Machine Learning for String Compactifications

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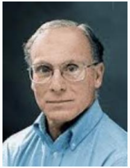
Collaborators



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Overview

Calabi–Yau metrics and hermitian Yang–Mills connections are crucial for string phenomenology

Numerical methods are the only way to access this data

Machine learning and neural networks provide a powerful set of tools to tackle geometric problems

Outline

Physics from geometry

Calabi–Yau metrics

Hermitian Yang–Mills connections

Machine learning and neural networks

Application: moduli dependence and the swampland

Physics from geometry

Motivation from physics

Does string theory describe our universe? Many semi-realistic MSSM-like string models from M-theory / F-theory / heterotic [...; Cole et al. '21; Abel et al. '21; Loges, Shiu '21, '22]

- Focus on models from heterotic string on Calabi–Yau

Coarse details: correct gauge group, matter spectrum, etc.

- Topological – do not need details of geometry

How many of these string vacua are physically reasonable?

- Predicted masses and couplings depend intricately on underlying geometry, i.e. metric and gauge connection
- No explicitly known (non-trivial) Calabi–Yau metrics or appropriate connections!

Calabi–Yau compactifications

Minimal supersymmetry on $\mathbb{R}^{1,3} \times X$ with $\mathrm{E}_8 \times \mathrm{E}_8$ bundle V [Candelas et al. '85]

- No H flux $\Rightarrow X$ equipped with Calabi–Yau metric g
- V admits hermitian Yang–Mills connection A
- Anomaly condition: $p_1(X) = p_1(V)$

Particle spectrum of low-energy theory determined by X and V

- e.g. standard embedding: SU(3) bundle gives E_6 GUT gauge group in 4d with $\frac{1}{2}\chi(X)$ particle generations
- Most interesting MSSM examples from non-standard embedding, but not so simple... [...;Donagi et al. '98; Braun et al. '05; Anderson et al. '11;...]

Low-energy physics

Compactification on X leads to 4d $N = 1$ effective theory with gauge + chiral multiplets.

- Chiral multiplets split into moduli fields and matter fields

Particle content comes from topology of X and V, e.g.

- SU(3) bundle V gives E_6 GUT group in 4d

$$\mathrm{E}_8 \to \mathrm{E}_6 \times \mathrm{SU}(3)$$
$$\underline{248} \to \bigoplus_{\underline{r},\underline{R}} (\underline{r}, \underline{R}) = (\underline{78}, \underline{1}) \oplus (\underline{1}, \underline{8}) \oplus (\underline{27}, \underline{3}) \oplus (\overline{\underline{27}}, \overline{\underline{3}})$$

- 4d multiplets transforming in $\underline{r}$ come from $H^1(X, \underline{R})$, e.g. matter fields from $C^I \in H^1(X, \underline{3})$

Yukawa couplings

Yukawa terms in Standard Model include $\mathcal{L}_{\text{SM}} \supset \mathcal{L}_{\text{Yuk}} = Y^d_{ij} H Q^i d^j + \dots$

4d $N = 1$ theory governed by superpotential and Kähler potential

$$W = \lambda_{IJK}(\phi) C^I C^J C^K + \dots \qquad K = G_{I\bar{J}}(\phi, \bar{\phi}) C^I \bar{C}^J + \dots$$

- Holomorphic superpotential from triple overlap of zero modes on X

$$\lambda_{IJK} = \int_X \Omega \wedge \text{tr}(C^I \wedge C^J \wedge C^K)$$

- Matter field Kähler potential gives normalisation where C^I are harmonic

$$G_{I\bar{J}} = \int_X C^I \wedge \star_V \bar{C}^J$$

A string model wish list

MSSM spectrum, three families, etc. ✓

- Reduces to topology / algebraic methods

Superpotential couplings λ_{IJK} ✓

- Holomorphic – can use algebraic / differential methods

Harmonic modes and Kähler metric G_{IJ} on field space ✗

- Numerical methods

Supersymmetry breaking, moduli stabilisation, etc. ✗

- Soft masses and couplings c.f. $N = 1$ Kähler potential and normalised zero modes [Kaplunovsky, Louis '93; Blumenhagen et al. '09; ...]

The missing ingredients

How do we calculate Calabi–Yau metrics or hermitian Yang–Mills connections?

Calabi–Yau metrics

Calabi–Yau geometry

Calabi–Yau manifolds are Kähler and admit Ricci-flat metrics

- Existence but no explicit constructions
- Kähler + $c_1(X) = 0 \Rightarrow$ there exists a Ricci-flat metric [Yau '77]

Kähler $\Rightarrow$ Kähler potential K gives metric g and closed two-form $J = \partial\bar{\partial}K$

$$\mathrm{vol}_g \equiv J \wedge J \wedge J$$

$c_1(X) = 0 \Rightarrow$ nowhere-vanishing holomorphic (3,0)-form Ω

$$\mathrm{vol}_\Omega \equiv \mathrm{i}\,\Omega \wedge \bar{\Omega}$$

Example: Fermat quintic

Calabi–Yau threefold is quintic hypersurface Q in $\mathbb{P}^4$

$$Q(z) \equiv z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 = 0$$

(3,0)-form Ω determined by Q, e.g. in $z_0 = 1$ patch

$$\Omega = \frac{dz_2 \wedge dz_3 \wedge dz_4}{\partial Q / \partial z_1}$$

Metric g and Kähler form J determined by Kähler potential

$$g_{i\bar{j}}(z, \bar{z}) = \partial_i \bar{\partial}_{\bar{j}} K(z, \bar{z}).$$

How do we measure accuracy?

The Ricci-flat metric is given by a K that satisfies (c.f. Monge–Ampère)

$$\left.\frac{\mathrm{vol}_g}{\mathrm{vol}_\Omega}\right|_p = 1 \quad \Rightarrow \quad R_{i\bar{j}} = 0$$

Define a functional of K [Douglas et al. '06]

$$\sigma(K) = \int_X \left| 1 - \frac{\mathrm{vol}_g}{\mathrm{vol}_\Omega} \right| \mathrm{vol}_\Omega$$

The exact CY metric has $\sigma(K) = 0$

- Finding the Ricci-flat metric reduces to finding $K(z, \bar{z})$ that minimises σ

Algebraic metrics [Tian '90; Donaldson '05]

Natural Kähler metric on $\mathbb{P}^4$ given by

$$K_{\mathrm{FS}} = \log \sum_{i=0}^{4} z_i \bar{z}_{\bar{i}}$$

Can generalise this with a hermitian matrix $h^{i\bar{j}}$

$$K(h) = \log \sum_{i,\bar{j}=0}^{4} z_i h^{i\bar{j}} \bar{z}_{\bar{j}}$$

Restricting to $Q \subset \mathbb{P}^4$ gives Kähler metric but not Ricci-flat

- 25 real parameters in $h^{i\bar{j}}$ that we can vary

Algebraic metrics [Tian '90; Donaldson '05]

Replace coordinates z_i with homogeneous monomials s_α of degree k

$$\text{e.g.} \quad k = 2: \qquad s_\alpha = (z_0^2, z_0 z_1, z_0 z_2, \ldots)$$

Kähler potential is then

$$K(h) = \log \sum_{\alpha,\beta=1}^{15} s_\alpha h^{\alpha\bar{\beta}} \bar{s}_{\bar{\beta}}, \qquad h^{\alpha\bar{\beta}} \sim 225 \text{ parameters}$$

At degree k have $\mathcal{O}(k^8)$ parameters, so can approximate the Ricci-flat metric to arbitrary precision

- Spectral method: $\frac{s_\alpha \bar{s}_{\bar{\beta}}}{(z_i \bar{z}_i)^k}$ gives basis for eigenspaces on $\mathbb{P}^4 = S^7 / U(1)$

How to fix K?

Finding the "best" approximation to the Ricci-flat metric amounts to finding K so that σ is minimised

Three approaches:

- Iterative procedure for $h^{\alpha\bar{\beta}}$ [Donaldson '05; Douglas '06; Braun '07]
- Minimise σ directly as function of $h^{\alpha\bar{\beta}}$ [Headrick, Nassar '09; Anderson et al. '20]
- Find K or $g_{i\bar{j}}$ directly by treating σ as a loss function for a neural network [Headrick, Wiseman '05; Douglas et al. 20; Anderson et al. '20; Jejjala et al. '20; Larfors et al. '21, '22]

In all cases, numerical integrals carried out by Monte Carlo [Shiffman, Zelditch '98]

Hermitian Yang–Mills connections

Hermitian Yang–Mills

A hermitian metric G on fibers of vector bundle V defines a connection and curvature

$$A_i = G^{-1}\partial_i G, \quad A_{\bar{i}} = 0 \quad \Rightarrow \quad F_{ij} = F_{\bar{i}\bar{j}} = 0, \quad F_{i\bar{j}} = \partial_{\bar{j}}(G^{-1}\partial_i G)$$

We say A is hermitian Yang–Mills if

$$g^{i\bar{j}} F_{i\bar{j}} = \mu(V)\,\mathrm{Id}$$

G is then known as a Hermite–Einstein metric on V

- Nonlinear PDE for G with no explicit solutions when X is Calabi–Yau
- HYM implies Yang–Mills: $\mathrm{d} \star F = 0$
- Supersymmetry in 10d requires HYM with $\mu(V) = 0$

Existence and stability

Existence of HYM solutions [Donaldson '85; Uhlenbeck, Yau '86]

A holomorphic vector bundle V over a compact Kähler manifold (X, g) admits a Hermite–Einstein metric iff V is slope polystable

Slope of V

$$\mu(V) \equiv \int_X c_1(V) \wedge J^{n-1}$$

V is stable if $\mu(\mathcal{F}) < \mu(V)$ for all $\mathcal{F} \subset V$ (or polystable if sum of stable bundles with same slope)

- Algebraic condition (like $c_1(X) = 0$), but not constructive!

How do we measure accuracy?

Defining $F_g \equiv g^{i\bar{j}} F_{i\bar{j}}$, the HYM equation is $F_g = \mu(V)\,\mathrm{Id}$

The average over the the Calabi–Yau is defined using the exact CY measure vol_Ω, e.g.

$$\langle \operatorname{tr} F_g \rangle \equiv \int_X \mathrm{vol}_\Omega \operatorname{tr} F_g$$

Suitable choice of accuracy measure is

$$E[F, g] = \langle \operatorname{tr} F_g^2 \rangle - \frac{1}{\operatorname{rank} V} \langle \operatorname{tr} F_g \rangle^2$$

$E[F, g]$ is positive semi-definite and vanishes on HYM solutions

$$\boxed{F \text{ solves HYM} \quad \Leftrightarrow \quad E[F, g] = 0}$$

Numerical approximations

Ansatz for hermitian metric [Wang '05; Douglas et al. '06; Anderson et al. '10]

$$(G^{-1})^{a\bar{b}} = \sum_{\alpha,\beta}^{N_k} S^a_\alpha H^{\alpha\bar{\beta}} \bar{S}^{\bar{b}}_{\bar{\beta}}$$

where $S^a_\alpha \in H^0(X, V \otimes L^k)$ and $H^{\alpha\bar{\beta}}$ is a hermitian matrix of parameters

Varying $H^{\alpha\bar{\beta}}$, one can find an approximate HYM connection for $V \otimes L^k$ and then on V itself

Increasing k increases the number of sections N_k and hence the number of parameters in $H^{\alpha\bar{\beta}}$

The goal

Train a neural network to find solutions to the hermitian Yang–Mills equation

Machine learning and neural networks

Overview

New era of big data in string theory

- Vacuum selection problem, huge number of CYs, larger number of flux vacua [Denef, Douglas '04; Taylor, Wang '15;...]

Many different types of machine learning

- Supervised – known inputs and outputs, e.g. recognise images, predict Hodge numbers [He '17; Bull et al. '18; Erbin, Finotello '20;...]
- Unsupervised – known inputs, e.g. looking for patterns or generate images
- Self-supervised – known inputs, output minimises a loss function, e.g. QM ground states, Ricci-flat metrics, HYM connections

Neural networks

Neural networks (NN) convert inputs to outputs: $\vec{x} \mapsto f(\vec{x}, \vec{w})$

- Network built from connected nodes called neurons
- Weights $\vec{w}$ are parameters in network (strength of connections)
- Non-linear activation functions
- Training attempts to minimise a loss function computed from NN
- Universal approximation theorem for NNs [Cybenko '89]

More interested in the network itself than the actual values!

- NN gives a variational ansatz for some function you want to find, e.g. Hermite–Einstein metric G that solves HYM equation

Line bundles on CY manifolds

Line bundles crucial in many string models [Anderson, Gray, Lukas, Palti '11;...]

Holomorphic line bundle L determined by $c_1(L)$. Given a basis of divisors $\mathcal{D}_I$ on X, denote by $\mathcal{O}_X(m^I)$ the line bundle with $c_1(L) = m^I \mathcal{D}_I$

Line bundles are automatically stable, so always admit a solution to HYM, $g^{i\bar{j}} F_{i\bar{j}} = \mu(L)$

We need the functional form of G to calculate harmonic representatives and the matter field Kähler metric

Bihomogenous networks [Douglas et al. '20]

$$\mathbb{C}^3 \longrightarrow \boxed{\text{Bihom}} \xrightarrow{\mathbb{R}^9} \boxed{\text{Square}} \xrightarrow{\mathbb{R}^{12}} \boxed{\text{Log}} \longrightarrow \mathbb{R}$$

$$\begin{array}{ccc} \mathbb{C}^3 \to \mathbb{R}^9 & \mathbb{R}^9 \to \mathbb{R}^{12} & \mathbb{R}^{12} \to \mathbb{R} \\ z_i \mapsto (\operatorname{re} z_j \bar{z}_k, \operatorname{im} z_j \bar{z}_k) & \vec{x} \mapsto (W_1 \vec{x})^2 & \vec{y} \mapsto \log(W_2 \vec{y}) \end{array}$$

Parameters in W_1 and W_2 are weights, collectively denoted by $\vec{w}$

First implemented for CY metrics in TensorFlow by [Douglas et al. '20] at

`https://github.com/yidiq7/MLGeometry`

A loss function

Network output is treated as $\log G^{-1}$, which defines F

- Together with approximate CY metric g, this gives $F_g[\vec{w}]$ as a function of the network weights $\vec{w}$
- Mimics previous ansatz for hermitian structure

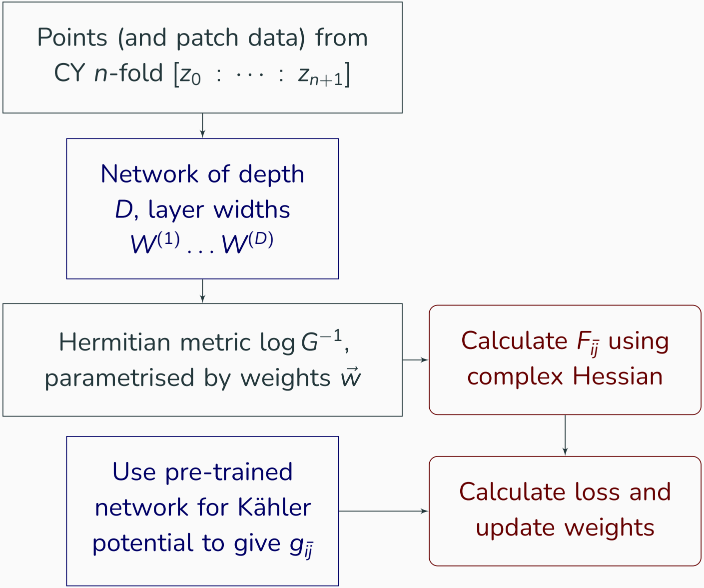
Loss function is

$$\text{Loss}[F,g] = E[F,g] \equiv \langle \operatorname{tr} F_g^2 \rangle - \frac{1}{\operatorname{rank} V} \langle \operatorname{tr} F_g \rangle^2$$

After training, the network gives a NN-based representation of the HYM connection

- Effectively the functional form of G (plus A or F as can take derivatives, etc.)

General strategy



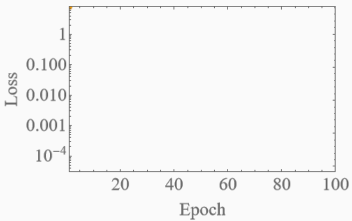
$\mathcal{O}_X(4)$ on elliptic curve

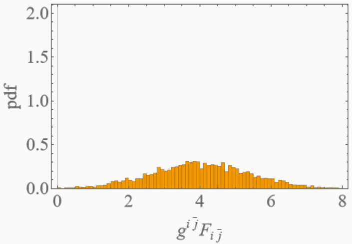
Line bundle $\mathcal{O}(4)$ over elliptic curve defined by

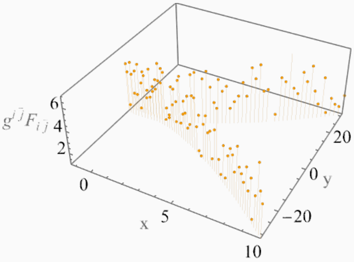
$$Q(z) \equiv z_1^3 - z_0^2 z_1 - z_0 z_2^2 + z_0^3 = 0 \qquad \subset \mathbb{P}^2$$

- Solution to HYM should give $g^{i\bar{j}} F_{i\bar{j}} = 4$ pointwise

Evolution of loss, pdf of $g^{i\bar{j}} F_{i\bar{j}}$ and values of $g^{i\bar{j}} F_{i\bar{j}}$ on elliptic curve







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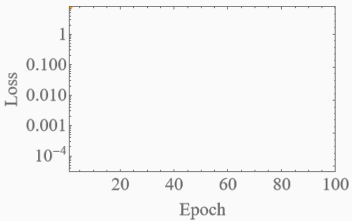
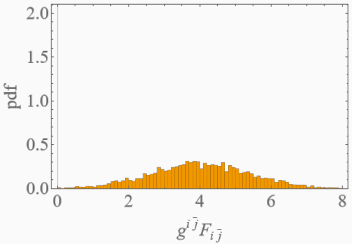
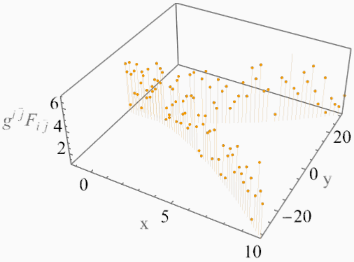






$\mathcal{O}_X(m)$ on quintic threefold

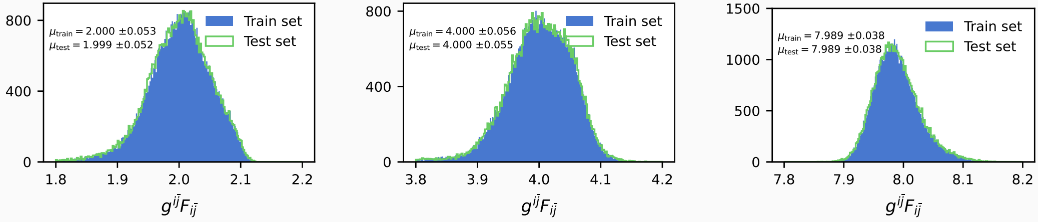
Dwork quintic defined by

$$Q(z) \equiv z_0^5 + \cdots + z_4^5 + \frac{1}{2} z_0 z_1 z_2 z_3 z_4 = 0 \qquad \subset \mathbb{P}^4$$

Approximate CY metric computed with $\sigma = 0.001$

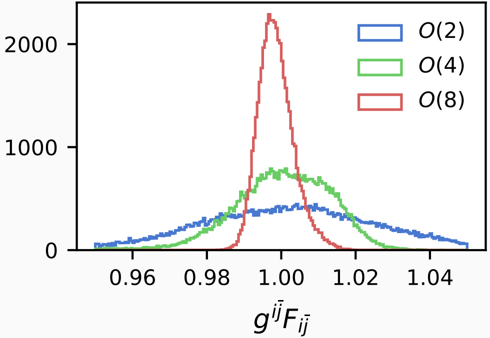
Neural networks of depth $D = 2, 3, 4$ with intermediate $W = 100$ layers

- Histogram of values of $g^{i\bar{j}} F_{i\bar{j}}$ – should be constant over X



$\mathcal{O}_X(1)$ on quintic threefold

$D = 2, 3, 4$ networks give connections on $\mathcal{O}_X(2)$, $\mathcal{O}_X(4)$ and $\mathcal{O}_X(8)$ – untwist to give connections on $V = \mathcal{O}_X(1)$



Loss curves show that $D = 2$ network is underparametrised, but all still within 5% of expected result $g^{i\bar{j}}F_{i\bar{j}} = 1$

Application: moduli dependence and the swampland

The swampland distance conjecture

As an application of these numerical techniques, we can compute the spectrum of the Laplacian on a Calabi–Yau. This will teach us something about its KK spectrum and effective field theory.

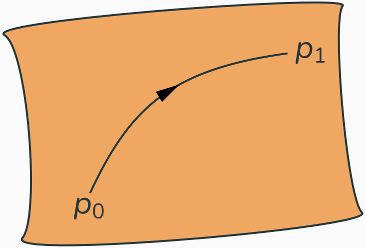
The Swampland Distance Conjecture [Ooguri, Vafa '06]

For an effective theory coupled to gravity with a moduli space parametrised by vevs of some fields, moving an infinite distance in moduli space brings down a tower of massless states that spoil the initial effective theory.

Finite version: there is a relation between the mass of a tower of states and the distance you move in moduli space

The swampland distance conjecture

Compare the effective theory at two points p_0 and p_1 in moduli space which are a geodesic distance $d(p_0, p_1)$ apart



Geodesic γ with $\gamma(\tau_i) = p_i$, distance along curve given by

$$d(p_0, p_1) = \int_{\tau_1}^{\tau_2} \sqrt{G_{ab}\dot{\gamma}^a\dot{\gamma}^b}$$

where G_{ab} is metric on moduli space

Conjecture implies that a tower of states becomes light on moving from p_0 to p_1 with masses

$$m(p_1) \sim m(p_0) e^{-\alpha\, d(p_0,p_1)}$$

where α is order one coefficient

The spectrum with varying complex structure [AA, Ruehle '21]

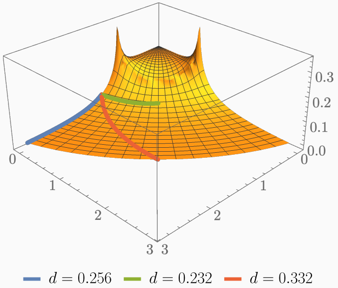
Focus on complex structure moduli space for quintic

$$Q_\psi \equiv z_0^5 + \cdots + z_4^5 - 5\psi z_0 \ldots z_4$$

1. Compute the moduli space metric [Candelas et al. '98; Keller, Lukic '09]

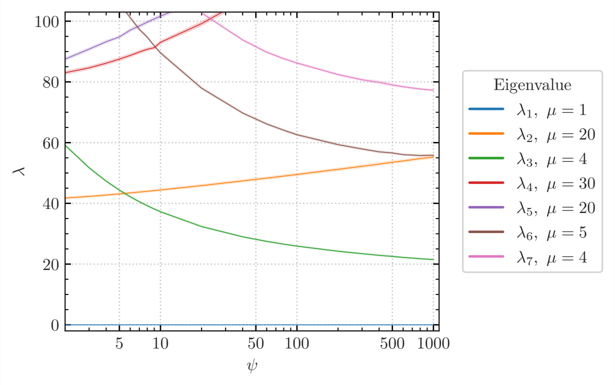
$$e^{-\mathcal{K}_{cs}} = i \int_{Q_\psi} \Omega \wedge \bar{\Omega}, \quad G_{a\bar{b}} = \partial_a \bar{\partial}_{\bar{b}} \mathcal{K}_{cs}$$

2. Compute geodesics and distances in moduli space for varying ψ
3. Compute the numerical Calabi–Yau metric and spectrum of the Laplacian Δ – scalars give spin-two modes



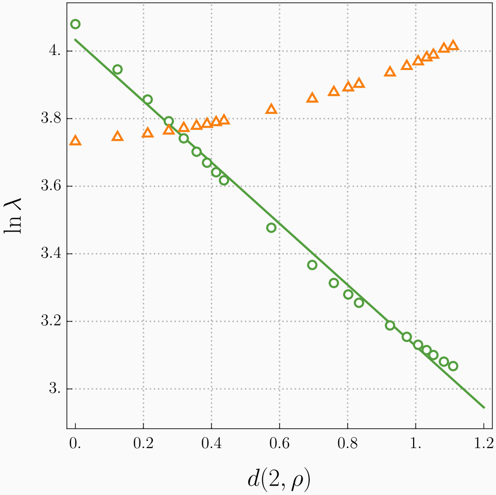
Scalar spectrum with varying ψ

Spectrum of $\Delta\phi = \lambda\phi$



1. Zero mode always present; massive modes appear with multiplicities given by irreps of $(S_5 \times \mathbb{Z}_2) \ltimes (\mathbb{Z}_5)^3$
2. Eigenvalues with small degeneracy become lighter
3. Local maximum in spectral gap for $\psi \approx 4$

A check of the distance conjecture



1. λ_3 falls exponentially as

$$\lambda_3 = 56.4\, e^{-(0.906 \pm 0.034)\, d(2,\rho)}$$
$$\sim e^{-2\alpha\, d(p_0, p_1)}$$

2. We see $\alpha \approx 0.45$ which is indeed order one
3. Almost saturates conjectured lower bound of $1/\sqrt{6} \approx 0.41$

   [Andriot et al. '20]

Summary and outlook

Calabi–Yau metrics and HYM connections are accessible with numerical methods and machine learning

- Can extend to other bundles, compute (p, q)-form Laplacian spectrum – source of new non-BPS data [AA '20]

Ongoing work: bundle-valued harmonic forms, compute Yukawa couplings at a given point in moduli space

- SYZ conjecture? Non-Kähler metrics? G_2 metrics? Neural networks as general PDE solvers?
- 2d CFTs? [Afkhami-Jeddi, AA, Córdova '21] Input for conformal bootstrap? [Lin et al. '15; Lin et al. '16]