

# High Dimensional Data Enrichment: Interpretable, Fast, and Data-Efficient\*

Amir Asiaee<sup>†</sup>, Samet Oymak<sup>‡</sup>, Kevin R. Coombes<sup>§</sup>, and Arindam Banerjee<sup>¶</sup>

**Abstract.** Given samples from a set of groups, a data-enriched model describes observations by a common and per-group individual parameters. In high-dimensional regime, each parameter has its own structure such as sparsity or group sparsity. In this paper, we consider the general form of data enrichment where data comes in a fixed but arbitrary number of groups  $G$  and any convex function, e.g., norm, can characterize the structure of both common and individual parameters. We propose an estimator for the high-dimensional data enriched model and investigate its statistical properties. We delineate sample complexity of our estimator and provide high probability non-asymptotic bound for estimation error of all parameters under a condition weaker than the state-of-the-art. We propose an iterative estimation algorithm with a geometric convergence rate and supplement our theoretical analysis with synthetic ~~data~~ experimental results. ~~In particular, we show the predictive power of data-enriched model along with its interpretable results in anti-cancer drug sensitivity analysis.~~ Overall, we present a first through statistical and computational analysis of inference in the data enriched model.

**Key words.** multi-task learning, superposition models, high-dimensional statistics, convergence rate analysis

**AMS subject classifications.** 62F10, 62J05, 90C25

**1. Introduction.** Over the past two decades, major advances have been made in estimating structured parameters, e.g., sparse, low-rank, etc., in high-dimensional small sample problems [11, 17, 18]. Such estimators consider a suitable (semi) parametric model of the response:  $y = \phi(\mathbf{x}, \beta^*) + w$  based on  $n$  samples  $\{(\mathbf{x}_i, y_i)\}_{i=1}^n$  and the parameter of interest,  $\beta^* \in \mathbb{R}^p$ . The unique aspect of such high-dimensional regime of  $n \ll p$  is that the structure of  $\beta^*$  makes the estimation possible for large enough samples  $n = m$  known as the sample complexity [9, 10, 36]. While the earlier developments in such high-dimensional estimation problems had focused on parametric linear models, the results have been widely extended to non-linear models, e.g., generalized linear models [1, 28], broad families of semi-parametric and single-index models [7, 33], non-convex models [5, 22], etc.

In several real world problems, the assumption that one global model parameter  $\beta_0^*$  is suitable for the entire population is unrealistic. We consider the more general setting where the population consists of sub-populations (groups) which are similar in many aspects but have unique differences. For example, in the context of anti-cancer drug sensitivity prediction where the goal is to predict responses of different tumor cells to a drug, using a same prediction model across cancer types (groups) ignores the unique properties of each cancer and leads to an uninterpretable global model. Alternatively, in such a setting, one can assume a separate model for each group  $g$  as  $y = \phi(\mathbf{x}, \beta_g^*) + w$  based on a group specific parameter  $\beta_g^*$ . Such a modeling choice fails to leverage the similarities across the sub-populations, and can only be estimated when sufficient number of samples are available for each

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<sup>†</sup>Mathematical Biosciences Institute, The Ohio State University, Columbus, OH ([asiaeetaheri.1@osu.edu](mailto:asiaeetaheri.1@osu.edu)).

<sup>‡</sup>Department of Electrical and Computer Engineering, UC Riverside, Riverside, CA ([oymak@ece.ucr.edu](mailto:oymak@ece.ucr.edu)).

<sup>§</sup>Departments of Biomedical Informatics, The Ohio State University, Columbus, OH ([coombes.3@osu.edu](mailto:coombes.3@osu.edu)).

<sup>¶</sup>Department of Computer Science and Engineering, University of Minnesota, Minneapolis, MN ([banerjee@cs.umn.edu](mailto:banerjee@cs.umn.edu)).

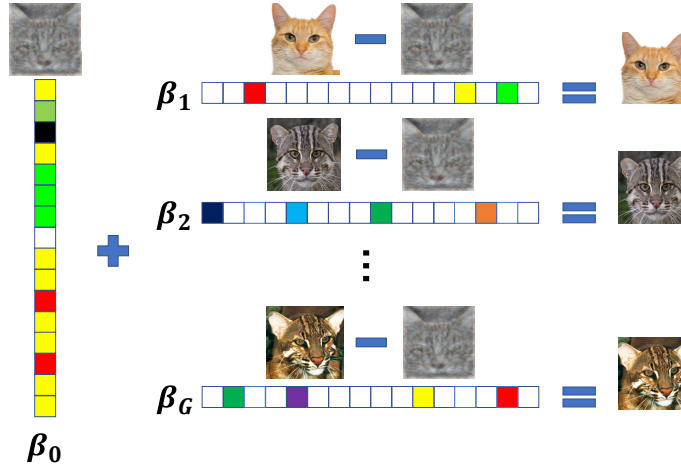


Figure 1: A conceptual illustration of data enrichment model for learning representation of cat species. The common parameter  $\beta_0$  captures a *generic cat* which consists of shared features among all cats.

group which is not the case in several problems, e.g., anti-cancer drug sensitivity prediction [3, 21].

The middle ground model for such a scenario is the *superposition* of common and individual parameters  $\beta_0^* + \beta_g^*$  which has been of recent interest [20, 25, 41]. Such a collection of *coupled* superposition models is known by multiple names in the statistical machine learning community. It is a form of multi-task learning [23, 42] when we consider regression in each group as a task. It is also called data sharing [19] since information contained in different groups is shared through the common parameter  $\beta_0^*$ . And finally, it has been called data enrichment [14] because we enrich our data set with pooling multiple samples from different but related sources.

Following the successful application of such a modeling scheme in recent years [16, 19, 30, 31], we consider the below *data enrichment* (DE) model:

$$(1.1) \quad y_{gi} = \phi(\mathbf{x}_{gi}, (\beta_0^* + \beta_g^*)) + w_{gi}, \quad g \in \{1, \dots, G\},$$

where  $g$  and  $i$  index the group and samples respectively. DE model (1.1) assumes that there is a *common* parameter  $\beta_0^*$  shared between all groups which models similarities between all samples. And there are *individual* per-group parameters  $\beta_g^*$ s each characterize the deviation of group  $g$ , Figure 1.

**The setting.** Our goal is to design an estimation procedure which consistently recovers all parameters of DE (1.1) fast and with small number of samples. We specifically focus on the high-dimensional small sample regime where the number of samples  $n_g$  for each group is much smaller than the ambient dimensionality, i.e.,  $\forall g : n_g \ll p$ . Similar to all other high-dimensional models, we assume that the parameters  $\beta_g$  are structured, i.e., for suitable convex functions  $f_g$ 's,  $f_g(\beta_g)$  is small. For example, when the structure is sparsity,  $f_g$ s are  $L_1$ -norms. Further, for the technical analysis and proofs, we focus on the case of linear models, i.e.,  $\phi(\mathbf{x}, \beta) = \mathbf{x}^T \beta$ . The results seamlessly extend to more general non-linear models, e.g., generalized linear models, broad families of semi-parametric and single-index models, non-convex models, etc., using existing results, i.e., how models like LASSO have been extended to these settings [27].

**1.1. Related Work.** In the context of *Multi-Task Learning* (MTL), similar models have been proposed which have the general form of  $y_{gi} = \mathbf{x}_{gi}^T(\beta_{1g}^* + \beta_{2g}^*) + w_{gi}$  where  $\mathbf{B}_1 = [\beta_{11}, \dots, \beta_{1G}]$  and  $\mathbf{B}_2 = [\beta_{21}, \dots, \beta_{2G}]$  are two parameter matrices [42]. To capture the relation of tasks, different types of constraints are assumed for parameter matrices. For example, [15] assumes  $\mathbf{B}_1$  and  $\mathbf{B}_2$  are sparse and low rank respectively. In this parameter matrix decomposition framework for MLT, the most related work to ours is the Dirty Model (DM) proposed in [23] where authors regularize the regression with  $\|\mathbf{B}_1\|_{1,\infty}$  and  $\|\mathbf{B}_2\|_{1,1}$  where norms are  $p, q$ -norms on rows of matrices, i.e.,  $\|\cdot\|_{p,q} = \|(\|\cdot\|_q, \dots, \|\cdot\|_q)\|_p$ .

If in our DE model we pick all structure inducing functions  $f_g$  to be  $l_1$ -norm, the resulting model is very similar to the DM where  $\|\mathbf{B}_1\|_{1,\infty}$  induces similarity between tasks and  $\|\mathbf{B}_2\|_{1,1}$  models their discrepancies. On the other hand, the degree of freedom of DM model is higher than DE because  $\|\mathbf{B}_1\|_{1,\infty}$  regularizer enforces shared support of  $\beta_{1g}^*$ s, i.e.,  $\text{supp}(\beta_{1i}^*) = \text{supp}(\beta_{1j}^*)$  but allows  $\beta_{1i}^* \neq \beta_{1j}^*$  while in DE we have a single common parameter  $\beta_0^*$ . So one would expect that DE estimators should have smaller sample complexity compared to their DM counterparts and our analysis confirm that our estimator is more data efficient than DM estimator of [23]. Mainly, they require every task  $i$  to have large enough samples to learn its own common parameters  $\beta_i$  but since DE shares the common parameter it only requires the *total dataset over all tasks* to be sufficiently large.

The linear DE model where  $\beta_g$ 's are sparse has recently gained attention because of its application in wide range of domains such as personalized medicine [16], sentiment analysis, banking strategy [19], single cell data analysis [31], road safety [30], and disease subtype analysis [16]. More generally, in any high-dimensional problem where the population consists of groups, data enrichment framework has the potential to boost the prediction accuracy and results in a more interpretable set of parameters.

**Motivation.** In spite of the recent surge in applying data enrichment framework to different domains, limited advances have been made in understanding the statistical and computational properties of suitable estimators for the DE model (1.1). In fact, non-asymptotic statistical properties, including sample complexity and statistical rates of convergence, of regularized estimators for the data enriched model is still an open question [19, 30]. To the best of our knowledge, the only theoretical guarantee for data enrichment is provided in [31] where authors prove sparsistency of their proposed method under the stringent irrepresentability condition of the design matrix for recovering supports of common and individual parameters. Existing support recovery guarantees [31], sample complexity and  $l_2$  consistency results [23] of related MTL models are restricted to sparsity and  $l_1$ -norm, while our estimator and *norm consistency* analysis work for *any* structure induced by arbitrary convex functions  $f_g$ . Moreover, no computational results, such as rates of convergence of the estimation procedures exist in the literature.

**1.2. Notation and Preliminaries.** We denote sets by curly  $\mathcal{V}$ , matrices by bold capital  $\mathbf{V}$ , random variables by capital  $V$ , and vectors by small bold  $\mathbf{v}$  letters. We take  $[G] = \{1, \dots, G\}$  and  $[G_+] = [G] \cup \{0\}$ . Throughout the manuscript  $c_i$  and  $C_i$  denote positive absolute constants. Given  $G$  groups and  $n_g$  samples in each as  $\{\{\mathbf{x}_{gi}, y_{gi}\}_{i=1}^{n_g}\}_{g=1}^G$ , we can form the per group design matrix  $\mathbf{X}_g \in \mathbb{R}^{n_g \times p}$  and output vector  $\mathbf{y}_g \in \mathbb{R}^{n_g}$ . The total number of samples is  $n = \sum_{g=1}^G n_g$  and the data enriched model takes the following vector form:

$$(1.2) \quad \mathbf{y}_g = \mathbf{X}_g(\beta_0^* + \beta_g^*) + \mathbf{w}_g, \quad \forall g \in [G]$$

where each row of  $\mathbf{X}_g$  is  $\mathbf{x}_{gi}^T$  and  $\mathbf{w}_g^T = (w_{g1}, \dots, w_{gn_g})$  is the noise vector. It is useful for indexing to consider the common parameter as the zeroth group and define  $n_0 \triangleq n$  and  $\mathbf{X}_0 \triangleq [\mathbf{X}_1^T, \dots, \mathbf{X}_G^T]^T$ .

**Sub-Gaussian random variable and vector.** A random variable  $V$  is sub-Gaussian if its moments satisfies  $\forall p \geq 1 : (\mathbb{E}|V|^p)^{1/p} \leq K_2 \sqrt{p}$ . The minimum value of  $K_2$  is called the sub-Gaussian norm of  $V$ , denoted by  $\|V\|_{\psi_2}$  [39]. A random vector  $\mathbf{v} \in \mathbb{R}^p$  is sub-Gaussian if the one-dimensional marginals  $\langle \mathbf{v}, \mathbf{u} \rangle$  are sub-Gaussian random variables for all  $\mathbf{u} \in \mathbb{R}^p$ . The sub-Gaussian norm of  $\mathbf{v}$  is defined [39] as  $\|\mathbf{v}\|_{\psi_2} = \sup_{\mathbf{u} \in \mathbb{S}^{p-1}} \|\langle \mathbf{v}, \mathbf{u} \rangle\|_{\psi_2}$ . For any set  $\mathcal{V} \in \mathbb{R}^p$  the Gaussian width of the set  $\mathcal{V}$  is defined as  $\omega(\mathcal{V}) = \mathbb{E}_{\mathbf{g}} [\sup_{\mathbf{u} \in \mathcal{V}} \langle \mathbf{g}, \mathbf{u} \rangle]$  [40], where the expectation is over  $\mathbf{g} \sim N(\mathbf{0}, \mathbf{I}_{p \times p})$ , a vector of independent zero-mean unit-variance Gaussian. The marginal tail function is defined as  $Q_\xi(\mathbf{u}) = \mathbb{P}(|\langle \mathbf{x}, \mathbf{u} \rangle| > \xi)$  for a fixed vector  $\mathbf{u}$ , random vector  $\mathbf{x}$  and  $\xi > 0$ .

**1.3. Our Contributions.** We propose the following Data Enrichment (DE) estimator  $\hat{\beta}$  for recovering the structured parameters where the structure is induced by *convex* functions  $f_g(\cdot)$ :

$$\hat{\beta} = (\hat{\beta}_0^T, \dots, \hat{\beta}_G^T) \in \underset{\beta_0, \dots, \beta_G}{\operatorname{argmin}} \frac{1}{n} \sum_{g=1}^G \|\mathbf{y}_g - \mathbf{X}_g(\beta_0 + \beta_g)\|_2^2, \text{ s.t. } \forall g \in [G] : f_g(\beta_g) \leq f_g(\beta_g^*). \quad (1.3)$$

We present several statistical and computational results for the DE estimator (1.3):

- The DE estimator (1.3) succeeds if a geometric condition that we call *Data EnRichment Incoherence Condition* (DERIC) is satisfied, Figure 2b. Compared to other known geometric conditions in the literature such as structural coherence [20] and stable recovery conditions [25], DERIC is a considerably weaker condition, Figure 2a.
- Assuming DERIC holds, we establish a high probability non-asymptotic bound on the weighted sum of parameter-wise estimation error,  $\delta_g = \hat{\beta}_g - \beta_g^*$  as:

$$\sum_{g=0}^G \sqrt{\frac{n_g}{n}} \|\delta_g\|_2 \leq \gamma O \left( \frac{\max_{g \in [G]} \omega(\mathcal{C}_g \cap \mathbb{S}^{p-1})}{\sqrt{n}} \right), \quad (1.3)$$

where  $n_0 \triangleq n$  is the total number of samples,  $\gamma \triangleq \max_{g \in [G]} \frac{n}{n_g}$  is the *sample condition number*, and  $\mathcal{C}_g$  is the error cone corresponding to  $\beta_g^*$  exactly defined in section 2. To the best of our knowledge, this is the first statistical estimation guarantee for the data enrichment.

- We also establish the sample complexity of the DE estimator for all parameters as  $\forall g \in [G_+] : n_g = O(\omega(\mathcal{C}_g \cap \mathbb{S}^{p-1}))^2$ . We emphasize that our result proves that the recovery of the common parameter  $\beta_0$  by DE estimator (1.3) benefits from *all* of the  $n$  pooled samples.
- We present an efficient projected block gradient descent algorithm DICER, to solve DE's objective (1.3) which converges geometrically to the statistical error bound of (1.3). To the best of our knowledge, this is the first rigorous computational result for the high-dimensional data-enriched regression.
- We illustrate promising empirical performance of the model on synthetic data as well as on the problem of finding bio-markers associated with drug sensitivity of cell lines from different cancer types, where the support of estimated individual parameters  $\operatorname{supp}(\hat{\beta}_g)$  for each cancer type  $g$  represents a different set of bio-markers per cancer type.

The rest of this paper is organized as follows: First, we characterize the error set of our estimator and provide a deterministic error bound in section 2. Then in section 3, we discuss the restricted eigenvalue condition and calculate the sample complexity required for the recovery of the true parameters by our estimator under DERIC condition. We close the statistical analysis in section 4 by providing

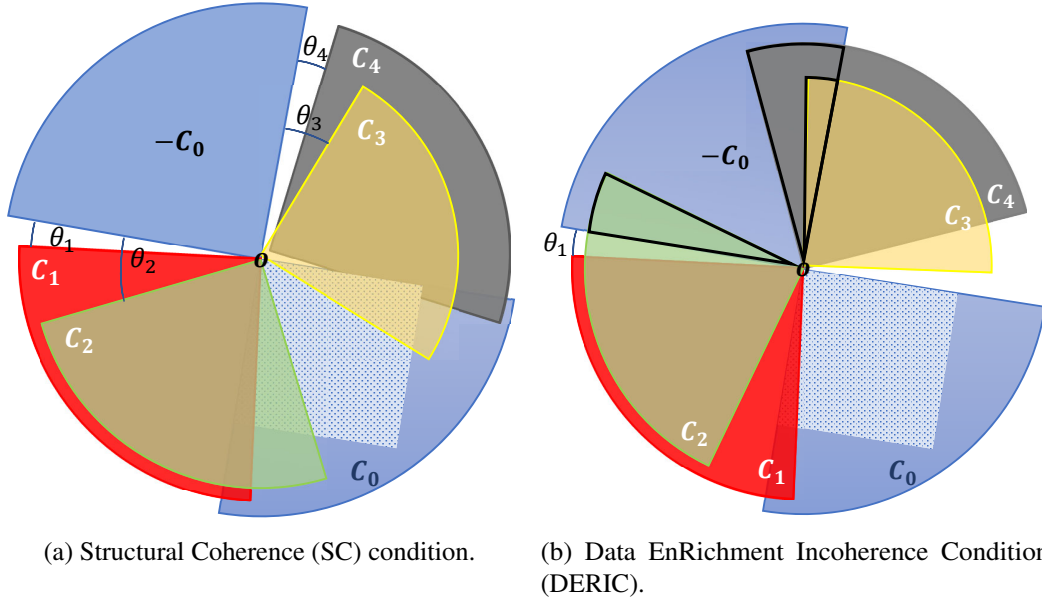


Figure 2: a) State-of-the-art condition for recovering common and individual parameters in superposition models where  $C_g = \text{Cone}(\mathcal{E}_g)$  are error cones and  $\mathcal{E}_g = \{\delta_g | f_g(\beta_g^* + \delta_g) \leq f_g(\beta_g^*)\}$  are the error sets for each parameter  $\beta_g^* \in [G]$  [20]. b) Our more relaxed recovery condition which allows *arbitrary non-zero fraction* of the error cones of individual parameters intersect with  $-C_0$ .

138 non-asymptotic high probability error bound for parameter recovery. We delineate our geometrically  
 139 convergent algorithm, DICER in section 5 and finally supplement our work with synthetic experiments  
 140 in section 7. *6? Talk about proofs. What are our three main thms?*

141 **2. The Data Enrichment Estimator.** A compact form of our proposed DE estimator (1.3) is:

$$142 \quad (2.1) \quad \hat{\beta} \in \underset{\beta}{\operatorname{argmin}} \frac{1}{n} \|\mathbf{y} - \mathbf{X}\beta\|_2^2, \quad \text{s.t. } \forall g \in [G] \setminus \{0\} : f_g(\beta_g) \leq f_g(\beta_g^*),$$

143 where  $\mathbf{y} = (\mathbf{y}_1^T, \dots, \mathbf{y}_G^T)^T \in \mathbb{R}^n$ ,  $\beta = (\beta_0^T, \dots, \beta_G^T)^T \in \mathbb{R}^{(G+1)p}$  and

$$144 \quad (2.2) \quad \mathbf{X} = \begin{pmatrix} \mathbf{X}_1 & \mathbf{X}_1 & 0 & \dots & 0 \\ \mathbf{X}_2 & 0 & \mathbf{X}_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \dots & \vdots \\ \mathbf{X}_G & 0 & \dots & \dots & \mathbf{X}_G \end{pmatrix} \in \mathbb{R}^{n \times (G+1)p}.$$

145 **Example 2.1. ( $L_1$ -norm)** When all parameters  $\beta_g$ s are  $s_g$ -sparse, i.e.,  $|\operatorname{supp}(\beta_g^*)| = s_g$  by using  
 146  $l_1$ -norm as the sparsity inducing function, our DE estimator of (2.1) becomes:

$$147 \quad (2.3) \quad \hat{\beta} \in \underset{\beta}{\operatorname{argmin}} \frac{1}{n} \|\mathbf{y} - \mathbf{X}\beta\|_2^2, \quad \text{s.t. } \forall g \in [G] \setminus \{0\} : \|\beta_g\|_1 \leq \|\beta_g^*\|_1.$$

We call [Example 2.1](#) *sparse DE* estimator and use it as the running example throughout the paper to illustrate outcomes of our analysis.

Consider the group-wise estimation error  $\delta_g = \hat{\beta}_g - \beta_g^*$ . Since  $\hat{\beta}_g = \beta_g^* + \delta_g$  is a feasible point of [\(2.1\)](#), the error vector  $\delta_g$  will belong to the following restricted error set:

$$(2.4) \quad \mathcal{E}_g = \{\delta_g | f_g(\beta_g^* + \delta_g) \leq f_g(\beta_g^*)\}, \quad g \in [G] \text{ ~~not~~ }.$$

We denote the cone of the error set as  $\mathcal{C}_g \triangleq \text{Cone}(\mathcal{E}_g)$  and the spherical cap corresponding to it as  $\mathcal{A}_g \triangleq \mathcal{C}_g \cap \mathbb{S}^{p-1}$ . Consider the set  $\mathcal{C} = \{\delta = (\delta_0^T, \dots, \delta_G^T)^T | \delta_g \in \mathcal{C}_g\}$ , following two subsets of  $\mathcal{C}$  play key roles in our analysis:

$$(2.5) \quad \mathcal{H} \triangleq \left\{ \delta \in \mathcal{C} \mid \sum_{g=0}^G \frac{n_g}{n} \|\delta_g\|_2 = 1 \right\}, \quad \bar{\mathcal{H}} \triangleq \left\{ \delta \in \mathcal{C} \mid \sum_{g=0}^G \sqrt{\frac{n_g}{n}} \|\delta_g\|_2 = 1 \right\}.$$

Starting from the optimality of  $\hat{\beta} = \beta^* + \delta$  as  $\frac{1}{n} \|\mathbf{y} - \mathbf{X}\hat{\beta}\|_2^2 \leq \frac{1}{n} \|\mathbf{y} - \mathbf{X}\beta^*\|_2^2$ , we have:  $\frac{1}{n} \|\mathbf{X}\delta\|_2^2 \leq \frac{1}{n} 2\mathbf{w}^T \mathbf{X}\delta$  where  $\mathbf{w} = [\mathbf{w}_1^T, \dots, \mathbf{w}_G^T]^T \in \mathbb{R}^n$  is the vector of all noises. Using this basic inequality, we can establish the following deterministic error bound.

**Theorem 2.2.** *For the DE estimator [\(2.1\)](#), assume there exist  $0 < \kappa \leq \inf_{\mathbf{u} \in \mathcal{H}} \frac{1}{n} \|\mathbf{X}\mathbf{u}\|_2^2$ . Then, for the sample condition number  $\gamma = \max_{g \in [G]} \frac{n}{n_g}$ , the following deterministic upper bounds holds:*

$$\sum_{g=0}^G \sqrt{\frac{n_g}{n}} \|\delta_g\|_2 \leq \frac{2\gamma \sup_{\mathbf{u} \in \bar{\mathcal{H}}} \mathbf{w}^T \mathbf{X}\mathbf{u}}{n\kappa}.$$

*Proof.* We lower bound the LHS and upper bound the RHS of the optimality inequality  $\frac{1}{n} \|\mathbf{X}\delta\|_2^2 \leq \frac{1}{n} 2\mathbf{w}^T \mathbf{X}\delta$  using the definition of the sets  $\mathcal{H}$  and  $\bar{\mathcal{H}}$  respectively. Starting with the lower bound using the definition of set  $\mathcal{H}$  [\(2.5\)](#) we have:

$$(2.6) \quad \begin{aligned} \frac{1}{n} \|\mathbf{X}\delta\|_2^2 &\geq \frac{1}{n} \inf_{\mathbf{u} \in \mathcal{H}} \|\mathbf{X}\mathbf{u}\|_2^2 \left( \sum_{g=0}^G \frac{n_g}{n} \|\delta_g\|_2 \right)^2 \geq \kappa \left( \sum_{g=0}^G \frac{n_g}{n} \|\delta_g\|_2 \right)^2 \\ &\geq \kappa \left( \min_{g \in [G]} \sqrt{\frac{n_g}{n}} \right)^2 \left( \sum_{g=0}^G \sqrt{\frac{n_g}{n}} \|\delta_g\|_2 \right)^2 \end{aligned}$$

where  $0 < \kappa \leq \frac{1}{n} \inf_{\mathbf{u} \in \mathcal{H}} \|\mathbf{X}\mathbf{u}\|_2^2$  is known as Restricted Eigenvalue (RE) condition. The upper bound factorizes as:

$$(2.7) \quad \frac{2}{n} \mathbf{w}^T \mathbf{X}\delta \leq \frac{2}{n} \sup_{\mathbf{u} \in \bar{\mathcal{H}}} \mathbf{w}^T \mathbf{X}\mathbf{u} \left( \sum_{g=0}^G \sqrt{\frac{n_g}{n}} \|\delta_g\|_2 \right), \quad \mathbf{u} \in \mathcal{H}$$

Putting together inequalities [\(2.6\)](#) and [\(2.7\)](#) completes the proof. ■



173 *Remark 2.3.* Consider the setting where  $n_g = \Theta(\frac{n}{G})$  so that each group has approximately  $\frac{1}{G}$   
 174 fraction of the samples. Then,  $\gamma = \Theta(G)$  and hence

$$175 \quad \frac{1}{G} \sum_{g=0}^G \|\delta_g\|_2 \leq O(G^{1/2}) \frac{\sup_{\mathbf{u} \in \mathcal{H}} \boldsymbol{\omega}^T \mathbf{X} \mathbf{u}}{n}.$$

176 **3. Restricted Eigenvalue Condition.** The main assumptions of [Theorem 2.2](#) is known as  
 177 Restricted Eigenvalue (RE) condition in the literature of high-dimensional statistics [2, 29, 34]:  
 178  $\inf_{\mathbf{u} \in \mathcal{H}} \frac{1}{n} \|\mathbf{X} \mathbf{u}\|_2^2 \geq \kappa > 0$ . The RE condition posits that the minimum eigenvalues of the matrix  
 179  $\mathbf{X}^T \mathbf{X}$  in directions restricted to  $\mathcal{H}$  is strictly positive. In this section, we show that for the design  
 180 matrix  $\mathbf{X}$  defined in (2.2), the RE condition holds with high probability under a suitable geometric  
 181 condition we call *Data EnRichment Incoherence Condition* (DERIC) and for enough number of samples.  
 182 We precisely characterize total and per-group sample complexities required for successful parameter  
 183 recovery. For the analysis, similar to existing work [20, 26, 37], we assume the design matrix to be  
 184 isotropic sub-Gaussian.<sup>1</sup>

185 **Definition 3.1.** We assume  $\mathbf{x}_{gi}$  are i.i.d. random vectors from a non-degenerate zero-mean,  
 186 isotropic sub-Gaussian distribution. In other words,  $\mathbb{E}[\mathbf{x}] = 0$ ,  $\mathbb{E}[\mathbf{x}^T \mathbf{x}] = \mathbf{I}_{p \times p}$ , and  $\|\mathbf{x}\|_{\psi_2} \leq k_x$ . As a  
 187 consequence,  $\exists \alpha > 0$  such that  $\forall \mathbf{u} \in \mathbb{S}^{p-1}$  we have  $\mathbb{E}|\langle \mathbf{x}, \mathbf{u} \rangle| \geq \alpha$ . Further, we assume noise  $\mathbf{w}_{gi}$  are  
 188 i.i.d. zero-mean, unit-variance sub-Gaussian with  $\|\mathbf{w}_{gi}\|_{\psi_2} \leq k_w$ .

189 **3.1. Geometric Condition of Recovery.** Unlike standard high-dimensional statistical esti-  
 190 mation, for RE condition to be true, parameters of superposition models need to satisfy geometric  
 191 conditions which limits the interaction of parameters with each other to make sure that recovery is  
 192 possible. In this section, we elaborate our sufficient geometric condition for recovery and compare it  
 193 with state-of-the-art condition for recovery of superposition models.

194 To intuitively illustrate the necessity of such a geometric condition, consider the simplest super-  
 195 position model i.e.,  $\beta_0^* + \beta_g^*$ . Without any restriction on parameter interactions, any estimates such  
 196 that  $\hat{\beta}_0 + \hat{\beta}_g = \beta_0^* + \beta_g^*$  are valid ones. To avoid such trivial solutions two error cones need to satisfy  
 197  $\delta_g \neq -\delta_0$ . In general, the RE condition of individual superposition models can be established under  
 198 the so-called Structural Coherence (SC) condition [20, 25] which is the generalization of this idea for  
 199 superposition of multiple parameters as  $\sum_{g=0}^G \beta_g^*$ .

200 **Definition 3.2 (Structural Coherence (SC) [20, 25]).** Consider a superposition model of the  
 201 form  $y = \mathbf{x}^T \sum_{g=0}^G \beta_g^* + w$ . The SC condition requires that

$$202 \quad \forall \delta_g \in \mathcal{C}_g, \exists \lambda \quad s.t. \quad \left\| \sum_{g=0}^G \delta_g \right\|_2 \geq \lambda \sum_{g=0}^G \|\delta_g\|_2,$$

203 and leads to the corresponding RE condition for the superposition model.

204 *Remark 3.3.* Note that the SC condition is satisfied if none of the individual error cones  $\mathcal{C}_g$  intersect  
 205 with the inverted error cone  $-\mathcal{C}_0$  [20, 37], i.e.,  $\forall g, \theta_g > 0$  in [Figure 2a](#) where

$$206 \quad \cos(\theta_g) = \sup_{\delta_0 \in \mathcal{C}_0, \delta_g \in \mathcal{C}_g} -\langle \delta_0 / \|\delta_0\|_2, \delta_g / \|\delta_g\|_2 \rangle.$$

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<sup>1</sup>Extension to an-isotropic sub-Gaussian case is straightforward by techniques developed in [2, 35].

Next, we introduce DERIC, a considerably weaker geometric condition compared to SC which leads to recovery of all parameters in the data enriched model.

**Definition 3.4 (Data EnRichment Incoherence Condition (DERIC)).** *There exists a non-empty set  $\mathcal{I} \subseteq [G]$  of groups where for some scalars  $0 < \bar{\rho} \leq 1$  and  $\lambda_{\min} > 0$  the following holds:*

1.  $\sum_{g \in \mathcal{I}} n_g \geq \lceil \bar{\rho} n \rceil$ .
2.  $\forall g \in \mathcal{I}, \forall \delta_g \in \mathcal{C}_g, \text{ and } \delta_0 \in \mathcal{C}_0: \|\delta_g + \delta_0\|_2 \geq \lambda_{\min}(\|\delta_0\|_2 + \|\delta_g\|_2)$

Observe that  $0 < \lambda_{\min}, \bar{\rho} \leq 1$  by definition.

**Remark 3.5.** Comparing to the SC conditions, DERIC holds even if only one of the  $\mathcal{C}_g$ s does not intersect with  $-\mathcal{C}_0$ . More specifically, DERIC holds if  $\exists g, \theta_g > 0$  in Figure 2b. Therefore, instead of SC stringent geometric condition, DERIC allows  $-\mathcal{C}_0$  to intersect with an arbitrarily large fraction of the  $\mathcal{C}_g$  cones and as the number of intersections increases, our final error bound becomes looser.

**3.2. Sample Complexity.** An alternative to our DE estimator (1.3) may be based on  $G$  isolated superposition model  $\mathbf{y}_g = \mathbf{X}_g(\beta_0^* + \beta_g^*) + \mathbf{w}_g$  each with two components. Now, if SC holds for at least one of the superposition models, i.e.,  $\exists g, -\mathcal{C}_0 \cap \mathcal{C}_g = \emptyset$ , one can recover  $\hat{\beta}_0$  and plug it in to the remaining  $G - 1$  superposition estimators to estimate the corresponding  $\hat{\beta}_g$ s. We call such an estimator, *plugin superposition* estimator. For such a trivial (based on existing literature) estimator, it seems that DERIC has no advantage over SC. But the downfall of such estimator is that it fails to utilize the true coupling structure in the data enriched model, where  $\beta_0^*$  is involved in all groups. In fact, below we show, the above combination of superposition and plug-in estimators using SC leads to a pessimistic estimates of the sample complexity for  $\beta_0^*$  recovery.

**Proposition 3.6.** *Assume observations distributed as defined in Definition 3.1 and pair-wise SC conditions are satisfied. Consider each superposition model (1.2) in isolation; to recover the common parameter  $\beta_0^*$  plugin superposition requires at least one group  $i$  to have  $n_i = O(\max(\omega^2(\mathcal{A}_0), \omega^2(\mathcal{A}_i)))$ . To recover the rest of individual parameters, it needs  $\forall g \neq i : n_g = O(\omega^2(\mathcal{A}_g))$  samples.*

In other words, by separate analysis of superposition estimators at least one problem needs to have sufficient samples for recovering the common parameter  $\beta_0$  and therefore the common parameter recovery does not benefit from the pooled  $n$  samples. But given the nature of coupling in the data enriched model, we hope to be able to get a better sample complexity specifically for the common parameter  $\beta_0$ . Using DERIC and the small ball method [26], a tool from empirical process theory, in the following theorem, we get a better sample complexity required for satisfying the RE condition:

**Theorem 3.7.** *Let  $\mathbf{x}_{gi}$ s be random vectors defined in Definition 3.1. Assume DERIC condition of Definition 3.4 holds for error cones  $\mathcal{C}_g$ s and  $\psi_{\mathcal{I}} = \min\{1/2, \lambda_{\min}\bar{\rho}/3\}$ . Then, for all  $\delta \in \mathcal{H}$ , when we have enough number of samples as  $\forall g \in [G_+] : n_g \geq m_g = O(k_x^6 \alpha^{-6} \psi_{\mathcal{I}}^{-2} \omega(\mathcal{A}_g)^2)$ , with probability at least  $1 - e^{-n\kappa_{\min}/4}$  we have:*

$$\inf_{\delta \in \mathcal{H}} \frac{1}{\sqrt{n}} \|\mathbf{X}\delta\|_2 \geq \frac{\kappa_{\min}}{2}$$

where  $\kappa_{\min} = \min_{g \in [G_+]} C\psi_{\mathcal{I}} \frac{\alpha^3}{k_x^2} - \frac{2c_g k_x \omega(\mathcal{A}_g)}{\sqrt{n_g}}$ .

**Remark 3.8.** Note that  $\kappa = \frac{\kappa_{\min}}{4}$  is the lower bound of the RE condition of Theorem 2.2, i.e.,  $0 < \kappa \leq \inf_{\mathbf{u} \in \mathcal{H}} \frac{1}{n} \|\mathbf{X}\mathbf{u}\|_2^2$  and is determined by the group with the worst individual RE condition.



	GI-LASSO	Dirty Stat. Model	Plugin Superposition	Sparse DE
$m_g$	$s_{0g} \log p$	$G \max_{g \in [G]} s_{0g} \log(p)$	$\exists i \in [G] : \max(s_0, s_i) \log p$ $\forall g \neq i : s_g \log p$	$s_g \log p$

Table 1: Comparison of the order of per group number of samples (sample complexities) of various methods for recovering sparse DE parameters. Let  $s_{0g} = |\text{support}(\beta_0^* + \beta_g^*)|$  be the superimposed support where  $s_0, s_g \leq s_{0g} \leq \max(s_0, s_g)$ .

**Example 3.9. ( $L_1$ -norm)** The Gaussian width of the spherical cap of a  $p$ -dimensional  $s$ -sparse vector is  $\omega(\mathcal{A}) = \Theta(\sqrt{s \log p})$  [2, 40]. Therefore, the number of samples per group and total required for satisfaction of the RE condition in the sparse DE estimator **Example 2.1** is  $\forall g \in [G] : n_g \geq m_g = \Theta(s_g \log p)$ . **Table 1** compares sample complexities of sparse-DE estimator with three baselines: plugin superposition estimator of **Proposition 3.6**, G Independent LASSO (GI-LASSO), and Jalali's Dirty Statistical Model (DSM) [23]. Note that GI-LASSO does not recover the common parameter and DSM needs all groups have same number of samples.

**3.3. Proof of Theorem 3.7.** Let's simplify the LHS of the RE condition:

$$\begin{aligned}
\frac{1}{\sqrt{n}} \|\mathbf{X}\boldsymbol{\delta}\|_2 &= \left( \frac{1}{n} \sum_{g=1}^G \sum_{i=1}^{n_g} |\langle \mathbf{x}_{gi}, \boldsymbol{\delta}_0 + \boldsymbol{\delta}_g \rangle|^2 \right)^{\frac{1}{2}} \\
&\stackrel{\text{(Lyapunov's inequality)}}{\geq} \frac{1}{n} \sum_{g=1}^G \sum_{i=1}^{n_g} |\langle \mathbf{x}_{gi}, \boldsymbol{\delta}_0 + \boldsymbol{\delta}_g \rangle| \\
&\geq \frac{1}{n} \sum_{g=1}^G \xi \|\boldsymbol{\delta}_0 + \boldsymbol{\delta}_g\|_2 \sum_{i=1}^{n_g} \mathbb{1}(|\langle \mathbf{x}_{gi}, \boldsymbol{\delta}_0 + \boldsymbol{\delta}_g \rangle| \geq \xi \|\boldsymbol{\delta}_0 + \boldsymbol{\delta}_g\|_2) \\
&= \frac{1}{n} \sum_{g=1}^G \xi_g \sum_{i=1}^{n_g} \mathbb{1}(|\langle \mathbf{x}_{gi}, \boldsymbol{\delta}_{0g} \rangle| \geq \xi_g),
\end{aligned}$$

where to avoid cluttering we denoted  $\boldsymbol{\delta}_{0g} = \boldsymbol{\delta}_0 + \boldsymbol{\delta}_g$  and  $\xi_g = \xi \|\boldsymbol{\delta}_{0g}\|_2 > 0$ . Now we add and subtract the corresponding per-group marginal tail function,  $Q_{\xi_g}(\boldsymbol{\delta}_{0g}) = \mathbb{P}(|\langle \mathbf{x}, \boldsymbol{\delta}_{0g} \rangle| > \xi_g)$  and take inf:

$$\begin{aligned}
\inf_{\boldsymbol{\delta} \in \mathcal{H}} \frac{1}{\sqrt{n}} \|\mathbf{X}\boldsymbol{\delta}\|_2 &\geq \inf_{\boldsymbol{\delta} \in \mathcal{H}} \sum_{g=1}^G \frac{n_g}{n} \xi_g Q_{2\xi_g}(\boldsymbol{\delta}_{0g}) - \sup_{\boldsymbol{\delta} \in \mathcal{H}} \frac{1}{n} \sum_{g=1}^G \xi_g \sum_{i=1}^{n_g} [Q_{2\xi_g}(\boldsymbol{\delta}_{0g}) - \mathbb{1}(|\langle \mathbf{x}_{gi}, \boldsymbol{\delta}_{0g} \rangle| \geq \xi_g)] \\
(3.1) \quad &= t_1(\mathbf{X}) - t_2(\mathbf{X})
\end{aligned}$$

For the ease of exposition we consider the LHS of (3.1) as the difference of two terms, i.e.,  $t_1(\mathbf{X}) - t_2(\mathbf{X})$  and in the followings we lower bound the first term  $t_1$  and upper bound the second term  $t_2$ .

**3.3.1. Lower Bounding the First Term.** Our main result is the following lemma which uses the DERIC condition of the **Definition 3.4** and provides a lower bound for the first term  $t_1(\mathbf{X})$ :

266 **Lemma 3.10.** Suppose DERIC holds. Let  $\psi_{\mathcal{I}} = \frac{\lambda_{\min} \bar{\rho}}{3}$ . For any  $\delta \in \mathcal{H}$ , we have:

$$267 \quad (3.2) \quad \sum_{g=1}^G \frac{n_g}{n} \xi_g Q_{2\xi_g}(\delta_{0g}) \geq \psi_{\mathcal{I}} \xi \frac{(\alpha - 2\xi)^2}{4ck_x^2} \sum_{g=0}^n \frac{n_g}{n} \|\delta_g\|_2,$$

is lower bounded by the

268 **Lemma 3.10** implies that  $t_1(\mathbf{X})$  satisfies ~~the same~~ RHS bound of (3.2).

269 **3.3.2. Upper Bounding the Second Term.** First we show  $t_2(\mathbf{X})$  satisfies the bounded dif-  
 270 ference property defined in Section 3.2. of [6], i.e., by changing each of  $\mathbf{x}_{gi}$  the value of  $t_2(\mathbf{X})$  at  
 271 most change by one. We rewrite  $t_2$  as  $t_2(\mathbf{X}) = \sup_{\delta \in \mathcal{H}} g_{\delta}(\mathbf{X})$  where  $g_{\delta}(\mathbf{X})$  is the argument of sup  
 272 in (3.1). Now we denote the design matrix resulted from replacement of  $k$ th sample from  $j$ th group  
 273  $\mathbf{x}_{jk}$  with another sample  $\mathbf{x}'_{jk}$  by  $\mathbf{X}'_{jk}$ . Then our goal is to show  $\forall j \in [G], k \in [n_j], \sup_{\mathbf{X}, \mathbf{x}'_{jk}} |t_2(\mathbf{X}) -$   
 274  $t_2(\mathbf{X}'_{jk})| \leq c_i$  for some constant  $c_i$ . Note that for bounded functions  $f, g : \mathcal{X} \rightarrow \mathbb{R}$ , we have  
 275  $|\sup_{\mathcal{X}} f - \sup_{\mathcal{X}} g| \leq \sup_{\mathcal{X}} |f - g|$ . Therefore:

$$\begin{aligned} 276 \quad \sup_{\mathbf{X}, \mathbf{x}'_{jk}} |t_2(\mathbf{X}) - t_2(\mathbf{X}'_{jk})| &\leq \sup_{\mathbf{X}, \mathbf{x}'_{jk}} \sup_{\delta \in \mathcal{H}} |g(\mathbf{X}) - g(\mathbf{X}'_{jk})| \\ 277 \quad &\leq \sup_{\mathbf{x}_{jk}, \mathbf{x}'_{jk}} \sup_{\delta \in \mathcal{H}} \frac{\xi_j}{n} |\mathbb{1}(|\langle \mathbf{x}'_{jk}, \delta_{0j} \rangle| \geq \xi_j) - \mathbb{1}(|\langle \mathbf{x}_{jk}, \delta_{0j} \rangle| \geq \xi_j)| \\ 278 \quad &\leq \sup_j \sup_{\delta \in \mathcal{H}} \frac{\xi_j}{n} = \frac{\xi}{n} \sup_j \sup_{\delta \in \mathcal{H}} \|\delta_0 + \delta_j\|_2 \\ 279 \quad &\leq \frac{\xi}{n} \sup_j \sup_{\delta \in \mathcal{H}} \|\delta_0\|_2 + \|\delta_j\|_2 \\ 280 \quad (\delta \in \mathcal{H}) &\leq \xi \left( \frac{1}{n} + \frac{1}{n_j} \right) \leq \frac{2\xi}{n}, \end{aligned}$$

281 Note that for  $\delta \in \mathcal{H}$  we have  $\|\delta_0\|_2 + \frac{n_g}{n} \|\delta_g\|_2 \leq 1$  which results in  $\|\delta_0\|_2 \leq 1$  and  $\|\delta_g\|_2 \leq \frac{n}{n_g}$ , which just  
 282 Now, we can invoke the bounded difference inequality from Theorem 6.2 of [6] which says that with  
 283 probability at least  $1 - e^{-\tau^2/2}$  we have:  $t_2(\mathbf{X}) \leq \mathbb{E} t_2(\mathbf{X}) + \frac{\tau}{\sqrt{n}}$ . Having this concentration bound, it is  
 284 enough to bound the expectation of  $t_2(\mathbf{X})$  using the following lemma:

285 **Lemma 3.11.** For the random vector  $\mathbf{x}$  of Definition 3.1, we have the following bound:

$$286 \quad \frac{2}{n} \mathbb{E} \sup_{\delta \in \mathcal{H}} \sum_{g=1}^G \xi_g \sum_{i=1}^{n_g} [Q_{2\xi_g}(\delta_{0g}) - \mathbb{1}(|\langle \mathbf{x}_{gi}, \delta_{0g} \rangle| \geq \xi_g)] \leq \frac{2}{\sqrt{n}} \sum_{g=0}^G \sqrt{\frac{n_g}{n}} c_g k \omega(\mathcal{A}_g) \|\delta_g\|_2$$

287 **3.3.3. Continuing the Proof of Theorem 3.7.** Define  $q \triangleq \frac{(\alpha-2\xi)^2}{4ck^2}$ . Putting back bounds of  
 288  $t_1(\mathbf{X})$  and  $t_2(\mathbf{X})$  together from Lemmas 3.10 and 3.11, with probability at least  $1 - e^{-\frac{\tau^2}{2}}$  we have:

$$\begin{aligned}
 289 \quad \inf_{\delta \in \mathcal{H}} \frac{1}{\sqrt{n}} \|\mathbf{X}\delta\|_2 &\leq \sum_{g=0}^G \frac{n_g}{n} \psi_{\mathcal{I}} \xi \|\delta_g\|_2 q - \frac{2}{\sqrt{n}} \sum_{g=0}^G \sqrt{\frac{n_g}{n}} k_x c_g \omega(\mathcal{A}_g) \|\delta_g\|_2 - \frac{\tau}{\sqrt{n}} \\
 290 &= n^{-1} \sum_{g=0}^G n_g \|\delta_g\|_2 (\psi_{\mathcal{I}} \xi q - 2c_g k_x \frac{\omega(\mathcal{A}_g)}{\sqrt{n_g}}) - \frac{\tau}{\sqrt{n}} \\
 291 \quad (\kappa_g = \psi_{\mathcal{I}} \xi q - \frac{2c_g k_x \omega(\mathcal{A}_g)}{\sqrt{n_g}}) &= \sum_{g=0}^G \frac{n_g}{n} \|\delta_g\|_2 \kappa_g - \frac{\tau}{\sqrt{n}} \\
 292 &\geq \kappa_{\min} \sum_{g=0}^G \frac{n_g}{n} \|\delta_g\|_2 - \frac{\tau}{\sqrt{n}} \\
 293 \quad (\delta \in \mathcal{H}) &= \kappa_{\min} - \frac{\tau}{\sqrt{n}}
 \end{aligned}$$

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294 where  $\kappa_{\min} = \min_{g \in [G]} \kappa_g$ . To conclude the proof, take  $\tau = \sqrt{n} \kappa_{\min} / 2$ .  
 295 ~~Note that all  $\kappa_g$ s should be bounded away from zero. To this end we need the following sample~~  
 296 complexities  $\forall g \in [G_+] : \left( \frac{2c_g k}{\psi_{\mathcal{I}} \xi q} \right)^2 \omega(\mathcal{A}_g)^2 \leq n_g$  where by taking  $\xi = \frac{\alpha}{6}$  simplifies to:

$$297 \quad \forall g \in [G_+] : O(k^6 \psi_{\mathcal{I}}^{-2} \alpha^{-6} \omega(\mathcal{A}_g)^2) \leq n_g$$

298 **4. General Error Bound.** In this section, we present our main statistical result which is a  
 299 non-asymptotic high probability upper bound for the estimation error of the common and individual  
 300 parameters.

301 **Theorem 4.1.** For  $\mathbf{x}_{gi}$  and  $w_{gi}$  described in Definition 3.1 when we have enough number of  
 302 samples  $\forall g \in [G_+] : n_g > m_g$  which lead to  $\kappa > 0$ , the following general error bound holds for  
 303 estimator (2.1) with probability at least  $1 - \sigma \exp(-\min[\nu \min_{g \in [G]} n_g - \log(G+1), \tau^2])$ :

$$304 \quad (4.1) \quad \sum_{g=0}^G \sqrt{\frac{n_g}{n}} \|\delta_g\|_2 \leq C \gamma \frac{\max_{g \in [G_+]} \omega(\mathcal{A}_g) + \sqrt{\log(G+1)} + \tau}{\kappa_{\min}^2 \sqrt{n}}$$

305 where  $\gamma = \max_{g \in [G]} n/n_g$  and  $\tau > 0$ . & 6, ν, and c are const. the

306 **Corollary 4.2.** ~~Note that from Theorem 4.1 one can immediately entail the error bound for estima-~~  
 307 ~~tion of the common and individual parameters as follows:~~  
all

$$308 \quad \forall g \in [G_+] : \|\delta_g\|_2 = O\left(\gamma \frac{\max_{g \in [G_+]} \omega(\mathcal{A}_g) + \sqrt{\log(G+1)}}{\sqrt{n_g}}\right)$$

309 **Example 4.3. ( $L_1$ -norm)** For sparse DE estimator of Example 2.1, results of Theorems 3.7 and 4.1  
 310 translates to the following: For enough number of samples as  $\forall g \in [G_+] : n_g \geq m_g = O(s_g \log p)$ ,

the upper bound of error simplifies to:

$$(4.2) \quad \sum_{g=0}^G \sqrt{\frac{n_g}{n}} \|\delta_g\|_2 = O \left( \gamma \sqrt{\frac{(\max_{g \in [G_+]} s_g) \log p}{n}} \right)$$

Therefore, individual errors are bounded as  $\|\delta_g\|_2 = O(\gamma \sqrt{(\max_{g \in [G]} s_g) \log p / n_g})$  which is slightly worse than  $O(\sqrt{s_g \log p / n_g})$ , the well-known error bound for recovering an  $s_g$ -sparse vector from  $n_g$  observations using LASSO or similar estimators [2, 12, 8, 13, 4]

**4.1. Proof of Theorem 4.1.** To avoid cluttering the notation, we rename the vector of all noises as  $\mathbf{w}_0 \triangleq \mathbf{w}$ . First, we massage the deterministic upper bound of Theorem 2.2 as follows:

$$\mathbf{w}^T \mathbf{X} \delta = \sum_{g=0}^G \langle \mathbf{X}_g^T \mathbf{w}_g, \delta_g \rangle = \sum_{g=0}^G \sqrt{\frac{n_g}{n}} \|\delta_g\|_2 \langle \mathbf{X}_g^T \frac{\mathbf{w}_g}{\|\mathbf{w}_g\|_2}, \frac{\delta_g}{\|\delta_g\|_2} \rangle \sqrt{\frac{n}{n_g}} \|\mathbf{w}_g\|_2$$

Assume  $q_g = \langle \mathbf{X}_g^T \frac{\mathbf{w}_g}{\|\mathbf{w}_g\|_2}, \frac{\delta_g}{\|\delta_g\|_2} \rangle \sqrt{\frac{n}{n_g}} \|\mathbf{w}_g\|_2$  and  $p_g = \sqrt{\frac{n_g}{n}} \|\delta_g\|_2$ . Then the above term is the inner product of two vectors  $\mathbf{p} = (p_0, \dots, p_G)$  and  $\mathbf{q} = (q_0, \dots, q_G)$  for which we have:

$$\sup_{\mathbf{p} \in \tilde{\mathcal{H}}} \mathbf{p}^T \mathbf{q} = \sup_{\|\mathbf{p}\|_1=1} \mathbf{p}^T \mathbf{q} \leq \|\mathbf{q}\|_\infty = \max_{g \in [G_+]} q_g,$$

where the inequality holds because of the definition of the dual norm. Now we can go back to the original form:

$$(4.3) \quad \sup_{\delta \in \mathcal{H}} \mathbf{w}^T \mathbf{X} \delta \leq \max_{g \in [G]} \langle \mathbf{X}_g^T \frac{\mathbf{w}_g}{\|\mathbf{w}_g\|_2}, \frac{\delta_g}{\|\delta_g\|_2} \rangle \sqrt{\frac{n}{n_g}} \|\mathbf{w}_g\|_2$$

$$\leq \max_{g \in [G]} \sqrt{\frac{n}{n_g}} \|\mathbf{w}_g\|_2 \sup_{\mathbf{u}_g \in \mathcal{C}_g \cap \mathbb{S}^{p-1}} \langle \mathbf{X}_g^T \frac{\mathbf{w}_g}{\|\mathbf{w}_g\|_2}, \mathbf{u}_g \rangle$$

To avoid cluttering we define a random quantity  $h_g(\mathbf{w}_g, \mathbf{X}_g) \triangleq \|\mathbf{w}_g\|_2 \sup_{\mathbf{u}_g \in \mathcal{A}_g} \langle \mathbf{X}_g^T \frac{\mathbf{w}_g}{\|\mathbf{w}_g\|_2}, \mathbf{u}_g \rangle$  and a corresponding constant  $e_g(\tau) \triangleq c_g \sqrt{(2k_w^2 + 1)k_x^2 n_g} (\omega(\mathcal{A}_g) + \sqrt{\log(G+1)} + \tau)$ . Then from (4.3), we have:

$$\mathbb{P} \left( \sup_{\delta \in \mathcal{H}} \mathbf{w}^T \mathbf{X} \delta > \max_{g \in [G]} \sqrt{\frac{n}{n_g}} e_g(\tau) \right) \leq \mathbb{P} \left( \max_{g \in [G]} \sqrt{\frac{n}{n_g}} h_g(\mathbf{w}_g, \mathbf{X}_g) > \max_{g \in [G]} \sqrt{\frac{n}{n_g}} e_g(\tau) \right)$$

$$\stackrel{\text{(Union Bound)}}{\leq} \sum_{g=0}^G \mathbb{P} \left( \sqrt{\frac{n}{n_g}} h_g(\mathbf{w}_g, \mathbf{X}_g) > \max_{g \in [G]} \sqrt{\frac{n}{n_g}} e_g(\tau) \right)$$

$$\leq \sum_{g=0}^G \mathbb{P} (h_g(\mathbf{w}_g, \mathbf{X}_g) > e_g(\tau))$$

$$\leq (G+1) \max_{g \in [G_+]} \mathbb{P} (h_g(\mathbf{w}_g, \mathbf{X}_g) > e_g(\tau))$$

$$\leq \sigma \exp \left( - \min \left[ \nu \min_{g \in [G]} n_g - \log(G+1), \tau^2 \right] \right)$$

where the last inequality is the result of the following lemma:

**Algorithm 5.1** DICER

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1: input:  $\mathbf{X}, \mathbf{y}$ , learning rates  $(\mu_0, \dots, \mu_G)$ , initialization  $\beta^{(1)} = \mathbf{0}$ 
2: output:  $\hat{\beta}$ 
3: for  $t = 1$  to  $T$  do
4:   for  $g=1$  to  $G$  do
5:      $\beta_g^{(t+1)} = \Pi_{\Omega_{f_g}} \left( \beta_g^{(t)} + \mu_g \mathbf{X}_g^T \left( \mathbf{y}_g - \mathbf{X}_g \left( \beta_0^{(t)} + \beta_g^{(t)} \right) \right) \right)$ 
6:   end for
7:    $\beta_0^{(t+1)} = \Pi_{\Omega_{f_0}} \left( \beta_0^{(t)} + \mu_0 \mathbf{X}_0^T \left( \mathbf{y} - \mathbf{X}_0 \beta_0^{(t)} - \begin{pmatrix} \mathbf{X}_1 \beta_1^{(t)} \\ \vdots \\ \mathbf{X}_G \beta_G^{(t)} \end{pmatrix} \right) \right)$ 
8: end for

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336 **Lemma 4.4.** For  $\mathbf{x}_{gi}$  and  $\omega_{gi}$  defined in [Definition 3.1](#) and  $\tau > 0$ , with probability at least  
337  $1 - \frac{\sigma_g}{(G+1)} \exp(-\min[\nu n_g - \log(G+1), \tau^2])$  we have:

$$338 \quad \|\mathbf{w}_g\|_2 \sup_{\mathbf{u}_g \in \mathcal{A}_g} \langle \mathbf{X}_g^T \frac{\mathbf{w}_g}{\|\mathbf{w}_g\|_2}, \mathbf{u}_g \rangle \leq c_g \sqrt{(2k_w^2 + 1)k_x^2 n_g} \left( \omega(\mathcal{A}_g) + \sqrt{\log(G+1)} + \tau \right),$$

339 where  $\sigma_g, \nu$  and  $c_g$  are constants.

340 The proof completes by replacing  $\max_{g \in [G]} \sqrt{\frac{n}{n_g}} e_g(\tau)$  as the upper bound of  $\sup_{\delta \in \mathcal{H}} \mathbf{w}^T \mathbf{X} \delta$  and  
341  $\kappa_{\min}^2/4$  as the lower bound of  $\kappa$  (from [Theorem 3.7](#)) both into the deterministic bound of [Theorem 2.2](#).

342 **5. Estimation Algorithm.** We propose *Data enrIchER* (DICER) a projected block gradient  
343 descent algorithm, [Algorithm 5.1](#), where  $\Pi_{\Omega_{f_g}}$  is the Euclidean projection onto the set  $\Omega_{f_g}(d_g) =$   
344  $\{f_g(\beta) \leq d_g\}$  where  $d_g = f_g(\beta_g^*)$  and is dropped to avoid cluttering. In practice,  $d_g$  can be determined  
345 by cross-validation.

346 To analysis convergence properties of DICER, we should upper bound the error of each iteration.  
347 Let's  $\delta^{(t)} = \beta^{(t)} - \beta^*$  be the error of iteration  $t$  of DICER, i.e., the distance from the true parameter (not  
348 the optimization minimum,  $\hat{\beta}$ ). We show that  $\|\delta^{(t)}\|_2$  decreases exponentially fast in  $t$  to the statistical  
349 error  $\|\delta\|_2 = \|\hat{\beta} - \beta^*\|_2$ . We first start with the required definitions for our analysis.

350 **Definition 5.1.** We define the following positive constants as functions of step sizes  $\mu_g > 0$ :

$$351 \quad \forall g \in [G_+] : \rho_g(\mu_g) = \sup_{\mathbf{u}, \mathbf{v} \in \mathcal{B}_g} \mathbf{v}^T (\mathbf{I}_g - \mu_g \mathbf{X}_g^T \mathbf{X}_g) \mathbf{u},$$

$$352 \quad \eta_g(\mu_g) = \mu_g \sup_{\mathbf{v} \in \mathcal{B}_g} \mathbf{v}^T \mathbf{X}_g^T \frac{\mathbf{w}_g}{\|\mathbf{w}_g\|_2},$$

$$353 \quad \forall g \in [G] : \phi_g(\mu_g) = \mu_g \sup_{\mathbf{v} \in \mathcal{B}_g, \mathbf{u} \in \mathcal{B}_0} -\mathbf{v}^T \mathbf{X}_g^T \mathbf{X}_g \mathbf{u},$$

354 where  $\mathcal{B}_g = \mathcal{C}_g \cap \mathbb{B}^p$  is the intersection of the error cone and the unit ball.

355 In the following theorem, we establish a deterministic bound on iteration errors  $\|\delta_g^{(t)}\|_2$  which depends  
356 on constants defined in [Definition 5.1](#) where to simplify the notation we drop  $\mu_g$  arguments.

357 **Theorem 5.2.** For *Algorithm 5.1* initialized by  $\beta^{(1)} = \mathbf{0}$ , we have the following deterministic  
 358 bound for the error at iteration  $t + 1$ :

$$359 \quad (5.1) \quad \sum_{g=0}^G \sqrt{\frac{n_g}{n}} \|\delta_g^{(t+1)}\|_2 \leq \rho^t \sum_{g=0}^G \sqrt{\frac{n_g}{n}} \|\beta_g^*\|_2 + \frac{1-\rho^t}{1-\rho} \sum_{g=0}^G \sqrt{\frac{n_g}{n}} \eta_g \|\omega_g\|_2,$$

$$360 \quad \text{where } \rho \triangleq \max \left( \rho_0 + \sum_{g=1}^G \sqrt{\frac{n_g}{n}} \phi_g, \max_{g \in [G]} \left[ \rho_g + \sqrt{\frac{n}{n_g} \frac{\mu_0}{\mu_g}} \phi_g \right] \right).$$

361 *Proof.* First using the following lemma, we establish a recursive relation between errors of consec-  
 362 utive iterations which leads to a bound for the  $t$ th iteration.

363 **Lemma 5.3.** We have the following recursive dependency between the error of  $t + 1$ th iteration  
 364 and  $t$ th iteration of DICER:

$$365 \quad \|\delta_g^{(t+1)}\|_2 \leq \left( \rho_g(\mu_g) \|\delta_g^{(t)}\|_2 + \xi_g(\mu_g) \|\omega_g\|_2 + \phi_g(\mu_g) \|\delta_0^{(t)}\|_2 \right)$$

$$366 \quad \|\delta_0^{(t+1)}\|_2 \leq \left( \rho_0(\mu_0) \|\delta_0^{(t)}\|_2 + \xi_0(\mu_0) \|\omega_0\|_2 + \mu_0 \sum_{g=1}^G \frac{\phi_g(\mu_g)}{\mu_g} \|\delta_g^{(t)}\|_2 \right)$$

367 By recursively applying results of **Lemma 5.3**, we get the following deterministic bound which depends  
 368 on constants defined in **Definition 5.1**:

$$369 \quad b_{t+1} = \sum_{g=0}^G \sqrt{\frac{n_g}{n}} \|\delta_g^{(t+1)}\|_2 \leq \left( \rho_0 + \sum_{g=1}^G \sqrt{\frac{n_g}{n}} \phi_g \right) \|\delta_0^{(t)}\|_2 + \sum_{g=1}^G \left( \sqrt{\frac{n_g}{n}} \rho_g + \mu_0 \frac{\phi_g}{\mu_g} \right) \|\delta_g^{(t)}\|_2 + \sum_{g=0}^G \sqrt{\frac{n_g}{n}} \xi_g \|\omega_g\|_2$$

$$370 \quad \leq \rho \sum_{g=0}^G \sqrt{\frac{n_g}{n}} \|\delta_g^{(t)}\|_2 + \sum_{g=0}^G \sqrt{\frac{n_g}{n}} \xi_g \|\omega_g\|_2$$

371 where  $\rho = \max \left( \rho_0 + \sum_{g=1}^G \sqrt{\frac{n_g}{n}} \phi_g, \max_{g \in [G]} \left[ \rho_g + \sqrt{\frac{n}{n_g} \frac{\mu_0}{\mu_g}} \phi_g \right] \right)$ . We have:

$$372 \quad b_{t+1} \leq \rho b_t + \sum_{g=0}^G \sqrt{\frac{n_g}{n}} \xi_g \|\omega_g\|_2$$

$$373 \quad \leq \rho^2 b_{t-1} + (\rho + 1) \sum_{g=0}^G \sqrt{\frac{n_g}{n}} \xi_g \|\omega_g\|_2$$

$$374 \quad \leq \rho^t b_1 + \left( \sum_{i=0}^{t-1} \rho^i \right) \sum_{g=0}^G \sqrt{\frac{n_g}{n}} \xi_g \|\omega_g\|_2$$

$$375 \quad = \rho^t \sum_{g=0}^G \sqrt{\frac{n_g}{n}} \|\beta_g^1 - \beta_g^*\|_2 + \left( \sum_{i=0}^{t-1} \rho^i \right) \sum_{g=0}^G \sqrt{\frac{n_g}{n}} \xi_g \|\omega_g\|_2$$

$$376 \quad (5.2) \quad (\beta^1 = 0) \leq \rho^t \sum_{g=0}^G \sqrt{\frac{n_g}{n}} \|\beta_g^*\|_2 + \frac{1-\rho^t}{1-\rho} \sum_{g=0}^G \sqrt{\frac{n_g}{n}} \xi_g \|\omega_g\|_2 \quad \blacksquare$$

$$377$$



The RHS of (5.2) consists of two terms. If we keep  $\rho < 1$ , the first term approaches zero fast, and the second term determines the bound. In the following, we show that for specific choices of step sizes  $\mu_g$ s we can keep  $\rho < 1$  with high probability and the second term can be upper bounded using the analysis of section 4. More specifically, the first term corresponds to the optimization error which shrinks in every iteration while the second term is of the same order of the upper bound of the statistical error characterized in Theorem 4.1.

One way for having  $\rho < 1$  is to keep all arguments of  $\max(\dots)$  defining  $\rho$  strictly below 1. To this end, we first establish high probability upper bound for  $\rho_g$ ,  $\eta_g$ , and  $\phi_g$  (in the subsection SM1.2) and then show that with enough number of samples and proper step sizes  $\mu_g$ ,  $\rho$  can be kept strictly below one with high probability. The high probability bounds for constants in Definition 5.1 and the deterministic bound of Theorem 5.2 leads to the following theorem which shows that for enough number of samples, of the same order as the statistical sample complexity of Theorem 3.7, we can keep  $\rho$  below one and have geometric convergence. *sparse! e?*

**Theorem 5.4.** Let  $\tau = \sqrt{\log(G+1)}/\zeta + s$  for  $s \zeta > 0$ . For the step sizes of:

$$\mu_0 = \frac{\min_{g \in [G]} h_g(\tau)^{-2}}{4n}, \forall g \in [G] : \mu_g = \frac{h_g(\tau)^{-1}}{2\sqrt{nn_g}}$$

where  $h_g(\tau) = \left(1 + c_{0g} \frac{\omega(\mathcal{A}_g) + \omega(\mathcal{A}_0) + 2\tau}{\sqrt{n_g}}\right)$  and sample complexities of  $\forall g \in [G_+] : n_g \geq C_g(\omega(\mathcal{A}_g) + \tau)^2$ , with probability at least  $1 - \sigma \exp(-\min(v \min_{g \in [G]} n_g - \log(G+1), s^2))$  updates of Algorithm 5.1 obey the following:

$$\sum_{g=0}^G \sqrt{\frac{n_g}{n}} \|\delta_g^{(t+1)}\|_2 \leq r(\tau)^t \sum_{g=0}^G \sqrt{\frac{n_g}{n}} \|\beta_g^*\|_2 + \frac{C(G+1)\sqrt{(2k_w^2+1)k_x^2}}{\sqrt{n}(1-r(\tau))} \left( \max_{g \in [G_+]} \omega(\mathcal{A}_g) + \tau \right)$$

where  $r(\tau) < 1$  and  $v, \zeta$ , and  $\sigma$  are constants.

**Corollary 5.5.** For enough number of samples, iterations of DE algorithm with step sizes  $\mu_0 = \Theta(\frac{1}{n})$  and  $\mu_g = \Theta(\frac{1}{\sqrt{nn_g}})$  geometrically converges to the following with high probability:

$$(5.3) \quad \sum_{g=0}^G \sqrt{\frac{n_g}{n}} \|\delta_g^\infty\|_2 \leq c \frac{\max_{g \in [G_+]} \omega(\mathcal{A}_g) + \sqrt{\log(G+1)}/\zeta + \theta}{\sqrt{n}(1-r(\tau))}$$

where  $c = C(G+1)\sqrt{(2k_w^2+1)k_x^2}$ .

It is instructive to compare RHS of (5.3) with that of Theorem 4.1:  $\kappa_{\min}$  defined in Theorem 3.7 corresponds to  $(1-r(\tau))$  and the extra  $G+1$  factor corresponds to the sample condition number  $\gamma = \max_{g \in [G]} \frac{n}{n_g}$ . Therefore, Corollary 5.5 shows that with the number of samples in the order of sample complexity determined in Theorem 3.7 DICER converges to the statistical error bound determined in Theorem 4.1.

**6. Proof Sketch of Theorem 5.4.** We want to determine  $r(\tau) < 1$  such that  $\rho < r(\tau)$  with high probability. Here, we provide a proof sketch using the below probabilistic bounds on constants of Definition 5.1 while ignoring detailed computation of subsequent probabilities in finding  $r(\tau)$ . The full probabilistic proof is provided in subsection SM1.2. First we need the following lemma to upper bound constants of Definition 5.1:

**Lemma 6.1.** Consider  $a_g \geq 1$  the following upper bounds hold:

$$\rho_g \left( \frac{1}{a_g n_g} \right) \leq \frac{1}{2} \left[ \left( 1 - \frac{1}{a_g} \right) + \sqrt{2} c_g \frac{2\omega_g + \tau}{a_g \sqrt{n_g}} \right], \quad \text{w.p. at least } 1 - 2 \exp(-\gamma_g(\omega(\mathcal{A}_g) + \tau)^2)$$

$$\eta_g \left( \frac{1}{a_g n_g} \right) \leq \frac{c_g k_x(\omega_g + \tau)}{a_g n_g}, \quad \text{w.p. at least } 1 - \pi_g \exp(-\tau^2)$$

$$\phi_g \left( \frac{1}{a_g n_g} \right) \leq \frac{1}{a_g} \left( 1 + c_{0g} \frac{\omega_{0g} + 2\tau}{\sqrt{n_g}} \right), \quad \text{w.p. at least } 1 - 2 \exp(-\gamma_g(\omega(\mathcal{A}_g) + \tau)^2)$$

where  $\omega_g = \omega(\mathcal{A}_g)$  and  $\omega_{0g} = \omega(\mathcal{A}_g) + \omega(\mathcal{A}_0)$ .

To keep  $\rho < 1$  in the deterministic bound of [Theorem 5.2](#) with the step sizes  $\mu_g = \frac{1}{n_g a_g}$  we need to find the number of samples which satisfy the following conditions:

- Condition 1:  $\rho_0(\mu_0) + \sum_{g=1}^G \sqrt{\frac{n_g}{n}} \phi_g(\mu_g) < 1$

- Condition 2:  $\forall g \in [G] : \rho_g(\mu_g) + \sqrt{\frac{n}{n_g} \frac{\mu_0}{\mu_g}} \phi_g(\mu_g) < 1$

where according to the step sizes determine in the Theorem  $a_0 \triangleq (4n \max_{g \in [G]} (1 + c_{0g} \frac{\omega_{0g} + 2\tau}{\sqrt{n_g}})^2)^{-1}$

and  $a_g \triangleq (2\sqrt{n/n_g} (1 + c_{0g} \frac{\omega_{0g} + 2\tau}{\sqrt{n_g}}))^{-1}$ . Condition 1 requires  $\rho_0 + \sum_{g=1}^G \sqrt{\frac{n_g}{n}} \phi_g$  to be strictly below 1 which is equivalent to:

$$\rho_0(\mu_0) + \sum_{g=1}^G \sqrt{\frac{n_g}{n}} \phi_g(\mu_g) \leq \frac{1}{2} \left[ \left( 1 - \frac{1}{a_0} \right) + \sqrt{2} c_0 \frac{2\omega_0 + \tau}{a_0 \sqrt{n}} \right] + \frac{1}{2} \sum_{g=1}^G \frac{2}{a_g} \sqrt{\frac{n_g}{n}} \left( 1 + c_{0g} \frac{\omega_{0g} + 2\tau}{\sqrt{n_g}} \right)$$

$$\text{(Substitute } a_g) = \frac{1}{2} \left[ \left( 1 - \frac{1}{a_0} \right) + \sqrt{2} c_0 \frac{2\omega_0 + \tau}{a_0 \sqrt{n}} \right] + \frac{1}{2} \sum_{g=1}^G \frac{n_g}{n}$$

$$= \frac{1}{2} \left[ \left( 2 - \frac{1}{a_0} \right) + \sqrt{2} c_0 \frac{2\omega_0 + \tau}{a_0 \sqrt{n}} \right] < 1$$

So Condition 1 reduces to  $n > 8c_0^2(\omega(\mathcal{A}_0) + \tau)^2$ .

Secondly in Condition 2, we want to bound all of  $\rho_g + \mu_0 \sqrt{\frac{n}{n_g} \frac{\phi_g}{\mu_g}}$  terms for  $\mu_g = \frac{1}{a_g n_g}$  by 1:

$$\rho_g(\mu_g) + \sqrt{\frac{n}{n_g} \frac{\mu_0}{\mu_g}} \phi_g(\mu_g) = \rho_g \left( \frac{1}{n_g a_g} \right) + \sqrt{\frac{n_g}{n} \frac{a_g}{a_0}} \phi_g \left( \frac{1}{n_g a_g} \right)$$

$$= \frac{1}{2} \left[ \left( 1 - \frac{1}{a_g} \right) + \sqrt{2} c_g \frac{2\omega_g + \tau}{a_g \sqrt{n_g}} \right] + \frac{2}{a_0} \sqrt{\frac{n_g}{n}} \left( 1 + c_{0g} \frac{\omega_{0g} + 2\tau}{\sqrt{n_g}} \right)$$

$$\leq 1$$

Condition 2 becomes:

$$\sqrt{2} c_g \frac{2\omega_g + \tau}{\sqrt{n_g}} \leq 1 + a_g - \sqrt{\frac{n_g}{n} \frac{2a_g}{a_0}} \left( 1 + c_{0g} \frac{\omega_{0g} + 2\tau}{\sqrt{n_g}} \right)$$

$$\text{(Substitute } a_g) = 1 + a_g - \frac{4}{a_0} \left( 1 + c_{0g} \frac{\omega_{0g} + 2\tau}{\sqrt{n_g}} \right)^2$$

$$\text{(Substitute } a_0) \leq 1 + a_g$$

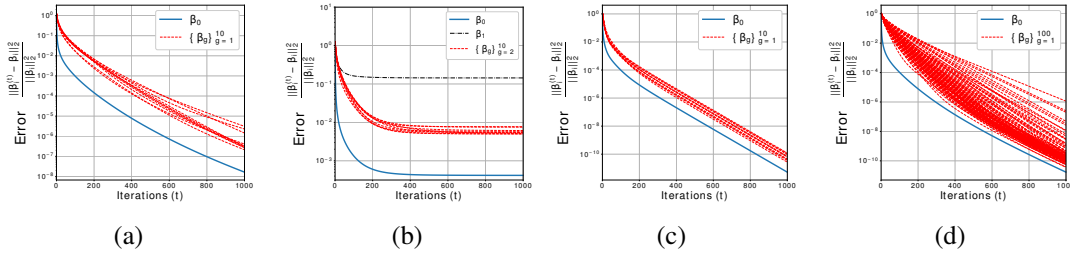


Figure 3: a) Noiseless fast convergence. b) Noise on the first group does not impact other groups as much. c) Increasing sample size improves rate of convergence. d) Our algorithm convergences fast even with a large number of groups  $G = 100$ .

436 So the sample complexity should be  $\sqrt{n_g} > \frac{\sqrt{2}c_g(2\omega_g+2\tau)}{1+a_g}$  and since  $a_g > 1$ , the final per group sample  
 437 complexity should be  $n_g > 8c_g(\omega(\mathcal{A}_g) + \tau)^2$ .

438 **7. Synthetic Experiments.** We considered sparsity based simulations with varying  $G$  and  
 439 sparsity levels. In our first set of simulations, we set  $p = 100$ ,  $G = 10$  and sparsity of the individual  
 440 parameters to be  $s = 10$ . We generated a dense  $\beta_0$  with  $\|\beta_0\| = p$  and did not impose any constraint.  
 441 Iterates  $\{\beta_g^{(t)}\}_{g=1}^G$  are obtained by projection onto the  $\ell_1$  ball  $\|\beta_g\|_1$ . Nonzero entries of  $\beta_g$  are  
 442 generated with  $\mathcal{N}(0, 1)$  and nonzero supports are picked uniformly at random. Inspired from our  
 443 theoretical step size choices, in all experiments, we used simplified learning rates of  $\frac{1}{n}$  for  $\beta_0$  and  $\frac{1}{\sqrt{nn_g}}$   
 444 for  $\beta_g$ ,  $g \in [G] \setminus \{1\}$ . Observe that, cones of the individual parameters intersect with that of  $\beta_0$  hence this  
 445 setup actually violates DERIC (which requires an arbitrarily small constant fraction of groups to be  
 446 non-intersecting). Our intuition is that the individual parameters are mostly incoherent with each other  
 447 and the existence of a nonzero perturbation over  $\beta_g$ 's that keeps all measurements intact is unlikely.  
 448 Remarkably, experimental results still show successful learning of all parameters from small amount  
 449 of samples. We picked  $n_g = 60$  for each group. Hence, in total, we have  $11p = 1100$  unknowns,  
 450  $200 = G \times 10 + 100$  degrees of freedom and  $G \times 60 = 600$  samples. In all figures, we study the  
 451 normalized squared error  $\frac{\|\beta_g^{(t)} - \beta_g\|_2^2}{\|\beta_g\|_2^2}$  and average 10 independent realization for each curve. **Figure 3a**  
 452 shows the estimation performance as a function of iteration number  $t$ . While each group might behave  
 453 slightly different, we do observe that all parameters are linear converging to ground truth.

454 In **Figure 3b**, we test the noise robustness of our algorithm. We add a  $\mathcal{N}(0, 1)$  noise to the  $n_1 = 60$   
 455 measurements of the first group *only*. The other groups are left untouched. While all parameters suffer  
 456 nonzero estimation error, we observe that, the global parameter  $\beta_0$  and noise-free groups  $\{\beta_g\}_{g=2}^G$  have  
 457 substantially less estimation error. This implies that noise in one group mostly affects itself rather than  
 458 the global estimation. In **Figure 3c**, we increased the sample size to  $n_g = 150$  per group. We observe  
 459 that, in comparison to **Figure 3a**, rate of convergence receives a boost from the additional samples as  
 460 predicted by our theory.

461 Finally, **Figure 3d** considers a very high-dimensional problem where  $p = 1000$ ,  $G = 100$ , individual  
 462 parameters are 10 sparse,  $\beta_0$  is 100 sparse and  $n_g = 150$ . The total degrees of freedom is 1100, number  
 463 of unknowns are 101000 and total number of datapoints are  $150 \times 100 = 15000$ . While individual  
 464 parameters have substantial variation in terms of convergence rate, at the end of 1000 iteration, all

parameters have relative reconstruction error below  $10^{-6}$ .

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