The Banzhaf Value for Network Games

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Cooperative Games

Definition

- There is a set of agents/players $N = \{1, 2, 3, ..., n\}$.
- Each subset (or coalition) S of agents can work together in various ways, leading to various utilities for the players — Makes Binding Agreement
- A cooperative game is a function $v : 2^N \rightarrow \mathbb{R}$ such that $v(\emptyset) = 0$.
- A value (or allocation rule) is a function $\Phi : \mathbb{R}^{2^N} \rightarrow \mathbb{R}^N$

The Shapley Value

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$$\Phi_i^{Sh}(v) = \frac{1}{n!} \sum_{\pi \in \Pi(N)} [v(P(\pi, i) \cup i) - v(P(\pi, i))]$$

$$\Phi_i^{Sh}(v) = \sum_{S \subseteq N \setminus i} \frac{s!(n-s-1)!}{n!} [v(S \cup i) - v(S)]$$

The Banzhaf value of the game (N, v) was introduced by Owen (1975) based on the Banzhaf power index in voting games (Banzhaf (1965)) and is defined as

$$\phi_i^B(v) = \frac{1}{2^{n-1}} \sum_{S \subseteq N-i} \left[v(S \cup i) - v(S) \right]. \tag{1}$$

- Denote a *network value function* by $v\colon \mathbb{G}^N \to \mathbb{R}$ such that $v(g_0) = 0$.
- A value function $v\colon \mathbb{G}^N \to \mathbb{R}$ is component additive if

$$v(g) = \sum_{h \in C(g)} v(h).$$

- The set of all network value functions on N to be defined as

$$\mathbb{V}^N = \{v \mid v\colon \mathbb{G}^N \to \mathbb{R} \text{ such that } v(g_0) = 0\}. \tag{2}$$

- It is clear that $\mathbb{V}^N$ is a $\left(2^{\frac{1}{2}n(n-1)} - 1\right)$ dimensional Euclidean space.
- The set of all component additive value functions is denoted by $\mathbb{V}^N_{ca}$.

- An *allocation rule* on the class of network situations is a mapping $Y : \mathbb{G}^N \times \mathbb{V}^N \to \mathbb{R}^N$ that for every network value function $v \in \mathbb{V}^N$ assigns to every player $i \in N$ in a network $g \in \mathbb{G}^N$ an allocated value $Y_i(g, v)$ such that $Y_i(g, v) = 0$ for every isolated player $i \in N_0(g)$.
- An allocation rule $Y : \mathbb{G}^N \times \mathbb{V}^N \to \mathbb{R}^N$ is *balanced* if for every network value function $v \in \mathbb{V}^N$ and every network $g \in \mathbb{G}^N$: $\sum_{i \in N} Y_i(g, v) = v(g)$.
- An allocation rule $Y : \mathbb{G}^N \times \mathbb{V}^N \to \mathbb{R}^N$ is *component balanced* if for every network value function $v \in \mathbb{V}^N_{ca}$, every network $g \in \mathbb{G}^N$ and all of its components $h \in C(g)$: $\sum_{i \in N(h)} Y_i(g, v) = v(h)$.

- An allocation rule $Y : \mathbb{G}^N \times \mathbb{V}^N \to \mathbb{R}^N$ satisfies *equal bargaining power* if $Y_i(g, v) - Y_i(g - ij, v) = Y_j(g, v) - Y_j(g - ij, v)$ for all $ij \in g$.
- An allocation rule $Y : \mathbb{G}^N \times \mathbb{V}^N \to \mathbb{R}^N$ is component decomposable if for every network value function $v \in \mathbb{V}^N_{ca}$, $Y_i(g, v) = Y_i(h, v)$ where $i \in N(h), h \in C(g)$.
- An allocation rule $Y : \mathbb{G}^N \times \mathbb{V}^N \to \mathbb{R}^N$ satisfies *balanced contributions property* if $Y_i(g, v) - Y_i(g - L_j(g), v) = Y_j(g, v) - Y_j(g - L_i(g), v)$ for all $i, j \in N$.

$$Y_i^{NBV}(g,v) = \frac{1}{2^{n-1}} \sum_{S \subseteq N - i} \left[v(g|S+i) - v(g|S) \right]. \qquad (3)$$

If $v \in \mathbb{V}_{ca}^N$,

$$v(g|S) = \sum_{h \in C(g|S)} v(h)$$

for all $S \subseteq N$.

Lemma

The network Banzhaf value is component decomposable.

An allocation rule $Y : \mathbb{G}^N \times \mathbb{V}^N \to \mathbb{R}^N$ satisfies component total power if and only if for all $v \in \mathbb{V}_{ca}^N$ and $h \in C(g)$, it holds that

$$\sum_{i \in N(h)} Y_i(g, v) = \frac{1}{2^{n(h)-1}} \sum_{iN(h)} \sum_{S \subseteq N(h)-i} \left[v(h|S+i) - v(h|S) \right].$$

Proposition

The allocation rule for network situations $Y : \mathbb{G}^N \times \mathbb{V}^N \to \mathbb{R}^N$ *satisfies component total power and equal bargaining power if and only if* $Y(g, v) = Y^{NBV}(g, v)$ *for all* $g \in \mathbb{G}^N$ *and* $v \in \mathbb{V}^N_{ca}$.

Proposition

The allocation rule for network situations $Y : \mathbb{G}^N \times \mathbb{V}^N \to \mathbb{R}^N$ *satisfies component total power and the balanced contributions property if and only if* $Y(g, v) = Y^{NBV}(g, v)$ *for all* $g \in \mathbb{G}^N$ *and* $v \in \mathbb{V}^N_{ca}$.

- An allocation rule $Y : \mathbb{U} \times \mathbb{G}^N \times \mathbb{V}^N \to \mathbb{R}^N$ satisfies the pairwise merging property if for all $g \in \mathbb{G}^N$, $v \in \mathbb{V}^N_{ca}$ and $i, j \in N$, it holds that

$$Y_i(N, g, v) + Y_j(N, g, v) = Y_p(N^{ij}, g^{ij}, v^{ij})$$

where

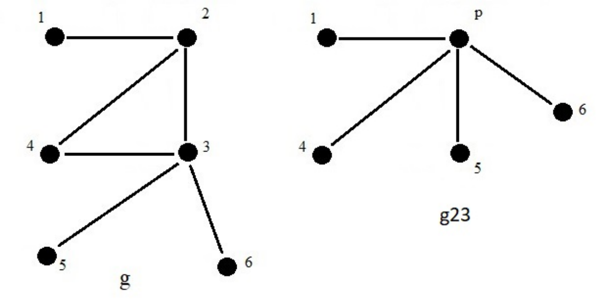
  - $N^{ij} = (N \backslash \{i, j\}) + p$,
  - $g^{ij} \in \mathbb{G}^{N^{ij}}$ is such that for all $l, k \in N^{ij}$, $lk \in g^{ij}$iff
    - either $lk \in g$ with $l, k \in N \backslash \{i, j\}$;
    - or $l = p$ and either $ik \in g$ or $jk \in g$;
    - or $k = p$ and either $li \in g$ or $lj \in g$;
  - $v^{ij} \in \mathbb{V}^{N^{ij}}$ is such that for each $g' \subseteq g^{ij} - L_p(g^{ij})$ and $g'' \subseteq L_p(g^{ij})$ and $N' = N(g'') - p$, $v^{ij}(g' + g'') =$

$$\begin{cases} v\left(g' + \left(L_i(g) + L_j(g)\right)\middle|\, N' \cup \{i, j\}\right) & \begin{array}{l}\text{if } g'' \neq \emptyset \\ \text{and either } g' \neq g^{ij} \\ \text{or } g'' \neq L_p(g^{ij});\end{array} \\ v(g') & \text{if } g'' = \emptyset; \\ v(g) + v(ij) & \text{if } g' = g^{ij} - L_p(g^{ij}) \end{cases} \quad (4)$$

Theorem

The allocation rule for network situations $Y : \mathbb{U} \times \mathbb{G}^N \times \mathbb{V}^N \to \mathbb{R}^N$ satisfies pairwise merging property and equal bargaining power (or the balanced contributions property) if and only if $Y(N, g, v) = Y^{NBV}(N, g, v)$ for all $g \in \mathbb{G}^N$ and $v \in \mathbb{V}^N_{ca}$.

Example



Consider the players' set $N = \{1, 2, 3, 4, 5, 6\}$ and let $\{12, 24, 34, 23, 35, 36\}$ be the network g on N. We merge the players 2 and 3. After deleting the link 23 replacing both players $2$and 3 by p, we have the network $g^{\{23\}}$ with the links' set $\{1p, 4p, 5p, 6p\}$.

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THANK YOU