Calculus I
Quiz 1 solutions

George Sparling

Laboratory of Axiomatics
University of Pittsburgh

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Question 1

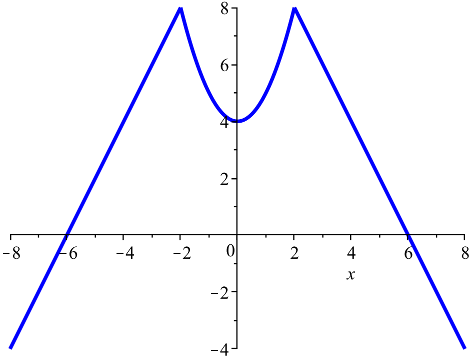
For x real, a function $f(x)$ is given by the following formulas:

- If $x \leq -2$, $f(x) = 2x + 12$,
- If $-2 < x < 2$, $f(x) = x^2 + 4$,
- If $x \geq 2$, $f(x) = 12 - 2x$.

Sketch the graph of the function and give its domain and range.
Is the function f continuous? Discuss.

- The part of the graph for $x \leq -2$, so $y = 2x + 12$ is a straight line of slope 2, crossing the x-axis at $(-6, 0)$ and ending at the point $(-2, 8)$, which is part of the graph.
- The part of the graph for $x \geq -2$, so $y = 12 - 2x$ is a straight line of slope -2, crossing the x-axis at $(6, 0)$ and ending at the point $(2, 8)$, which is part of the graph.
- The part of the graph for $-2 < x < 2$, so $y = x^2 + 4$ is a concave up parabola, symmetrical about the y-axis, crossing the y-axis at $(0, 4)$ and ending at the points $(\pm 2, 8)$, which are points of the straight line parts of the graph.

The plot



Assembling the three graphs, we get a continuous curve, defined for all x. The domain is all reals, the range is $(-\infty, 8]$.

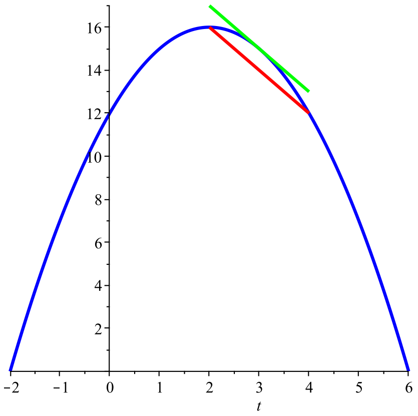
Question 2

The vertical height y feet, above ground level, at time t seconds of a ball tossed in the air is given by the equation $y = -t^2 + 4t + 12$.

- At which times is the height zero? What is the maximum height of the ball and when does it occur? Sketch the trajectory of the ball.

  The height is zero when $-t^2 + 4t + 12 = 0$, so when $0 = t^2 - 4t - 12 = (t - 6)(t + 2)$, so when $t = 6$ or $t = -2$ seconds.
  The maximum height is then at the midpoint of the interval $[-2, 6]$, so when $t = \frac{6-2}{2} = \frac{4}{2} = 2$. The height is then $y(2) = -2^2 + 4(2) + 12 = -4 + 8 + 12 = 16$ feet.
  Alternatively we can complete the square: $-t^2 + 4t + 12 = -(t^2 - 4t - 12) = -((t - 2)^2 - 16) = 16 - (t - 2)^2$.
  The maximum height is then when $t - 2 = 0$, so when $t = 2$ and the maximum height is 16 feet, as before.

The plot



The blue parabola is the trajectory. The slope of the red chord is -2 and gives the average velocity from $t = 2$ to $t = 4$. The slope of the green tangent line at $(3, 15)$ is also -2 and gives the instantaneous velocity at time $t = 3$.

- Find the average velocity of the projectile over the period from $t = 2$ seconds to $t = 4$ seconds and determine the instantaneous velocity of the projectile at time $t = 3$ seconds.
  We have $y = -t^2 + 4t + 12$.
  When $t = 2$, we have $y = -2^2 + 4(2) + 12 = -4 + 8 + 12 = 16$. When $t = 4$, we have $y = -4^2 + 4(4) + 12 = -16 + 16 + 12 = 12$.
  So the chord slope between $(2, 16)$ and $(4, 12)$ is:

$$\frac{\Delta y}{\Delta t} = \frac{12 - 16}{4 - 2} = \frac{-4}{2} = -2.$$

  The instantaneous velocity at time $t = 3$, where $y = -3^2 + 4(3) + 12 = -9 + 12 + 12 = 15$ is the limit: $\lim_{t\to 3} \frac{y(t)-y(3)}{t-3} = \lim_{t\to 3} \frac{-t^2+4t+12-15}{t-3} = \lim_{t\to 3} \frac{-t^2+4t-3}{t-3} = \lim_{t\to 3} \frac{-(t-3)(t-1)}{t-3} = \lim_{t\to 3} -(t-1) = -(3-1) = -2$.

Question 3a

Determine the following limits, or explain why the limit in question does not exist:

- $\lim_{x \to 5} \frac{x^2 - 2x - 15}{x - 5}$.

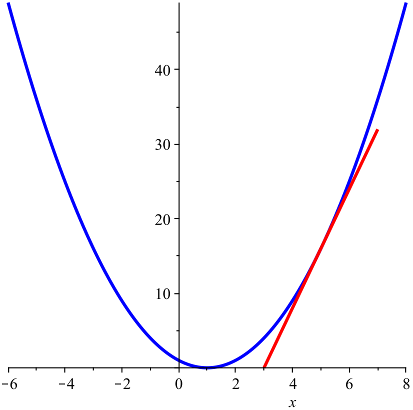
We have:

$$\lim_{x \to 5} \frac{x^2 - 2x - 15}{x - 5} = \lim_{x \to 5} \frac{(x-5)(x+3)}{x - 5}$$

$$= \lim_{x \to 5}(x+3) = 5 + 3 = 8.$$

Geometrically, this is the slope of the tangent line to the parabola $y = x^2 - 2x + 1$, or $y = (x-1)^2$ at the point $(5, 16)$: this line has the point-slope equation, $y - 16 = 8(x - 5)$, or $y = 8x - 24$, so it crosses the x-axis, where $y = 0$ at $x = 3$.

The parabola and the tangent line



The blue curve is the parabola $y = (x - 1)^2$. The red line is the tangent line to the parabola at the point $(5, 16)$, whose slope is 8.

Question 3b

- $\lim_{x\to 4} \dfrac{\sqrt{x}-2}{x-4}$.

We use the factorization of the difference of two squares $a^2 - b^2 = (a-b)(a+b)$ to simplify the numerator:

$$\lim_{x\to 4} \frac{\sqrt{x}-2}{x-4} = \lim_{x\to 4} \frac{(\sqrt{x}-2)(\sqrt{x}+2)}{(x-4)(\sqrt{x}+2)} = \lim_{x\to 4} \frac{\left((\sqrt{x})^2 - 2^2\right)}{(x-4)(\sqrt{x}+2)}$$

$$= \lim_{x\to 4} \frac{(x-4)}{(x-4)(\sqrt{x}+2)} = \lim_{x\to 4} \frac{1}{(\sqrt{x}+2)} = \frac{1}{(\sqrt{4}+2)} = \frac{1}{(2+2)} = \frac{1}{4}.$$

Geometrically, this is the slope of the tangent line to the half-parabola $y = \sqrt{x}$ at the point $(4, 2)$: this line has the point-slope equation, $y - 2 = \frac{1}{4}(x-4)$, or $y = \frac{x}{4} + 1$, so it crosses the y-axis, where $x = 0$ at $y = 1$.

Question 3b

- $\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4}$.

We can also evaluate this limit in a slightly different way.
We use the fact that we can write $x - 4$ as the difference of two squares and then use the factorization $a^2 - b^2 = (a - b)(a + b)$!

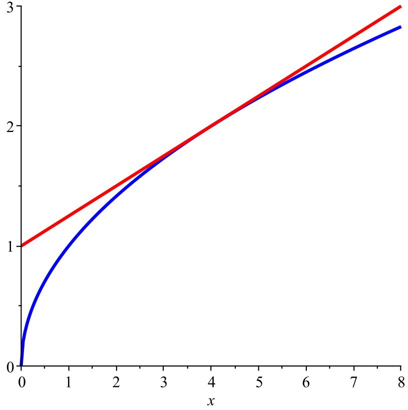
$$x - 4 = \left(\sqrt{x}\right)^2 - 2^2 = (\sqrt{x} - 2)(\sqrt{x} + 2).$$

So we have:

$$\lim_{x \to 4} \frac{\sqrt{x} - 2}{x - 4} = \lim_{x \to 4} \frac{(\sqrt{x} - 2)}{(\sqrt{x} - 2)(\sqrt{x} + 2)}$$

$$= \lim_{x \to 4} \frac{1}{(\sqrt{x} + 2)} = \frac{1}{(\sqrt{4} + 2)} = \frac{1}{2 + 2} = \frac{1}{4}.$$

The half-parabola and the tangent line



The blue curve is the half-parabola $y = \sqrt{x}$. The red line is the tangent line to the parabola at the point $(4, 2)$, whose slope is $\frac{1}{4}$.

Question 4

Let $f(x) = 4 - x^2$, defined for any real x and $g(x) = \sqrt{x}$, defined for any real $x \geq 0$.
Give formulas for the compositions $f \circ f$, $f \circ g$, $g \circ f$ and $g \circ g$ and for each of these functions give its appropriate domain.

- $(f \circ f)(x) = f(f(x)) = 4 - (f(x))^2 = 4 - (4 - x^2)^2 = 4 - (16 + x^4 - 8x^2) = -x^4 + 8x^2 - 12.$
  This composition is defined for all real x, since f is.
- $(f \circ g)(x) = f(g(x)) = 4 - (g(x))^2 = 4 - \left(\sqrt{x}\right)^2 = 4 - x.$
  This composition is defined for all real $x \geq 0$, since these are the values for which $g(x)$ is defined.
- $(g \circ f)(x) = g(f(x)) = \sqrt{f(x)} = \sqrt{4 - x^2}.$
  This composition is defined for x real and $-2 \leq x \leq 2$ since we need $x^2 \leq 4$, for $\sqrt{4 - x^2}$ to be well-defined.
- $(g \circ g)(x) = g(g(x)) = \sqrt{g(x)} = \sqrt{\sqrt{x}} = x^{\frac{1}{4}}.$
  This composition is defined for all real $x \geq 0$, since then both $\sqrt{x}$ and $\sqrt{\sqrt{x}}$ are well-defined.