# Strategic Network Formation: Stability Vs Efficiency

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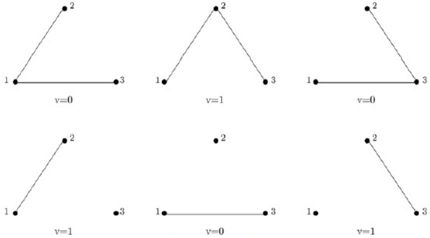
# Outline

Communication situation and the Myerson value
Network games and its allocation rules

- the Myerson value
- the Position value
- the Egalitarian value
- the Link-based Egalitarian value
- Component-wise Egalitarian value
- the Network Banzhaf value
- the Link-based Banzhaf value
- the Multilateral Interactive value
- the Core Solution

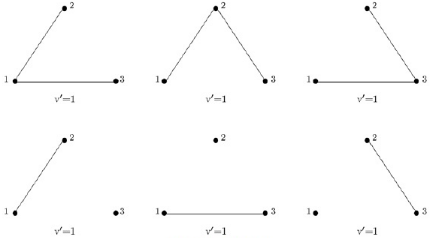
# Flexible Network: Criticism of the Myerson Value






Myerson value under $v$

*Consider another value function $v'$ defined by $v'(g) = 1$ for all $g \neq \emptyset$. That is, under $v'$ the value of every nonempty network is 1.*




Myerson value under $v'$

$$Y^{MV}(\{12,23\},v) = Y^{MV}(\{12,23\},v') = (\tfrac{1}{6},\tfrac{2}{3},\tfrac{1}{6}).$$

Given a value function $v$, its monotonic cover $\hat{v}$ is defined as

$$\hat{v}(g) = \max_{g' \subset g} v(g')$$

Let three players $1$, $2$ and $3$ forming the complete network $g = \{12, 23, 31\}$.
$v(\{12, 23, 31\}) = 2$, $v(\{12, 23\}) = 4$ and
$v(\{13, 23\}) = 1 = v(\{12, 13\})$.
$\hat{v}(\{12, 23, 31\}) = 4.$

An allocation rule $Y$ satisfies equal treatment of vital players, weak additivity and is a flexible network rule iff it is defined by

$$Y_i^{PBFN}(g,v) = \frac{v(g)}{\hat{v}(g^N)} \sum_{S \subseteq N \setminus \{i\}} (\hat{v}(g^{S \cup \{i\}}) - \hat{v}(g^S))(\frac{s!(n-s-1)!}{n!})$$

for all $v \in V$ and each $g \in G$ that is efficient relative to $v$.

Let $N = \{1, 2, 3\}$ and $g^N = \{12, 23, 13\}$. The value functions $v$ and $v'$ are defined as follows.

- $v(12) = v(23) = 1$, $v(12, 23) = w > 1$, and $v(g) = 0$ for all other networks;
- $v'(g) = w$ for all $g$ with at least two links and $v'(g) = 1$ on $g$ with one link.

$$Y^{PBFN}(\{12, 23\}, v) = (\frac{w}{3} - \frac{1}{6}, \frac{w}{3} + \frac{1}{3}, \frac{w}{3} - \frac{1}{6})$$

$$Y^{PBFN}(\{12, 23\}, v') = (\frac{w}{3}, \frac{w}{3}, \frac{w}{3})$$

Q. How do such network structures form and what their characteristics are likely to be?
Q. When do the private incentives of individuals to form ties with one another lead to network structures that maximize some appropriate a measure of efficiency?

# DEFINING EFFICIENCY

- A network g is *efficient* relative to v if $v(g) \geq v(g'), \forall g' \in G$.
- A network g is *Pareto efficient* relative to v and Y if no $g' \in G$ exists such that $Y_i(g', v) \geq Y_i(g, v), \forall i$.
- A network $g$ is *constrained efficient* relative to v if there does not exist any $g' \in G$ and a component balanced and anonymous Y such that $Y_i(g', v) \geq Y_i(g, v)$ for all $i \in N$.

The relationship between the three definitions of efficiency we consider here is as follows.
Let $PE(v, Y)$ denote the Pareto efficient networks relative to v and Y, and similarly let $CE(v)$ and $E(v)$ denote the constrained efficient and efficient networks relative to v, respectively.
Remark: If Y is component balanced and anonymous, then $E(v) \subset CE(v) \subset PE(v, Y)$.

An allocation Y is anonymous if $Y_{\pi(i)}(g^{\pi}, v^{\pi}) = Y_i(g, v), \forall \pi$, where $v^{\pi}$ is defined as $v^{\pi}(g^{\pi}) = v(g)$.
Anonymity states that if all that has changed is the names of the agents, then the allocations they receive should not change

# MODELLING: NETWORK FORMATION

A network g is pairwise stable with respect to an allocation rule Y and v if

- for all $ij \in g$, $Y_i(g,v) \geq Y_i(g-ij,v)$ and $Y_j(g,v) \geq Y_j(g-ij,v)$
- for all $ij \notin g$, $Y_i(g+ij,v) > Y_i(g,v)$ and $Y_j(g+ij,v) < Y_j(g,v)$

No player wishes to delete a link that he or she is involved in.
If some link is not in the network and one of the involved players would benefit from adding it, then it must be that the other player would suffer from the addition of the link.

# Existence of pairwise Stable networks

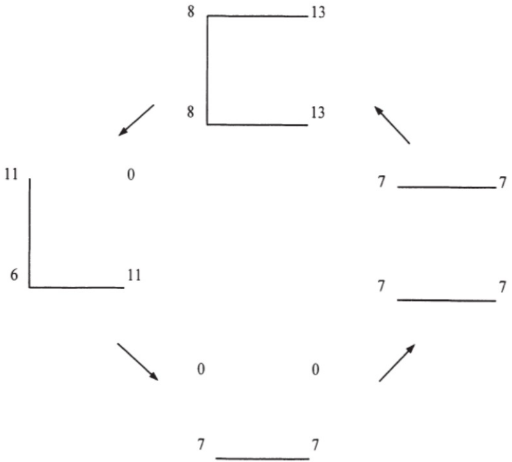



Fig. 4.

A cycle in this example is $\{12, 34\}$ is defeated by $\{12, 23, 34\}$ which is defeated by $\{12, 23\}$ which is defeated by $\{12\}$ which is defeated by $\{12, 34\}$.

# Existence of pairwise Stable networks



- Pairwise stable networks may not exist in some settings for some allocation rules, there are interesting allocation rules for which pairwise stable networks always exist.
- Under the Myerson value allocation rule there always exists a pairwise stable network.
- Under the egalitarian rule, any efficient network will be pairwise stable.
- Under the component-wise egalitarian rule, one can also always find a pairwise stable network.

# The Compatibility of Efficiency and Stability



Theorem

There is no allocation rule Y which is anonymous and component balanced such that for each v at least one efficient network is pairwise stable.

- Def1: A link i j is *critical link* to the graph g if $g \setminus ij$ has more components than g.
- Def2: Let h denote a component that contains a critical link, and let $h_1$ and $h_2$ denote the components obtained from h by severing that link. The pair (g, v) satisfies *critical link monotonicity* if, for any critical link in g and its associated components h, $h_1$ and $h_2$, we have that $v(h) \leq v(h_1) + v(h_2)$ implies that $v(h)/n(h) \leq max[v(h1)/n(h_1), v(h2)/n(h_2)]$.
- Theorem: If g is efficient relative to a component additive v, then g is pairwise stable for $Y^{ce}$ relative to v if and only if (g, v) satisfies critical-link monotonicity.

# Several aspects of pairwise stability deserve discussion



- First, it is a very weak notion in that it only considers deviations on a single link at a time.
- An individual would not benefit from severing any single link but would benefit from severing several links simultaneously, yet the network would still be pairwise stable.
- Some groups of individuals could all be made better off by some more complicated reorganization of their links, which is not accounted for under pairwise stability.
- It satisfies a very narrow class of allocation rules.

# Alternative approach of network formation: Strongly Stable



- A network $g' \in G$ is obtainable from $g \in G$ via deviations by $S$ if
  - $ij \in g'$ and $ij \notin g$ implies $\{i,j\} \subset S$ and
  - $ij \in g$ and $ij \notin g'$ implies $\{i,j\} \cap S = \emptyset$.
- A network $g$ is strongly stable with respect to allocation rule Y and value function $v$ if for any $S \subseteq N, g'$ that is obtainable from $g$ via deviations by $S$, and $i \in S$ such that $Y_i(g', v) > Y_i(g, v)$, there exists $j \in S$ such that $Y_j(g', v) < Y_j(g, v)$.
- Strong stability of networks is a very demanding property, as it means that no set of players could benefit through any rearranging of the links that they are involved with (including those linking them to players outside the coalition).
- It implies pair-wise stability.

# Alternative approach of network formation: Strongly Stable



- Theorem 1. Consider any anonymous and component additive value function $v \in V$. If Y is an anonymous, component decomposable, and component balanced allocation rule and $g \in G$ with $\Pi(g) \neq N$ is a network that is strongly stable with respect to Y and v, then $Y(g, v) = Y^{ce}(g, v)$ and $Y_i(g, v) = v(g)/n$ for each $i \in N$.
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A network $g$ induces a partition $\Pi(g)$ of the player set $N$, where two players are in the same portion iff there is a path between them.
$Y^{ce}$ is the component-wise egalitarian rule

# Alternative approach of network formation: Stongly Stablity

Theorem 2. Consider any anonymous and component additive value function $v \in V$. Some efficient $g \in G$ with respect to v is strongly stable with respect to $Y^{ce}(g, v)$ if and only if the core of $w^v$ is nonempty.
Where the cooperative game $(N, w^v)$ is defined by

$$w^v(S) = max_{g \in g^S} v(g)$$

# Alternative approach of network formation: Non-cooperative approach



- One is to explicitly model a game by which links form and then solve for an equilibrium of that game.
- Aumann and Myerson (1988) take such an approach in the context of communication games, where individuals sequentially propose links which are then accepted or rejected.
- Pairwise stability can be thought of as a condition that identifies networks that are the only ones that could emerge at the end of any well-defined game where players where the process does not artificially end, but only ends when no player(s) wish to make further changes to the network.

# Alternative approach of network formation



- Dutta and Mutuswami (1997) analyze the equilibria of a link formation game under various solution concepts and outline the relationship between pairwise stability and equilibria of that game.
- Individuals simultaneously announce all the links they wish to be involved in. Links form if both individuals involved have announced that link.
- While such games have a multiplicity of unappealing Nash equilibria (e.g., nobody announces any links), using strong equilibrium and coalition-proof Nash equilibrium, and variations on strong equilibrium where only pairs of individuals might deviate, lead to nicer classes of equilibria.
- The networks arising in variations of the strong equilibrium are in fact subsets of the pairwise stable networks.

# Open Question



Finally, there is another aspect of network formation that deserves attention. The above definitions (including some of the game theoretic approaches) are both static and myopic. Individuals do not forecast how others might react to their actions. For instance, the adding or severing of one link might lead to the subsequent addition or severing of another link.

Dynamic (but still myopic) network formation processes are studied by Watts (2001) and Jackson and Watts (1998), but a fully dynamic and forward-looking analysis of network formation is still missing.

# Some important Open Questions



- To develop a fuller understanding of how the specifics of the process matter, and tie this to different sorts of applications to get some sense of what modelling techniques fit different sorts of problems.
- All models are theoretical in nature. Many of them provide very pointed predictions regarding various aspects of network formation. Some of these predictions can be tested both in experiments, as well as being brought directly to the data.