Spring Tuning: Math Appendix

The mathematical derivations in this document are intended to supplement the appendix provided in *Playing with Tuning in bitKlavier*, published in the Computer Music Journal.

1 Spring System with One Note Fixed

1.1 Minimizing the Energy

We want to minimize the energy in the spring system, given by

$$E = \sum_{\substack{i,j \\ j>i}} \frac{1}{2}\omega(x_j - x_i - I_{ij})^2 \tag{1}$$

where x_i is the pitch of the ith note (in cents), I_{ij} is the desired interval between the ith and jth note, and ω is the spring strength or spring constant.

There are N notes, so the i and j indices range from 0 to N-1. To avoid double counting, we are only considering terms in the sum where j exceeds i. For simplicity, we are also assuming here that all the springs are of equal strength, i.e. $\omega_{ij} = \omega$.

To minimize the energy, we take the derivative of (1) with respect to each coordinate and set it equal to zero, giving a set of N equations. We can interpret these equations physically by noting that $-\frac{\partial E}{\partial x_k}$ is the force exerted by the spring system on each note, and so we are solving for the equilibrium condition where the force one each note is zero.

Taking the derivative,

$$\frac{\partial E}{\partial x_k} = \sum_{\substack{i,j \\ j>i}} \omega(x_j - x_i - I_{ij})(\delta_{jk} - \delta_{ik}) = 0$$

where we have used that $\frac{\partial x_i}{\partial x_k} = \delta_{ik}$, and δ_{ik} is the Kronecker Delta function which equals 1 when i=k, and 0 otherwise.

Simplifying further,

$$\begin{aligned}
0 &= \sum_{i<k}(x_k - x_i - I_{ik}) - \sum_{j>k}(x_j - x_k - I_{kj}) \\
&= \sum_{i<k}(x_k - x_i - I_{ik}) - \sum_{i>k}(x_i - x_k - I_{ki}) \\
&= (N-1)x_k - \sum_{i\neq k} x_i - \sum_{i<k} I_{ik} + \sum_{i>k} I_{ki} \\
&= Nx_k - \sum_i x_i + \phi_k \\
\implies x_k &= \frac{\sum_i x_i - \phi_k}{N} \qquad (2)
\end{aligned}$$

where ϕ_k are a set of constants defined in terms of the provided intervals:

$$\phi_k \equiv -\sum_{i<k} I_{ik} + \sum_{i>k} I_{ki} \qquad (3)$$

Summing (2) over the k index, we see that

$$\begin{aligned}
0 &= \sum_k (Nx_k - \sum_i x_i + \phi_k) \\
&= N\sum_i x_i - N\sum_i x_i + \sum_k \phi_k \\
\implies \sum_k \phi_k &= 0 \qquad (4)
\end{aligned}$$

Notice that since we have made no assumptions about the intervals, Eqn. (4) is always true, irrespective of the specific choice of intervals. This identity will be useful later.

1.2 Adding a Constraint

The set of equations in (2) are not uniquely solvable unless we add an additional constraint. This is because we have not yet set a reference pitch. We can do this by simply fixing the value of a single note (i.e. setting its pitch to a constant). Physically, this corresponds to a spring system where one mass is locked in place.

Say we fix x_0. Rearranging (2) for $k = 0$,

$$\sum_i x_i = N x_0 + \phi_0$$

Substituting this value of $\sum_i x_i$ in (2) gives

$$\boxed{x_k = x_0 + \frac{\phi_0 - \phi_k}{N}} \tag{5}$$

which is the solution that minimizes the energy. Here x_k is the pitch of each note, in cents, x_0 is the fixed referenced note, and ϕ_k are constants defined in (3) or (11) in terms of the provided intervals.

Some algebraic manipulation (equation 12) shows that

$$\phi_0 - \phi_k = \sum_{i=1}^{k} (I_i + I_{N-i})$$

Which allows us to re-express the solution as

$$x_k = x_0 + \frac{1}{N} \sum_{i=1}^{k} (I_i + I_{N-i})$$

If the intervals are chosen to be symmetric, then complementary intervals add to an octave or 1200 cents, i.e. $I_i + I_{N-i} = 1200$. In this case, the solution simplifies to $x_k = x_0 + \frac{1200k}{N}$, i.e. we recover equal temperament.

2 Spring System With Tethers to N Reference Notes

In this section, instead of locking a single note as a constraint, we add a set of 'tether springs' that connect each note to a corresponding reference pitch (e.g. the equal temperament pitch of each note). We index these reference pitches by x_i^{ET}.

So we have a new term in our energy equation for these tether springs.

$$E = \sum_{\substack{i,j\\ j>i}} \frac{1}{2}\omega(x_j - x_i - I_{ij})^2 + \sum_i \frac{1}{2}\epsilon(x_i - x_i^{ET})^2 \tag{6}$$

Once again, we assume that the interval springs all have the same strength ω, and the tether springs have the same strength ϵ. Typically we are working in the regime where $\omega >> \epsilon$.

Minimizing the energy (or setting forces to zero) as before,

$$\frac{\partial E}{\partial x_k} = \sum_{\substack{i,j\\ j>i}} \omega(x_j - x_i - I_{ij})(\delta_{jk} - \delta_{ik}) + \sum_i \epsilon(x_i - x_i^{ET})\delta_{ik} = 0$$

Simplifying,

$$0 = \omega\big(\sum_{i<k}(x_k - x_i - I_{ik}) - \sum_{j>k}(x_j - x_k - I_{kj})\big) + \epsilon(x_k - x_k^{ET})$$

Expanding out the sums and simplifying as in (2), we get

$$0 = (Nx_k - \sum_i x_i + \phi_k) + \frac{\epsilon}{\omega}(x_k - x_k^{ET}) \tag{7}$$

where ϕ_k is a known function of the intervals, defined earlier in (3).

To solve, we sum over k.

$$\begin{aligned} 0 &= N\sum_i x_i - N\sum_i x_i + \sum_k \phi_k + \frac{\epsilon}{\omega}(\sum_k x_k - \sum_k x_k^{ET}) \\ &= \sum_k \phi_k + \frac{\epsilon}{\omega}(\sum_k x_k - \sum_k x_k^{ET}) \end{aligned}$$

Recall that $\sum_k \phi_k = 0$ (equation 4). So, this reduces to

$$\sum_i x_i = \sum_i x_i^{ET}$$

i.e. the constraint ensures that, once the energy is minimized, the average value of the notes equals the average value of the reference notes. Substituting this into (7), we arrive at the solution

$$\boxed{x_k = x_k^* - \frac{\phi_k}{N + \frac{\epsilon}{\omega}}} \tag{8}$$

where

$$x_k^* = \frac{\sum_i x_i^{ET} + \frac{\epsilon}{\omega} x_k^{ET}}{N + \frac{\epsilon}{\omega}} \tag{9}$$

are a set of constants defined in terms of the reference notes x_i^{ET}. Notice that if the tether springs are not very weak, the resultant intervals will shrink due to the ϵ/ω term in the denominator.

In the limit of very weak tethers ($\epsilon << \omega$ or $\lim_{\frac{\epsilon}{\omega} \to 0}$), this reduces to

$$x_k = \overline{x^{ET}} - \frac{\phi_k}{N}$$

where $\overline{x^{ET}}$ is the average of the reference notes. If the intervals are symmetric, $\phi_k = \phi_0 - 1200k$ (see Section 4), so we recover equal temperament.

3 Spring System With Tethers to a Single Reference Note

Here, instead of tethering each note to a separate reference pitch, we tether them all to the same reference pitch x^*, following the approach of Stange et al. This is a special case of the previous configuration (6), with $x_i^{ET} = x^*$. Substituting this into (9) gives

$$x_k^* = \frac{\sum_i x^* + \frac{\epsilon}{\omega} x^*}{N + \frac{\epsilon}{\omega}} = x^*$$

yielding the solution

$$\boxed{x_k = x^* - \frac{\phi_k}{N + \frac{\epsilon}{\omega}}} \tag{10}$$

Once again we have an interval stretching if the tethers are not very weak.

In the limit of very weak tethers ($\epsilon << \omega$ or $\lim_{\frac{\epsilon}{\omega} \to 0}$), this reduces to

$$x_k = x^* - \frac{\phi_k}{N}$$

In this limit, we recover the same solution as the previous section. If the intervals are symmetric, $\phi_k = \phi_0 - 1200k$, and we recover equal temperament.

4 Interval Algebra

All of our solutions are expressed in terms of ϕ_k, a set of constants defined in terms of the provided intervals.

$$\phi_k = -\sum_{i<k} I_{ik} + \sum_{i>k} I_{ki}$$

We can simplify the above expression by noting that our intervals have the form $I_{ij} = I_{j-i}$, i.e. the intervals depend only on the difference of indices. Making this change,

$$\begin{aligned}\phi_k &= -\sum_{i=0}^{k-1} I_{k-i} + \sum_{i=k+1}^{N-1} I_{i-k} \\ &= -\sum_{j=1}^{k} I_j + \sum_{j=1}^{N-k-1} I_j \end{aligned} \tag{11}$$

where for clarity we have switched to indices $j = k - i$ for the first sum and $j = i - k$ for the second sum.

We can see from the above equation that for $k = 0$,

$$\phi_0 = \sum_{j=1}^{N-1} I_j$$

So, subtracting ϕ_0 from ϕ_k,

$$\begin{aligned}\phi_k - \phi_0 &= -\sum_{j=1}^{k} I_j + \sum_{j=1}^{N-k-1} I_j - \sum_{j=1}^{N-1} I_j \\ &= -\Big(\sum_{j=1}^{k} I_j + \sum_{j=N-k}^{N-1} I_j\Big) \\ &= -\sum_{j=1}^{k} (I_j + I_{N-j}) \end{aligned} \tag{12}$$

If the intervals are symmetric, $I_j + I_{N-j} = 1200$, so $\phi_k = \phi_0 - 1200k$.