - y^2$ with initial condition $y = 1$, when $x_0 = 0$. Find the value of y for ~~$x = 0.1$~~. $x = 0.1$

$\Rightarrow$ Soln:

$y' = x - y^2 \Rightarrow dy/dx$

$y'' = \cancel{\frac{\partial f}{\partial x}}\ f_x + f_y . f$

$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} . f$

$= \frac{\partial (x - y^2)}{\partial x} + \frac{\partial (x - y^2)}{\partial y} (x - y^2)$

$= 1 - 2y(x - y^2)$

$= 1 - 2y y'$

$y''' = \frac{d}{dx}(1 - 2y y')$

$= 0 - 2\frac{dy}{dx} \times y' = -\left[2y\left(\frac{dy'}{dx}\right) + y' . \frac{d(2y)}{dx}\right]$

$= -2y' \times y' = -2yy'' - 2y'^2$

$= -2y'^2$

$= -2y(1 - 2y(x - y^2)) - 2(x - y^2)^2$

when $x_0 = 0, y = 1$

$y' = 0 - 1^2 = -1$

$y'' = 1 - 2 \times 1 \times -1 = 1 + 2 = 3$

$y''' = -2 \times 1 \times 3 - 2 \times (-1)^2 = -6 - 2 = -8$

When ~~$x = 0$ and~~ $y = 1$ $x = 0.1$

$y(0.1) = y(x_0) + (x - x_0) y'(x_0) + (x - x_0)^2 \frac{y''(x_0)}{2!} + (x - x_0)^3 \frac{y'''(x_0)}{3!}$

$y(0) = 1$

$= 1 + (0.1 - 0) \times -1 + (0.1 - 0)^2 \times \frac{3}{2} + (0.1)^3 \times \frac{-8}{6}$

$= 1 + 0.1 + 0.015 - 0.00133$

$= 0.913$

8. Given, $y' = x^2 + y^2$ with initial condition $y(0.5) = 1$ when $x_0 = 0.5$. Find y for $x = 0.25$

⟹ Soln:

$y' = x^2 + y^2$

$y'' = \frac{d(x^2 + y^2)}{dx}$

$= \frac{d(x^2)}{dx} + \frac{d(y^2)}{dx} \cdot \frac{dy}{dx}$

$= 2x + 2y \cdot y'$

$y''' = \frac{d(2x + 2y \cdot y')}{dx}$

$= \frac{d(2x)}{dx} + 2y \frac{d(y')}{dx} + y' \frac{d(2y)}{dx}$

$= 2 + 2y \cdot y'' + 2y'^2$

When $x_0 = 0.5$ & $y = 1$

$y' = 0.5^2 + 1^2 = 1.25$

$y'' = 2 \times 0.5 + 2 \times 1 \times 1.25 = 3.5$

$y''' = 2 + 2 \times 1 \times 3.5 + 2 \times (1.25)^2 = 12.125$

Now, $x = 0.25$ $(0.25 - 0.5)$

$y(0.25) = 1 + (-0.25) \times 1.25 + 0.0625 \times \frac{3.5}{2} + (-0.015625) \times \frac{12.125}{6} = 0.765$

Picard's Method :-

Consider a differential equation $y' = \frac{dy}{dx} = f(x,y)$ — ① with Initial condition $y = y_0$ for $x = x_0$

Integrating the differential eqⁿ,

$$\int_{x_0}^{x_1} y' = \int_{x_0}^{x_1} f(x,y)\,dx = y(x_1) - y(x_0)$$

$$= \int_{x_0}^{x_1} f(x,y)\,dx$$

$$\Rightarrow y(x_1) = y(x_0) + \int_{x_0}^{x_1} f(x,y)\,dx \quad — ②$$

in Equation ② y appears under the Integral sign, so Integration can not be done directly, so either we need to replace y by constant or function of x, we know that,

$$y = y_0 \text{ for } x = x_0$$

Thus, the first approximation $y^{(1)}$ is obtained by replacing $f(x,y)$ in RHS of eqⁿ ② y by y_0 in and integrating wrt x.

$$y^{(1)} = y_0 + \int_{x_0}^{x_1} f(x, y_0)\,dx$$

The second approximation $y^{(2)}$ is obtained by replacing $f(x,y)$ in RHS of eqⁿ ② y by $y^{(1)}$ in and Integrating wrt x, we get

$$y^{(2)} = y_0 + \int_{x_0}^{x_1} f(x, y^{(1)})\,dx$$

proceeding in the same way we obtain

$y^{(3)}, y^{(4)}, \ldots y^{(n)}$ where,

$$y^{(n)} = y_0 + \int_{x_0}^{x_1} f(x, y^{(n-1)})\, dx \quad \text{--- (3)}$$

with $y^0 = y_0$.

Example:-

Solve the equation $y' = x^2 + y^2$ by using Picard's method with initial condition $y(0) = 0$. Find value of $y(0.2), y(0.1), y(1)$.

⇒ Solⁿ:

$x_0 = 0, \; y_0 = 0$

$f(x, y_0) = x^2 + 0^2 = x^2$

~~Computing~~

$$y' = y_0 + \int_{x_0}^{x_1} f(x, y_0)\, dx$$

$$= 0 + \int_{x_0}^{x_1} x^2\, dx$$

$$= \left[\frac{x_1^3}{3}\right]_{x_0}^{x_1}$$

$$= \frac{x_1^3}{3}.$$

$f(x, y_1)$

$$= x^2 + \left(\frac{x_1^3}{3}\right)^2$$

$$= x^2 + \frac{x_1^6}{9}$$

$$y' = \frac{x_1^3}{3}$$

$$y^2 = y_0 + \int_{x_0}^{x_1} f(x, y^{(1)})\,dx = 0 + \int_{x_0}^{x_1} x^2 + \left(\frac{x_1^3}{3}\right)^2 dx$$

$$= \int_{x_0}^{x_1} x^2 + \frac{x_1^6}{9}\,dx$$

$$= \int_{x_0}^{x_1} x^2\,dx + \int_{x_0}^{x_1} \frac{x_1^6}{9}\,dx$$

$$= \left[\frac{x^3}{3}\right]_{x_0}^{x_1} = \frac{x_1^3}{3} + \frac{x_1^7}{63}$$

For $y(0.2)$

$$y^{(1)} = \frac{x_1^3}{3} = \frac{(0.2)^3}{3} = 0.0026$$

$$y^{(2)} = \frac{x_1^3}{3} + \frac{x_1^7}{63} = \frac{(0.2)^3}{3} + \frac{(0.2)^7}{63} = 0.0026$$

For $y(0.1)$

$$y^{(1)} = \frac{(0.1)^3}{3} = 0.00033$$

$$y^{(2)} = \frac{(0.1)^3}{3} + \frac{(0.1)^7}{63} = 0.00033$$

For $y(1)$

$$y^{(1)} = \frac{1^3}{3} = 0.3333$$

$$y^{(2)} = \frac{1^3}{3} + \frac{1^7}{63} = 0.3491$$

Q. Use the Picards process of successive approximations, obtain a solution upto the fifth approximation of the eq^n $\frac{dy}{dx} = y + x$, such that $y=1$ when $x=0$.

⇒ Sol^n:

$$f(x, y_0) = 1 + x$$

$$y^{(1)} = y_0 + \int_{x_0}^{x_1} f(x, y_0)\, dx$$

$$= 1 + \int_{x_0}^{x_1} 1 + x \quad dx$$

$$= 1 + \left[x + \frac{x^2}{2} \right]_{x_0}^{x_1}$$

$$= 1 + x_1 + \frac{x_1^2}{2}$$

$$y^{(2)} = y_0 + \int_{x_0}^{x_1} f(x, y^{(1)})\, dx$$

$$= 1 + \int_{x_0}^{x_1} \left(x + 1 + x_1 + \frac{x_1^2}{2} \right) dx$$

$$= 1 + \left[\frac{x^2}{2} + x + \frac{x_1^2}{2} + \frac{x_1^3}{2 \times 3} \right]_0^{x_1} \cancel{dx}$$

$$= 1 + \frac{x_1^2}{2} + x_1 + \frac{x_1^2}{2} + \frac{x_1^3}{6}$$

$$= 1 + x_1 + x_1^2 + \frac{x_1^3}{6}$$

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$$y^{(3)} = y_0 + \int_{x_0}^{x_1} f(x_1, y^{(2)})\, dx$$

$$= 1 + \int_{x_0}^{x_1} \left(x + 1 + x_1 + x_1^2 + \frac{x_1^3}{6}\right) dx$$

$$= 1 + \left[\frac{x^2}{2} + x + \frac{x_1^2}{2} + \frac{x_1^3}{6} + \frac{x_1^4}{24}\right]_0^{x_1}$$

$$= 1 + x_1 + \frac{x_1^2}{2} + \frac{x_1^2}{2} + \frac{x_1^3}{6} + \frac{x_1^4}{24}$$

$$= 1 + x_1 + x_1^2 + \frac{x_1^3}{6} + \frac{x_1^4}{24}$$

$$y^{(4)} = y_0 + \int_{x_0}^{x_1} f(x, y^{(3)})\, dx$$

$$= 1 + \int_{x_0}^{x_1} \left(x + 1 + x_1 + x_1^2 + \frac{x_1^3}{6} + \frac{x_1^4}{24}\right) dx$$

$$= 1 + x_1 + \frac{x_1^2}{2} + \frac{x_1^2}{2} + \frac{x_1^3}{3} + \frac{x_1^4}{12} + \frac{x_1^5}{120}$$

$$= 1 + x_1 + x_1^2 + \frac{x_1^3}{3} + \frac{x_1^4}{12} + \frac{x_1^5}{120}$$

$$y^{(5)} = y_0 + \int_{x_0}^{x_1} f(x, y^{(4)})\, dx$$

$$= 1 + \int_{x_0}^{x_1} \left(x + 1 + x_1 + x_1^2 + \frac{x_1^3}{3} + \frac{x_1^4}{12} + \frac{x_1^5}{120}\right) dx$$

$$= 1 + x_1 + \frac{x_1^2}{2} + \frac{x_1^2}{2} + \frac{x_1^3}{3} + \frac{x_1^4}{12} + \frac{x_1^5}{60} + \frac{x_1^6}{720}$$

$$= 1 + x_1 + x_1^2 + \frac{x_1^3}{3} + \frac{x_1^4}{12} + \frac{x_1^5}{60} + \frac{x_1^6}{720}$$

Euler's Method:

This is simplest and oldest method. It illustrates the basic idea of those numerical methods which seek to determine the change Δy in y corresponding to a small increase in the arguments x.

Consider ODE of the form,

$$y' = \frac{dy}{dx} = f(x, y) \text{ with } y(0) = y_0 \quad \text{—} (1)$$

We wish to solve eq^n (1) for the values of y at $x = x_i$ where $x_i = x_0 + ih$, $i = 1, 2, 3 \ldots$

Now, Integrate eq^n (1) we get

$$\int_{x_0}^{x_1} y' = \int_{x_0}^{x_1} f(x, y)\, dx$$

$$\Rightarrow y(x_1) = y(x_0) + \int_{x_0}^{x_1} f(x, y)\, dx \quad \text{—} (2)$$

Assuming that $f(x, y) \simeq f(x_0, y_0)$ in the range $x_0 \le x \le x_1$ eq^n (2) can be written as,

$$y(x_1) \simeq y(x_0) + \underbrace{f(x_0, y_0)}_{\text{const.}} \int_{x_0}^{x_1} dx$$

$$= y(x_0) + f(x_0, y_0)(x_1 - x_0)$$

$$= y(x_0) + hf(x_0, y_0), \quad \text{where } h = x_1 - x_0$$

Similarly for the range $x_1 \le x \le x_2$ eq^n (2) can be written as,

$$y(x_2) \simeq y(x_1) + hf(x_1, y_1)$$

Generalizing this we can get,

$$y(x_{i+1}) = y(x_i) + hf(x_i, y_i) \quad \text{—} \ (3)$$

This eqn (3) is called Euler's eqn.

Q. Example:

Approximate the solution of IVP, $y' = 2x + y$, $y(0) = 1$ by using Euler method with step size 0.1. Calculate upto 5^{th} step.

Soln: Given that,

$\Rightarrow$ $x_0 = 0$, $y_0 = 1$, $h = 0.1$, $y(x_0) = 1$

$y' = 2x + y$

Now,

$$y(x_1) = y(x_0) + \int_{x_0}^{x_1} f(x, y)\, dx$$

$$= 1 + \int_{x_0}^{x_1} (2\times 0 + 1)\, dx$$

$$= 1$$

$y(x_1) = y(x_0) + hf(x_0, y_0)$, $\quad x_1 = x_0 + h$

$= 1 + 0.1(2\times 0 + 1)$ $\quad = 0 + 0.1$

$= 1 + 0.1$ $\quad = 0.1$

$= 1.1$

$0+0.1, 1.1$

$y(x_2) = y(x_1) + hf(x_1, y_1)$

$= 1.1 + 0.1(2\times 0.1 + 1.1)$

$= 1.23$

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$x_2 = x_1 + h = 0.1 + 0.1 = 0.2$

$y(x_3) = y(x_2) + h f(x_2, y_2)$
$= 1.23 + 0.1 (2 \times 0.2 + 1.23)$
$= 1.393$

$x_3 = x_2 + h = 0.2 + 0.1 = 0.3$

$y(x_4) = y(x_3) + h f(x_3, y_3)$
$= 1.393 + 0.1 (2 \times 0.3 + 1.393)$
$= 1.5923$

$x_4 = x_3 + h = 0.3 + 0.1 = 0.4$

$y(x_5) = y(x_4) + h f(x_4, y_4)$
$= 1.5923 + 0.1 (2 \times 0.4 + 1.5923)$
$= 1.83153$

Heun's Method:

It is clear that in Euler's method the slope of (x_i, y_i) is used to estimate the value of $y(x_{i+1})$ as:

$y(x_{i+1}) = y(x_i) + M_1 h$, where $M_1 = f(x_i, y_i)$, (1)

Alternatively, we can use the line that is parallel to the tangent at (x_{i+1}, y_{i+1}) estimate the value of $y(x_{i+1})$ as,

$y(x_{i+1}) = y(x_i) + M_2 h$, where $M_2 = f(x_{i+1}, y_{i+1})$ (2)

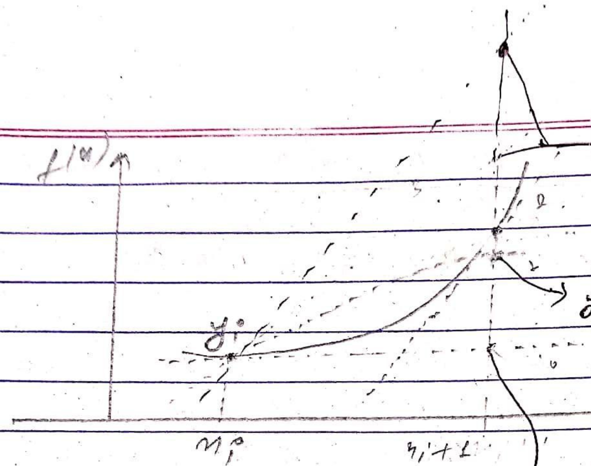

fig: Geometrical illustration of Heun's method

From figure, estimate given by the eqⁿ ① appears to be underestimated whereas the estimate given by eqⁿ ② seems to be overestimated. Better estimate of $y(x_{i+1})$ is given by Heun's method.

The main idea behind the Heun's method is to use the average of the slopes computed at the beginning and at the end of interval (i.e. average of slopes of m_1, m_2).

Thus by using Heun's method we can estimate the value of $y(x_{i+1})$ as

$$y(x_{i+1}) = y(x_i) + \frac{h}{2}(m_1 + m_2)$$

$$\Rightarrow y(x_{i+1}) = y(x_i) + \frac{h}{2}\left(f(x_i, y_i) + f(x_{i+1}, y_{i+1})\right) \quad ③$$

Since, $y(x_{i+1})$ appears on both sides of eqn ③ it cannot be evaluated until the value of $y(x_{i+1})$ inside the function $f(x_{i+1}, y_{i+1})$ is available. This value $y(x_{i+1})$ inside the function $f(x_{i+1}, y_{i+1})$ can be predicted by using Euler's formula as,

$$y(x_{i+1}) = y(x_i) + h f(x_i, y_i)$$

Thus, the Heun's formula given in eqn ③ becomes.

$$y(x_{i+1}) = y(x_i) + \frac{h}{2}\left[f(x_i, y_i) + f\{x_{i+1}, y(x_i) + h f(x_i, y_i)\}\right] \quad \text{—④}$$

Example:

Approximate the solution of the IVP, $y' = 2x + y$, $y(0) = 1$, by using Heun's method with step size of $\underline{0.1}$ upto 4^{th} iteration.

⇒ Sol^n: Iteration 1:

$y_0 = 1$

$x_0 = 0$

$y' = 2x + y$

$h = 0.1$

$y_1 = y(x_1) + h f(x_0, y_0)$
$= 1 + 0.1 \times 1$
$= 1 + 0.1$
$= 1.1$

$x_1 = 0.1$

Now, $m_1 = f(x_i, y_i) = f(x_0, y_0) =$
$= 1$

$m_2 = f(x_1, y_1) = 2 \times 0.1 + 1.1$ $[x_1 = x_0 + h = 0.1]$
$= 1.3$

$f(x_{i+1}, y_{i+1})$

$y(x_1) = y(x_0) + \frac{h}{2}(m_1 + m_2)$

$= 1 + \frac{0.1}{2}(1 + 1.3)$

$= 1.115$

Iteration 2:

$x_1 = 0.1, \; y_1 = 1.115$

$m_1 = f(x_1, y_1) = f(0.1, 1.115)$

$= 1.315$

$m_2 =$

$x_2 = 0.1 + 0.1 = 0.2 \; [\because x_1 + h]$

$y_2 = y(x_1) + h f(x_1, y_1)$

$= 1.115 + 0.1 \times 1.315$

$= 1.2465$

$m_2 = f(x_2, y_2) = f(0.2, 1.2465) = 2 \times 0.2 + 1.2465$

$= 1.6465$

$\therefore y(x_2) = y(x_1) + \frac{h}{2}(m_1 + m_2)$

$= 1.115 + \frac{0.1}{2}(1.315 + 1.6465)$

$= 1.2630$

Iteration 3:

$x_2 = 0.2, \; y_2 = 1.2465$

$m_1 = f(x_2, y_2) = 2 \times 0.2 + 1.2465 = 1.6465$

$x_3 = \cancel{0.1}\, x_2 + h = 0.2 + 0.1 = 0.3$

$y_3 = y(x_2) + h f(x_2, y_2) = 1.2630 + 0.1 \times 1.6465$

$= 1.4276$

$m_2 = f(x_3, y_3) = 2 \times 0.3 + 1.4276$

$= 2.0276$

$\therefore y(x_3) = y(x_2) + \frac{h}{2}(m_1 + m_2)$

$= 1.2630 + \frac{0.1}{2}(1.6465 + 2.0276)$

$= 1.4467$

Iteration 4:

$x_3 = 0.3 \quad , y_3 = 1.4276$

$m_1 = f(x_3, y_3) = 2 \times 0.3 + 1.4276 = 2.0276$

$x_4 = x_3 + h = 0.3 + 0.1 = 0.4$

$y_4 = y(x_3) + h\, f(x_3, y_3)$

$= 1.4467 + 0.1 \times 2.0276$

$= 1.6494$

$m_2 = f(x_4, y_4) = 2 \times 0.4 + 1.6494$

$= 2.4494$

$\therefore y(x_4) = y(x_3) + \frac{h}{2}(m_1 + m_2)$

$y(0.4)$

$= 1.4467 + \frac{0.1}{2}(2.0276 + 2.4494)$

$= 1.6698$

(RK fourth order method)

1. Runge-Kutta Method :-

Heun's method can be further refined by replacing the average slope of two points with a slope that is average of $f(x,y)$ at four points within the interval. This is refinement in the Heun's method improves the order of approximation from h^2 to h^4.

Let, Given initial-value problem,

$y' = f(x,y)$, $y(x_0) = y_0$, for a fixed constant value of $h = y(x_n+h)$ can be approximated by,

$$y(x_n+h) = y_{n+1} = y_n + \frac{1}{6}h(m_1+2m_2+2m_3+m_4)$$

where,

$m_1 = f(x_i, y_i)$ [slope at (x_i, y_i)]

$m_2 = f(x_i + \frac{1}{2}h, y_i + \frac{h}{2}m_1)$ [slope at midpoint of the interval along the line connecting (x_i, y_i) and x_i+h, y_i+hm_1]

$m_3 = f(x_i + \frac{1}{2}h, y_i + \frac{h}{2}m_2)$ [slope at the midpoint of the interval along the line connecting (x_i, y_i) and x_i+h, y_i+hm_2]

$m_4 = f(x_i+h, y_i+hm_3)$ [slope at (x_i+h, y_i+hm_3)]

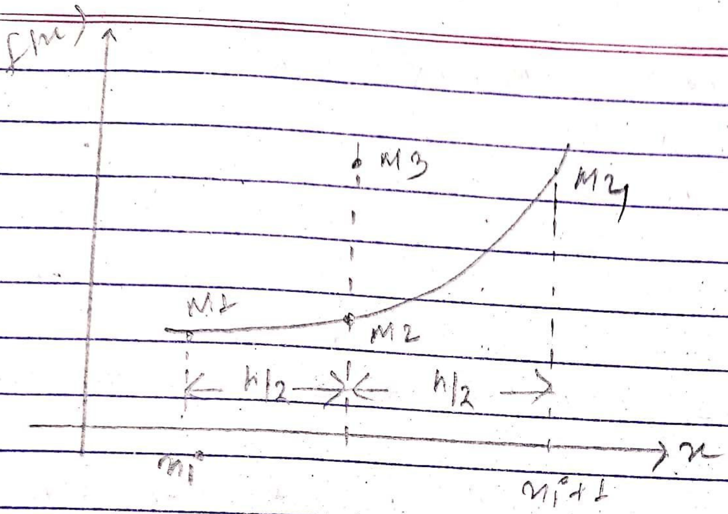

fig: Geometrical representation of RK method.

Q. Use Runge-kutta method to estimate $y(0.4)$ if $y' = 2x+y$, $y(0) = 1$

⇒ Soln:

$y_0 = 1$, $x_0 = 0$, $h = 0.4$

$y' = 2x + y$

Now,

$m_1 = f(x_0, y_0) = 1$

$m_2 = f(x_0 + \frac{1}{2} \times 0.4,\ y_0 + \frac{0.4}{2} \times 1)$

$= f(0+0.2,\ 1+0.2)$

$= f(0.2, 1.2)$

$= 2 \times 0.2 + 1.2$

$= 1.6$

$m_3 = f(0 + \frac{1}{2} \times 0.4,\ 1 + \frac{0.4}{2} \times 1.6)$

$= f(0.2, 1.32)$

$= 2 \times 0.2 + 1.32$

$= 1.72$

$m_4 = f(0+0.4, 1+0.4\times 1.72)$
$= f(0.4, 1.688)$
$= 2\times 0.4 + 1.688$
$= 2.488$

$y(x_0+h)$ ~~$y(x_0+$~~

$y(x_1) = \overset{y_1}{\cancel{y(0.4)}} = y_0 + \frac{1}{6}h(m_1 + 2m_2 + 2m_3 + m_4)$

$= 1 + \frac{0.4}{6}(1 + 2\times 1.6 + 2\times 1.72 + 2.488)$

$= 1.6752$

Q. Use Rk method of order four to find y at $x = 1.1$ and 1.2 by solving $y' = x^2 + y^2$, $y(1) = 2.3$

⇒ Sol^n:

$x_0 = 1$, $y_0 = 2.3$
$h = 0.1$ $[\because 1.1 - 1 = 0.1, 1.2 - 1.1 = 0.1]$
$y' = x^2 + y^2$

For $x = 1.1$

$m_1 = f(x_0, y_0) = 1^2 + (2.3)^2 = 6.29$
$m_2 = f(x_0 + \frac{1}{2}\times 0.1, y_0 + \frac{0.1}{2}\times 6.29)$

$= f(1.05, 2.6145)$
$= 7.93$
$m_3 = f(1.05, 2.696) =$
$= 8.37$

$m_4 = f(x_0 + 0.1,\ 2.3 + 0.1 \times 8.37)$

$= f(1.1,\ 3.137)$

$= 11.05$

$y(1.1) = y_0 + \frac{1}{6} \times 0.1\ (6.29 + 2 \times 7.93 + 2 \times 8.37 + 11.05)$

$= 3.13$

For $x = 1.2$

$m_1 = f(x_1, y_1)$

$= f(1.1,\ 3.13)$

$= 11.0069$

$m_2 = f(1.15,\ 3.6803)$

$= 14.86$

$m_3 = f(1.15,\ 3.873)$

$= 16.32$

$m_4 = f(1.2,\ 4.762)$

$= 24.11$

$y(1.2) = 3.13 + \frac{1}{6} \times 0.1\ (11.0069 + 2 \times 14.86 + 2 \times 16.32 + 24.11)$

$= 4.75$ //

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1 Q. Apply RK fourth order method to find an approximate value of y when $x = 0.2$, given that $\frac{dy}{dx} = x+y$ and $y=1$ when $x=0$

2 Q. Using RK method of fourth order, solve $\frac{dy}{dx} = \frac{y^2-x^2}{y^2+x^2}$ with $y(0)=1$ at $x=0.2, 0.4$

1 => Soln:

$y_0 = 1$, $x_0 = 0$

$h = 0.2$

$\frac{dy}{dx} = y' = x+y$

Now,

$m_1 = f(x_0, y_0)$

$= 1$

$m_2 = f(x_0 + \frac{1}{2} \times h, y_0 + \frac{h}{2} \times m_1)$

$= f(0 + \frac{0.2}{2}, 1 + \frac{0.2}{2} \times 1)$

$= f(0.1, 1.1)$

$= 1.2$

$m_3 = f(0.1, 1.12)$

$= 1.22$

$m_4 = f(0.2, 1.244)$

$= 1.444$

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$y(0.2) = \cancel{y_0}1 + \frac{0.2}{6}(1 + 2\times1.2 + 2\times1.22 + 1.444)$

$= 1.2428$

2.2) Soln:

$x_0 = 0,\ y_0 = 1,\ h = 0.2$

$y' = \frac{y^2 - x^2}{y^2 + x^2}$

Now,

$m_1 = f(0, 1) = \frac{1^2 - 0^2}{1^2 + 0^2} = 1$

$m_2 = f(0 + \frac{0.2}{2}, 1 + \frac{0.2\times1}{2})$

$= f(0.1, 1.1)$

$= 0.9836$

$m_3 = f(0.1, 1.09836)$

$= 0.9835$

$m_4 = f(0 + 0.2, 1 + 0.2\times0.9835)$

$= (0.2, 1.1967)$

$= 0.9456$

$y(0.2) = 1 + \frac{0.2}{6}(1 + (0.9836 + 0.9835)2 + 0.9456)$

$= 1.196$

For $x = 0.4$

$m_1 = f(0.2, 1.196)$
$= 0.9455$

$m_2 = f\left(0.2 + \frac{0.2}{2}, 1.196 + \frac{0.2}{2} \times 0.9455\right)$

$= f(0.3, 1.29055)$
$= 0.897$

$m_3 = f(0.3, 1.2857)$
$= 0.896$

$m_4 = f(0.3 + 0.2, 1.2857 + 0.2 \times 0.896)$
$= f(0.4, 1.4649)$
$= 1$

$y(0.4) = 1.196 + \frac{0.2}{6}(0.9455 + 2 \times 0.897 + 2 \times 0.896 + 1)$

$= 1.380$

System of ordinary Differential Equation

Many mathematical problems require solving the system of several first order differential equations represented as:

$$\frac{dy_1}{dx} = f_1(x, y_1, y_2, \ldots, y_n)$$

$$\frac{dy_2}{dx} = f_2(x, y_1, y_2, \ldots, y_n)$$

$$\vdots \qquad \vdots$$

$$\frac{dy_n}{dx} = f_n(x, y_1, y_2, \ldots, y_n)$$

The solution of such a system requires that n initial conditions be known as starting value of x. Single method can be used to solve systems of ODE's as well.

Example:- Solve following two simultaneous first order differential equations.

$$\frac{dy}{dx} = z = f_1(x, y, z), \quad y(0) = 1$$

$$\frac{dz}{dx} = e^{-x} - 2z - y = f_2(x, y, z), \quad z(0) = 2$$

Use Euler method to find $y(0.75)$ & $z(0.75)$ with step size 0.25 upto iteration 3

Soln:

$$[\because y(x_{i+1}) = y(x_i) + h f(x_i, y_i, z_i)]$$

~~Iteration 1:~~

$x_0 = 0$, $y_0 = 1$, $h = 0.25$

$x_0 = 0$, $z_0 = 2$

~~$y' = z$~~ $y' = z$, $z' = e^{-x} - 2z - y$

From Euler's formula:

$y(x_{i+1}) = y(x_i) + h f(x_i, y_i, z_i)$

Iteration 1:

$y_1 = y(x_0) + h f(x_0, y_0, z_0)$

$= 1 + 0.25 f(0, 1, 2)$ [$\because y' = z$]

$= 1 + 0.25 \times 2$

$y(x_1) = 1.5$

$z_1 = z(x_0) + h f(x_0, y_0, z_0)$

$= 2 + 0.25 f(0, 1, 2)$

$= 2 + 0.25 (e^{-0} - 2 \times 2 - 1)$

$= 2 + 0.25 \times -4$

$z(0.25) = 1$

Iteration 2: ~~$h = 0.25$~~ $h = 0.25$, $x_1 = 0.25$

$y_2 = y(x_1) + h f(x_1, y_1, z_1)$

$y(x_2) = 1.5 + 0.25 f(0.25, 1.5, 1)$

$= 1.5 + 0.25 \times 1$

$y(0.5) = 1.75$

$z_2 = z(x_1) + h f(x_1, y_1, z_1)$

$= 1 + 0.25 f(0.25, 1.5, 1)$

$= 1 + 0.25 (e^{-0.25} - 2 \times 1 - 1.5)$

$= 1 + 0.25 \times -2.72$

$z(0.5) = 0.32$

Iteration 3: $h = 0.25$, $x_2 = 0.5$

$y_3 = y(x_2) + hf(x_2, y_2, z_2)$

$= 1.75 + 0.25(0.5, 1.75, 0.32)$

$= 1.75 + 0.25 \times 0.32$

$y(0.75) = 1.83$

$z_3 = z(x_2) + hf(x_2, y_2, z_2)$

$= 0.32 + 0.25 f(0.5, 1.75, 0.32)$

$= 0.32 + 0.25 (e^{-0.5} - 2 \times 0.32 - 1.75)$

$= 0.32 + 0.25 \times -1.78$

$z(0.75) = -0.125$

Higher order differential equations:

A n^{th} order differential eq^n has the form $a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots +$

$a_1 \frac{dy}{dx} + a_0 y = f(x)$ with $n-1$

Initial conditions can be solved assuming $y(x_0) = a_1$, $y'(x_0) = a_2$,

$y''(x_0) = a_3, \dots, y^{n-1}(x_0) = a_n$

Let us denote,

$y = z_1$, $\frac{dy}{dx} = z_2$, $\frac{d^2y}{dx^2} = z_3, \dots \frac{d^{n-1}y}{dx^{n-1}} = z_n$

Then, above equations represent n first order differential equation as:

$\frac{dz_1}{dx} = z_2$, $z_1(x_0) = a_1$

$\frac{dz_2}{dx} = z_3$, $z_2(x_0) = a_2$

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$$\frac{dz_n}{dx} = f(x, z_1, z_2, \dots, z_m), \quad z_n(x_0) = a_n$$

Each of the n first order ODE are associated with one initial condition. These first order ODE are simultaneous in nature and hence can be solved by the methods used for solving system of first order ODE.

Example: Re-write the following differential equation as a set of first order differential eqn.

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = e^{-x}, \quad y(0) = 1, \quad \frac{dy}{dx}(0) = 2$$

Find using Heun's method, $y(0.5)$ with step size 0.25.

Soln:

Let, $\frac{dy}{dx} = z$

Then, $\frac{dz}{dx} + 2z + y = e^{-x}$

$$\frac{dz}{dx} = e^{-x} - 2z - y$$

So, the simultaneous first order differential eqns are,

$$\frac{dy}{dx} = z, \quad y(0) = 1 \quad \text{— (i)}$$

$$\frac{dz}{dx} = e^{-x} - 2z - y, \quad z(0) = 2 \quad \text{— (ii)}$$

$y_{i+1} = y_i + \frac{h}{2}(m_1(1) + m_2(1))$, $z_{i+1} = z_i + \frac{h}{2}(m_1(2) + m_2(2))$

For $i=0$, $x_0 = 0$, $y_0 = 1$, $z_0 = 2$, $h = 0.25$

Now, Using Heun's method on eqn (I) & (I')

Iteration 1:

$m_1(1) = f(x_0, y_0, z_0)$
$= f_1(0, 1, 2)$ [$\because y' = z$]
$= 2$

$m_2 = \cancel{f(x_0, y_0, z_0)}$

$x_1 = 0.25$, $y_1 = y_0 + 0.25 \times 2$
$= 1 + 0.5$
$= 1.5$

$x_1 = 0.25$
$y_1 = y_0 + 0.25 \times m_1$
$= 1 + 0.5 = 1.5$
$z_1 = z_0 + 0.25 \times 2$
$= 2 + 0.5$
$= 2.5$

$(-4$

$m_1(2) = f_2(x_0, y_0, z_0) = f_2(0, 1, 2)$
$= \cancel{f_2(\ ) = -4}$
$= e^{-0} - 2 \times 2 - 1$
$= -4$

$m_2(1) = f_1(x_0 + h, y_0 + h m_1(1), z_0 + h m_1(2))$
$= f_1(0 + 0.25, 1 + 0.25 \times 2, 2 + 0.25 \times -4)$
$= f_1(0.25, 1.5, 1)$
$= \cancel{e^{-0.25} - 2 \times 1 - 1.5} = 1$
$= \cancel{-2.721}$

$m_2(2) = f_2(x_0 + h, y_0 + h m_1(1), z_0 + h m_1(2))$
$= f_2(0 + 0.25, 1 + 0.25 \times 2, 2 + 0.25 \times -4)$
$= e^{-0.25} - 2 \times 1 - 1.5$
$= -2.721$

$y' = z$

$= e^{-x} - 2z - y$

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$y(0.25) = y_0 + \frac{h}{2}(m_1(1) + m_2(1))$

$y_1 = 1 + \frac{0.25}{2}(2+1)$

$= 1.375$

$z(0.25) = z_0 + \frac{h}{2}(m_1(2) + m_2(2))$

$= 2 + \frac{0.25}{2}(-4 + (-2.727))$

$\therefore z_1 = 1.159$

Iteration 2:

$m_1(1) = f_1(x_1, y_1, z_1)$
$= f(0.25, 1.375, 1.159)$
$= 1.159$

$m_1(2) = f_2(x_1, y_1, z_1)$
$= f(0.25, 1.375, 1.159)$
$= -2.916$

$m_2(1) = f_1(x_1 + h, y_1 + h m_1(1), z_1 + h m_1(2))$
$= 0.431$

$m_2(2) = f_2(0.5, 1.665, 0.431)$
$= -1.92$

$y(0.5) = y_1 + h/2 (m_1(1) + m_2(1))$
$= 1.375 + \frac{0.25}{2}(1.16 + 0.431)$
$= 1.572$

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0.555

$z(0.5) = 1.16 + \frac{0.25}{2}(-2.916 - 1.92)$

$= 0.555$

Boundary Value Problem:

A boundary value problem is a system of ordinary differential eq^n with solution and derivative values, specified at more than one point.

Shooting method:

In this method, given boundary value problem is first transformed into equivalent IVP and then it is solved by using any one of the method for solving IVP.

The main steps involved in shooting method are:

1) Transformation of boundary value problem into equivalent Initial value problem.
2) Solution of IVP by using existing method.
3) Solution of boundary value problem.

Consider the boundary value problem,

$y'' = f(x, y, y')$, $y(a) = u$, $y(b) = v$

let $y' = z$

& $z' = f(x, y, z)$

To solve above IVP, we need to have two conditions at $x = a$, we have given one condition $y(a) = u$.

Lets guess another condition is $z(a) = g_1$. Here g_1 represents slope of $y(x)$ at $x = a$. Thus the problem can be written as system of two first order eqn as

$$y' = z, \quad y(a) = u$$
$$z' = f(x, y, z), \quad z(a) = g_1$$

Now, these equations can be solved by using any method for solving IVP until the solution at $x = b$ reaches to specified accuracy level. Suppose first estimated value of $y(x)$ at $x = b$ is given by $y(b) = v_1$. If $v_1 = v$ then we are done and it is the required solution otherwise, we should repeat the same process by taking second guess as g_2, suppose v_2 is the estimated value at $y(b)$ for second guess. If solution is not achieved by from second guess, we can obtain better approximation by using,

$$\frac{g_3 - g_2}{v - v_2} = \frac{g_2 - g_1}{v_2 - v_1}$$

$$\Rightarrow g_3 = g_2 - \frac{v_2 - v}{v_2 - v_1} \cdot (g_2 - g_1)$$

Now, with $z(a) = g_3$, solution of $y(x)$ can be obtained.

Q. Example: Solve the ODE $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = 0$, $y(5) = 0.0038731$, $y(8) = 0.0030770$ with step size 0.75 (use 4 segments between 5 & 8)

⇒ Soln:

Let $\frac{dy}{dx} = z$

Then $\frac{dz}{dx} + \frac{1}{x} z - \frac{y}{x^2} = 0$

Giving us two first order differential equations,

$\frac{dy}{dx} = z$, $y(5) = 0.0038731$

$\frac{dz}{dx} = -\frac{z}{x} + \frac{y}{x^2}$, $z(5) = ?$

Let us assume,

$z(5) = \frac{dy}{dx}(5) = \frac{y(8) - y(5)}{8-5} = \frac{0.0030770 - 0.0038731}{3}$

$= -0.00026538$

Now, set up initial value problem,

$\frac{dy}{dx} = z = f_1(x, y, z)$, $y(5) = 0.0038731$

$\frac{dz}{dx} = -\frac{z}{x} + \frac{y}{x^2} = f_2(x, y, z)$, $z(5) = -0.00026538$

From Euler method, we have

$y_{i+1} = y_i + h f_1(x_i, y_i, z_i)$

$z_{i+1} = z_i + h f_2(x_i, y_i, z_i)$

$-\frac{2}{x} + \frac{y}{x^2}$

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First Iteration: $h = 0.75$

$i=0,\ x_0 = 5,\ y_0 = 0.0038731,\ z_0 = -0.00026538$

$y_{i+1} = y_i + h f_1(x_i, y_i, z_i)$

$y_1 = 0.0038731 + 0.75(-0.00026538)$
$= 0.003674065$

$z_1 = -0.00026538 + 0.75(5, 0.0038731, -0.00026538)$
$= -0.00026538 + (0.75 \times 0.000208)$
$= -0.00010938$

Second Iteration:

$i=1,\ x_1 = 5.75,\ y_1 = 0.003674065,\ z_1 = -0.00010938$

$y_2 = 0.003674065 + 0.75(-0.00010938)$
$= 0.00359203$

$z_2 = -0.00010938 + 0.75(5.75, 0.003674065, -0.00010938)$
$= -0.00010938 + (0.75 \times 0.00013014)$
$= 0.0000011775$

Third iteration

$x_2 = 6.5,\ y_2 = 0.00359203,\ z_2 = -0.0000011775$

$y_3 = 0.00359203 + 0.75(-0.0000011775$
$= 0.0003583$

$z_3 = -0.0000011775 + 0.75(x_2, y_2, z_2)$
$= -0.0000011775 + 0.0000868 3 \times 0.75$
$= 0.0000 5334$

0.003475

Fourth Iteration

$x_3 = 7.25$, $z_3 = 0.00005334$, $y_3 = 0.0003583$

$y_4 = 0.0003583 + 0.75 \times 0.00005334$

$= 0.00039833$

$z_4 = 0.00005334 + 0.75 (x_3, y_3, z_3)$

$= 0.00005334 + 0.75 \times -0.00000054$

$= 0.0000989$

Thus at $x = x_4 = x_3 + h = 7.25 + 0.75 = 8$

we have,

$y_4 = y(8) = 0.003623$, while the given value of $y_4 = y(8) = 0.003077$.

Let us assume a new value for $\frac{dy}{dx}(5)$. Based on the first assumed value, may be twice the value of initial guess,

$$z(5) = \frac{dy}{dx}(5) = 2 * \frac{y(8) - y(5)}{8 - 5} = g_2$$

$= -0.00053076$

First Iteration: $x_0 = 5$

$y_1 = y_0 + h f_1 (x_0, y_0, z_0)$

$= 0.003475$

$z_1 = z_0 + h f_2 (x_0, y_0, z_0)$

$= -0.000335$

Second Iteration:

$y_2 = y_1 + h f_1(x_1, y_1, z_1)$

$= 0.0032238$

$z_2 = z_1 + h f_2(x_1, y_1, z_1)$

$= -0.0002125$

Third iteration:

$y_3 = y_2 + h f_1(x_2, y_2, z_2)$

$= 0.0030644$

$z_3 = z_2 + h f_2(x_2, y_2, z_2)$

$= -0.0001308$7

Fourth Iteration: $x_3 = 7.25$

$y_4 = y_3 + h f_1(x_3, y_3, z_3)$

$= 0.0030644 + 0.75 \times -0.0001308$

$= 0.0030644$

$z_4 = z_3 + h f_2(x_3, y_3, z_3)$

$= -0.0013057 + 0.74 \times (-0.00001357)$

$= 0.0030644$

while, the given value of this boundary is $y_4 = y(8) = 0.003077$

Now, we can use the result obtained from this previous iteration to get a better estimate of the assumed initial condition of $\frac{dy}{dx}(5)$.

we have,

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$y(8) \approx 0.003623 = v_1$

and with,

$\frac{dy(5)}{dx} = -0.0053076 = g_2$

we obtained $y(8) = 0.0030644 = v_2$

so, better starting values of $\frac{dy}{dx}(5)$.

knowing that the actual value at $y(8) = 0.003077$, we get

$$\frac{dy}{dx}(5) = g_3 = g_2 - \frac{v_2 - v}{v_2 - v_1}(g_2 - g_1)$$

$$= -0.0053076 - \frac{0.0030644 - 0.003077}{0.0030644 - 0.003623}(g_2 - g_1)$$

$g_3 = -0.0051938$

$$M_3 = M_2 - \frac{B_2 - V}{B_2 - B_1}(M_2 - M_1)$$

first Iteration,

$y_1 = y_0 + hf_1(x_0, y_0, z_0)$

$= 0.003508$

$z_1 = z_0 + hf_2(x_0, y_0, z_0)$

$= -0.00291$

second iteration

$y_1 = y_1 + hf_1(x_1, y_1, z_1)$

$=$