Checking the Penrose condition for the Alber equation

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1 The (in)stability condition

Consider the Alber equation with homogeneous background $\Gamma(k)$,

$$\begin{gathered} i\partial_t u + p(\Delta_x - \Delta_y)u + q\Big(V(x,t) - V(y,t)\Big)\Big(\Gamma(x-y) + u(x,y,t)\Big) = 0, \\ V(x,t) = u(x,x,t), \qquad u(x,y,0) = u_0(x,y). \end{gathered} \tag{1}$$

The autocorrelation Γ is related to the power spectrum P through

$$P(k) = \mathcal{F}_{y\to k}[\Gamma(y)], \tag{2}$$

assumed to be smooth and of compact support. Moreover, the divided difference of the spectrum P is defined as

$$D_X P(k) = \begin{cases} \frac{P\left(k+\frac{X}{2}\right) - P\left(k-\frac{X}{2}\right)}{X}, & X \neq 0, \\ P'(k), & X = 0. \end{cases} \tag{3}$$

The Alber equation (1) is known [1, 2] to exhibit modulation instability if

$$\exists X_* \qquad \exists \omega_* \text{ with } \mathrm{Re}(\omega_*) > 0 \qquad \mathbb{H}[D_{X_*}P](\omega_*) = 4\pi\frac{p}{q}. \tag{4}$$

Any wavenumber X for which (4) holds is called an unstable wavenumber. In [2] it was shown that condition (4) is equivalent to

$$\text{the curve } \gamma_X := \{\mathbb{H}[D_X P](t) - iD_X P(t),\ t \in \bar{\mathbb{R}}\} \text{ is winding around the point } 4\pi\frac{p}{q}. \tag{5}$$

Remarks:

- $\bar{\mathbb{R}} = [-\infty, +\infty]$, and by continuous extension

$$\mathbb{H}[D_X P](+\infty) - iD_X P(+\infty) = \mathbb{H}[D_X P](-\infty) - iD_X P(-\infty) = 0$$

and γ_X is always a bounded, closed curve.

- The critical case of $4\pi p/q$ being on the curve γ_X corresponds to a condition of the form (4) being satisfied but with $\mathrm{Re}(\omega_*) = 0$. That case does not lead to exponential growth of the instability, but neither to dispersion of the instability.

The stability condition for (1), often called Penrose condition, is

$$\exists \kappa > 0 \;\text{ such that }\; \inf_{\substack{\mathrm{Re}\omega>0,\\ X\in\mathbb{R}}} \left| \frac{4\pi p}{q} - \mathbb{H}[D_X P](\omega) \right| \geqslant \kappa, \tag{6}$$

which is equivalent to

$$\exists \kappa > 0 \;\text{ such that }\; d\left(\frac{4\pi p}{q}, \bigcup_{X\in\mathbb{R}} \overline{\gamma_X} \right) \tag{7}$$

where $\overline{\gamma_X}$ is the closed set circumscribed by γ_X.

2 This code

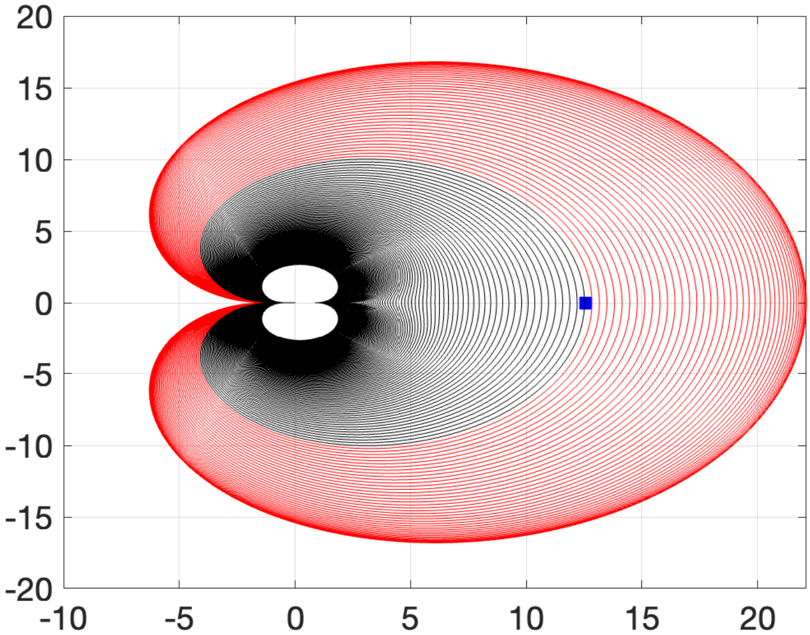
Inputs required by the code:

- the "target point on the complex plane," `target` $= 4\pi p/q$. The parameters p, q appear in the equation (1). Default value is 4π.
- the spectrum `P` is required as the handle of a function that can return both scalar and array values, for both positive and negative values `k`.
- the `plot_flag` determines whether a plot is generated (default value is `1`.)
- the array `X` determines the wavenumbers to be checked for stability. Large wavenumbers are always stable, so when in doubt check for stability wavenumbers close to 0 (e.g. $k = O(10^{-3})$ up to $O(1)$). Default value is an array of 160 linearly spaced wavenumbers between `4e-3` and `1.5`.
- the parameter `Cutoff` is the largest wavenumber assumed to be in the support of the spectrum `P`. Default value is `9`, for heavy tailed spectra like JONSWAP values of `30` or larger can be used.

Outputs:

- `Xstar` the largest unstable wavenumber. Will return an empty array if all wavenumbers checked are stable. Due to the symmetry of the condition (if X is unstable, $-X$ is also unstable) the bandwidth of unstable wavenumbers is `2Xstar`.

- `fig1` (optional) the handle to a figure with the plots of the curves γ_X for the wavenumbers specified in the array `X`. Curves circumscribing the target, i.e., corresponding to unstable wavenumbers, appear red. Curves corresponding to stable waveumbers appear black. The target is shown as a large blue square.
- `S` (optional) returns the last curve γ_X that was generated.



We use the adaptive spectral implementation of the Hilbert transform from

`https://github.com/aathanas/HilbertTransform`

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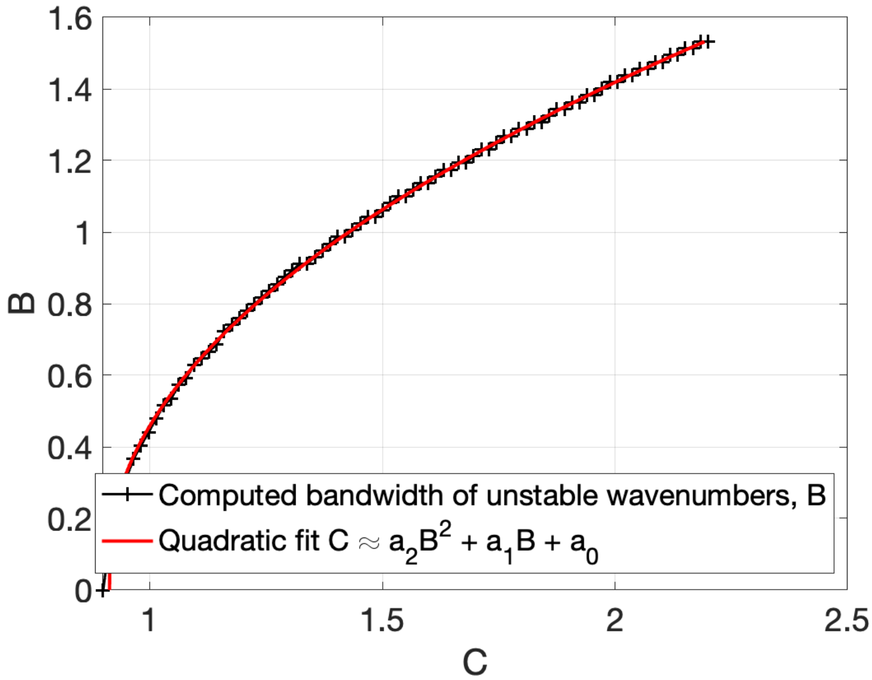
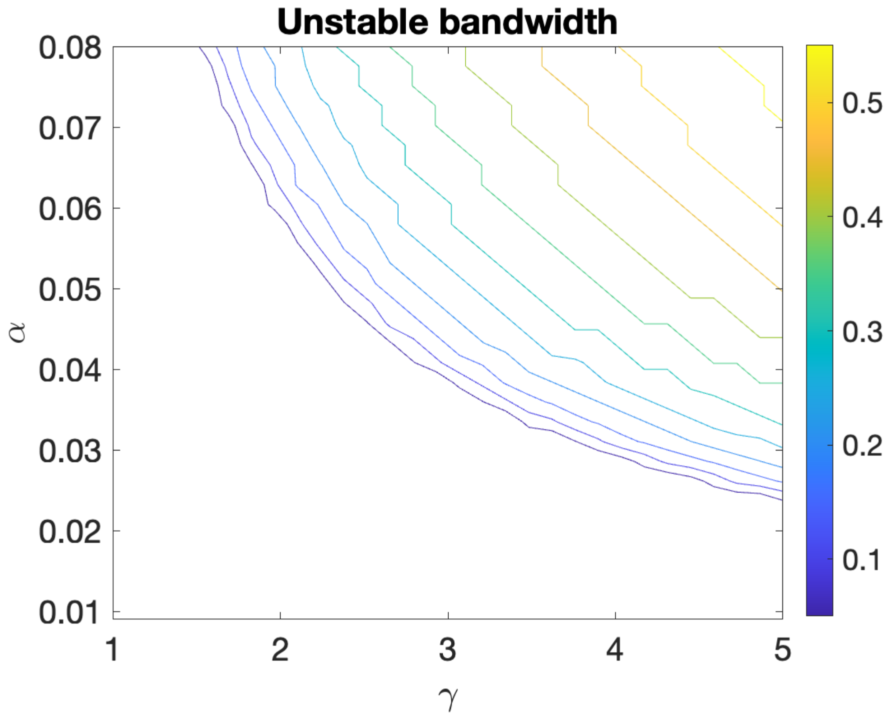
3 Demos

There are two demos using the function `CheckPenroseCondition.m` included:

- `demo_Gaussian` investigates first a single Gaussian spectrum for stability. Then it proceeds to investigate what happens as the intensity C of the spectrum grows. The bandwidth of unstable wavenumbers is found to grow like $\sqrt{C - C_*}$, where C_* is the threshold for the appearance of modulation instability.



- `demo_JONSWAP` investigates the stability of a single JONSWAP spectrum. Then it proceeds to investigate JONSWAP spectra with different values of the parameters α (for intensity) and γ (for peakedness) as in [2]. A $2D$ plot of bandwidth of unstable wavenumbers against α, γ is produced in the end. The non-dimensionalization of the JONSWAP spectrum used in the JONSWAP demo is described in [2, 3]. It amounts to a problem with nominal $k_0 = 1$ and target $1/(4\pi)$ (equivalent to $p/q = 1/(16\pi^2)$).
- Moreover, if `CheckPenroseCondition.m` is ran with no inputs, it will run a basic demo of stability analysis for a single Gaussian spectrum.



4 Computation on domains of finite length L

If the Alber equation (1) is to be periodized and solved numerically on a domain $(x, y) \in [-\frac{L}{2}, \frac{L}{2}]^2$, then the relevant wavenumbers are the discrete wavenumbers of that domain,

$$X_n = \frac{2\pi n}{L}, \qquad n \in \mathbb{N}.$$

In particular, if modulation instability is present, but the highest unstable wavenumber X_* is $X_* < \frac{2\pi}{L}$, then the problem becomes artificially stabilized by the short computational domain. As a rule of thumb, if modulation instability is present, L should be chosen large enough so that at $4\pi/L$ or even $6\pi/L$ is unstable.

References

[1] Alber, I.E., 1978. The effects of randomness on the stability of two-dimensional surface wavetrains. Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences, 363(1715), pp.525-546.

[2] Athanassoulis, A., Athanassoulis, G., Ptashnyk, M. and Sapsis, T., 2020. Strong solutions for the Alber equation and stability of unidirectional wave spectra. Kinetic and Related Models, 13(4), pp.703-737.

[3] Athanassoulis, A.G. and Gramstad, O., 2021. Modelling of ocean waves with the Alber equation: application to non-parametric spectra and generalisation to crossing seas. Fluids, 6(8), p.291.