Problem 1

a) $D_c u_n = \dfrac{u_{n+1} - u_{n-1}}{2h}$

$$\frac{d(uv)}{dx} = u\frac{d(v)}{dx} + v\frac{d(u)}{dx}$$

does $D_c(u_n v_n) = u_n D_c v_n + v_n D_c u_n$ ?

$$\frac{u_n \cdot (v_{n+1} - v_{n-1})}{2h} + \frac{v_n(u_{n+1} - u_{n-1})}{2h} = u_n D_c v_n + v_n D_c u_n$$

$$\Rightarrow \boxed{\frac{1}{2h}\left[u_n(v_{n+1} - v_{n-1}) + v_n(u_{n+1} - u_{n-1})\right] \neq u_n D_c v_n + v_n D_c u_n \quad \text{Does not hold}}$$

b) $D_c(u_n v_n) = \bar{u}_n D_c v_n + \bar{v}_n D_c u_n$ $\quad \bar{u}_n = \frac{1}{2}(u_{n+1} + u_{n-1})$, $\bar{v}_n = \frac{1}{2}(v_{n+1} + v_{n-1})$

$$\frac{1}{2}(u_{n+1} + u_{n-1}) D_c v_n + \frac{1}{2}(v_{n+1} + v_{n-1}) D_c u_n \qquad D_c u_n = \frac{u_{n+1} - u_{n-1}}{2h}$$

$$\Rightarrow \frac{1}{2}(u_{n+1} + u_{n-1})\frac{(v_{n+1} - v_{n-1})}{2h} + \frac{1}{2}(v_{n+1} + v_{n-1})\frac{(u_{n+1} - u_{n-1})}{2h} \qquad D_c v_n = \frac{v_{n+1} - v_{n-1}}{2h}$$

$$\Rightarrow \frac{1}{4h}\left[(u_{n+1} + u_{n-1})(v_{n+1} - v_{n-1}) + (v_{n+1} + v_{n-1})(u_{n+1} - u_{n-1})\right]$$

$$\Rightarrow \frac{1}{4h}\left[u_{n+1}v_{n+1} + \cancel{u_{n-1}v_{n+1}} - \cancel{u_{n+1}v_{n-1}} - u_{n-1}v_{n-1} + u_{n+1}v_{n+1} + \cancel{v_{n-1}u_{n+1}} - v_{n-1}u_{n-1} - \cancel{v_{n+1}u_{n-1}}\right]$$

$$\Rightarrow \frac{1}{4h}\left[2u_{n+1}v_{n+1} - 2u_{n-1}v_{n-1}\right] = \frac{1}{2h}\left[u_{n+1}v_{n+1} - u_{n-1}v_{n-1}\right]$$

Problem 2

$$f_j' + \sum_{k=-1}^{1} a_k f_{j+k} = O(?)$$ (central-differences)

$$f''(x_0) = \frac{f_1 - 2f_0 - f_{-1}}{h^2}$$

trying to replicate this

Taylor Table

| | $f_j$ | $f_j'$ | $f_j''$ | $f_j^{(3)}$ | $f_j^{(4)}$ |
|---|---|---|---|---|---|
| $f_j''$ | 0 | 0 | 1 | 0 | 0 |
| $a_{-1}f_{j-1}$ | $a_{-1}$ | $-a_{-1}h$ | $a_{-1}h^2/2$ | $-a_{-1}h^3/6$ | $a_{-1}\frac{h^4}{24}$ |
| $a_0 f_j$ | $a_0$ | 0 | 0 | 0 | 0 |
| $a_1 f_{j+1}$ | $a_1$ | $a_1 h$ | $a_1 h^2/2$ | $a_1 h^3/6$ | $a_1 h^4/24$ |

$$\cancel{f_j'} + \sum_{k=-1}^{1} a_k f_{j+k} = (a_{-1} + a_0 + a_1) f_j + (-a_{-1}h + a_1 h) f_j'$$

$$+ (a_{-1} h^2/2 + a_1 h^2/2) f_j'' + (a_1 h^3/6 - a_{-1} h^3/6) f_j'''$$

$$(a_{-1} h^2/2 + a_1 h^2/2) f_j'' = (a_{-1} + a_0 + a_1) f_j + (a_1 h - a_{-1} h) f_j' + (a_1 h^3/6 - a_{-1} h^3/6) f_j'''$$

$$(a_{-1} + a_1) f_j'' = \frac{2}{h^2}(a_{-1} + a_0 + a_1) f_j + \frac{2}{h}(a_1 - a_{-1}) f_j' + \frac{h}{3}(a_1 - a_{-1}) f_j'''$$

$$\boxed{f_j'' = \frac{2}{h^2}\frac{(a_{-1} + a_0 + a_1)}{(a_{-1} + a_1)} f_j + \frac{2}{h}\frac{(a_1 - a_{-1})}{(a_{-1} + a_1)} f_j' + \frac{h}{3}\frac{(a_1 - a_{-1})}{(a_1 + a_1)} f_j''' + \frac{h^2}{12}\frac{\cancel{(a_1 + a_{-1})}}{\cancel{(a_{-1} + a_1)}} f_j^{(4)}}$$

$$a_1 = a_{-1}$$

$$a_{-1} + a_0 + a_1 = 0$$

I am missing a third equation to solve for $a_{-1}, a_0, a_1$

Problem 3

given $x_j$, $x_{j+1}$, $x_{j+2}$, get accurate $f_j'$ using one-sided derivative formula.

$$f_j' + \sum_{k=0}^{2} a_k f_{j+k} = O(?)$$

Taylor Table

| | $f_j$ | $f_j'$ | $f_j''$ | $f_j'''$ |
|---|---|---|---|---|
| $f_j'$ | 0 | 1 | 0 | 0 |
| $a_0 f_j$ | $a_0$ | 0 | 0 | 0 |
| $a_1 f_{j+1}$ | $a_1$ | $a_1 h$ | $a_1 \frac{h^2}{2}$ | $a_1 \frac{h^3}{6}$ |
| $a_2 f_{j+2}$ | $a_2$ | $2h a_2$ | $a_2 \frac{(2h)^2}{2}$ | $a_2 \frac{(2h)^3}{6}$ |

$$f_j' + \sum_{k=0}^{2} a_k f_{j+k} = (a_0 + a_1 + a_2) f_j + (1 + a_1 h + 2h a_2) f_j'$$

$$+ \left(a_1 \frac{h^2}{2} + a_2 \frac{(2h)^2}{2}\right) f_j'' + \left(a_1 \frac{h^3}{6} + a_2 \frac{(2h)^3}{6}\right) f_j''' + \ldots$$

solve for $a_1, a_2, a_0$ set low-order terms to 0

$a_0 + a_1 + a_2 = 0$

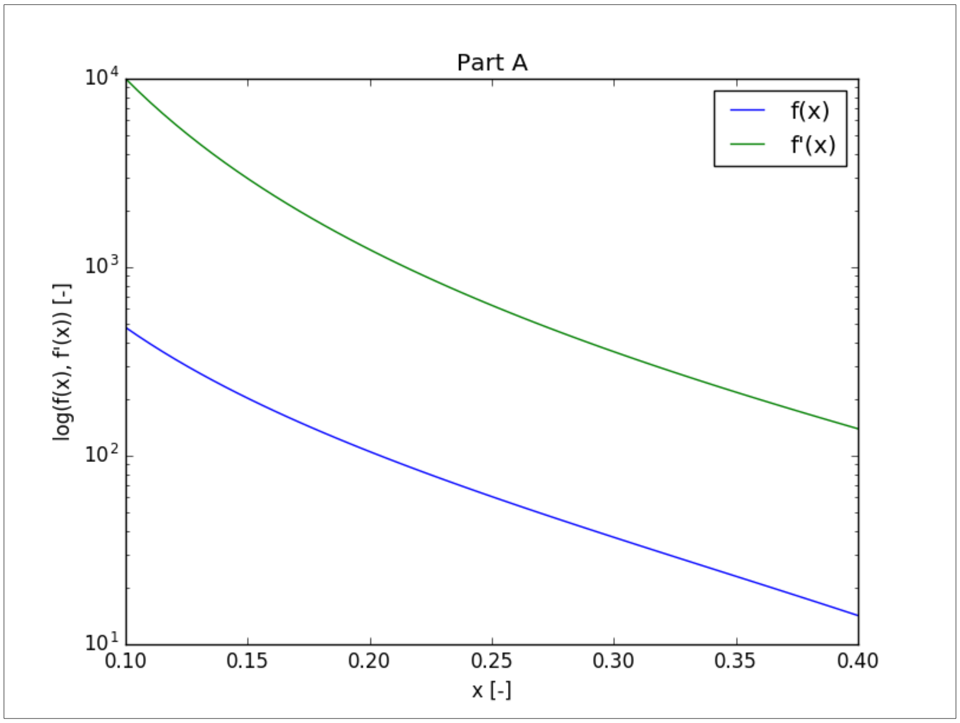
$1 + a_1 h + 2h a_2 = 0$

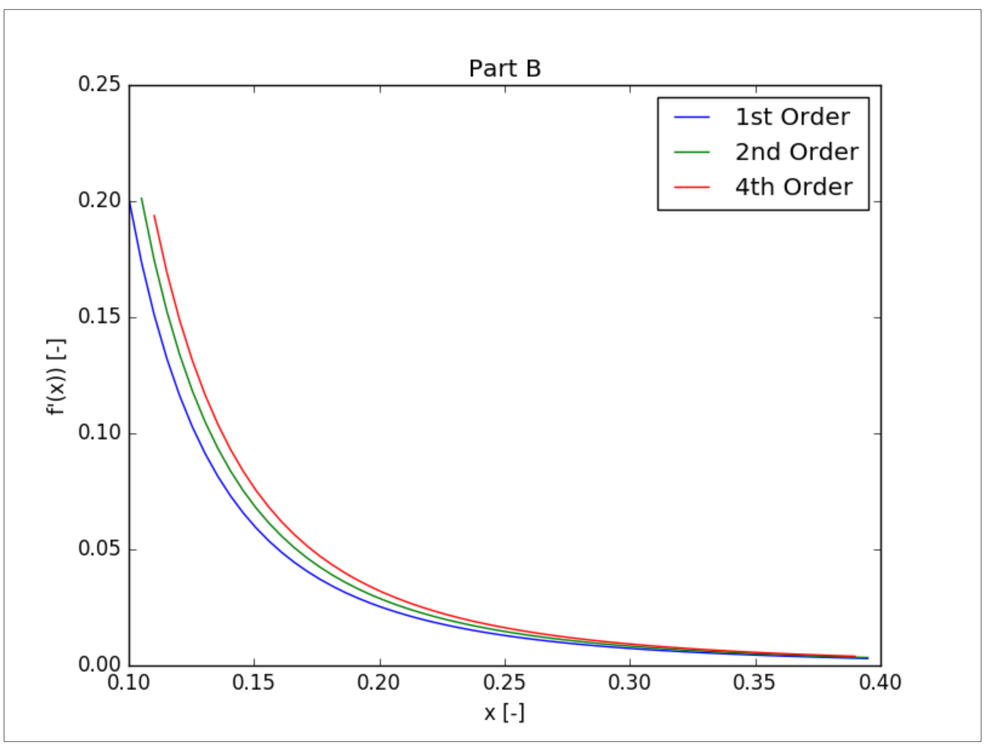
$a_1 h^2/2 + 2 a_2 h^2 = 0$

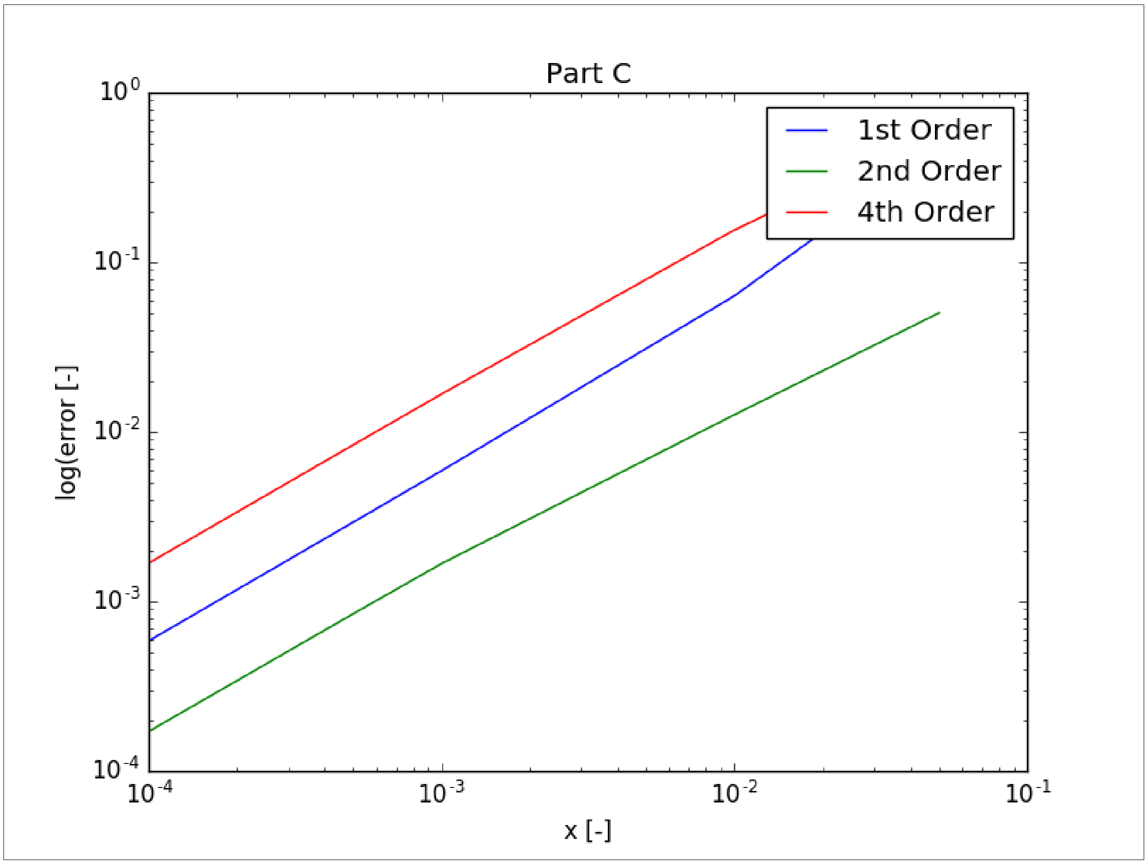
$a_1 = -\frac{2}{h}$ $a_2 = \frac{1}{2h}$ $a_0 = \frac{3}{2h}$

$$\boxed{f_j' = \frac{-3f_j + 4_{j+1} - f_{j+2}}{2h} + O(h^2)}$$

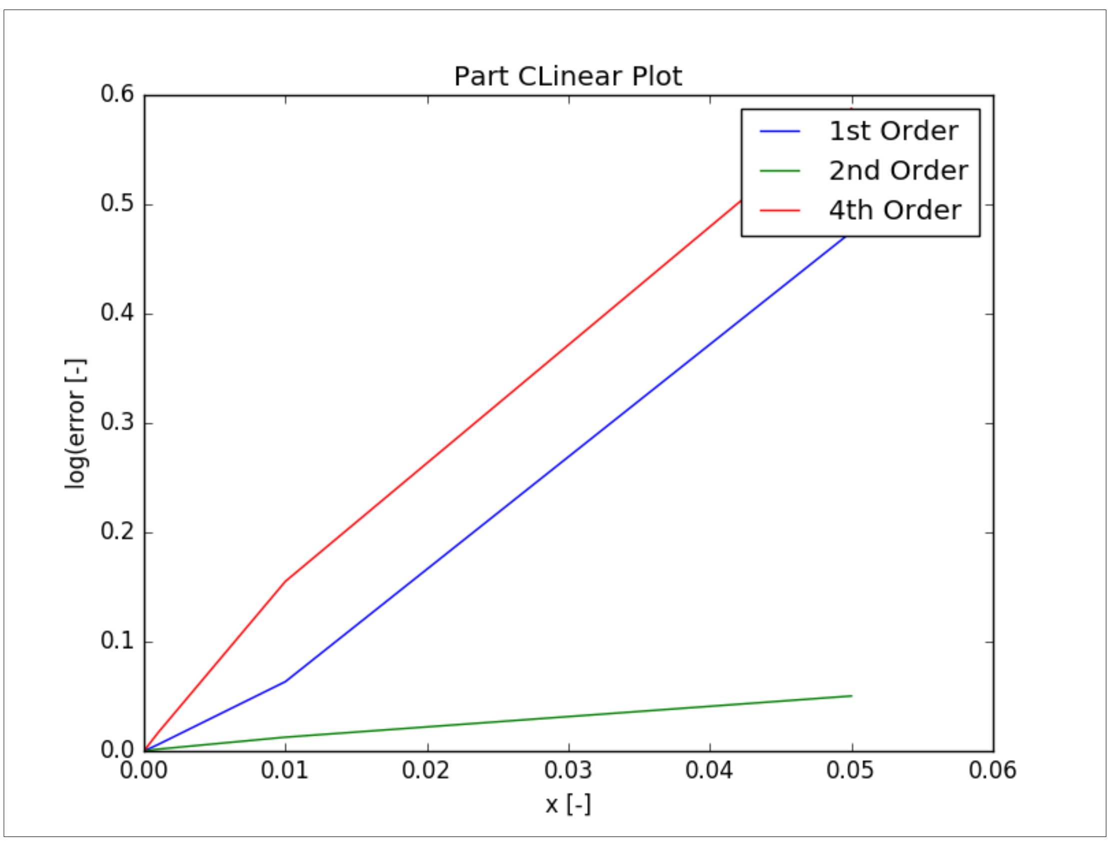
leading truncation term is $\frac{h^2}{3} f_j'''$


Part A
f(x)
f'(x)
log(f(x), f'(x)) [-]
x [-]



Part B
1st Order
2nd Order
4th Order
f'(x)) [-]
x [-]



Part C
1st Order
2nd Order
4th Order
log(error [-]
x [-]
$10^0$
$10^{-1}$
$10^{-2}$
$10^{-3}$
$10^{-4}$
$10^{-4}$
$10^{-3}$
$10^{-2}$
$10^{-1}$

Part CLinear Plot
1st Order
2nd Order
4th Order
log(error [-]
x [-]
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.00
0.01
0.02
0.03
0.04
0.05
0.06