

Homework #0

Due: January 26, 2023 at 11:59 PM
(with free, no-questions-asked
extension, see below)

Welcome to CS181! The purpose of this assignment is to help assess your readiness for this course. **This assignment will be graded for completion and effort.** If you encounter any difficulty with these problems, fear not! We will have sections in the first week of class reviewing the math, statistics, and coding pre-requisites for this course. TFs will also directly discuss relevant problems from this HW0 in these sections. You are, of course, more than welcome to swing by office hours and post questions on Ed.

1. Please type your solutions after the corresponding problems using this L^AT_EX template, and start each problem on a new page.
2. Please submit the **writeup PDF to the Gradescope assignment ‘HW0’**. Remember to assign pages for each question.
3. Please submit your **L^AT_EX file and code files (i.e., anything ending in .py, .ipynb, or .tex) to the Gradescope assignment ‘HW0 - Supplemental’**.

Free, No-Questions-Asked Extension:

- The official deadline is January 26, in order to not overlap too much with HW1, which covers substantial new course material. With teaching staff discussing these problems in Week 1 sections, a reasonably prepared (in terms of pre-requisites) student should be able to complete this assignment quite comfortably in a week.
- However, given the uncertainty and frenzy of the start of the new semester, we will also give free, no-questions-asked extensions to **February 2, 2023 at 11:59 PM** if you need some extra breathing room and / or if you need to brush up a little more on the pre-requisites. Do note, however, that HW1 will be due a week later on February 9.
- You do not need to formally request this free, no-questions-asked extension (i.e., no Ed post or email is necessary). Please simply submit your assignment before the late deadline of **February 2, 2023 at 11:59 PM** in Gradescope. Given that this is a free extension, it does not count into your late days for the semester.

Problem 1 (Modeling Linear Trends - Linear Algebra Review)

In this class we will be exploring the question of “how do we model the trend in a dataset” under different guises. In this problem, we will explore the algebra of modeling a linear trend in data. We call the process of finding a model that capture the trend in the data, “fitting the model.”

Learning Goals: In this problem, you will practice translating machine learning goals (“modeling trends in data”) into mathematical formalism using linear algebra. You will explore how the right mathematical formalization can help us express our modeling ideas unambiguously and provide ways for us to analyze different pathways to meeting our machine-learning goals.

Let’s consider a dataset consisting of two points $\mathcal{D} = \{(x_1, y_1), (x_2, y_2)\}$, where x_n, y_n are scalars for $n = 1, 2$. Recall that the equation of a line in 2-dimensions can be written: $y = w_0 + w_1x$.

1. Write a system of linear equations determining the coefficients w_0, w_1 of the line passing through the points in our dataset \mathcal{D} and analytically solve for w_0, w_1 by solving this system of linear equations (i.e., using substitution). Please show your work.
2. Write the above system of linear equations in matrix notation, so that you have a matrix equation of the form $\mathbf{y} = \mathbf{X}\mathbf{w}$, where $\mathbf{y}, \mathbf{w} \in \mathbb{R}^2$ and $\mathbf{X} \in \mathbb{R}^{2 \times 2}$. For full credit, it suffices to write out what \mathbf{X} , \mathbf{y} , and \mathbf{w} should look like in terms of $x_1, x_2, y_1, y_2, w_0, w_1$, and any other necessary constants. Please show your reasoning and supporting intermediate steps.
3. Using properties of matrices, characterize exactly when an unique solution for $\mathbf{w} = (w_0 \ w_1)^T$ exists. In other words, what must be true about your dataset in order for there to be a unique solution for \mathbf{w} ? When the solution for \mathbf{w} exists (and is unique), write out, as a matrix expression, its analytical form (i.e., write \mathbf{w} in terms of \mathbf{X} and \mathbf{y}).

Hint: What special property must our \mathbf{X} matrix possess? What must be true about our data points in \mathcal{D} for this special property to hold?

4. Compute \mathbf{w} by hand via your matrix expression in (3) and compare it with your solution in (1). Do your final answers match? What is one advantage for phrasing the problem of fitting the model in terms of matrix notation?
5. In real-life, we often work with datasets that consist of hundreds, if not millions, of points. In such cases, does our analytical expression for \mathbf{w} that we derived in (3) apply immediately to the case when \mathcal{D} consists of more than two points? Why or why not?

Solutions: (Modeling Linear Trends - Linear Algebra Review)

1. Our system of linear equations:

$$\begin{cases} y_1 = w_0 + w_1x_1 \\ y_2 = w_0 + w_1x_2 \end{cases}$$

Solve the second equation for w_0 :

$$y_2 = w_0 + w_1x_2$$

$$w_0 = y_2 - w_1x_2$$

Substitute w_0 in first equation and solve for w_1 :

$$y_1 = y_2 - w_1x_2 + w_1x_1$$

$$y_1 = y_2 + w_1(x_1 - x_2)$$

$$w_1 = \frac{y_1 - y_2}{x_1 - x_2}$$

Substitute w_1 in the second equation to solve for w_0 :

$$y_1 = w_0 + \left(\frac{y_1 - y_2}{x_1 - x_2}\right)x_1$$

$$w_0 = y_1 - \left(\frac{y_1 - y_2}{x_1 - x_2}\right)x_1$$

By solving the system of equations using substitution, $w_0 = y_1 - \left(\frac{y_1 - y_2}{x_1 - x_2}\right)x_1$, $w_1 = \frac{y_1 - y_2}{x_1 - x_2}$.

2. The system of linear equations in matrix notation:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \end{bmatrix}$$

where $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, $\mathbf{X} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} w_0 \\ w_1 \end{bmatrix}$.

3. In a set of linear simultaneous equations, a unique solution exists if and only if, (i) the number of unknowns is equal to the number of equations, (ii) all equations are consistent, meaning each equation has at least one solution, and (iii) there is no linear dependence between any two or more equations. When a solution exists for \mathbf{w} , it can be written in matrix expression as $\mathbf{w} = \mathbf{X}^{-1}\mathbf{y}$

4. To solve the matrix expression, first, we solve for \mathbf{X}^{-1} using RREF (Gauss-Jordan):

$$\begin{array}{l} \left[\begin{array}{cc|cc} 1 & x_1 & 1 & 0 \\ 1 & x_2 & 0 & 1 \end{array} \right] \\ \left[\begin{array}{cc|cc} 1 & x_1 & 1 & 0 \\ 0 & x_2 - x_1 & -1 & 1 \end{array} \right] \quad R_2 \leftarrow R_2 - R_1 \\ \left[\begin{array}{cc|cc} 1 & x_1 & 1 & 0 \\ 0 & 1 & \frac{-1}{x_2 - x_1} & \frac{1}{x_2 - x_1} \end{array} \right] \quad R_2 \leftarrow \frac{R_2}{x_2 - x_1} \\ \left[\begin{array}{cc|cc} 1 & 0 & 1 + \frac{x_1}{x_2 - x_1} & \frac{-x_1}{x_2 - x_1} \\ 0 & 1 & \frac{-1}{x_2 - x_1} & \frac{1}{x_2 - x_1} \end{array} \right] \quad R_1 \leftarrow R_1 - x_1 R_2 \end{array}$$

Therefore,

$$\mathbf{X}^{-1} = \begin{bmatrix} 1 + \frac{x_1}{x_2 - x_1} & \frac{-x_1}{x_2 - x_1} \\ \frac{-1}{x_2 - x_1} & \frac{1}{x_2 - x_1} \end{bmatrix}$$

Now, we can solve the matrix expression $\mathbf{w} = \mathbf{X}^{-1}\mathbf{y}$.

$$\begin{bmatrix} 1 + \frac{x_1}{x_2 - x_1} & \frac{-x_1}{x_2 - x_1} \\ \frac{-1}{x_2 - x_1} & \frac{1}{x_2 - x_1} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_1 + \frac{y_1 x_1}{x_2 - x_1} - \frac{y_2 x_1}{x_2 - x_1} \\ \frac{-y_1}{x_2 - x_1} + \frac{y_2}{x_2 - x_1} \end{bmatrix}$$

$$\mathbf{w} = \begin{bmatrix} y_1 + \frac{y_1 x_1}{x_2 - x_1} - \frac{y_2 x_1}{x_2 - x_1} \\ \frac{-y_1}{x_2 - x_1} + \frac{y_2}{x_2 - x_1} \end{bmatrix}$$

This solution matches the solution from (1). One advantage of fitting the model using matrix expression is that this method is much more efficient in solving systems and requires fewer steps than solving the system using substitution, especially in cases when the order of equations is larger than two or three.

5. In this problem, we found the coefficients for a 2D line that passes through two points. However, if we were given more than these two points, our analytical expression for \mathbf{w} would not apply.

Problem 2 (Optimizing Objectives - Calculus Review)

In this class, we will write real-life goals we want our model to achieve into a mathematical expression and then find the optimal settings of the model that achieves these goals. The formal framework we will employ is that of mathematical optimization. Although the mathematics of optimization can be quite complex and deep, we have all encountered basic optimization problems in our first calculus class!

Learning Goals: In this problem, we will explore how to formalize real-life goals as mathematical optimization problems. We will also investigate under what conditions these optimization problems have solutions.

In her most recent work-from-home shopping spree, Nari decided to buy several house plants. *Her goal is to make them to grow as tall as possible.* After perusing the internet, Nari learns that the height y in mm of her Weeping Fig plant can be directly modeled as a function of the oz of water x she gives it each week:

$$y = -3x^2 + 72x + 70.$$

1. Based on the above formula, is Nari's goal achievable: does the plant have a maximum height? Why or why not? Does her goal have a unique solution - i.e. is there one special watering schedule that would achieve the maximum height (if it exists)?

Hint: plot this function. In your solution, words like “convex” and “concave” may be helpful.

2. Using calculus, find how many oz per week should Nari water her plant in order to maximize its height. With this much water, how tall will her plant grow?

Hint: solve analytically for the critical points of the height function (i.e., where the derivative of the function is zero). For each critical point, use the second-derivative test to identify if each point is a max or min point, and use arguments about the global structure (e.g., concavity or convexity) of the function to argue whether this is a local or global optimum.

Now suppose that Nari want to optimize both the amount of water x_1 (in oz) *and* the amount of direct sunlight x_2 (in hours) to provide for her plants. After extensive research, she decided that the height y (in mm) of her plants can be modeled as a two variable function:

$$y = f(x_1, x_2) = \exp(-(x_1 - 2)^2 - (x_2 - 1)^2)$$

3. Using `matplotlib`, visualize in 3D the height function as a function of x_1 and x_2 using the `plot_surface` utility for $(x_1, x_2) \in (0, 6) \times (0, 6)$. Use this visualization to argue why there exists a unique solution to Nari's optimization problem on the specified intervals for x_1 and x_2 .

Remark: in this class, we will learn about under what conditions do *multivariate* optimization problems have unique global optima (and no, the second derivative test doesn't exactly generalize directly). Looking at the visualization you produced and the expression for $f(x_1, x_2)$, do you have any ideas for why this problem is guaranteed to have a global maxima? You do not need to write anything responding to this – this is simply food for thought and a preview for the semester.

Solutions: (Optimizing Objectives - Calculus Review))

1. We are given a quadratic function with a negative coefficient, meaning the parabola concaves downward. This also indicates there is a maximum value for the function which is unique.
2. To find how many oz per week Nari should water her plant, we first need to find critical points of the function using the first derivative.

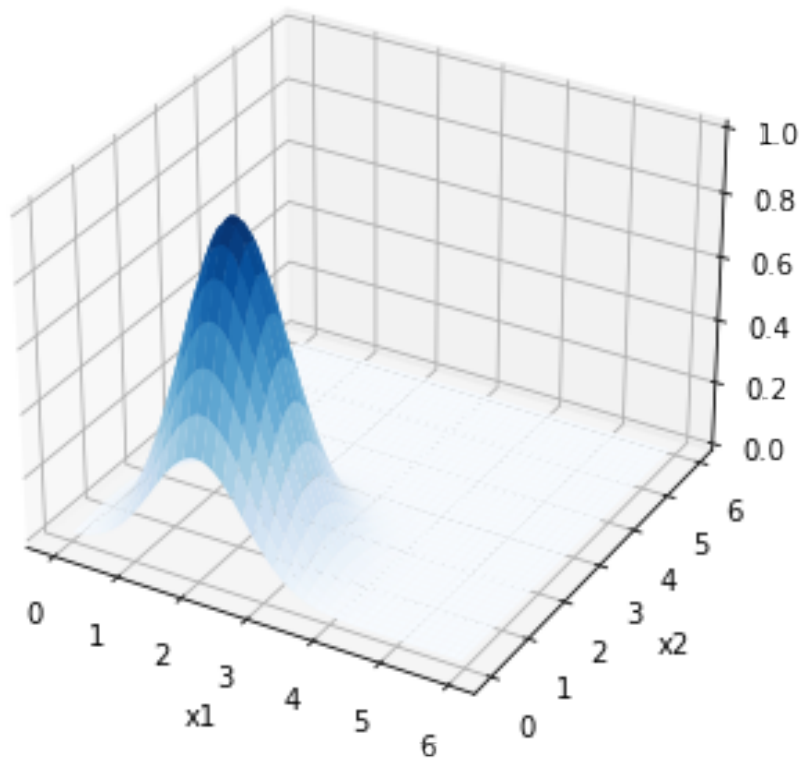
$$\begin{aligned}y &= -3x^2 + 72x + 70 \\ \frac{dy}{dx} &= -6x + 72 \\ 0 &= -6x + 72 \\ x &= 12\end{aligned}$$

To identify if the critical point is a maximum or minimum, we use the second derivative test.

$$\frac{d^2y}{d^2x} = -6$$

The second derivative of the function is -6. Because the second derivative is negative, there is a maximum at the critical point. Additionally, there is only one critical point, this maximum is a global maximum. Nari should water her plant with 12 ounces of water a week so that her plant reaches a maximum height of 502 feet.

3. From the graph, it seems like the function has a maximum which is a unique solution because one part of the function is concave downward.



Problem 3 (Reasoning about Randomness - Probability and Statistics Review)

In this class, one of our main focuses is to model the unexpected variations in real-life phenomena using the formalism of random variables. In this problem, we will use random variables to model how much time it takes an USPS package processing system to process packages that arrive in a day.

Learning Goals: In this problem, you will analyze random variables and their distributions both analytically and computationally. You will also practice drawing connections between said analytical and computational conclusions.

Consider the following model for packages arriving at the US Postal Service (USPS):

- Packages arrive randomly in any given hour according to a Poisson distribution. That is, the number of packages in a given hour N is distributed $Pois(\lambda)$, with $\lambda = 3$.
- Each package has a random size S (measured in in^3) and weight W (measured in pounds), with joint distribution

$$(S, W)^T \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}), \text{ with } \boldsymbol{\mu} = \begin{bmatrix} 120 \\ 4 \end{bmatrix} \text{ and } \boldsymbol{\Sigma} = \begin{bmatrix} 1.5 & 1 \\ 1 & 1.5 \end{bmatrix}.$$

- Processing time T (in seconds) for each package is given by $T = 60 + 0.6W + 0.2S + \epsilon$, where ϵ is a random noise variable with Gaussian distribution $\epsilon \sim \mathcal{N}(0, 5)$.

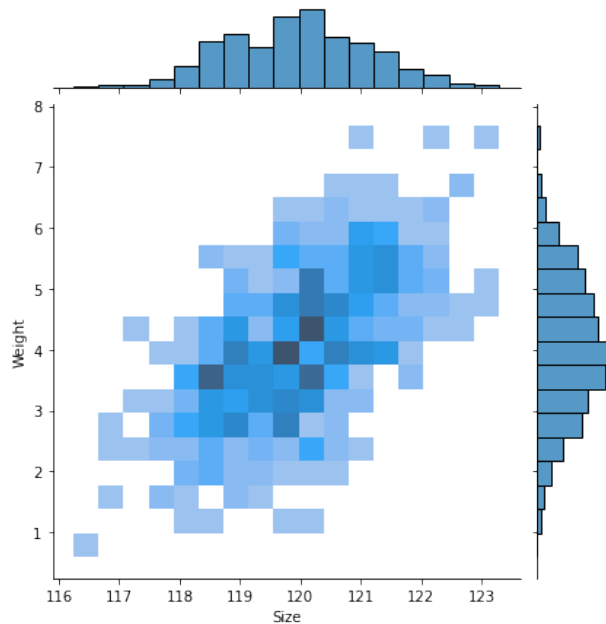
For this problem, you may find the `multivariate.normal` module from `scipy.stats` especially helpful. You may also find the `seaborn.histplot` function quite helpful.

1. Perform the following tasks:

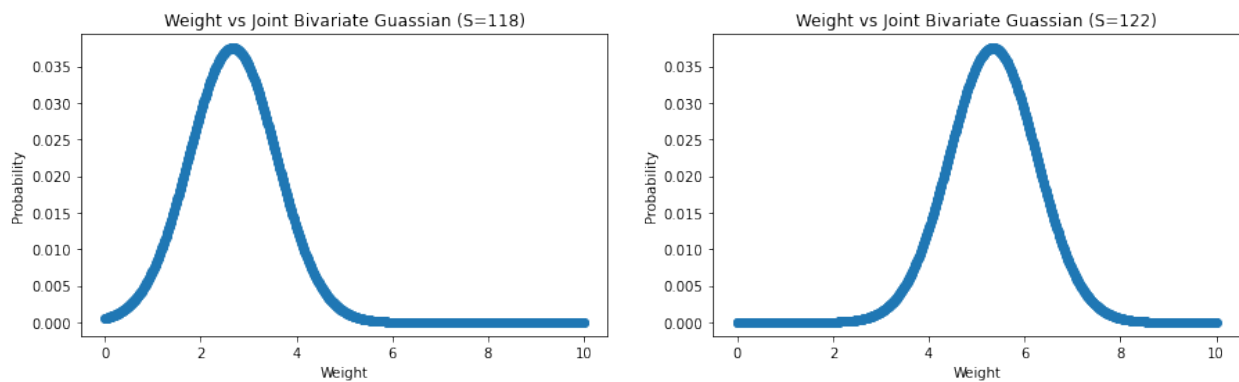
- (a) Visualize the Bivariate Gaussian distribution for the size S and weight W of the packages by sampling 500 times from the joint distribution of S and W and generating a bivariate histogram of your S and W samples.
 - (b) Empirically estimate the most likely combination of size and weight of a package by finding the bin of your bivariate histogram (i.e., specify both a value of S and a value of W) with the highest frequency. A visual inspection is sufficient – you do not need to be incredibly precise. How close are these empirical values to the theoretical expected size and expected weight of a package, according to the given Bivariate Gaussian distribution?
2. For 1001 evenly-spaced values of W between 0 and 10, plot W versus the joint Bivariate Gaussian PDF $p(W, S)$ with S fixed at $S = 118$. Repeat this procedure for S fixed at $S = 122$. Comparing these two PDF plots, what can you say about the correlation of random variables S and W ?
 3. Give one reason for why the Gaussian distribution is an appropriate model for the size and weight of packages. Give one reason for why it may not be appropriate.
 4. Because T is a linear combination of random variables, it itself is a random variable. Using properties of expectations and variance, please compute $\mathbb{E}(T)$ and $\text{Var}(T)$ analytically.
 5. Let us treat the *total* amount of time it takes to process *all* packages received at the USPS office within *an entire day* (assuming a single day is 24 hours long) as a random variable T^* .
 - (a) Write a function to simulate draws from the distribution of T^* .
 - (b) Using your function, empirically estimate the mean and standard deviation of T^* by generating 1000 samples from the distribution of T^* .

Solutions: (Optimizing Objectives - Calculus Review))

1. (a) Bivariate histogram of your S and W samples.



- (b) The most likely combination of size and weight of a package is 120 cubic inches, 4 pounds. The empirical values are the theoretically expected size and weight of a package, according to the given Bivariate Gaussian distribution.
2. The random variables S and W are correlated. As the size of a package increases, the probability the weight is higher also increases.



3. One reason the Gaussian distribution is an appropriate model for size and weight is due to the variables being independent and linearly correlated. However, a Gaussian distribution could not be an appropriate model because the variables (size, weight) can have extreme values and may not follow a bell-shaped curve of a Gaussian distribution.

4. Using the properties and expectations and variance, the $E(T)$ and $Var(T)$ are:

$$E(T) = 60 + 0.6E(W) + 0.2E(S) + E(\epsilon)$$

$$E(T) = 60 + 0.6 * 4 + 0.2 * 120 + 0$$

$$E(T) = 86.4$$

$$Var(T) = 0.6^2 Var(W) + (0.2)^2 Var(S) + Var(\epsilon)$$

$$Var(T) = 0.36 * 1.5 + 0.6 * 4 + 0.04 * 1.5 + 5$$

$$Var(T) = 5.6$$

5. (a) `def draw_from_T():`

```
    t_hat = 0
```

```
    num_packages = np.random.poisson(3, 24)
```

```
    for n in num_packages:
```

```
        if n == 0:
```

```
            T = np.array(0)
```

```
        else:
```

```
            rv = multivariate_normal([120, 4],
                                     [[1.5, 1], [1, 1.5]])
```

```
            data = rv.rvs(size = n).reshape(n,-n)
```

```
            noise = np.random.normal(0, 5, n)
```

```
            T = 60 + 0.6*data[:,1] + 0.2*data[:,0] + noise.reshape(n,-n)[0]
```

```
            t_hat += T.sum()
```

```
    return(t_hat)
```

- (b) The empirical estimation of the mean and standard deviation of T^* is 6198 seconds and 742, respectively.