

# SUPPLEMENTARY MATERIALS: High Dimensional Data Sharing: Multi-Task Learning with Theoretical Guarantee \*

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**SM1. Proofs of Theorems.** Here, we present proof of [Proposition 3.6](#) and full proof for [Theorem 5.4](#) and each lemma used during the proofs of theorems.

## SM1.1. Proof of [Proposition 3.6](#).

*Proof.* Consider only one group for regression in isolation. Note that  $\mathbf{y}_g = \mathbf{X}_g(\beta_g^* + \beta_0^*) + \mathbf{w}_g$  is a superposition model and as shown in [\[SM3\]](#) the sample complexity required for the RE condition and subsequently recovering  $\beta_0^*$  and  $\beta_g^*$  is  $n_g \geq c(\max(\omega(\mathcal{A}_0), \omega(\mathcal{A}_g)) + \sqrt{\log 2})^2$ . ■

**SM1.2. Proof of [Theorem 5.4](#).** Now we rewrite the same analysis using the tail bounds for the coefficients to clarify the probabilities. To simplify the notation, we define the following functions of  $\tau$ :

$$\begin{aligned} r_{g1}(\tau) &\triangleq \frac{1}{2} \left[ \left(1 - \frac{1}{a_g}\right) + \sqrt{2c_g} \frac{2\omega(\mathcal{A}_g) + \tau}{a_g \sqrt{n_g}} \right], \forall g \in [G_+] \\ r_{g2}(\tau) &\triangleq \frac{1}{a_g} \left( 1 + c_{0g} \frac{\omega(\mathcal{A}_0) + \omega(\mathcal{A}_g) + \tau}{\sqrt{n_g}} \right), \forall g \in [G] \\ r_0(\tau) &\triangleq r_{01}(\tau) + \sum_{g=1}^G \sqrt{\frac{n_g}{n}} r_{g2}(\tau) \\ r_g(\tau) &\triangleq r_{g1}(\tau) + \sqrt{\frac{n_g}{n}} \frac{a_g}{a_0} r_{g2}(\tau), \forall g \in [G] \\ \text{(SM1.1)} \quad r(\tau) &\triangleq \max_{g \in [G_+]} r_g(\tau) \end{aligned}$$

All of which are computed using  $a_g$ s specified in the proof sketch of [subsection 5.1](#). Basically  $r(\tau)$  is an instantiation of an high probability upper bound of the  $\rho$  defined in [Theorem 5.2](#). We are interested

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20 in upper bounding the following probability:

$$\begin{aligned}
21 \quad & (\text{SM1.}\mathbb{P}) \left( \sum_{g=0}^G \sqrt{\frac{n_g}{n}} \|\delta_g^{(t+1)}\|_2 \geq r(\tau)^t \sum_{g=0}^G \sqrt{\frac{n_g}{n}} \|\beta_g^*\|_2 + \frac{C(G+1)\sqrt{(2k_w^2+1)k_x^2}}{(1-r(\tau))\sqrt{n}} (\max_{g \in [G_+]} \omega(\mathcal{A}_g) + \tau) \right) \\
22 \quad & \leq \mathbb{P} \left( \rho^t \sum_{g=0}^G \sqrt{\frac{n_g}{n}} \|\beta_g^*\|_2 + \frac{1-\rho^t}{1-\rho} \sum_{g=0}^G \sqrt{\frac{n_g}{n}} \eta_g(\mu_g) \|\mathbf{w}_g\|_2 \geq \right. \\
23 \quad & \left. r(\tau)^t \sum_{g=0}^G \sqrt{\frac{n_g}{n}} \|\beta_g^*\|_2 + \frac{C(G+1)\sqrt{(2k_w^2+1)k_x^2}}{(1-r(\tau))\sqrt{n}} (\max_{g \in [G_+]} \omega(\mathcal{A}_g) + \tau) \right) \\
24 \quad & \leq \mathbb{P}(\rho \geq r(\tau)) + \mathbb{P} \left( \frac{1}{1-\rho} \sum_{g=0}^G \sqrt{n_g} \eta_g(\mu_g) \|\mathbf{w}_g\|_2 \geq \frac{C(G+1)\sqrt{(2k_w^2+1)k_x^2}}{(1-r(\tau))} (\max_{g \in [G_+]} \omega(\mathcal{A}_g) + \tau) \right)
\end{aligned}$$

25 where the first inequality comes from the deterministic bound of [Theorem 5.2](#) and the second one is  
 26 based on the law of total probability.

27 We first focus on bounding the first term  $\mathbb{P}(\rho \geq r(\tau))$ :

$$\begin{aligned}
28 \quad & \mathbb{P}(\rho \geq r(\tau)) = \mathbb{P} \left( \max \left( \rho_0(\mu_0) + \sum_{g=1}^G \sqrt{\frac{n_g}{n}} \phi_g(\mu_g), \max_{g \in [G]} \rho_g(\mu_g) + \sqrt{\frac{n}{n_g}} \frac{\mu_0}{\mu_g} \phi_g(\mu_g) \right) \geq \max_{g \in [G_+]} r(\tau) \right) \\
29 \quad & (\text{Union Bound}) \leq \mathbb{P} \left( \rho_0(\mu_0) + \sum_{g=1}^G \sqrt{\frac{n_g}{n}} \phi_g(\mu_g) \geq r_0 \right) + \sum_{g=1}^G \mathbb{P} \left( \rho_g(\mu_g) + \sqrt{\frac{n}{n_g}} \frac{\mu_0}{\mu_g} \phi_g(\mu_g) \geq r_g \right) \\
30 \quad & \leq \mathbb{P}(\rho_0(\mu_0) \geq r_{01}) + \sum_{g=1}^G \mathbb{P}(\phi_g(\mu_g) \geq r_{g2}) + \sum_{g=1}^G [\mathbb{P}(\rho_g(\mu_g) \geq r_{g1}) + \mathbb{P}(\phi_g(\mu_g) \geq r_{g2})] \\
31 \quad & \leq \sum_{g=0}^G \mathbb{P}(\rho_g(\mu_g) \geq r_{g1}) + 2 \sum_{g=1}^G \mathbb{P}(\phi_g(\mu_g) \geq r_{g2}) \\
32 \quad & \leq \sum_{g=0}^G 2 \exp(-\gamma_g(\omega(\mathcal{A}_g) + \tau)^2) + 2 \sum_{g=1}^G 2 \exp(-\gamma_g(\omega(\mathcal{A}_g) + \tau)^2) \\
33 \quad & (\gamma = \min_{g \in [G_+]} \gamma_g) \leq 2(G+1) \exp \left( -\gamma \min_{g \in [G_+]} (\omega(\mathcal{A}_g) + \tau)^2 \right) + 4G \exp \left( -\gamma \min_{g \in [G]} (\omega(\mathcal{A}_g) + \tau)^2 \right) \\
34 \quad & (\text{SM1.3}) \leq 6(G+1) \exp \left( -\gamma \min_{g \in [G_+]} (\omega(\mathcal{A}_g) + \tau)^2 \right) \\
35 \quad &
\end{aligned}$$

36 Now we focus on bounding the second term:

$$\begin{aligned}
 37 \quad & \mathbb{P} \left( \frac{1}{1-\rho} \sum_{g=0}^G \sqrt{n_g} \eta_g(\mu_g) \|\mathbf{w}_g\|_2 \geq \frac{C(G+1)\sqrt{(2k_w^2+1)k_x^2}}{(1-r(\tau))} (\max_{g \in [G_+]} \omega(\mathcal{A}_g) + \tau) \right) \\
 38 \quad & \leq \mathbb{P} \left( \frac{1}{1-\rho} \sum_{g=0}^G \sqrt{n_g} \eta_g(\mu_g) \|\mathbf{w}_g\|_2 \geq \frac{C}{(1-r(\tau))} \sum_{g=0}^G \sqrt{(2k_w^2+1)k_x^2} (\max_{g \in [G_+]} \omega(\mathcal{A}_g) + \tau) \right) \\
 39 \quad & \leq \mathbb{P} \left( \sum_{g=0}^G \sqrt{n_g} \eta_g(\mu_g) \|\mathbf{w}_g\|_2 \geq \sum_{g=0}^G c_g \sqrt{(2k_w^2+1)k_x^2} (\max_{g \in [G_+]} \omega(\mathcal{A}_g) + \tau) \right) + \mathbb{P}(\rho \geq r(\tau)) \\
 40 \quad (\text{SM1.4}) \quad & \leq \sum_{g=0}^G \mathbb{P} \left( \sqrt{n_g} \eta_g(\mu_g) \|\mathbf{w}_g\|_2 \geq c_g \sqrt{(2k_w^2+1)k_x^2} (\max_{g \in [G_+]} \omega(\mathcal{A}_g) + \tau) \right) + \mathbb{P}(\rho \geq r(\tau)) \\
 41
 \end{aligned}$$

42 Focusing on the first term, since  $\eta_g(\frac{1}{a_g n_g}) = \frac{1}{a_g n_g} \sup_{\mathbf{v} \in \mathcal{B}_g} \mathbf{v}^T \mathbf{X}_g^T \frac{\mathbf{w}_g}{\|\mathbf{w}_g\|_2}, \forall g \in [G]$ :

$$\begin{aligned}
 43 \quad (\text{SM1.5}) \quad & \mathbb{P} \left( \|\mathbf{w}_g\|_2 \sup_{\mathbf{v} \in \mathcal{B}_g} \mathbf{v}^T \mathbf{X}_g^T \frac{\mathbf{w}_g}{\|\mathbf{w}_g\|_2} \geq a_g c_g \sqrt{(2k_w^2+1)k_x^2 n_g} (\max_{g \in [G_+]} \omega(\mathcal{A}_g) + \tau) \right) \\
 44 \quad & (a_g \geq 1) \leq \mathbb{P} \left( \|\mathbf{w}_g\|_2 \sup_{\mathbf{v} \in \mathcal{B}_g} \mathbf{v}^T \mathbf{X}_g^T \frac{\mathbf{w}_g}{\|\mathbf{w}_g\|_2} \geq c_g \sqrt{(2k_w^2+1)k_x^2 n_g} (\max_{g \in [G_+]} \omega(\mathcal{A}_g) + \tau) \right) \\
 45 \quad ((\text{SM1.9}) \text{ and } (\text{SM1.10})) \quad & \leq 2 \exp(-\nu n_g) + \pi_g \exp(-\tau^2) \\
 46 \quad (\sigma_g = \max(\pi_g, 2)) \quad & \leq \sigma_g \exp(-\min(\nu n_g, \tau^2))
 \end{aligned}$$

48 where we used the intermediate form of [Lemma 4.5](#) for  $\tau > 0$ . Putting all of the bounds [\(SM1.3\)](#),  
 49 [\(SM1.4\)](#), and [\(SM1.5\)](#) back as the upper bound of [subsection SM1.2](#):

$$\begin{aligned}
 50 \quad & (\pi = \max_{g \in [G]} \pi_g) \leq 2(G+1) \exp(-\nu \min_{g \in [G]} n_g) + \pi(G+1) \exp(-\tau^2) + \\
 51 \quad & 6(G+1) \exp \left( -\gamma \min_{g \in [G_+]} (\omega(\mathcal{A}_g) + \tau)^2 \right) \\
 52 \quad (v = \max(6, \pi), \zeta = \min(1, \gamma)) \quad & \leq 2 \exp(-\nu \min_{g \in [G]} n_g + \log(G+1)) + v(G+1) \exp(-\zeta \tau^2) \\
 53 \quad (\tau = \epsilon + \sqrt{\log(G+1)}/\zeta) \quad & \leq 2 \exp(-\nu \min_{g \in [G]} n_g + \log(G+1)) + v \exp(-\zeta \epsilon^2) \\
 54 \quad (\sigma = \max(2, v)) \quad & \leq \sigma \exp(-\min(\nu \min_{g \in [G]} n_g - \log(G+1), \zeta \epsilon^2))
 \end{aligned}$$

55 Note that setting  $\tau = \epsilon + \sqrt{\log(G+1)}/\zeta$  increases the sample complexities to the followings:

$$56 \quad n > 2c_0^2 \left( 2\omega(\mathcal{A}_0) + \sqrt{\log(G+1)}/\zeta + \epsilon \right)^2, \forall g \in [G] : n_g \geq 2c_g^2 (2\omega(\mathcal{A}_g) + C\sqrt{\log(G+1)}/\zeta + \epsilon)^2$$

57 And it also affects step sizes as follows:

$$58 \quad \mu_0 = \frac{1}{4n} \times \min_{g \in [G]} \left( 1 + c_{0g} \frac{\omega_{0g} + \sqrt{\log(G+1)}/\zeta + \epsilon}{\sqrt{n_g}} \right)^{-2}, \mu_g = \frac{1}{2\sqrt{n n_g}} \left( 1 + c_{0g} \frac{\omega_{0g} + \sqrt{\log(G+1)}/\zeta + \epsilon}{\sqrt{n_g}} \right)^{-1}$$

59

### SM1.3. Proof of Lemma 3.10.

*Proof.* LHS of (3.2) is the weighted summation of  $\xi_g Q_{2\xi_g}(\delta_{0g}) = \|\delta_{0g}\|_2 \xi \mathbb{P}(|\langle \mathbf{x}, \delta_{0g} / \|\delta_{0g}\|_2 \rangle| > 2\xi) = \|\delta_{0g}\|_2 \xi Q_{2\xi}(\mathbf{u})$  where  $\xi > 0$  and  $\mathbf{u} = \delta_{0g} / \|\delta_{0g}\|_2$  is a unit length vector. So we can rewrite the LHS of (3.2) as:

$$\sum_{g=1}^G \frac{n_g}{n} \xi_g Q_{2\xi_g}(\delta_{0g}) = \sum_{g=1}^G \frac{n_g}{n} \|\delta_0 + \delta_g\|_2 \xi Q_{2\xi}(\mathbf{u}), \quad \mathbf{u} \in \mathbb{S}^{p-1}$$

With this observation, the lower bound of the Lemma 3.10 is a direct consequence of the following two results:

**Lemma SM1.1.** *Let  $\mathbf{u}$  be any unit length vector and suppose  $\mathbf{x}$  obeys Definition 3.1. Then for any  $\mathbf{u}$ , we have*

$$(SM1.6) \quad Q_{2\xi}(\mathbf{u}) \geq \frac{(\alpha - 2\xi)^2}{4ck_x^2}.$$

**Lemma SM1.2.** *Suppose Definition 3.4 holds. Then, we have:*

$$(SM1.7) \quad \sum_{g=1}^G \frac{n_g}{n} \|\delta_0 + \delta_g\|_2 \geq \frac{\bar{\rho}\lambda_{\min}}{3} \left( G\|\delta_0\|_2 + \sum_{g=1}^G \frac{n_g}{n} \|\delta_g\|_2 \right), \quad \forall g \in [G_+] : \delta_g \in \mathcal{C}_g.$$

### SM1.4. Proof of Lemma 3.11.

*Proof.* Consider the following soft indicator function which we use in our derivation:

$$\psi_a(s) = \begin{cases} 0, & |s| \leq a \\ (|s| - a)/a, & a \leq |s| \leq 2a \\ 1, & 2a < |s| \end{cases}$$

Now using the definition of the marginal tail function we have:

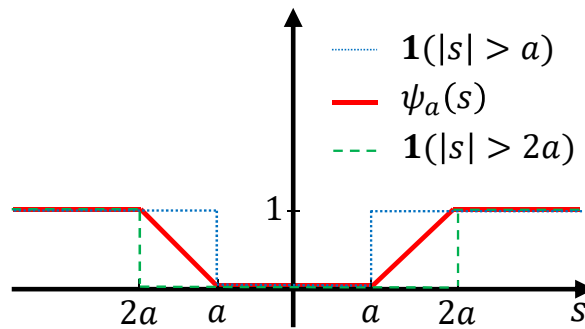


Figure SM1:  $\mathbb{1}(|s| > 2a) \leq \psi_a(s) \leq \mathbb{1}(|s| > a)$

75

$$\begin{aligned}
& \mathbb{E} \sup_{\delta \in \mathcal{H}} \sum_{g=1}^G \xi_g \sum_{i=1}^{n_g} [Q_{2\xi_g}(\delta_{0g}) - \mathbb{1}(|\langle \mathbf{x}_{gi}, \delta_{0g} \rangle| \geq \xi_g)] \\
&= \mathbb{E} \sup_{\delta \in \mathcal{H}} \sum_{g=1}^G \xi_g \sum_{i=1}^{n_g} [\mathbb{E} \mathbb{1}(|\langle \mathbf{x}_{gi}, \delta_{0g} \rangle| \geq 2\xi_g) - \mathbb{1}(|\langle \mathbf{x}_{gi}, \delta_{0g} \rangle| \geq \xi_g)] \\
&\quad (\text{Figure Figure SM1}) \leq \mathbb{E} \sup_{\delta \in \mathcal{H}} \sum_{g=1}^G \xi_g \sum_{i=1}^{n_g} [\mathbb{E} \psi_{\xi_g}(\langle \mathbf{x}, \delta_{0g} \rangle) - \psi_{\xi_g}(\langle \mathbf{x}_{gi}, \delta_{0g} \rangle)] \\
&\quad (\text{Symmetrization [SM7]}) \leq 2\mathbb{E} \sup_{\delta \in \mathcal{H}} \sum_{g=1}^G \xi_g \sum_{i=1}^{n_g} \epsilon_{gi} \psi_{\xi_g}(\langle \mathbf{x}_{gi}, \delta_{0g} \rangle) \\
&\quad (\text{Rademacher comparison [SM4]}) \leq 2\mathbb{E} \sup_{\delta \in \mathcal{H}} \sum_{g=1}^G \sum_{i=1}^{n_g} \epsilon_{gi} \langle \mathbf{x}_{gi}, \delta_{0g} \rangle
\end{aligned}$$

where  $\epsilon_{gi}$  are iid copies of Rademacher random variable which are independent of every other random variables and themselves.

Now we add back  $\frac{1}{n}$  and expand  $\delta_{0g} = \delta_0 + \delta_g$ . Also, we substitute  $\delta \in \mathcal{H}$  constraint with  $\delta \in \mathcal{C}$  because  $\mathcal{H} \subseteq \mathcal{C}$  where  $\mathcal{C} = \{\delta = (\delta_0^T, \dots, \delta_G^T)^T \mid \delta_g \in \mathcal{C}_g\}$ :

$$\begin{aligned}
& \frac{2}{n} \mathbb{E} \sup_{\delta \in \mathcal{C}} \sum_{g=1}^G \sum_{i=1}^{n_g} \epsilon_{gi} \langle \mathbf{x}_{gi}, \delta_{0g} \rangle = \frac{2}{n} \mathbb{E} \sup_{\delta_0 \in \mathcal{C}_0} \sum_{i=1}^n \epsilon_i \langle \mathbf{x}_i, \delta_0 \rangle + \frac{2}{n} \mathbb{E} \sup_{\forall g \in [G]: \delta_g \in \mathcal{C}_g} \sum_{g=1}^G \sum_{i=1}^{n_g} \epsilon_{gi} \langle \mathbf{x}_{gi}, \delta_g \rangle \\
&= \frac{2}{\sqrt{n}} \mathbb{E} \sup_{\delta_0 \in \mathcal{C}_0} \sum_{i=1}^n \langle \frac{1}{\sqrt{n}} \epsilon_i \mathbf{x}_i, \delta_0 \rangle + \frac{2}{\sqrt{n}} \mathbb{E} \sup_{\forall g \in [G]: \delta_g \in \mathcal{C}_g} \sum_{g=1}^G \sqrt{\frac{n_g}{n}} \sum_{i=1}^{n_g} \langle \frac{1}{\sqrt{n_g}} \epsilon_{gi} \mathbf{x}_{gi}, \delta_g \rangle \\
&\quad (n_0 := n, \epsilon_{0i} := \epsilon_0, \mathbf{x}_{0i} := \mathbf{x}_i) = \frac{2}{\sqrt{n}} \mathbb{E} \sup_{\forall g \in [G+]: \delta_g \in \mathcal{C}_g} \sum_{g=0}^G \sqrt{\frac{n_g}{n}} \sum_{i=1}^{n_g} \langle \frac{1}{\sqrt{n_g}} \epsilon_{gi} \mathbf{x}_{gi}, \delta_g \rangle \\
&\quad (\mathbf{h}_g := \frac{1}{\sqrt{n_g}} \sum_{i=1}^{n_g} \epsilon_{gi} \mathbf{x}_{gi}) = \frac{2}{\sqrt{n}} \mathbb{E} \sup_{\forall g \in [G+]: \delta_g \in \mathcal{C}_g} \sum_{g=0}^G \sqrt{\frac{n_g}{n}} \langle \mathbf{h}_g, \delta_g \rangle \\
&\quad (\mathbf{u}_g \in \frac{\delta_g}{\|\delta_g\|_2}, \mathcal{A}_g \in \mathcal{C}_g \cap \mathbb{S}^{p-1}) \leq \frac{2}{\sqrt{n}} \mathbb{E} \sup_{\forall g \in [G+]: \mathbf{u}_g \in \mathcal{A}_g} \sum_{g=0}^G \sqrt{\frac{n_g}{n}} \langle \mathbf{h}_g, \mathbf{u}_g \rangle \|\delta_g\|_2 \\
&\quad (\sup \sum < \sum \sup) \leq \frac{2}{\sqrt{n}} \sum_{g=0}^G \sqrt{\frac{n_g}{n}} \mathbb{E} \mathbf{h}_g \sup_{\mathbf{u}_g \in \mathcal{A}_g} \langle \mathbf{h}_g, \mathbf{u}_g \rangle \|\delta_g\|_2 \\
&\leq \frac{2}{\sqrt{n}} \sum_{g=0}^G \sqrt{\frac{n_g}{n}} c_g k_x \omega(\mathcal{A}_g) \|\delta_g\|_2
\end{aligned}$$

Note that the  $\mathbf{h}_{gi}$  is a sub-Gaussian random vector which let us bound the  $\mathbb{E} \sup$  using the Gaussian width [SM6] in the last step. ■

### SM1.5. Proof of Lemma 4.5.

*Proof.* To avoid cluttering let  $h_g(\mathbf{w}_g, \mathbf{X}_g) \triangleq \|\mathbf{w}_g\|_2 \sup_{\mathbf{u}_g \in \mathcal{A}_g} \langle \mathbf{X}_g^T \frac{\mathbf{w}_g}{\|\mathbf{w}_g\|_2}, \mathbf{u}_g \rangle$  be a random quantity and  $e_g(\tau) \triangleq c_g k_x(\omega(\mathcal{A}_g) + \sqrt{\log(G+1)} + \tau)$ , and  $s_g \triangleq \sqrt{(2k_w^2 + 1)n_g}$  constants. From the law of total probability, we have:

$$\begin{aligned} \mathbb{P}(h_g(\mathbf{w}_g, \mathbf{X}_g) > s_g e_g(\tau)) &= \mathbb{P}\left(h_g(\mathbf{w}_g, \mathbf{X}_g) > s_g e_g(\tau) \mid \|\mathbf{w}_g\|_2 > s_g\right) \mathbb{P}(\|\mathbf{w}_g\|_2 > s_g) \\ &\quad + \mathbb{P}\left(h_g(\mathbf{w}_g, \mathbf{X}_g) > s_g e_g(\tau) \mid \|\mathbf{w}_g\|_2 < s_g\right) \mathbb{P}(\|\mathbf{w}_g\|_2 < s_g) \\ &\leq \mathbb{P}(\|\mathbf{w}_g\|_2 > s_g) + \mathbb{P}\left(\|\mathbf{w}_g\|_2 \sup_{\mathbf{u}_g \in \mathcal{A}_g} \langle \mathbf{X}_g^T \frac{\mathbf{w}_g}{\|\mathbf{w}_g\|_2}, \mathbf{u}_g \rangle > s_g e_g(\tau) \mid \|\mathbf{w}_g\|_2 < s_g\right) \\ &\leq \mathbb{P}\left(\|\mathbf{w}_g\|_2 > \sqrt{(2k_w^2 + 1)n_g}\right) + \mathbb{P}\left(\sup_{\mathbf{u}_g \in \mathcal{C}_g \cap \mathbb{S}^{p-1}} \langle \mathbf{X}_g^T \frac{\mathbf{w}_g}{\|\mathbf{w}_g\|_2}, \mathbf{u}_g \rangle > e_g(\tau)\right) \\ &\stackrel{\text{(SM1.8)}}{\leq} \mathbb{P}\left(\|\mathbf{w}_g\|_2 > \sqrt{(2k_w^2 + 1)n_g}\right) + \sup_{\mathbf{v} \in \mathbb{S}^{p-1}} \mathbb{P}\left(\sup_{\mathbf{u}_g \in \mathcal{C}_g \cap \mathbb{S}^{p-1}} \langle \mathbf{X}_g^T \mathbf{v}, \mathbf{u}_g \rangle > e_g(\tau)\right) \end{aligned}$$

Let's focus on the first term. Since  $\mathbf{w}_g$  consists of i.i.d. centered unit-variance sub-Gaussian elements with  $\|w_{gi}\|_{\psi_2} < k_w$ ,  $w_{gi}^2$  is sub-exponential with  $\|\mathbf{w}_{gi}\|_{\psi_1} < 2k_w^2$ . Let's apply the Bernstein's inequality [SM2] to  $\|\mathbf{w}_g\|_2^2 = \sum_{i=1}^{n_g} \mathbf{w}_{gi}^2$ :

$$\mathbb{P}(|\|\mathbf{w}_g\|_2^2 - \mathbb{E}\|\mathbf{w}_g\|_2^2| > \tau) \leq 2 \exp\left(-\nu \min\left[\frac{\tau^2}{4k_w^4 n_g}, \frac{\tau}{2k_w^2}\right]\right)$$

We also know that  $\mathbb{E}\|\mathbf{w}_g\|_2^2 \leq n_g$  [SM8] which gives us:

$$\mathbb{P}(\|\mathbf{w}_g\|_2 > \sqrt{n_g + \tau}) \leq 2 \exp\left(-\nu \min\left[\frac{\tau^2}{4k_w^4 n_g}, \frac{\tau}{2k_w^2}\right]\right)$$

Finally, we set  $\tau = 2k_w^2 n_g$ :

$$\mathbb{P}\left(\|\mathbf{w}_g\|_2 > \sqrt{(2k_w^2 + 1)n_g}\right) \leq 2 \exp(-\nu n_g) = \frac{2}{(G+1)} \exp(-\nu n_g + \log(G+1))$$

Now we upper bound the second term of (SM1.8). Given any fixed  $\mathbf{v} \in \mathbb{S}^{p-1}$ ,  $\mathbf{X}_g \mathbf{v}$  is a sub-Gaussian random vector with  $\|\mathbf{X}_g^T \mathbf{v}\|_{\psi_2} \leq C_g k_x$  [SM1]. From Theorem 9 of [SM1] for any  $\mathbf{v} \in \mathbb{S}^{p-1}$  we have:

$$\mathbb{P}\left(\sup_{\mathbf{u}_g \in \mathcal{A}_g} \langle \mathbf{X}_g^T \mathbf{v}, \mathbf{u}_g \rangle > v_g C_g k_x \omega(\mathcal{A}_g) + t\right) \leq \pi_g \exp\left(-\left(\frac{t}{\theta_g C_g k_x \phi_g}\right)^2\right)$$

where  $\phi_g = \sup_{\mathbf{u}_g \in \mathcal{A}_g} \|\mathbf{u}_g\|_2$  and in our problem  $\phi_g = 1$ . To simplify we take  $c_g = C_g \max(v_g, \theta_g)$  and then substitute  $t = c_g k_x(\tau + \sqrt{\log(G+1)})$ :

$$\begin{aligned} \mathbb{P}\left(\sup_{\mathbf{u}_g \in \mathcal{A}_g} \langle \mathbf{X}_g^T \mathbf{v}, \mathbf{u}_g \rangle > c_g k_x \left(\omega(\mathcal{A}_g) + \sqrt{\log(G+1)} + \tau\right)\right) &\leq \pi_g \exp\left(-\left(\tau + \sqrt{\log(G+1)}\right)^2\right) \\ &\leq \pi_g \exp\left(-\log(G+1) - \tau^2\right) \\ &\leq \frac{\pi_g}{(G+1)} \exp(-\tau^2) \end{aligned}$$

121 Now we put back results to the original inequality (SM1.8):

$$\begin{aligned}
 122 \quad & \mathbb{P} \left( h_g(\mathbf{w}_g, \mathbf{X}_g) > \sqrt{(2k_w^2 + 1)n_g} \times c_g k_x \left( \omega(\mathcal{A}_g) + \sqrt{\log(G+1)} + \tau \right) \right) \\
 123 \quad & \leq \frac{2}{(G+1)} \exp(-\nu n_g + \log(G+1)) + \frac{\pi_g}{(G+1)} \exp(-\tau^2) \\
 124 \quad & \leq \frac{\sigma_g}{(G+1)} \exp(-\min[\nu n_g - \log(G+1), \tau^2])
 \end{aligned}$$

125 where  $\sigma_g = \pi_g + 2$ . ■

### 126 SM1.6. Proof of Lemma 5.3.

127 *Proof.* We upper bound the individual error  $\|\delta_g^{(t+1)}\|_2$  and the common one  $\|\delta_0^{(t+1)}\|_2$  in the  
 128 followings:

$$\begin{aligned}
 129 \quad & \|\delta_g^{(t+1)}\|_2 = \|\beta_g^{(t+1)} - \beta_g^*\|_2 \\
 130 \quad & = \left\| \Pi_{\Omega_{f_g}} \left( \beta_g^{(t)} + \mu_g \mathbf{X}_g^T (\mathbf{y}_g - \mathbf{X}_g (\beta_0^{(t)} + \beta_g^{(t)})) \right) - \beta_g^* \right\|_2 \\
 131 \quad & \text{(Lemma 6.3 of [SM5])} = \left\| \Pi_{\Omega_{f_g} - \{\beta_g^*\}} \left( \beta_g^{(t)} + \mu_g \mathbf{X}_g^T (\mathbf{y}_g - \mathbf{X}_g (\beta_0^{(t)} + \beta_g^{(t)})) \right) - \beta_g^* \right\|_2 \\
 132 \quad & = \left\| \Pi_{\mathcal{E}_g} \left( \delta_g^{(t)} + \mu_g \mathbf{X}_g^T (\mathbf{y}_g - \mathbf{X}_g (\beta_0^{(t)} + \beta_g^{(t)}) - \mathbf{X}_g (\beta_0^* + \beta_g^*) + \mathbf{X}_g (\beta_0^* + \beta_g^*)) \right) \right\|_2 \\
 133 \quad & = \left\| \Pi_{\mathcal{E}_g} \left( \delta_g^{(t)} + \mu_g \mathbf{X}_g^T (\omega_g - \mathbf{X}_g (\delta_0^{(t)} + \delta_g^{(t)})) \right) \right\|_2 \\
 134 \quad & \text{(Lemma 6.4 of [SM5])} \leq \left\| \Pi_{\mathcal{C}_g} \left( \delta_g^{(t)} + \mu_g \mathbf{X}_g^T (\omega_g - \mathbf{X}_g (\delta_0^{(t)} + \delta_g^{(t)})) \right) \right\|_2 \\
 135 \quad & \text{(Lemma 6.2 of [SM5])} \leq \sup_{\mathbf{v} \in \mathcal{C}_g \cap \mathbb{B}^p} \mathbf{v}^T \left( \delta_g^{(t)} + \mu_g \mathbf{X}_g^T (\omega_g - \mathbf{X}_g (\delta_0^{(t)} + \delta_g^{(t)})) \right) \\
 136 \quad & (\mathcal{B}_g = \mathcal{C}_g \cap \mathbb{B}^p) = \sup_{\mathbf{v} \in \mathcal{B}_g} \mathbf{v}^T \left( \delta_g^{(t)} + \mu_g \mathbf{X}_g^T (\omega_g - \mathbf{X}_g (\delta_0^{(t)} + \delta_g^{(t)})) \right) \\
 137 \quad & \leq \sup_{\mathbf{v} \in \mathcal{B}_g} \mathbf{v}^T (\mathbf{I}_g - \mu_g \mathbf{X}_g^T \mathbf{X}_g) \delta_g^{(t)} + \mu_g \sup_{\mathbf{v} \in \mathcal{B}_g} \mathbf{v}^T \mathbf{X}_g^T \omega_g + \mu_g \sup_{\mathbf{v} \in \mathcal{B}_g} -\mathbf{v}^T \mathbf{X}_g^T \mathbf{X}_g \delta_0^{(t)} \\
 138 \quad & \leq \left\| \delta_g^{(t)} \right\|_2 \sup_{\mathbf{u}, \mathbf{v} \in \mathcal{B}_g} \mathbf{v}^T (\mathbf{I}_g - \mu_g \mathbf{X}_g^T \mathbf{X}_g) \mathbf{u} + \mu_g \|\omega_g\|_2 \sup_{\mathbf{v} \in \mathcal{B}_g} \mathbf{v}^T \mathbf{X}_g^T \frac{\omega_g}{\|\omega_g\|_2} \\
 139 \quad & + \mu_g \|\delta_0^{(t)}\|_2 \sup_{\mathbf{v} \in \mathcal{B}_g, \mathbf{u} \in \mathcal{B}_0} -\mathbf{v}^T \mathbf{X}_g^T \mathbf{X}_g \mathbf{u} \\
 140 \quad & = \rho_g(\mu_g) \|\delta_g^{(t)}\|_2 + \xi_g(\mu_g) \|\omega_g\|_2 + \phi_g(\mu_g) \|\delta_0^{(t)}\|_2
 \end{aligned}$$

141 So the final bound becomes:

$$142 \quad \text{(SM1.11)} \quad \|\delta_g^{(t+1)}\|_2 \leq \rho_g(\mu_g) \|\delta_g^{(t)}\|_2 + \xi_g(\mu_g) \|\omega_g\|_2 + \phi_g(\mu_g) \|\delta_0^{(t)}\|_2$$

143 Now we upper bound the error of common parameter. Remember common parameter's update:

$$144 \quad \beta_0^{(t+1)} = \Pi_{\Omega_{f_0}} \left( \beta_0^{(t)} + \mu_0 \mathbf{X}_0^T \begin{pmatrix} (\mathbf{y}_1 - \mathbf{X}_1(\beta_0^{(t)} + \beta_1^{(t)})) \\ \vdots \\ (\mathbf{y}_G - \mathbf{X}_G(\beta_0^{(t)} + \beta_G^{(t)})) \end{pmatrix} \right).$$

$$145 \quad \|\delta_0^{(t+1)}\|_2 = \|\beta_0^{(t+1)} - \beta_0^*\|_2$$

146

$$147 \quad = \left\| \Pi_{\Omega_{f_0}} \left( \beta_0^{(t)} + \mu_0 \sum_{g=1}^G \mathbf{X}_g^T (\mathbf{y}_g - \mathbf{X}_g(\beta_0^{(t)} + \beta_g^{(t)})) \right) - \beta_0^* \right\|_2$$

$$148 \quad (\text{Lemma 6.3 of [SM5]}) = \left\| \Pi_{\Omega_{f_0} - \{\beta_0^*\}} \left( \beta_0^{(t)} + \mu_0 \sum_{g=1}^G \mathbf{X}_g^T (\mathbf{y}_g - \mathbf{X}_g(\beta_0^{(t)} + \beta_g^{(t)})) - \beta_0^* \right) \right\|_2$$

$$149 \quad = \left\| \Pi_{\mathcal{E}_0} \left( \delta_0^{(t)} + \mu_0 \sum_{g=1}^G \mathbf{X}_g^T (\mathbf{y}_g - \mathbf{X}_g(\beta_0^{(t)} + \beta_g^{(t)})) \right) \right\|_2$$

$$150 \quad (\text{Lemma 6.4 of [SM5]}) \leq \left\| \Pi_{\mathcal{C}_0} \left( \delta_0^{(t)} + \mu_0 \sum_{g=1}^G \mathbf{X}_g^T (\omega_g - \mathbf{X}_g(\delta_0^{(t)} + \delta_g^{(t)})) \right) \right\|_2$$

$$151 \quad (\text{Lemma 6.2 of [SM5]}) \leq \sup_{\mathbf{v} \in \mathcal{B}_0} \mathbf{v}^T \left( \delta_0^{(t)} + \mu_0 \sum_{g=1}^G \mathbf{X}_g^T (\omega_g - \mathbf{X}_g(\delta_0^{(t)} + \delta_g^{(t)})) \right)$$

$$152 \quad \leq \sup_{\mathbf{v} \in \mathcal{B}_0} \mathbf{v}^T (\mathbf{I} - \mu_0 \sum_{g=1}^G \mathbf{X}_g^T \mathbf{X}_g) \delta_0^{(t)} + \mu_0 \sup_{\mathbf{v} \in \mathcal{B}_0} \mathbf{v}^T \sum_{g=1}^G \mathbf{X}_g^T \omega_g$$

$$153 \quad + \mu_0 \sup_{\mathbf{v} \in \mathcal{B}_0} -\mathbf{v}^T \sum_{g=1}^G \mathbf{X}_g^T \mathbf{X}_g \delta_g^{(t)}$$

$$154 \quad \leq \|\delta_0^{(t)}\|_2 \sup_{\mathbf{u}, \mathbf{v} \in \mathcal{B}_0} \mathbf{v}^T (\mathbf{I} - \mu_0 \mathbf{X}_0^T \mathbf{X}_0) \mathbf{u} + \mu_0 \sup_{\mathbf{v} \in \mathcal{B}_0} \mathbf{v}^T \mathbf{X}_0^T \frac{\omega_0}{\|\omega_0\|_2} \|\omega_0\|_2$$

$$155 \quad + \mu_0 \sum_{g=1}^G \sup_{\mathbf{v}_g \in \mathcal{B}_0, \mathbf{u}_g \in \mathcal{B}_g} -\mathbf{v}_g^T \mathbf{X}_g^T \mathbf{X}_g \mathbf{u}_g \|\delta_g^{(t)}\|_2$$

$$156 \quad (\text{SM1.12}) \leq \rho_0(\mu_0) \|\delta_0^{(t)}\|_2 + \xi_0(\mu_0) \|\omega_0\|_2 + \mu_0 \sum_{g=1}^G \frac{\phi_g(\mu_g)}{\mu_g} \|\delta_g^{(t)}\|_2$$

157

■

159 To avoid cluttering we drop  $\mu_g$  as the arguments. Putting together (SM1.11) and (SM1.12)



inequalities we reach to the followings:

$$\begin{aligned} \|\delta_g^{(t+1)}\|_2 &\leq \rho_g \|\delta_g^{(t)}\|_2 + \xi_g \|\omega_g\|_2 + \phi_g \|\delta_0^{(t)}\|_2 \\ \|\delta_0^{(t+1)}\|_2 &\leq \rho_0 \|\delta_0^{(t)}\|_2 + \xi_0 \|\omega_0\|_2 + \mu_0 \sum_{g=1}^G \frac{\phi_g}{\mu_g} \|\delta_g^{(t)}\|_2 \end{aligned}$$

**SM1.7. Proof of Lemma 5.6.** We will need the following lemma in our proof. It establishes the RE condition for individual isotropic sub-Gaussian designs and provides us with the essential tool for proving high probability bounds.

**Lemma SM1.3 (Theorem 11 of [SM1]).** *For all  $g \in [G]$ , for the matrix  $\mathbf{X}_g \in \mathbb{R}^{n_g \times p}$  with independent isotropic sub-Gaussian rows, i.e.,  $\|\mathbf{x}_{gi}\|_{\psi_2} \leq k_x$  and  $\mathbb{E}[\mathbf{x}_{gi}\mathbf{x}_{gi}^T] = \mathbf{I}$ , the following result holds with probability at least  $1 - 2 \exp(-\gamma_g(\omega(\mathcal{A}_g) + \tau)^2)$  for  $\tau > 0$ :*

$$\forall \mathbf{u}_g \in \mathcal{C}_g : n_g \left(1 - c_g \frac{\omega(\mathcal{A}_g) + \tau}{\sqrt{n_g}}\right) \|\mathbf{u}_g\|_2^2 \leq \|\mathbf{X}_g \mathbf{u}_g\|_2^2 \leq n_g \left(1 + c_g \frac{\omega(\mathcal{A}_g) + \tau}{\sqrt{n_g}}\right) \|\mathbf{u}_g\|_2^2$$

where  $c_g > 0$  is constant.

The statement of **Lemma SM1.3** characterizes the distortion in the Euclidean distance between points  $\mathbf{u}_g \in \mathcal{C}_g$  when the matrix  $\mathbf{X}_g/n_g$  is applied to them and states that any sub-Gaussian design matrix is approximately isometry, with high probability:

$$(1 - \alpha) \|\mathbf{u}_g\|_2^2 \leq \frac{1}{n_g} \|\mathbf{X}_g \mathbf{u}_g\|_2^2 \leq (1 + \alpha) \|\mathbf{u}_g\|_2^2$$

where  $\alpha = c_g \frac{\omega(\mathcal{A}_g)}{\sqrt{n_g}}$ . Now the proof for **Lemma 5.6**:

**SM1.7.1. Bounding  $\rho_g(\mu_g)$ .**

*Proof.* First we upper bound each of the coefficients  $\forall g \in [G]$ :

$$\rho_g(\mu_g) = \sup_{\mathbf{u}, \mathbf{v} \in \mathcal{B}_g} \mathbf{v}^T (\mathbf{I}_g - \mu_g \mathbf{X}_g^T \mathbf{X}_g) \mathbf{u}$$

We upper bound the argument of the sup as follows:

$$\begin{aligned}
\mathbf{v}^T (\mathbf{I}_g - \mu_g \mathbf{X}_g^T \mathbf{X}_g) \mathbf{u} &= \frac{1}{4} [(\mathbf{u} + \mathbf{v})^T (\mathbf{I} - \mu_g \mathbf{X}_g^T \mathbf{X}_g) (\mathbf{u} + \mathbf{v}) - (\mathbf{u} - \mathbf{v})^T (\mathbf{I} - \mu_g \mathbf{X}_g^T \mathbf{X}_g) (\mathbf{u} - \mathbf{v})] \\
&= \frac{1}{4} [\|\mathbf{u} + \mathbf{v}\|_2^2 - \mu_g \|\mathbf{X}_g (\mathbf{u} + \mathbf{v})\|_2^2 - \|\mathbf{u} - \mathbf{v}\|_2^2 + \mu_g \|\mathbf{X}_g (\mathbf{u} - \mathbf{v})\|_2^2] \\
&\stackrel{(\text{Lemma SM1.3})}{\leq} \frac{1}{4} \left[ \left( 1 - \mu_g n_g \left( 1 - c_g \frac{2\omega(\mathcal{A}_g) + \tau}{\sqrt{n_g}} \right) \right) \|\mathbf{u} + \mathbf{v}\|_2 \right. \\
&\quad \left. - \left( 1 - \mu_g n_g \left( 1 + c_g \frac{2\omega(\mathcal{A}_g) + \tau}{\sqrt{n_g}} \right) \right) \|\mathbf{u} - \mathbf{v}\|_2 \right] \\
&\left( \mu_g = \frac{1}{a_g n_g} \right) \leq \frac{1}{4} \left[ \left( 1 - \frac{1}{a_g} \right) (\|\mathbf{u} + \mathbf{v}\|_2 - \|\mathbf{u} - \mathbf{v}\|_2) + c_g \frac{2\omega(\mathcal{A}_g) + \tau}{a_g \sqrt{n_g}} (\|\mathbf{u} + \mathbf{v}\|_2 + \|\mathbf{u} - \mathbf{v}\|_2) \right] \\
&\leq \frac{1}{4} \left[ \left( 1 - \frac{1}{a_g} \right) 2\|\mathbf{v}\|_2 + c_g \frac{2\omega(\mathcal{A}_g) + \tau}{a_g \sqrt{n_g}} 2\sqrt{2} \right]
\end{aligned}$$

where the last line follows from the triangle inequality and the fact that  $\|\mathbf{u} + \mathbf{v}\|_2 + \|\mathbf{u} - \mathbf{v}\|_2 \leq 2\sqrt{2}$  which itself follows from  $\|\mathbf{u} + \mathbf{v}\|_2^2 + \|\mathbf{u} - \mathbf{v}\|_2^2 \leq 4$ . Note that we applied the [Lemma SM1.3](#) for bigger sets of  $\mathcal{A}_g + \mathcal{A}_g$  and  $\mathcal{A}_g - \mathcal{A}_g$  where Gaussian width of both of them are upper bounded by  $2\omega(\mathcal{A}_g)$ .

The above holds with high probability which is computed as follows. Let's set  $\mu_g = \frac{1}{a_g n_g}$ ,  $d_g := \frac{1}{2} \left( 1 - \frac{1}{a_g} \right) + \sqrt{2} c_g \frac{\omega(\mathcal{A}_g) + \tau/2}{a_g \sqrt{n_g}}$  and name the bad events of  $\|\mathbf{X}_g (\mathbf{u} + \mathbf{v})\|_2^2 < n_g \left( 1 - c_g \frac{2\omega(\mathcal{A}_g) + \tau}{\sqrt{n_g}} \right)$  and  $\|\mathbf{X}_g (\mathbf{u} - \mathbf{v})\|_2^2 > n_g \left( 1 + c_g \frac{2\omega(\mathcal{A}_g) + \tau}{\sqrt{n_g}} \right)$  as  $\mathcal{E}_1$  and  $\mathcal{E}_2$  respectively. Then from the law of total probability we have:

$$\begin{aligned}
\mathbb{P}(\rho_g(\mu_g) \geq d_g) &\leq \mathbb{P}(\rho_g(\mu_g) \geq d_g | \neg \mathcal{E}_1, \neg \mathcal{E}_2) + \mathbb{P}(\mathcal{E}_1, \mathcal{E}_2) \\
&\leq 0 + \mathbb{P}(\mathcal{E}_1 | \mathcal{E}_2) \mathbb{P}(\mathcal{E}_2) \leq \mathbb{P}(\mathcal{E}_2) \\
&\stackrel{(\text{Lemma SM1.3})}{\leq} 2 \exp(-\gamma_g(\omega(\mathcal{A}_g) + \tau)^2)
\end{aligned}$$

which concludes the proof. ■

### SM1.7.2. Bounding $\eta_g(\mu_g)$ .

*Proof.* The proof of this bound has been worked out during the proof of [Lemma 4.5](#) where we show the following in equations [\(SM1.8\)](#) and [\(SM1.10\)](#)

$$\eta_g(\mu_g) = \mu_g \sup_{\mathbf{v} \in \mathcal{B}_g} \mathbf{v}^T \mathbf{X}_g^T \frac{\mathbf{w}_g}{\|\mathbf{w}_g\|_2} = \mu_g c_g k_x(\omega(\mathcal{A}_g) + \tau), \quad \text{w.p. at least } 1 - \pi_g \exp(-\tau^2)$$
■

### SM1.7.3. Bounding $\phi_g(\mu_g)$ .

*Proof.* The following holds for any  $\mathbf{u}$  and  $\mathbf{v}$  because of  $\|\mathbf{X}_g (\mathbf{u} + \mathbf{v})\|_2^2 \geq 0$ :

$$-\mathbf{v}^T \mathbf{X}_g^T \mathbf{X}_g \mathbf{u} \leq \frac{1}{2} (\|\mathbf{X}_g \mathbf{u}\|_2^2 + \|\mathbf{X}_g \mathbf{v}\|_2^2)$$

Now we can bound  $\phi_g$  as follows:

$$\phi_g(\mu_g) = \mu_g \sup_{\mathbf{v} \in \mathcal{B}_g, \mathbf{u} \in \mathcal{B}_0} -\mathbf{v}^T \mathbf{X}_g^T \mathbf{X}_g \mathbf{u} \leq \frac{\mu_g}{2} \left( \sup_{\mathbf{u} \in \mathcal{B}_0} \|\mathbf{X}_g \mathbf{u}\|_2^2 + \sup_{\mathbf{v} \in \mathcal{B}_g} \|\mathbf{X}_g \mathbf{v}\|_2^2 \right)$$

So we have:

$$\begin{aligned} \phi_g \left( \frac{1}{a_g n_g} \right) &\leq \frac{1}{2a_g} \left( \frac{1}{n_g} \sup_{\mathbf{u} \in \mathcal{B}_0} \|\mathbf{X}_g \mathbf{u}\|_2^2 + \frac{1}{n_g} \sup_{\mathbf{v} \in \mathcal{B}_g} \|\mathbf{X}_g \mathbf{v}\|_2^2 \right) \\ (\text{Lemma SM1.3}) &\leq \frac{1}{a_g} \left( 1 + c_{0g} \frac{\omega(\mathcal{A}_g) + \omega(\mathcal{A}_0) + 2\tau}{2\sqrt{n_g}} \right) \\ (\omega_{0g} = \max(\omega(\mathcal{A}_0), \omega(\mathcal{A}_g))) &\leq \frac{1}{a_g} \left( 1 + c_{0g} \frac{\omega_{0g} + 2\tau}{\sqrt{n_g}} \right) \end{aligned}$$

where  $c_{0g} = \max(c_0, c_g)$ .

To compute the exact probabilities let's define  $s_g := \frac{1}{a_g} \left( 1 + c_{0g} \frac{\omega(\mathcal{A}_g) + \omega(\mathcal{A}_0) + 2\tau}{2\sqrt{n_g}} \right)$  and name the bad events of  $\frac{1}{n_g} \sup_{\mathbf{u} \in \mathcal{B}_0} \|\mathbf{X}_g \mathbf{u}\|_2^2 > 1 + c_0 \frac{\omega(\mathcal{A}_0) + \tau}{\sqrt{n_g}}$  and  $\frac{1}{n_g} \sup_{\mathbf{v} \in \mathcal{B}_g} \|\mathbf{X}_g \mathbf{v}\|_2^2 > 1 + c_g \frac{\omega(\mathcal{A}_g) + \tau}{\sqrt{n_g}}$  as  $\mathcal{E}_1$  and  $\mathcal{E}_2$  respectively. Then from the law of total probability we have:

$$\begin{aligned} \mathbb{P}(\phi_g(\mu_g) > s_g) &\leq \mathbb{P}(\phi_g(\mu_g) > s_g | \neg \mathcal{E}_1, \neg \mathcal{E}_2) + \mathbb{P}(\mathcal{E}_1, \mathcal{E}_2) \\ &\leq 0 + \mathbb{P}(\mathcal{E}_1 | \mathcal{E}_2) \mathbb{P}(\mathcal{E}_2) \leq \mathbb{P}(\mathcal{E}_2) \\ (\text{Lemma SM1.3}) &\leq 2 \exp(-\gamma_g(\omega(\mathcal{A}_g) + \tau)^2) \end{aligned}$$

which concludes the proof. ■

### SM1.8. Proof of Lemma SM1.1.

*Proof.* To obtain lower bound, we use the Paley–Zygmund inequality for the zero-mean, non-degenerate ( $0 < \alpha \leq \mathbb{E}|\langle \mathbf{x}, \mathbf{u} \rangle|$ ,  $\mathbf{u} \in \mathbb{S}^{p-1}$ ) sub-Gaussian random vector  $\mathbf{x}$  with  $\|\mathbf{x}\|_{\psi_2} \leq k_x$  [SM6].

$$Q_{2\xi}(\mathbf{u}) \geq \frac{(\alpha - 2\xi)^2}{4ck_x^2}.$$

### SM1.9. Proof of Lemma SM1.2.

*Proof.* We split  $[G] - \mathcal{I}$  into two groups  $\mathcal{J}, \mathcal{K}$ .  $\mathcal{J}$  consists of  $\delta_g$ 's with  $\|\delta_g\|_2 \geq 2\|\delta_0\|_2$  and  $\mathcal{K} = [G] - \mathcal{I} - \mathcal{J}$ . We use the bounds

$$\|\delta_0 + \delta_g\|_2 \geq \begin{cases} \lambda_{\min}(\|\delta_g\|_2 + \|\delta_0\|_2) & \text{if } g \in \mathcal{I} \\ \|\delta_g\|_2/2 & \text{if } g \in \mathcal{J} \\ 0 & \text{if } g \in \mathcal{K} \end{cases}$$

This implies

$$\sum_{g=1}^G n_g \|\delta_0 + \delta_g\|_2 \geq \sum_{g \in \mathcal{J}} \frac{n_g}{2} \|\delta_g\|_2 + \lambda_{\min} \sum_{g \in \mathcal{I}} n_g (\|\delta_g\|_2 + \|\delta_0\|_2).$$
■

231 Let  $S_{\mathcal{S}} = \sum_{g \in \mathcal{S}} n_g \|\delta_g\|_2$  for  $\mathcal{S} = \mathcal{I}, \mathcal{J}, \mathcal{K}$ . We know that over  $\mathcal{K}$ ,  $\|\delta_g\|_2 \leq 2\|\delta_0\|_2$  which implies  
 232  $S_{\mathcal{K}} = \sum_{g \in \mathcal{K}} n_g \|\delta_g\|_2 \leq 2 \sum_{g \in \mathcal{K}} n_g \|\delta_0\|_2 \leq 2n\|\delta_0\|_2$ . Set  $\psi_{\mathcal{I}} = \min\{1/2, \lambda_{\min}\bar{\rho}/3\}$ . Using  
 233  $1/2 \geq \psi_{\mathcal{I}}$ , we write:

$$\begin{aligned}
 234 \quad & \sum_{g=1}^G n_g \|\delta_0 + \delta_g\|_2 \geq \psi_{\mathcal{I}} S_{\mathcal{J}} + \lambda_{\min} \sum_{g \in \mathcal{I}} n_g (\|\delta_g\|_2 + \|\delta_0\|_2) \\
 235 \quad & (S_{\mathcal{K}} \leq 2n\|\delta_0\|_2) \geq \psi_{\mathcal{I}} S_{\mathcal{J}} + \psi_{\mathcal{I}} S_{\mathcal{K}} - 2\psi_{\mathcal{I}} n \|\delta_0\|_2 + \left( \sum_{g \in \mathcal{I}} n_g \right) \lambda_{\min} \|\delta_0\|_2 + \lambda_{\min} S_{\mathcal{I}} \\
 236 \quad & (\lambda_{\min} \geq \psi_{\mathcal{I}}) \geq \psi_{\mathcal{I}} (S_{\mathcal{I}} + S_{\mathcal{J}} + S_{\mathcal{K}}) + \left( \left( \sum_{g \in \mathcal{I}} n_g \right) \lambda_{\min} - 2\psi_{\mathcal{I}} n \right) \|\delta_0\|_2.
 \end{aligned}$$

237 Now, observe that, assumption of the [Definition 3.4](#),  $\sum_{g \in \mathcal{I}} n_g \geq \bar{\rho}n$  implies:

$$238 \quad \left( \sum_{g \in \mathcal{I}} n_g \right) \lambda_{\min} - 2\psi_{\mathcal{I}} n \geq (\bar{\rho}\lambda_{\min} - 2\psi_{\mathcal{I}})n \geq \psi_{\mathcal{I}} n.$$

239 Combining all, we obtain:

$$240 \quad \sum_{g=1}^G n_g \|\delta_0 + \delta_g\|_2 \geq \psi_{\mathcal{I}} (S_{\mathcal{I}} + S_{\mathcal{J}} + S_{\mathcal{K}} + \|\delta_0\|_2) = \psi_{\mathcal{I}} (n\|\delta_0\|_2 + \sum_{g=1}^G n_g \|\delta_g\|_2).$$

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