High Dimensional Data Enrichment:
Interpretable, Fast, and Data-Efficient

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Abstract

Given samples from a set of groups, a data-enriched model describes observations by a common and per-group individual parameters. In high-dimensional regime, each parameter has its own structure such as sparsity or group sparsity. In this paper, we consider the general form of data enrichment where data comes in a fixed but arbitrary number of groups G and any convex function, e.g., norm, can characterize the structure of both common and individual parameters. We propose an estimator for the high-dimensional data enriched model and investigate its statistical properties. We delineate sample complexity of our estimator and provide high probability non-asymptotic bound for estimation error of all parameters under a condition weaker than the state-of-the-art. We propose an iterative estimation algorithm with a linear convergence rate and supplement our theoretical analysis with synthetic and real experimental results. In particular, we show the predictive power of data-enriched model along with its interpretable results in anticancer drug sensitivity analysis. Overall, we present a first through statistical and computational analysis of inference in the data enriched model.

1. Introduction

Over the past two decades, major advances have been made in estimating structured parameters, e.g., sparse, low-rank, etc., in high-dimensional small sample problems (Donoho, 2006; Candès & Tao, 2010; Friedman et al., 2008). Such estimators consider a suitable (semi) parametric model of the response: $y = \phi(\mathbf{x}, \boldsymbol{\beta}^*) + \omega$ based on n samples $\{(\mathbf{x}_i, y_i), i = 1, \ldots, n\}$ and $\beta \in \mathbb{R}^p$ is the parameter of interest. The unique aspect of such high-dimensional setup

is that the number of samples $n < p$, and the structure in $\boldsymbol{\beta}^*$, e.g., sparsity, low-rank, makes the estimation possible (Tibshirani, 1996; Candès et al., 2006; Candès & Recht, 2009). While the earlier developments in such high dimensional estimation problems had focused on parametric linear models, the results have been widely extended to non-linear models, e.g., generalized linear models (Negahban et al., 2009; Bach et al., 2012), broad families of semi-parametric and single-index models (Plan et al., 2017; Boufounos & Baraniuk, 2008), non-convex models (Blumensath & Davies, 2009; Jain et al., 2013), etc.

In several real world problems, the assumption that one global model parameter β is suitable for the entire population is unrealistic. We consider the more general setting where the population consists of sub-populations (groups) which are similar is many aspects but are different in certain unique aspects. For example, in the context of anti-cancer drug sensitivity prediction where the goal is to predict responses of different tumor cells to a drug, using a same prediction model for all cancer types (groups) ignores the specific aspects of each group and leads to an uninterpretable global model. In such a setting, one can assume a model for each group based on a group specific parameter $\boldsymbol{\beta}_g^*$, i.e., $y = \phi(\mathbf{x}, \boldsymbol{\beta}_g^*) + \omega$ for group g. Such a modeling approach fails to leverage the similarities across the sub-populations, and can only be estimated when sufficient number of samples are available for each sub-population which is not the case in several problems, e.g., anti-cancer drug sensitivity prediction (Barretina et al., 2012; Iorio et al., 2016).

In this paper, we consider the *data enrichment* strategy recently suggested in the literature (Dondelinger & Mukherjee, 2016; Gross & Tibshirani, 2016; Ollier & Viallon, 2014; 2015) for the above problem setup. A data enriched model *enriches* above models by taking their middle ground. It assumes that there is a *common* parameter $\boldsymbol{\beta}_0^*$ shared between all groups and *individual* per-group parameters $\boldsymbol{\beta}_g^*$ which characterize the deviation of group g, i.e.,

$$y_{gi} = \phi(\mathbf{x}_{gi}, (\boldsymbol{\beta}_0^* + \boldsymbol{\beta}_g^*)) + \omega_{gi}, \quad g \in \{1, \ldots, G\} \quad (1)$$

where g and i index the group and samples respectively. In (1), we have G models coupled by the common parameter $\boldsymbol{\beta}_0^*$ which models similarities between all samples.

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Individual parameters $\boldsymbol{\beta}_g$s capture unique aspects of each group. We specifically focus on the high-dimensional small sample regime for (1) where the number of samples n_g for each group is much smaller than the ambient dimensionality, i.e., $\forall g : n_g \ll p$. Similar to all other high-dimensional models, we assume that the parameters $\boldsymbol{\beta}_g$ are structured, i.e., for suitable convex functions f_g's, $f_g(\boldsymbol{\beta}_g)$ is small. For example, when the structure is sparsity, f_gs are L_1-norms. Further, for the technical analysis and proofs, we focus on the case of linear models, i.e., $\phi(\mathbf{x}, \boldsymbol{\beta}) = \mathbf{x}^T \boldsymbol{\beta}$. The results seamlessly extend to more general non-linear models, e.g., generalized linear models, broad families of semi-parametric and single-index models, non-convex models, etc., using existing results, i.e., how models like Lasso have been extended to these settings (e.g. employing ideas such as restricted strong convexity (Negahban & Wainwright, 2012)). it is OK I think(**Arindam Says: Is the last line correct? Lets check this.**)

Note that the models in (1) for each group is a superposition (Gu & Banerjee, 2016; McCoy & Tropp, 2013), or dirty statistical model (Yang & Ravikumar, 2013). (DE) model is *a system of coupled superposition models*. A related model is proposed by (Jalali et al., 2010) in the context of multi-task learning, where for each task g the output is coming from $y_{gi} = \mathbf{x}_{gi}^T (\boldsymbol{\beta}_{0g}^* + \boldsymbol{\beta}_g^*) + \omega_{gi}$. As emphasized by the subscript of $\boldsymbol{\beta}_{0g}^*$ the common parameters are different in every task but they share a same support (index of non-zero values), i.e., $\boldsymbol{\beta}_{0i}^* \neq \boldsymbol{\beta}_{0j}^*$ but $\mathrm{supp}(\boldsymbol{\beta}_{0i}^*) = \mathrm{supp}(\boldsymbol{\beta}_{0j}^*)$. This setup requires that every task should have enough samples to learn their own common parameters. In contrast, our model shares the common parameters and only requires that *total dataset over all tasks* is sufficiently large. (**Arindam Says: need a line on what Jalali et al. show, and how our results are (qualitatively) different**) (**Arindam Says: we don't have a discussion on 'related work' – including hierarchical models, multi-task learning; should we have a sub-section on these, or otherwise discuss these related developments**) At a high level, the DE model can be viewed as a of hierarchical additive model (Hastie, 2017; Rigby & Stasinopoulos, 2005) couldn't find good ref, we might want to remove this, where $\boldsymbol{\beta}_0^*$ corresponds to the root and $\boldsymbol{\beta}_g^*$ correspond to the refinements in the leaves.

The DE model where $\boldsymbol{\beta}_g$'s are sparse has recently gained attention because of its application in wide range of domains such as personalized medicine (Dondelinger & Mukherjee, 2016), sentiment analysis, banking strategy (Gross & Tibshirani, 2016), single cell data analysis (Ollier & Viallon, 2015), road safety (Ollier & Viallon, 2014), and disease subtype analysis (Dondelinger & Mukherjee, 2016). More generally, in any high-dimensional problem where the population consists of groups, data enrichment framework has the potential to boost the prediction accuracy and results in a more interpretable set of parameters.
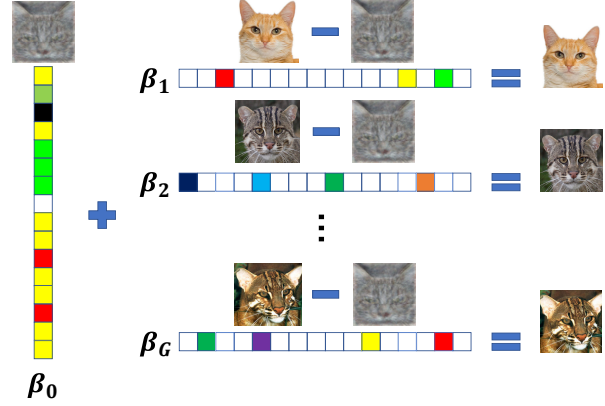

Figure 1. A conceptual illustration of data enrichment model for learning representation of different cat species. The common parameter $\boldsymbol{\beta}_0$ captures a *generic cat* which consists of shared features among all cats.

In spite of the recent surge in applying data enrichment framework to different domains, limited advances have been made in understanding the statistical and computational properties of suitable estimators for the data enriched model. In fact, non-asymptotic statistical properties, including sample complexity and statistical rates of convergence, of regularized estimators for the data enriched model is still an open question (Gross & Tibshirani, 2016; Ollier & Viallon, 2014). To the best of our knowledge, the only theoretical guarantee for data enrichment is provided in (Ollier & Viallon, 2015) where authors prove sparsistency of their proposed method under the stringent irrepresentability condition of the design matrix for recovering supports of common and individual parameters. In contrast to (Ollier & Viallon, 2015) where the analysis is restricted to sparsity and l_1-norm, our estimator and *norm consistency* analysis work for any structure induced by arbitrary convex functions. Moreover, no computational results, such as rates of convergence of the optimization algorithms associated with proposed estimators, exist in the literature.

Notation and Preliminaries: We denote sets by curly $\mathcal{V}$, matrices by bold capital $\mathbf{V}$, random variables by capital V, and vectors by small bold $\mathbf{v}$ letters. We take $[G] = \{0, \ldots, G\}$ and $[G]_{\setminus} = [G] \setminus \{0\}$.

Given G groups and n_g samples in each as $\{\{\mathbf{x}_{gi}, y_{gi}\}_{i=1}^{n_g}\}_{g=1}^{G}$, we can form the per group design matrix $\mathbf{X}_g \in \mathbb{R}^{n_g \times p}$ and output vector $\mathbf{y}_g \in \mathbb{R}^{n_g}$. The total number of samples is $n = \sum_{g=1}^{G} n_g$. The data enriched model takes the following vector form:

$$\mathbf{y}_g = \mathbf{X}_g(\boldsymbol{\beta}_0^* + \boldsymbol{\beta}_g^*) + \boldsymbol{\omega}_g, \quad \forall g \in [G]_{\setminus} \qquad (2)$$

where each row of $\mathbf{X}_g$ is $\mathbf{x}_{gi}^T$ and $\boldsymbol{\omega}_g^T = (\omega_{g1}, \ldots, \omega_{gn_g})$ is the noise vector.

A random variable V is sub-Gaussian if its moments satisfies $\forall p \geq 1 : (\mathbb{E}|V|^p)^{1/p} \leq K_2\sqrt{p}$. The minimum value of K_2 is called the sub-Gaussian norm of V, denoted by $\|V\|_{\psi_2}$ (Vershynin, 2012). A random vector $\mathbf{v} \in \mathbb{R}^p$ is sub-Gaussian if the one-dimensional marginals $\langle \mathbf{v}, \mathbf{u} \rangle$ are sub-Gaussian random variables for all $\mathbf{u} \in \mathbb{R}^p$. The sub-Gaussian norm of $\mathbf{v}$ is defined (Vershynin, 2012) as $\|\mathbf{v}\|_{\psi_2} = \sup_{\mathbf{u} \in \mathbb{S}^{p-1}} \|\langle \mathbf{v}, \mathbf{u} \rangle\|_{\psi_2}$. For any set $\mathcal{V} \in \mathbb{R}^p$ the Gaussian width of the set $\mathcal{V}$ is defined as $\omega(\mathcal{V}) = \mathbb{E}_{\mathbf{g}}\left[\sup_{\mathbf{u} \in \mathcal{V}} \langle \mathbf{g}, \mathbf{u} \rangle\right]$ (Vershynin, 2018), where the expectation is over $\mathbf{g} \sim N(\mathbf{0}, \mathbf{I}_{p \times p})$, a vector of independent zero-mean unit-variance Gaussian.

Contributions: We propose the following Data Enrichment (DE) estimator $\hat{\boldsymbol{\beta}}$ for recovering the structured parameters where the structure is induced by *convex* functions $f_g(\cdot)$:

$$\hat{\boldsymbol{\beta}} = (\hat{\boldsymbol{\beta}}_0^T, \ldots, \hat{\boldsymbol{\beta}}_G^T) \in \operatorname*{argmin}_{\boldsymbol{\beta}_0, \ldots, \boldsymbol{\beta}_G} \frac{1}{n}\sum_{g=1}^{G} \|\mathbf{y}_g - \mathbf{X}_g(\boldsymbol{\beta}_0 + \boldsymbol{\beta}_g)\|_2^2, \quad (3)$$

$$\text{s.t.} \quad \forall g \in [G] : f_g(\boldsymbol{\beta}_g) \leq f_g(\boldsymbol{\beta}_g^*).$$

We present several statistical and computational results for the DE estimator (3) of the data enriched model:

- The DE estimator (3) succeeds if a geometric condition that we call *Data EnRichment Incoherence Condition* (DERIC) is satisfied. Compared to other known geometric conditions in the literature such as structural coherence (Gu & Banerjee, 2016) and stable recovery conditions (McCoy & Tropp, 2013), DERIC is a considerably weaker condition.

- Assuming DERIC holds, we establish a high probability non-asymptotic bound on the weighted sum of parameter-wise estimation error, $\boldsymbol{\delta}_g = \hat{\boldsymbol{\beta}}_g - \boldsymbol{\beta}_g^*$ as:

$$\sum_{g=0}^{G} \sqrt{\frac{n_g}{n}} \|\boldsymbol{\delta}_g\|_2 \leq C\gamma \frac{\max_{g \in [G]} \omega(\mathcal{C}_g \cap \mathbb{S}^{p-1}) + \sqrt{\log(G+1)}}{\sqrt{n}}, \quad (4)$$

  where n_g is number of samples per group, $n_0 \triangleq n$ is the total number of samples, $\gamma \triangleq \max_{g \in [G]} \frac{n}{n_g}$ is the *sample condition number*, and $\mathcal{C}_g$ is the error cone corresponding to $\boldsymbol{\beta}_g^*$ exactly defined in Section 2. To the best of our knowledge, this is the first statistical estimation guarantee for the data enriched model.

- We also establish the sample complexity of the DE estimator for all parameters as $\forall g \in [G] : n_g = O(\omega(\mathcal{C}_g \cap \mathbb{S}^{p-1}))^2$. We emphasize that our result proofs that the recovery of the common parameter $\boldsymbol{\beta}_0$ by DE estimator benefits from *all* of the n pooled samples.

- We present an efficient projected block gradient descent algorithm DICER, to solve DE's objective (3) which

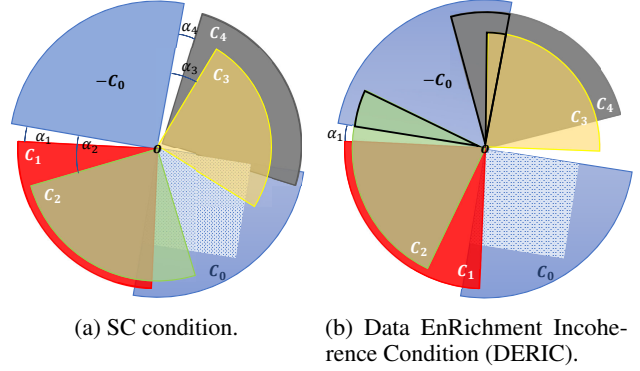(a) SC condition.　　(b) Data EnRichment Incoherence Condition (DERIC).

Figure 2. Geometric recovery conditions on error cones $\mathcal{C}_g, g = 0, \ldots, G$: (a) Structured coherence (SC) condition (Gu & Banerjee, 2016) requires $\mathcal{C}_g \cap -\mathcal{C}_0 = \emptyset$, i.e., no intersection between any $\mathcal{C}_g$ and $-\mathcal{C}_0$; and (b) proposed Data EnRichment Incoherence Condition (DERIC) condition allows intersections between some (not all) $\mathcal{C}_g$ and $-\mathcal{C}_0$.

converges geometrically to the statistical error bound of (4). To the best of our knowledge, this is the first rigorous computational result for the high-dimensional data-enriched regression.

- We illustrate promising empirical performance of the model on synthetic data as well as on the problem of finding bio-markers associated with drug sensitivity of cell lines from different cancer types, where the support of individual parameters $\hat{\boldsymbol{\beta}}_g^T$ for each cancer g shows a different set of bio-markers per cancer type.

The rest of this paper is organized as follows: First, we characterize the error set of our estimator and provide a deterministic error bound in Section 2. Then in Section 3, we discuss the restricted eigenvalue condition and calculate the sample complexity required for the recovery of the true parameters by our estimator under DERIC condition. We close the statistical analysis in Section 4 by providing non-asymptotic high probability error bound for parameter recovery. We delineate our geometrically convergent algorithm, DICER in Section 5 and finally supplement our work with synthetic and real data experiments in Sections **??** and **??**.

2. The Data Enrichment Estimator

A compact form of our proposed DE estimator (3) is:

$$\hat{\boldsymbol{\beta}} \in \operatorname*{argmin}_{\boldsymbol{\beta}} \frac{1}{n} \|\mathbf{y} - \mathbf{X}\boldsymbol{\beta}\|_2^2, \forall g \in [G] : f_g(\boldsymbol{\beta}_g) \leq f_g(\boldsymbol{\beta}_g^*), \quad (5)$$

where $\mathbf{y} = (\mathbf{y}_1^T, \ldots \mathbf{y}_G^T)^T \in \mathbb{R}^n, \boldsymbol{\beta} = (\boldsymbol{\beta}_0{}^T, \ldots, \boldsymbol{\beta}_G{}^T)^T \in \mathbb{R}^{(G+1)p}$ and

$$\mathbf{X} = \begin{pmatrix} \mathbf{X}_1 & \mathbf{X}_1 & 0 & \cdots & 0 \\ \mathbf{X}_2 & 0 & \mathbf{X}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ \mathbf{X}_G & 0 & \cdots & \cdots & \mathbf{X}_G \end{pmatrix} \in \mathbb{R}^{n \times (G+1)p} . \quad (6)$$

Example 1. (L_1-**norm**) When all parameters $\boldsymbol{\beta}_g$s are s_g-sparse, i.e.,$|\text{supp}(\boldsymbol{\beta}_g^*)| = s_g$ by using l_1-norm as the sparsity inducing function, our DE estimator of (5) becomes:

$$\hat{\boldsymbol{\beta}} \in \underset{\boldsymbol{\beta}}{\operatorname{argmin}} \frac{1}{n} \|\mathbf{y} - \mathbf{X}\boldsymbol{\beta}\|_2^2, \forall g \in [G] : \|\boldsymbol{\beta}_g\|_1 \le \|\boldsymbol{\beta}_g^*\|_1. \quad (7)$$

We call (7) *sparse DE* estimator and use it as a running example throughout the paper to illustrate outcomes of our analysis.

Consider the group-wise estimation error $\boldsymbol{\delta}_g = \hat{\boldsymbol{\beta}}_g - \boldsymbol{\beta}_g^*$. Since $\hat{\boldsymbol{\beta}}_g = \boldsymbol{\beta}_g^* + \boldsymbol{\delta}_g$ is a feasible point of (5), the error vector $\boldsymbol{\delta}_g$ will belong to the following restricted error set:

$$\mathcal{E}_g = \left\{ \boldsymbol{\delta}_g | f_g(\boldsymbol{\beta}_g^* + \boldsymbol{\delta}_g) \le f_g(\boldsymbol{\beta}_g^*) \right\}, \quad g \in [G] . \quad (8)$$

We denote the cone of the error set as $\mathcal{C}_g \triangleq \text{Cone}(\mathcal{E}_g)$ and the spherical cap corresponding to it as $\mathcal{A}_g \triangleq \mathcal{C}_g \cap \mathbb{S}^{p-1}$. Consider the set $\mathcal{C} = \left\{ \boldsymbol{\delta} = (\boldsymbol{\delta}_0^T, \ldots, \boldsymbol{\delta}_G^T)^T \middle| \boldsymbol{\delta}_g \in \mathcal{C}_g \right\}$, following two subsets of $\mathcal{C}$ play key roles in our analysis:

$$\mathcal{H} \triangleq \left\{ \boldsymbol{\delta} \in \mathcal{C} | \sum_{g=0}^{G} \frac{n_g}{n} \|\boldsymbol{\delta}_g\|_2 = 1 \right\}, \quad (9)$$

$$\bar{\mathcal{H}} \triangleq \left\{ \boldsymbol{\delta} \in \mathcal{C} | \sum_{g=0}^{G} \sqrt{\frac{n_g}{n}} \|\boldsymbol{\delta}_g\|_2 = 1 \right\}. \quad (10)$$

Starting from the optimality of $\hat{\boldsymbol{\beta}} = \boldsymbol{\beta}^* + \boldsymbol{\delta}$ as $\frac{1}{n}\|\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}}\|_2^2 \le \frac{1}{n}\|\mathbf{y} - \mathbf{X}\boldsymbol{\beta}^*\|_2^2$, we have: $\frac{1}{n}\|\mathbf{X}\boldsymbol{\delta}\|_2^2 \le \frac{1}{n}2\boldsymbol{\omega}^T\mathbf{X}\boldsymbol{\delta}$ where $\boldsymbol{\omega} = [\boldsymbol{\omega}_1^T, \ldots, \boldsymbol{\omega}_G^T]^T \in \mathbb{R}^n$ is the vector of all noises. Using this basic inequality, we can establish the following deterministic error bound.

Theorem 1. *For the proposed estimator (5), assume there exist* $0 < \kappa \le \inf_{\mathbf{u} \in \mathcal{H}} \frac{1}{n}\|\mathbf{X}\mathbf{u}\|_2^2$. *Then, for the sample condition number* $\gamma = \max_{g \in [G]_\backslash} \frac{n}{n_g}$, *the following deterministic upper bounds holds:*

$$\sum_{g=0}^{G} \sqrt{\frac{n_g}{n}} \|\boldsymbol{\delta}_g\|_2 \le \frac{2\gamma \sup_{\mathbf{u} \in \bar{\mathcal{H}}} \boldsymbol{\omega}^T\mathbf{X}\mathbf{u}}{n\kappa} .$$

Remark 1. Consider the setting where $n_g = \Theta(\frac{n}{G})$ so that each group has approximately $\frac{1}{G}$ fraction of the samples. Then, $\gamma = \Theta(G)$ and hence

$$\frac{1}{G} \sum_{g=0}^{G} \|\delta_g\|_2 \le O(G^{1/2}) \frac{\sup_{\mathbf{u} \in \bar{\mathcal{H}}} \boldsymbol{\omega}^T\mathbf{X}\mathbf{u}}{n} . \quad (11)$$

3. Restricted Eigenvalue Condition

The main assumptions of Theorem 1 is known as Restricted Eigenvalue (RE) condition in the literature of high dimensional statistics (Banerjee et al., 2014; Negahban et al., 2012; Raskutti et al., 2010): $\inf_{\mathbf{u} \in \mathcal{H}} \frac{1}{n}\|\mathbf{X}\mathbf{u}\|_2^2 \ge \kappa > 0$. The RE condition posits that the minimum eigenvalues of the matrix $\mathbf{X}^T\mathbf{X}$ in directions restricted to $\mathcal{H}$ is strictly positive. In this section, we show that for the design matrix $\mathbf{X}$ defined in (6), the RE condition holds with high probability under a suitable geometric condition we call *Data EnRichment Incoherence Condition* (DERIC) and enough number of samples. We precisely characterize total and per-group sample complexities required for successful parameter recovery. For the analysis, similar to existing work (Tropp, 2015; Mendelson, 2014; Gu & Banerjee, 2016), we assume the design matrix to be isotropic sub-Gaussian.[1]

Definition 1. *We assume* $\mathbf{x}_{gi}$ *are i.i.d. random vectors from a non-degenerate zero-mean, isotropic sub-Gaussian distribution. In other words,* $\mathbb{E}[\mathbf{x}] = 0$, $\mathbb{E}[\mathbf{x}^T\mathbf{x}] = \mathbf{I}_{p \times p}$, *and* $\|\mathbf{x}\|_{\psi_2} \le k$. *As a consequence,* $\exists \alpha > 0$ *such that* $\forall \mathbf{u} \in \mathbb{S}^{p-1}$ *we have* $\mathbb{E}|\langle \mathbf{x}, \mathbf{u} \rangle| \ge \alpha$. *Further, we assume noise* $\boldsymbol{\omega}_{gi}$ *are i.i.d. zero-mean, unit-variance sub-Gaussian with* $\|\boldsymbol{\omega}_{gi}\|_{\psi_2} \le K$.

Unlike standard high-dimensional statistical estimation, for RE condition to be true, parameters of the data enriched model (2) needs to satisfy a geometric condition under which trivial solutions such as $\boldsymbol{\delta}_g = -\boldsymbol{\delta}_0$ for all $g \in [G]_\backslash$ are avoided. To derive this condition, first note that each of the linear models in (2) is a superposition model (Gu & Banerjee, 2016) or dirty statistical model (Yang & Ravikumar, 2013). RE condition of individual superposition models can be established under the so-called Structural Coherence (SC) condition (Gu & Banerjee, 2016; McCoy & Tropp, 2013).

Definition 2 (Structural Coherence (SC) (Gu & Banerjee, 2016; McCoy & Tropp, 2013))**.** *Consider a superposition model of the form (2). The SC condition is sufficient for satisfying the RE condtion and requires for* $\boldsymbol{\delta}_0 \in \mathcal{C}_0$ *and each* $\boldsymbol{\delta}_g \in \mathcal{C}_g$ *there exist* $\lambda > 0$ *such that:* $\|\boldsymbol{\delta}_0 + \boldsymbol{\delta}_g\|_2 \ge \lambda (\|\boldsymbol{\delta}_0\|_2 + \|\boldsymbol{\delta}_g\|_2)$.

The SC condition on each individual problem fails to utilize the true coupling structure in the data enriched model, where $\boldsymbol{\beta}_0^*$ is involved in all groups. In fact, below we show, using SC on each individual model leads to radically pessimistic estimates of the sample complexity for $\boldsymbol{\beta}_0^*$ recovery.

Proposition 1. *Assume observations distributed as defined in Definition 1 and pair-wise SC conditions are satisfied. Consider each superposition model (2) in isolation; to reco-*

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[1]Extension to an-isotropic sub-Gaussian case is straightforward by techniques developed in (Banerjee et al., 2014; Rudelson & Zhou, 2013).

ver the common parameter $\boldsymbol{\beta}_0^*$ requires at least one group i to have $n_i = O(\omega^2(\mathcal{A}_0))$. To recover the rest of individual parameters, we need $\forall g \neq i : n_g = O(\omega^2(\mathcal{A}_g))$ samples.

In other words, by separate analysis of superposition estimators one problem needs to have sufficient samples for recovering the common parameter $\boldsymbol{\beta}_0$ and therefore the common parameter recovery does not benefit from the pooled n samples. But given the nature of coupling in the data enriched model, we hope to be able to get a better sample complexity specifically for the common parameter $\boldsymbol{\beta}_0$.

Here, we introduce DERIC, a considerably weaker geometric condition compared to SC of (Gu & Banerjee, 2016; McCoy & Tropp, 2013).

Definition 3 (Data EnRichment Incoherence Condition (DE-RIC))**.** *There exists a set $\mathcal{I} \subseteq [G]_\backslash$ of groups where for some scalars $0 \leq \bar{\rho} \leq 1$ and $\lambda_{\min} > 0$ the following holds:*

1. *$\sum_{i \in \mathcal{I}} n_i \geq \lceil \bar{\rho} n \rceil$.*

2. *$\forall i \in \mathcal{I}$, $\forall \boldsymbol{\delta}_i \in \mathcal{C}_i$, and $\boldsymbol{\delta}_0 \in \mathcal{C}_0$: $\|\boldsymbol{\delta}_i + \boldsymbol{\delta}_0\|_2 \geq \lambda_{\min}(\|\boldsymbol{\delta}_0\|_2 + \|\boldsymbol{\delta}_i\|_2)$*

Observe that $0 \leq \lambda_{\min}, \bar{\rho} \leq 1$ by definition.

(Arindam Says: why would $\bar{\rho} = 0$ work? also, can $\mathcal{I}$ be empty? pls update as needed)

Remark 2. Clearly DERIC and SC conditions are satisfied if the error cones $\mathcal{C}_g$ and $\mathcal{C}_0$ does not have a ray in common, i.e., $\sup\langle \boldsymbol{\delta}_0/\|\boldsymbol{\delta}_0\|_2, \boldsymbol{\delta}_g/\|\boldsymbol{\delta}_g\|_2 \rangle < 1$ (Tropp, 2015; Gu & Banerjee, 2016), Figure 2a. However, DERIC condition also allows for a large fraction of cones to intersect with $-\mathcal{C}_0$. In particular, SC requires that none of the individual error cones $\mathcal{C}_g$ intersect with the inverted error cone $-\mathcal{C}_0$. Instead of this stringent geometric condition, we allow $-\mathcal{C}_0$ to intersect with an arbitrarily large fraction of the $\mathcal{C}_g$ cones, Figure 2b. As the number of intersections increases, our bound becomes looser.

Using DERIC and the small ball method (Mendelson, 2014), a recent tool from empirical process theory, we get a better sample complexity for satisfying the RE condition:

Theorem 2. *Let $\mathbf{x}_{gi}s$ be random vectors defined in Definition 1. Assume DERIC condition of Definition 3 holds for error cones $\mathcal{C}_g s$ and $\psi_{\mathcal{I}} = \lambda_{\min}\bar{\rho}/3$. Then, for all $\boldsymbol{\delta} \in \mathcal{H}$, when we have enough number of samples as $\forall g \in [G]_\backslash : n_g \geq m_g = O(k^6 \alpha^{-6} \psi_{\mathcal{I}}^{-2} \omega(\mathcal{A}_g)^2)$, with probability at least $1 - e^{-n\kappa_{\min}/4}$ we have:*

$$\inf_{\boldsymbol{\delta} \in \mathcal{H}} \frac{1}{\sqrt{n}} \|\mathbf{X}\boldsymbol{\delta}\|_2 \geq \frac{\kappa_{\min}}{2}$$

where $\kappa_{\min} = \min_{g \in [G]_\backslash} C\psi_{\mathcal{I}} \frac{\alpha^3}{k^2} - \frac{2c_g k\omega(\mathcal{A}_g)}{\sqrt{n_g}}$ and $\kappa = \frac{\kappa_{\min}^2}{4}$ is the lower bound of the RE condition.

Example 2. (L_1-**norm**) The Gaussian width of the spherical cap of a p-dimensional s-sparse vector is $\omega(\mathcal{A}) = \Theta(\sqrt{s \log p})$ (Banerjee et al., 2014; Vershynin, 2018). Therefore, the number of samples per group and total required for satisfaction of the RE condition in the sparse DE estimator (7) is $\forall g \in [G] : n_g \geq m_g = \Theta(s_g \log p)$.

4. General Error Bound

In this section, we provide a high probability upper bound for the estimation error of the common and individual parameters. To avoid cluttering the notation, we rename the vector of all noises as $\boldsymbol{\omega}_0 \triangleq \boldsymbol{\omega}$. First, we massage the deterministic upper bound of Theorem 1 as follows:

$$\boldsymbol{\omega}^T \mathbf{X} \boldsymbol{\delta} = \sum_{g=0}^{G} \langle \mathbf{X}_g^T \boldsymbol{\omega}_g, \boldsymbol{\delta}_g \rangle$$

$$= \sum_{g=0}^{G} \sqrt{\frac{n_g}{n}} \|\boldsymbol{\delta}_g\|_2 \langle \mathbf{X}_g^T \frac{\boldsymbol{\omega}_g}{\|\boldsymbol{\omega}_g\|_2}, \frac{\boldsymbol{\delta}_g}{\|\boldsymbol{\delta}_g\|_2} \rangle \sqrt{\frac{n}{n_g}} \|\boldsymbol{\omega}_g\|_2$$

Assume $b_g = \langle \mathbf{X}_g^T \frac{\boldsymbol{\omega}_g}{\|\boldsymbol{\omega}_g\|_2}, \frac{\boldsymbol{\delta}_g}{\|\boldsymbol{\delta}_g\|_2} \rangle \sqrt{\frac{n}{n_g}} \|\boldsymbol{\omega}_g\|_2$ and $a_g = \sqrt{\frac{n_g}{n}} \|\boldsymbol{\delta}_g\|_2$. Then the above term is the inner product of two vectors $\mathbf{a} = (a_0, \ldots, a_G)$ and $\mathbf{b} = (b_0, \ldots, b_G)$ for which we have:

$$\sup_{\mathbf{a} \in \mathcal{H}} \mathbf{a}^T \mathbf{b} = \sup_{\|\mathbf{a}\|_1 = 1} \mathbf{a}^T \mathbf{b} \leq \|\mathbf{b}\|_\infty = \max_{g \in [G]} b_g,$$

where the inequality holds because of the definition of the dual norm. We can upper bounds b_gs with high probability and then by union bound below theorem establishes a high probability upper bound for the deterministic bound of Theorem 1, i.e., $\frac{2}{n} \boldsymbol{\omega}^T \mathbf{X} \mathbf{u}$.

Theorem 3. *Assume $\mathbf{x}_{gi}$ and ω_{gi} distributed according to Definition 1, then with probability at least $1 - \sigma \exp\left(-\min_{g \in [G]}\left[\nu_g n_g - \log(G+1), \frac{\tau^2}{\eta_g^2 k^2}\right]\right)$ we have:*

$$\frac{2}{n} \boldsymbol{\omega}^T \mathbf{X} \boldsymbol{\delta} \leq \sqrt{\frac{8K^2 + 4}{n}} \max_{g \in [G]} \left(\zeta_g k\omega(\mathcal{A}_g) + \epsilon_g \sqrt{\log(G+1)} + \tau \right)$$

where $\sigma = \max_{g \in [G]} \sigma_g$ and $\tau > 0$.

The following corollary characterizes the general error bound and results from the direct combination of Theorem 1, Theorem 2, and Theorem 3.

Corollary 1. *For $\mathbf{x}_{gi}$ and ω_{gi} described in Definition 1 when we have enough number of samples $\forall g \in [G] : n_g > m_g$ which lead to $\kappa > 0$, the following general error bound holds with high probability for estimator (5):*

$$\sum_{g=0}^{G} \sqrt{\frac{n_g}{n}} \|\boldsymbol{\delta}_g\|_2 \leq C\gamma \frac{k\zeta \max_{g \in [G]} \omega(\mathcal{A}_g) + \epsilon \sqrt{\log(G+1)} + \tau}{\kappa_{\min}^2 \sqrt{n}} \quad (12)$$

where $C = 8\sqrt{2K^2 + 1}$, $\zeta = \max_{g \in [G]} \zeta_g$, $\epsilon = \max_{g \in [G]} \epsilon_g$, $\gamma = \max_{g \in [G]_\backslash} n/n_g$ and $\tau > 0$.

Algorithm 1 DICER

1: **input:** $\mathbf{X}, \mathbf{y}$, learning rates $(\mu_0, \ldots, \mu_G)$, initialization $\boldsymbol{\beta}^{(1)} = \mathbf{0}$
2: **output:** $\hat{\boldsymbol{\beta}}$
3: **for** t = 1 **to** T **do**
4:   **for** g=1 **to** G **do**
5:     $\boldsymbol{\beta}_g^{(t+1)} = \Pi_{\Omega_{f_g}} \left(\boldsymbol{\beta}_g^{(t)} + \mu_g \mathbf{X}_g^T \left(\mathbf{y}_g - \mathbf{X}_g \left(\boldsymbol{\beta}_0^{(t)} + \boldsymbol{\beta}_g^{(t)} \right) \right) \right)$
6:   **end for**
7:   $\boldsymbol{\beta}_0^{(t+1)} = \Pi_{\Omega_{f_0}} \left(\boldsymbol{\beta}_0^{(t)} + \mu_0 \mathbf{X}_0^T \left(\mathbf{y} - \mathbf{X}_0 \boldsymbol{\beta}_0^{(t)} - \begin{pmatrix} \mathbf{X}_1 \boldsymbol{\beta}_1^{(t)} \\ \vdots \\ \mathbf{X}_G \boldsymbol{\beta}_G^{(t)} \end{pmatrix} \right) \right)$
8: **end for**

Example 3. (L_1-**norm**) For sparse DE estimator of (7), results of Theorem 2 and 3 translates to the following: For enough number of samples as $\forall g \in [G] : n_g \geq m_g = \Theta(s_g \log p)$, the error bound of (12) simplifies to:

$$\sum_{g=0}^{G} \sqrt{\frac{n_g}{n}} \|\boldsymbol{\delta}_g\|_2 = O\left(\sqrt{\frac{(\max_{g \in [G]} s_g) \log p}{n}} \right) \quad (13)$$

Therefore, individual errors are bounded as $\|\boldsymbol{\delta}_g\|_2 = O(\sqrt{(\max_{g \in [G]} s_g) \log p / n_g})$ which is slightly worse than $O(\sqrt{s_g \log p / n_g})$, the well-known error bound for recovering s_g-sparse vector from n_g observations using LASSO or similar estimators (Banerjee et al., 2014; Chandrasekaran et al., 2012; Candes et al., 2007; Chatterjee et al., 2014; Bickel et al., 2009). Note that $\max_{g \in [G]} s_g$ (instead of s_g) is the price we pay to recover the common parameter $\boldsymbol{\beta}_0$.

5. Estimation Algorithm

We propose *Data enrIChER* (DICER) a projected block gradient descent algorithm, Algorithm 1, where $\Pi_{\Omega_{f_g}}$ is the Euclidean projection onto the set $\Omega_{f_g}(d_g) = \{f_g(\boldsymbol{\beta}) \leq d_g\}$ where $d_g = f_g(\boldsymbol{\beta}_g^*)$ and is dropped to avoid cluttering. In practice, d_g can be determined by cross-validation.

To analysis convergence properties of DICER, we should upper bound the error of each iteration. Let's $\boldsymbol{\delta}^{(t)} = \boldsymbol{\beta}^{(t)} - \boldsymbol{\beta}^*$ be the error of iteration t of DICER, i.e., the distance from the true parameter (not the optimization minimum, $\hat{\boldsymbol{\beta}}$). We show that $\|\boldsymbol{\delta}^{(t)}\|_2$ decreases exponentially fast in t to the statistical error $\|\boldsymbol{\delta}\|_2 = \|\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}^*\|_2$. We first start with the required definitions for our analysis.

Definition 4. *We define the following positive constants as functions of step sizes $\mu_g > 0$:*

$$\forall g \in [G] \quad : \quad \rho_g(\mu_g) = \sup_{\mathbf{u}, \mathbf{v} \in \mathcal{B}_g} \mathbf{v}^T \left(\mathbf{I}_g - \mu_g \mathbf{X}_g^T \mathbf{X}_g \right) \mathbf{u},$$

$$\eta_g(\mu_g) = \mu_g \sup_{\mathbf{v} \in \mathcal{B}_g} \mathbf{v}^T \mathbf{X}_g^T \frac{\boldsymbol{\omega}_g}{\|\boldsymbol{\omega}_g\|_2},$$

$$\forall g \in [G]_{\setminus} \quad : \quad \phi_g(\mu_g) = \mu_g \sup_{\mathbf{v} \in \mathcal{B}_g, \mathbf{u} \in \mathcal{B}_0} -\mathbf{v}^T \mathbf{X}_g^T \mathbf{X}_g \mathbf{u},$$

where $\mathcal{B}_g = \mathcal{C}_g \cap \mathbb{B}^p$ is the intersection of the error cone and the unit ball.

In the following theorem, we establish a deterministic bound on iteration errors $\|\boldsymbol{\delta}_g^{(t)}\|_2$ which depends on constants defined in Definition 4.

Theorem 4. *For Algorithm 1 initialized by $\boldsymbol{\beta}^{(1)} = \mathbf{0}$, we have the following deterministic bound for the error at iteration $t + 1$:*

$$\sum_{g=0}^{G} \sqrt{\frac{n_g}{n}} \|\boldsymbol{\delta}_g^{(t+1)}\|_2 \quad (14)$$

$$\leq \rho^t \sum_{g=0}^{G} \sqrt{\frac{n_g}{n}} \|\boldsymbol{\beta}_g^*\|_2 + \frac{1 - \rho^t}{1 - \rho} \sum_{g=0}^{G} \sqrt{\frac{n_g}{n}} \eta_g \|\boldsymbol{\omega}_g\|_2,$$

where $\rho \triangleq \max \left(\rho_0 + \sum_{g=1}^{G} \sqrt{\frac{n_g}{n}} \phi_g, \max_{g \in [G]} \left[\rho_g + \sqrt{\frac{n}{n_g}} \frac{\mu_0}{\mu_g} \phi_g \right] \right)$.

The RHS of (14) consists of two terms. If we keep $\rho < 1$, the first term approaches zero fast, and the second term determines the bound. In the following, we show that for specific choices of step sizes μ_gs, the second term can be upper bounded using the analysis of Section 4. More specifically, the first term corresponds to the optimization error which shrinks in every iteration while the second term is constant times the upper bound of the statistical error characterized in Corollary 1. Therefore, if we keep ρ below one, the estimation error of DE algorithm geometrically converges to the approximate statistical error bound.

One way for having $\rho < 1$ is to keep all arguments of $\max(\cdots)$ defining ρ strictly below 1. To this end, we first establish high probability upper bound for $\rho_g, \eta_g,$ and ϕ_g (in the Appendix A.6) and then show that with enough number of samples and proper step sizes μ_g, ρ can be kept strictly below one with high probability. The high probability bounds for constants in Definition 4 and the deterministic bound of Theorem 4 leads to the following theorem which shows for enough number of samples, of the same order as the statistical sample complexity, we can keep ρ below one and have geometric convergence.

Theorem 5. *Let $\tau = C\sqrt{\log(G+1)} + b$ for $b > 0$ and $\omega_{0g} = \omega(\mathcal{A}_0) + \omega(\mathcal{A}_g)$. For the per-group step sizes of:*

$$\mu_0 = \frac{1}{4n} \times \min_{g \in [G]_{\setminus}} \left(1 + c_{0g} \frac{\omega_{0g} + \tau}{\sqrt{n_g}} \right)^{-2},$$

$$\mu_g = \frac{1}{2\sqrt{nn_g}} \left(1 + c_{0g} \frac{\omega_{0g} + \tau}{\sqrt{n_g}} \right)^{-1}$$

and sample complexities of $\forall g \in [G] : n_g \geq 2c_g^2(2\omega(\mathcal{A}_g) + \tau)^2$, updates of the Algorithm 1 obey the following with high

probability:

$$\sum_{g=0}^{G} \sqrt{\frac{n_g}{n}} \|\boldsymbol{\delta}_g^{(t+1)}\|_2 \leq r(\tau)^t \sum_{g=0}^{G} \sqrt{\frac{n_g}{n}} \|\boldsymbol{\beta}_g^*\|_2$$
$$+ \quad \frac{(G+1)\sqrt{(2K^2+1)}}{\sqrt{n}(1-r(\tau))} \left(\zeta k \max_{g \in [G]} \omega(\mathcal{A}_g) + \tau \right),$$

where $r(\tau) < 1$.

Corollary 2. *For enough number of samples, iterations of DE algorithm with step sizes $\mu_0 = \Theta(\frac{1}{n})$ and $\mu_g = \Theta(\frac{1}{\sqrt{n n_g}})$ geometrically converges to the following with high probability:*

$$\sum_{g=0}^{G} \sqrt{\frac{n_g}{n}} \|\boldsymbol{\delta}_g^{\infty}\|_2 \leq c \frac{\zeta k \max_{g \in [G]} \omega(\mathcal{A}_g) + C\sqrt{\log(G+1)} + b}{\sqrt{n}(1-r(\tau))} \quad (15)$$

where $c = (G+1)\sqrt{(2K^2+1)}$.

It is instructive to compare RHS of (15) with that of (12): $\kappa_{\min}$ defined in Theorem 2 corresponds to $(1 - r(\tau))$ and the extra $G + 1$ factor corresponds to the sample condition number $\gamma = \max_{g \in [G]} \frac{n}{n_g}$. Therefore, Corollary 2 shows that DE converges to a scaled variant of statical error bound determined in Corollary 1.

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A. Proofs of Theorems

In this Section we present detail proof for each theorem and proposition. To avoid cluttering, during our proofs, we state some needed results as lemmas and provide their proof in the next Section B.

A.1. Proof of Theorem 1

Proof. Starting from the optimality inequality, for the lower bound with the set $\mathcal{H}$ we get:

$$
\frac{1}{n}\|\mathbf{X}\boldsymbol{\delta}\|_2^2 \geq \frac{1}{n}\inf_{\mathbf{u}\in\mathcal{H}}\|\mathbf{X}\mathbf{u}\|_2^2\left(\sum_{g=0}^{G}\frac{n_g}{n}\|\boldsymbol{\delta}_g\|_2\right)^2 \tag{16}
$$

$$
\geq \kappa\left(\sum_{g=0}^{G}\frac{n_g}{n}\|\boldsymbol{\delta}_g\|_2\right)^2
$$

$$
\geq \kappa\left(\min_{g\in[G]}\frac{n_g}{n}\right)\left(\sum_{g=0}^{G}\sqrt{\frac{n_g}{n}}\|\boldsymbol{\delta}_g\|_2\right)^2
$$

where $0 < \kappa \leq \frac{1}{n}\inf_{\mathbf{u}\in\mathcal{H}}\|\mathbf{X}\mathbf{u}\|_2^2$ is known as Restricted Eigenvalue (RE) condition. The upper bound will factorize as:

$$
\frac{2}{n}\boldsymbol{\omega}^T\mathbf{X}\boldsymbol{\delta} \leq \frac{2}{n}\sup_{\mathbf{u}\in\bar{\mathcal{H}}}\boldsymbol{\omega}^T\mathbf{X}\mathbf{u}\left(\sum_{g=0}^{G}\sqrt{\frac{n_g}{n}}\|\boldsymbol{\delta}_g\|_2\right), \quad \mathbf{u}\in\mathcal{H} \tag{17}
$$

Putting together inequalities (16) and (17) completes the proof. ∎

A.2. Proof of Proposition 1

Proof. Consider only one group for regression in isolation. Note that $\mathbf{y}_g = \mathbf{X}_g(\boldsymbol{\beta}_g^* + \boldsymbol{\beta}_0^*) + \boldsymbol{\omega}_g$ is a superposition model and as shown in (Gu & Banerjee, 2016) the sample complexity required for the RE condition and subsequently recovering $\boldsymbol{\beta}_0^*$ and $\boldsymbol{\beta}_g^*$ is $n_g \geq c(\max_{g\in[G]}\omega(\mathcal{A}_g) + \sqrt{\log 2})^2$. ∎

A.3. Proof of Theorem 2

Let's simplify the LHS of the RE condition:

$$
\frac{1}{\sqrt{n}}\|\mathbf{X}\boldsymbol{\delta}\|_2 = \left(\frac{1}{n}\sum_{g=1}^{G}\sum_{i=1}^{n_g}|\langle\mathbf{x}_{gi}, \boldsymbol{\delta}_0 + \boldsymbol{\delta}_g\rangle|^2\right)^{\frac{1}{2}}
$$

$$
\geq \frac{1}{n}\sum_{g=1}^{G}\sum_{i=1}^{n_g}|\langle\mathbf{x}_{gi}, \boldsymbol{\delta}_0 + \boldsymbol{\delta}_g\rangle|
$$

$$
\geq \frac{1}{n}\sum_{g=1}^{G}\xi\|\boldsymbol{\delta}_0 + \boldsymbol{\delta}_g\|_2\sum_{i=1}^{n_g}\mathbb{1}\left(|\langle\mathbf{x}_{gi}, \boldsymbol{\delta}_0 + \boldsymbol{\delta}_g\rangle| \geq \xi\|\boldsymbol{\delta}_0 + \boldsymbol{\delta}_g\|_2\right),
$$

where the first inequality is due to Lyapunov's inequality. To avoid cluttering we denote $\boldsymbol{\delta}_{0g} = \boldsymbol{\delta}_0 + \boldsymbol{\delta}_g$ where $\boldsymbol{\delta}_0 \in \mathcal{C}_0$ and $\boldsymbol{\delta}_g \in \mathcal{C}_g$. Now we add and subtract the corresponding per-group marginal tail function, $Q_{\xi_g}(\boldsymbol{\delta}_{0g}) = \mathbb{P}(|\langle\mathbf{x}, , \boldsymbol{\delta}_{0g}\rangle| > \xi_g)$ where $\xi_g > 0$. Let $\xi_g = \|\boldsymbol{\delta}_{0g}\|_2\xi$ then the LHS of the RE condition reduces to:

$$
\inf_{\boldsymbol{\delta}\in\mathcal{H}}\frac{1}{\sqrt{n}}\|\mathbf{X}\boldsymbol{\delta}\|_2 \geq \inf_{\boldsymbol{\delta}\in\mathcal{H}}\sum_{g=1}^{G}\frac{n_g}{n}\xi_g Q_{2\xi_g}(\boldsymbol{\delta}_{0g}) \tag{18}
$$

$$
- \sup_{\boldsymbol{\delta}\in\mathcal{H}}\frac{1}{n}\sum_{g=1}^{G}\xi_g\sum_{i=1}^{n_g}\left[Q_{2\xi_g}(\boldsymbol{\delta}_{0g}) - \mathbb{1}(|\langle\mathbf{x}_{gi}, \boldsymbol{\delta}_{0g}\rangle| \geq \xi_g)\right]
$$

$$
= t_1(\mathbf{X}) - t_2(\mathbf{X})
$$

For the ease of exposition we have written the LHS of (18) as the difference of two terms, i.e., $t_1(\mathbf{X}) - t_2(\mathbf{X})$ and in the followings we lower bound the first term t_1 and upper bound the second term t_2.

A.3.1. LOWER BOUNDING THE FIRST TERM

Our main result is the following lemma which uses the DERIC condition of the Definition 3 and provides a lower bound for the first term $t_1(\mathbf{X})$:

Lemma 1. *Suppose DERIC holds. Let* $\psi_{\mathcal{I}} = \frac{\lambda_{\min} \bar{\rho}}{3}$. *For any* $\boldsymbol{\delta} \in \mathcal{H}$, *we have:*

$$\sum_{g=1}^{G} \frac{n_g}{n} \xi_g Q_{2\xi_g}(\boldsymbol{\delta}_{0g}) \geq \psi_{\mathcal{I}} \xi \frac{(\alpha - 2\xi)^2}{4ck^2} \left(\|\boldsymbol{\delta}_0\|_2 + \sum_{g=1}^{n} \frac{n_g}{n} \|\boldsymbol{\delta}_g\|_2 \right), \tag{19}$$

which implies that $t_1(\mathbf{X}) = \inf_{\boldsymbol{\delta} \in \mathcal{H}} \sum_{g=1}^{G} \frac{n_G}{n} \xi_g Q_{2\xi_g}(\boldsymbol{\delta}_{0g})$ *satisfies the same RHS bound of* (19).

A.3.2. UPPER BOUNDING THE SECOND TERM

Let's focus on the second term, i.e., $t_2(\mathbf{X})$. First we want to show that the second term satisfies the bounded difference property defined in Section 3.2. of (Boucheron et al., 2013). In other words, by changing each of $\mathbf{x}_{gi}$ the value of $t_2(\mathbf{X})$ at most change by one. First, we rewrite t_2 as follows:

$$h\left(\mathbf{x}_{11}, \ldots, \mathbf{x}_{jk}, \ldots, \mathbf{x}_{Gn_G}\right) = t_2\left(\mathbf{x}_{11}, \ldots, \mathbf{x}_{jk}, \ldots, \mathbf{x}_{Gn_G}\right) = \sup_{\boldsymbol{\delta} \in \mathcal{H}} g\left(\mathbf{x}_{11}, \ldots, \mathbf{x}_{jk}, \ldots, \mathbf{x}_{Gn_G}\right)$$

where $g\left(\mathbf{x}_{11}, \ldots, \mathbf{x}_{jk}, \ldots, \mathbf{x}_{Gn_G}\right) = \sum_{g=1}^{G} \frac{\xi_g}{n} \sum_{i=1}^{n_g} \left[Q_{2\xi_g}(\boldsymbol{\delta}_{0g}) - \mathbb{1}(|\langle \mathbf{x}_{gi}, \boldsymbol{\delta}_{0g} \rangle| \geq \xi_g)\right]$. To avoid cluttering let's $\mathcal{X} = \{\mathbf{x}_{11}, \ldots, \mathbf{x}_{jk}, \ldots, \mathbf{x}_{Gn_G}\}$. We want to show that t_2 has the bounded difference property, meaning:

$$\sup_{\mathcal{X}, \mathbf{x}'_{jk}} |h\left(\mathbf{x}_{11}, \ldots, \mathbf{x}_{jk}, \ldots, \mathbf{x}_{Gn_G}\right) - h\left(\mathbf{x}_{11}, \ldots, \mathbf{x}'_{jk}, \ldots, \mathbf{x}_{Gn_G}\right)| \leq c_i$$

for some constant c_i. Note that for bounded functions $f, g : \mathcal{X} \to \mathbb{R}$, we have $|\sup_{\mathcal{X}} f - \sup_{\mathcal{X}} g| \leq \sup_{\mathcal{X}} |f - g|$. Therefore:

$$
\begin{aligned}
&\sup_{\mathcal{X}, \mathbf{x}'_{jk}} |h\left(\mathbf{x}_{11}, \ldots, \mathbf{x}_{jk}, \ldots, \mathbf{x}_{Gn_G}\right) - h\left(\mathbf{x}_{11}, \ldots, \mathbf{x}'_{jk}, \ldots, \mathbf{x}_{Gn_G}\right)| \\
&\leq \sup_{\mathcal{X}, \mathbf{x}'_{jk}} \sup_{\boldsymbol{\delta} \in \mathcal{H}} |g\left(\mathbf{x}_{11}, \ldots, \mathbf{x}_{jk}, \ldots, \mathbf{x}_{Gn_G}\right) - g\left(\mathbf{x}_{11}, \ldots, \mathbf{x}'_{jk}, \ldots, \mathbf{x}_{Gn_G}\right)| \\
&\leq \sup_{\mathcal{X}, \mathbf{x}'_{jk}} \sup_{\boldsymbol{\delta} \in \mathcal{H}} \sup_{\mathbf{x}_{jk}, \mathbf{x}'_{jk}} \frac{\xi_j}{n} \left(\mathbb{1}(|\langle \mathbf{x}'_{jk}, \boldsymbol{\delta}_{0j} \rangle| \geq \xi_j) - \mathbb{1}(|\langle \mathbf{x}_{jk}, \boldsymbol{\delta}_{0j} \rangle| \geq \xi_j)\right) \\
&\leq \sup_{\mathcal{X}, \mathbf{x}'_{jk}} \sup_{\boldsymbol{\delta} \in \mathcal{H}} \frac{\xi_j}{n} \\
&= \frac{\xi}{n} \sup_{\boldsymbol{\delta} \in \mathcal{H}} \|\boldsymbol{\delta}_0 + \boldsymbol{\delta}_g\|_2 \\
&= \frac{\xi}{n} \sup_{\boldsymbol{\delta} \in \mathcal{H}} \|\boldsymbol{\delta}_0\|_2 + \|\boldsymbol{\delta}_g\|_2 \\
(\boldsymbol{\delta} \in \mathcal{H}) \quad &= \xi \left(\frac{1}{n} + \frac{1}{n_g}\right) \\
&\leq \frac{2\xi}{n}
\end{aligned}
$$

Note that for $\boldsymbol{\delta} \in \mathcal{H}$ we have $\|\boldsymbol{\delta}_0\|_2 + \frac{n_g}{n} \|\boldsymbol{\delta}_g\|_2 \leq 1$ which results in $\|\boldsymbol{\delta}_0\|_2 \leq 1$ and $\|\boldsymbol{\delta}_g\|_2 \leq \frac{n}{n_g}$. Now, we can invoke the bounded difference inequality from Theorem 6.2 of (Boucheron et al., 2013) which says that with probability at least $1 - e^{-\tau^2/2}$ we have: $t_2(\mathbf{X}) \leq \mathbb{E} t_2(\mathbf{X}) + \frac{\tau}{\sqrt{n}}$.

Having this concentration bound, it is enough to bound the expectation of the second term. Following lemma provides us with the bound on the expectation.

Lemma 2. *For the random vector* $\mathbf{x}$ *of Definition 1, we have the following bound:*

$$\frac{2}{n}\mathbb{E}\sup_{\boldsymbol{\delta}_{[G]}}\sum_{g=1}^{G}\xi_g\sum_{i=1}^{n_g}\left[Q_{2\xi_g}(\boldsymbol{\delta}_{0g}) - \mathbb{1}(|\langle\mathbf{x}_{gi},\boldsymbol{\delta}_{0g}\rangle| \geq \xi_g)\right] \leq \frac{2}{\sqrt{n}}\sum_{g=0}^{G}\sqrt{\frac{n_g}{n}}c_g k\omega(\mathcal{A}_g)\|\boldsymbol{\delta}_g\|_2$$

A.3.3. CONTINUING THE PROOF OF THEOREM 2

Set $n_0 = n$. Putting back bounds of $t_1(\mathbf{X})$ and $t_2(\mathbf{X})$ together from Lemma 1 and 2, with probability at least $1 - e^{-\frac{\tau^2}{2}}$ we have:

$$\inf_{\boldsymbol{\delta}\in\mathcal{H}}\frac{1}{\sqrt{n}}\|\mathbf{X}\boldsymbol{\delta}\|_2 \geq \sum_{g=0}^{G}\frac{n_g}{n}\psi_{\mathcal{I}}\xi\|\boldsymbol{\delta}_g\|_2\frac{(\alpha-2\xi)^2}{4ck^2} - \frac{2}{\sqrt{n}}\sum_{g=0}^{G}\sqrt{\frac{n_g}{n}}c_g k\omega(\mathcal{A}_g)\|\boldsymbol{\delta}_g\|_2 - \frac{\tau}{\sqrt{n}}$$

$$\left(q = \frac{(\alpha-2\xi)^2}{4ck^2}\right) = \sum_{g=0}^{G}\frac{n_g}{n}\psi_{\mathcal{I}}\xi\|\boldsymbol{\delta}_g\|_2 q - \frac{2c}{\sqrt{n}}\sum_{g=0}^{G}\sqrt{\frac{n_g}{n}}k\omega(\mathcal{A}_g)\|\boldsymbol{\delta}_g\|_2 - \frac{\tau}{\sqrt{n}}$$

$$= n^{-1}\sum_{g=0}^{G}n_g\|\boldsymbol{\delta}_g\|_2(\psi_{\mathcal{I}}\xi q - 2ck\frac{\omega(\mathcal{A}_g)}{\sqrt{n_g}}) - \frac{\tau}{\sqrt{n}}$$

$$\left(\kappa_g = \psi_{\mathcal{I}}\xi q - \frac{2ck\omega(\mathcal{A}_g)}{\sqrt{n_g}}\right) = \sum_{g=0}^{G}\frac{n_g}{n}\|\boldsymbol{\delta}_g\|_2\kappa_g - \frac{\tau}{\sqrt{n}}$$

$$\geq \kappa_{\min}\sum_{g=0}^{G}\frac{n_g}{n}\|\boldsymbol{\delta}_g\|_2 - \frac{\tau}{\sqrt{n}}$$

$$(\boldsymbol{\delta}\in\mathcal{H}) = \kappa_{\min} - \frac{\tau}{\sqrt{n}}$$

where $\kappa_{\min} = \mathrm{argmin}_{g\in[G]}\kappa_g$. Note that all κ_gs should be bounded away from zero. To this end we need the follow sample complexities:

$$\forall g \in [G]: \quad \left(\frac{2ck}{\psi_{\mathcal{I}}\xi q}\right)^2 \omega(\mathcal{A}_g)^2 \leq n_g \tag{20}$$

Taking $\xi = \frac{\alpha}{6}$ we can simplify the sample complexities to the followings:

$$\forall g \in [G]: \quad \left(\frac{Ck^3}{\psi_{\mathcal{I}}\alpha^3}\right)^2 \omega(\mathcal{A}_g)^2 \leq n_g \tag{21}$$

Finally, to conclude, we take $\tau = \sqrt{n}\kappa_{\min}/2$. ∎

A.4. Proof of Theorem 3

Proof. From now on, to avoid cluttering the notation assume $\boldsymbol{\omega} = \boldsymbol{\omega}_0$. We massage the equation as follows:

$$\boldsymbol{\omega}^T\mathbf{X}\boldsymbol{\delta} = \sum_{g=0}^{G}\langle\mathbf{X}_g^T\boldsymbol{\omega}_g, \boldsymbol{\delta}_g\rangle = \sum_{g=0}^{G}\sqrt{\frac{n_g}{n}}\|\boldsymbol{\delta}_g\|_2\langle\mathbf{X}_g^T\frac{\boldsymbol{\omega}_g}{\|\boldsymbol{\omega}_g\|_2}, \frac{\boldsymbol{\delta}_g}{\|\boldsymbol{\delta}_g\|_2}\rangle\sqrt{\frac{n}{n_g}}\|\boldsymbol{\omega}_g\|_2$$

Assume $b_g = \langle\mathbf{X}_g^T\frac{\boldsymbol{\omega}_g}{\|\boldsymbol{\omega}_g\|_2}, \frac{\boldsymbol{\delta}_g}{\|\boldsymbol{\delta}_g\|_2}\rangle\sqrt{\frac{n}{n_g}}\|\boldsymbol{\omega}_g\|_2$ and $a_g = \sqrt{\frac{n_g}{n}}\|\boldsymbol{\delta}_g\|_2$. Then the above term is the inner product of two vectors $\mathbf{a} = (a_0, \ldots, a_G)$ and $\mathbf{b} = (b_0, \ldots, b_G)$ for which we have:

$$\sup_{\mathbf{a}\in\mathcal{H}}\mathbf{a}^T\mathbf{b} = \sup_{\|\mathbf{a}\|_1=1}\mathbf{a}^T\mathbf{b}$$

$$(\text{definition of the dual norm}) \leq \|\mathbf{b}\|_\infty$$

$$= \max_{g\in[G]}b_g$$

Now we can go back to the original form:

$$
\begin{aligned}
\sup_{\boldsymbol{\delta}\in\mathcal{H}} \boldsymbol{\omega}^T \mathbf{X}\boldsymbol{\delta} &\leq \max_{g\in[G]}\langle \mathbf{X}_g^T \frac{\boldsymbol{\omega}_g}{\|\boldsymbol{\omega}_g\|_2}, \frac{\boldsymbol{\delta}_g}{\|\boldsymbol{\delta}_g\|_2}\rangle \sqrt{\frac{n}{n_g}}\|\boldsymbol{\omega}_g\|_2 \qquad (22)\\
&\leq \max_{g\in[G]} \sqrt{\frac{n}{n_g}}\|\boldsymbol{\omega}_g\|_2 \sup_{\mathbf{u}_g\in\mathcal{C}_g\cap\mathbb{S}^{p-1}}\langle \mathbf{X}_g^T \frac{\boldsymbol{\omega}_g}{\|\boldsymbol{\omega}_g\|_2}, \mathbf{u}_g\rangle
\end{aligned}
$$

To avoid cluttering we name $h_g(\boldsymbol{\omega}_g, \mathbf{X}_g) = \|\boldsymbol{\omega}_g\|_2 \sup_{\mathbf{u}_g\in\mathcal{A}_g}\langle \mathbf{X}_g^T \frac{\boldsymbol{\omega}_g}{\|\boldsymbol{\omega}_g\|_2}, \mathbf{u}_g\rangle$ and $e_g(\tau) = \sqrt{(2K^2+1)n_g}\left(v_g C_g k\omega(\mathcal{A}_g) + \epsilon_g\sqrt{\log G} + \tau\right)$. Then from (22), we have:

$$
\mathbb{P}\left(\frac{2}{n}\sup_{\boldsymbol{\delta}\in\mathcal{H}}\boldsymbol{\omega}^T\mathbf{X}\boldsymbol{\delta} > \frac{2}{n}\max_{g\in[G]}\sqrt{\frac{n}{n_g}}e_g(\tau)\right) \leq \mathbb{P}\left(\frac{2}{n}\max_{g\in[G]}\sqrt{\frac{n}{n_g}}h_g(\boldsymbol{\omega}_g, \mathbf{X}_g) > \frac{2}{n}\max_{g\in[G]}\sqrt{\frac{n}{n_g}}e_g(\tau)\right)
$$

To simplify the notation, we drop arguments of h_g for now. From the union bound we have:

$$
\begin{aligned}
\mathbb{P}\left(\frac{2}{n}\max_{g\in[G]}\sqrt{\frac{n}{n_g}}h_g > \frac{2}{n}\max_{g\in[G]}\sqrt{\frac{n}{n_g}}e_g(\tau)\right) &\leq \sum_{g=0}^{G}\mathbb{P}\left(h_g > \max_{g\in[G]}e_g(\tau)\right)\\
&\leq \sum_{g=0}^{G}\mathbb{P}\left(h_g > e_g(\tau)\right)\\
&\leq (G+1)\max_{g\in[G]}\mathbb{P}\left(h_g > e_g(\tau)\right)\\
&\leq \sigma\exp\left(-\min_{g\in[G]}\left[\nu_g n_g - \log(G+1), \frac{\tau^2}{\eta_g^2 k^2}\right]\right)
\end{aligned}
$$

where $\sigma = \max_{g\in[G]}\sigma_g$ and the last inequality is a result of the following lemma:

Lemma 3. *For $\mathbf{x}_{gi}$ and ω_{gi} defined in Definition 1 and $\tau > 0$, with probability at least $1 - \frac{\sigma_g}{(G+1)}\exp\left(-\min\left[\nu_g n_g - \log(G+1), \frac{\tau^2}{\eta_g^2 k^2}\right]\right)$ we have:*

$$
\sqrt{\frac{n}{n_g}}\|\boldsymbol{\omega}_g\|_2 \sup_{\mathbf{u}_g\in\mathcal{A}_g}\langle \mathbf{X}_g^T \frac{\boldsymbol{\omega}_g}{\|\boldsymbol{\omega}_g\|_2}, \mathbf{u}_g\rangle \leq \sqrt{(2K^2+1)n}\left(\zeta_g k\omega(\mathcal{A}_g) + \epsilon_g\sqrt{\log(G+1)} + \tau\right),
$$

where $\sigma_g, \eta_g, \zeta_g$ and ϵ_g are group dependent constants.

∎

A.5. Proof of Theorem 4

Proof. In the following lemma we establish a recursive relation between errors of consecutive iterations which leads to a bound for the tth iteration.

Lemma 4. *We have the following recursive dependency between the error of $t+1$th iteration and tth iteration of DE:*

$$
\begin{aligned}
\|\boldsymbol{\delta}_g^{(t+1)}\|_2 &\leq \left(\rho_g(\mu_g)\|\boldsymbol{\delta}_g^{(t)}\|_2 + \xi_g(\mu_g)\|\boldsymbol{\omega}_g\|_2 + \phi_g(\mu_g)\|\boldsymbol{\delta}_0^{(t)}\|_2\right)\\
\|\boldsymbol{\delta}_0^{(t+1)}\|_2 &\leq \left(\rho_0(\mu_0)\|\boldsymbol{\delta}_0^{(t)}\|_2 + \xi_0(\mu_0)\|\boldsymbol{\omega}_0\|_2 + \mu_0\sum_{g=1}^{G}\frac{\phi_g(\mu_g)}{\mu_g}\|\boldsymbol{\delta}_g^{(t)}\|_2\right)
\end{aligned}
$$

By recursively applying the result of Lemma 4, we get the following deterministic bound which depends on constants defined in Definition 4:

$$b_{t+1} = \sum_{g=0}^{G} \sqrt{\frac{n_g}{n}} \|\boldsymbol{\delta}_g^{(t+1)}\|_2 \quad \leq \quad \left(\rho_0 + \sum_{g=1}^{G} \sqrt{\frac{n_g}{n}} \phi_g\right) \|\boldsymbol{\delta}_0^{(t)}\|_2 + \sum_{g=1}^{G} \left(\sqrt{\frac{n_g}{n}} \rho_g + \mu_0 \frac{\phi_g}{\mu_g}\right) \|\boldsymbol{\delta}_g^{(t)}\|_2 + \sum_{g=0}^{G} \sqrt{\frac{n_g}{n}} \xi_g \|\boldsymbol{\omega}_g\|_2$$

$$\leq \quad \rho \sum_{g=0}^{G} \sqrt{\frac{n_g}{n}} \|\boldsymbol{\delta}_g^{(t)}\|_2 + \sum_{g=0}^{G} \sqrt{\frac{n_g}{n}} \xi_g \|\boldsymbol{\omega}_g\|_2 \tag{23}$$

where $\rho = \max\left(\rho_0 + \sum_{g=1}^{G} \sqrt{\frac{n_g}{n}} \phi_g, \max_{g\in[G]} \left[\rho_g + \sqrt{\frac{n}{n_g}} \frac{\mu_0}{\mu_g} \phi_g\right]\right)$. We have:

$$b_{t+1} \quad \leq \quad \rho b_t + \sum_{g=0}^{G} \sqrt{\frac{n_g}{n}} \xi_g \|\boldsymbol{\omega}_g\|_2$$

$$\leq \quad (\rho)^2 b_{t-1} + (\rho+1) \sum_{g=0}^{G} \sqrt{\frac{n_g}{n}} \xi_g \|\boldsymbol{\omega}_g\|_2$$

$$\leq \quad (\rho)^t b_1 + \left(\sum_{i=0}^{t-1} (\rho)^i\right) \sum_{g=0}^{G} \sqrt{\frac{n_g}{n}} \xi_g \|\boldsymbol{\omega}_g\|_2$$

$$= \quad (\rho)^t \sum_{g=0}^{G} \sqrt{\frac{n_g}{n}} \|\boldsymbol{\beta}_g^1 - \boldsymbol{\beta}_g^*\|_2 + \left(\sum_{i=0}^{t-1} (\rho)^i\right) \sum_{g=0}^{G} \sqrt{\frac{n_g}{n}} \xi_g \|\boldsymbol{\omega}_g\|_2$$

$$(\boldsymbol{\beta}^1 = 0) \quad \leq \quad (\rho)^t \sum_{g=0}^{G} \sqrt{\frac{n_g}{n}} \|\boldsymbol{\beta}_g^*\|_2 + \frac{1 - (\rho)^t}{1 - \rho} \sum_{g=0}^{G} \sqrt{\frac{n_g}{n}} \xi_g \|\boldsymbol{\omega}_g\|_2$$

■

A.6. Proof of Theorem 5

Proof. First we need following two lemmas which are proved separately in the following sections.

Lemma 5. *Consider $a_g \geq 1$, with probability at least $1 - 6\exp\left(-\gamma_g(\omega(\mathcal{A}_g) + \tau)^2\right)$ the following upper bound holds:*

$$\rho_g\left(\frac{1}{a_g n_g}\right) \leq \frac{1}{2}\left[\left(1 - \frac{1}{a_g}\right) + \sqrt{2} c_g \frac{2\omega(\mathcal{A}_g) + \tau}{a_g \sqrt{n_g}}\right] \tag{24}$$

Lemma 6. *Consider $a_g \geq 1$, with probability at least $1 - 4\exp\left(-\gamma_g(\omega(\mathcal{A}_g) + \tau)^2\right)$ the following upper bound holds:*

$$\phi_g\left(\frac{1}{a_g n_g}\right) \leq \frac{1}{a_g}\left(1 + c_{0g} \frac{\omega(\mathcal{A}_g) + \omega(\mathcal{A}_0) + 2\tau}{\sqrt{n_g}}\right) \tag{25}$$

Note that Lemma 3 readily provides a high probability upper bound for $\eta_g(1/(a_g n_g))$ as $\sqrt{(2K^2 + 1)}\left(\zeta_g k\omega(\mathcal{A}_g) + \epsilon_g\sqrt{\log G} + \tau\right)/(a_g\sqrt{n_g})$.

Starting from the deterministic form of the bound in Theorem 4 and putting in the step sizes as $\mu_g = \frac{1}{n_g a_g}$:

$$\sum_{g=0}^{G} \sqrt{\frac{n_g}{n}} \|\boldsymbol{\delta}_g^{(t+1)}\|_2 \leq (\rho)^t \sum_{g=0}^{G} \|\boldsymbol{\beta}_g^*\|_2 + \frac{1 - (\rho)^t}{1 - \rho} \sum_{g=0}^{G} \sqrt{\frac{n_g}{n}} \eta_g\left(\frac{1}{n_g a_g}\right) \|\boldsymbol{\omega}_g\|_2, \tag{26}$$

where

$$\rho(a_0, \cdots, a_G) = \max\left(\rho_0\left(\frac{1}{na_0}\right) + \sum_{g=1}^{G} \sqrt{\frac{n_g}{n}} \phi_g\left(\frac{1}{n_g a_g}\right), \max_{g\in[G]} \rho_g\left(\frac{1}{n_g a_g}\right) + \sqrt{\frac{n}{n_g}} \frac{\mu_0}{\mu_g} \phi_g\left(\frac{1}{n_g a_g}\right)\right) \tag{27}$$

Remember the following two results to upper bound ρ_gs and ϕ_gs from Lemmas 5 and 6:

$$\rho_g\left(\frac{1}{a_g n_g}\right) \leq \frac{1}{2}\left[\left(1-\frac{1}{a_g}\right)+\sqrt{2}c_g\frac{2\omega(\mathcal{A}_g)+\tau}{a_g\sqrt{n_g}}\right], \quad \text{w.p.} \quad 1-6\exp\left(-\gamma_g(\omega(\mathcal{A}_g)+\tau)^2\right)$$

$$\phi_g\left(\frac{1}{a_g n_g}\right) \leq \frac{1}{a_g}\left(1+c_{0g}\frac{\omega_{0g}+\tau}{\sqrt{n_g}}\right), \quad \text{w.p.} \quad 1-4\exp\left(-\gamma_g(\omega(\mathcal{A}_g)+\tau)^2\right)$$

First we want to keep $\rho_0 + \sum_{g=1}^{G}\sqrt{\frac{n_g}{n}}\phi_g$ of (27) strictly below 1.

$$\rho_0\left(\frac{1}{a_0 n}\right)+\sum_{g=1}^{G}\sqrt{\frac{n_g}{n}}\phi_g\left(\frac{1}{a_g n_g}\right) \leq \frac{1}{2}\left[\left(1-\frac{1}{a_0}\right)+\sqrt{2}c_0\frac{2\omega_0+\tau}{a_0\sqrt{n}}\right]$$

$$+ \frac{1}{2}\sum_{g=1}^{G}\frac{2}{a_g}\sqrt{\frac{n_g}{n}}\left(1+c_{0g}\frac{\omega_{0g}+\tau}{\sqrt{n_g}}\right)$$

Remember that $a_g \geq 1$ was arbitrary. So we pick it as $a_g = 2\sqrt{\frac{n}{n_g}}\left(1+c_{0g}\frac{\omega_{0g}+\tau}{\sqrt{n_g}}\right)/b_g$ where $b_g \leq 2\sqrt{\frac{n}{n_g}}\left(1+c_{0g}\frac{\omega_{0g}+\tau}{\sqrt{n_g}}\right)$ (because we need $a_g \geq 1$) and the condition becomes:

$$\rho_0\left(\frac{1}{a_0 n}\right)+\sum_{g=1}^{G}\sqrt{\frac{n_g}{n}}\phi_g\left(\frac{1}{a_g n_g}\right) \leq \frac{1}{2}\left[\left(1-\frac{1}{a_0}\right)+\sqrt{2}c_0\frac{2\omega(\mathcal{A}_0)+\tau}{a_0\sqrt{n}}\right]+\frac{1}{2}\sum_{g=1}^{G}\frac{n_g}{n}b_g \leq 1$$

We want to upper bound the RHS by $1/\theta_f$ which will determine the sample complexity for the shared component:

$$\sqrt{2}c_0\frac{2\omega(\mathcal{A}_0)+\tau}{\sqrt{n}} \leq a_0\left(1-\sum_{g=1}^{G}\frac{n_g}{n}b_g\right)+1 \tag{28}$$

Note that any lower bound on the RHS of (28) will lead to the correct sample complexity for which the coefficient of $\|\boldsymbol{\delta}_0^{(t)}\|_2$ (determined in (27)) will be below one. Since $a_0 \geq 1$ we can ignore the first term by assuming $\max_{g \in [G]_\backslash} b_g \leq 1$ and the condition becomes:

$$n > 2c_0^2(2\omega(\mathcal{A}_0)+\tau)^2, \forall g \in [G]_\backslash : a_g = 2b_g^{-1}\sqrt{\frac{n}{n_g}}\left(1+c_{0g}\frac{\omega_{0g}+\tau}{\sqrt{n_g}}\right),$$

$$a_0 \geq 1, 0 < b_g \leq 2\sqrt{\frac{n}{n_g}}\left(1+c_{0g}\frac{\omega_{0g}+\tau}{\sqrt{n_g}}\right), \max_{g \in [G]_\backslash} b_g \leq 1,$$

which can be simplified to:

$$n > 2c_0^2(2\omega(\mathcal{A}_0)+\tau)^2, a_0 \geq 1, \tag{29}$$

$$\forall g \in [G]_\backslash : a_g = 2b_g^{-1}\sqrt{\frac{n}{n_g}}\left(1+c_{0g}\frac{\omega_{0g}+\tau}{\sqrt{n_g}}\right), 0 < b_g \leq 1$$

Secondly, we want to bound all of $\rho_g + \mu_0\sqrt{\frac{n}{n_g}}\frac{\phi_g}{\mu_g}$ terms of (27) for $\mu_g = \frac{1}{a_g n_g}$ by 1:

$$\rho_g\left(\frac{1}{n_g a_g}\right)+\sqrt{\frac{n}{n_g}}\frac{\mu_0}{\mu_g}\phi_g\left(\frac{1}{n_g a_g}\right) = \rho_g\left(\frac{1}{n_g a_g}\right)+\sqrt{\frac{n_g}{n}}\frac{a_g}{a_0}\phi_g\left(\frac{1}{n_g a_g}\right) \tag{30}$$

$$= \frac{1}{2}\left[\left[\left(1-\frac{1}{a_g}\right)+\sqrt{2}c_g\frac{2\omega_g+\tau}{a_g\sqrt{n_g}}\right]\right.$$

$$+ \left.\frac{2}{a_0}\sqrt{\frac{n_g}{n}}\left(1+c_{0g}\frac{\omega_{0g}+\tau}{\sqrt{n_g}}\right)\right]$$

$$\leq 1$$

The condition becomes:

$$\sqrt{2}c_g \frac{2\omega_g + \tau}{\sqrt{n_g}} \le a_g + 1 - \sqrt{\frac{n_g}{n}} \frac{2a_g}{a_0} \left(1 + c_{0g} \frac{\omega_{0g} + \tau}{\sqrt{n_g}}\right) \tag{31}$$

Remember that we chose $a_g = 2b_g^{-1} \sqrt{\frac{n}{n_g}} \left(1 + c_{0g} \frac{\omega_{0g} + \tau}{\sqrt{n_g}}\right)$. We substitute the value of a_g by keeping in mind the constraints for the b_g and the condition reduces to:

$$\sqrt{2}c_g \frac{2\omega_g + \tau}{d_g} \le \sqrt{n_g}, \quad d_g := a_g + 1 - \frac{4}{b_g a_0} \left(1 + c_{0g} \frac{\omega_{0g} + \tau}{\sqrt{n_g}}\right)^2 \tag{32}$$

for $d_g > 0$. Note that any positive lower bound of the d_g will satisfy the condition in (32) and the result is a valid sample complexity. In the following we show that $d_g > 1$. We have $a_0 \ge 1$ condition from (29), so we take $a_0 = 4\max_{g \in [G]_\backslash} \left(1 + c_{0g} \frac{\omega_{0g} + \tau}{\sqrt{n_g}}\right)^2$ and look for a lower bound for d_g:

$$\begin{aligned}
d_g &\ge a_g + 1 - b_g^{-1} \tag{33} \\
(a_g \text{ from (29)}) &= 2b_g^{-1} \sqrt{\frac{n}{n_g}} \left(1 + c_{0g} \frac{\omega_{0g} + \tau}{\sqrt{n_g}}\right) + 1 - b_g^{-1} \\
&= 1 + b_g^{-1} \left[2\sqrt{\frac{n}{n_g}} \left(1 + c_{0g} \frac{\omega_{0g} + \tau}{\sqrt{n_g}}\right) - 1\right] \tag{34}
\end{aligned}$$

The term inside of the last bracket (34) is always positive and therefore a lower bound is one, i.e., $d_g \ge 1$. From the condition (32) we get the following sample complexity:

$$n_g > 2c_g^2 (2\omega_g + \tau)^2 \tag{35}$$

Now we need to determine b_g from previous conditions (29), knowing that $a_0 = 4\max_{g \in [G]_\backslash} \left(1 + c_{0g} \frac{\omega_{0g} + \tau}{\sqrt{n_g}}\right)^2$. We have $0 < b_g \le 1$ in (29) and we take the largest step by setting $b_g = 1$.

Here we summarize the setting under which we have the linear convergence:

$$\begin{aligned}
&n > 2c_0^2 \left(2\omega(\mathcal{A}_0) + \tau\right)^2, \forall g \in [G]_\backslash : n_g \ge 2c_g^2 (2\omega(\mathcal{A}_g) + \tau)^2 \\
&a_0 = 4 \max_{g \in [G]_\backslash} \left(1 + c_{0g} \frac{\omega_{0g} + \tau}{\sqrt{n_g}}\right)^2, a_g = 2\sqrt{\frac{n}{n_g}} \left(1 + c_{0g} \frac{\omega_{0g} + \tau}{\sqrt{n_g}}\right) \\
&\mu_0 = \frac{1}{4n} \times \frac{1}{\max_{g \in [G]_\backslash} \left(1 + c_{0g} \frac{\omega_{0g} + \tau}{\sqrt{n_g}}\right)^2}, \mu_g = \frac{1}{2\sqrt{nn_g}} \left(1 + c_{0g} \frac{\omega_{0g} + \tau}{\sqrt{n_g}}\right)^{-1}
\end{aligned} \tag{36}$$

Now we rewrite the same analysis using the tail bounds for the coefficients to clarify the probabilities. To simplify the notation, let $r_{g1} = \frac{1}{2}\left[\left(1 - \frac{1}{a_g}\right) + \sqrt{2}c_g \frac{2\omega(\mathcal{A}_g) + \tau}{a_g \sqrt{n_g}}\right]$ and $r_{g2} = \frac{1}{a_g}\left(1 + c_{0g} \frac{\omega_{0g} + \tau}{\sqrt{n_g}}\right)$ and $r_0(\tau) = r_{01} + \sum_{g=1}^{G} \sqrt{\frac{n_g}{n}} r_{g2}$ and $r_g(\tau) = r_{g1} + \sqrt{\frac{n_g}{n} \frac{a_g}{a_0}} r_{g2}, \forall g \in [G]_\backslash$, and $r(\tau) = \max_{g \in [G]} r_g$. All of which are computed using a_gs specified in (36). Basically r is an instantiation of an upper bound of the ρ defined in (27) using a_gs in (36).

We are interested to upper bound the following probability:

$$\mathbb{P}\left(\sum_{g=0}^{G}\sqrt{\frac{n_g}{n}}\|\boldsymbol{\delta}_g^{(t+1)}\|_2 \geq r(\tau)^t \sum_{g=0}^{G}\sqrt{\frac{n_g}{n}}\|\boldsymbol{\beta}_g^*\|_2 + \frac{(G+1)\sqrt{(2K^2+1)}}{(1-r(\tau))\sqrt{n}}\left(\zeta k \max_{g\in[G]}\omega(\mathcal{A}_g)+\tau\right)\right)$$

$$\leq \mathbb{P}\left((\rho)^t\sum_{g=0}^{G}\sqrt{\frac{n_g}{n}}\|\boldsymbol{\beta}_g^*\|_2 + \frac{1-(\rho)^t}{1-\rho}\sum_{g=0}^{G}\sqrt{\frac{n_g}{n}}\eta_g\left(\frac{1}{n_g a_g}\right)\|\boldsymbol{\omega}_g\|_2\right.$$

$$\geq r(\tau)^t\sum_{g=0}^{G}\sqrt{\frac{n_g}{n}}\|\boldsymbol{\beta}_g^*\|_2 + \frac{(G+1)\sqrt{(2K^2+1)}}{(1-r(\tau))\sqrt{n}}\left.\left(\zeta k \max_{g\in[G]}\omega(\mathcal{A}_g)+\tau\right)\right)$$

$$\leq \mathbb{P}\left(\rho \geq r(\tau)\right)$$

$$+ \mathbb{P}\left(\frac{1}{1-\rho}\sum_{g=0}^{G}\sqrt{n_g}\eta_g\left(\frac{1}{n_g a_g}\right)\|\boldsymbol{\omega}_g\|_2 \geq \frac{(G+1)\sqrt{(2K^2+1)}}{(1-r(\tau))}\left(\zeta k \max_{g\in[G]}\omega(\mathcal{A}_g)+\tau\right)\right) \tag{37}$$

where the first inequality comes from the deterministic bound of (26), We first focus on bounding the first term $\mathbb{P}\left(\rho \geq r(\tau)\right)$:

$$\mathbb{P}\left(\rho \geq r(\tau)\right)$$

$$= \mathbb{P}\left(\max\left(\rho_0\left(\frac{1}{na_0}\right) + \sum_{g=1}^{G}\sqrt{\frac{n_g}{n}}\phi_g\left(\frac{1}{n_g a_g}\right), \max_{g\in[G]}\rho_g\left(\frac{1}{n_g a_g}\right) + \sqrt{\frac{n}{n_g}\frac{\mu_0}{\mu_g}}\phi_g\left(\frac{1}{n_g a_g}\right)\right) \geq \max_{g\in[G]}r(\tau)\right)$$

$$\leq \mathbb{P}\left(\rho_0\left(\frac{1}{na_0}\right) + \sum_{g=1}^{G}\sqrt{\frac{n_g}{n}}\phi_g\left(\frac{1}{n_g a_g}\right) \geq r_0\right) + \sum_{g=1}^{G}\mathbb{P}\left(\rho_g\left(\frac{1}{n_g a_g}\right) + \sqrt{\frac{n}{n_g}\frac{\mu_0}{\mu_g}}\phi_g\left(\frac{1}{n_g a_g}\right) \geq r_g\right)$$

$$\leq \mathbb{P}\left(\rho_0\left(\frac{1}{na_0}\right) \geq r_{01}\right) + \sum_{g=1}^{G}\mathbb{P}\left(\phi_g\left(\frac{1}{n_g a_g}\right) \geq r_{g2}\right) + \sum_{g=1}^{G}\left[\mathbb{P}\left(\rho_g\left(\frac{1}{n_g a_g}\right) \geq r_{g1}\right) + \mathbb{P}\left(\phi_g\left(\frac{1}{n_g a_g}\right) \geq r_{g2}\right)\right]$$

$$\leq \sum_{g=0}^{G}\mathbb{P}\left(\rho_g\left(\frac{1}{n_g a_g}\right) \geq r_{g1}\right) + 2\sum_{g=1}^{G}\mathbb{P}\left(\phi_g\left(\frac{1}{n_g a_g}\right) \geq r_{g2}\right)$$

$$\leq \sum_{g=0}^{G}6\exp\left(-\gamma_g(\omega(\mathcal{A}_g)+\tau)^2\right) + 2\sum_{g=1}^{G}4\exp\left(-\gamma_g(\omega(\mathcal{A}_g)+\tau)^2\right)$$

$$\leq 6(G+1)\exp\left(-\gamma\min_{g\in[G]}(\omega(\mathcal{A}_g)+\tau)^2\right) + 8G\exp\left(-\gamma\min_{g\in[G]\setminus}(\omega(\mathcal{A}_g)+\tau)^2\right)$$

$$\leq 14(G+1)\exp\left(-\gamma\min_{g\in[G]}(\omega(\mathcal{A}_g)+\tau)^2\right) \tag{38}$$

Now we focus on bounding the second term:

$$\mathbb{P}\left(\frac{1}{1-\rho}\sum_{g=0}^{G}\sqrt{n_g}\eta_g\left(\frac{1}{n_g a_g}\right)\|\boldsymbol{\omega}_g\|_2 \geq \frac{(G+1)\sqrt{(2K^2+1)}}{(1-r(\tau))}\left(\zeta k \max_{g\in[G]}\omega(\mathcal{A}_g)+\tau\right)\right)$$

$$\leq \mathbb{P}\left(\frac{1}{1-\rho}\sum_{g=0}^{G}\sqrt{n_g}\eta_g\left(\frac{1}{n_g a_g}\right)\|\boldsymbol{\omega}_g\|_2 \geq \frac{1}{(1-r(\tau))}\sum_{g=0}^{G}\sqrt{(2K^2+1)}\left(\zeta_g k\omega(\mathcal{A}_g)+\tau\right)\right)$$

$$\leq \mathbb{P}\left(\sum_{g=0}^{G}\sqrt{n_g}\eta_g\left(\frac{1}{n_g a_g}\right)\|\boldsymbol{\omega}_g\|_2 \geq \sum_{g=0}^{G}\sqrt{(2K^2+1)}\left(\zeta_g k\omega(\mathcal{A}_g)+\tau\right)\right) + \mathbb{P}\left(\rho \geq r(\tau)\right)$$

$$\leq \sum_{g=0}^{G}\mathbb{P}\left(\sqrt{n_g}\eta_g\left(\frac{1}{n_g a_g}\right)\|\boldsymbol{\omega}_g\|_2 \geq \sqrt{(2K^2+1)}\left(\zeta_g k\omega(\mathcal{A}_g)+\tau\right)\right) + \mathbb{P}\left(\rho \geq r(\tau)\right) \tag{39}$$

Focusing on the summand of the first term, remember from Definition 4 that $\eta_g(\mu_g) = \frac{1}{a_g n_g}\sup_{\mathbf{v}\in\mathcal{B}_g}\mathbf{v}^T\mathbf{X}_g^T\frac{\boldsymbol{\omega}_g}{\|\boldsymbol{\omega}_g\|_2}$, $\quad g \in$

$[G]$ and $a_g \geq 1$:

$$\mathbb{P}\left(\|\boldsymbol{\omega}_g\|_2 \sup_{\mathbf{v}\in\mathcal{B}_g} \mathbf{v}^T\mathbf{X}_g^T \frac{\boldsymbol{\omega}_g}{\|\boldsymbol{\omega}_g\|_2} \geq a_g\sqrt{(2K^2+1)n_g}\left(\zeta_g k\omega(\mathcal{A}_g)+\tau\right)\right) \leq \sigma_g \exp\left(-\min\left[\nu_g n_g, \frac{\tau^2}{\eta_g^2 k^2}\right]\right) \quad (40)$$

where we used the intermediate form of Lemma 3 for $\tau > 0$. Putting all of the bounds (38), (39), and (40) back into the (37):

$$\sigma_g(G+1)\exp\left(-\min_{g\in[G]}\left(\min\left[\nu_g n_g, \frac{\tau^2}{\eta_g^2 k^2}\right]\right)\right) + 28(G+1)\exp\left(-\gamma \min_{g\in[G]}(\omega(\mathcal{A}_g)+\tau)^2\right)$$

$$\leq \upsilon\exp\left[\min_{g\in[G]}\left(-\min\left[\nu_g n_g - \log G, \gamma(\omega(\mathcal{A}_g)+t)^2, \frac{t^2}{\eta_g^2 k^2}\right]\right)\right]$$

where $\upsilon = \max(28, \sigma)$ and $\gamma = \min_{g\in[G]}\gamma_g$ and $\tau = t + \max(\epsilon, \gamma^{-1/2})\sqrt{\log(G+1)}$ where $\epsilon = k\max_{g\in[G]}\eta_g$. Note that $\tau = t + C\sqrt{\log(G+1)}$ increases the sample complexities to the followings:

$$n > 2c_0^2\left(2\omega(\mathcal{A}_0) + C\sqrt{\log(G+1)} + t\right)^2, \forall g \in [G]_\backslash : n_g \geq 2c_g^2(2\omega(\mathcal{A}_g) + C\sqrt{\log(G+1)} + t)^2$$

and it also affects step sizes as follows:

$$\mu_0 = \frac{1}{4n} \times \min_{g\in[G]_\backslash}\left(1 + c_{0g}\frac{\omega_{0g} + C\sqrt{\log(G+1)} + t}{\sqrt{n_g}}\right)^{-2}, \mu_g = \frac{1}{2\sqrt{nn_g}}\left(1 + c_{0g}\frac{\omega_{0g} + C\sqrt{\log(G+1)} + t}{\sqrt{n_g}}\right)^{-1}$$

∎

B. Proofs of Lemmas

Here, we present proofs of each lemma used during the proofs of theorems in Section A.

B.1. Proof of Lemma 1

Proof. LHS of (19) is the weighted summation of $\xi_g Q_{2\xi_g}(\boldsymbol{\delta}_{0g}) = \|\boldsymbol{\delta}_{0g}\|_2 \xi \mathbb{P}(|\langle\mathbf{x}, ,\boldsymbol{\delta}_{0g}/\|\boldsymbol{\delta}_{0g}\|_2\rangle| > 2\xi) = \|\boldsymbol{\delta}_{0g}\|_2 \xi Q_{2\xi}(\mathbf{u})$ where $\xi > 0$ and $\mathbf{u} = \boldsymbol{\delta}_{0g}/\|\boldsymbol{\delta}_{0g}\|_2$ is a unit length vector. So we can rewrite the LHS of (19) as:

$$\sum_{g=1}^{G}\frac{n_g}{n}\xi_g Q_{2\xi_g}(\boldsymbol{\delta}_{0g}) = \sum_{g=1}^{G}\frac{n_g}{n}\|\boldsymbol{\delta}_0 + \boldsymbol{\delta}_g\|_2 \xi Q_{2\xi}(\mathbf{u})$$

With this observation, the lower bound of the Lemma 1 is a direct consequence of the following two results:

Lemma 7. *Let $\mathbf{u}$ be any unit length vector and suppose $\mathbf{x}$ obeys Definiton 1. Then for any $\mathbf{u}$, we have*

$$Q_{2\xi}(\mathbf{u}) \geq \frac{(\alpha - 2\xi)^2}{4ck^2}. \quad (41)$$

Lemma 8. *Suppose Definition 3 holds. Then, we have:*

$$\sum_{i=1}^{G} n_i\|\boldsymbol{\delta}_0 + \boldsymbol{\delta}_i\|_2 \geq \frac{\bar{\rho}\lambda_{\min}}{3}\left(Gn\|\boldsymbol{\delta}_0\|_2 + \sum_{i=1}^{G} n_i\|\boldsymbol{\delta}_i\|_2\right), \quad \forall i \in [G] : \boldsymbol{\delta}_i \in \mathcal{C}_i. \quad (42)$$

∎

B.2. Proof of Lemma 2

Proof. Consider the following soft indicator function which we use in our derivation:

$$\psi_a(s) = \begin{cases} 0, & |s| \leq a \\ (|s| - a)/a, & a \leq |s| \leq 2a \\ 1, & 2a < |s| \end{cases}$$

Now:

$$\mathbb{E} \sup_{\boldsymbol{\delta}_{[G]}} \sum_{g=1}^{G} \xi_g \sum_{i=1}^{n_g} \left[Q_{2\xi_g}(\boldsymbol{\delta}_{0g}) - \mathbb{1}(|\langle \mathbf{x}_{gi}, \boldsymbol{\delta}_{0g}\rangle| \geq \xi_g) \right]$$

$$= \mathbb{E} \sup_{\boldsymbol{\delta}_{[G]}} \sum_{g=1}^{G} \xi_g \sum_{i=1}^{n_g} \left[\mathbb{E} \mathbb{1}(|\langle \mathbf{x}_{gi}, \boldsymbol{\delta}_{0g}\rangle| \geq 2\xi_g) - \mathbb{1}(|\langle \mathbf{x}_{gi}, \boldsymbol{\delta}_{0g}\rangle| \geq \xi_g) \right]$$

$$\leq \mathbb{E} \sup_{\boldsymbol{\delta}_{[G]}} \sum_{g=1}^{G} \xi_g \sum_{i=1}^{n_g} \left[\mathbb{E} \psi_{\xi_g}(\langle \mathbf{x}, \boldsymbol{\delta}_{0g}\rangle) - \psi_{\xi_g}(\langle \mathbf{x}_{gi}, \boldsymbol{\delta}_{0g}\rangle) \right]$$

$$\leq 2\mathbb{E} \sup_{\boldsymbol{\delta}_{[G]}} \sum_{g=1}^{G} \xi_g \sum_{i=1}^{n_g} \epsilon_{gi} \psi_{\xi_g}(\langle \mathbf{x}_{gi}, \boldsymbol{\delta}_{0g}\rangle)$$

$$\leq 2\mathbb{E} \sup_{\boldsymbol{\delta}_{[G]}} \sum_{g=1}^{G} \sum_{i=1}^{n_g} \epsilon_{gi} \langle \mathbf{x}_{gi}, \boldsymbol{\delta}_{0g}\rangle$$

where ϵ_{gi} are iid copies of Rademacher random variable which are independent of every other random variables and themselves. Now we add back $\frac{1}{n}$ and expand $\boldsymbol{\delta}_{0g} = \boldsymbol{\delta}_0 + \boldsymbol{\delta}_g$:

$$\frac{2}{n}\mathbb{E} \sup_{\boldsymbol{\delta}_{[G]} \in \mathcal{C}_{[G]}} \sum_{g=1}^{G} \sum_{i=1}^{n_g} \epsilon_{gi} \langle \mathbf{x}_{gi}, \boldsymbol{\delta}_{0g}\rangle = \frac{2}{n}\mathbb{E} \sup_{\boldsymbol{\delta}_0 \in \mathcal{C}_0} \sum_{i=1}^{n} \epsilon_i \langle \mathbf{x}_i, \boldsymbol{\delta}_0\rangle + \frac{2}{n}\mathbb{E} \sup_{\boldsymbol{\delta}_{[G]} \setminus \in \mathcal{C}_{[G]} \setminus} \sum_{g=1}^{G} \sum_{i=1}^{n_g} \epsilon_{gi} \langle \mathbf{x}_{gi}, \boldsymbol{\delta}_g\rangle$$

$$= \frac{2}{\sqrt{n}}\mathbb{E} \sup_{\boldsymbol{\delta}_0 \in \mathcal{C}_0} \sum_{i=1}^{n} \langle \frac{1}{\sqrt{n}}\epsilon_i \mathbf{x}_i, \boldsymbol{\delta}_0\rangle + \frac{2}{\sqrt{n}}\mathbb{E} \sup_{\boldsymbol{\delta}_{[G]} \setminus \in \mathcal{C}_{[G]} \setminus} \sum_{g=1}^{G} \sqrt{\frac{n_g}{n}} \sum_{i=1}^{n_g} \langle \frac{1}{\sqrt{n_g}}\epsilon_{gi} \mathbf{x}_{gi}, \boldsymbol{\delta}_g\rangle$$

$$(n_0 := n, \epsilon_{0i} := \epsilon_0, \mathbf{x}_{0i} := \mathbf{x}_i) = \frac{2}{\sqrt{n}}\mathbb{E} \sup_{\boldsymbol{\delta}_{[G]} \in \mathcal{C}_{[G]}} \sum_{g=0}^{G} \sqrt{\frac{n_g}{n}} \sum_{i=1}^{n_g} \langle \frac{1}{\sqrt{n_g}}\epsilon_{gi} \mathbf{x}_{gi}, \boldsymbol{\delta}_g\rangle$$

$$\left(\mathbf{h}_g := \frac{1}{\sqrt{n_g}} \sum_{i=1}^{n_g} \epsilon_{gi} \mathbf{x}_{gi}\right) = \frac{2}{\sqrt{n}}\mathbb{E} \sup_{\boldsymbol{\delta}_{[G]} \in \mathcal{C}_{[G]}} \sum_{g=0}^{G} \sqrt{\frac{n_g}{n}} \langle \mathbf{h}_g, \boldsymbol{\delta}_g\rangle$$

$$(\mathcal{A}_g \in \mathcal{C}_g \cap \mathbb{S}^{p-1}) \leq \frac{2}{\sqrt{n}}\mathbb{E} \sup_{\boldsymbol{\delta}_{[G]} \in \mathcal{A}_{[G]}} \sum_{g=0}^{G} \sqrt{\frac{n_g}{n}} \langle \mathbf{h}_g, \boldsymbol{\delta}_g\rangle \|\boldsymbol{\delta}_g\|_2$$

$$\leq \frac{2}{\sqrt{n}} \sum_{g=0}^{G} \sqrt{\frac{n_g}{n}} \mathbb{E}_{\mathbf{h}_g} \sup_{\boldsymbol{\delta}_g \in \mathcal{A}_g} \langle \mathbf{h}_g, \boldsymbol{\delta}_g\rangle \|\boldsymbol{\delta}_g\|_2$$

$$\leq \frac{2}{\sqrt{n}} \sum_{g=0}^{G} \sqrt{\frac{n_g}{n}} c_g k \omega(\mathcal{A}_g) \|\boldsymbol{\delta}_g\|_2$$

Note that the $\mathbf{h}_{gi}$ is a sub-Gaussian random vector which let us bound the $\mathbb{E} \sup$ using the Gaussian width (Tropp, 2015) in the last step. ∎

B.3. Proof of Lemma 3

Proof. To avoid cluttering let $h_g(\boldsymbol{\omega}_g, \mathbf{X}_g) = \sqrt{\frac{n}{n_g}} \|\boldsymbol{\omega}_g\|_2 \sup_{\mathbf{u}_g \in \mathcal{A}_g} \langle \mathbf{X}_g^T \frac{\boldsymbol{\omega}_g}{\|\boldsymbol{\omega}_g\|_2}, \mathbf{u}_g \rangle$, $e_g = \zeta_g k \omega(\mathcal{A}_g) + \epsilon_g \sqrt{\log G} + \tau$, where $s_g = \sqrt{\frac{n}{n_g}} \sqrt{(2K^2+1)n_g}$.

$$
\begin{aligned}
\mathbb{P}\left(h_g(\boldsymbol{\omega}_g, \mathbf{X}_g) > e_g s_g\right) &= \mathbb{P}\left(h_g(\boldsymbol{\omega}_g, \mathbf{X}_g) > e_g s_g \Big| \sqrt{\frac{n}{n_g}}\|\boldsymbol{\omega}_g\|_2 > s_g\right) \mathbb{P}\left(\sqrt{\frac{n}{n_g}}\|\boldsymbol{\omega}_g\|_2 > s_g\right) \qquad (43) \\
&\quad + \mathbb{P}\left(h_g(\boldsymbol{\omega}_g, \mathbf{X}_g) > e_g s_g \Big| \sqrt{\frac{n}{n_g}}\|\boldsymbol{\omega}_g\|_2 < s_g\right) \mathbb{P}\left(\sqrt{\frac{n}{n_g}}\|\boldsymbol{\omega}_g\|_2 < s_g\right) \\
&\leq \mathbb{P}\left(\sqrt{\frac{n}{n_g}}\|\boldsymbol{\omega}_g\|_2 > s_g\right) + \mathbb{P}\left(h_g(\boldsymbol{\omega}_g, \mathbf{X}_g) > e_g s_g \Big| \sqrt{\frac{n}{n_g}}\|\boldsymbol{\omega}_g\|_2 < s_g\right) \\
&\leq \mathbb{P}\left(\|\boldsymbol{\omega}_g\|_2 > \sqrt{(2K^2+1)n_g}\right) + \mathbb{P}\left(\sup_{\mathbf{u}_g \in \mathcal{C}_g \cap \mathbb{S}^{p-1}} \langle \mathbf{X}_g^T \frac{\boldsymbol{\omega}_g}{\|\boldsymbol{\omega}_g\|_2}, \mathbf{u}_g \rangle > e_g\right) \\
&\leq \mathbb{P}\left(\|\boldsymbol{\omega}_g\|_2 > \sqrt{(2K^2+1)n_g}\right) + \sup_{\mathbf{v} \in \mathbb{S}^{p-1}} \mathbb{P}\left(\sup_{\mathbf{u}_g \in \mathcal{C}_g \cap \mathbb{S}^{p-1}} \langle \mathbf{X}_g^T \mathbf{v}, \mathbf{u}_g \rangle > e_g\right)
\end{aligned}
$$

Let's focus on the first term. Since $\boldsymbol{\omega}_g$ consists of i.i.d. centered unit-variance sub-Gaussian elements with $\|\omega_{gi}\|_{\psi_2} < K$, ω_{gi}^2 is sub-exponential with $\|\omega_{gi}\|_{\psi_1} < 2K^2$. Let's apply the Bernstein's inequality to $\|\boldsymbol{\omega}_g\|_2^2 = \sum_{i=1}^{n_g} \omega_{gi}^2$:

$$
\mathbb{P}\left(\left|\|\boldsymbol{\omega}_g\|_2^2 - \mathbb{E}\|\boldsymbol{\omega}_g\|_2^2\right| > \tau\right) \leq 2\exp\left(-\nu_g \min\left[\frac{\tau^2}{4K^4 n_g}, \frac{\tau}{2K^2}\right]\right)
$$

We also know that $\mathbb{E}\|\omega_g\|_2^2 \leq n_g$ (Banerjee et al., 2014) which gives us:

$$
\mathbb{P}\left(\|\boldsymbol{\omega}_g\|_2 > \sqrt{n_g + \tau}\right) \leq 2\exp\left(-\nu_g \min\left[\frac{\tau^2}{4K^4 n_g}, \frac{\tau}{2K^2}\right]\right)
$$

Finally, we set $\tau = 2K^2 n_g$:

$$
\mathbb{P}\left(\|\boldsymbol{\omega}_g\|_2 > \sqrt{(2K^2+1)n_g}\right) \leq 2\exp\left(-\nu_g n_g\right) = \frac{2}{(G+1)}\exp\left(-\nu_g n_g + \log(G+1)\right)
$$

Now we upper bound the second term of (43). Given any fixed $\mathbf{v} \in \mathbb{S}^{p-1}$, $\mathbf{X}_g \mathbf{v}$ is a sub-Gaussian random vector with $\|\mathbf{X}_g^T \mathbf{v}\|_{\psi_2} \leq C_g k$ (Banerjee et al., 2014). From Theorem 9 of (Banerjee et al., 2014) for any $\mathbf{v} \in \mathbb{S}^{p-1}$ we have:

$$
\mathbb{P}\left(\sup_{\mathbf{u}_g \in \mathcal{A}_g} \langle \mathbf{X}_g^T \mathbf{v}, \mathbf{u}_g \rangle > \upsilon_g C_g k \omega(\mathcal{A}_g) + t\right) \leq \pi_g \exp\left(-\left(\frac{t}{\theta_g C_g k \phi_g}\right)^2\right)
$$

where $\phi_g = \sup_{\mathbf{u}_g \in \mathcal{A}_g} \|\mathbf{u}_g\|_2$ and in our problem $\phi_g = 1$. We now substitute $t = \tau + \epsilon_g \sqrt{\log(G+1)}$ where $\epsilon_g = \theta_g C_g k$.

$$
\begin{aligned}
\mathbb{P}\left(\sup_{\mathbf{u}_g \in \mathcal{A}_g} \langle \mathbf{X}_g^T \mathbf{v}, \mathbf{u}_g \rangle > \upsilon_g C_g k \omega(\mathcal{A}_g) + \epsilon_g \sqrt{\log(G+1)} + \tau\right) &\leq \pi_g \exp\left(-\left(\frac{\tau + \epsilon_g \sqrt{\log(G+1)}}{\epsilon_g}\right)^2\right) \\
&\leq \pi_g \exp\left(-\log G - \left(\frac{\tau}{\theta_g C_g k}\right)^2\right) \\
&\leq \frac{\pi_g}{(G+1)} \exp\left(-\left(\frac{\tau}{\theta_g C_g k}\right)^2\right)
\end{aligned}
$$

Now we put back results to the original inequality (43):

$$\mathbb{P}\left(h_g(\boldsymbol{\omega}_g, \mathbf{X}_g) > \sqrt{\frac{n}{n_g}}\sqrt{(2K^2+1)n_g} \times \left(v_g C_g k\omega(\mathcal{A}_g) + \epsilon_g\sqrt{\log(G+1)} + \tau\right)\right)$$

$$\leq \quad \frac{\sigma_g}{(G+1)}\exp\left(-\min\left[\nu_g n_g - \log(G+1), \frac{\tau^2}{\theta_g^2 C_g^2 k^2}\right]\right)$$

$$\leq \quad \frac{\sigma_g}{(G+1)}\exp\left(-\min\left[\nu_g n_g - \log(G+1), \frac{\tau^2}{\eta_g^2 k^2}\right]\right)$$

where $\sigma_g = \pi_g + 2$, $\zeta_g = v_g C_g$, $\eta_g = \theta_g C_g$. ∎

B.4. Proof of Lemma 4

Proof. We upper bound the individual error $\|\boldsymbol{\delta}_g^{(t+1)}\|_2$ and the common one $\|\boldsymbol{\delta}_0^{(t+1)}\|_2$ in the followings:

$$
\begin{aligned}
\|\boldsymbol{\delta}_g^{(t+1)}\|_2 &= \|\boldsymbol{\beta}_g^{(t+1)} - \boldsymbol{\beta}_g^*\|_2 \\
&= \left\|\Pi_{\Omega_{f_g}}\left(\boldsymbol{\beta}_g^{(t)} + \mu_g \mathbf{X}_g^T\left(\mathbf{y}_g - \mathbf{X}_g\left(\boldsymbol{\beta}_0^{(t)} + \boldsymbol{\beta}_g^{(t)}\right)\right)\right) - \boldsymbol{\beta}_g^*\right\|_2 \\
\text{(Lemma 6.3 of (Oymak et al., 2015))} \quad &= \left\|\Pi_{\Omega_{f_g} - \{\boldsymbol{\beta}_g^*\}}\left(\boldsymbol{\beta}_g^{(t)} + \mu_g \mathbf{X}_g^T\left(\mathbf{y}_g - \mathbf{X}_g\left(\boldsymbol{\beta}_0^{(t)} + \boldsymbol{\beta}_g^{(t)}\right)\right) - \boldsymbol{\beta}_g^*\right)\right\|_2 \\
&= \left\|\Pi_{\mathcal{E}_g}\left(\boldsymbol{\delta}_g^{(t)} + \mu_g \mathbf{X}_g^T\left(\mathbf{y}_g - \mathbf{X}_g\left(\boldsymbol{\beta}_0^{(t)} + \boldsymbol{\beta}_g^{(t)}\right) - \mathbf{X}_g\left(\boldsymbol{\beta}_0^* + \boldsymbol{\beta}_g^*\right) + \mathbf{X}_g\left(\boldsymbol{\beta}_0^* + \boldsymbol{\beta}_g^*\right)\right)\right)\right\|_2 \\
&= \left\|\Pi_{\mathcal{E}_g}\left(\boldsymbol{\delta}_g^{(t)} + \mu_g \mathbf{X}_g^T\left(\boldsymbol{\omega}_g - \mathbf{X}_g\left(\boldsymbol{\delta}_0^{(t)} + \boldsymbol{\delta}_g^{(t)}\right)\right)\right)\right\|_2 \\
\text{(Lemma 6.4 of (Oymak et al., 2015))} \quad &\leq \left\|\Pi_{\mathcal{C}_g}\left(\boldsymbol{\delta}_g^{(t)} + \mu_g \mathbf{X}_g^T\left(\boldsymbol{\omega}_g - \mathbf{X}_g\left(\boldsymbol{\delta}_0^{(t)} + \boldsymbol{\delta}_g^{(t)}\right)\right)\right)\right\|_2 \\
\text{(Lemma 6.2 of (Oymak et al., 2015))} \quad &\leq \sup_{\mathbf{v} \in \mathcal{C}_g \cap \mathbb{B}^p} \mathbf{v}^T\left(\boldsymbol{\delta}_g^{(t)} + \mu_g \mathbf{X}_g^T\left(\boldsymbol{\omega}_g - \mathbf{X}_g\left(\boldsymbol{\delta}_0^{(t)} + \boldsymbol{\delta}_g^{(t)}\right)\right)\right) \\
(\mathcal{B}_g = \mathcal{C}_g \cap \mathbb{B}^p) \quad &= \sup_{\mathbf{v} \in \mathcal{B}_g} \mathbf{v}^T\left(\boldsymbol{\delta}_g^{(t)} + \mu_g \mathbf{X}_g^T\left(\boldsymbol{\omega}_g - \mathbf{X}_g\left(\boldsymbol{\delta}_0^{(t)} + \boldsymbol{\delta}_g^{(t)}\right)\right)\right) \\
&\leq \sup_{\mathbf{v} \in \mathcal{B}_g} \mathbf{v}^T(\mathbf{I}_g - \mu_g \mathbf{X}_g^T \mathbf{X}_g)\boldsymbol{\delta}_g^{(t)} + \mu_g \sup_{\mathbf{v} \in \mathcal{B}_g} \mathbf{v}^T \mathbf{X}_g^T \boldsymbol{\omega}_g + \mu_g \sup_{\mathbf{v} \in \mathcal{B}_g} -\mathbf{v}^T \mathbf{X}_g^T \mathbf{X}_g \boldsymbol{\delta}_0^{(t)} \\
&\leq \left\|\boldsymbol{\delta}_g^{(t)}\right\|_2 \sup_{\mathbf{u},\mathbf{v} \in \mathcal{B}_g} \mathbf{v}^T(\mathbf{I}_g - \mu_g \mathbf{X}_g^T \mathbf{X}_g)\mathbf{u} + \mu_g \|\boldsymbol{\omega}_g\|_2 \sup_{\mathbf{v} \in \mathcal{B}_g} \mathbf{v}^T \mathbf{X}_g^T \frac{\boldsymbol{\omega}_g}{\|\boldsymbol{\omega}_g\|_2} \\
&\quad + \mu_g \|\boldsymbol{\delta}_0^{(t)}\|_2 \sup_{\mathbf{v} \in \mathcal{B}_g, \mathbf{u} \in \mathcal{B}_0} -\mathbf{v}^T \mathbf{X}_g^T \mathbf{X}_g \mathbf{u} \\
&= \rho_g(\mu_g)\|\boldsymbol{\delta}_g^{(t)}\|_2 + \xi_g(\mu_g)\|\boldsymbol{\omega}_g\|_2 + \phi_g(\mu_g)\|\boldsymbol{\delta}_0^{(t)}\|_2
\end{aligned}
$$

So the final bound becomes:

$$\|\boldsymbol{\delta}_g^{(t+1)}\|_2 \quad \leq \quad \rho_g(\mu_g)\|\boldsymbol{\delta}_g^{(t)}\|_2 + \xi_g(\mu_g)\|\boldsymbol{\omega}_g\|_2 + \phi_g(\mu_g)\|\boldsymbol{\delta}_0^{(t)}\|_2 \tag{44}$$

Now we upper bound the error of common parameter. Remember common parameter's update: $\boldsymbol{\beta}_0^{(t+1)} =$

$$\Pi_{\Omega_{f_0}}\left(\boldsymbol{\beta}_0^{(t)} + \mu_0\mathbf{X}_0^T\begin{pmatrix}(\mathbf{y}_1 - \mathbf{X}_1(\boldsymbol{\beta}_0^{(t)} + \boldsymbol{\beta}_1^{(t)}))\\ \vdots \\ (\mathbf{y}_G - \mathbf{X}_G(\boldsymbol{\beta}_0^{(t)} + \boldsymbol{\beta}_G^{(t)}))\end{pmatrix}\right).$$

$$\|\boldsymbol{\delta}_0^{(t+1)}\|_2 = \|\boldsymbol{\beta}_0^{(t+1)} - \boldsymbol{\beta}_0^*\|_2$$

$$= \left\|\Pi_{\Omega_{f_0}}\left(\boldsymbol{\beta}_0^{(t)} + \mu_0\sum_{g=1}^G\mathbf{X}_g^T\left(\mathbf{y}_g - \mathbf{X}_g(\boldsymbol{\beta}_0^{(t)} + \boldsymbol{\beta}_g^{(t)})\right)\right) - \boldsymbol{\beta}_0^*\right\|_2$$

(Lemma 6.3 of (Oymak et al., 2015)) $= \left\|\Pi_{\Omega_{f_0}-\{\boldsymbol{\beta}_0^*\}}\left(\boldsymbol{\beta}_0^{(t)} + \mu_0\sum_{g=1}^G\mathbf{X}_g^T\left(\mathbf{y}_g - \mathbf{X}_g(\boldsymbol{\beta}_0^{(t)} + \boldsymbol{\beta}_g^{(t)})\right) - \boldsymbol{\beta}_0^*\right)\right\|_2$

$$= \left\|\Pi_{\mathcal{E}_0}\left(\boldsymbol{\delta}_0^{(t)} + \mu_0\sum_{g=1}^G\mathbf{X}_g^T\left(\mathbf{y}_g - \mathbf{X}_g(\boldsymbol{\beta}_0^{(t)} + \boldsymbol{\beta}_g^{(t)})\right)\right)\right\|_2$$

(Lemma 6.4 of (Oymak et al., 2015)) $\leq \left\|\Pi_{\mathcal{C}_0}\left(\boldsymbol{\delta}_0^{(t)} + \mu_0\sum_{g=1}^G\mathbf{X}_g^T\left(\boldsymbol{\omega}_g - \mathbf{X}_g(\boldsymbol{\delta}_0^{(t)} + \boldsymbol{\delta}_g^{(t)})\right)\right)\right\|_2$

(Lemma 6.2 of (Oymak et al., 2015)) $\leq \sup_{\mathbf{v}\in\mathcal{B}_0}\mathbf{v}^T\left(\boldsymbol{\delta}_0^{(t)} + \mu_0\sum_{g=1}^G\mathbf{X}_g^T\left(\boldsymbol{\omega}_g - \mathbf{X}_g(\boldsymbol{\delta}_0^{(t)} + \boldsymbol{\delta}_g^{(t)})\right)\right)$

$$\leq \sup_{\mathbf{v}\in\mathcal{B}_0}\mathbf{v}^T\left(\mathbf{I} - \mu_0\sum_{g=1}^G\mathbf{X}_g^T\mathbf{X}_g\right)\boldsymbol{\delta}_0^{(t)} + \mu_0\sup_{\mathbf{v}\in\mathcal{B}_0}\mathbf{v}^T\sum_{g=1}^G\mathbf{X}_g^T\boldsymbol{\omega}_g$$

$$+ \mu_0\sup_{\mathbf{v}\in\mathcal{B}_0}-\mathbf{v}^T\sum_{g=1}^G\mathbf{X}_g^T\mathbf{X}_g\boldsymbol{\delta}_g^{(t)}$$

$$\leq \|\boldsymbol{\delta}_0^{(t)}\|_2\sup_{\mathbf{u},\mathbf{v}\in\mathcal{B}_0}\mathbf{v}^T\left(\mathbf{I} - \mu_0\mathbf{X}_0^T\mathbf{X}_0\right)\mathbf{u} + \mu_0\sup_{\mathbf{v}\in\mathcal{B}_0}\mathbf{v}^T\mathbf{X}_0^T\frac{\boldsymbol{\omega}_0}{\|\boldsymbol{\omega}_0\|_2}\|\boldsymbol{\omega}_0\|_2$$

$$+ \mu_0\sum_{g=1}^G\sup_{\mathbf{v}_g\in\mathcal{B}_0,\mathbf{u}_g\in\mathcal{B}_g}-\mathbf{v}_g^T\mathbf{X}_g^T\mathbf{X}_g\mathbf{u}_g\|\boldsymbol{\delta}_g^{(t)}\|_2$$

$$\leq \rho_0(\mu_0)\|\boldsymbol{\delta}_0^{(t)}\|_2 + \xi_0(\mu_0)\|\boldsymbol{\omega}_0\|_2 + \mu_0\sum_{g=1}^G\frac{\phi_g(\mu_g)}{\mu_g}\|\boldsymbol{\delta}_g^{(t)}\|_2 \qquad (45)$$

To avoid cluttering we drop μ_g as the arguments. Putting together (44) and (45) inequalities we reach to the followings:

$$\|\boldsymbol{\delta}_g^{(t+1)}\|_2 \leq \rho_g\|\boldsymbol{\delta}_g^{(t)}\|_2 + \xi_g\|\boldsymbol{\omega}_g\|_2 + \phi_g\|\boldsymbol{\delta}_0^{(t)}\|_2$$

$$\|\boldsymbol{\delta}_0^{(t+1)}\|_2 \leq \rho_0\|\boldsymbol{\delta}_0^{(t)}\|_2 + \xi_0\|\boldsymbol{\omega}_0\|_2 + \mu_0\sum_{g=1}^G\frac{\phi_g}{\mu_g}\|\boldsymbol{\delta}_g^{(t)}\|_2$$

$\blacksquare$

B.5. Proof of Lemma 5

We will need the following lemma in our proof. It establishes the RE condition for individual isotropic sub-Gaussian designs and provides us with the essential tool for proving high probability bounds.

Lemma 9 (Theorem 11 of (Banerjee et al., 2014)). *For all $g \in [G]$, for the matrix $\mathbf{X}_g \in \mathbb{R}^{n_g \times p}$ with independent isotropic sub-Gaussian rows, i.e., $\|\mathbf{x}_{gi}\|_{\psi_2} \leq k$ and $\mathbb{E}[\mathbf{x}_{gi}\mathbf{x}_{gi}^T] = \mathbf{I}$, the following result holds with probability at least*

$1 - 2\exp\left(-\gamma_g(\omega(\mathcal{A}_g) + \tau)^2\right)$ *for $\tau > 0$:*

$$\forall \mathbf{u}_g \in \mathcal{C}_g : n_g\left(1 - c_g\frac{\omega(\mathcal{A}_g) + \tau}{\sqrt{n_g}}\right)\|\mathbf{u}_g\|_2^2 \leq \|\mathbf{X}_g\mathbf{u}_g\|_2^2 \leq n_g\left(1 + c_g\frac{\omega(\mathcal{A}_g) + \tau}{\sqrt{n_g}}\right)\|\mathbf{u}_g\|_2^2$$

where $c_g > 0$ is constant.

The statement of Lemma 9 characterizes the distortion in the Euclidean distance between points $\mathbf{u}_g \in \mathcal{C}_g$ when the matrix $\mathbf{X}_g/n_g$ is applied to them and states that any sub-Gaussian design matrix is approximately isometry, with high probability:

$$(1 - \alpha)\|\mathbf{u}_g\|_2^2 \leq \frac{1}{n_g}\|\mathbf{X}_g\mathbf{u}_g\|_2^2 \leq (1 + \alpha)\|\mathbf{u}_g\|_2^2$$

where $\alpha = c_g\frac{\omega(\mathcal{A}_g)}{\sqrt{n_g}}$.

Now the proof for Lemma 5:

Proof. First we upper bound each of the coefficients $\forall g \in [G]$:

$$\rho_g(\mu_g) \quad = \quad \sup_{\mathbf{u},\mathbf{v} \in \mathcal{B}_g} \mathbf{v}^T\left(\mathbf{I}_g - \mu_g\mathbf{X}_g^T\mathbf{X}_g\right)\mathbf{u}$$

We upper bound the argument of the sup as follows:

$$\mathbf{v}^T\left(\mathbf{I}_g - \mu_g\mathbf{X}_g^T\mathbf{X}_g\right)\mathbf{u} \quad = \quad \frac{1}{4}\left[(\mathbf{u} + \mathbf{v})^T(\mathbf{I} - \mu_g\mathbf{X}_g^T\mathbf{X}_g)(\mathbf{u} + \mathbf{v}) - (\mathbf{u} - \mathbf{v})^T(\mathbf{I} - \mu_g\mathbf{X}_g^T\mathbf{X}_g)(\mathbf{u} - \mathbf{v})\right]$$

$$= \quad \frac{1}{4}\left[\|\mathbf{u} + \mathbf{v}\|_2^2 - \mu_g\|\mathbf{X}_g(\mathbf{u} + \mathbf{v})\|_2^2 - \|\mathbf{u} - \mathbf{v}\|_2^2 + \mu_g\|\mathbf{X}_g(\mathbf{u} - \mathbf{v})\|_2^2\right]$$

$$\text{(Lemma 9)} \quad \leq \quad \frac{1}{4}\left[\left(1 - \mu_g n_g\left(1 - c_g\frac{2\omega(\mathcal{A}_g) + \tau}{\sqrt{n_g}}\right)\right)\|\mathbf{u} + \mathbf{v}\|_2\right.$$

$$\left. - \left(1 - \mu_g n_g\left(1 + c_g\frac{2\omega(\mathcal{A}_g) + \tau}{\sqrt{n_g}}\right)\right)\|\mathbf{u} - \mathbf{v}\|_2\right]$$

$$\left(\mu_g = \frac{1}{a_g n_g}\right) \quad \leq \quad \frac{1}{4}\left[\left(1 - \frac{1}{a_g}\right)(\|\mathbf{u} + \mathbf{v}\|_2 - \|\mathbf{u} - \mathbf{v}\|_2) + c_g\frac{2\omega(\mathcal{A}_g) + \tau}{a_g\sqrt{n_g}}(\|\mathbf{u} + \mathbf{v}\|_2 + \|\mathbf{u} - \mathbf{v}\|_2)\right]$$

$$\leq \quad \frac{1}{4}\left[\left(1 - \frac{1}{a_g}\right)2\|\mathbf{v}\|_2 + c_g\frac{2\omega(\mathcal{A}_g) + \tau}{a_g\sqrt{n_g}}2\sqrt{2}\right]$$

where the last line follows from the triangle inequality and the fact that $\|\mathbf{u} + \mathbf{v}\|_2 + \|\mathbf{u} - \mathbf{v}\|_2 \leq 2\sqrt{2}$ which itself follows from $\|\mathbf{u} + \mathbf{v}\|_2^2 + \|\mathbf{u} - \mathbf{v}\|_2^2 \leq 4$. Note that we applied the Lemma 9 for bigger sets of $\mathcal{A}_g + \mathcal{A}_g$ and $\mathcal{A}_g - \mathcal{A}_g$ where Gaussian width of both of them are upper bounded by $2\omega(\mathcal{A}_g)$. The above holds with high probability (computed below). Now we set :

$$\mathbf{v}^T\left(\mathbf{I}_g - \frac{1}{a_g n_g}\mathbf{X}_g^T\mathbf{X}_g\right)\mathbf{u} \quad \leq \quad \frac{1}{2}\left[\left(1 - \frac{1}{a_g}\right) + \sqrt{2}c_g\frac{2\omega(\mathcal{A}_g) + \tau}{a_g\sqrt{n_g}}\right] \tag{46}$$

To keep the upper bound of ρ_g in (46) below any arbitrary $\frac{1}{b} < 1$ we need $n_g = O(b^2(\omega(\mathcal{A}_g) + \tau)^2)$ samples.

Now we rewrite the same analysis using the tail bounds for the coefficients to clarify the probabilities. Let's set $\mu_g = \frac{1}{a_g n_g}$, $d_g := \frac{1}{2}\left(1 - \frac{1}{a_g}\right) + \sqrt{2}c_g\frac{\omega(\mathcal{A}_g) + \tau/2}{a_g\sqrt{n_g}}$ and name the bad events of $\|\mathbf{X}_g(\mathbf{u} + \mathbf{v})\|_2^2 < n_g\left(1 - c_g\frac{2\omega(\mathcal{A}_g) + \tau}{\sqrt{n_g}}\right)$ and $\|\mathbf{X}_g(\mathbf{u} - \mathbf{v})\|_2^2 > n_g\left(1 + c_g\frac{2\omega(\mathcal{A}_g) + \tau}{\sqrt{n_g}}\right)$ as $\mathcal{E}_1$ and $\mathcal{E}_2$ respectively:

$$\mathbb{P}(\rho_g \geq d_g) \quad \leq \quad \mathbb{P}(\rho_g \geq d_g | \neg\mathcal{E}_1, \neg\mathcal{E}_2) + 2\mathbb{P}(\mathcal{E}_1) + \mathbb{P}(\mathcal{E}_2)$$

$$\text{Lemma 9} \quad \leq \quad 0 + 6\exp\left(-\gamma_g(\omega(\mathcal{A}_g) + \tau)^2\right)$$

which concludes the proof. ∎

B.6. Proof of Lemma 6

Proof. The following holds for any $\mathbf{u}$ and $\mathbf{v}$ because of $\|\mathbf{X}_g(\mathbf{u}+\mathbf{v})\|_2^2 \geq 0$:

$$-\mathbf{v}^T\mathbf{X}_g^T\mathbf{X}_g\mathbf{u} \leq \frac{1}{2}\left(\|\mathbf{X}_g\mathbf{u}\|_2^2 + \|\mathbf{X}_g\mathbf{v}\|_2^2\right) \tag{47}$$

Now we can bound ϕ_g as follows:

$$\phi_g(\mu_g) = \mu_g \sup_{\mathbf{v}\in\mathcal{B}_g, \mathbf{u}\in\mathcal{B}_0} -\mathbf{v}^T\mathbf{X}_g^T\mathbf{X}_g\mathbf{u} \leq \frac{\mu_g}{2}\left(\sup_{\mathbf{u}\in\mathcal{B}_0}\|\mathbf{X}_g\mathbf{u}\|_2^2 + \sup_{\mathbf{v}\in\mathcal{B}_g}\|\mathbf{X}_g\mathbf{v}\|_2^2\right) \tag{48}$$

So we have:

$$\phi_g\left(\frac{1}{a_g n_g}\right) \leq \frac{1}{2a_g}\left(\frac{1}{n_g}\sup_{\mathbf{u}\in\mathcal{B}_0}\|\mathbf{X}_g\mathbf{u}\|_2^2 + \frac{1}{n_g}\sup_{\mathbf{v}\in\mathcal{B}_g}\|\mathbf{X}_g\mathbf{v}\|_2^2\right) \tag{49}$$

$$(\text{Lemma 9}) \leq \frac{1}{a_g}\left(1 + c_{0g}\frac{\omega(\mathcal{A}_g) + \omega(\mathcal{A}_0) + 2\tau}{2\sqrt{n_g}}\right)$$

$$(\omega_{0g} = \max(\omega(\mathcal{A}_0), \omega(\mathcal{A}_g)) \leq \frac{1}{a_g}\left(1 + c_{0g}\frac{\omega_{0g} + \tau}{\sqrt{n_g}}\right)$$

where $c_{0g} = \max(c_0, c_g)$.

To compute the exact probabilities lets define $s_g := \frac{1}{a_g}\left(1 + c_{0g}\frac{\omega(\mathcal{A}_g)+\omega(\mathcal{A}_0)+2\tau}{2\sqrt{n_g}}\right)$ and name the bad events of $\frac{1}{n_g}\sup_{\mathbf{u}\in\mathcal{B}_0}\|\mathbf{X}_g\mathbf{u}\|_2^2 > 1 + c_0\frac{\omega(\mathcal{A}_0)+\tau}{\sqrt{n_g}}$ and $\frac{1}{n_g}\sup_{\mathbf{v}\in\mathcal{B}_g}\|\mathbf{X}_g\mathbf{v}\|_2^2 > 1 + c_g\frac{\omega(\mathcal{A}_g)+\tau}{\sqrt{n_g}}$ as $\mathcal{E}_1$ and $\mathcal{E}_2$ respectively.

$$\mathbb{P}(\phi_g > s_g) \leq \mathbb{P}(\phi_g > s_g | \neg\mathcal{E}_1)\mathbb{P}(\neg\mathcal{E}_1) + \mathbb{P}(\mathcal{E}_1) \tag{50}$$

$$\leq \mathbb{P}(\mathcal{E}_2) + \mathbb{P}(\mathcal{E}_1)$$

$$\leq 4\exp\left(-\gamma_g(\omega(\mathcal{A}_g)+\tau)^2\right)$$

∎

B.7. Proof of Lemma 7

Proof. To obtain lower bound, we use the Paley–Zygmund inequality for the zero-mean, non-degenerate ($0 < \alpha \leq \mathbb{E}|\langle\mathbf{x}, \mathbf{u}\rangle|, \mathbf{u} \in \mathbb{S}^{p-1}$) sub-Gaussian random vector $\mathbf{x}$ with $\||\mathbf{x}|\|_{\psi_2} \leq k$ (Tropp, 2015).

$$Q_{2\xi}(\mathbf{u}) \geq \frac{(\alpha - 2\xi)^2}{4ck^2}.$$

∎

B.8. Proof of Lemma 8

Proof. We split $[G]_\backslash - \mathcal{I}$ into two groups $\mathcal{J}, \mathcal{K}$. $\mathcal{J}$ consists of $\boldsymbol{\delta}_i$'s with $\|\boldsymbol{\delta}_i\|_2 \geq 2\|\boldsymbol{\delta}_0\|_2$ and $\mathcal{K} = [G]_\backslash - \mathcal{I} - \mathcal{J}$. We use the bounds

$$\|\boldsymbol{\delta}_0 + \boldsymbol{\delta}_i\|_2 \geq \begin{cases} \lambda_{\min}(\|\boldsymbol{\delta}_i\|_2 + \|\boldsymbol{\delta}_0\|_2) & \text{if } i \in \mathcal{I} \\ \|\boldsymbol{\delta}_i\|_2/2 & \text{if } i \in \mathcal{J} \\ 0 & \text{if } i \in \mathcal{K} \end{cases} \tag{51}$$

This implies

$$\sum_{i=1}^{G} n_i\|\boldsymbol{\delta}_0 + \boldsymbol{\delta}_i\|_2 \geq \sum_{i\in\mathcal{J}}\frac{n_i}{2}\|\boldsymbol{\delta}_i\|_2 + \lambda_{\min}\sum_{i\in\mathcal{I}}n_i(\|\boldsymbol{\delta}_i\|_2 + \|\boldsymbol{\delta}_0\|_2).$$

Let $S_{\mathcal{S}} = \sum_{i \in \mathcal{S}} n_i \|\boldsymbol{\delta}_i\|_2$ for $\mathcal{S} = \mathcal{I}, \mathcal{J}, \mathcal{K}$. We know that over $\mathcal{K}$, $\|\boldsymbol{\delta}_i\|_2 \leq 2\|\boldsymbol{\delta}_0\|_2$ which implies $S_{\mathcal{K}} = \sum_{i \in \mathcal{K}} n_i \|\boldsymbol{\delta}_i\|_2 \leq 2 \sum_{i \in \mathcal{K}} n_i \|\boldsymbol{\delta}_0\|_2 \leq 2n\|\boldsymbol{\delta}_0\|_2$. Set $\psi_{\mathcal{I}} = \min\{1/2, \lambda_{\min}\bar{\rho}/3\} = \lambda_{\min}\bar{\rho}/3$. Using $1/2 \geq \psi_{\mathcal{I}}$, we write:

$$\sum_{i=1}^{G} n_i \|\boldsymbol{\delta}_0 + \boldsymbol{\delta}_i\|_2 \quad \geq \quad \psi_{\mathcal{I}} S_{\mathcal{J}} + \lambda_{\min} \sum_{i \in \mathcal{I}} n_i (\|\boldsymbol{\delta}_i\|_2 + \|\boldsymbol{\delta}_0\|_2)$$

$$(S_{\mathcal{K}} \leq 2n\|\boldsymbol{\delta}_0\|_2) \quad \geq \quad \psi_{\mathcal{I}} S_{\mathcal{J}} + \psi_{\mathcal{I}} S_{\mathcal{K}} - 2\psi_{\mathcal{I}} n \|\boldsymbol{\delta}_0\|_2 + \left(\sum_{i \in \mathcal{I}} n_i\right) \lambda_{\min} \|\boldsymbol{\delta}_0\|_2 + \lambda_{\min} S_{\mathcal{I}}$$

$$(\lambda_{\min} \geq \psi_{\mathcal{I}}) \quad \geq \quad \psi_{\mathcal{I}}(S_{\mathcal{I}} + S_{\mathcal{J}} + S_{\mathcal{K}}) + \left(\left(\sum_{i \in \mathcal{I}} n_i\right) \lambda_{\min} - 2\psi_{\mathcal{I}} n\right) \|\boldsymbol{\delta}_0\|_2.$$

Now, observe that, assumption of the Definition 3, $\sum_{i \in \mathcal{I}} n_i \geq \bar{\rho} n$ implies:

$$\left(\sum_{i \in \mathcal{I}} n_i\right) \lambda_{\min} - 2\psi_{\mathcal{I}} n \geq (\bar{\rho}\lambda_{\min} - 2\psi_{\mathcal{I}})n \geq \psi_{\mathcal{I}} n.$$

Combining all, we obtain:

$$\sum_{i=1}^{G} n_i \|\boldsymbol{\delta}_0 + \boldsymbol{\delta}_i\|_2 \geq \psi_{\mathcal{I}}(S_{\mathcal{I}} + S_{\mathcal{J}} + S_{\mathcal{K}} + \|\boldsymbol{\delta}_0\|_2) = \psi_{\mathcal{I}}\left(n\|\boldsymbol{\delta}_0\|_2 + \sum_{i=1}^{G} n_i \|\boldsymbol{\delta}_i\|_2\right).$$

∎