Dynamical processes on complex networks

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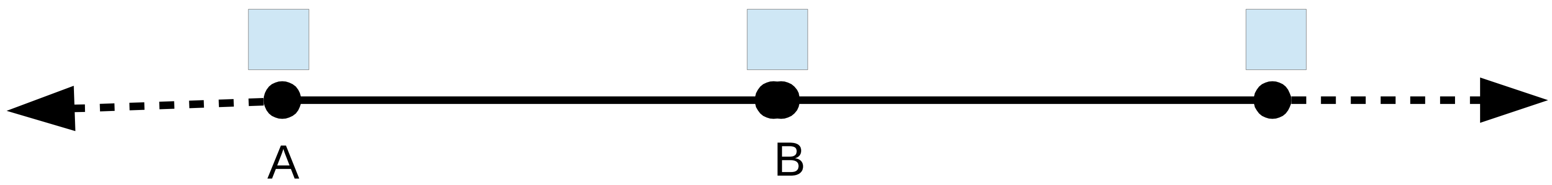
Dynamics on networks



Transport on networks :

Traffic flow,
Epidemic spreading,
IP packet flow on internet,
Animal migration & Foraging,
Search problems on networks,
Spread of misinformation,
Power distribution network,
And many others.

Blackout 2012 : Extreme event of sorts

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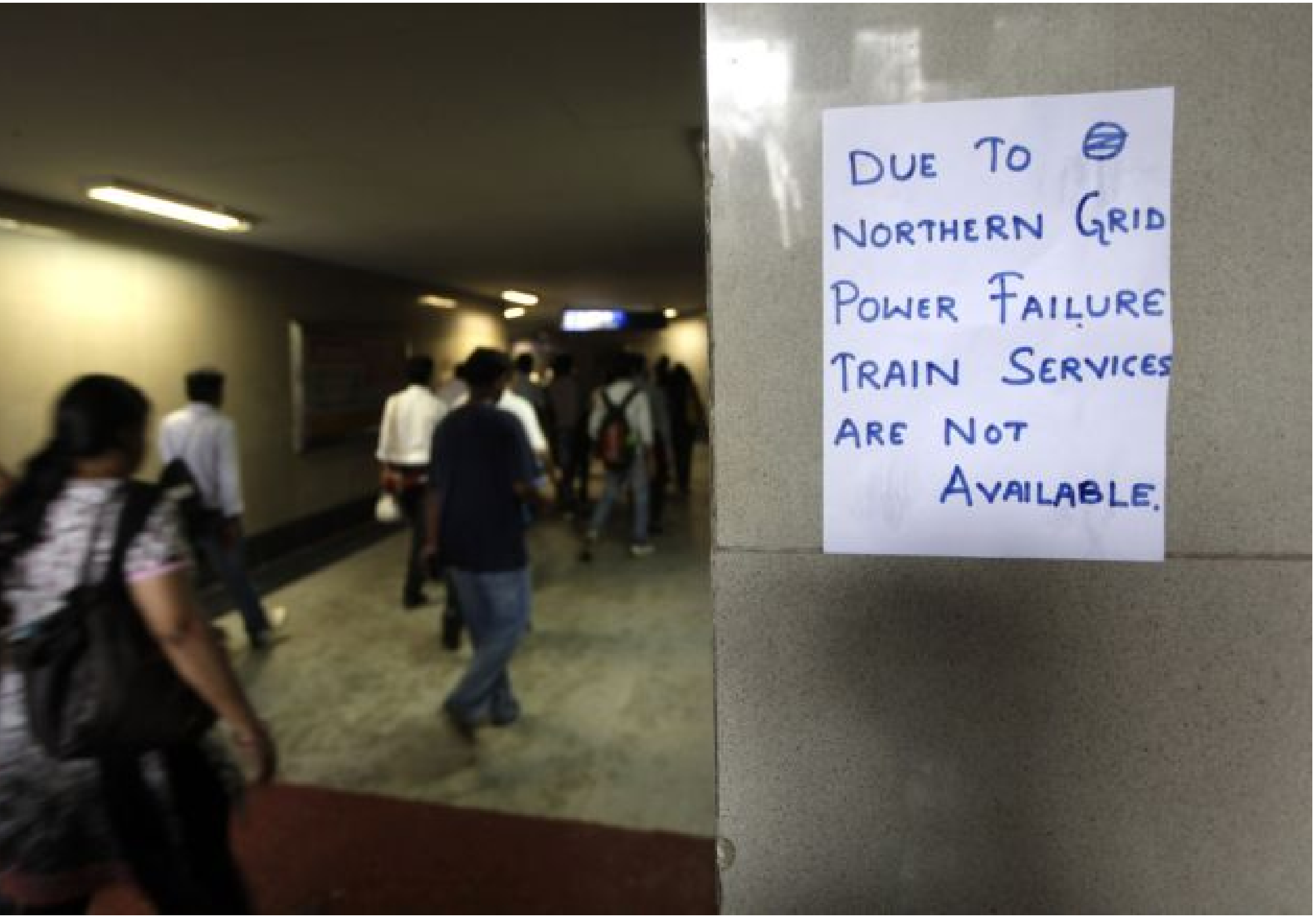
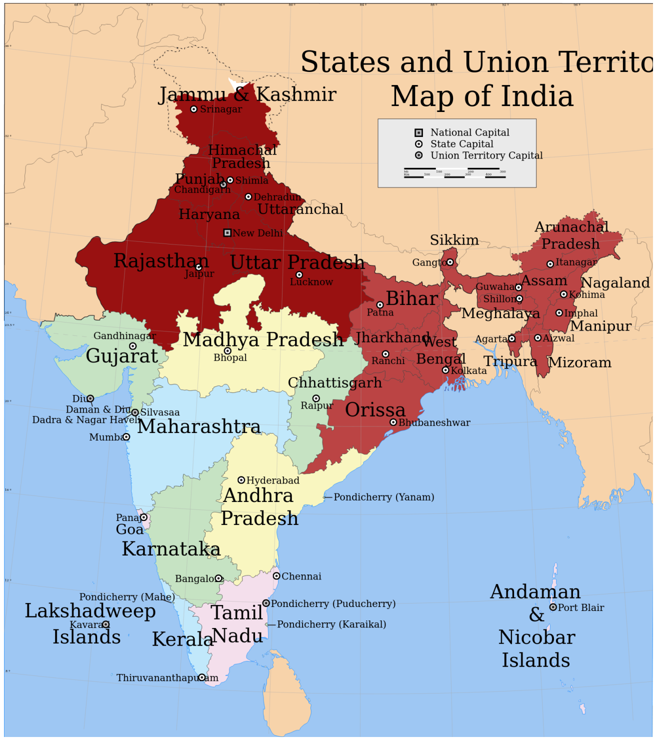
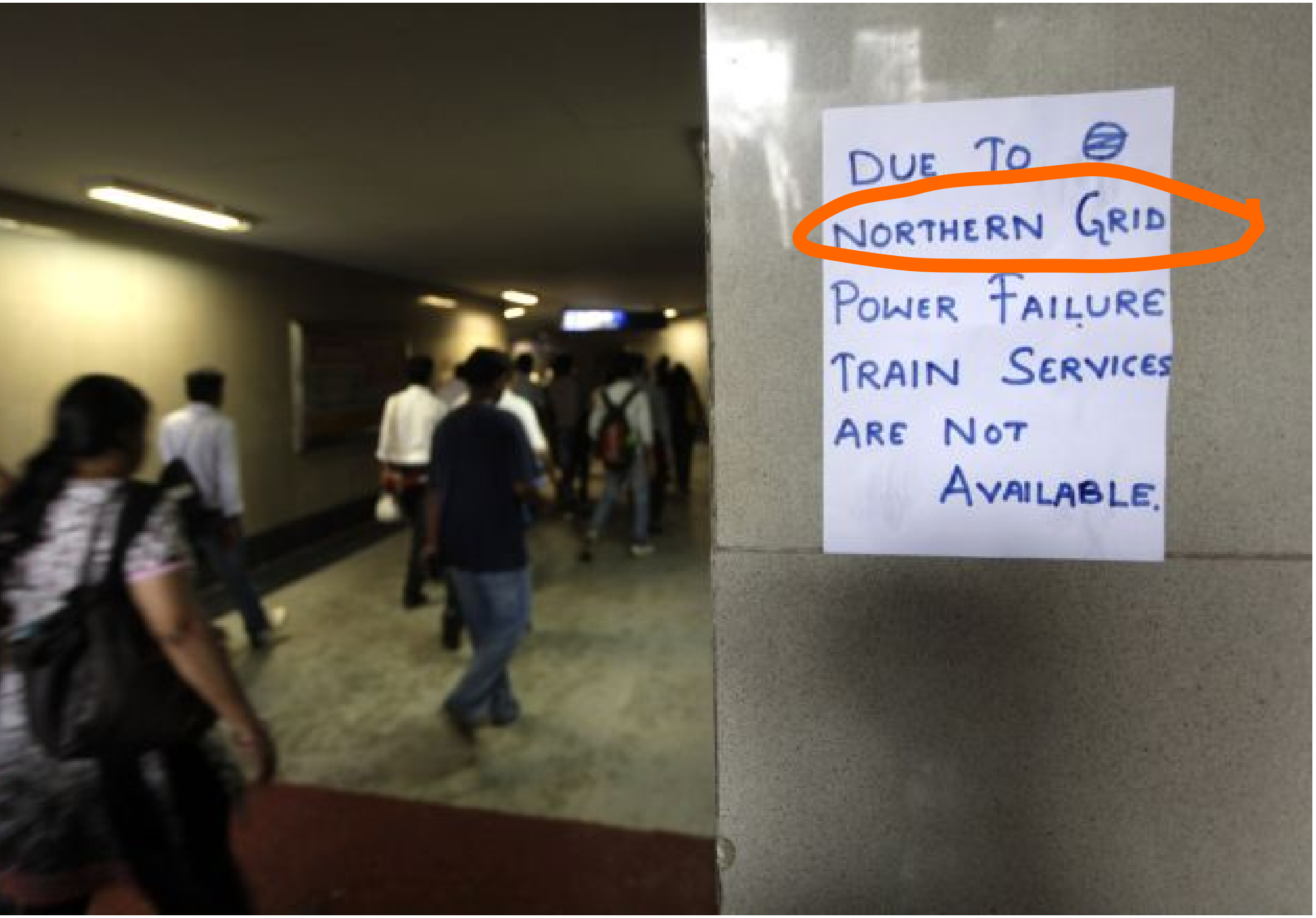


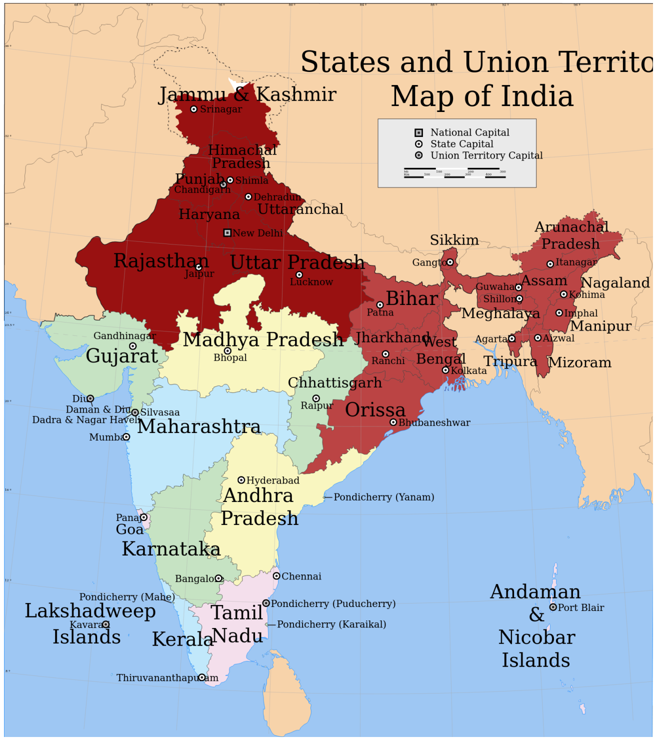


Source : Yahoo News

Lasted for more than two days (July 31- Aug 1, 2012). Affected 600 million people.

Blackout 2012 : Extreme event of sorts





Source : Yahoo News

Lasted for more than two days (July 31- Aug 1, 2012). Affected 600 million people due to GRID FAILURE.

Times of India 14.8.2008

THE TIMES OF INDIA, PUNE
THURSDAY, AUGUST 14, 2008

PIPELINE WORK CHOKES ROADS

Traffic Snarls At Shimla Office Chowk Have A Cascading Effect Across The City

Manish Umbrajkar
& Nikhil Deshmukh | TNN

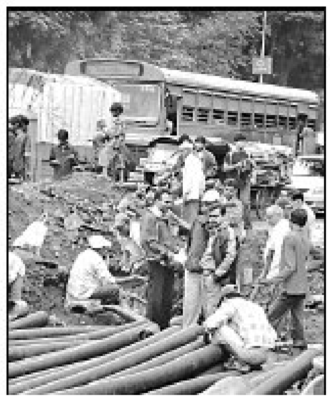
Pune: Traffic congestions are not new to the city but Wednesday proved to be a nightmare even for the seasoned sufferers. Thousands of commuters were stranded on several arterial roads because of PMC's day-long work of laying the water pipeline at Shimla chowk in Shivajinagar.

Closure of one side of the road, which goes towards Shivajinagar railway station, had a cascading effect on the movement of vehicular traffic on several other roads, such as the entire University road, Fergusson College road and the Mumbai-Pune highway stretch, near the College of Engineering Pune.

Traffic on most roads in Shivajingar area was either diverted or faced movement restrictions. This resulted in long queues of vehicles, waiting endlessly at traffic signals at Sancheti chowk, Engineering College chowk, S.G. Barve chowk and below the flyover near Agriculture College chowk.

Although the traffic police tried their best to regulate traffic, chaos reigned supreme. The gravity of the situation was such that deputy commissioner of police (traffic) Manoj Patil himself spent about half-an-hour in the afternoon at the Shimla office junction, giving instructions to his officers and staff manning the traffic.

Speaking to TOI at the junction, Patil said that completion of the work was necessary and there was bound to be some inconvenience. "Works are going on all over the city for the Commonwealth Youth Games and there is bound to be some inconvenience," he said. Asked if the Pune Municipal Corporation had given prior intimation about laying of the pipeline at Shiva-



HOW IT ALL GOT CLOGGED

- Laying of new water pipeline work at Shimla junction was started at about 11 pm on Tuesday
- As a result of this work, traffic at Shimla office chowk moving towards the Shivajinagar railway station was closed
- Citizens were caught unawares by the closure and vehicles started piling up in the Shivajinagar area from 8 am onwards on Wednesday
- Adding to this chaos, was an overturned dumper and an old pipeline that had burst

chowk and below the flyover near Agriculture College chowk

- By 4 pm, traffic snarls went out of control and spread to the nearby roads like University road, Fergusson College road, Jangli Maharaj road, Shivaji road and the highway stretch near CoEP
- By 6 pm, the office crowd returning home added to the growing numbers stranded on the road with nothing to do but wait for roads to clear
- The contractor said that the situation was likely to ease by 8 pm

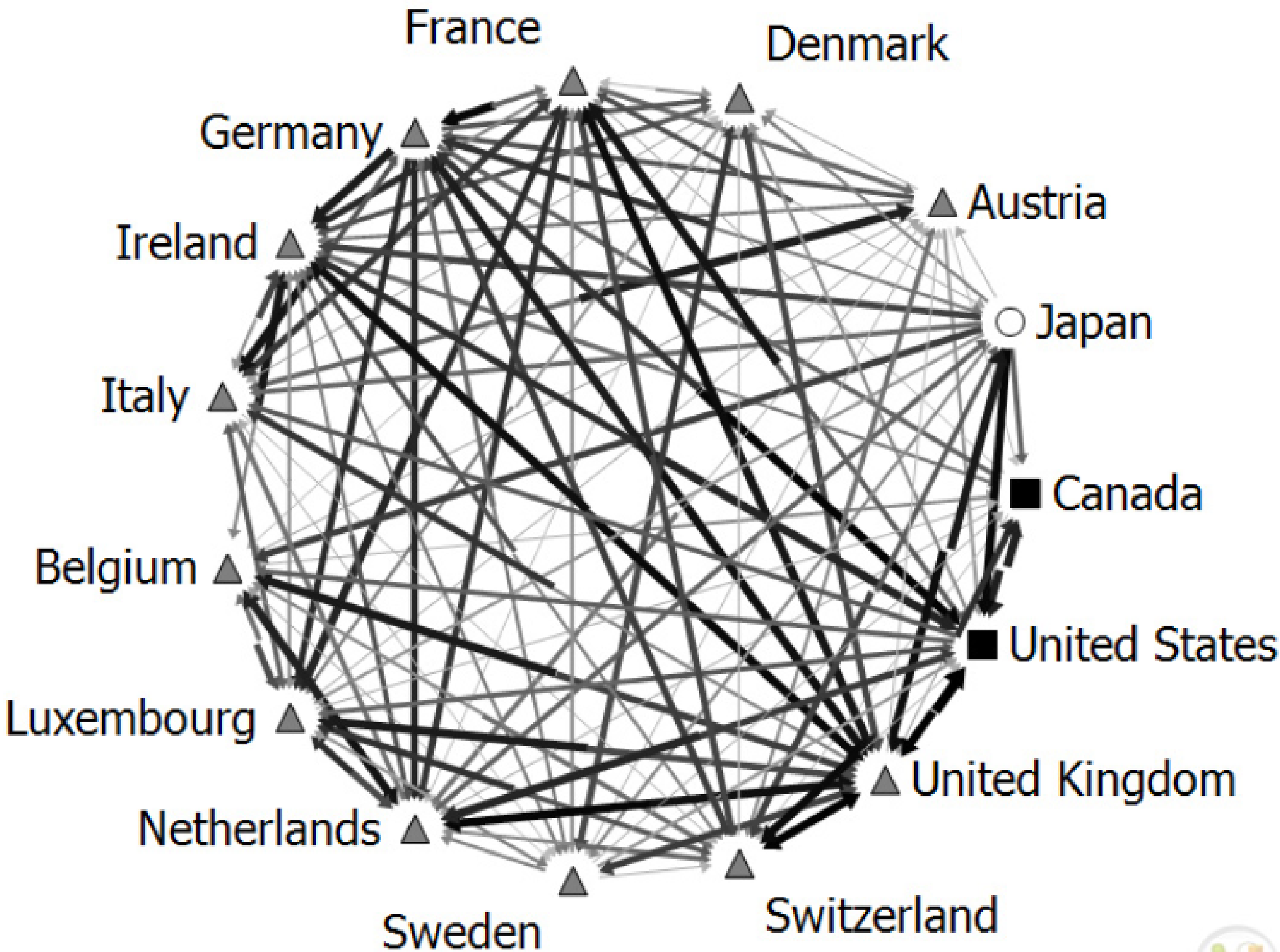
Big snarl has cascading effect

15/6/2010 | Author : DNA Correspondent

Vehicular traffic in the city was disrupted following the heavy rain on Monday evening. The snarl at the Regional Transport Office (RTO) Chowk had a cascading effect and caused jams on Ambedkar Road, at Engineering College Chowk, Sancheti Hospital Chowk, SG Barve Chowk, on Jungli

DNA **15.6.2010**

International financial network in 2007



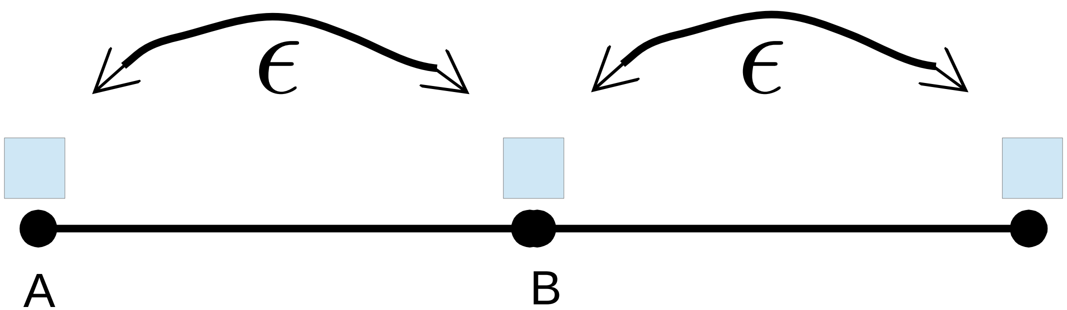
How to construct graphs where there is none ?

(IMF Working paper 2010)

Synchronisation on networks

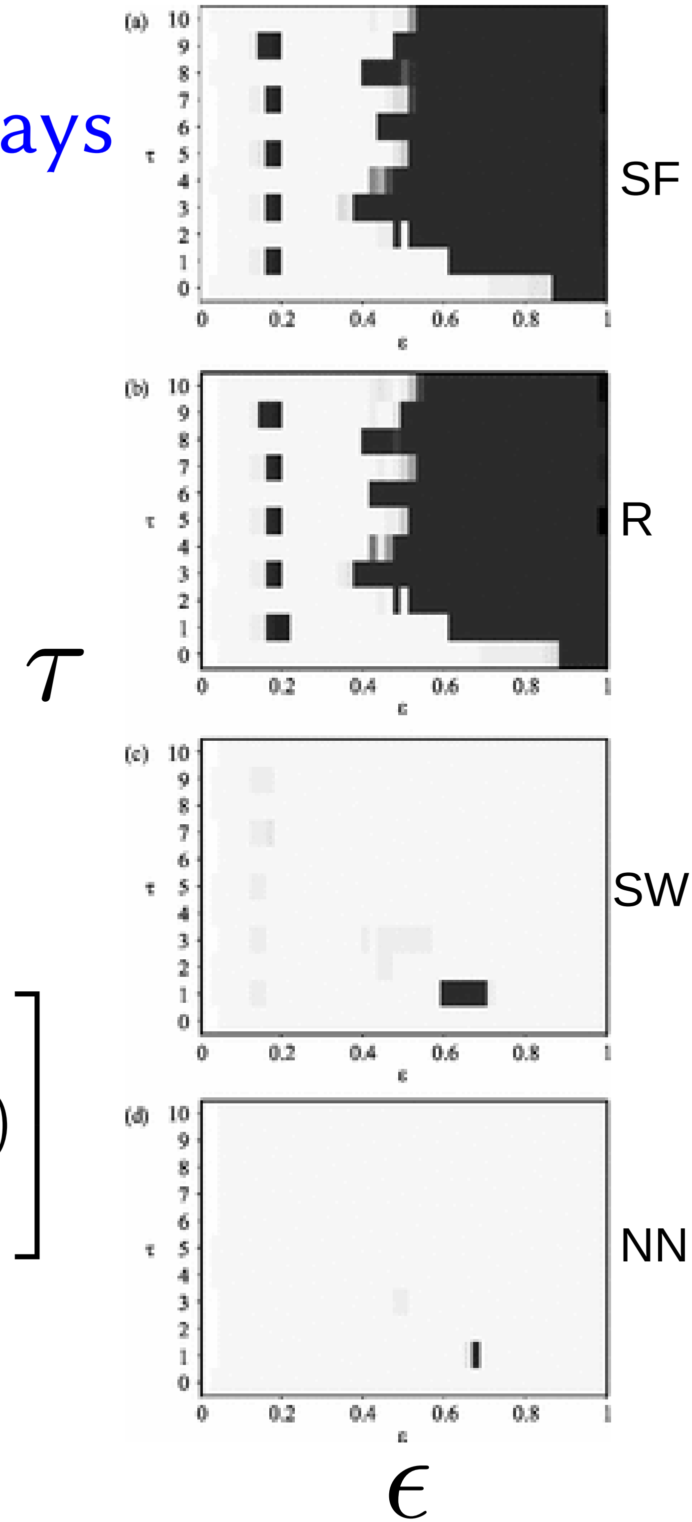
Source : BBC

Synchronisation on networks with delays

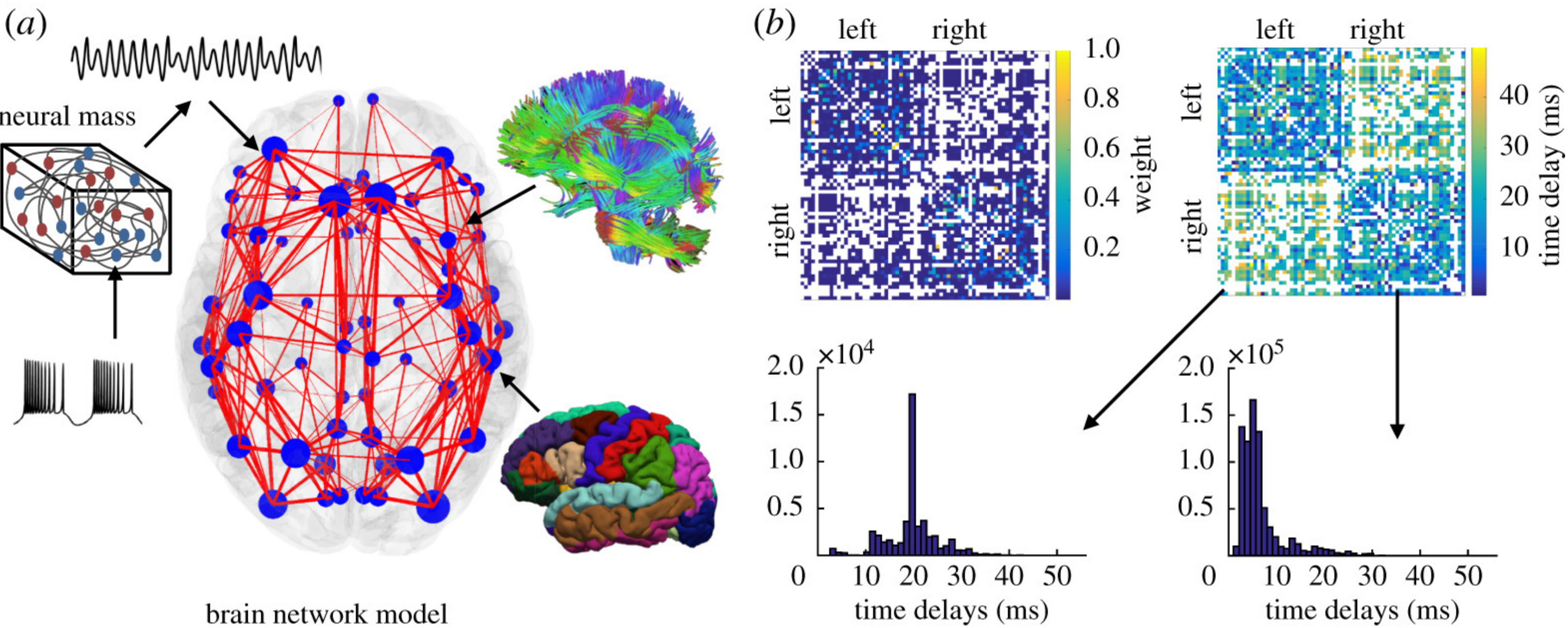


$$x_i(t+1) = f(x_i(t)) + \epsilon\left[\frac{1}{n_i}\sum_j f(x_i(t-\tau)) - f(x_i(t))\right]$$

Atay, Jost and Wende, Phys. Rev. Lett. **92**, 144101 (2004).



Brain synchronisation and time delays



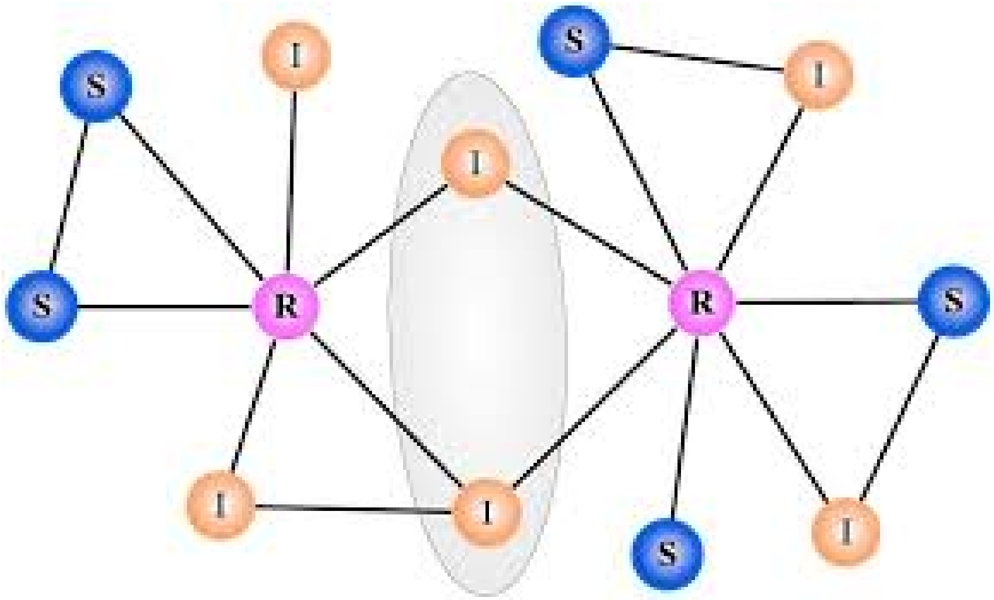
S. Petkoski and V. K. Jirsa
Phil. Trans. Roy. Soc. A **377** (2019).

https://royalsocietypublishing.org/doi/10.1098/rsta.2018.0132#d3e2318

Infection spreading models

SIR model

$$\left.\begin{aligned}\frac{\mathrm{d}S}{\mathrm{d}t} &= bN - \lambda S - \mathrm{d}S,\\ \frac{\mathrm{d}I}{\mathrm{d}t} &= \lambda S - gI - \mathrm{d}I,\\ \frac{\mathrm{d}R}{\mathrm{d}t} &= gI - \mathrm{d}R,\end{aligned}\right\}$$



Birth rate : b

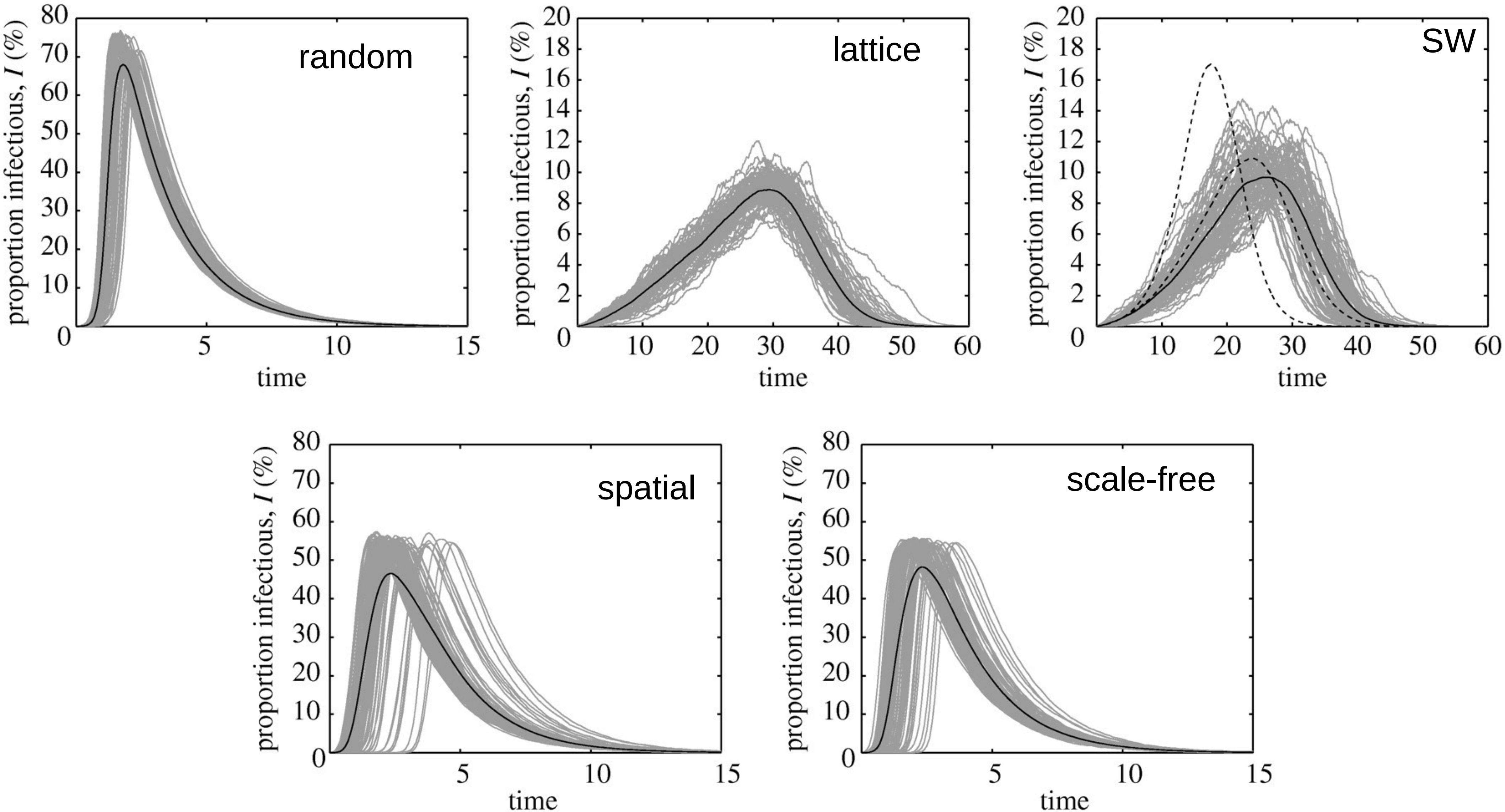
Death rate : d

Rate of recovery from infection : g

Force of infection : λ

If each one can come into contact any other person, (equal mixing)

$$\lambda = \beta I/N$$



M. J. Keeling and K. T. D. Eames,
J. Roy. Soc. Inter. **2**, 295 (2005).

PNAS
Proceedings of the National Academy of Sciences of the United States of America

RESEARCH ARTICLE

Local structure can identify and quantify influential global spreaders in large scale social networks

Yanqing Hu, Shenggong Ji, Yuliang Jin, Ling Feng, H. Eugene Stanley, and Shlomo Havlin

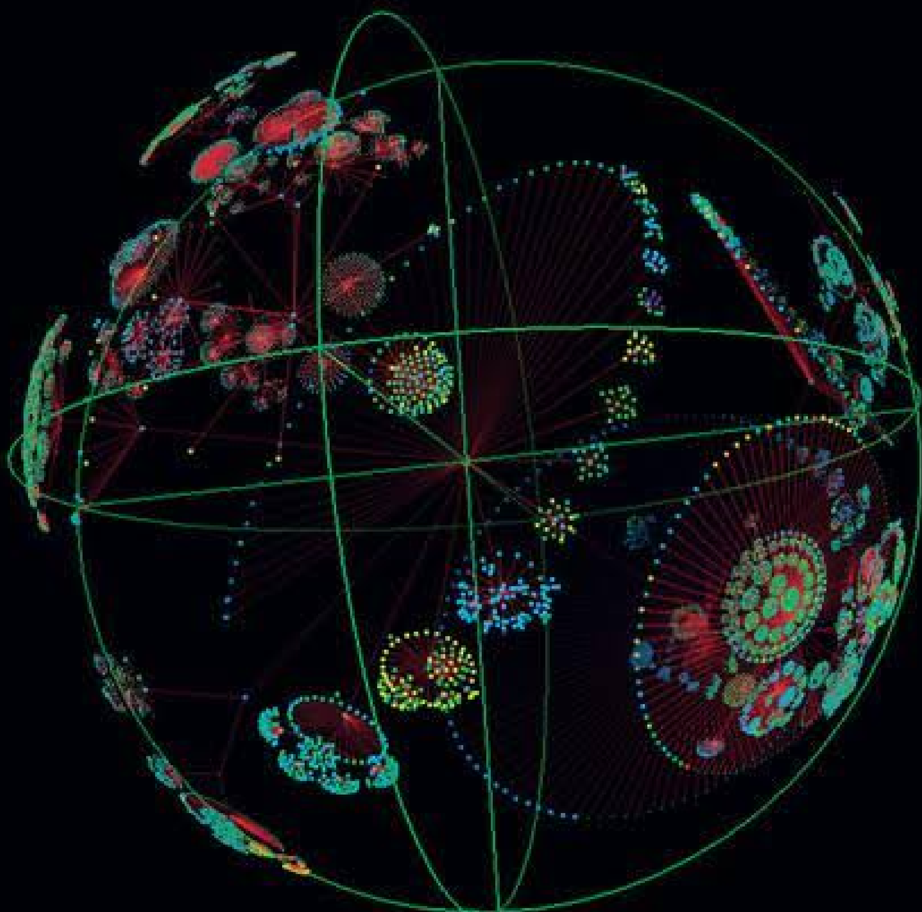
PNAS July 17, 2018 115 (29) 7468-7472; first published July 3, 2018 https://doi.org/10.1073/pnas.1710547115

Contributed by H. Eugene Stanley, December 31, 2017 (sent for review August 31, 2017; reviewed by Marc Barthelemy and Zoltán Toroczkai)

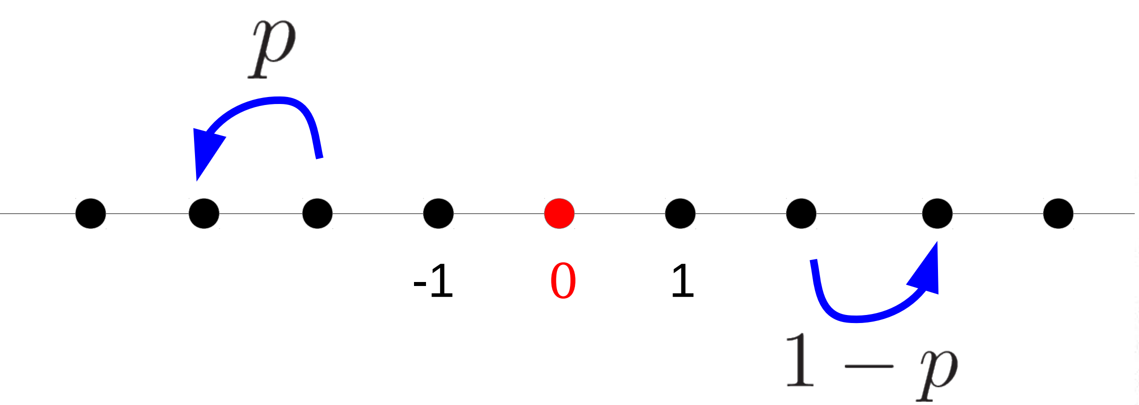
Hu, Ji, Jin, Feng, Stanley and Havlin
PNAS 29, 7466 (2018).

Dynamical Processes on Complex Networks

Alain Barrat, Marc Barthélemy, Alessandro Vespignani



CAMBRIDGE



- Mean position : $\langle r \rangle = 0$

- Mean 'distance' covered :

$$\langle r^2 \rangle = p(1-p)\sqrt{N}$$

- Normal diffusion

$$q_t \sim t^{1/2}$$

The Problem of the Random Walk.

CAN any of your readers refer me to a work wherein I should find a solution of the following problem, or failing the knowledge of any existing solution provide me with an original one? I should be extremely grateful for aid in the matter.

A man starts from a point O and walks l yards in a straight line; he then turns through any angle whatever and walks another l yards in a second straight line. He repeats this process n times. I require the probability that after these n stretches he is at a distance between r and $r+\delta r$ from his starting point, O.

The problem is one of considerable interest, but I have only succeeded in obtaining an integrated solution for *two* stretches. I think, however, that a solution ought to be found, if only in the form of a series in powers of $1/n$, when n is large.

KARL PEARSON.

The Gables, East Ilsley, Berks.

The Problem of the Random Walk.

THIS problem, proposed by Prof. Karl Pearson in the current number of NATURE, is the same as that of the composition of n iso-periodic vibrations of unit amplitude and of phases distributed at random, considered in *Phil. Mag.*, x., p. 73, 1880; xlvii., p. 246, 1899; ("Scientific Papers," i., p. 491, iv., p. 370). If n be very great, the probability sought is

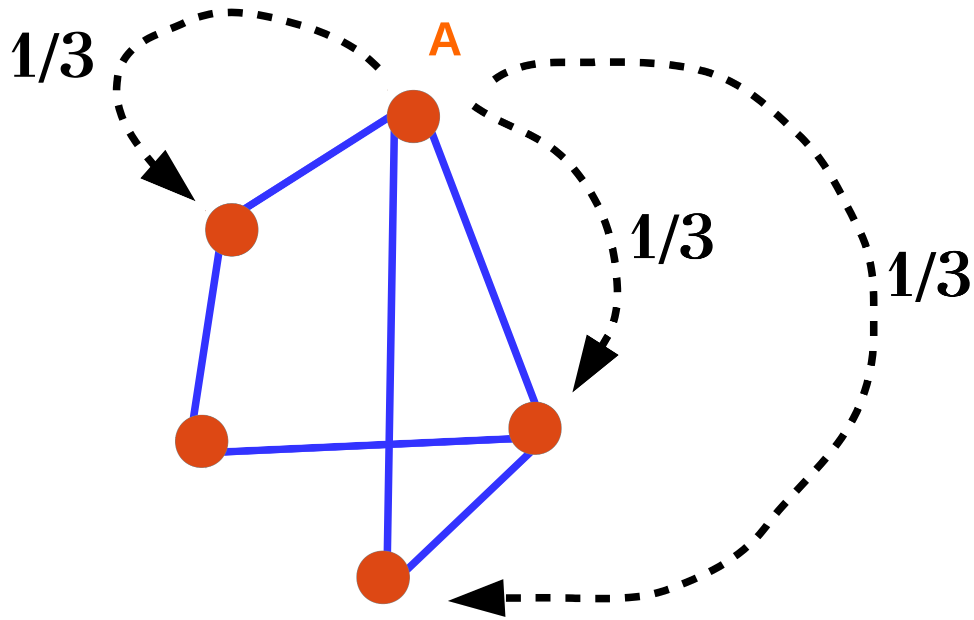
$$\frac{2}{n} e^{-r^2/n} r\,dr.$$

Probably methods similar to those employed in the papers referred to would avail for the development of an approximate expression applicable when n is only moderately great.

RAYLEIGH.

Terling Place, July 29.

Random walks on networks :



Transition probability : $$p_i \propto \frac{1}{k_i}$$

On 1-D lattice : $$p_i = p$$

Random walks : what can we do with them ?

Modelling stock prices : RW model for stock prices. Louis Bachelier.
Geometric random walk model.

Computer sciences : Information discovery and diffusion.
Queuing theory.
Web search. Centrality measures and Ranking.
Recommender systems.

Biology : Movement of species. Random walks to Levy flight models.
Cell locomotion and migration.

Physics / Complex networks :
Statistical physics, Diffusion problems. Brownian motion.
Physics of polymers. Traffic models.
Characterisation of networks, flow on networks.
Extreme events on networks.

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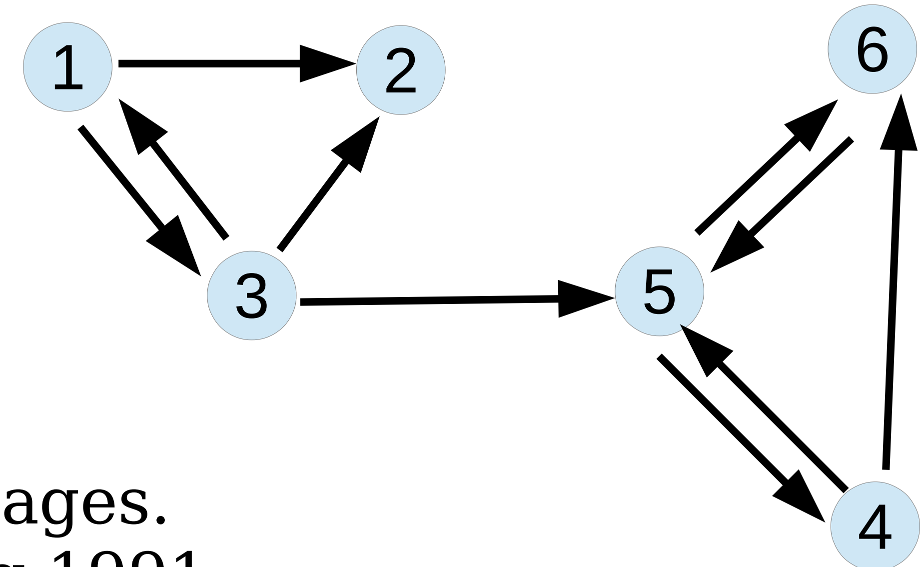
Random ~~walker~~ surfer : PageRank Algorithm



www : 45 bn pages.
1 in Aug 1991.

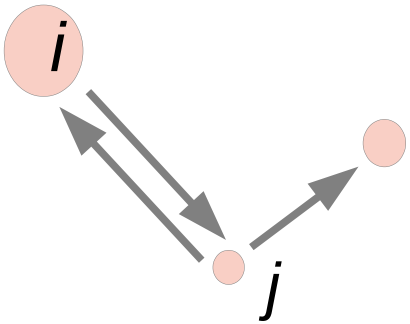
Central idea : A page is important if pointed to by other webpages.

- Follow the links. (*random walk*)
- Occasionally, teleport to a new webpage. (*RW with restart*)

PageRank

$$r(P_i) = \sum_{P_j \in B} \frac{r(P_j)}{|p_j|}$$

$r(P_i)$: Pagerank of i-th page $|p_i|$= Number of outlinks from page i



Central idea : A page is important if it is pointed to by other important pages.

Circular argument ! But it works.
Pagerank of inlinking pages unknown.

$$r(P_i) = \frac{r(P_j)}{2}$$

Assumption : Initially, all pages have equal pagerank.

At k=0, $$r(P_i) = \frac{1}{n}$$ n : number of web pages

Start iterations : $$r_{k+1}(P_i) = \sum_{P_j \in B} \frac{r_k(P_j)}{|p_j|}$$

This can be written as matrix equation.

$$\pi^{(k+1)^{\mathsf{T}}} = \pi^{(k)^{\mathsf{T}}} \mathbf{H}$$

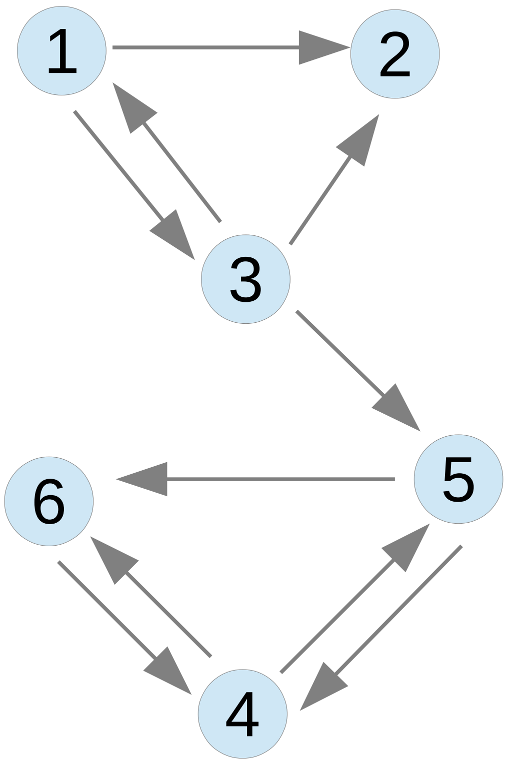
PageRank vector

Hyperlink matrix

What do we expect of this equation ?

The PageRank vector should converge in few iterations.

Problem 1 : *Dangling nodes*



probability

$$\mathbf{H} = \begin{pmatrix} 0 & 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1/3 & 1/3 & 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

transition probability matrix for Markov Chain

$$\mathbf{H} = \begin{pmatrix} 0 & 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1/3 & 1/3 & 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

1/6

$$\mathbf{S} = \mathbf{H} + \mathbf{a}(1/n\mathsf{e}^{\mathsf{T}})$$

Solution to problem 1 : *Stochasticity adjustment. Rank-1 update to* **H**.

Problem 2 : *Irreducibility*

Random surfer model :

- Follow the links randomly.
- Occasionally, teleport to a new site with equal apriori probability.

Solution to problem 2 : *Primitivity adjustment*

$$\mathbf{G} = \alpha \mathbf{S} + (1-\alpha)\frac{1}{n}\mathbf{e}\mathbf{e}^{\mathbf{T}}$$

Google matrix

probability of following the links

G is a stochastic matrix.

G is irreducible.

Then, theory of stochastic processes assures us that the eigenvalue equation (*Perron-Frobenius theorem*)

$$\pi^{\mathsf{T}} = \pi^{\mathsf{T}}\mathbf{G}$$

converges to a stationary, unique eigenvector (pagerank vector).

$$\mathbf{G} = \alpha\ \mathbf{S} + (1-\alpha)\ 1/n\mathbf{e}\mathbf{e}^T$$

Google matrix

Prob. of following the links

Transition prob matrix

Prob. of restart

The Anatomy of a Large-Scale Hypertextual Web Search Engine

Sergey Brin and Lawrence Page

Computer Science Department,
Stanford University, Stanford, CA 94305, USA
sergey@cs.stanford.edu and page@cs.stanford.edu

Abstract

In this paper, we present Google, a prototype of a large-scale search engine which makes heavy use of the structure present in hypertext. Google is designed to crawl and index the Web efficiently and produce much more satisfying search results than existing systems. The prototype with a full text and hyperlink database of at least 24 million pages is available at http://google.stanford.edu/

Computer Networks **30** 107 (1998).

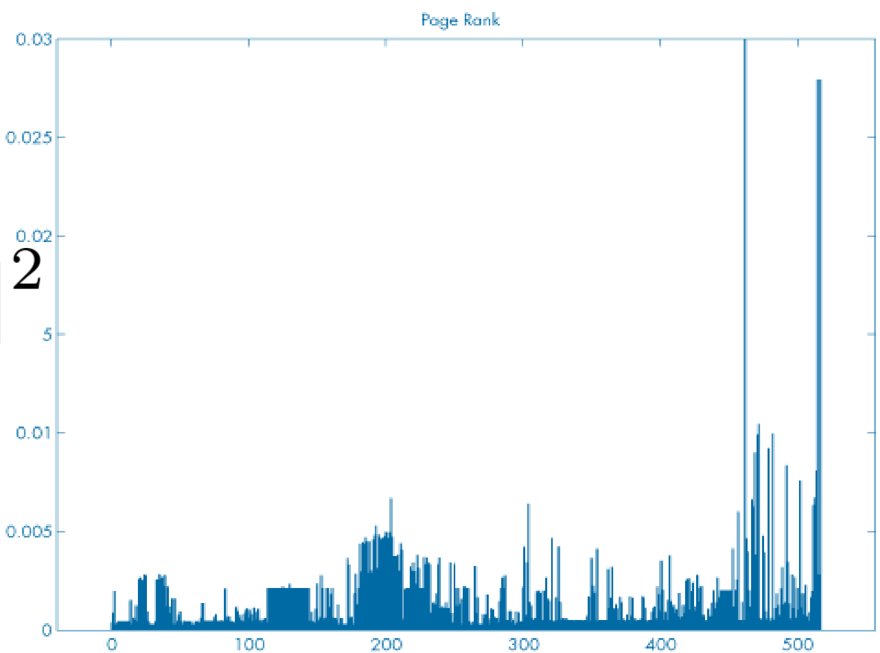
G is aperiodic, irreducible stochastic matrix.
Perron-Frobenius theorem applies.

$$\pi^{\mathsf{T}} = \pi^{\mathsf{T}} \mathbf{G}$$

stationary, unique eigenvector.

$\alpha = 0.85$ is suitable.

$|x_j|^2$



j

Localisation – Delocalisation transition in eigenvector π of Google matrix.

Methods of MIT / quantum chaos can be used.

Phys. Rev. E 80, 026107 (2009)

Indexing : Bots follow the links and scourge for information.

- Around the year 2000, search engines were indexing only about 16% of www.
  (Lawrence and Giles, *Nature* (1999))

Translated in the language of physics :

- On complex networks, how many *distinct nodes* are visited by N random walkers in finite time ?
- Relaxation to equilibrium on complex networks.

Some results for distinct nodes visited by random walkers :

$$S_1(t) \sim \sqrt{t}$$ (in 1D)

$$S_1(t) \sim t, \qquad t \to \infty$$ (in >2D) Dvoretzsky and Erdos (1950).

$$S_N(t) \sim At^d$$ (d dimensional lattice) Larraide et. al. Nature (1992).

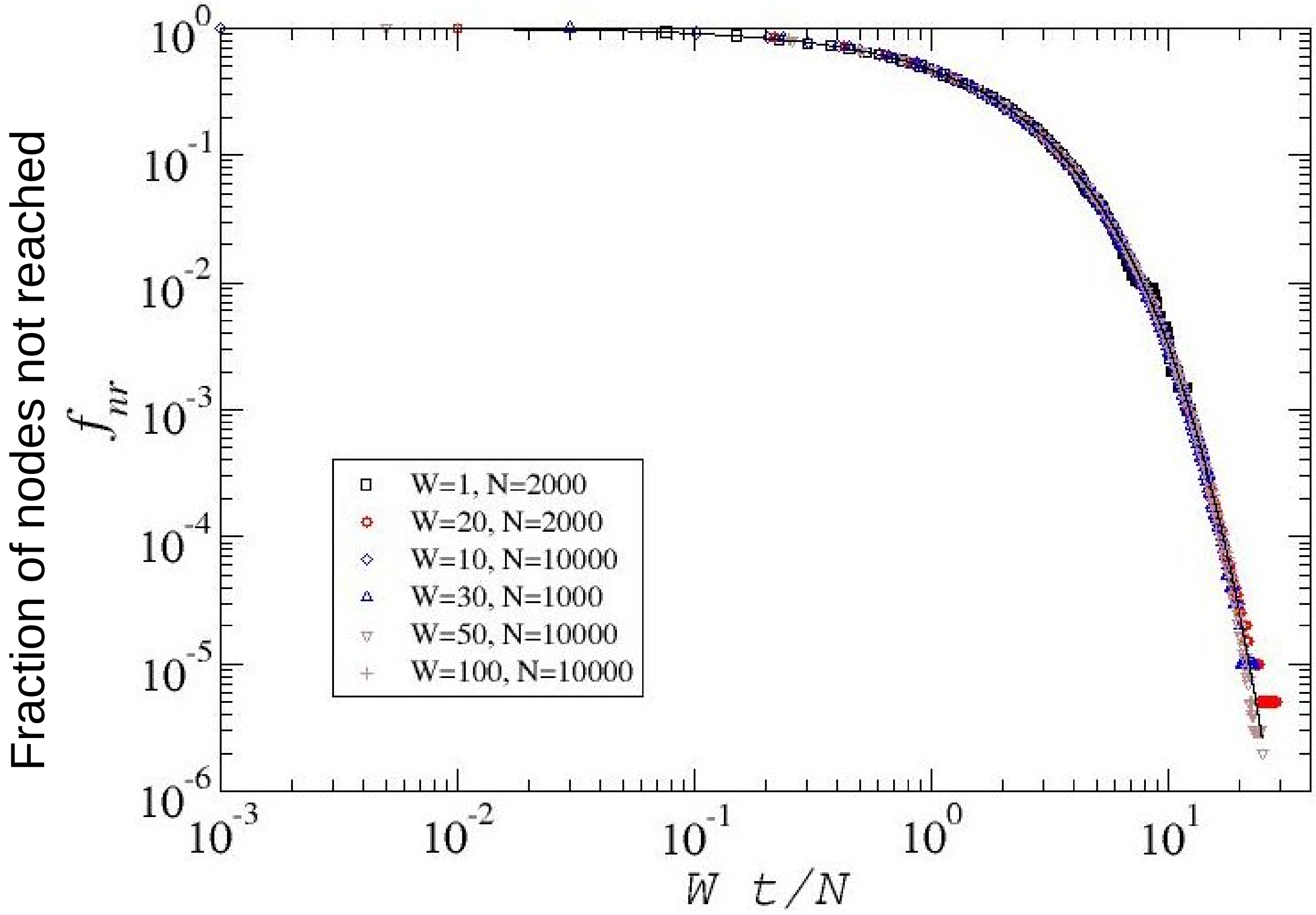
$$S_N(t) \sim \sqrt{t}$$
$$\sim t$$
(on small world network) Almass et. al., PRE (2003)

$$S_1(t) \sim t$$ (on scale-free network) Lee et. al. Physica A (2008)

$$S_1(t)$$ On random graphs Majumdar et. al., 2015

No results as yet for scale-free networks.

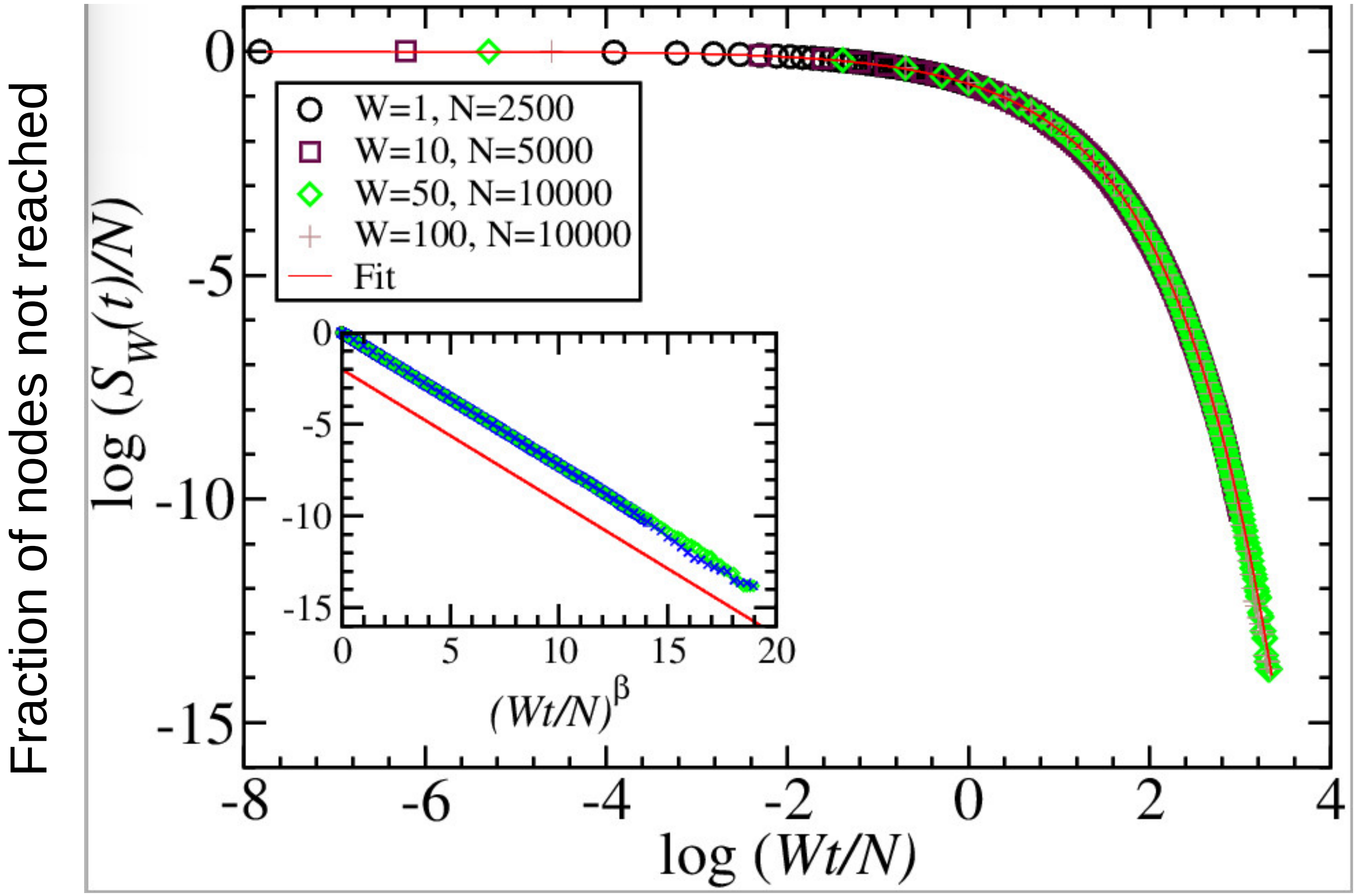
Exploring network by N random walkers :



BA Network
N : nodes
W : walkers

Averaged over 500 realisations.

Yagyik Goswami & Santhanam (2015)

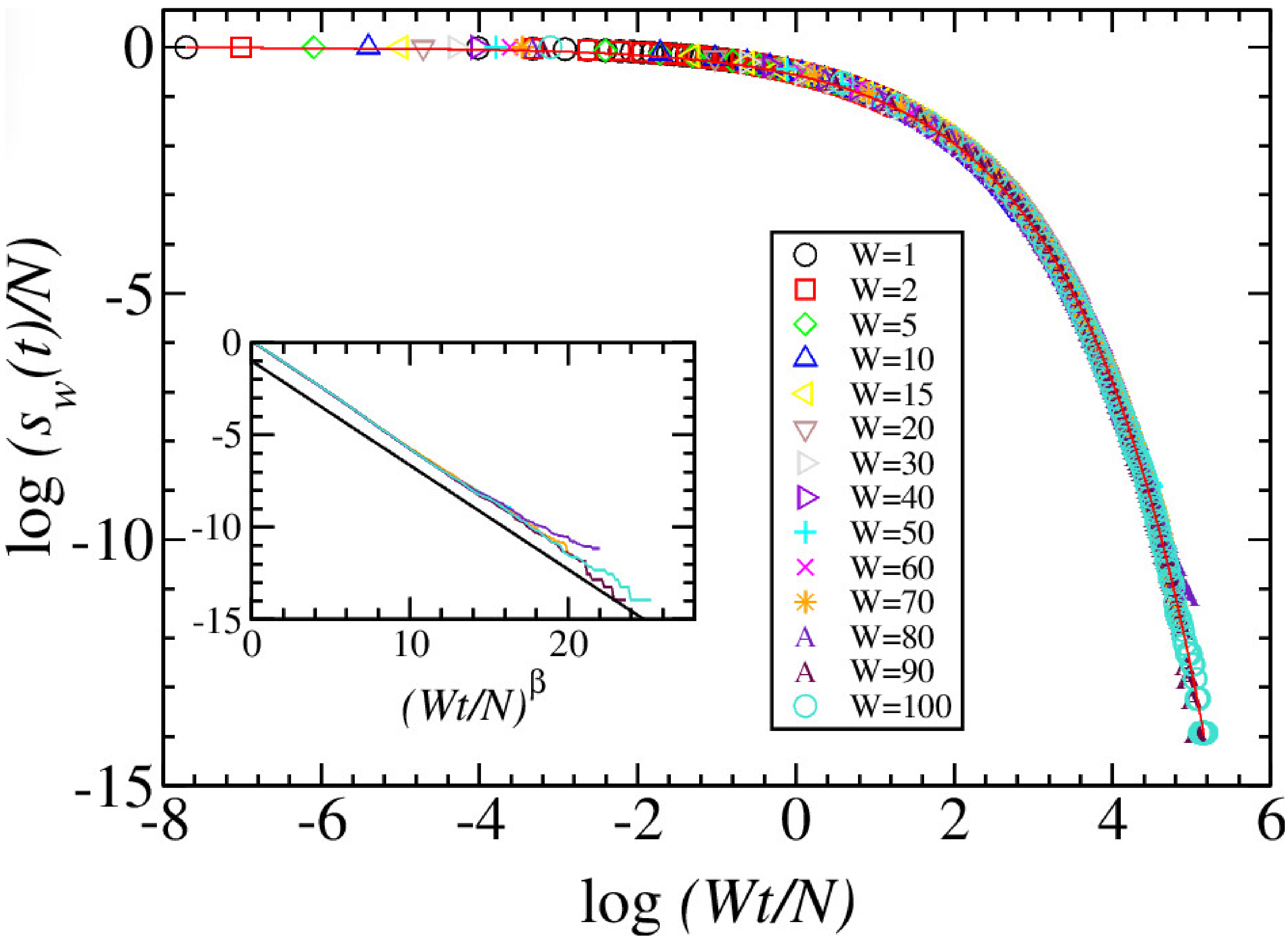


Scale-free network
N nodes
W Walkers

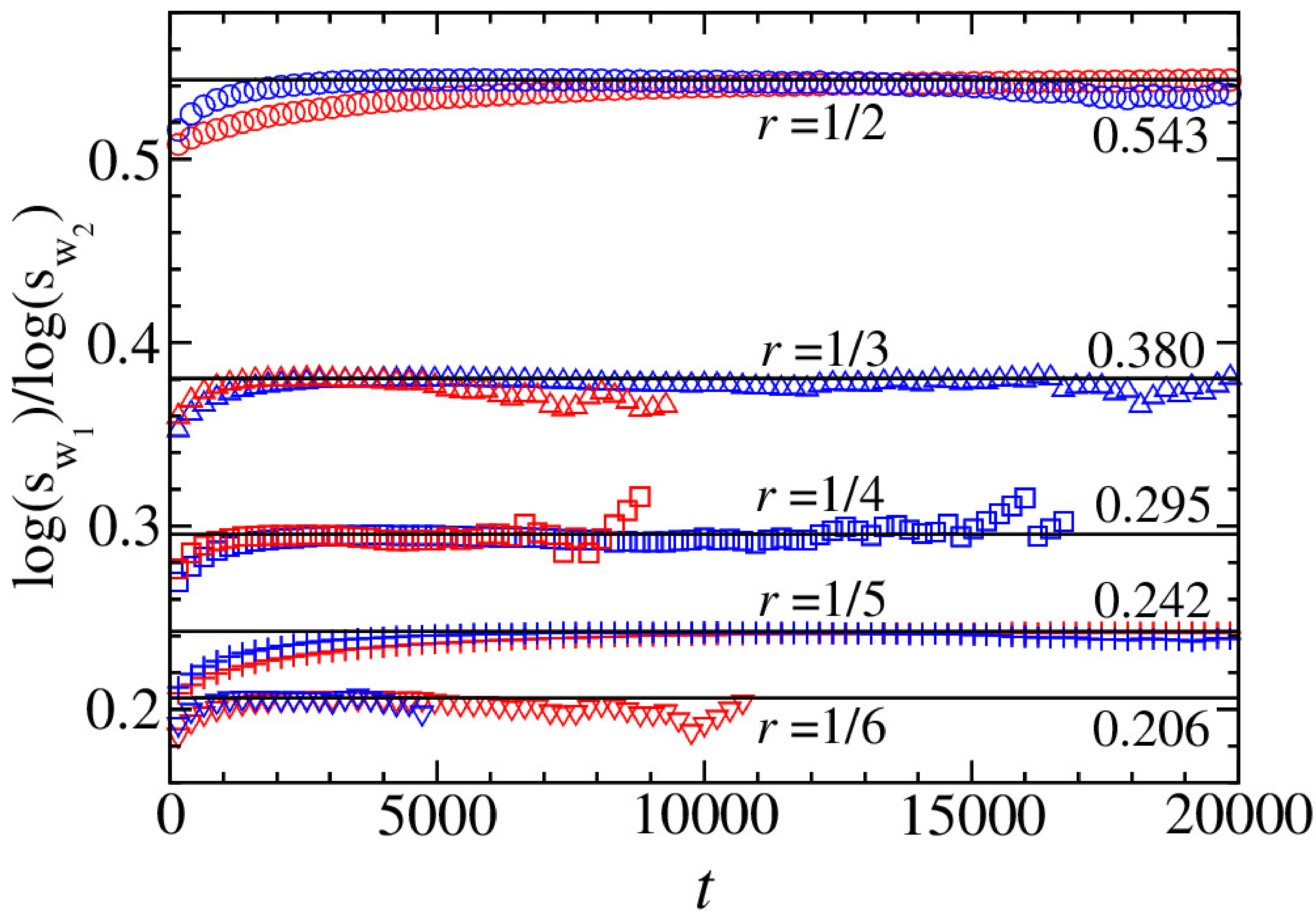
$$s_W(t) = \left(\frac{N-1}{N}\right) e^{-A\ x(t)^\beta}$$

$$x = Wt/N$$ Scaled time

Protein interaction network of yeast (N=2361)

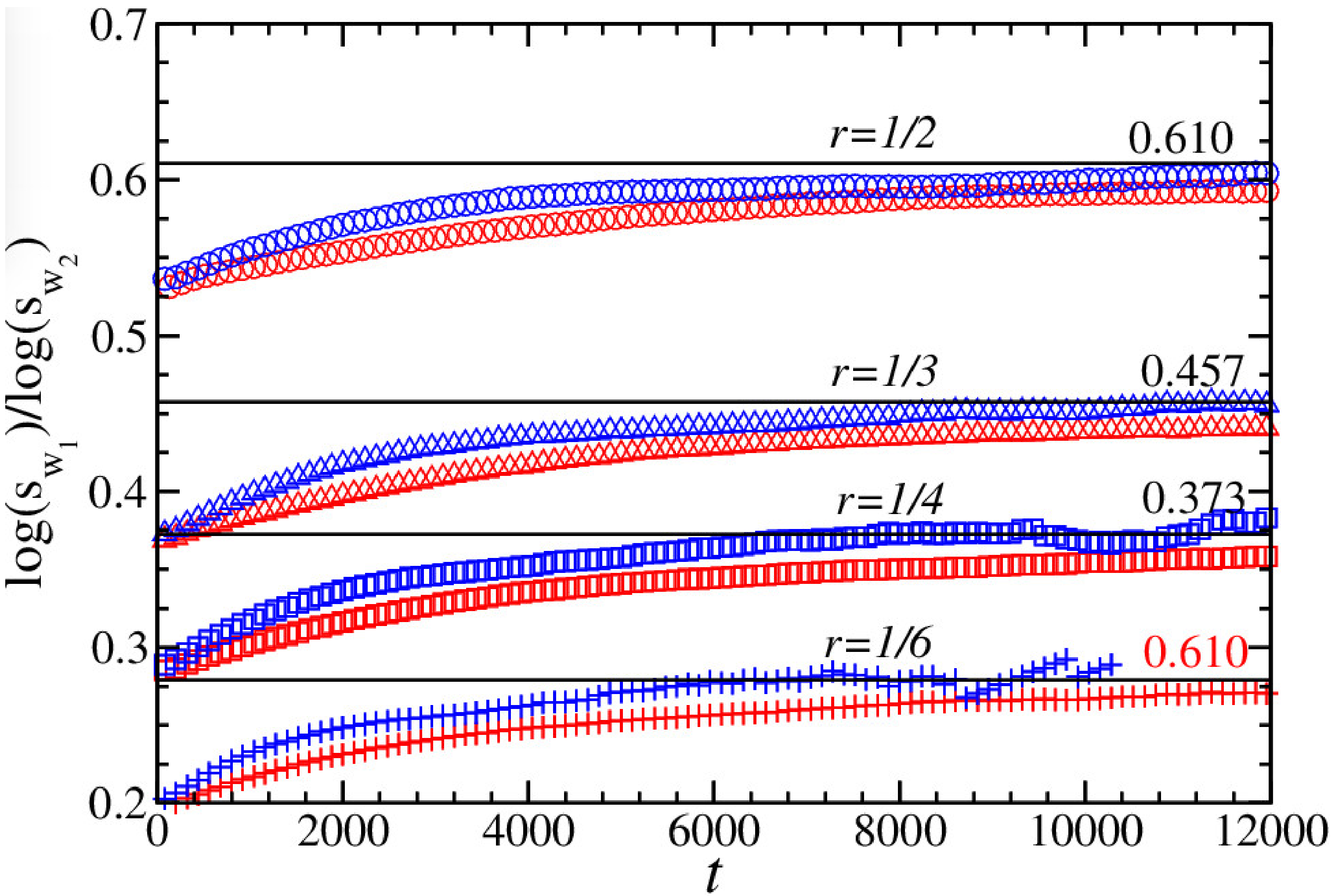


BA model of Scale free network (N=5000)

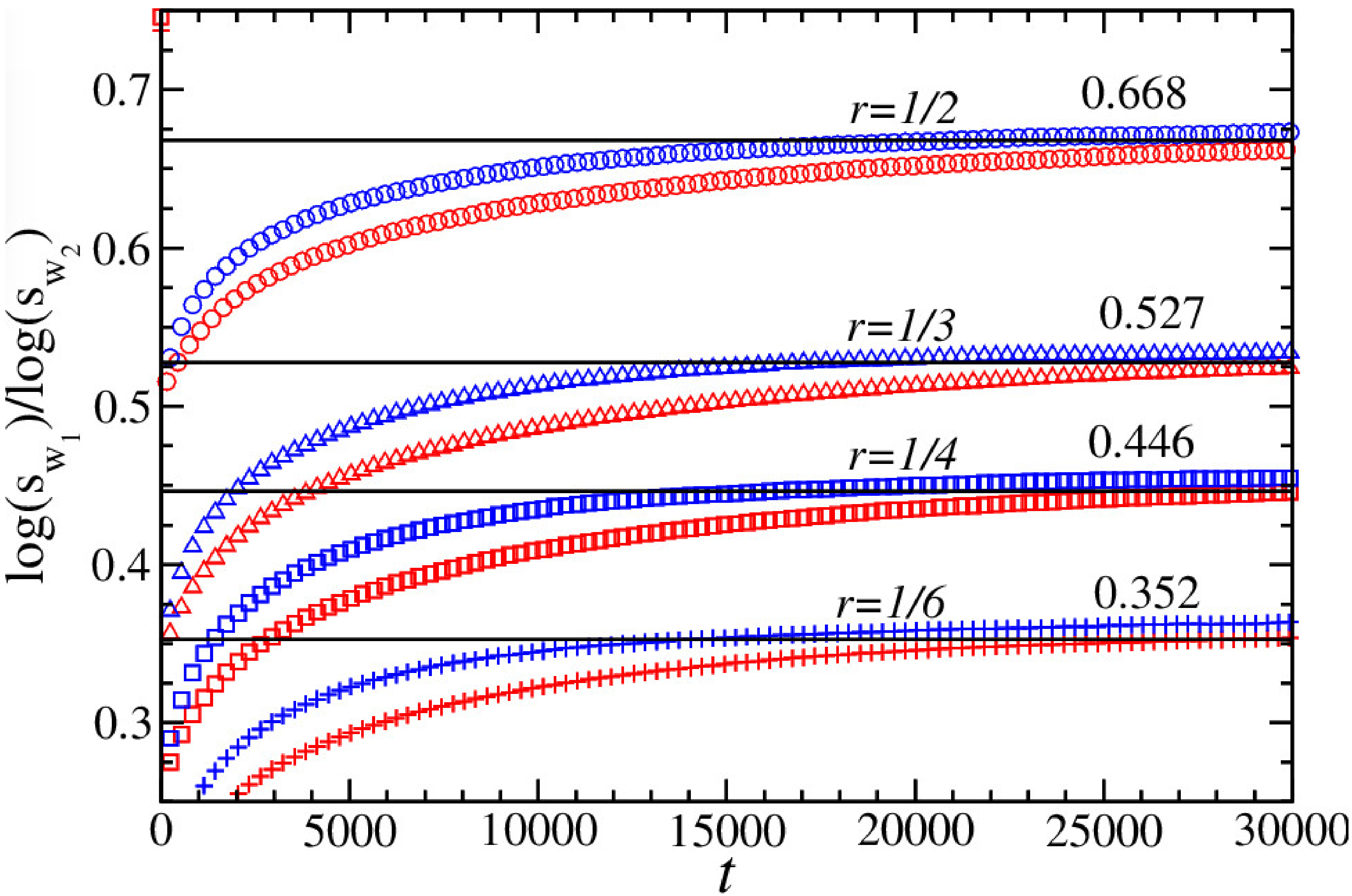


$$\log s_{W_2}(t) = \left(\frac{W_2}{W_1}\right)^{\beta} \log s_{W_1}(t),$$

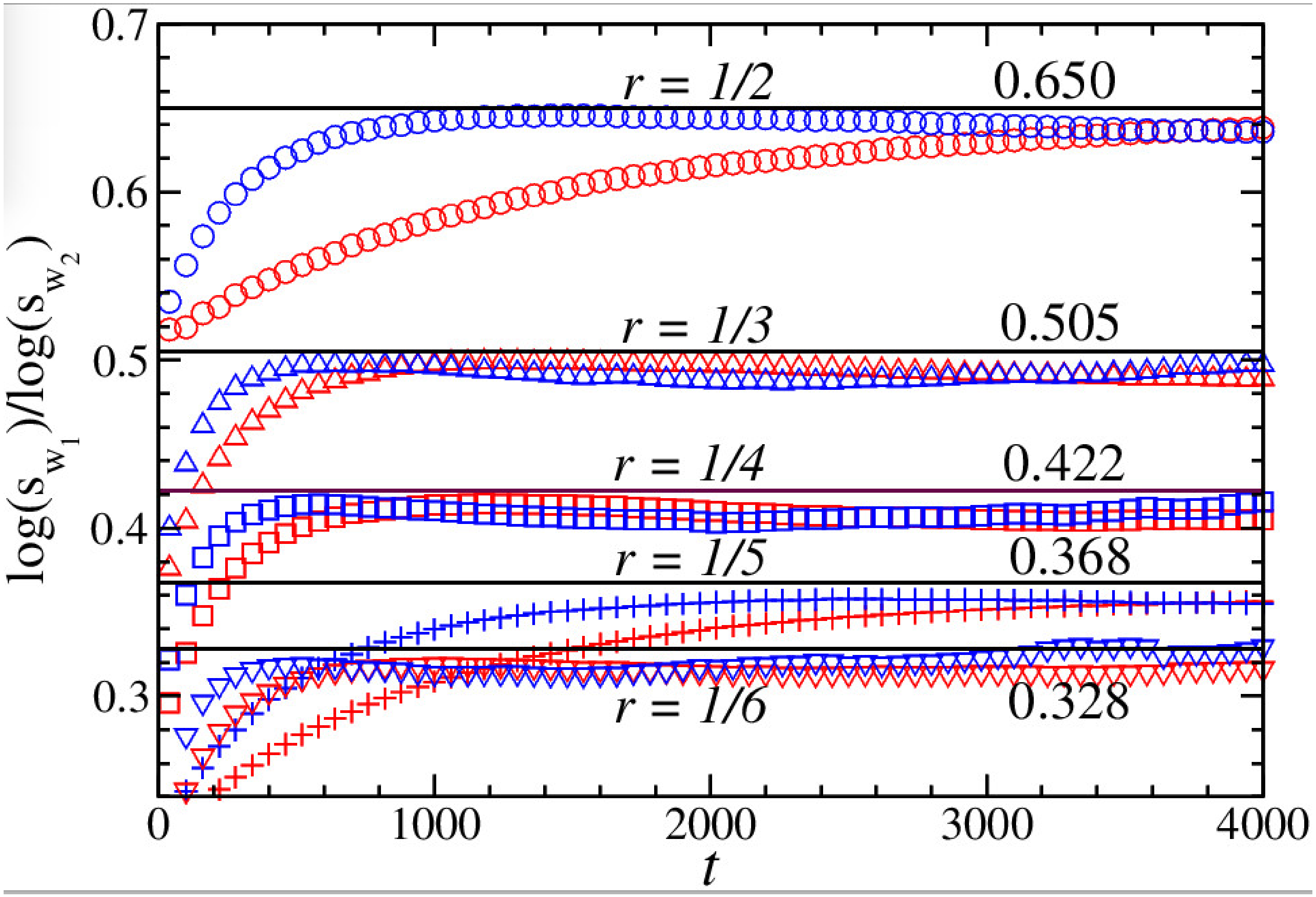
Internet at the level of autonomous systems (N=11174)



Citation network of cond-mat papers (N=23133)

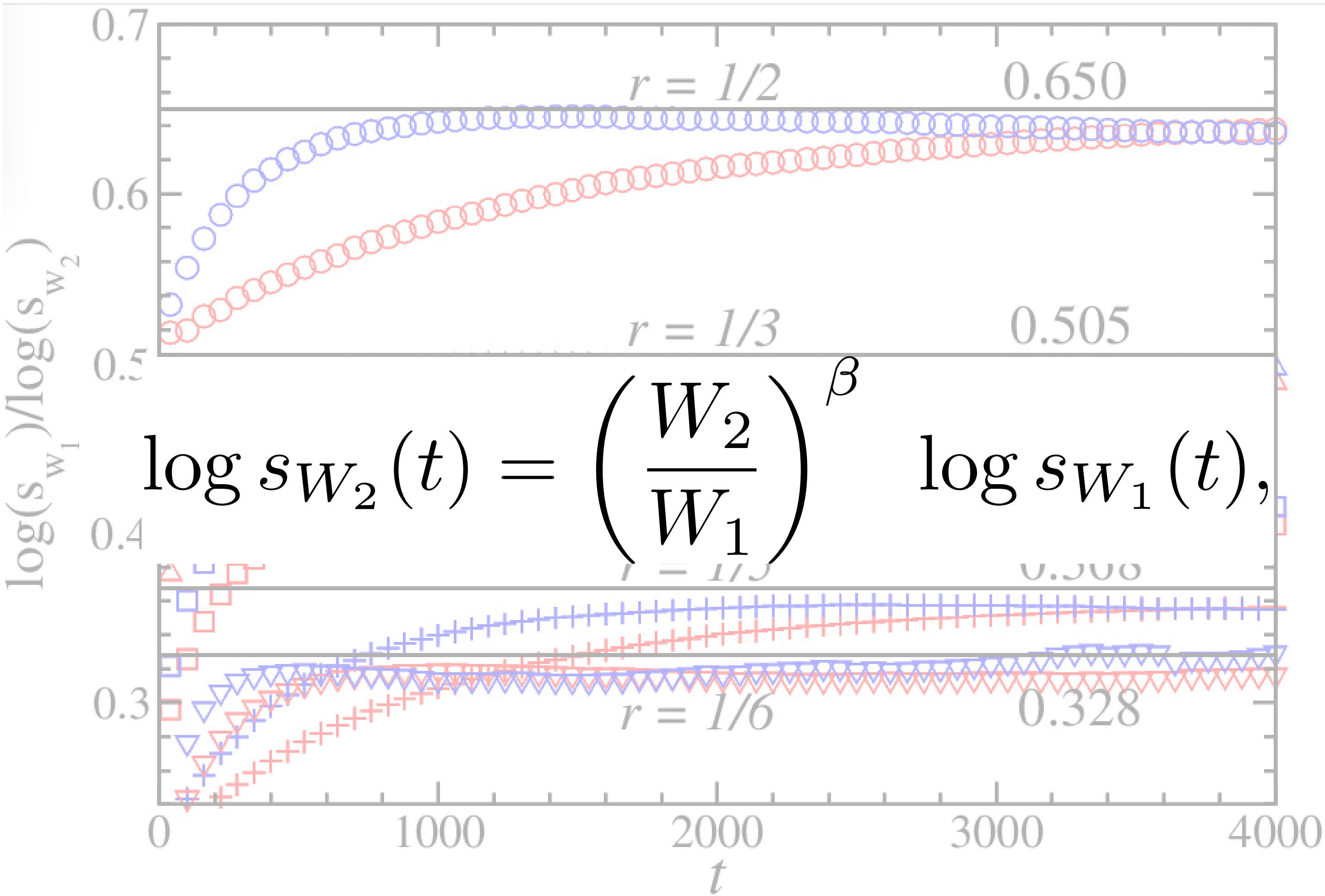


Yeast network (N=2361)



Finding distinct sites visited by N walkers --> Finding distinct sites visited by 1 walker.

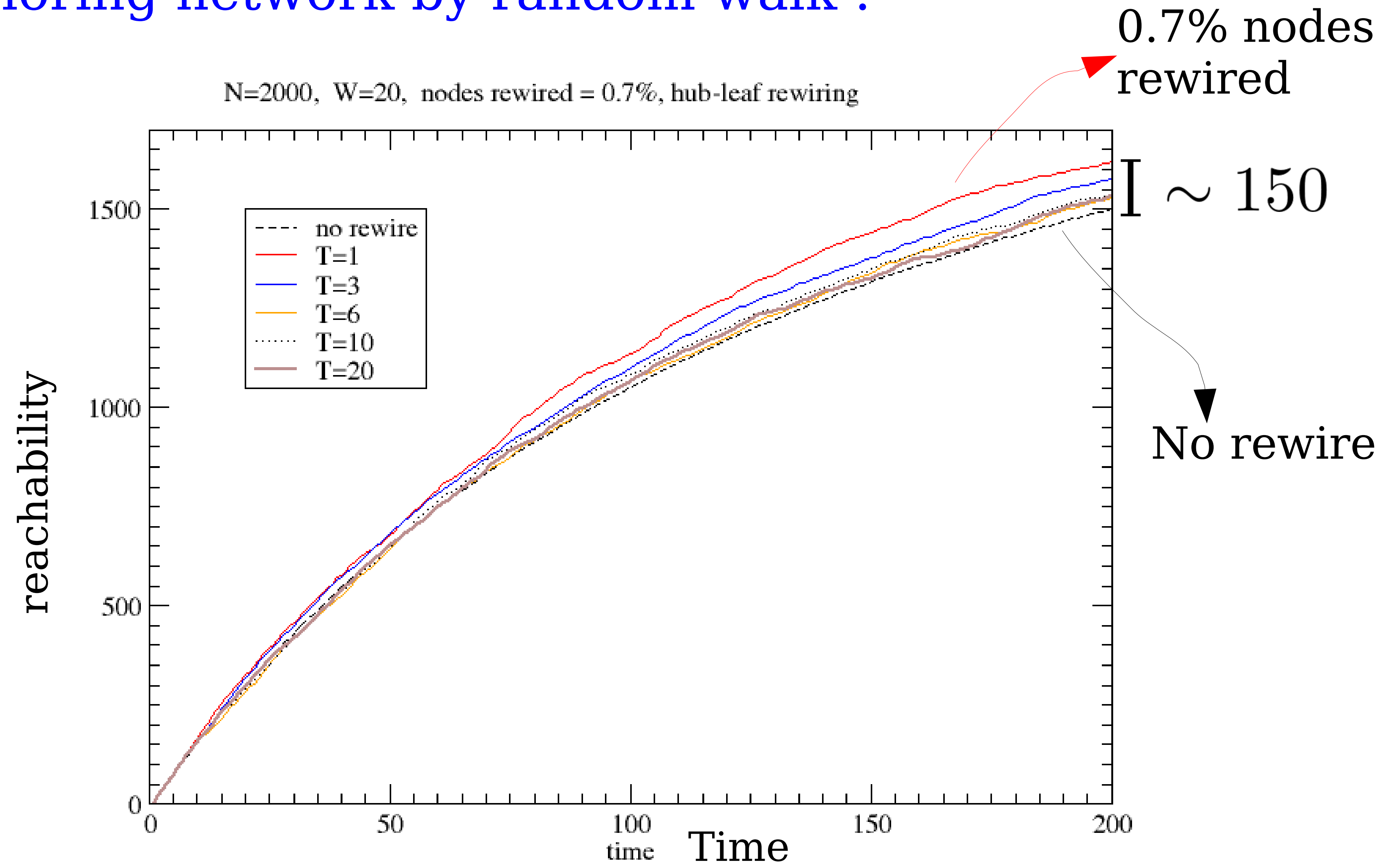
Yeast network



$$\log s_{W_2}(t) = \left(\frac{W_2}{W_1}\right)^{\beta} \log s_{W_1}(t),$$

Finding distinct sites visited by N walkers **-->** Finding distinct sites visited by 1 walker.

Exploring network by random walk :



For RW, rewiring enhances reachability on networks.

Yagyik Goswami & Santhanam (2015)

Random walks : what can we do with them ?

Modelling stock prices : RW model for stock prices. Louis Bachelier.
Geometric random walk model.

Computer sciences : Information discovery and diffusion.
Queuing theory.
Search engines. PageRank algorithm. Ranking.
Recommender systems.

Biology : Movement of species. Random walks to Levy flight models.
Cell locomotion and migration.

Physics / Complex networks :
Statistical physics, Diffusion problems. Brownian motion.
Physics of polymers.
Characterisation of networks, flow on networks.
Traffic models and extreme events on networks.

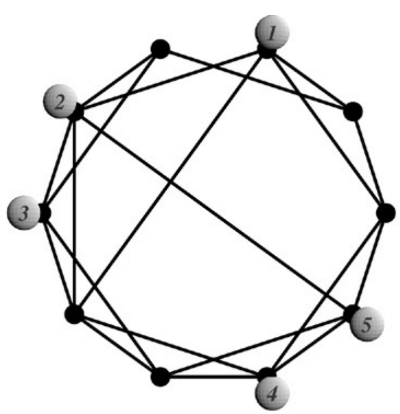
Random walkers on networks

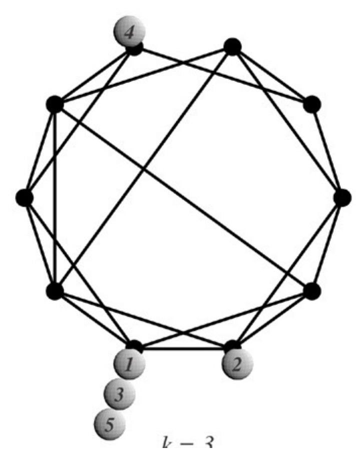
- <u>Basic model</u> : *Random walks on networks.*

- Random walkers on networks are the events.

  Packets of information flowing through a network.
  Vehicles on highways.

- What's the probability of seeing *m* walkers on a node ?

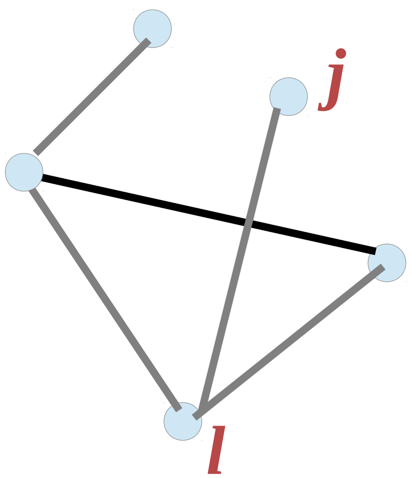




- Walker number changes due to natural fluctuations.

Transition probability for a walker to go from l to j

$$w_{lj} \propto k_j^{\alpha}$$



$\alpha = 0$ Standard random walk

$\alpha > 0$ Random walk biased towards hubs

$\alpha < 0$ Random walk biased towards small degree nodes

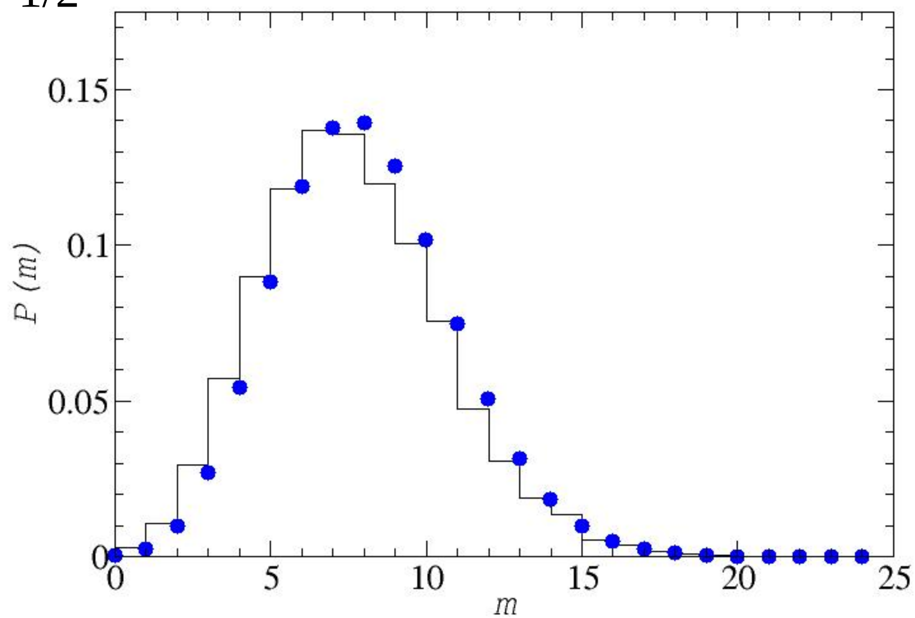
Probability of m walkers on node i : Binomial distribution

$$F_i(m) = \binom{W}{m} \bar{p}_i^{m} \; (1 - \bar{p}_i)^{W-m}$$

Mean flux and variance

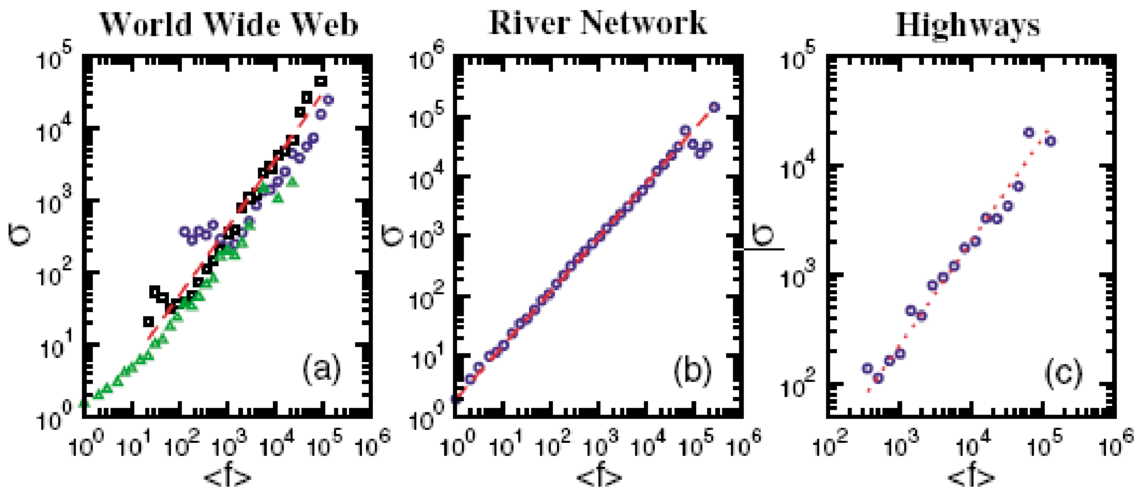
$$\langle f_i \rangle = \frac{W k_i}{K} \qquad\qquad \sigma_i^2 = W \frac{k_i}{K} \left(1 - \frac{k_i}{K} \right)$$

$$\text{If, } \; k_i << K, \;\; \text{then } \; \sigma_i \approx \langle f_i \rangle^{1/2}$$



W = 15560
N = 1000
10^8 time steps
Scale free network

'Universality' in random walk on networks ....

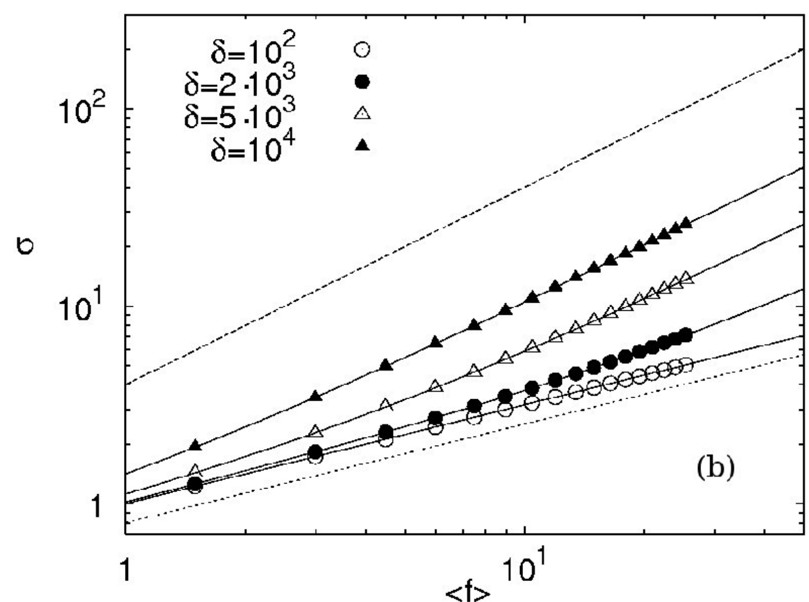


$$\sigma \propto \langle f \rangle^{\alpha} \qquad\qquad \alpha = 1/2 \;\; \text{or} \;\; 1$$

Menezes and Barabasi, PRL **92**, 028701 (2004).

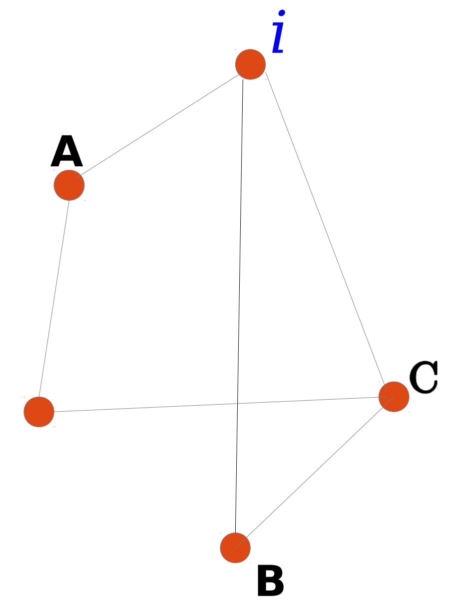
.......And its break down

$$\frac{k_i M \delta}{2KM} << 1$$



Meloni *et. al.* PRL **100**, 208701 (2008).

Random walks as dynamics on networks



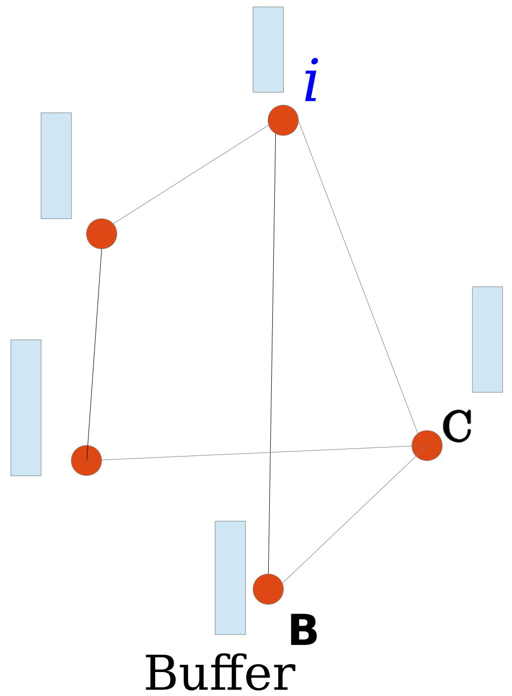
- RW on networks (*conservative model*)
  N nodes, E edges, W walkers

  Walking rule : RW

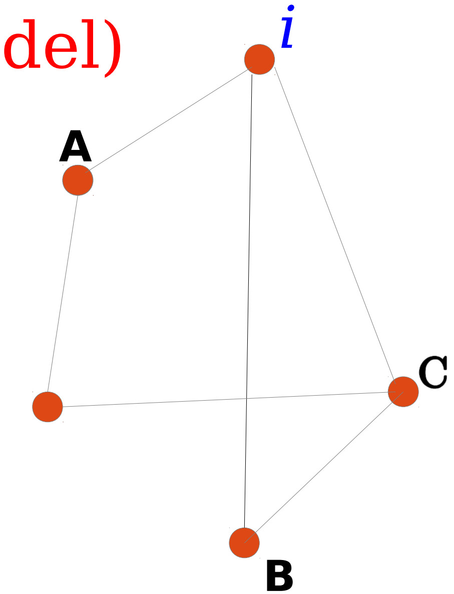
- RW on networks (*non-equilibrium model*)
  N nodes, E edges, 0 walkers,
  n buffer size.

  Walking rule : RW + additional rules.
  p Birth prob, μ absorption prob.

  η Rejection probability, ϕ outflux.



Random walks on networks (Conservative model)



- Occupation probability : $$P_i = \frac{K_i}{\sum_i K_i}$$

- First passage time : $$\langle T_{ij} \rangle \propto \frac{\sum_i K_i}{K_j}$$

- RWC related to second largest eigenvalue of time evolution operator.

Noh and Reiger, *PRL* **92**, 118701 (2004).

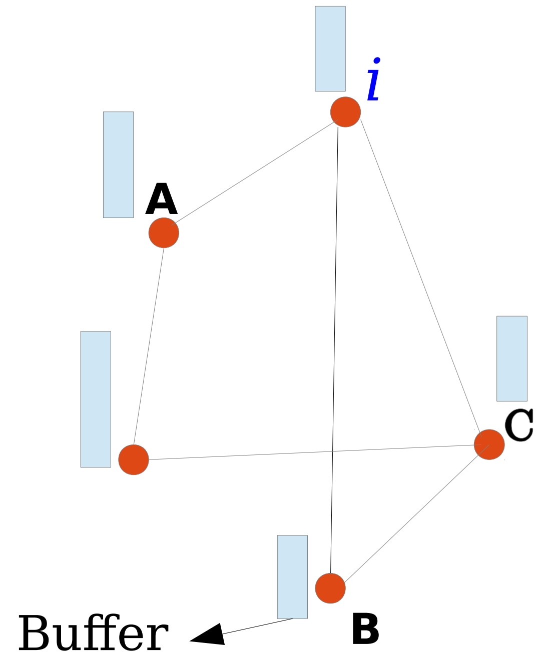
- Extreme event probability : $$F(K) \propto K^{1-\tau(q)}$$

Kishore, Santhanam, Amritkar *PRL* **106**, 188701 (2011).

- Flux-fluctuation law : $$\langle \sigma_i \rangle \propto f_i^{\beta}$$ **(?)**

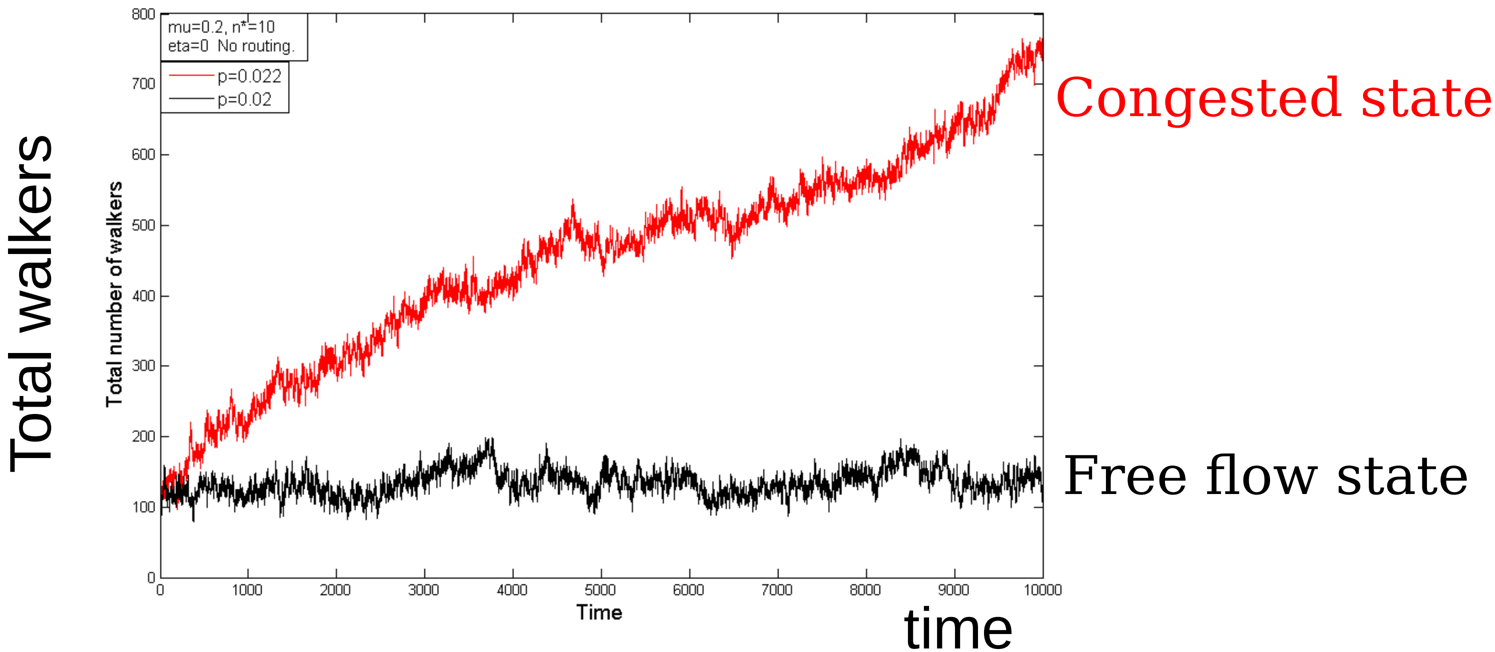
Y. C. Lai *et. al.*, *Sci. Rep* (2014), Y. Moreno *et. al. PRL* (2008), Menezes and Barabasi *PRL* (2004).

RW on networks (non-equilibrium model)



- RW on networks (non-equilibrium model)
  N nodes, E edges, 0 walkers,
  n buffer size.

- Walking rule (RW + additional rules)
  Birth and death prob.,
  Rejection probability, outflux



Y. Moreno *et. al.*, *EPL* (2005), De Martino *et. al.*, *PRE(Rapid)* (2009),
Ezaki *et al PRE* (2014).

outflux=1

Order parameter : $$\frac{\Delta W}{\Delta t}$$

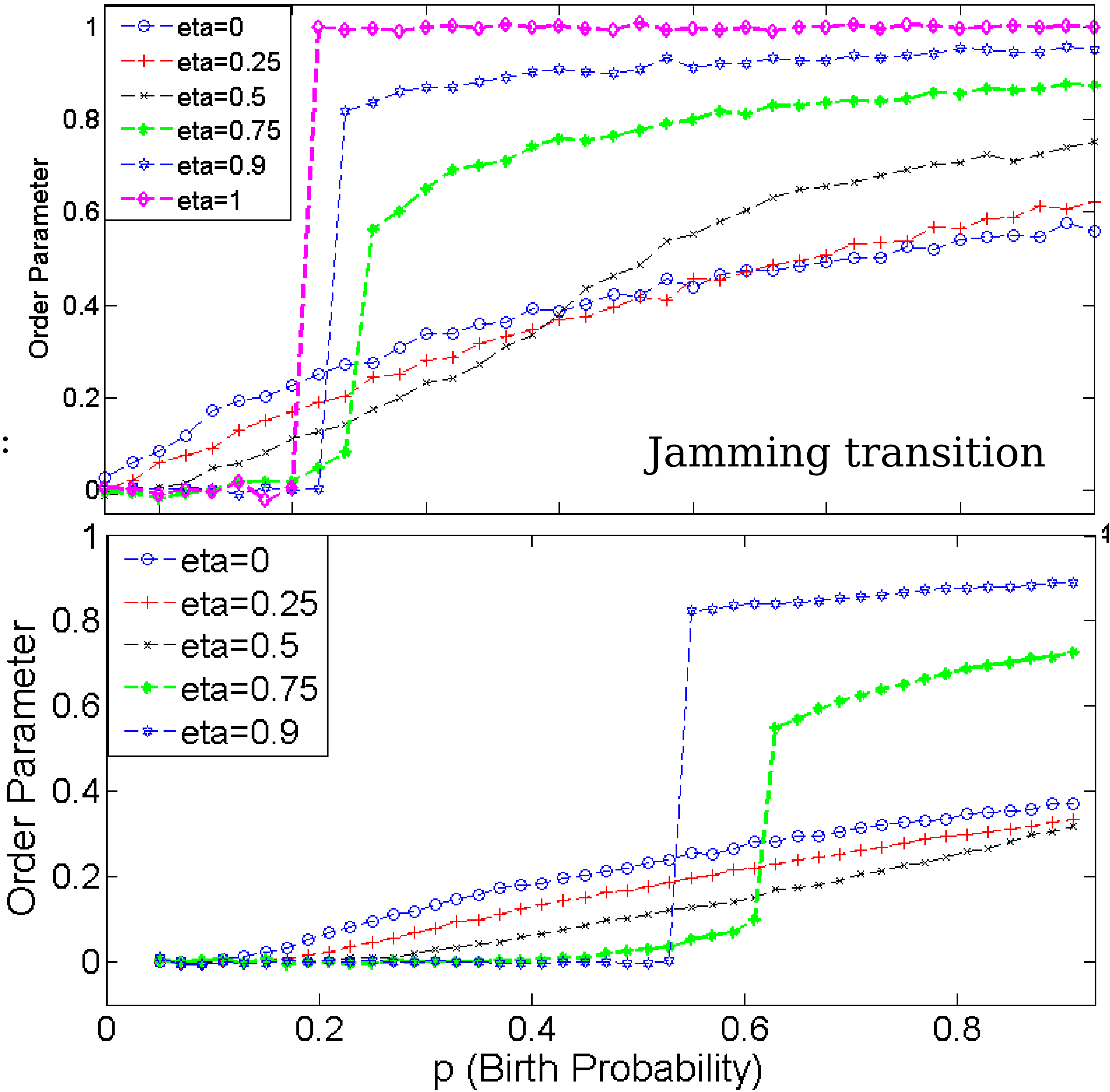
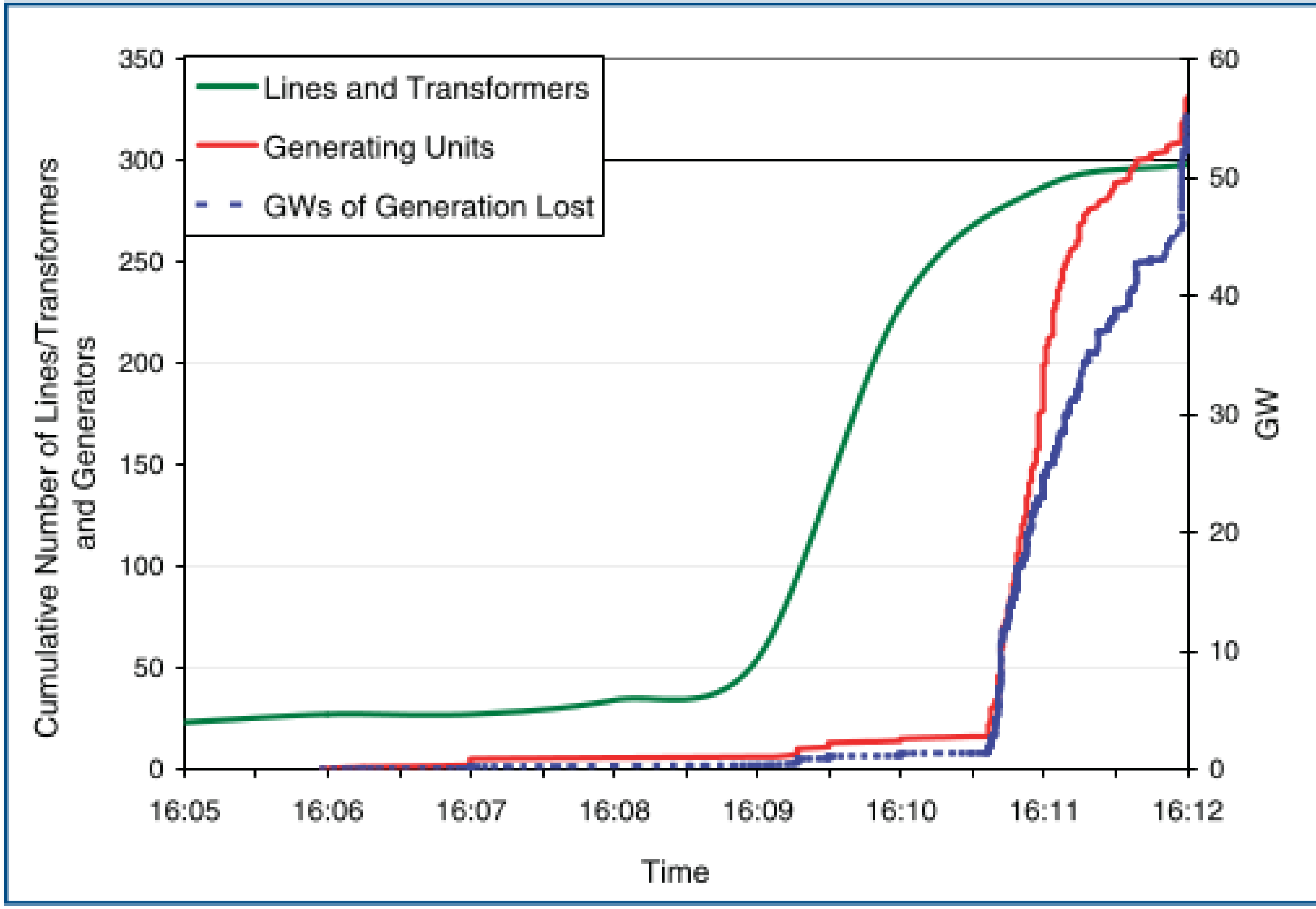


Figure 6.1. Rate of Line and Generator Trips During the Cascade

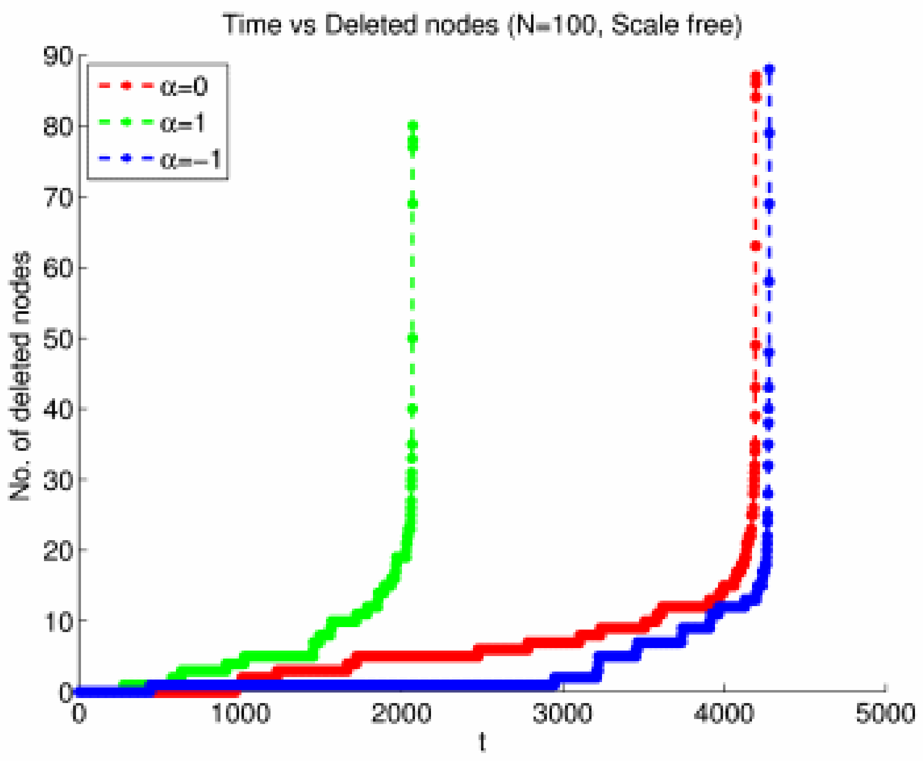


~ 1 minute

Source : Report on US power blackout in 2003.

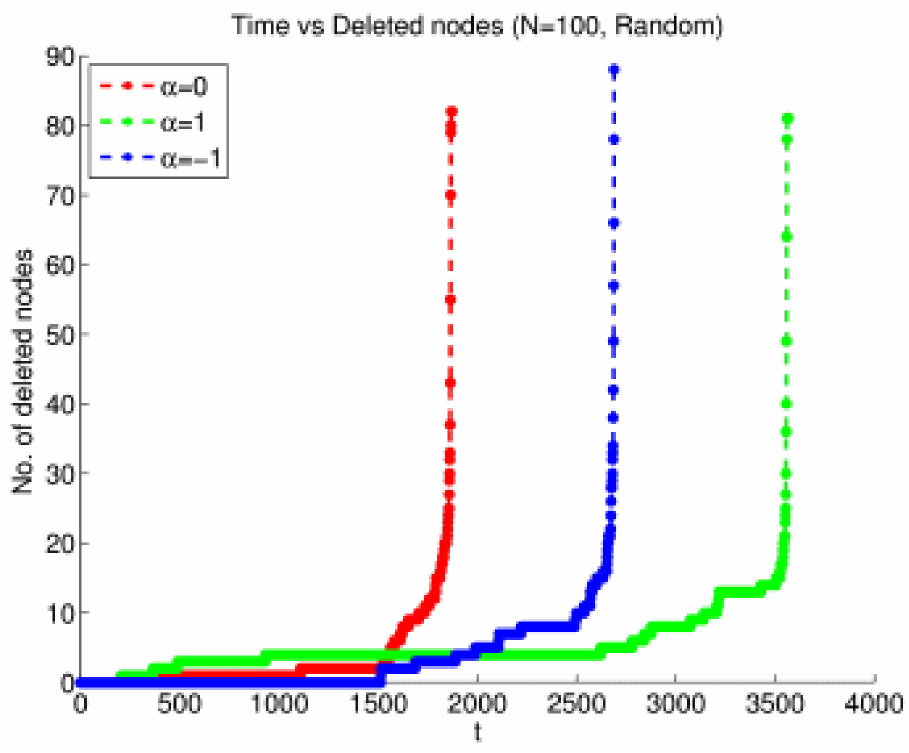
Extreme events can quickly paralyse a network

Scale free network



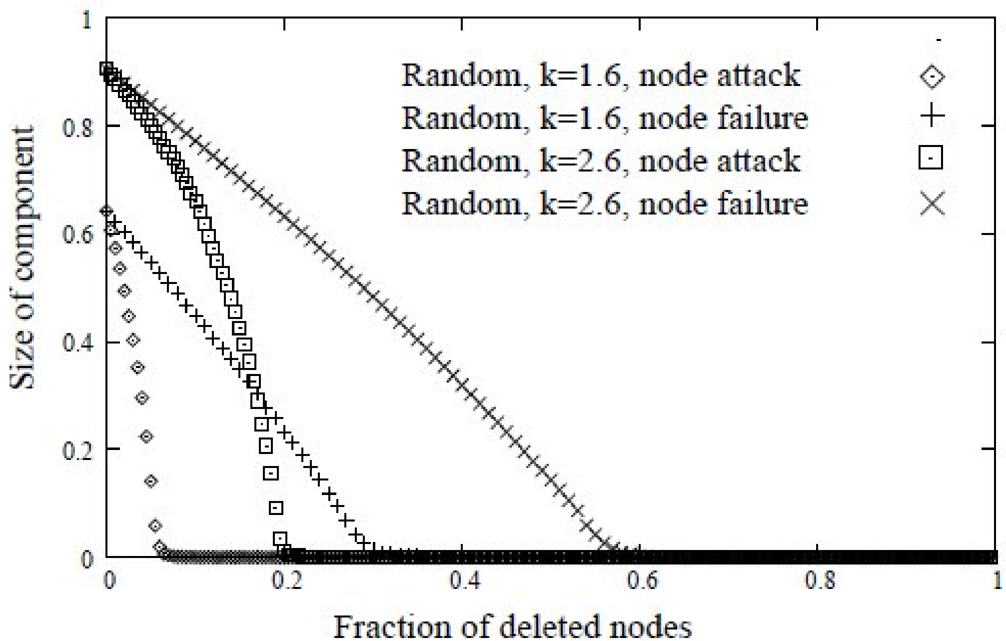
Time

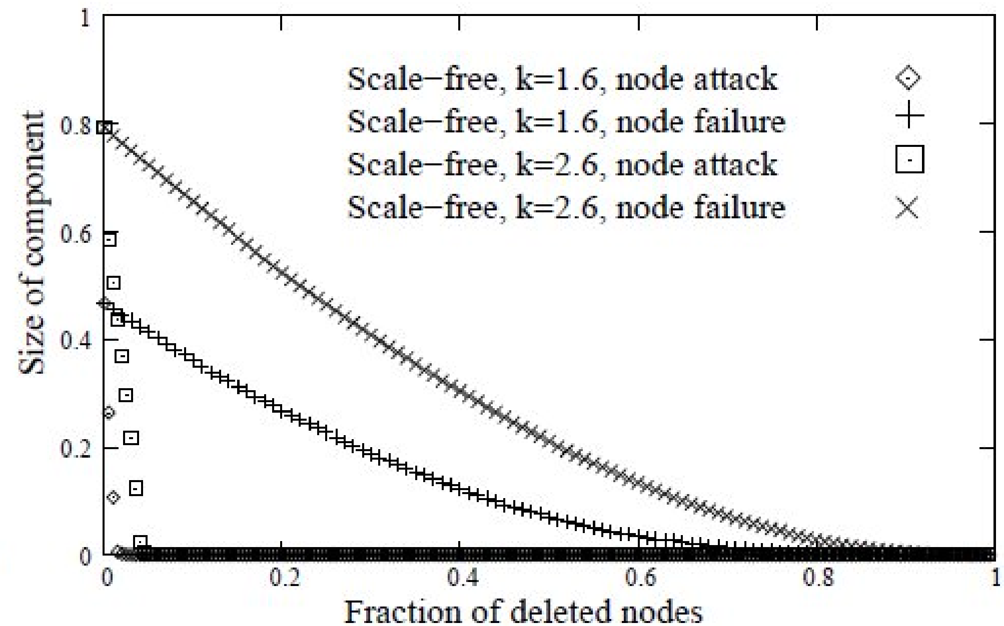
Random network



Time

How vulnerable are networks to attacks ?





LNCS **3544**, (2005)

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New model of disease contagion ranks U.S. airports in terms of their spreading influence

Airports in New York, Los Angeles and Honolulu are judged likeliest to play a significant role in the growth of a pandemic.

Denise Brehm, Civil and Environmental Engineering

Summary

Random walks on networks are interesting for its inherent properties and applications.

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Aanjaneya Kumar (*Reachability on scalefree networks*)