Problem 1

Consider Constant Coefficient PDE

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial u}{\partial x} + cu$$

Sub $u(x,t) = Ae^{Ikx - I\omega t}$

$$\frac{\partial u}{\partial t} = -AI\omega e^{Ikx - I\omega t}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial x} AIke^{Ikx - I\omega t}$$

$$= AI^2k^2e^{Ikx - I\omega t}$$

$$-\cancel{A}I\omega \cancel{e^{Ikx - I\omega t}} = a(\cancel{A}I^2k^2\cancel{e^{Ikx - I\omega t}}) + b\cancel{A}\cancel{I}k\cancel{e^{Ikx - I\omega t}} + c\cancel{A}\cancel{I}\omega\cancel{e^{Ikx - I\omega t}}$$

$$-\omega = aIk^2 + bk + c \Rightarrow \boxed{\omega = -aIk^2 + bk + c}$$

b) $\omega = -aIk^2$

ω is a real value. The solution will oscillate in time.

c) $\omega = bk$

ω is an imaginary value. The solution will reach steady-state

Problem 2

$u(x) = \iint f(x)$

$$f(x) = \frac{4}{\pi} \sum_{k=1}^{\infty} \sin^2\left(\frac{k\pi}{4}\right) \sin(2kx)$$

$$\iint f(x) = -\frac{1}{\pi} \sin^2\left(\frac{\pi k}{4}\right) \sin(2kx) + C_1 x + C$$

$$u(x) = -\frac{1}{\pi} \sin^2\left(\frac{\pi k}{4}\right) \sin(2kx) + C_1 x + C_2$$

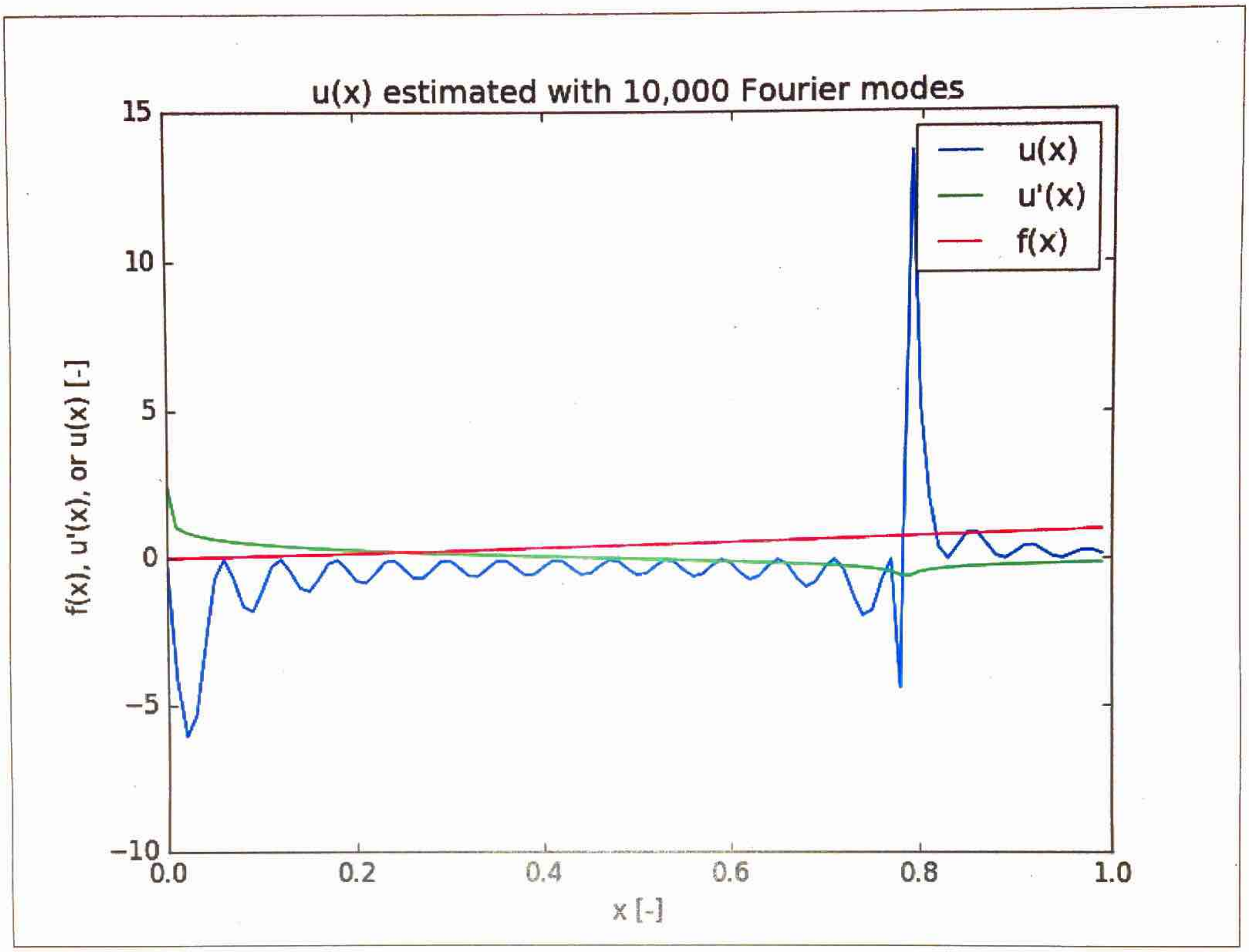
$u(0) = 0 \rightarrow C_2 = 0$

$u(\pi) = 0 \rightarrow$

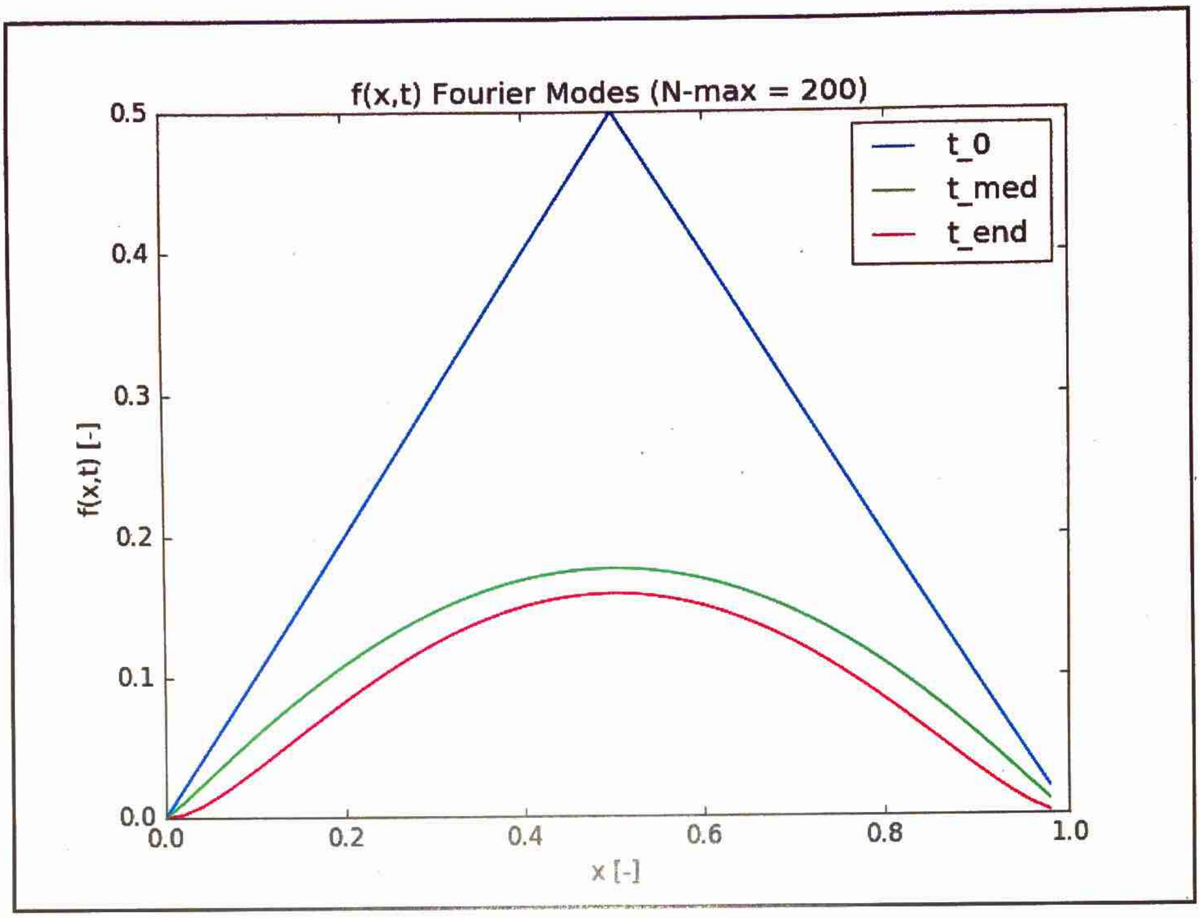
$$= -\frac{1}{\pi} \sin^2\left(\frac{\pi k}{4}\right) \underbrace{\sin(2k\pi)}_{0} + C_1 \pi = 0$$

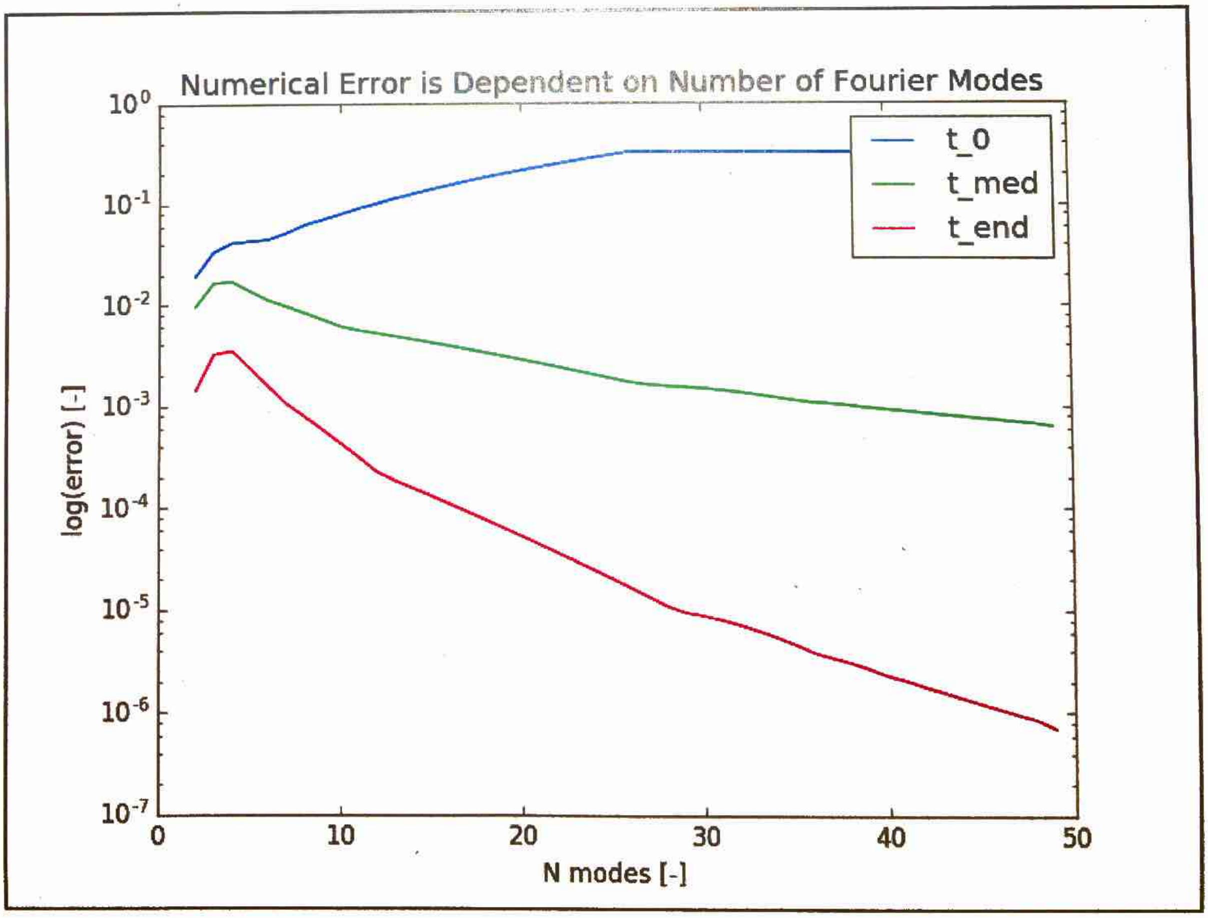
$C_1 = 0$

$$\boxed{\begin{aligned} u(x) &= -\frac{1}{\pi} \sin^2\left(\frac{\pi k}{4}\right) \sin(2kx) \\ u'(x) &= \frac{\sin^2\left(\frac{\pi k}{4}\right) \cos(2kx)}{2k} \end{aligned}}$$



u(x) estimated with 10,000 Fourier modes
u(x)
u'(x)
f(x)
f(x), u'(x), or u(x) [-]
x [-]
15
10
5
0
−5
−10
0.0
0.2
0.4
0.6
0.8
1.0


f(x,t) Fourier Modes (N-max = 200)
f(x,t) [-]
x [-]
t_0
t_med
t_end
0.0
0.1
0.2
0.3
0.4
0.5
0.0
0.2
0.4
0.6
0.8
1.0


Numerical Error is Dependent on Number of Fourier Modes
log(error) [-]
N modes [-]
t_0
t_med
t_end
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
0
10
20
30
40
50