Strategic Network Formation: Stability Vs Efficiency

Loyimee Gogoi

Amrut Mody School of Management
Ahmedabad University

12/12/2023

OUTLINE

GAME THEORY

- “Game: It’s defined as any situation involving interactions among multiple agents that are governed by a set of rules specifying the possible moves for each participant and a set of outcomes for each possible combination of moves."
- Conflict & cooperation situations: competition among firms, war and peace negotiation among countries, the conflict between management and labour, merging two companies: VI, TATAGroup/AirIndia etc.
- Types of games
  - Cooperative vs non-cooperative
  - Symmetric vs asymmetric
  - Zero-sum vs non-zero-sum
  - Static vs dynamic
- Representations of games
  - Strategic/normal form
  - Extensive form
  - Characteristic form

Cooperative Game

What is a Cooperative game?

- von Neumann and Morgenstern distinguish between two approaches to the theory of games: Cooperative and Non-cooperative.
- Classification of Harsanyi: In Cooperative games commitments are fully binding and enforceable.
- There is a set of agents/players $N = \{1, 2, 3, ..., n\}$.
- Each subset (or coalition) S of agents can work together in various ways, leading to various utilities for the agents.
- Key assumptions:
  - Rationality: Player always makes decision in pursuit of her own objective.
  - Intelligence: Players knows everything about the game.

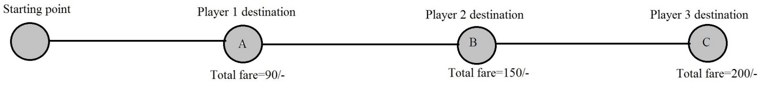
Cooperative Game Formally

- Let $N = \{1, ..., n\}$ be a finite set of players.
- $v : 2^N \to \mathbb{R}$, is the characteristic function from the set of all possible coalitions of players that satisfies $v(\emptyset) = 0$.
- A Cooperative game (transferable utility game) is characterized by two main factors:
  - the player set N and
  - the characteristic function $v : 2^N \to \mathbb{R}$.
- Let $\mathcal{G}(N)$ denote the universal game space consisting of all TU Cooperative games.
- It is easy to show that $\mathcal{G}(N)$ is a linear space of dimension $2^{|N|-1}$, with the set $\{u_T | \emptyset \neq T \subseteq N\}$ as the standard basis.

Solutions

- Key criteria:
  - Stability: No coalition of agents should want to deviate from the solution and go their own way.
  - Fairness: Agents should be rewarded for what they contribute to the group.
- Main assumption:-"the grand coalition $v(N)$ always forms".
- The challenge is to allocate the payoff among the players N in some fair way (or Stability ensured).
- A solution concept evaluates how much will be paid to a player for participating in a game.
- A solution concept for TU Cooperative game is a function that assigns a set of payoff vectors to each player in a Cooperative game.
- The core solution (Gillies, 1952), The Shapley value (Shapley, 1953), Banzhaf value (Banzhaf, 1965), Compromise value (Tijs,1993), Nucleolus (Schmeidler, 1969), Aumann-Shapley value (Aumann, 1995) and many more...

An Example of a Cooperative Game



- 3 friends $\{1, 2, 3\}$.
- Hire a cab.
- $v(S) =$ total cost (Cab fare)i.e., $v(1) = 90, v(2) = 150, v(3) = 200, v(1, 2) = 150, v(1, 3) = 200, v(2, 3) = 200, v(1, 2, 3) = 200$.
- How to divide the total fare $200/-$ among $\{1, 2, 3\}$?
- Fair division is
  - 90 is divided among 3 players.
  - $150 - 90 = 60$ is divided between 2 and 3.
  - Remaining 50 is paid by 3 only.
  - $(30, 60, 110)$.

Shapley Value

The Shapley value is the aggregation of the marginal contributions of a player in each coalition and is given by,

$$\Phi_i^{Sh}(v) = \sum_{S \subseteq N \,:\, i \in S} \frac{(s-1)!\,(n-s)!}{n!}[v(S) - v(S \backslash i)] \tag{1}$$

Theorem:
The Shapley value is the unique value satisfying the following axioms.

- *Efficiency* : $\sum_{i \in N} \Phi(v) = v(N)$
- *Linearity*: $\Phi_i(\alpha u + \beta v) = \alpha \Phi_i(u) + \beta \Phi_i(v)$
- *Null player property*: $\Phi_i(v) = 0$ for every null player $i \in N$
- *Anonymity* : $\Phi_i(v) = \Phi_{\pi i}(\pi v)$.
- *Monotonicity* : $v \preceq w \Rightarrow \Phi_i(v) \preceq \Phi_i(w)$.

The Shapley value for the previous example is

$$\Phi^{Sh}(v) = ?$$

The Core solution

The core of a game (N, v) denoted by $C(N, v)$ is defined by

$$C(N, v) = \{\mathbf{y} \in \mathbb{R}^n | v(N) \geq \mathbf{y}(N), v(S) \leq \mathbf{y}(S) \ \ \forall \ S \subseteq N\}$$

Each member of the core is a highly stable payoff distribution where no player has incentives to ask more than what she gets. A game $v \in G^N$ is said to be convex if

$$v(S_1 \cup S_2) + v(S_1 \cap S_2) \geq v(S_1) + v(S_2) \text{ for all } S_1, S_2 \in 2^N.$$

A weighted majority game:
each player $i \in N$ has a number of votes w,
a coalition of players needs a total of q votes (the quota) to obtain the surplus, which is normalized to equal 1.
The tuple $(N, q, (w_i)_{i \in N})$ is a weighted majority situation. The characteristic function of $(N, q, (w_i)_{i \in N})$ is defined by

$$v(S) = \begin{cases} 1 & \text{if} \sum_{i \in S} w_i > q \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

for all $S \subseteq N$.
The Banzhaf Value and its Characterizations...

The Egalitarian value: Definition and Characterizations

- Myerson (1977) introduced the concept of a Communication situation as a special type of Cooperative game where the players are restricted to cooperate only through existing links.
- A Communication situation is a triple (N, U, g), where (N, U) is a Cooperative game and (N, g) is an undirected network.
- $S/g = \{\{i | i \text{ and } j \text{ are connected in } S \text{ by } g\} | j \in S\}$.
- An allocation rule is a function $Y : G \times G^N \to \mathbb{R}^n$ such that for all $g \in G, S \in N/g$, $i \in S$ we have $\sum_{i \in N} Y_i(g, U) = U(S)$.

Let $N = \{1,2,3,4,5,6\}$ and $g = \{12,23,45,56,46\}$. Let $S = \{1,2,4\}$. Then $S/g = \{\{1,2\},\{4\}\}$ and $N/g = \{\{1,2,3\},\{4,5,6\}\}$. Fig 2 illustrates the structure of g.

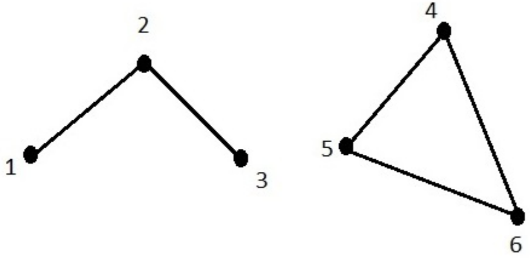


Fig 2: An Example of a Communication Situation

Given a characteristic function U and a network g, the associated Cooperative game U/g is defined for all $S \subseteq N$ as follows.

$$(U/g)(S) = \sum_{T \in S/g} U(T) \tag{3}$$

THEOREM

(Myerson, 1977, pg. 227) Given a characteristic function U, The Myerson value Y^{MV} is the unique fair allocation rule $Y : G \rightarrow \mathbb{R}^n$, which satisfies component balance and equal bargaining power. Moreover we have $Y(g, U) = \Phi(U/g), \forall g \in G$, where $\Phi(.)$ is the Shapley value operator.

- Given the player set $N = \{1, 2, ..., n\}$, let $G = \{g | g \subseteq g^N\}$ denote the set of all possible networks on N.
- The network obtained by adding two networks g and g' is denoted by $g + g'$ and the network obtained by deleting subnetwork g' from an existing network g is denoted $g \setminus g'$.
- For $g \in G$, $L(g)$ denotes the set of all links in g and $l(g)$, the total number of such links.
- $N(g) = \{i \mid \exists\ j \text{ such that } ij \in g\}$
- $L_i(g) = \{ij \mid \exists\ j\ :\ ij \in g\}g|_S = \{ij \mid ij \in g \text{ and } i \in S, j \in S\}$
- Let $n(g) = \#N(g)$ denote the number of players involved in g. We denote by $l_i(g)$ the number of links in player i's link set. It follows that $l(g) = \frac{1}{2} \sum_i l_i(g)$.

A component of a network g, is a non-empty subnetwork $g' \subset g$, such that

- (A) if $i \in N(g')$ and $j \in N(g')$ where $j \neq i$, then there exists a path in g' between i and j, and
- (B) if $i \in N(g')$ and $ij \in g$, then $ij \in g'$.

A value function is a function $v : G \to \mathbb{R}$ such that $v(\emptyset) = 0$, where $\emptyset$ represents the empty network i.e. network without links. The set of all possible value functions is denoted by V.

A value function $v \in V$ is component additive if

$$v(g) = \sum_{g' \in C(g)} v(g'), \quad \text{for any} \quad g \in G.$$

Given a network $g \in G$, each of the following special value functions is a basis for V.

$$v_g(g') = \begin{cases} 1 & \text{if } g \subseteq g' \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

$$\hat{v}_g(g') = \begin{cases} 1 & \text{if } g \subsetneq g' \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

and,

$$v^*_g(g') = \begin{cases} 1 & \text{if } g = g' \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

A Network game is a pair, (N, v), consisting of a set of players and a value function. If N is fixed and no confusion arises about this, we denote the Network game (N, v) by only v.
Given a Network game (N, v), the associated Cooperative game $U_{g,v}$ for all $S \subseteq N$ is defined by,

$$U_{g,v}(S) = \sum_{g' \in C(g|_S)} v(g')$$

An allocation rule is a function $Y : G \times V \to \mathbb{R}^n$ such that $Y_i(g, v)$ represents the payoff to player i with respect to v and g and $\sum_i Y_i(g, v) = v(g), \ \forall \ v$ and g.

An allocation rule Y is component balanced if for any component additive $v,\ g \in G$, and $g' \in C(g)$,

$$\sum_{i \in N(g')} Y_i(g, v) = v(g').$$

An allocation rule Y satisfies equal bargaining power if for any component additive v , $g \in G$ and $i, j \in N$, it holds that,

$$Y_i(g, v) - Y_i(g - ij, v) = Y_j(g, v) - Y_j(g - ij, v).$$

THEOREM

(Jackson, 2005, pg. 134) Y satisfies component balance and equal bargaining power if and only if $Y(g, v) = Y^{NMV}(g, v)$ for all $g \in G$ and any component additive v.

The Myerson value for Network games $Y^{NMV} : G \times V \to \mathbb{R}^n$ is obtained as follows.

$$Y_i^{NMV}(g, v) = \sum_{S \subset N \setminus \{i\}} \frac{s!(n-s-1)!}{n!} (v(g|_{S \cup i}) - v(g|_S))$$

EXAMPLE

Let $N = \{1, 2, 3\}$ and $g = \{12, 23, 13\}$. Let $v(g) = 3$, $v(12, 23) = v(13, 23) = 2$, $v(12, 13) = 4$, $v(12) = 2$, $v(23) = 1$, $v(13) = 0$. The Myerson value Y^{MV} is

The Position value $Y^{PV} : G \times V \to \mathbb{R}^n$ is formally given by,

$$Y_i^{PV}(g,v) = \sum_{\substack{i\neq j,\\ ij\in g}} \Big(\sum_{g'\subset N\setminus ij} \frac{1}{2}(v(g'+ij)-v(g'))\Big) \frac{l(g')!(l(g)-l(g')-1)!}{l(g)!}$$

Example

$N = \{1,2,3\}$, $g = \{12,23,13\}$, $v(g) = 3$, $v(12,23) = v(13,23) = 2$, $v(12,13) = 4$, $v(12) = 2$, $v(23) = 1$ and $v(13) = 0$. After simple calculations, the Position value is obtained as $Y^{PV}(g,v) = ?$.

Egalitarian Allocation Rule
Component-wise Egalitarian Allocation Rule
Banzhaf Allocation Rule
The Core in Network Settings...

A network g is pairwise stable with respect to allocation rule Y and value function v if

The first part of the definition of pairwise stability requires that no player wish to delete a link that he or she is involved in. Implicitly, any player has the discretion to unilaterally terminate relationships in which he or she is involved. The second part of the definition requires that if some link is not in the network and one of the involved players would benefit from adding it, then it must be that the other player would suffer from the addition of the link. Here it is implicit that the consent of both players is needed for adding a link.

A network g â G is obtainable from g â G via deviations by S if (i) i j â g and i j / â g implies i j â S, and (ii) i j â g and i j / â g implies i j â© S = â. This definition identifies changes in a network that can be made by a coalition S without the consent of any players outside of S. Part (i) requires that any new links that are added can only be between players in S. This requirement arises because the consent of both players is needed to add a link. Part (ii) requires that at least one player of any deleted link be in S. This is because either player in a link can unilaterally sever the relationship. A network g is strongly stable with respect to allocation rule Y and value function v if for any that is obtainable from g via deviations by S, and i â S such that
Strong stability provides a powerful refinement of pairwise stability. The concept of strong stability mainly makes sense in smaller network situations in which players have substantial information about the overall structure and potential payoffs and can coordinate their actions. Thus, for instance, strong stability might be more applicable to agreements between firms in an

A network $g \in G$ is called efficient network relative to a value function v if $v(g) \geq v(g'),\ \forall\ g' \in G$.