A Simulation Preorder for Koopman-like Lifted Control Systems

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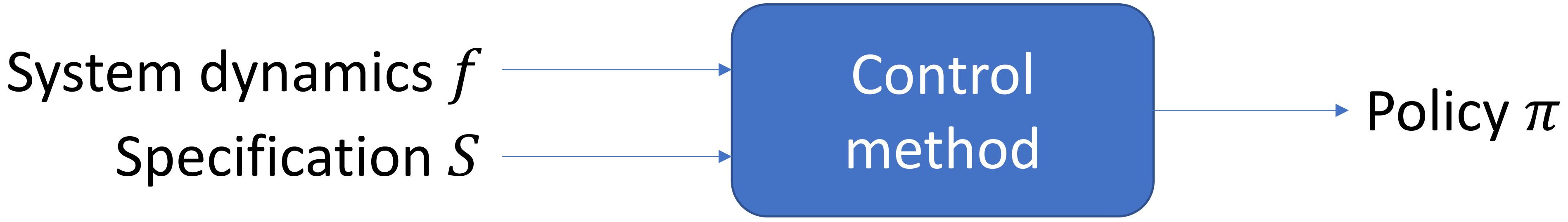
Goal

Given a system and a specification, find a policy such that the closed-loop system satisfies the specification.



$$f \vDash_{\pi} S$$

Systems and specifications

- A discrete-time system

$$x(t+1) = f_X\big(x(t), u(t)\big)$$

with state $x(t) \in X$ and $u(t) \in U$

- A **specification** is a set of sequences of (x, u) pairs: $S \subseteq (X \times U)^\infty$
- The **behavior** of a system under a policy π is the set
  $\boldsymbol{B}_\pi[f_X] \coloneqq \{(\boldsymbol{x}, \boldsymbol{u}) |\ (\boldsymbol{x}, \boldsymbol{u}) \text{ is a max. solution } \&\ u(t) = \pi(x(0), \dots, x(t))\}$
- The specification S is **satisfied** by the system f_X under the policy π if $\boldsymbol{B}_\pi[f_X] \subseteq S$. This is written $f_X \vDash_\pi S$

Few words on simulation relations

Simulation-based approach:

1. Define a notion of simulation between systems $f_X \preccurlyeq f_Y$
2. Prove that $f_X \preccurlyeq f_Y$ implies $\boldsymbol{B}_\pi[f_X] \subseteq \boldsymbol{B}_\pi[f_Y]$

Why is it useful?

1. Replace the system f_X by a simpler (e.g., affine) one f_Y s.t. $f_X \preccurlyeq f_Y$
2. Synthesize a policy π s.t. $f_Y \vDash_\pi S$
3. It follows that $f_X \vDash_\pi S$

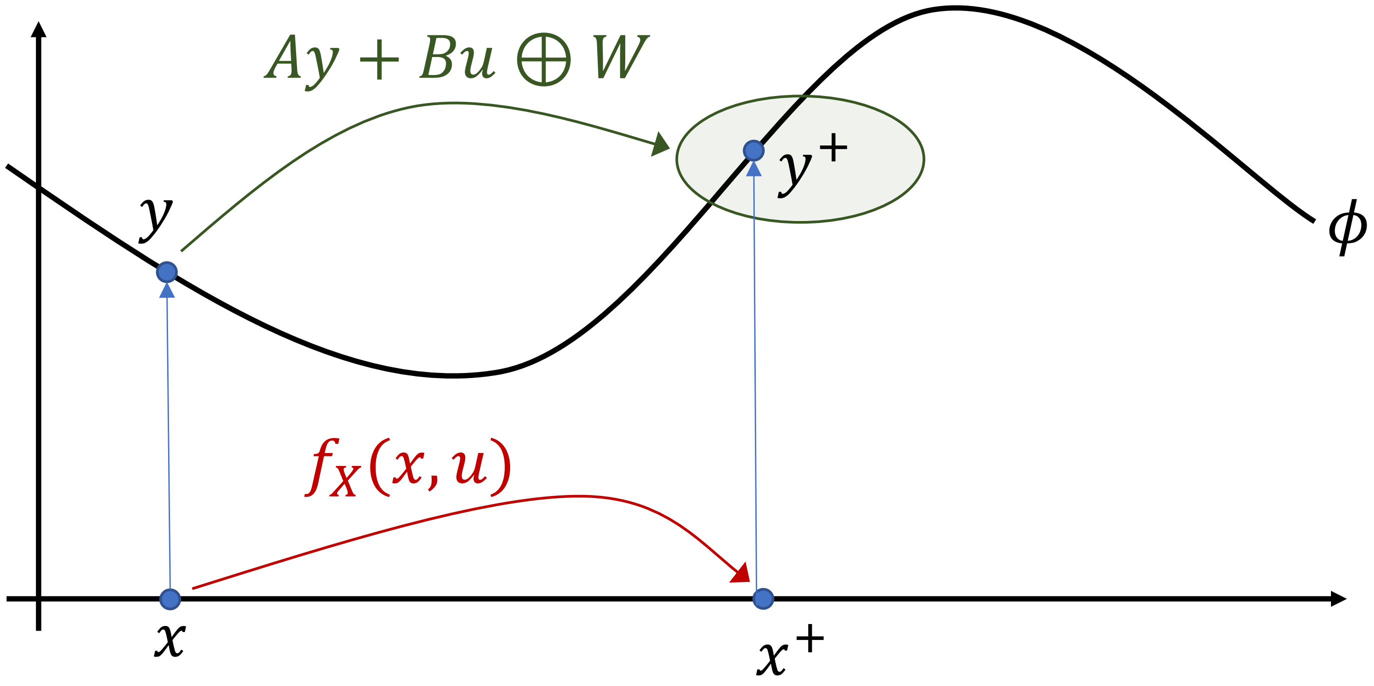
Koopman approach

$$x^+ = f_X(x,u)$$

$$y = \psi(x)$$

$$x = Cy$$

$$y^+ \in Ay + Bu \oplus W$$
$$x = Cy$$



$$\psi\big(f_X(x,u)\big) \in A\psi(x) + Bu \oplus W$$

$$\psi(x) = \begin{bmatrix} x \\ \phi(x) \end{bmatrix}$$

$$C = [I \quad 0]$$

<u>Question:</u> How to choose ψ ?

Lifted systems

A **lifted system** is a tuple $LS_Y = (X, U, \psi_Y, f_Y, g_Y)$

- X is a set of outputs
- U is a set of inputs
- $\psi_Y: \mathbb{R}^{n_X} \to \mathbb{R}^{n_Y}$ is a lifting function
- $f_Y: \mathbb{R}^{n_Y} \times U \rightrightarrows \mathbb{R}^{n_Y}$ is a set-valued dynamics
- $g_Y: \mathbb{R}^{n_Y} \to \mathbb{R}^{n_X}$ is an output map such that $g_Y\big(\psi_Y(x)\big) = x$

A **solution** of LS_Y is a tuple $(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{y}) \in X^{[0,T[} \times U^{[0,T[} \times (\mathbb{R}^{n_Y})^{[0,T[}$ s.t.

- $y(0) = \psi_Y\big(x(0)\big)$
- $y(t+1) \in f_{\mathrm{Y}}\big(y(t), u(t)\big)$
- $x(t) = g_Y(y(t))$

The **behavior** of LS_Y under a policy π is the set

$$\boldsymbol{B}_\pi[LS_Y] = \{\ (\boldsymbol{x}, \boldsymbol{u}) \mid\ \exists \boldsymbol{y} \text{ s.t. } (\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{y}) \text{ is a max. solution } \&\ u(t) = \pi\big(x(0), \dots, x(t)\big)\ \}$$

Important classes of lifted systems

- **Unlifted** (i.e., “classical”) systems

$$x(t+1) = f_X\big(x(t), u(t)\big)$$

are lifted systems with $n_Y = n_X$ and $\psi_Y = g_Y = id$

- **Affine** lifted systems

$$y(t+1) \in Ay(t) + Bu(t) \oplus W$$
$$x(t) = Cy(t)$$

for which linear control methods can be used

- **Picewise affine** lifted systems

Simulation relation for lifted systems

LS_Y is **simulated** by LS_Z (denoted $LS_Y \preccurlyeq LS_Z$) if there exists a set-valued map $\rho: \mathbb{R}^{n_Z} \rightrightarrows \mathbb{R}^{n_Y}$ such that

- $\forall x \in X: \psi_Y(x) \in \rho\big(\psi_Z(x)\big)$ $\quad$ $\color{red}{(Relation\ between\ liftings)}$
- $\forall (z, u) \in \mathbb{R}^{n_Z} \times U: f_Y(\rho(z), u) \subseteq \rho\big(f_Z(z, u)\big)$ $\quad$ $\color{red}{(between\ dynamics)}$
- $\forall z \in \mathbb{R}^{n_Z}: g_Y\big(\rho(z)\big) \subseteq \{g_Z(z)\}$ $\quad$ $\color{red}{(between\ outputs)}$

Simulation implies behavioral inclusion

THEOREM
Given two lifted systems LS_Y and LS_Z and a policy π, if LS_Y is simulated by LS_Z, then the closed loop behavior of LS_Y under π is included in the closed loop behavior of LS_Z under π, i.e.,

$$LS_Y \preccurlyeq LS_Z \quad \Longrightarrow \quad \boldsymbol{B}_\pi[LS_Y] \subseteq \boldsymbol{B}_\pi[LS_Z]$$

<u>Proof idea:</u> If $(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{y})$ is a max. sol. of LS_Y, it exists $\boldsymbol{z}$ s.t. $(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{z})$ is a max. sol. of LS_Z

- $\forall x \in X: \psi_Y(x) \in \rho\big(\psi_Z(x)\big)$

  → $y(0) \in \rho(z(0))$

- $\forall (z, u) \in \mathbb{R}^{n_Z} \times U: f_Y(\rho(z), u) \subseteq \rho\big(f_Z(z, u)\big)$

  → $y(t) \in \rho\big(z(t)\big) \Longrightarrow y(t+1) \in \rho\big(z(t+1)\big)$

- $\forall z \in \mathbb{R}^{n_Z}: g_Y\big(\rho(z)\big) \subseteq \{g_Z(z)\}$

  → $y(t) \in \rho\big(z(t)\big) \Longrightarrow g_Y\big(y(t)\big) = g_Z(z(t))$

Why is it useful?

COROLLARY

$$LS_Y \preccurlyeq LS_Z \vDash_\pi S \quad \Longrightarrow \quad LS_Y \vDash_\pi S$$

LS_Y is simulated by LS_Z

The specification S is satisfied by the lifted system LS_Z under the policy π

The specification S is satisfied by the lifted system LS_Y under the policy π

Given an unlifted system LS_X, and two affine lifted systems LS_Y and LS_Z, if

$$LS_X \preccurlyeq LS_Y \preccurlyeq LS_Z$$

then LS_Y is a « better » representation of LS_X than LS_Z

Some special cases of $LS_Y \preccurlyeq LS_Z$

If LS_Y is unlifted and

- LS_Z is affine

  → Koopman over-approximation in (Balim, Aspeel, Liu, Ozay, 2023)

- LS_Z is affine and all systems are autonomous (i.e., $U = \{0\}$)
  → approximate immersion in (Wang, Jungers, Ong, 2023)

- LS_Z is picewise affine and unlifted
  → Hybridization in (Girard, Martin, 2011)

Computational aspects

Verifying if $LS_Y \preccurlyeq LS_Z$ is a feasibility problem:

Find $\boxed{\rho}$ s.t. $\rho\colon\ \mathbb{R}^{n_Z} \rightrightarrows \mathbb{R}^{n_Y}$ Infinite number of variables

- $\forall x \in X: \psi_Y(x) \in \rho\big(\psi_Z(x)\big)$
- $\forall (z,u) \in \mathbb{R}^{n_Z} \times U: f_Y(\rho(z),u) \subseteq \rho\big(f_Z(z,u)\big)$
- $\forall z \in \mathbb{R}^{n_Z}: g_Y\big(\rho(z)\big) \subseteq \{g_Z(z)\}$

Infinite number of constraints

We derived **finite-dimensional** sufficient conditions in two cases:

1. LS_Y is unlifted and LS_Z is (picewise) affine
2. LS_Y and LS_Z are both (picewise) affine

In practice…

Given a classical (i.e., unlifted) system LS_X

1. Pick K lifting functions $\psi_1, \ldots, \psi_K$
2. For each, compute an affine lifted system: LS_k s.t. $LS_X \preccurlyeq LS_k$
3. If $LS_i \preccurlyeq LS_j$, delete LS_j

Experiments with backward reachable sets

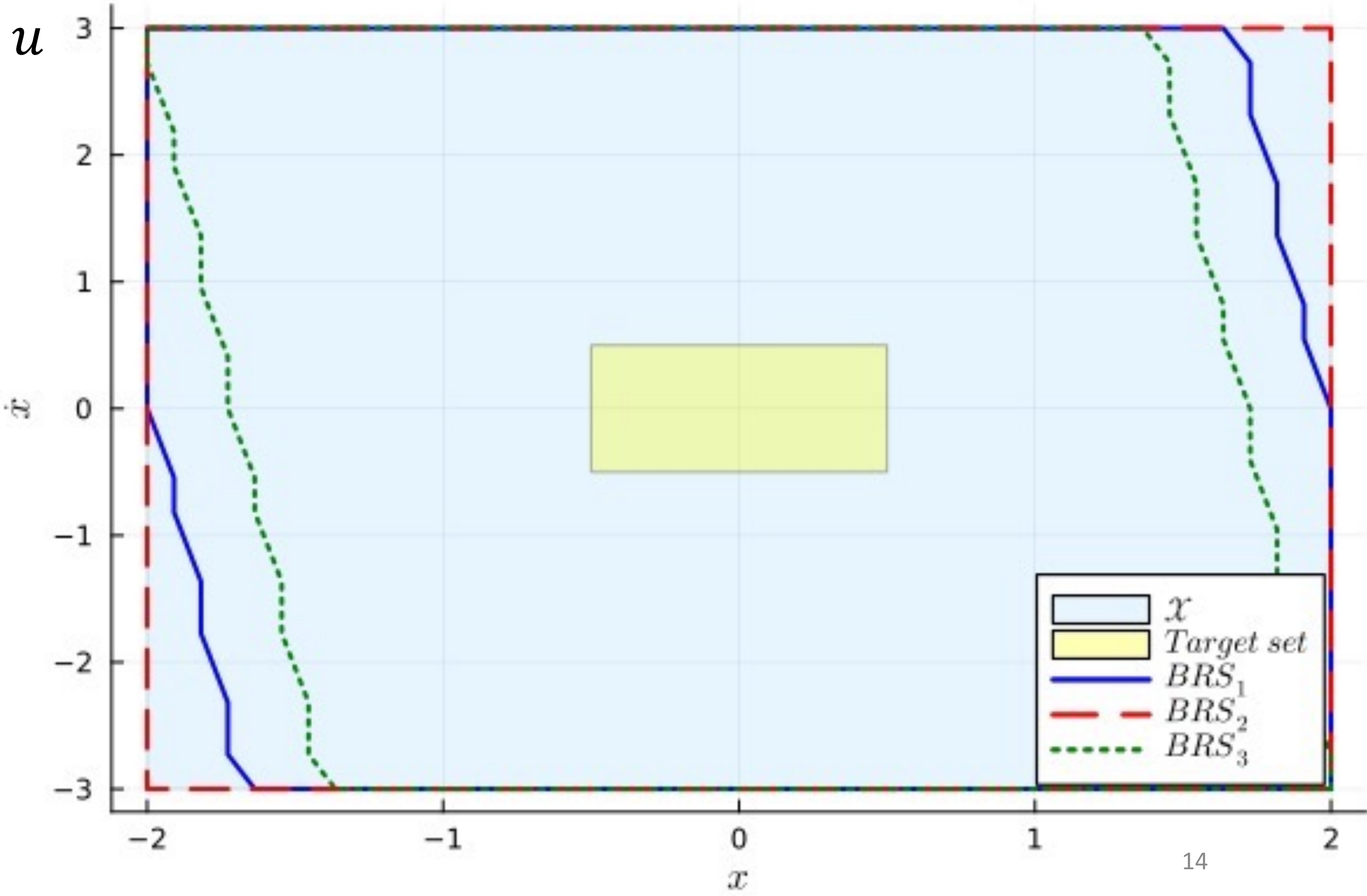
$$\ddot{x} = 2x - 2x^3 - 0.5\,\dot{x} + u$$

$$\psi_1(x) = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$

$$\psi_2(x) = \begin{bmatrix} x \\ \dot{x} \\ x^3 \end{bmatrix}$$

$$\psi_3(x) = \begin{bmatrix} x \\ \dot{x} \\ x\dot{x} \end{bmatrix}$$

$$LS_1 \preccurlyeq LS_3$$



Take home message

Simulation relations between lifted systems

- Contribution to abstraction/hybridization theory

  → More general class of simulations

- Contribution to finite-dimensional Koopman theory

  → Tools to select the lifting function ψ

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