ME573 Homework Set # 9

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1 Problem 1

Show that for FTBS:

$$\frac{f_i^{n+1} - f_i^n}{\Delta t} + U\frac{f_i^n - f_{i-1}^n}{\Delta x} = 0 \tag{1}$$

$$|\frac{A^{n+1}}{A^n}|^2 = 1 - 2C_0(1 - C_0)(1 - cos\theta) \tag{2}$$

From equation 1:

$$f_i^{n+1} = f_i^n - C_0(f_i^n - f_{i-1}^n) \tag{3}$$

And from Von-Neumann Stability Analysis:

$$f_i^n = A^n e^{Ii\theta} \tag{4}$$

Substitute 4 into 3:

$$A^{n+1}e^{Ii\theta} = A^n e^{Ii\theta} - C_0[A^n e^{Ii\theta} - A^n e^{I(i-1)\theta}] \tag{5}$$

$$\frac{A^{n+1}}{A^n}e^{Ii\theta} = e^{Ii\theta} - C_0[e^{Ii\theta} - e^{I(i-1)\theta}] \tag{6}$$

$$\frac{A^{n+1}}{A^n} = 1 - C_0[1 - e^{-I\theta}] \tag{7}$$

Which matches the Pozrikis's definition for $\frac{A^{n+1}}{A^n}$. Now we square the definition.

$$|\frac{A^{n+1}}{A^n}|^2 = -2C_0^2 e^{i\theta} + C_0^2 e^{2i\theta} + C_0^2 + 2C_0 e^{i\theta} - 2C_0 + 1 \tag{8}$$

$$|\frac{A^{n+1}}{A^n}|^2 = C_0[-2C_0 e^{i\theta} + C_0 e^{2i\theta} + C_0 + 2e^{i\theta} - 2] + 1 \tag{9}$$

$$|\frac{A^{n+1}}{A^n}|^2 = 2C_0[-C_0 e^{i\theta} + \frac{C_0 e^{2i\theta}}{2} + \frac{C_0}{2} + e^{i\theta} - 2] + 1 \tag{10}$$

$$|\frac{A^{n+1}}{A^n}|^2 = 2C_0[-C_0(cos\theta + isin\theta) + \frac{C_0(cos(2\theta) + isin(2\theta))}{2} + \frac{C_0}{2} + (cos\theta + isin\theta) - 2] + 1 \quad (11)$$

$$|\frac{A^{n+1}}{A^n}|^2 = 2C_0[(1 - C_0)(cos\theta + isin\theta) + \frac{C_0}{2}(cos(2\theta) + isin(2\theta) - 1)] + 1 \quad (12)$$

$$|\frac{A^{n+1}}{A^n}|^2 = (1 - 2C_0)[(1 - C_0)(cos\theta + isin\theta) + \frac{C_0}{2}(cos(2\theta) + isin(2\theta))] \quad (13)$$

The final step to reach Equation 2 is unclear.

2 Problem 2

Figure 2 displays results of the FTCS and FTBS schemes compared to the exact solution. This implementation of the scheme (Δt= 0.01 and Δx= 0.05) was stable ($C_0 = 0.628$)

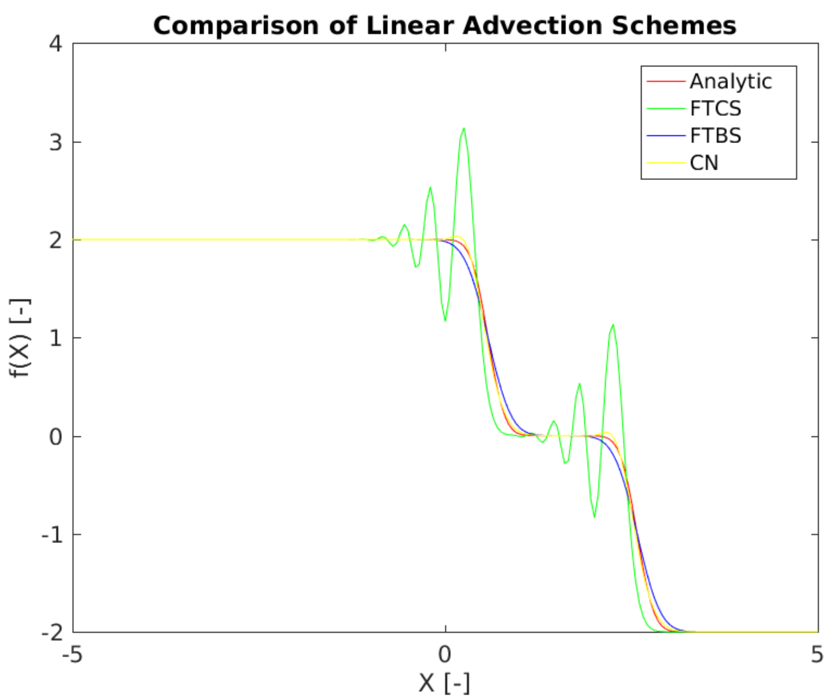

Figure 1:

The FTBS scheme was re-implemented in an unstable condition (Δt= 0.02 and Δx= 0.025), the Courant number in this state is 2.51. Figure 2 shows the results of this unstable solution.

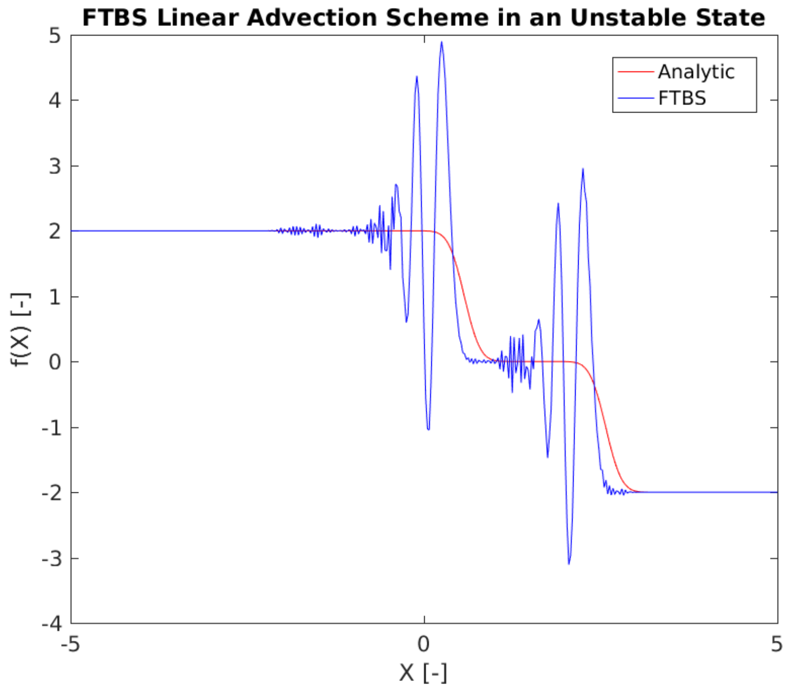

Figure 2:

3 Problem 3

Finally, the Crank-Nicolson Scheme was implemented to solve the same pure advection problem. Figure 3 shows the results of the CN approach compared to the FTBS scheme and Figure 3 shows the spatial error.

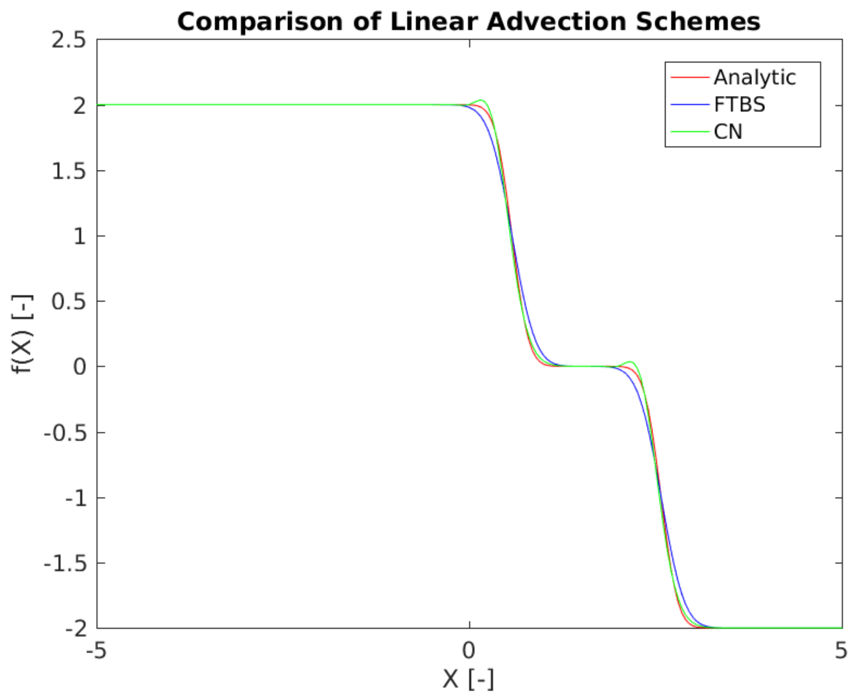

Figure 3:

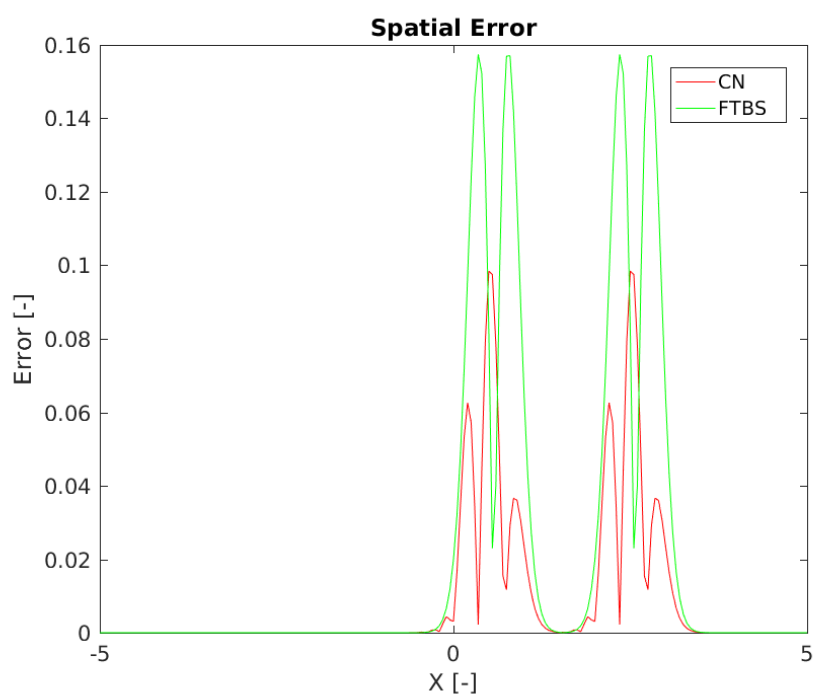

Figure 4: