STRATEGIC NETWORK FORMATION: STABILITY VS EFFICIENCY

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OUTLINE

- Myerson (1977) introduced the concept of a Communication situation as a special type of Cooperative game where the players are restricted to cooperate only through existing links.
- A Communication situation is a triple (N, U, g), where (N, U) is a Cooperative game and (N, g) is an undirected network.
- $S/g = \{\{i | i \text{ and } j \text{ are connected in } S \text{ by } g\} | j \in S\}$.
- An allocation rule is a function $Y : G \times G^N \to \mathbb{R}^n$ such that for all $g \in G, S \in N/g$, $i \in S$ we have $\sum_{i \in N} Y_i(g, U) = U(S)$.

Let $N = \{1, 2, 3, 4, 5, 6\}$ and $g = \{12, 23, 45, 56, 46\}$. Let $S = \{1, 2, 4\}$. Then $S/g = \{\{1, 2\}, \{4\}\}$ and $N/g = \{\{1, 2, 3\}, \{4, 5, 6\}\}$. Fig 2 illustrates the structure of g.

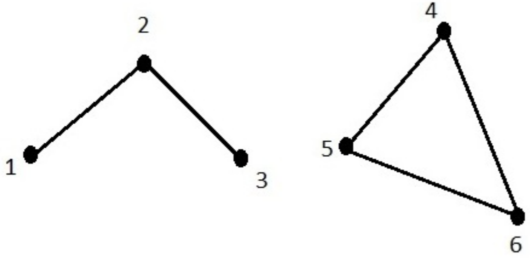

Fig 2: An Example of a Communication Situation

Given a characteristic function U and a network g, the associated Cooperative game U/g is defined for all $S \subseteq N$ as follows.

$$(U/g)(S) = \sum_{T \in S/g} U(T) \tag{1}$$

THEOREM

(Myerson, 1977, pg. 227) Given a characteristic function U, The Myerson value Y^{MV} is the unique fair allocation rule $Y : G \to \mathbb{R}^n$, which satisfies component balance and equal bargaining power. Moreover we have $Y(g, U) = \Phi(U/g), \forall g \in G$, where $\Phi(.)$ is the Shapley value operator.

Let $N = \{1,2,3\}$ and define a game v : $v(1) = v(2) = v(3) = 0, v(1,2) = v(1,2,3) = 12, v(2,3) = v(1,3) = 6$.
Thus

$$Y(N, v, \emptyset) = (0,0,0),\ Y(N, v, \{12\}) = (6,6,0)$$

$$Y(N, v, \{23\}) = (0,3,3)$$

$$Y(N, v, \{13\}) = (3,0,3)$$

$$Y(N, v, \{12, 23\}) = (4,7,1)$$

$$Y(N, v, \{13, 23\}) = (3,3,6)$$

$$Y(N, v, \{12, 13\}) = (7,4,1)$$

$$Y(N, v, \{12, 23, 13\}) = (5,5,2)$$

Criticism of communication situations

Is the value for $\{12, 23\}$ and $\{12, 23, 13\}$ the same?
The buyer-seller model $\{s, b_1, b_2\}$...

- Given the player set $N = \{1, 2, ..., n\}$, let $G = \{g | g \subseteq g^N\}$ denote the set of all possible networks on N.
- The network obtained by adding two networks g and g' is denoted by $g + g'$ and the network obtained by deleting subnetwork g' from an existing network g is denoted $g \setminus g'$.
- For $g \in G$, $L(g)$ denotes the set of all links in g and $l(g)$, the total number of such links.
- $N(g) = \{i \mid \exists\ j \text{ such that } ij \in g\}$
- $L_i(g) = \{ij \mid \exists\ j\ :\ ij \in g\}g|_S = \{ij \mid ij \in g \text{ and } i \in S, j \in S\}$
- Let $n(g) = \#N(g)$ denote the number of players involved in g. We denote by $l_i(g)$ the number of links in player i's link set. It follows that $l(g) = \frac{1}{2} \sum_i l_i(g)$.

A component of a network g, is a non-empty subnetwork $g' \subset g$, such that

- (A) if $i \in N(g')$ and $j \in N(g')$ where $j \neq i$, then there exists a path in g' between i and j, and
- (B) if $i \in N(g')$ and $ij \in g$, then $ij \in g'$.

A value function is a function $v : G \to \mathbb{R}$ such that $v(\emptyset) = 0$, where $\emptyset$ represents the empty network i.e. network without links. The set of all possible value functions is denoted by V.

A value function $v \in V$ is component additive if

$$v(g) = \sum_{g' \in C(g)} v(g'), \quad \text{for any} \quad g \in G.$$

Given a network $g \in G$, each of the following special value functions is a basis for V.

$$v_g(g') = \begin{cases} 1 & \text{if } g \subseteq g' \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

$$\hat{v}_g(g') = \begin{cases} 1 & \text{if } g \subsetneq g' \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

and,

$$v^*_g(g') = \begin{cases} 1 & \text{if } g = g' \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

A Network game is a pair, (N, v), consisting of a set of players and a value function. If N is fixed and no confusion arises about this, we denote the Network game (N, v) by only v.
Given a Network game (N, v), the associated Cooperative game $U_{g,v}$ for all $S \subseteq N$ is defined by,

$$U_{g,v}(S) = \sum_{g' \in C(g|_S)} v(g')$$

An allocation rule is a function $Y : G \times V \to \mathbb{R}^n$ such that $Y_i(g, v)$ represents the payoff to player i with respect to v and g and $\sum_i Y_i(g, v) = v(g), \ \forall \ v$ and g.

The Coauthor Model (Jackson and Wolinsky 1996)

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- Agent:researcher who spends time working on research projects.
- If two researchers are connected, then they are working on a project together.
- Each player has a fixed amount of time to spend on research, and so the time that researcher i spends on a given project is inversely related to the number of projects, n_i , that she is involved in.
- For each i, $n_i > 0$
- The synergy between two researchers depends on how much time they spend together, and this is captured by a term $1/n_in_j$.
- The $v(g) = \sum_{i\in N(g)} \sum_{j\in N(g):ij\in g} 1/n_i + 1/n_j + 1/n_in_j$.

An allocation rule Y is component balanced if for any component additive $v,\ g \in G$, and $g' \in C(g)$,

$$\sum_{i \in N(g')} Y_i(g, v) = v(g').$$

An allocation rule Y satisfies equal bargaining power if for any component additive v , $g \in G$ and $i, j \in N$, it holds that,

$$Y_i(g, v) - Y_i(g - ij, v) = Y_j(g, v) - Y_j(g - ij, v).$$

An allocation rule Y satisfies balanced contributions if for any component additive v , $g \in G$ and $i, j \in N$, it holds that,

$$Y_i(g, v) - Y_i(g_{-j}, v) = Y_j(g, v) - Y_j(g_{-i}, v).$$

PLAYER-BASED ALLOCATION RULE

THEOREM

(Jackson, 2005, pg. 134) Y satisfies component balance and equal bargaining power if and only if $Y(g,v) = Y^{NMV}(g,v)$ for all $g \in G$ and any component additive v.

The Myerson value for Network games $Y^{NMV} : G \times V \to \mathbb{R}^n$ is obtained as follows.

$$Y_i^{NMV}(g,v) = \sum_{S \subset N \setminus \{i\}} \frac{s!(n-s-1)!}{n!} (v(g|_{S \cup i}) - v(g|_S))$$

EXAMPLE

Let $N = \{1,2,3\}$ and $g = \{12,23,13\}$. Let $v(g) = 3$, $v(12,23) = v(13,23) = 2$, $v(12,13) = 4$, $v(12) = 2$, $v(23) = 1$, $v(13) = 0$. The Myerson value Y^{MV} is

Link-based allocation rule

The Position value $Y^{PV} : G \times V \to \mathbb{R}^n$ is formally given by,

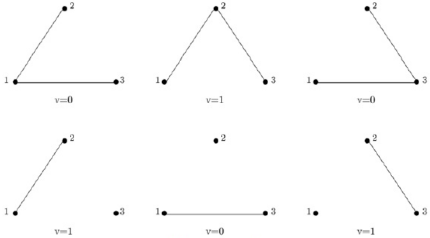
$$Y_i^{PV}(g,v) = \sum_{\substack{i \neq j, \\ ij \in g}} \Big(\sum_{g' \subset N \setminus ij} \frac{1}{2}(v(g'+ij) - v(g'))\Big) \frac{l(g')!(l(g)-l(g')-1)!}{l(g)!}$$

Example

$N = \{1,2,3\}$, $g = \{12,23,13\}$, $v(g) = 3$, $v(12,23) = v(13,23) = 2$, $v(12,13) = 4$, $v(12) = 2$, $v(23) = 1$ and $v(13) = 0$. After simple calculations, the Position value is obtained as $Y^{PV}(g,v) = ?$.

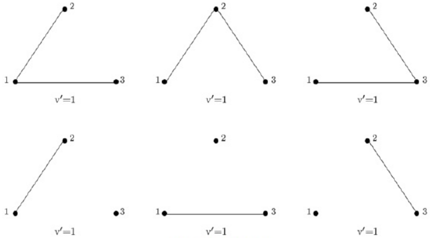
Egalitarian Allocation Rule
Link-based Egalitarian Allocation Rule
Component-wise Egalitarian Allocation Rule
Banzhaf Allocation Rule... Pair-wise merging property
Link-based Banzhaf Allocation Rule
Multilateral Interactive allocation rule
The Core in Network Settings...

Flexible Network: Criticism of the Myerson Value



Myerson value under v

Consider another value function v' defined by $v'(g) = 1$ for all $g \neq \emptyset$. That is, under v' the value of every nonempty network is 1.



Myerson value under v'

$$Y^{MV}(\{12,23\},v) = Y^{MV}(\{12,23\},v') = (\tfrac{1}{6},\tfrac{2}{3},\tfrac{1}{6}).$$

Given a value function v, its monotonic cover $\hat{v}$ is defined as

$$\hat{v}(g) = \max_{g' \subset g} v(g')$$

The monotonic cover associated with a given Network game represents a value function that gives the best possible value to a network choosing from its subnetworks.
Let three players 1, 2 and 3 forming the complete network $g = \{12, 23, 31\}$.
$v(\{12, 23, 31\}) = 2$, $v(\{12, 23\}) = 4$ and
$v(\{13, 23\}) = 1 = v(\{12, 13\})$.
$\hat{v}(\{12, 23, 31\}) = 4.$

An allocation rule Y satisfies equal treatment of vital players, weak additivity and is a flexible network rule iff it is defined by

$$Y_i^{PBFN}(g,v) = \frac{v(g)}{\hat{v}(g^N} \sum_{S \subseteq N \setminus \{i\}} (\hat{v}(g^{S \cup \{i\}}) - \hat{v}(g^S))(\frac{s!(n-s-1)!}{n!})$$

for all $v \in V$ and each $g \in G$ that is efficient relative to v.

A network g is efficient relative to v if $v(g) \geq v(g')$, $\forall g' \in G$. maximization of the overall total value among all possible networks.

A network g is pareto efficient relative to v and Y if no $g' \in G$ exists such that $Y_i(g', v) \geq Y_i(g, v), \forall i$

A network g is pairwise stable with respect to an allocation rule Y and v if

- for all $ij \in g$, $Y_i(g,v) \geq Y_i(g-ij,v)$ and $Y_j(g,v) \geq Y_j(g-ij,v)$
- for all $ij \notin g$, $Y_i(g+ij,v) > Y_i(g,v)$ and $Y_j(g+ij,v) < Y_j(g,v)$

no player wish to delete a link that he or she is involved in and if some link is not in the network and one of the involved players would benefit from adding it, then it must be that the other player would suffer from the addition of the link.

- Pairwise stable networks may not exist in some settings for some allocation rules, there are interesting allocation rules for which pairwise stable networks always exist.
- Under the Myerson value allocation rule there always exists a pairwise stable network.
- Under the egalitarian rule, any efficient network will be pairwise stable.
- Under the component-wise egalitarian rule, one can also always find a pairwise stable network.

An allocation Y is anonymous if $Y_{\pi(i)}(g^{\pi}, v^{\pi}) = Y_i(g, v), \forall \pi$, where v^{π} is defined as $v^{\pi}(g^{\pi}) = v(g)$.
Anonymity states that if all that has changed is the names of the agents, then the allocations they receive should not change

Theorem
There is no allocation rule Y which is anonymous and component balanced such that for each v at least one efficient network is pairwise stable.

- A link i j is critical to the graph g if $g\hat{a}ij$ has more components than g.
- Let h denote a component that contains a critical link, and let h_1 and h_2 denote the components obtained from h by severing that link.
- The pair (g, v) satisfies critical link monotonicity if, for any critical link in g and its associated components h, h_1 and h_2, we have that $v(h) \leq v(h_1) + v(h_2)$ implies that $v(h)/n(h) \leq max[v(h1)/n(h_1), v(h2)/n(h_2)]$.
- If g is efficient relative to a component additive v, then g is pairwise stable for Y^{ce} relative to v if and only if (g, v) satisfies critical-link monotonicity.