

# Lecture 20: Generative Models, Part 2

Admin: A4

A4 due yesterday, many people still working

# Admin: A5

A5 Released last night

Recurrent networks, image captioning, Transformers

Due Tuesday April 12th at 11:59pm ET

# Admin: Project Proposal

If you want to propose your own project:

Need to submit a project proposal by tomorrow, 4/1 on Piazza

# Last Time: Supervised vs Unsupervised Learning

## Supervised Learning

**Data:**  $(x, y)$

$x$  is data,  $y$  is label

**Goal:** Learn a *function* to map  $x \rightarrow y$

**Examples:** Classification, regression, object detection, semantic segmentation, image captioning, etc.

## Unsupervised Learning

**Data:**  $x$

Just data, no labels!

**Goal:** Learn some underlying hidden *structure* of the data

**Examples:** Clustering, dimensionality reduction, feature learning, density estimation, etc.

# Last Time: Discriminative vs Generative Models

## **Discriminative Model:**

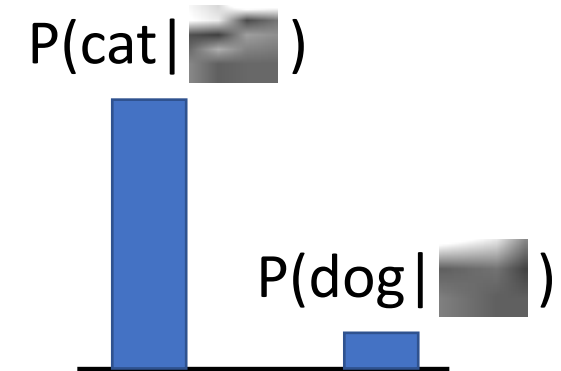
Learn a probability distribution  $p(y|x)$

## **Generative Model:**

Learn a probability distribution  $p(x)$

**Conditional Generative Model:** Learn  $p(x|y)$

**Data:  $x$**



## **Density Function**

$p(x)$  assigns a positive number to each possible  $x$ ; higher numbers mean  $x$  is more likely

Density functions are **normalized**:

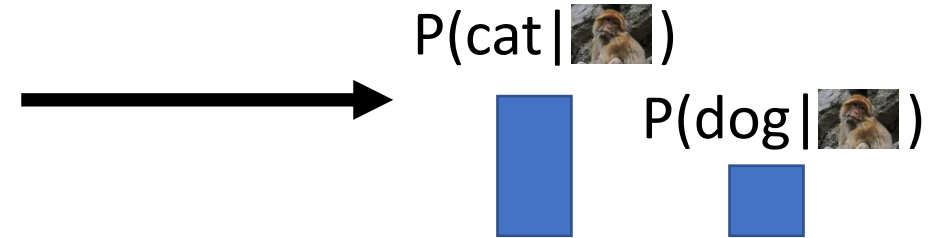
$$\int_X p(x)dx = 1$$

Different values of  $x$  **compete** for density

# Last Time: Discriminative vs Generative Models

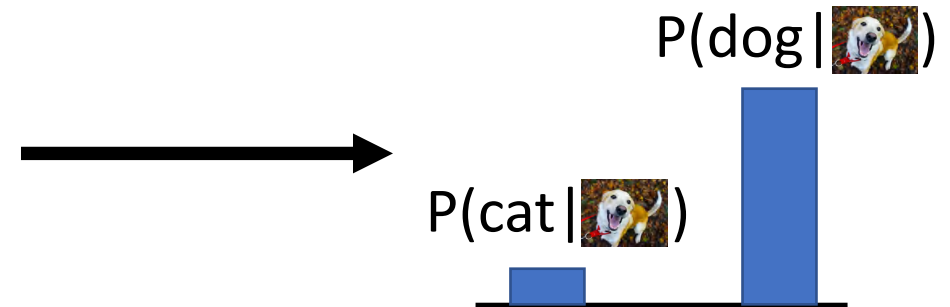
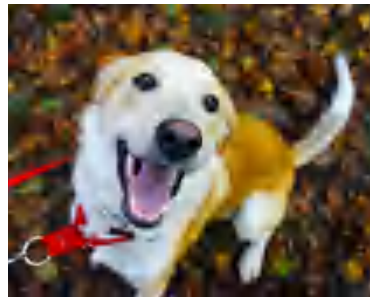
## Discriminative Model:

Learn a probability distribution  $p(y|x)$



## Generative Model:

Learn a probability distribution  $p(x)$



## Conditional Generative Model: Learn $p(x|y)$

Discriminative model: No way for the model to handle unreasonable inputs; it must give label distributions for all images

[Monkey image](#) is CC0 Public Domain

# Last Time: Discriminative vs Generative Models

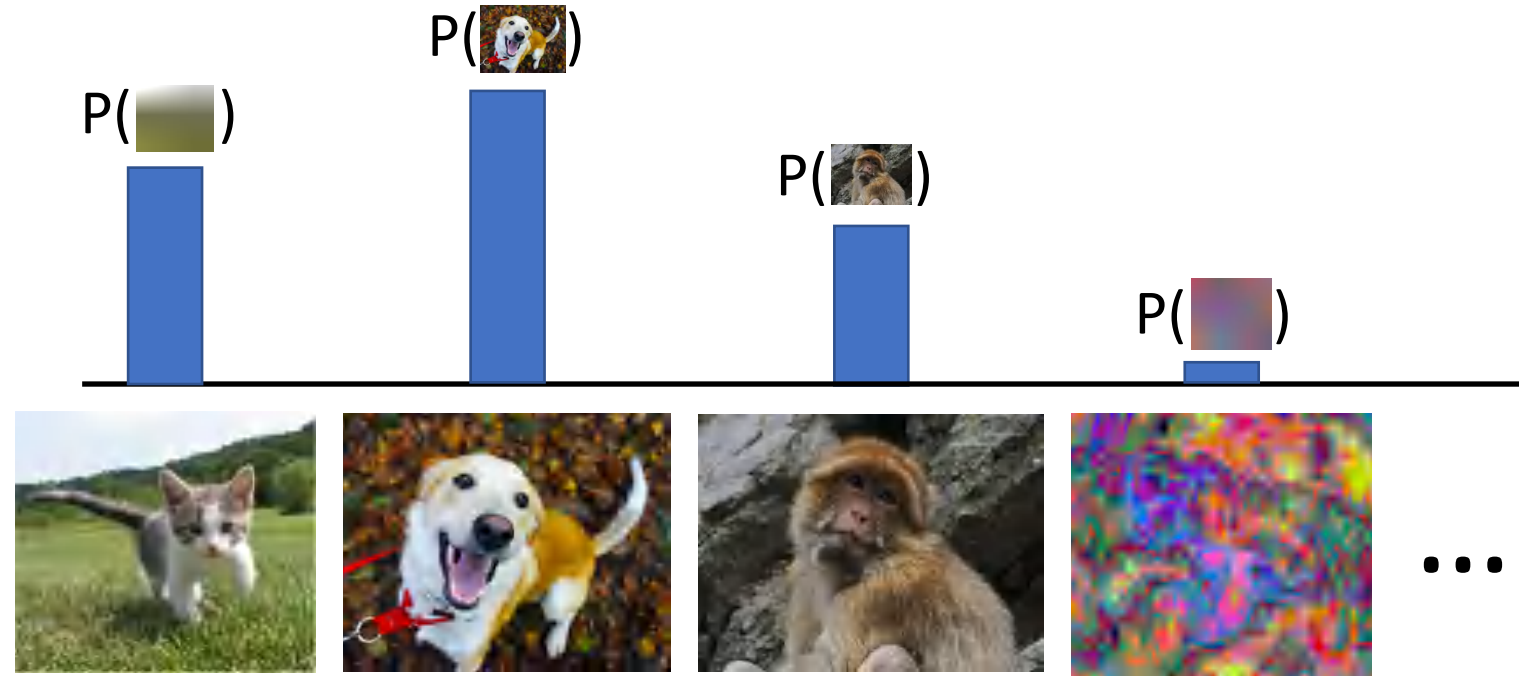
## Discriminative Model:

Learn a probability distribution  $p(y|x)$

## Generative Model:

Learn a probability distribution  $p(x)$

**Conditional Generative Model:** Learn  $p(x|y)$



Generative model: All possible images compete with each other for probability mass

Requires deep image understanding! Is a dog more likely to sit or stand? How about 3-legged dog vs 3-armed monkey?



# Last Time: Discriminative vs Generative Models

## Discriminative Model:

Learn a probability distribution  $p(y|x)$

## Generative Model:

Learn a probability distribution  $p(x)$

**Conditional Generative Model:** Learn  $p(x|y)$

Recall **Bayes' Rule**:

$$\underbrace{P(x|y)}_{\text{Conditional Generative Model}} = \frac{\underbrace{P(y|x)}_{\text{Discriminative Model}}}{\underbrace{P(y)}_{\text{Prior over labels}}} \underbrace{P(x)}_{\text{(Unconditional) Generative Model}}$$

We can build a conditional generative model from other components!

# Last Time: Taxonomy of Generative Models

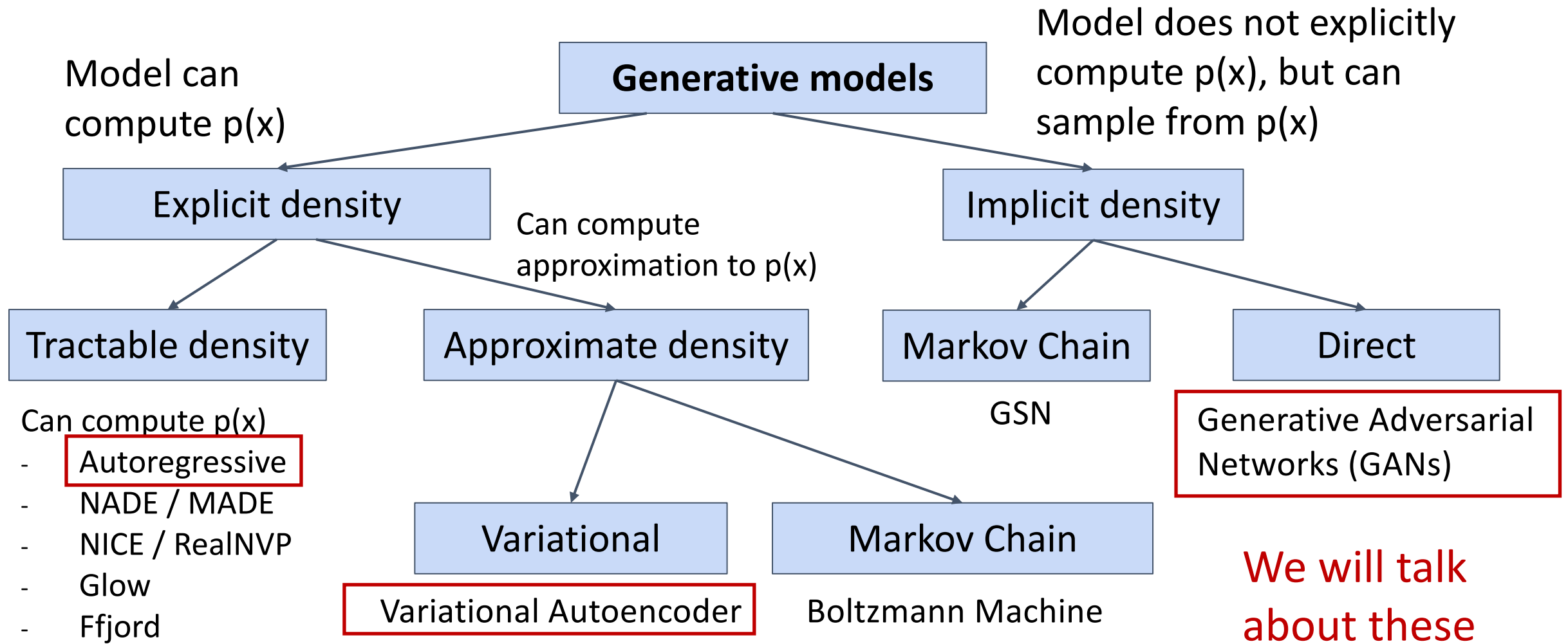


Figure adapted from Ian Goodfellow, Tutorial on Generative Adversarial Networks, 2017.

# Last Time: Autoregressive Models

## Explicit Density Function

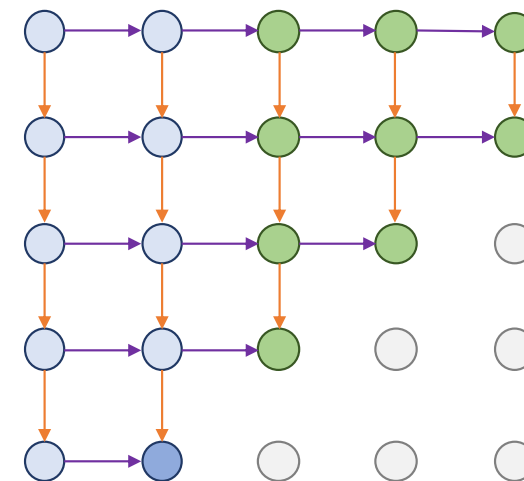
$$\begin{aligned} p(x) &= p(x_1, x_2, x_3, \dots, x_T) \\ &= p(x_1)p(x_2 | x_1)p(x_3 | x_1, x_2) \dots \\ &= \prod_{t=1}^T p(x_t | x_1, \dots, x_{t-1}) \end{aligned}$$

Train by maximizing  
log-likelihood of  
training data

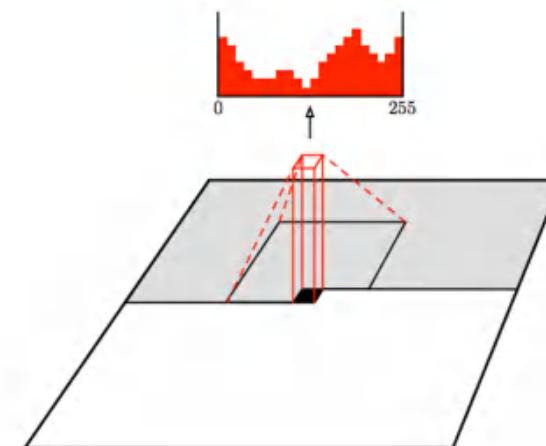
Van den Oord et al, "Pixel Recurrent Neural Networks", ICML 2016

Van den Oord et al, "Conditional Image Generation with PixelCNN Decoders", NeurIPS 2016

PixelRNN



PixelCNN



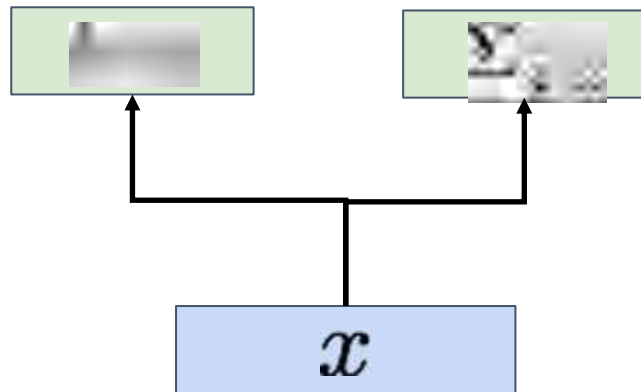
# Last Time: Variational Autoencoders

Jointly train **encoder**  $q$  and **decoder**  $p$  to maximize the **variational lower bound** on the data likelihood

$$\log p_{\theta}(x) \geq E_{z \sim q_{\phi}(z|x)} [\log p_{\theta}(x|z)] - D_{KL} \left( q_{\phi}(z|x), p(z) \right)$$

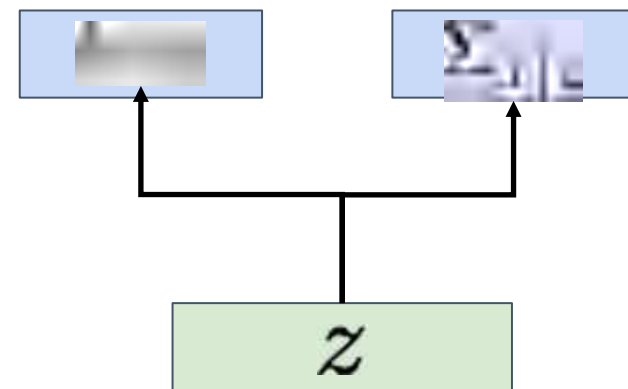
**Encoder Network**

$$q_{\phi}(z | x) = N(\mu_{z|x}, \Sigma_{z|x})$$



**Decoder Network**

$$p_{\theta}(x | z) = N(\mu_{x|z}, \Sigma_{x|z})$$



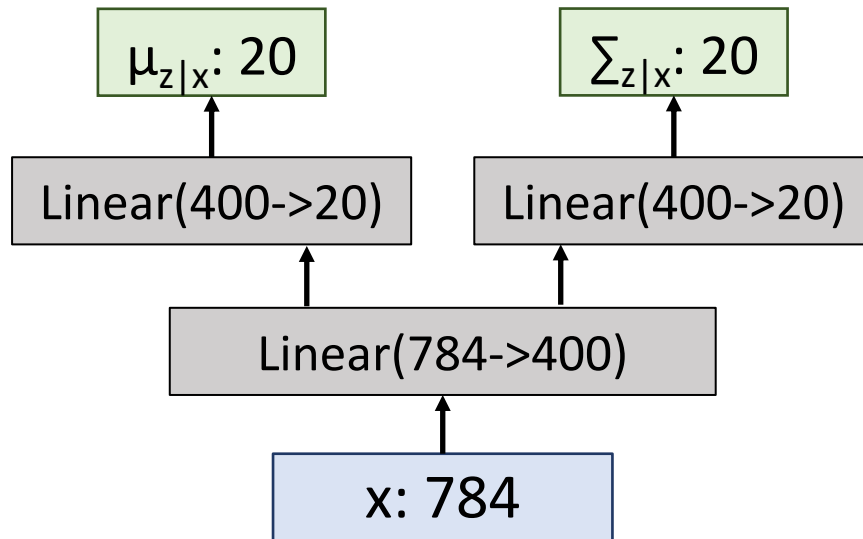
# Example: Fully-Connected VAE

x: 28x28 image, flattened to 784-dim vector

z: 20-dim vector

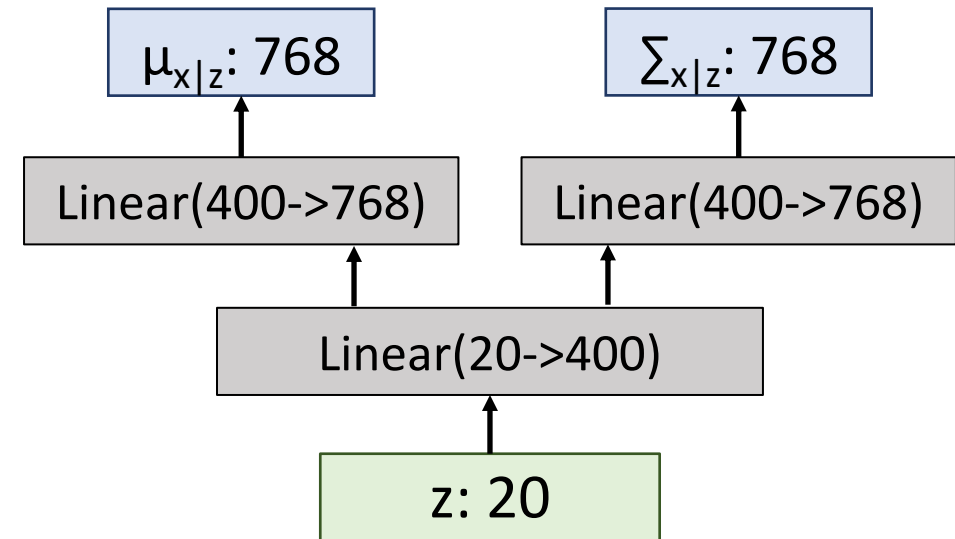
## Encoder Network

$$q_{\phi}(z | x) = N(\mu_{z|x}, \Sigma_{z|x})$$



## Decoder Network

$$p_{\theta}(x | z) = N(\mu_{x|z}, \Sigma_{x|z})$$



# Variational Autoencoders

Train by maximizing the  
**variational lower bound**

$$E_{z \sim q_{\phi}(z|x)} [\log p_{\theta}(x|z)] - D_{KL} (q_{\phi}(z|x), p(z))$$

Input  
Data



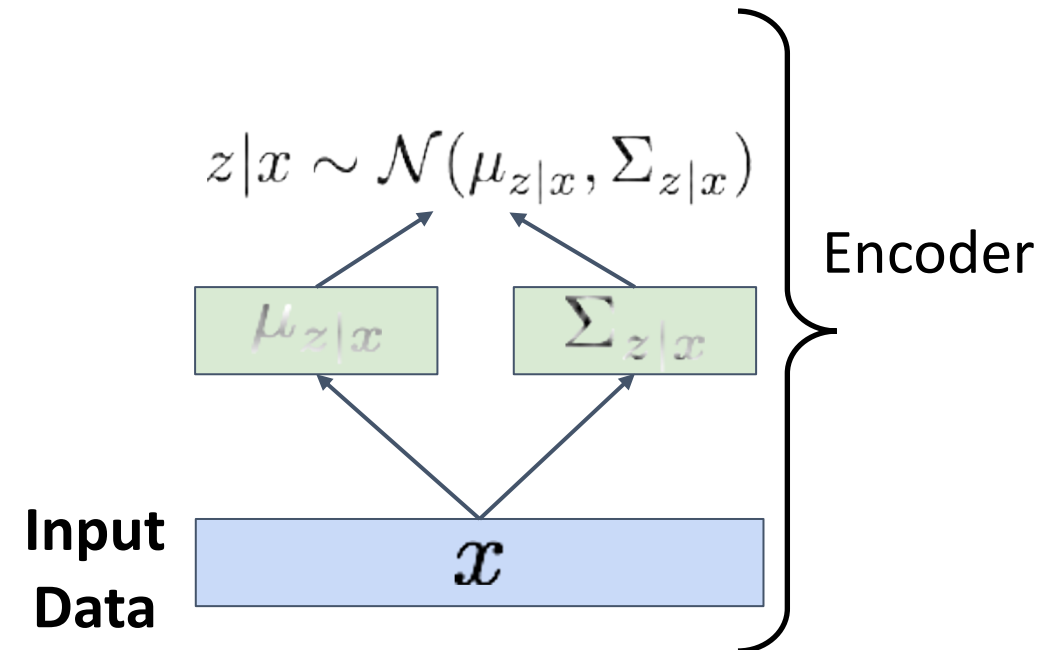
$x$

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1. Run input data through **encoder** to get a distribution over latent codes

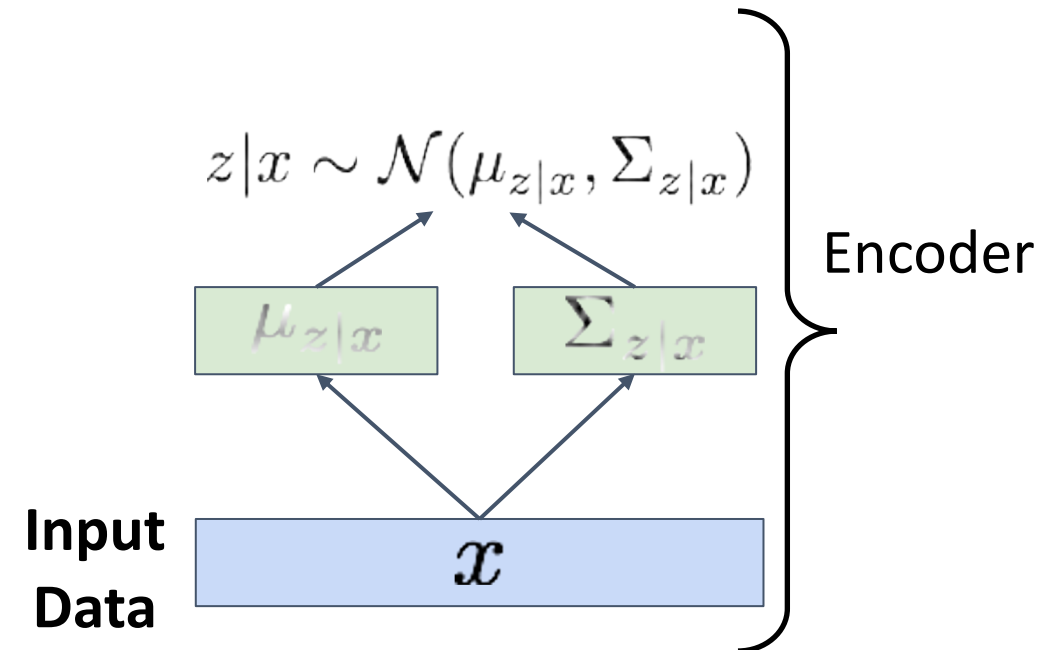


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2. **Encoder output should match the prior  $p(z)$ !**





# Variational Autoencoders

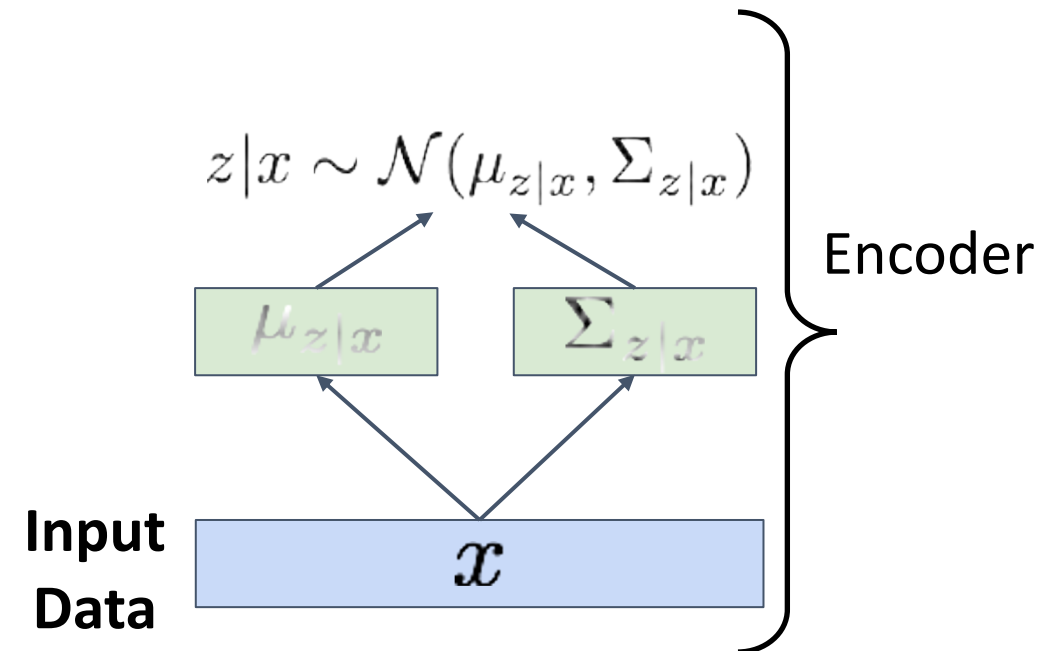
Train by maximizing the  
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$$E_{z \sim q_\phi(z|x)} [\log p_\theta(x|z)] - \boxed{D_{KL}(q_\phi(z|x), p(z))}$$

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$$\begin{aligned} -D_{KL}(q_\phi(z|x), p(z)) &= \int_z q_\phi(z|x) \log \frac{p(z)}{q_\phi(z|x)} dz \\ &= \int_z N(z; \mu_{z|x}, \Sigma_{z|x}) \log \frac{N(z; 0, I)}{N(z; \mu_{z|x}, \Sigma_{z|x})} dz \\ &= \frac{1}{2} \sum_{j=1}^J \left( 1 + \log \left( (\Sigma_{z|x})_j^2 \right) - (\mu_{z|x})_j^2 - (\Sigma_{z|x})_j^2 \right) \end{aligned}$$

Closed form solution when  
 $q_\phi$  is diagonal Gaussian and  
 $p$  is unit Gaussian!  
(Assume  $z$  has dimension  $J$ )

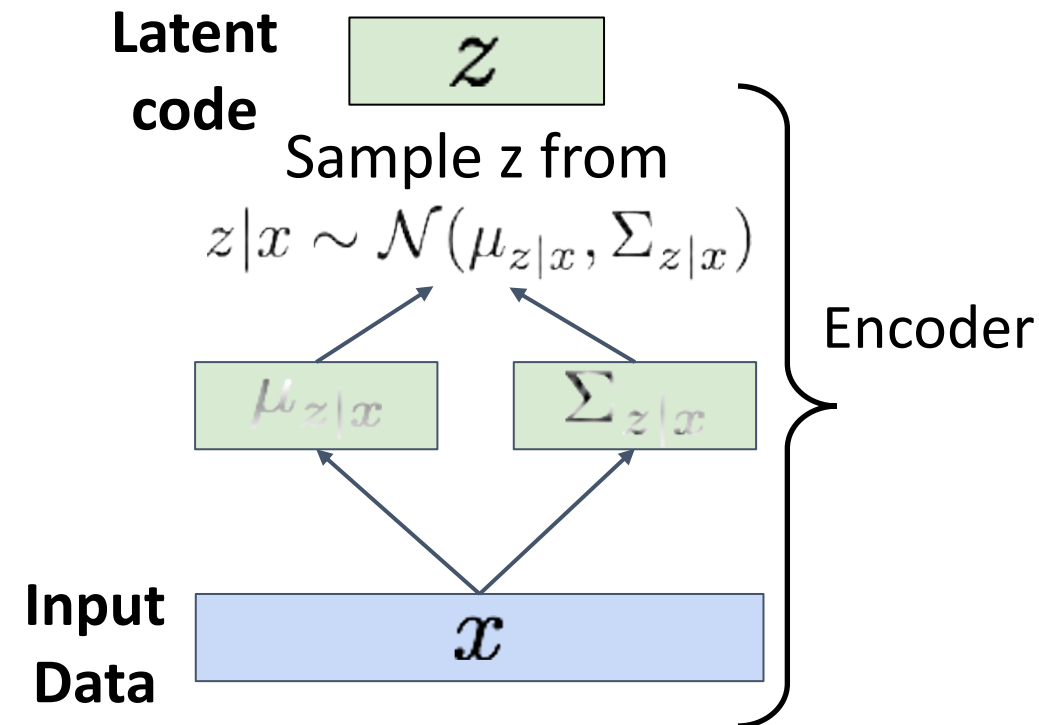


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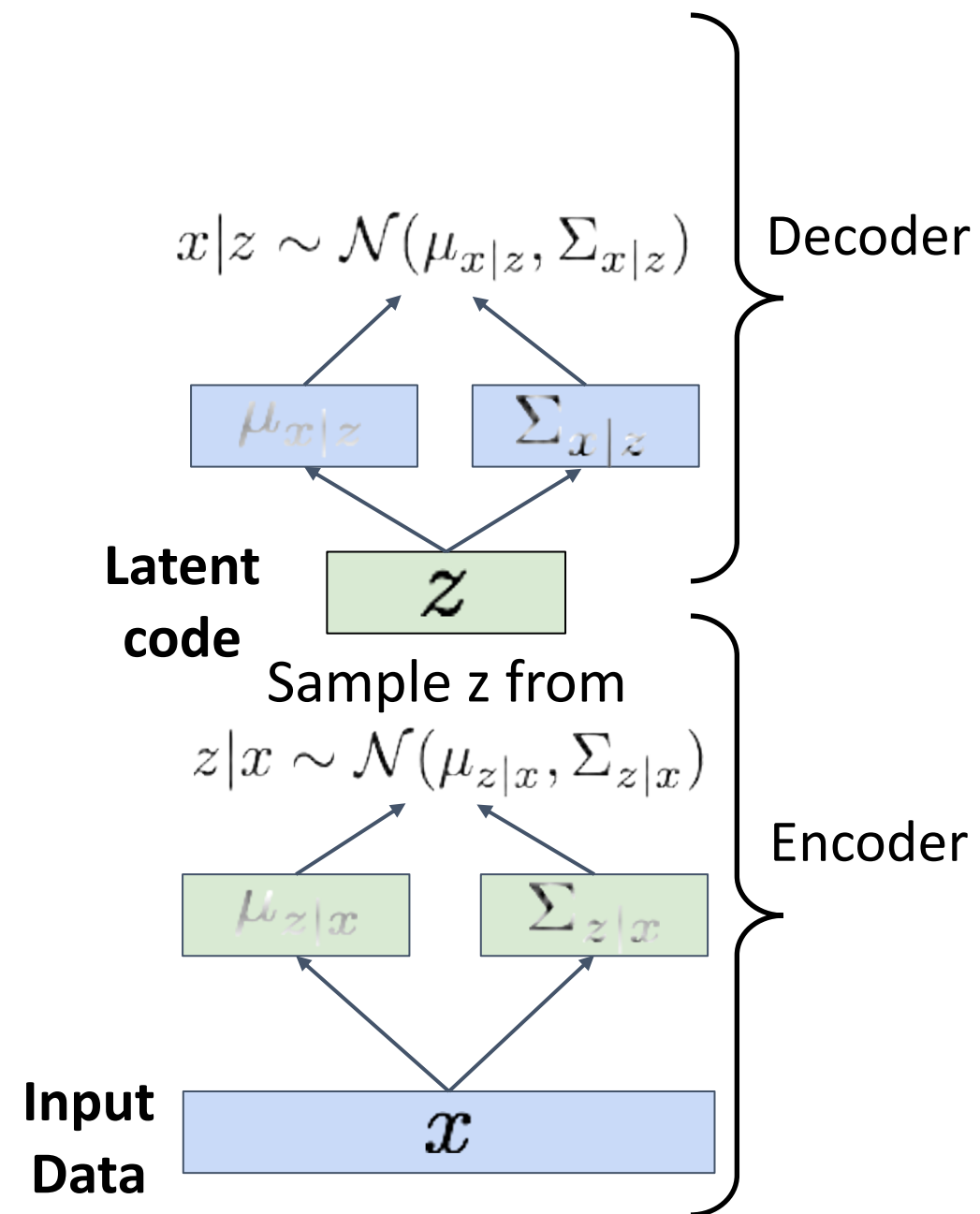


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4. Run sampled code through **decoder** to get a distribution over data samples

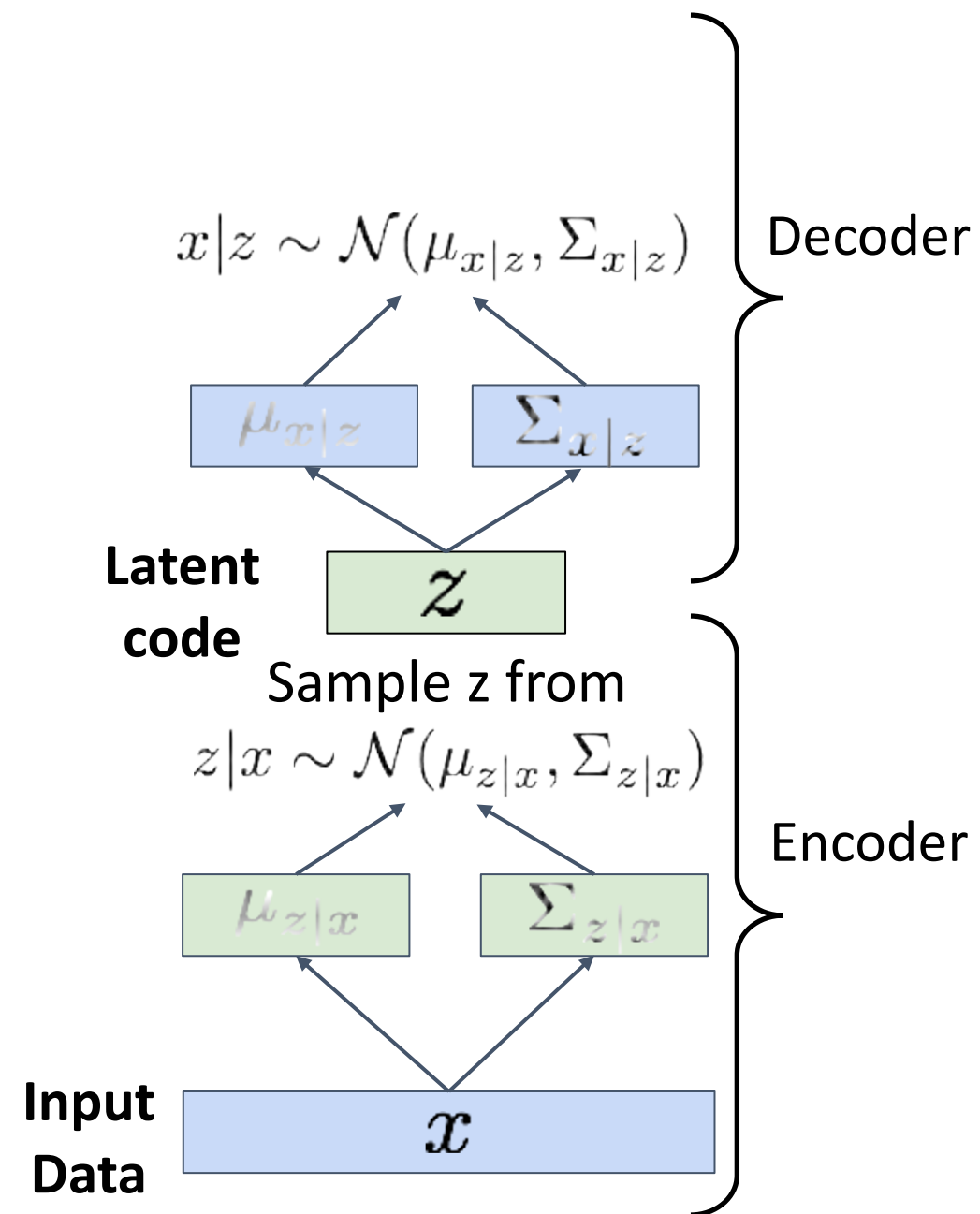


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5. **Original input data should be likely under the distribution output from (4)!**



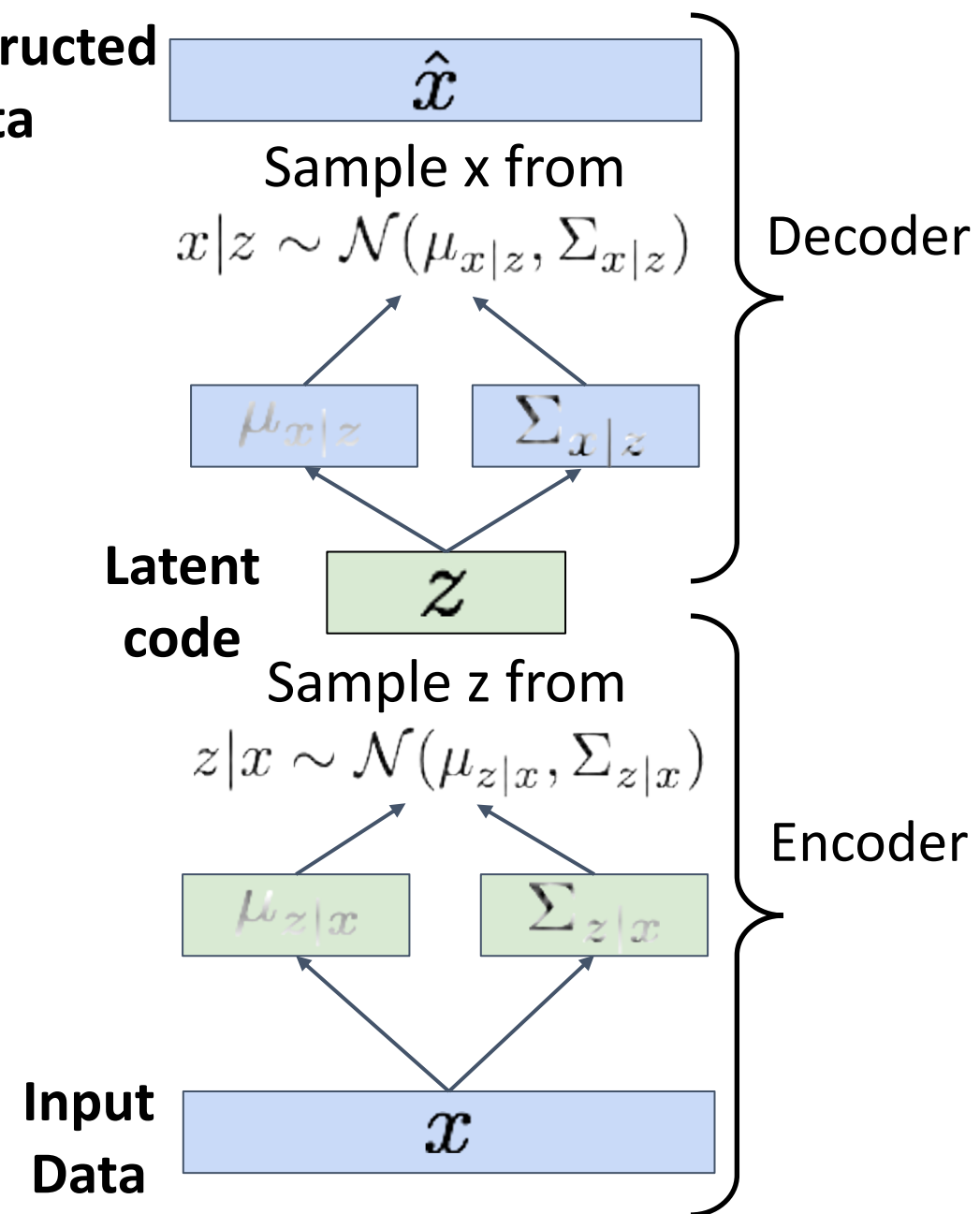
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6. Can sample a reconstruction from (4)

Reconstructed  
data

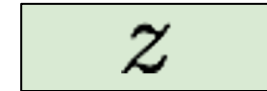


# Variational Autoencoders: Generating Data

After training we can  
generate new data!

1. Sample  $z$  from prior  $p(z)$

**Latent  
code**

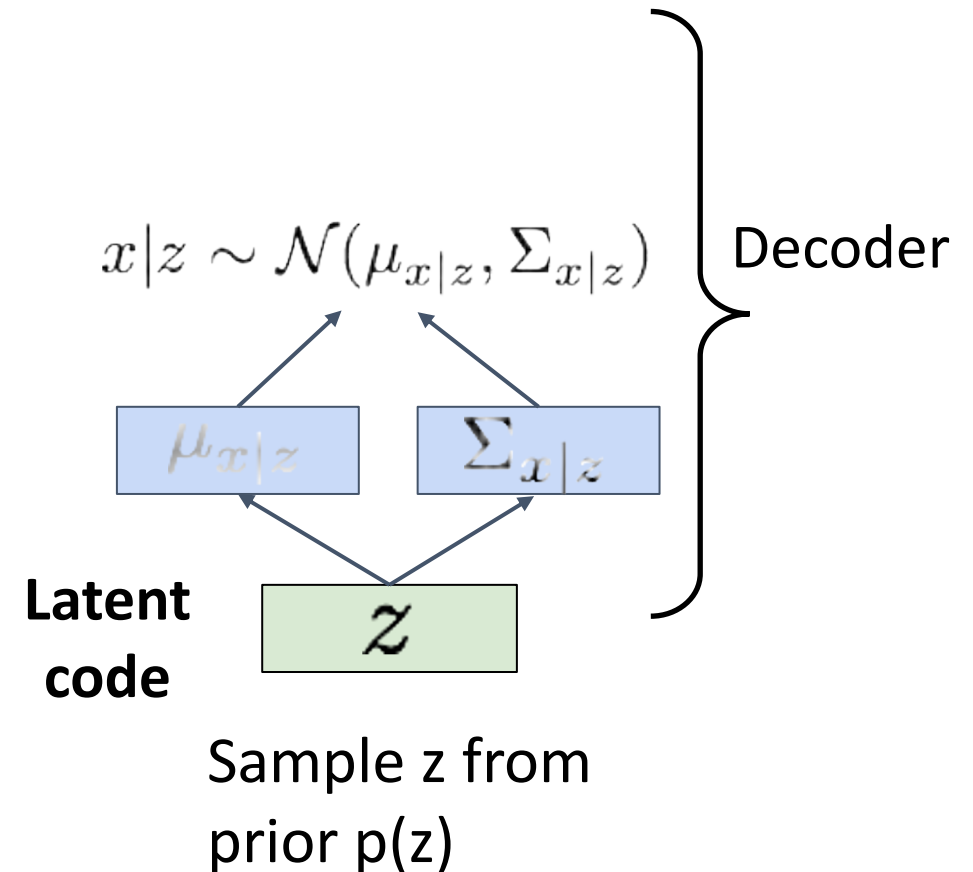


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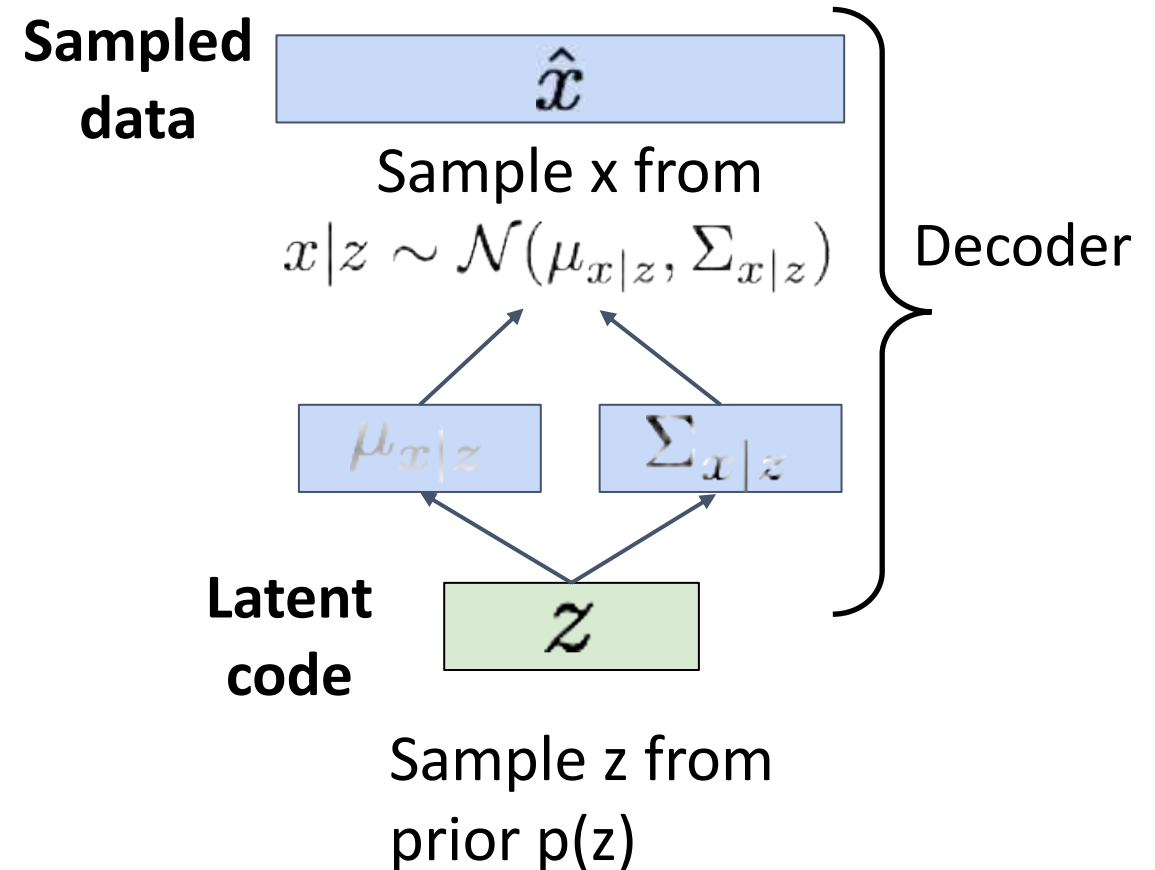
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2. Run sampled  $z$  through decoder to get distribution over data  $x$
3. Sample from distribution in (2) to generate data





# Variational Autoencoders: Generating Data

32x32 CIFAR-10



Labeled Faces in the Wild

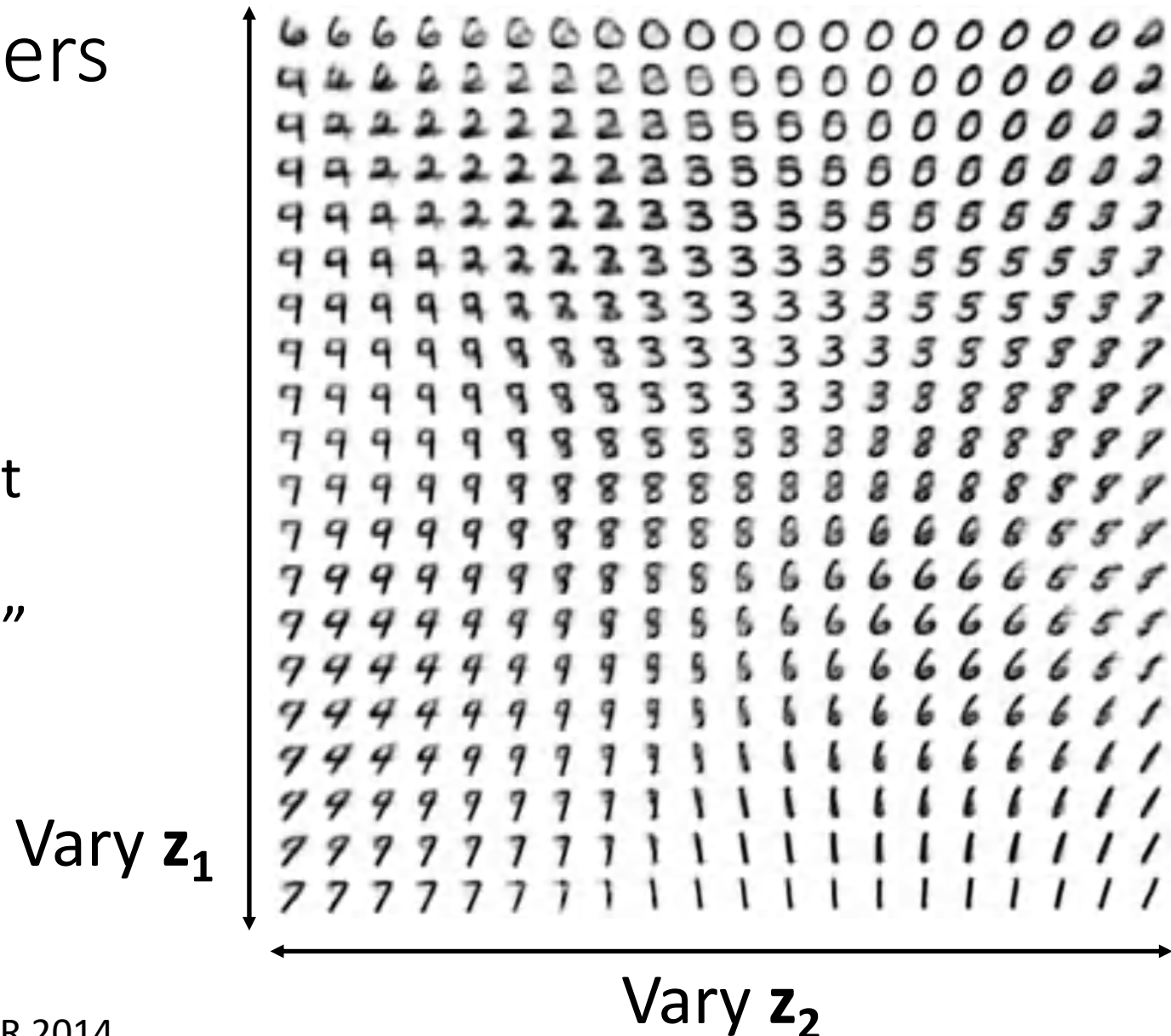


Figures from (L) Dirk Kingma et al. 2016; (R) Anders Larsen et al. 2017.

# Variational Autoencoders

The diagonal prior on  $p(z)$  causes dimensions of  $z$  to be independent

“Disentangling factors of variation”

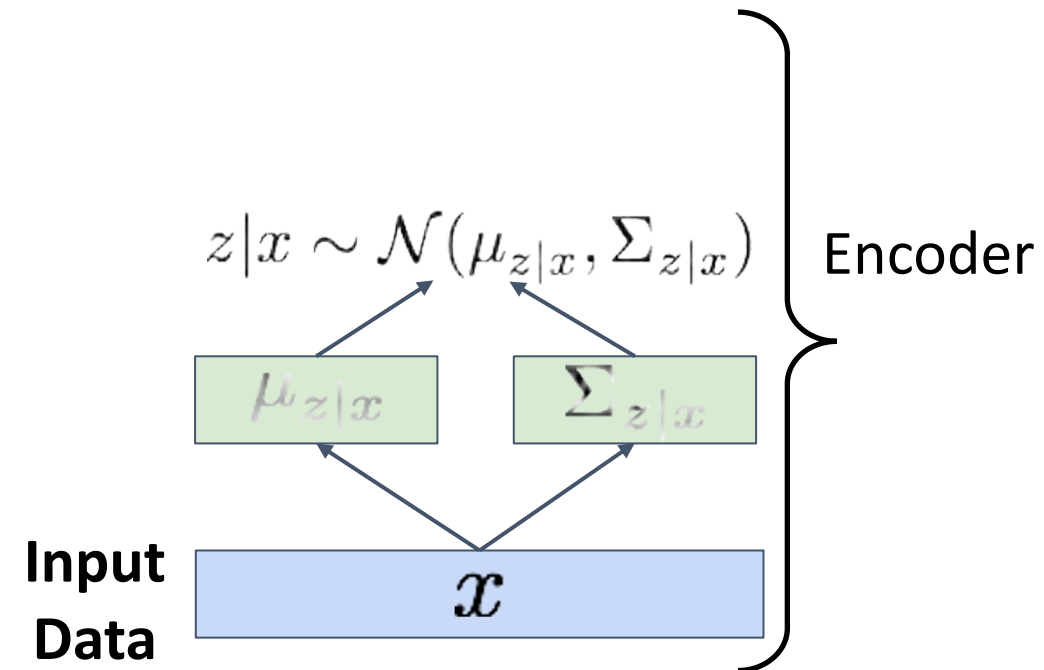


Kingma and Welling, Auto-Encoding Variational Bayes, ICLR 2014

# Variational Autoencoders

After training we can **edit images**

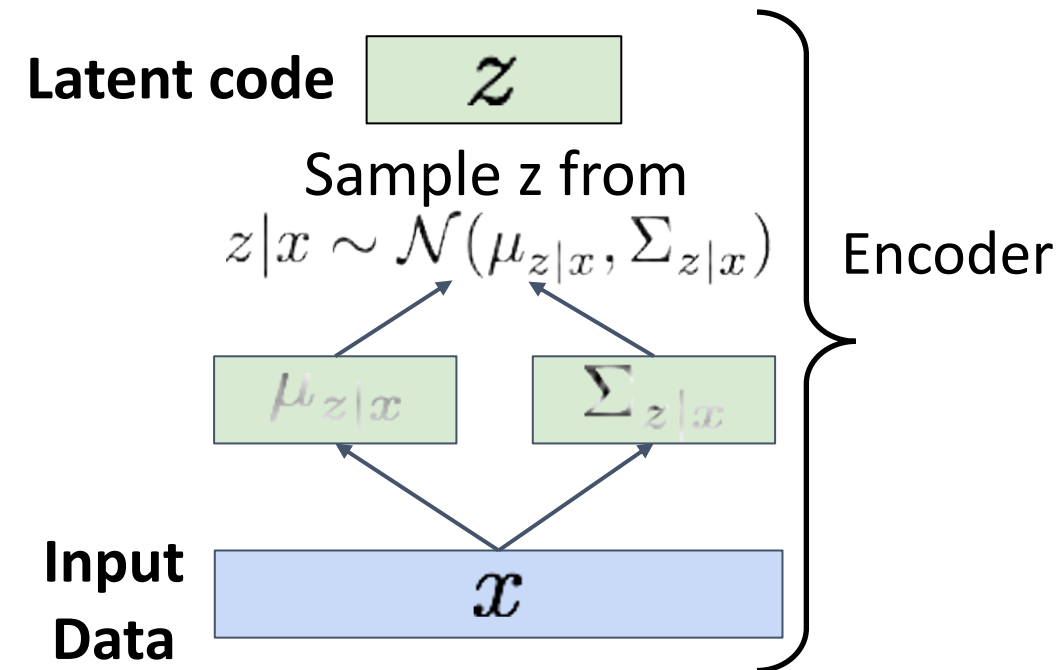
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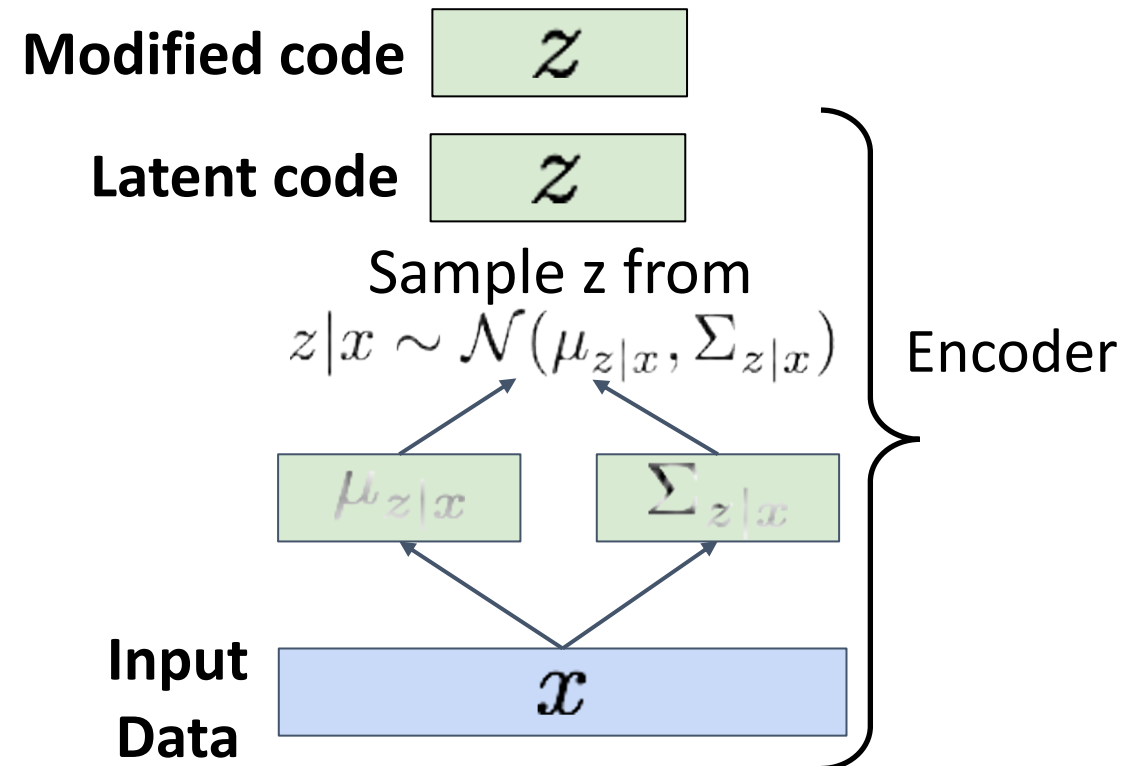
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2. Sample code  $z$  from encoder output



# Variational Autoencoders

After training we can **edit images**

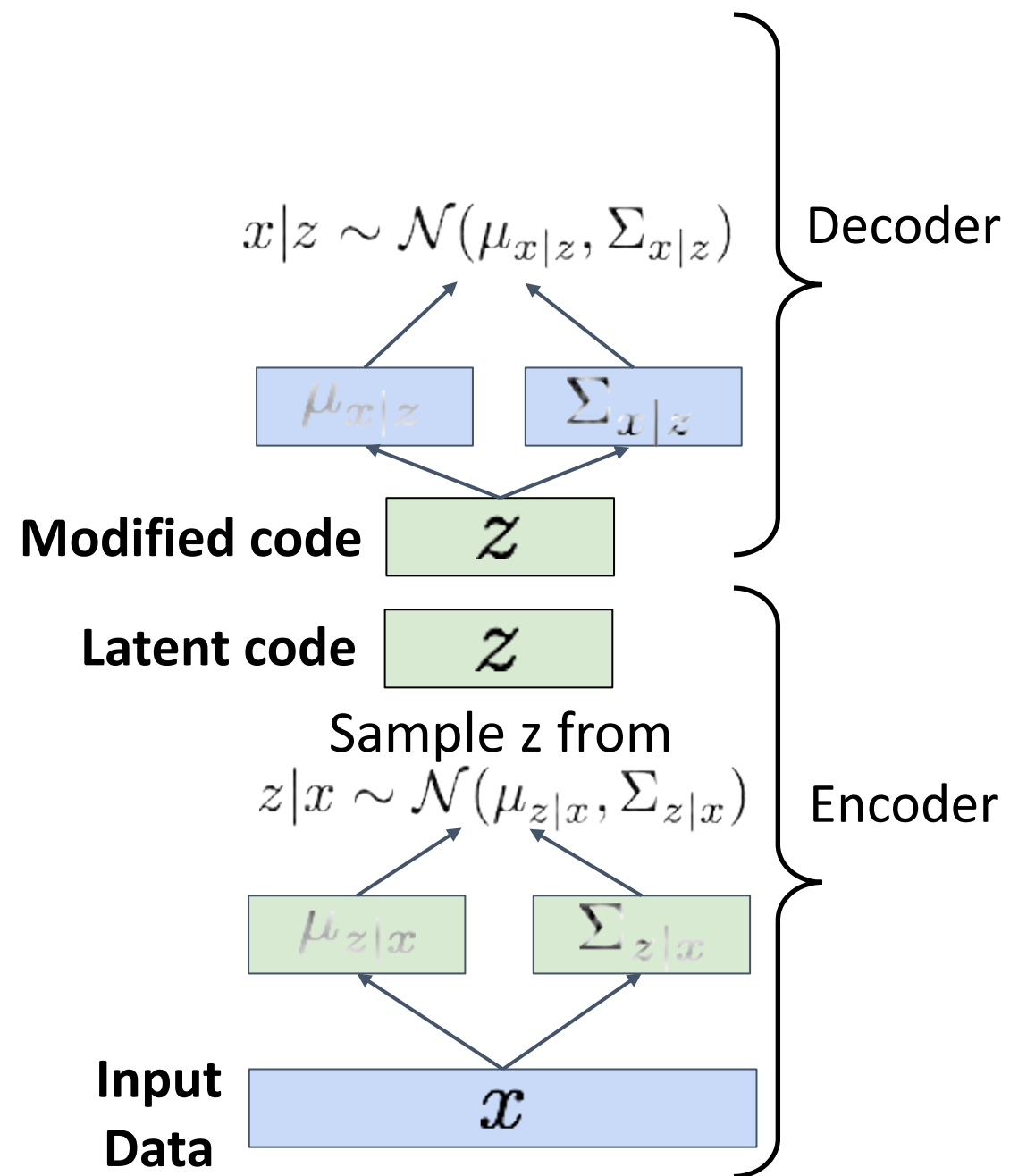
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3. Modify some dimensions of sampled code



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After training we can **edit images**

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4. Run modified  $z$  through **decoder** to get a distribution over data sample

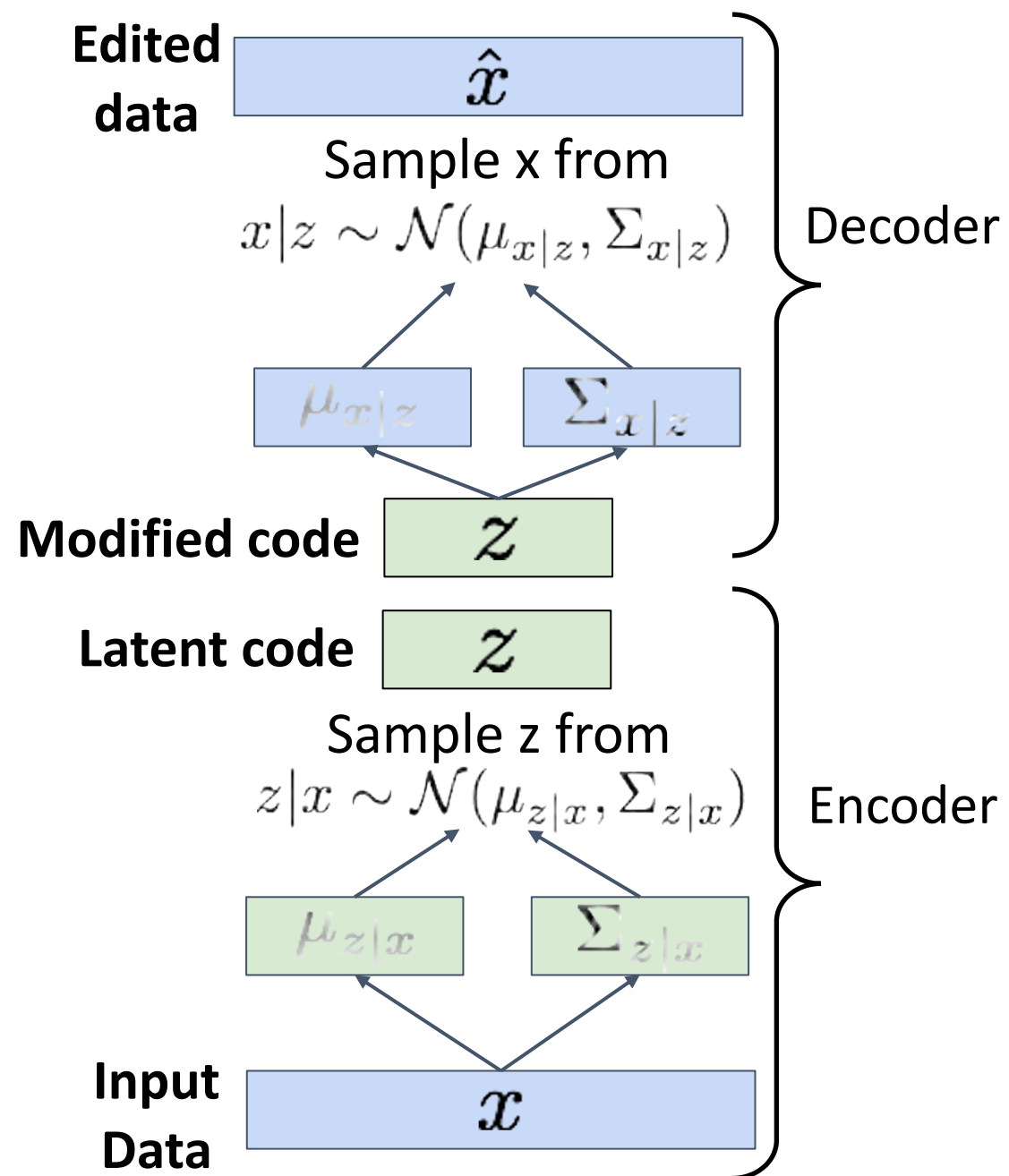




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5. Sample new data from (4)



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The diagonal prior on  $p(z)$  causes dimensions of  $z$  to be independent

“Disentangling factors of variation”

Degree of smile

Vary  $z_1$

Head pose

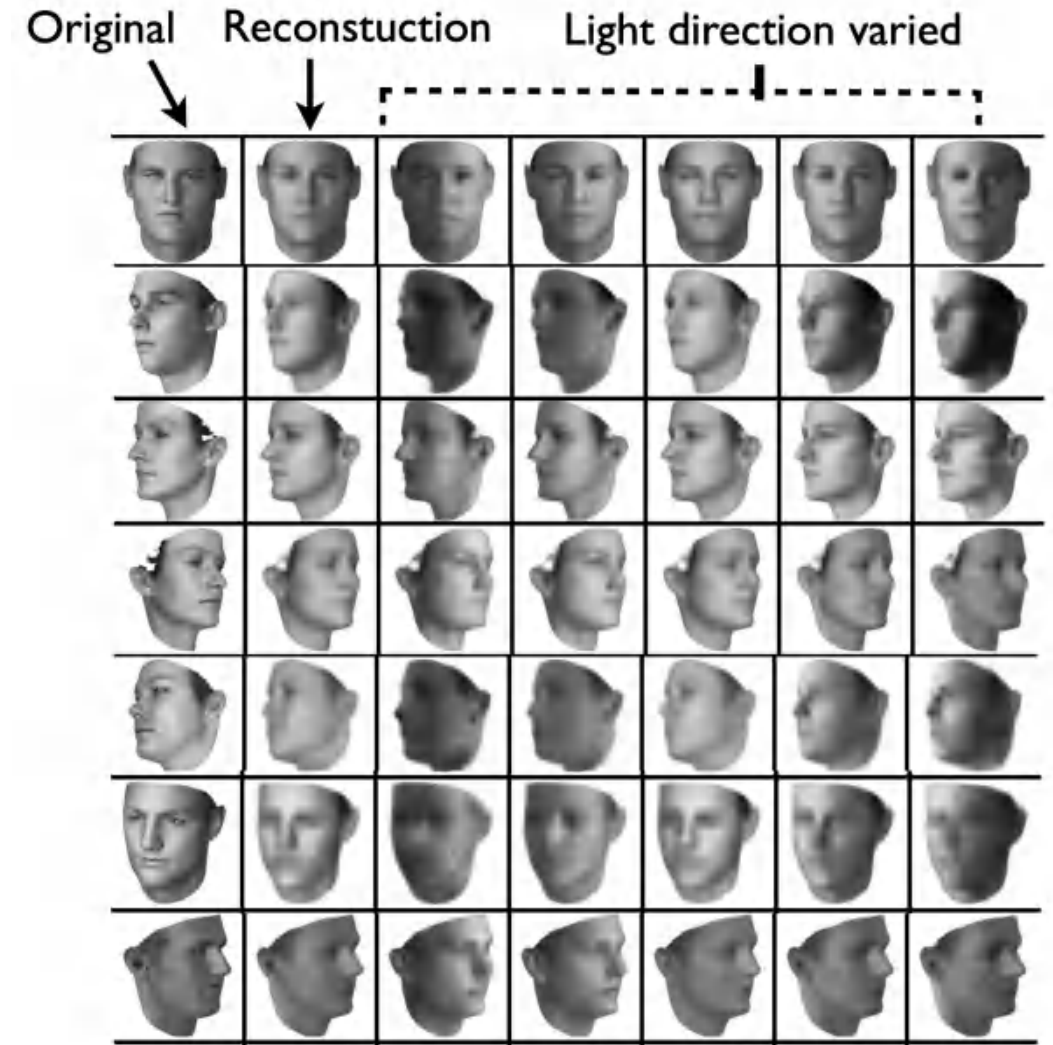
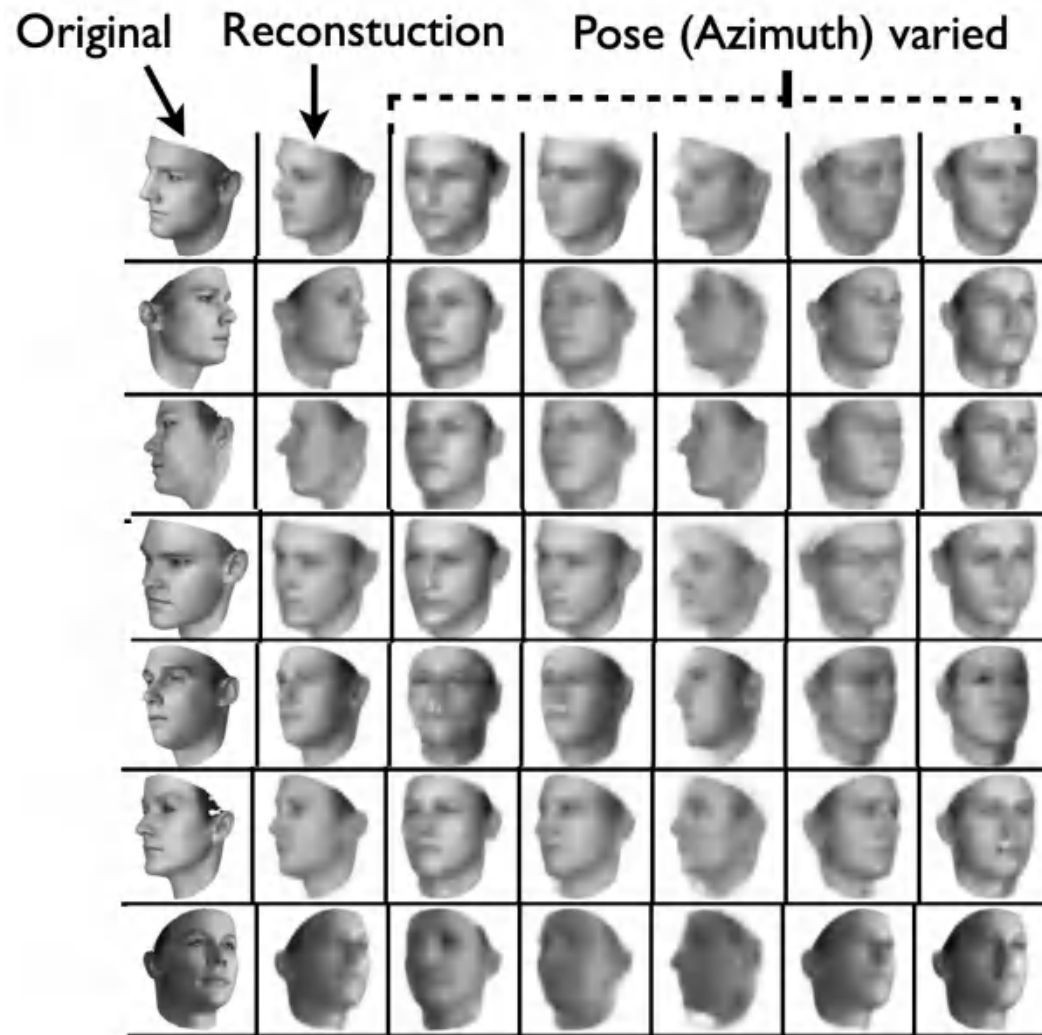
Vary  $z_2$



Kingma and Welling, Auto-Encoding Variational Bayes, ICLR 2014



# Variational Autoencoders: Image Editing



Kulkarni et al, "Deep Convolutional Inverse Graphics Networks", NeurIPS 2014

# Variational Autoencoder: Summary

Probabilistic spin to traditional autoencoders => allows generating data

Defines an intractable density => derive and optimize a (variational) lower bound

## Pros:

- Principled approach to generative models
- Allows inference of  $q(z|x)$ , can be useful feature representation for other tasks

## Cons:

- Maximizes lower bound of likelihood: okay, but not as good evaluation as PixelRNN/PixelCNN
- Samples blurrier and lower quality compared to state-of-the-art (GANs)

## Active areas of research:

- More flexible approximations, e.g. richer approximate posterior instead of diagonal Gaussian, e.g., Gaussian Mixture Models (GMMs)
- Incorporating structure in latent variables, e.g., Categorical Distributions

# So far: Two types of generative models

## Autoregressive models

- Directly maximize  $p(\text{data})$
- High-quality generated images
- Slow to generate images
- No explicit latent codes

## Variational models

- Maximize lower-bound on  $p(\text{data})$
- Generated images often blurry
- Very fast to generate images
- Learn rich latent codes

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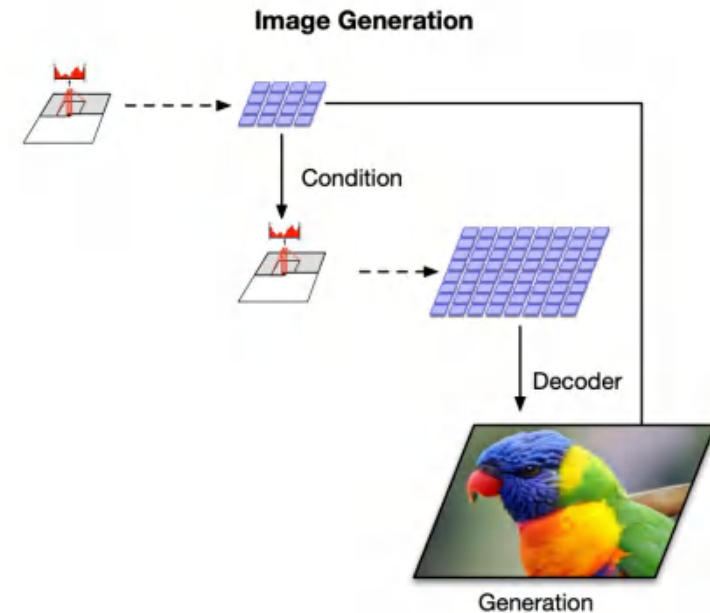
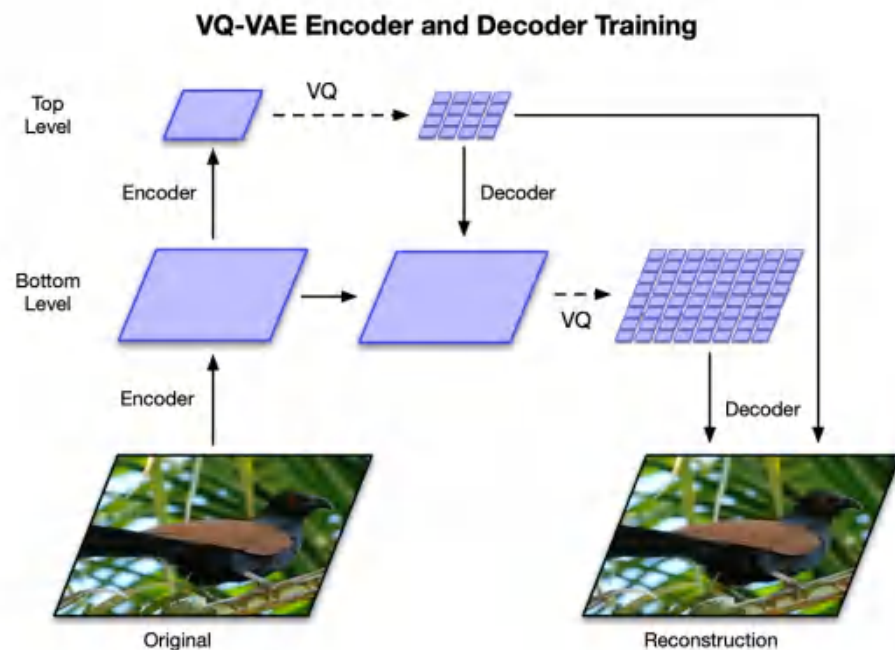
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Can we combine them and get the best of both worlds?

# Combining VAE + Autoregressive: Vector-Quantized Variational Autoencoder (VQ-VAE2)

Train a VAE-like model to generate multiscale grids of latent codes

Use a multiscale PixelCNN to sample in latent code space



Razavi et al, "Generating Diverse High-Fidelity Images with VQ-VAE-2", NeurIPS 2019

# Combining VAE + Autoregressive: VQ-VAE2

256 x 256 class-conditional samples, trained on ImageNet



Razavi et al, "Generating Diverse High-Fidelity Images with VQ-VAE-2", NeurIPS 2019



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1024 x 1024 generated faces, trained on FFHQ



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# Generative Models So Far:

**Autoregressive Models** directly maximize likelihood of training data:

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**Generative Adversarial Networks** give up on modeling  $p(x)$ , but allow us to draw samples from  $p(x)$

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**Setup:** Assume we have data  $x_i$  drawn from distribution  $p_{\text{data}}(x)$ . Want to sample from  $p_{\text{data}}$ .

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Sample  $z \sim p(z)$  and pass to a **Generator Network**  $x = G(z)$

Then  $x$  is a sample from the **Generator distribution**  $p_G$ . Want  $p_G = p_{\text{data}}$ !

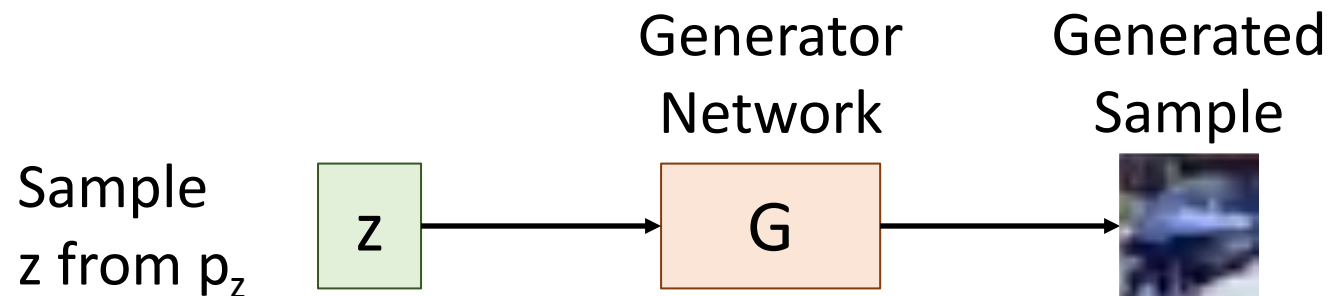
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Train **Generator Network**  $G$  to convert  $z$  into fake data  $x$  sampled from  $p_G$

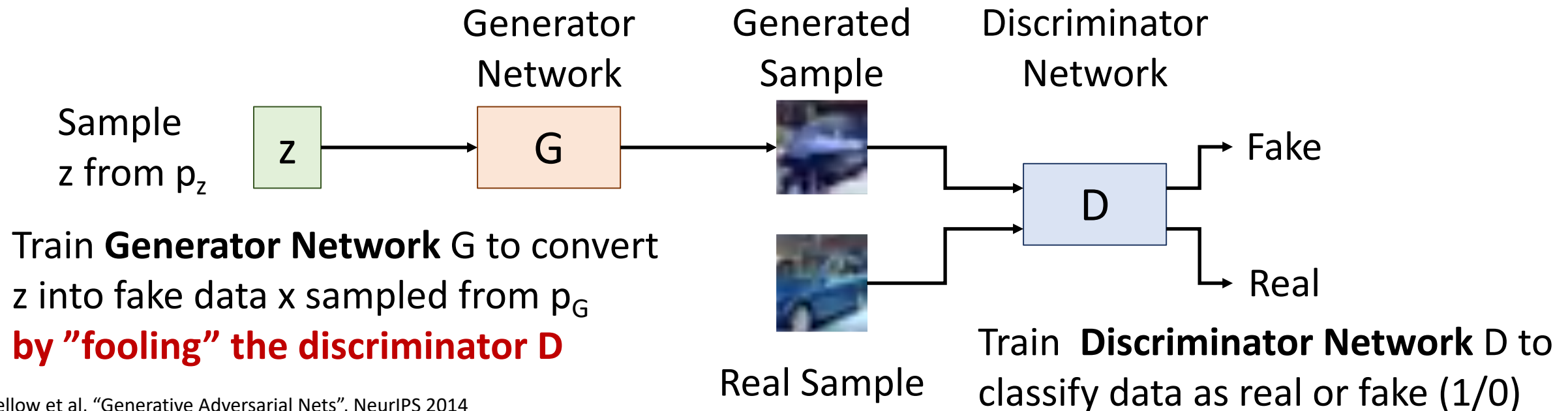
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Goodfellow et al, "Generative Adversarial Nets", NeurIPS 2014



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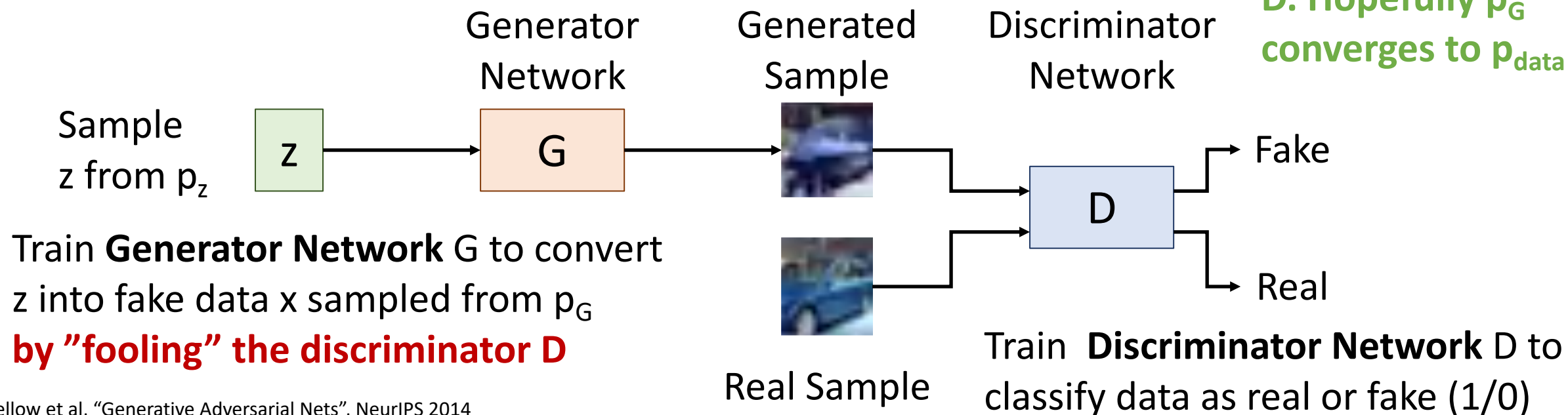
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**Jointly train G and D. Hopefully  $p_G$  converges to  $p_{\text{data}}$ !**



Goodfellow et al, "Generative Adversarial Nets", NeurIPS 2014

# Generative Adversarial Networks: Training Objective

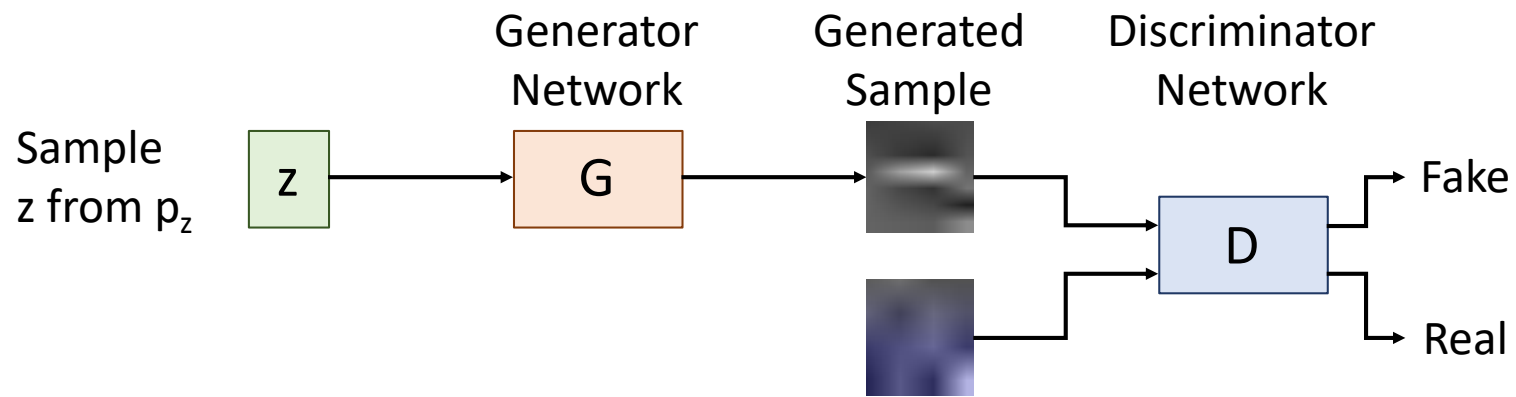
Jointly train generator G and discriminator D with a **minimax game**

$$\min_G \max_D \left( E_{x \sim p_{data}} [\log D(x)] + E_{z \sim p(z)} \left[ \log \left( 1 - D(G(z)) \right) \right] \right)$$

# Generative Adversarial Networks: Training Objective

Jointly train generator G and discriminator D with a **minimax game**

$$\min_{\mathbf{G}} \max_{\mathbf{D}} \left( E_{x \sim p_{data}} [\log \mathbf{D}(x)] + E_{\mathbf{z} \sim p(\mathbf{z})} \left[ \log \left( 1 - \mathbf{D}(\mathbf{G}(\mathbf{z})) \right) \right] \right)$$



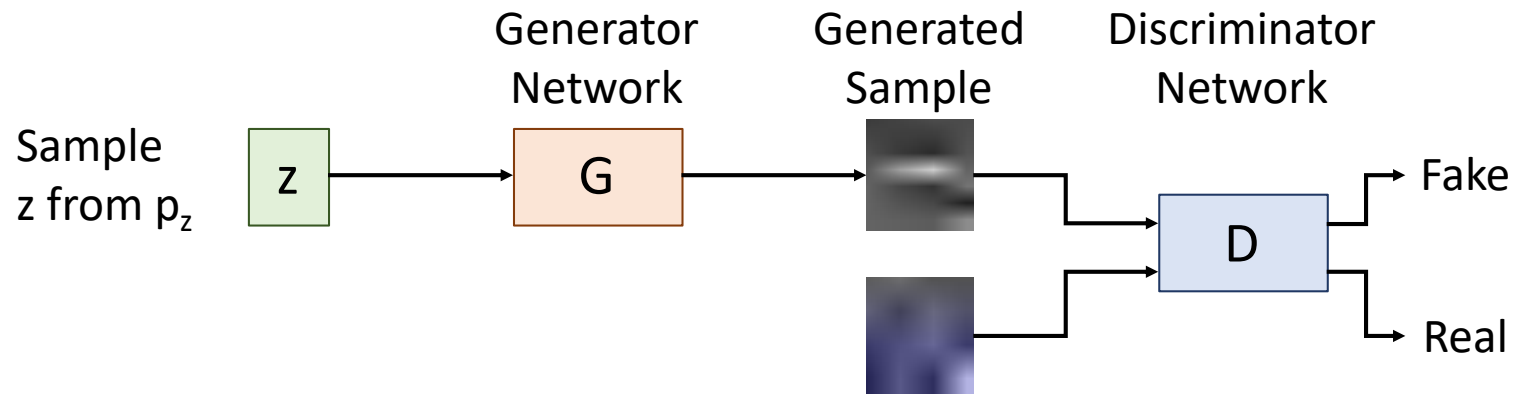
Goodfellow et al, "Generative Adversarial Nets", NeurIPS 2014

# Generative Adversarial Networks: Training Objective

Jointly train generator G and discriminator D with a **minimax game**

Discriminator wants  
 $D(x) = 1$  for real data

$$\min_G \max_D \left( E_{x \sim p_{data}} [\log D(x)] + E_{z \sim p(z)} \left[ \log \left( 1 - D(G(z)) \right) \right] \right)$$



Goodfellow et al, "Generative Adversarial Nets", NeurIPS 2014

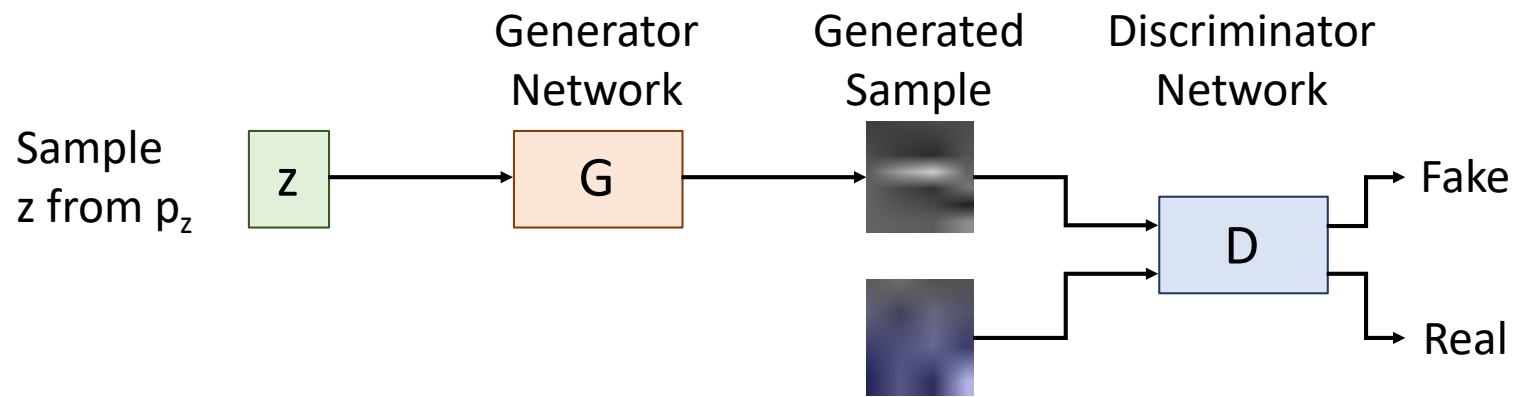
# Generative Adversarial Networks: Training Objective

Jointly train generator G and discriminator D with a **minimax game**

Discriminator wants  
 $D(x) = 1$  for real data

Discriminator wants  
 $D(x) = 0$  for fake data

$$\min_G \max_D \left( E_{x \sim p_{data}} [\log D(x)] + E_{z \sim p(z)} \left[ \log \left( 1 - D(G(z)) \right) \right] \right)$$



Goodfellow et al, "Generative Adversarial Nets", NeurIPS 2014

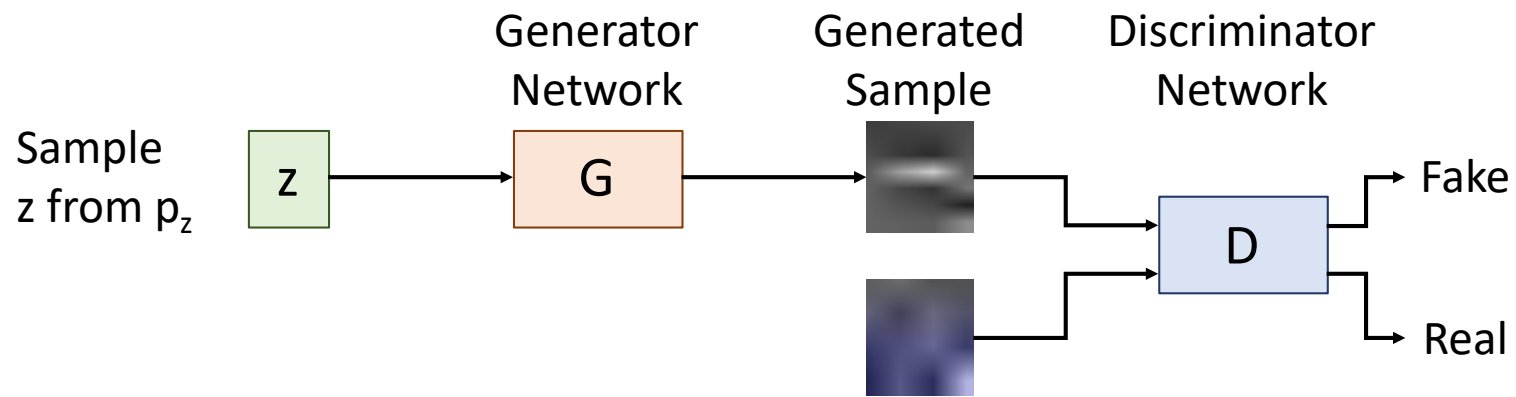
# Generative Adversarial Networks: Training Objective

Jointly train generator G and discriminator D with a **minimax game**

Discriminator wants  
 $D(x) = 1$  for real data

Discriminator wants  
 $D(x) = 0$  for fake data

$$\min_G \max_D \left( E_{x \sim p_{data}} [\log D(x)] + E_{z \sim p(z)} \left[ \log \left( 1 - D(G(z)) \right) \right] \right)$$



Generator wants  
 $D(x) = 1$  for fake data

# Generative Adversarial Networks: Training Objective

Jointly train generator G and discriminator D with a **minimax game**

Train G and D using alternating gradient updates

$$\min_{\textcolor{brown}{G}} \max_{\textcolor{blue}{D}} \left( E_{x \sim p_{data}} [\log \textcolor{blue}{D}(x)] + E_{\textcolor{green}{z} \sim p(\textcolor{green}{z})} \left[ \log \left( 1 - \textcolor{blue}{D}(\textcolor{brown}{G}(\textcolor{green}{z})) \right) \right] \right)$$

# Generative Adversarial Networks: Training Objective

Jointly train generator G and discriminator D with a **minimax game**

Train G and D using alternating gradient updates

$$\min_{\mathbf{G}} \max_{\mathbf{D}} \left( E_{x \sim p_{data}} [\log \mathbf{D}(x)] + E_{\mathbf{z} \sim p(\mathbf{z})} \left[ \log \left( 1 - \mathbf{D}(\mathbf{G}(\mathbf{z})) \right) \right] \right)$$
$$= \min_{\mathbf{G}} \max_{\mathbf{D}} V(\mathbf{G}, \mathbf{D})$$



# Generative Adversarial Networks: Training Objective

Jointly train generator  $G$  and discriminator  $D$  with a **minimax game**

Train  $G$  and  $D$  using alternating gradient updates

$$\min_{\mathbf{G}} \max_{\mathbf{D}} \left( E_{x \sim p_{data}} [\log \mathbf{D}(x)] + E_{\mathbf{z} \sim p(\mathbf{z})} \left[ \log \left( 1 - \mathbf{D}(\mathbf{G}(\mathbf{z})) \right) \right] \right)$$

$$= \min_{\mathbf{G}} \max_{\mathbf{D}} V(\mathbf{G}, \mathbf{D})$$

For  $t$  in  $1, \dots T$ :

1. (Update  $\mathbf{D}$ )  $\mathbf{D} = \mathbf{D} + \alpha_{\mathbf{D}} \frac{\partial V}{\partial \mathbf{D}}$
2. (Update  $\mathbf{G}$ )  $\mathbf{G} = \mathbf{G} - \alpha_{\mathbf{G}} \frac{\partial V}{\partial \mathbf{G}}$

# Generative Adversarial Networks: Training Objective

Jointly train generator  $G$  and discriminator  $D$  with a **minimax game**

Train  $G$  and  $D$  using alternating gradient updates

$$\min_{\mathbf{G}} \max_{\mathbf{D}} \left( E_{x \sim p_{data}} [\log \mathbf{D}(x)] + E_{\mathbf{z} \sim p(\mathbf{z})} \left[ \log \left( 1 - \mathbf{D}(\mathbf{G}(\mathbf{z})) \right) \right] \right)$$

$$= \min_{\mathbf{G}} \max_{\mathbf{D}} V(\mathbf{G}, \mathbf{D})$$

We are not minimizing any overall loss! No training curves to look at!

For  $t$  in  $1, \dots T$ :

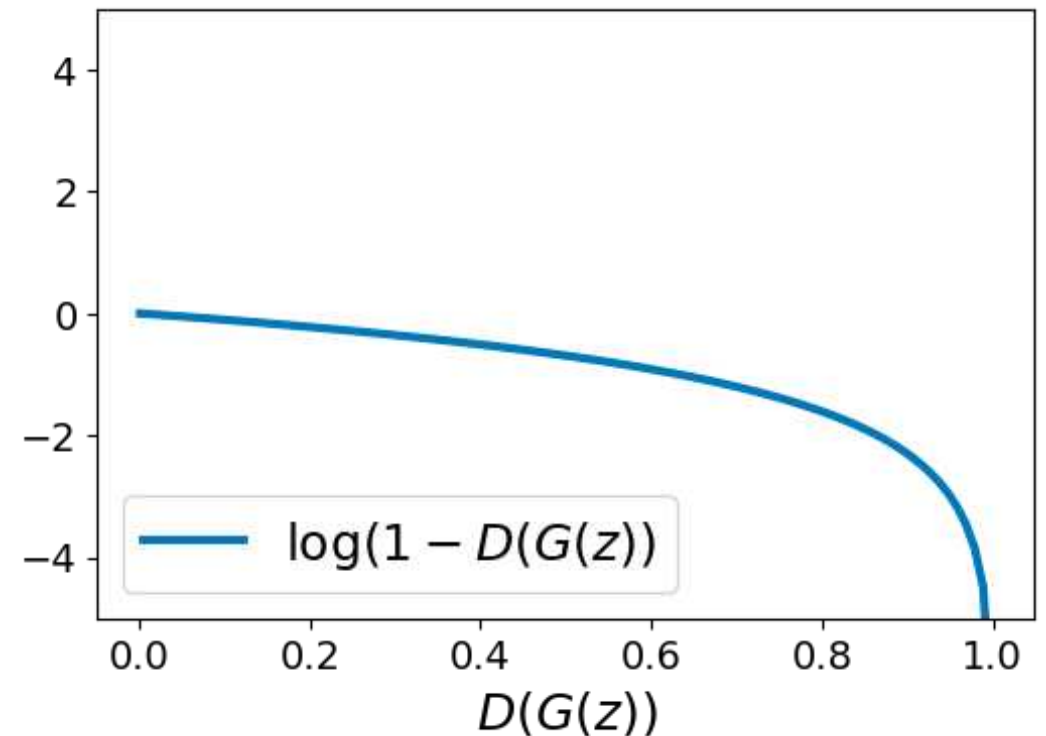
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2. (Update  $\mathbf{G}$ )  $\mathbf{G} = \mathbf{G} - \alpha_{\mathbf{G}} \frac{\partial V}{\partial \mathbf{G}}$

# Generative Adversarial Networks: Training Objective

Jointly train generator  $G$  and discriminator  $D$  with a **minimax game**

$$\min_G \max_D \left( E_{x \sim p_{data}} [\log D(x)] + E_{z \sim p(z)} \left[ \log \left( 1 - D(G(z)) \right) \right] \right)$$

At start of training, generator is very bad  
and discriminator can easily tell apart  
real/fake, so  $D(G(z))$  close to 0



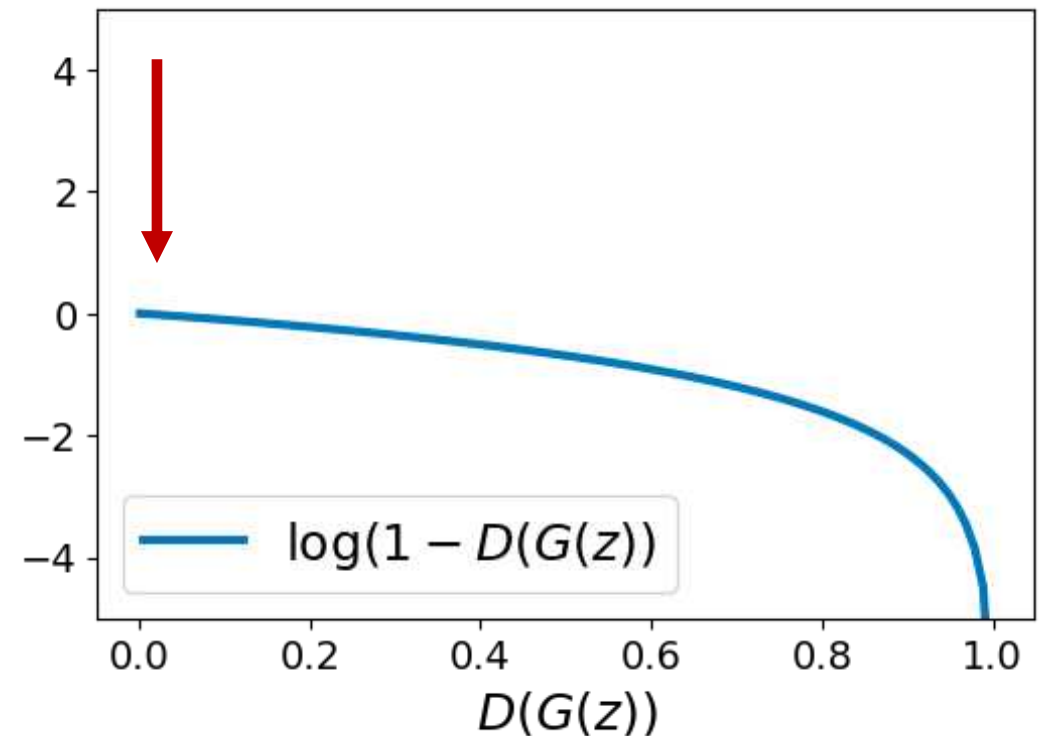
# Generative Adversarial Networks: Training Objective

Jointly train generator  $G$  and discriminator  $D$  with a **minimax game**

$$\min_G \max_D \left( E_{x \sim p_{data}} [\log D(x)] + E_{z \sim p(z)} \left[ \log \left( 1 - D(G(z)) \right) \right] \right)$$

At start of training, generator is very bad and discriminator can easily tell apart real/fake, so  $D(G(z))$  close to 0

**Problem:** Vanishing gradients for  $G$



# Generative Adversarial Networks: Training Objective

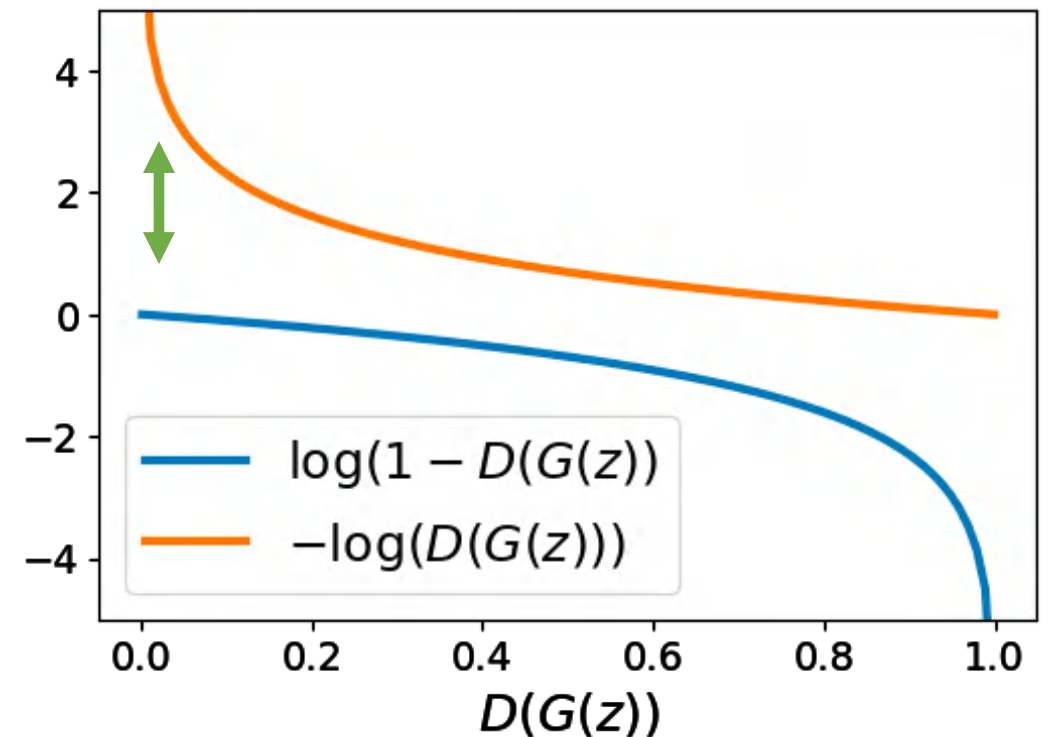
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$$\min_G \max_D \left( E_{x \sim p_{data}} [\log D(x)] + E_{z \sim p(z)} \left[ \log \left( 1 - D(G(z)) \right) \right] \right)$$

At start of training, generator is very bad and discriminator can easily tell apart real/fake, so  $D(G(z))$  close to 0

**Problem:** Vanishing gradients for  $G$

**Solution:** Right now  $G$  is trained to minimize  $\log(1-D(G(z)))$ . Instead, train  $G$  to minimize  $-\log(D(G(z)))$ . Then  $G$  gets strong gradients at start of training!



# Generative Adversarial Networks: Optimality

Jointly train generator G and discriminator D with a **minimax game**

Why is this particular objective a good idea?

$$\min_{\mathbf{G}} \max_{\mathbf{D}} \left( E_{x \sim p_{data}} [\log \mathbf{D}(x)] + E_{\mathbf{z} \sim p(\mathbf{z})} \left[ \log \left( 1 - \mathbf{D}(\mathbf{G}(\mathbf{z})) \right) \right] \right)$$

This minimax game achieves its global minimum when  $p_G = p_{data}$ !

# Generative Adversarial Networks: Optimality

$$\min_{\textcolor{brown}{G}} \max_{\textcolor{blue}{D}} \left( E_{x \sim p_{data}} [\log \textcolor{blue}{D}(x)] + E_{z \sim p(z)} \left[ \log \left( 1 - \textcolor{blue}{D}(\textcolor{brown}{G}(\textcolor{green}{z})) \right) \right] \right)$$

(Our objective so far)

# Generative Adversarial Networks: Optimality

$$\min_{\textcolor{brown}{G}} \max_{\textcolor{blue}{D}} \left( E_{x \sim p_{data}} [\log \textcolor{blue}{D}(x)] + E_{z \sim p(z)} \left[ \log \left( 1 - \textcolor{blue}{D}(\textcolor{brown}{G}(z)) \right) \right] \right)$$

$$= \min_{\textcolor{brown}{G}} \max_{\textcolor{blue}{D}} \left( E_{x \sim p_{data}} [\log \textcolor{blue}{D}(x)] + E_{x \sim p_{\textcolor{brown}{G}}} [\log(1 - \textcolor{blue}{D}(x))] \right)$$

(Change of variables on second term)



# Generative Adversarial Networks: Optimality

$$\begin{aligned} & \min_{\textcolor{brown}{G}} \max_{\textcolor{blue}{D}} \left( E_{x \sim p_{data}} [\log \textcolor{blue}{D}(x)] + E_{z \sim p(z)} \left[ \log \left( 1 - \textcolor{blue}{D}(\textcolor{brown}{G}(\textcolor{green}{z})) \right) \right] \right) \\ &= \min_{\textcolor{brown}{G}} \max_{\textcolor{blue}{D}} \left( E_{x \sim p_{data}} [\log \textcolor{blue}{D}(x)] + E_{x \sim p_{\textcolor{brown}{G}}} [\log(1 - \textcolor{blue}{D}(x))] \right) \\ &= \min_{\textcolor{brown}{G}} \max_{\textcolor{blue}{D}} \int_X (p_{data}(x) \log \textcolor{blue}{D}(x) + p_{\textcolor{brown}{G}}(x) \log(1 - \textcolor{blue}{D}(x))) dx \end{aligned}$$

(Definition of expectation)

# Generative Adversarial Networks: Optimality

$$\begin{aligned} & \min_{\textcolor{brown}{G}} \max_{\textcolor{blue}{D}} \left( E_{x \sim p_{data}} [\log \textcolor{blue}{D}(x)] + E_{z \sim p(z)} \left[ \log \left( 1 - \textcolor{blue}{D}(\textcolor{brown}{G}(\textcolor{green}{z})) \right) \right] \right) \\ &= \min_{\textcolor{brown}{G}} \max_{\textcolor{blue}{D}} \left( E_{x \sim p_{data}} [\log \textcolor{blue}{D}(x)] + E_{x \sim p_{\textcolor{brown}{G}}} [\log(1 - \textcolor{blue}{D}(x))] \right) \\ &= \min_{\textcolor{brown}{G}} \int_X \max_{\textcolor{blue}{D}} (p_{data}(x) \log \textcolor{blue}{D}(x) + p_{\textcolor{brown}{G}}(x) \log(1 - \textcolor{blue}{D}(x))) dx \end{aligned}$$

(Push  $\max_D$  inside integral)

# Generative Adversarial Networks: Optimality

$$\min_{\textcolor{brown}{G}} \max_{\textcolor{blue}{D}} \left( E_{x \sim p_{data}} [\log \textcolor{blue}{D}(x)] + E_{z \sim p(z)} \left[ \log \left( 1 - \textcolor{blue}{D}(\textcolor{brown}{G}(z)) \right) \right] \right)$$

$$= \min_{\textcolor{brown}{G}} \max_{\textcolor{blue}{D}} \left( E_{x \sim p_{data}} [\log \textcolor{blue}{D}(x)] + E_{x \sim p_{\textcolor{brown}{G}}} [\log(1 - \textcolor{blue}{D}(x))] \right)$$

$$= \min_{\textcolor{brown}{G}} \int_X \max_{\textcolor{blue}{D}} (\textcolor{red}{p}_{data}(x) \log \textcolor{purple}{D}(x) + \textcolor{orange}{p}_G(x) \log(1 - \textcolor{purple}{D}(x))) dx$$

$$f(y) = \textcolor{red}{a} \log \textcolor{purple}{y} + \textcolor{orange}{b} \log(1 - \textcolor{purple}{y})$$

(Side computation to compute max)

# Generative Adversarial Networks: Optimality

$$\min_G \max_D \left( E_{x \sim p_{data}} [\log D(x)] + E_{z \sim p(z)} \left[ \log \left( 1 - D(G(z)) \right) \right] \right)$$

$$= \min_G \max_D \left( E_{x \sim p_{data}} [\log D(x)] + E_{x \sim p_G} [\log(1 - D(x))] \right)$$

$$= \min_G \int_X \max_D \left( p_{data}(x) \log D(x) + p_G(x) \log(1 - D(x)) \right) dx$$

$$f(y) = a \log y + b \log(1 - y)$$

$$f'(y) = \frac{a}{y} - \frac{b}{1 - y}$$

# Generative Adversarial Networks: Optimality

$$\min_G \max_D \left( E_{x \sim p_{data}} [\log D(x)] + E_{z \sim p(z)} \left[ \log \left( 1 - D(G(z)) \right) \right] \right)$$

$$= \min_G \max_D \left( E_{x \sim p_{data}} [\log D(x)] + E_{x \sim p_G} [\log(1 - D(x))] \right)$$

$$= \min_G \int_X \max_D (p_{data}(x) \log D(x) + p_G(x) \log(1 - D(x))) dx$$

$$f(y) = a \log y + b \log(1 - y) \quad f'(y) = 0 \iff y = \frac{a}{a+b} \text{ (local max)}$$

$$f'(y) = \frac{a}{y} - \frac{b}{1-y}$$

# Generative Adversarial Networks: Optimality

$$\min_G \max_D \left( E_{x \sim p_{data}} [\log D(x)] + E_{z \sim p(z)} \left[ \log \left( 1 - D(G(z)) \right) \right] \right)$$

$$= \min_G \max_D \left( E_{x \sim p_{data}} [\log D(x)] + E_{x \sim p_G} [\log(1 - D(x))] \right)$$

$$= \min_G \int_X \max_D \left( p_{data}(x) \log D(x) + p_G(x) \log(1 - D(x)) \right) dx$$

$$f(y) = a \log y + b \log(1 - y) \quad f'(y) = 0 \iff y = \frac{a}{a+b} \text{ (local max)}$$

$$f'(y) = \frac{a}{y} - \frac{b}{1-y} \quad \text{Optimal Discriminator: } D_G^*(x) = \frac{p_{data}(x)}{p_{data}(x) + p_G(x)}$$

# Generative Adversarial Networks: Optimality

$$\begin{aligned} & \min_G \max_D \left( E_{x \sim p_{data}} [\log D(x)] + E_{z \sim p(z)} \left[ \log \left( 1 - D(G(z)) \right) \right] \right) \\ &= \min_G \max_D \left( E_{x \sim p_{data}} [\log D(x)] + E_{x \sim p_G} [\log(1 - D(x))] \right) \\ &= \min_G \int_X \max_D \left( p_{data}(x) \log D(x) + p_G(x) \log(1 - D(x)) \right) dx \end{aligned}$$

**Optimal Discriminator:**  $D_G^*(x) = \frac{p_{data}(x)}{p_{data}(x) + p_G(x)}$

# Generative Adversarial Networks: Optimality

$$\begin{aligned} & \min_{\textcolor{brown}{G}} \max_{\textcolor{blue}{D}} \left( E_{x \sim p_{data}} [\log \textcolor{blue}{D}(x)] + E_{z \sim p(z)} \left[ \log \left( 1 - \textcolor{blue}{D}(\textcolor{brown}{G}(z)) \right) \right] \right) \\ &= \min_{\textcolor{brown}{G}} \max_{\textcolor{blue}{D}} \left( E_{x \sim p_{data}} [\log \textcolor{blue}{D}(x)] + E_{x \sim p_{\textcolor{brown}{G}}} [\log(1 - \textcolor{blue}{D}(x))] \right) \\ &= \min_{\textcolor{brown}{G}} \int_X \left( p_{data}(x) \log \textcolor{blue}{D}_{\textcolor{brown}{G}}^*(x) + p_{\textcolor{brown}{G}}(x) \log(1 - \textcolor{blue}{D}_{\textcolor{brown}{G}}^*(x)) \right) dx \end{aligned}$$

**Optimal Discriminator:**  $\textcolor{blue}{D}_{\textcolor{brown}{G}}^*(x) = \frac{p_{data}(x)}{p_{data}(x) + p_{\textcolor{brown}{G}}(x)}$



# Generative Adversarial Networks: Optimality

$$\begin{aligned} & \min_G \max_D \left( E_{x \sim p_{data}} [\log D(x)] + E_{z \sim p(z)} \left[ \log \left( 1 - D(G(z)) \right) \right] \right) \\ &= \min_G \max_D \left( E_{x \sim p_{data}} [\log D(x)] + E_{x \sim p_G} [\log(1 - D(x))] \right) \\ &= \min_G \int_X \left( p_{data}(x) \log D_G^*(x) + p_G(x) \log(1 - D_G^*(x)) \right) dx \\ &= \min_G \int_X \left( p_{data}(x) \log \frac{p_{data}(x)}{p_{data}(x) + p_G(x)} + p_G(x) \log \frac{p_G(x)}{p_{data}(x) + p_G(x)} \right) dx \end{aligned}$$

$$\textbf{Optimal Discriminator: } D_G^*(x) = \frac{p_{data}(x)}{p_{data}(x) + p_G(x)}$$

# Generative Adversarial Networks: Optimality

$$\min_{\mathcal{G}} \max_{\mathcal{D}} \left( E_{x \sim p_{data}} [\log \mathcal{D}(x)] + E_{z \sim p(z)} \left[ \log \left( 1 - \mathcal{D}(\mathcal{G}(z)) \right) \right] \right)$$
$$= \min_{\mathcal{G}} \int_X \left( p_{data}(x) \log \frac{p_{data}(x)}{p_{data}(x) + p_{\mathcal{G}}(x)} + p_{\mathcal{G}}(x) \log \frac{p_{\mathcal{G}}(x)}{p_{data}(x) + p_{\mathcal{G}}(x)} \right) dx$$

# Generative Adversarial Networks: Optimality

$$\begin{aligned} & \min_G \max_D \left( E_{x \sim p_{data}} [\log D(x)] + E_{z \sim p(z)} \left[ \log \left( 1 - D(G(z)) \right) \right] \right) \\ &= \min_G \int_X \left( p_{data}(x) \log \frac{p_{data}(x)}{p_{data}(x) + p_G(x)} + p_G(x) \log \frac{p_G(x)}{p_{data}(x) + p_G(x)} \right) dx \\ &= \min_G \left( E_{x \sim p_{data}} \left[ \log \frac{p_{data}(x)}{p_{data}(x) + p_G(x)} \right] + E_{x \sim p_G} \left[ \log \frac{p_G(x)}{p_{data}(x) + p_G(x)} \right] \right) \end{aligned}$$

(Definition of expectation)

# Generative Adversarial Networks: Optimality

$$\begin{aligned} & \min_G \max_D \left( E_{x \sim p_{data}} [\log D(x)] + E_{z \sim p(z)} \left[ \log \left( 1 - D(G(z)) \right) \right] \right) \\ &= \min_G \int_X \left( p_{data}(x) \log \frac{p_{data}(x)}{p_{data}(x) + p_G(x)} + p_G(x) \log \frac{p_G(x)}{p_{data}(x) + p_G(x)} \right) dx \\ &= \min_G \left( E_{x \sim p_{data}} \left[ \log \frac{2}{2} \frac{p_{data}(x)}{p_{data}(x) + p_G(x)} \right] + E_{x \sim p_G} \left[ \log \frac{2}{2} \frac{p_G(x)}{p_{data}(x) + p_G(x)} \right] \right) \\ & \quad \text{(Multiply by a constant)} \end{aligned}$$

# Generative Adversarial Networks: Optimality

$$\begin{aligned} & \min_G \max_D \left( E_{x \sim p_{data}} [\log D(x)] + E_{z \sim p(z)} \left[ \log \left( 1 - D(G(z)) \right) \right] \right) \\ &= \min_G \int_X \left( p_{data}(x) \log \frac{p_{data}(x)}{p_{data}(x) + p_G(x)} + p_G(x) \log \frac{p_G(x)}{p_{data}(x) + p_G(x)} \right) dx \\ &= \min_G \left( E_{x \sim p_{data}} \left[ \log \frac{2}{2} \frac{p_{data}(x)}{p_{data}(x) + p_G(x)} \right] + E_{x \sim p_G} \left[ \log \frac{2}{2} \frac{p_G(x)}{p_{data}(x) + p_G(x)} \right] \right) \\ &= \min_G \left( E_{x \sim p_{data}} \left[ \log \frac{2 * p_{data}(x)}{p_{data}(x) + p_G(x)} \right] + E_{x \sim p_G} \left[ \log \frac{2 * p_G(x)}{p_{data}(x) + p_G(x)} \right] - \log 4 \right) \end{aligned}$$

# Generative Adversarial Networks: Optimality

$$\min_{\textcolor{brown}{G}} \max_{\textcolor{blue}{D}} \left( E_{x \sim p_{data}} [\log \textcolor{blue}{D}(x)] + E_{z \sim p(z)} \left[ \log \left( 1 - \textcolor{blue}{D}(\textcolor{brown}{G}(z)) \right) \right] \right)$$

$$= \min_{\textcolor{brown}{G}} \left( E_{x \sim p_{data}} \left[ \log \frac{\textcolor{violet}{2} * p_{data}(x)}{p_{data}(x) + p_{\textcolor{brown}{G}}(x)} \right] + E_{x \sim p_{\textcolor{brown}{G}}} \left[ \log \frac{\textcolor{violet}{2} * p_{\textcolor{brown}{G}}(x)}{p_{data}(x) + p_{\textcolor{brown}{G}}(x)} \right] - \log 4 \right)$$

# Generative Adversarial Networks: Optimality

$$\min_G \max_D \left( E_{x \sim p_{data}} [\log D(x)] + E_{z \sim p(z)} \left[ \log \left( 1 - D(G(z)) \right) \right] \right)$$
$$= \min_G \left( E_{x \sim p_{data}} \left[ \log \frac{2 * p_{data}(x)}{p_{data}(x) + p_G(x)} \right] + E_{x \sim p_G} \left[ \log \frac{2 * p_G(x)}{p_{data}(x) + p_G(x)} \right] - \log 4 \right)$$

**Kullback-Leibler Divergence:**

$$KL(p, q) = E_{x \sim p} \left[ \log \frac{p(x)}{q(x)} \right]$$

# Generative Adversarial Networks: Optimality

$$\min_G \max_D \left( E_{x \sim p_{data}} [\log D(x)] + E_{z \sim p(z)} \left[ \log \left( 1 - D(G(z)) \right) \right] \right)$$

$$= \min_G \left( E_{x \sim p_{data}} \left[ \log \frac{2 * p_{data}(x)}{p_{data}(x) + p_G(x)} \right] + E_{x \sim p_G} \left[ \log \frac{2 * p_G(x)}{p_{data}(x) + p_G(x)} \right] - \log 4 \right)$$

**Kullback-Leibler Divergence:**

$$KL(p, q) = E_{x \sim p} \left[ \log \frac{p(x)}{q(x)} \right]$$



# Generative Adversarial Networks: Optimality

$$\begin{aligned} & \min_G \max_D \left( E_{x \sim p_{data}} [\log D(x)] + E_{z \sim p(z)} \left[ \log \left( 1 - D(G(z)) \right) \right] \right) \\ &= \min_G \left( E_{x \sim p_{data}} \left[ \log \frac{2 * p_{data}(x)}{p_{data}(x) + p_G(x)} \right] + E_{x \sim p_G} \left[ \log \frac{2 * p_G(x)}{p_{data}(x) + p_G(x)} \right] - \log 4 \right) \\ &= \min_G \left( KL \left( p_{data}, \frac{p_{data} + p_G}{2} \right) + KL \left( p_G, \frac{p_{data} + p_G}{2} \right) - \log 4 \right) \end{aligned}$$

**Kullback-Leibler Divergence:**

$$KL(p, q) = E_{x \sim p} \left[ \log \frac{p(x)}{q(x)} \right]$$

# Generative Adversarial Networks: Optimality

$$\min_{\textcolor{brown}{G}} \max_{\textcolor{blue}{D}} \left( E_{x \sim p_{data}} [\log \textcolor{blue}{D}(x)] + E_{z \sim p(z)} \left[ \log \left( 1 - \textcolor{blue}{D}(\textcolor{brown}{G}(z)) \right) \right] \right)$$

$$= \min_{\textcolor{brown}{G}} \left( E_{x \sim p_{data}} \left[ \log \frac{2 * p_{data}(x)}{p_{data}(x) + p_{\textcolor{brown}{G}}(x)} \right] + E_{x \sim p_{\textcolor{brown}{G}}} \left[ \log \frac{2 * p_{\textcolor{brown}{G}}(x)}{p_{data}(x) + p_{\textcolor{brown}{G}}(x)} \right] - \log 4 \right)$$

$$= \min_{\textcolor{brown}{G}} \left( KL \left( p_{data}, \frac{p_{data} + p_{\textcolor{brown}{G}}}{2} \right) + KL \left( p_{\textcolor{brown}{G}}, \frac{p_{data} + p_{\textcolor{brown}{G}}}{2} \right) - \log 4 \right)$$

# Generative Adversarial Networks: Optimality

$$\begin{aligned} & \min_G \max_D \left( E_{x \sim p_{data}} [\log D(x)] + E_{z \sim p(z)} \left[ \log \left( 1 - D(G(z)) \right) \right] \right) \\ &= \min_G \left( E_{x \sim p_{data}} \left[ \log \frac{2 * p_{data}(x)}{p_{data}(x) + p_G(x)} \right] + E_{x \sim p_G} \left[ \log \frac{2 * p_G(x)}{p_{data}(x) + p_G(x)} \right] - \log 4 \right) \\ &= \min_G \left( KL \left( p_{data}, \frac{p_{data} + p_G}{2} \right) + KL \left( p_G, \frac{p_{data} + p_G}{2} \right) - \log 4 \right) \end{aligned}$$

**Jensen-Shannon Divergence:**

$$JSD(p, q) = \frac{1}{2} KL \left( p, \frac{p + q}{2} \right) + \frac{1}{2} KL \left( q, \frac{p + q}{2} \right)$$

# Generative Adversarial Networks: Optimality

$$\begin{aligned} & \min_G \max_D \left( E_{x \sim p_{data}} [\log D(x)] + E_{z \sim p(z)} \left[ \log \left( 1 - D(G(z)) \right) \right] \right) \\ &= \min_G \left( E_{x \sim p_{data}} \left[ \log \frac{2 * p_{data}(x)}{p_{data}(x) + p_G(x)} \right] + E_{x \sim p_G} \left[ \log \frac{2 * p_G(x)}{p_{data}(x) + p_G(x)} \right] - \log 4 \right) \\ &= \min_G \left( KL \left( p_{data}, \frac{p_{data} + p_G}{2} \right) + KL \left( p_G, \frac{p_{data} + p_G}{2} \right) - \log 4 \right) \end{aligned}$$

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$$JSD(p, q) = \frac{1}{2} KL \left( p, \frac{p + q}{2} \right) + \frac{1}{2} KL \left( q, \frac{p + q}{2} \right)$$

# Generative Adversarial Networks: Optimality

$$\min_G \max_D \left( E_{x \sim p_{data}} [\log D(x)] + E_{z \sim p(z)} \left[ \log \left( 1 - D(G(z)) \right) \right] \right)$$

$$= \min_G \left( E_{x \sim p_{data}} \left[ \log \frac{2 * p_{data}(x)}{p_{data}(x) + p_G(x)} \right] + E_{x \sim p_G} \left[ \log \frac{2 * p_G(x)}{p_{data}(x) + p_G(x)} \right] - \log 4 \right)$$

$$= \min_G \left( KL \left( p_{data}, \frac{p_{data} + p_G}{2} \right) + KL \left( p_G, \frac{p_{data} + p_G}{2} \right) - \log 4 \right)$$

$$= \min_G (2 * JSD(p_{data}, p_G) - \log 4)$$

JSD is always nonnegative, and zero only when the two distributions are equal!

Thus  $p_{data} = p_G$  is the global min, QED

**Jensen-Shannon Divergence:**

$$JSD(p, q) = \frac{1}{2} KL \left( p, \frac{p + q}{2} \right) + \frac{1}{2} KL \left( q, \frac{p + q}{2} \right)$$

# Generative Adversarial Networks: Optimality

$$\min_{\underset{G}{G}} \max_{\underset{D}{D}} \left( E_{x \sim p_{data}} [\log D(x)] + E_{z \sim p(z)} \left[ \log \left( 1 - D(G(z)) \right) \right] \right)$$
$$= \min_{\underset{G}{G}} (2 * JSD(p_{data}, p_G) - \log 4)$$

# Generative Adversarial Networks: Optimality

$$\min_G \max_D \left( E_{x \sim p_{data}} [\log D(x)] + E_{z \sim p(z)} \left[ \log \left( 1 - D(G(z)) \right) \right] \right) \\ = \min_G (2 * JSD(p_{data}, p_G) - \log 4)$$

**Summary:** The global minimum of the minimax game happens when:

1.  $D_G^*(x) = \frac{p_{data}(x)}{p_{data}(x) + p_G(x)}$  (Optimal discriminator for any G)
2.  $p_G(x) = p_{data}(x)$  (Optimal generator for optimal D)



# Generative Adversarial Networks: Optimality

$$\min_G \max_D \left( E_{x \sim p_{data}} [\log D(x)] + E_{z \sim p(z)} \left[ \log \left( 1 - D(G(z)) \right) \right] \right) \\ = \min_G (2 * JSD(p_{data}, p_G) - \log 4)$$

**Summary:** The global minimum of the minimax game happens when:

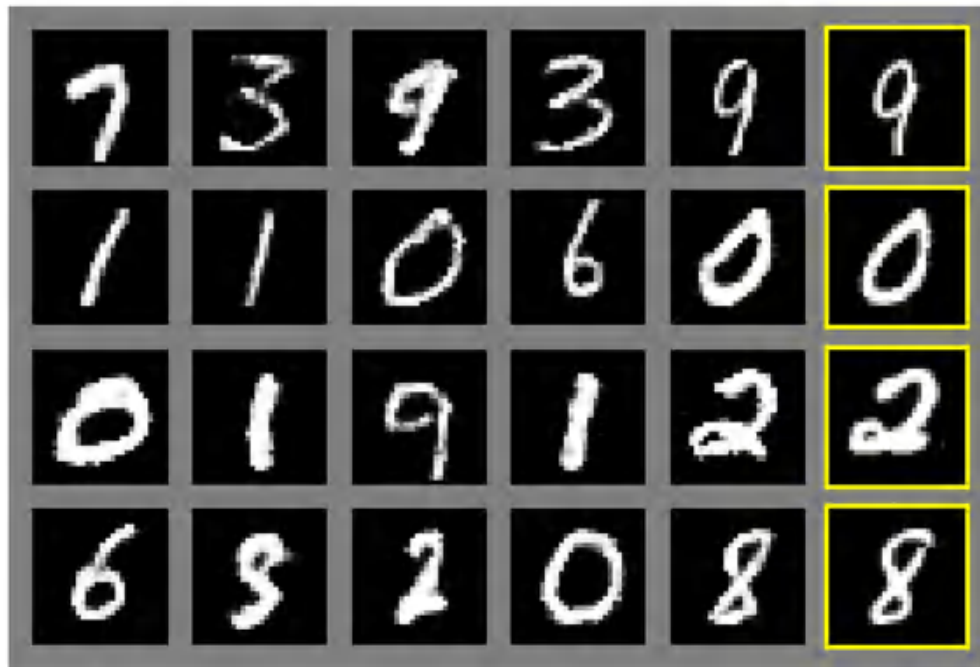
1.  $D_G^*(x) = \frac{p_{data}(x)}{p_{data}(x) + p_G(x)}$  (Optimal discriminator for any G)
2.  $p_G(x) = p_{data}(x)$  (Optimal generator for optimal D)

## Caveats:

1. G and D are neural nets with fixed architecture. We don't know whether they can actually represent the optimal D and G.
2. This tells us nothing about convergence to the optimal solution

# Generative Adversarial Networks: Results

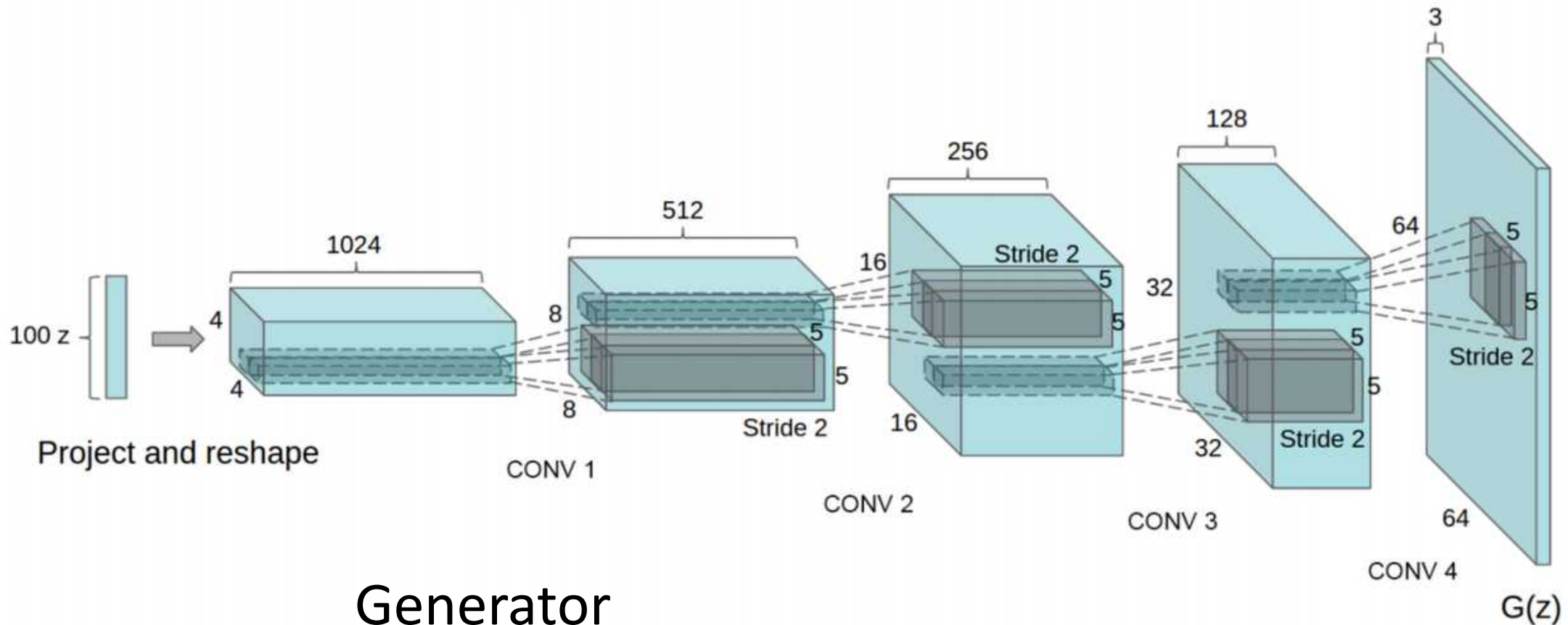
Generated samples



Nearest neighbor from training set

Goodfellow et al, "Generative Adversarial Nets", NeurIPS 2014

# Generative Adversarial Networks: DC-GAN



Radford et al, "Unsupervised Representation Learning with Deep Convolutional Generative Adversarial Networks", ICLR 2016

# Generative Adversarial Networks: DC-GAN

Samples from the model look much better!



Radford et al,  
ICLR 2016



# Generative Adversarial Networks: Interpolation

Interpolating  
between  
points in  
latent  $z$   
space

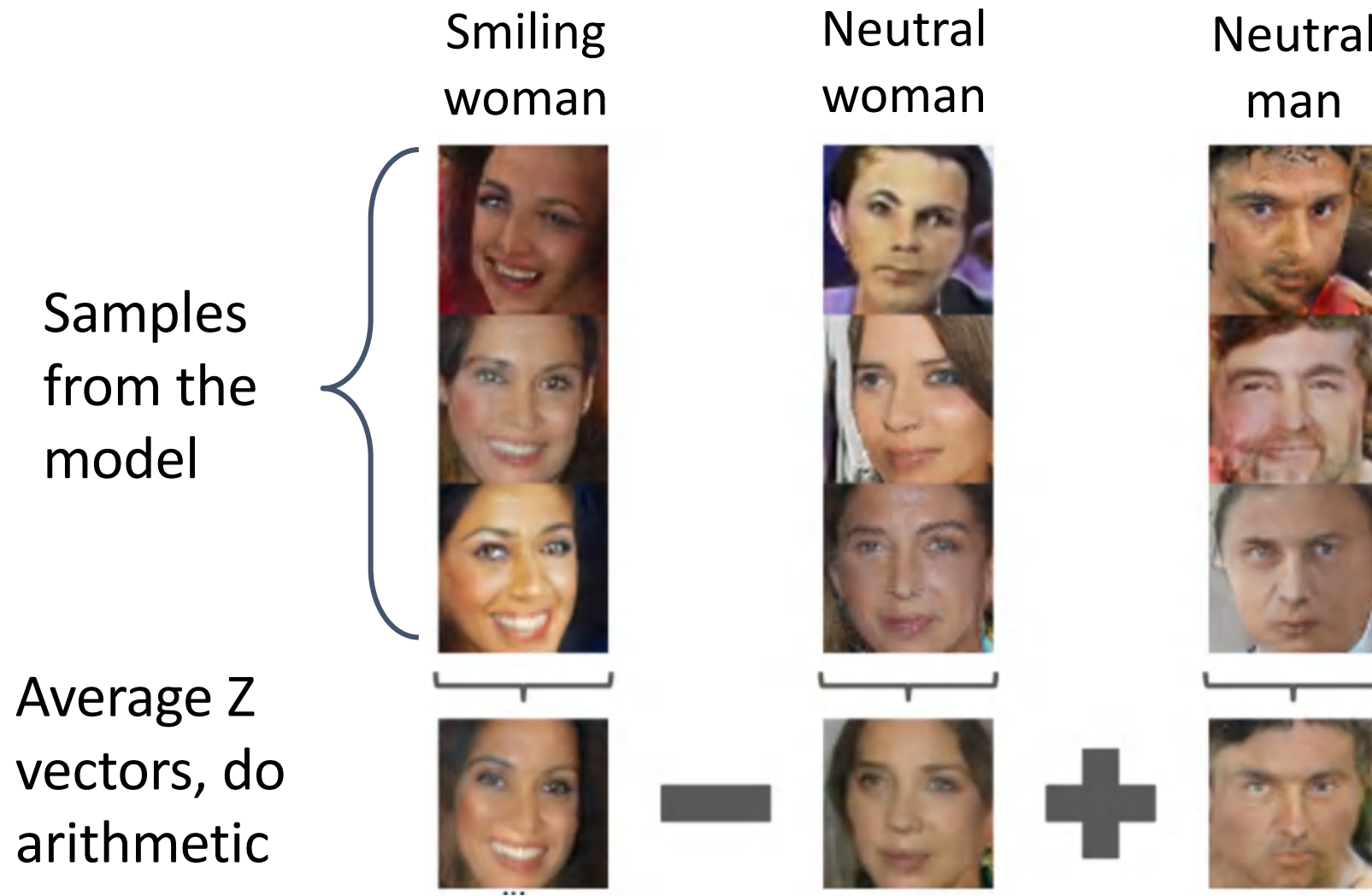


Radford et al,  
ICLR 2016

# Generative Adversarial Networks: Vector Math



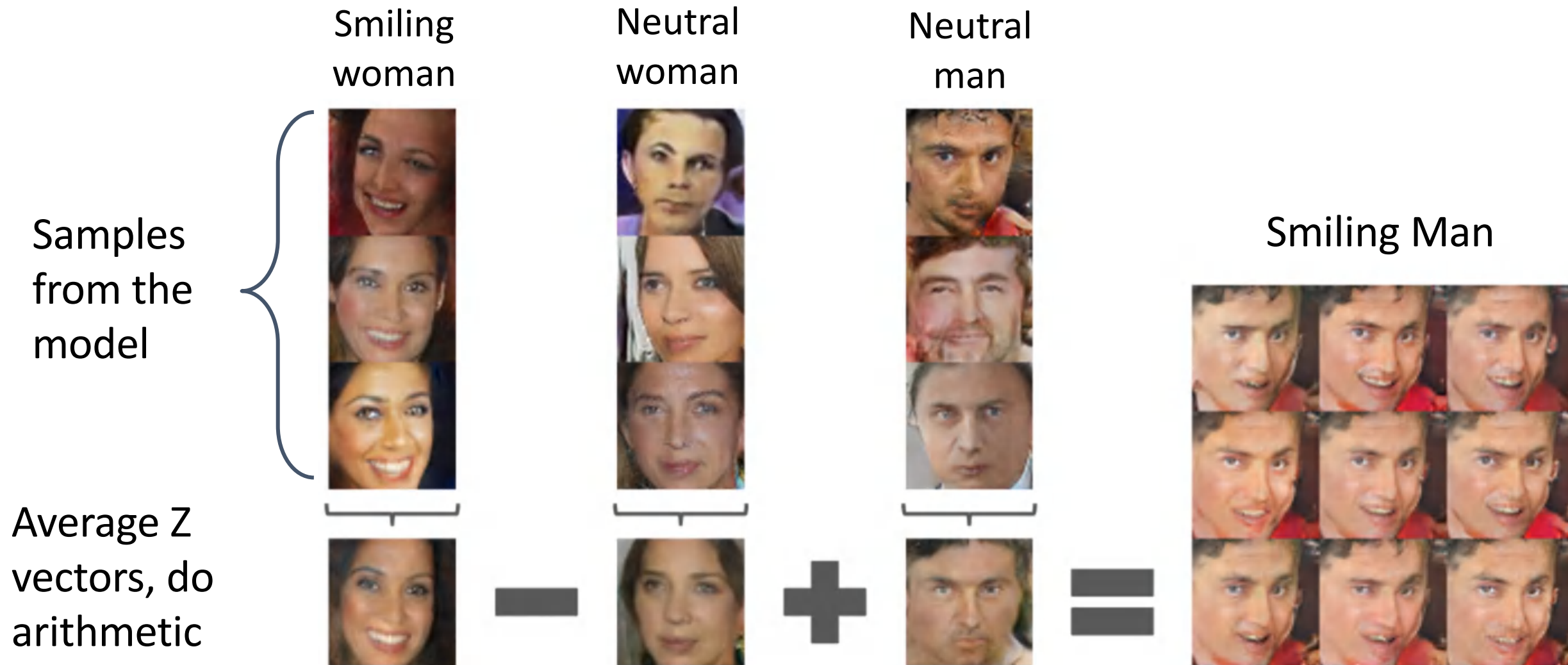
# Generative Adversarial Networks: Vector Math



Radford et al, ICLR 2016



# Generative Adversarial Networks: Vector Math



Radford et al, ICLR 2016



# Generative Adversarial Networks: Vector Math

Man with  
glasses



Man w/o  
glasses



Woman  
w/o glasses



Samples  
from the  
model

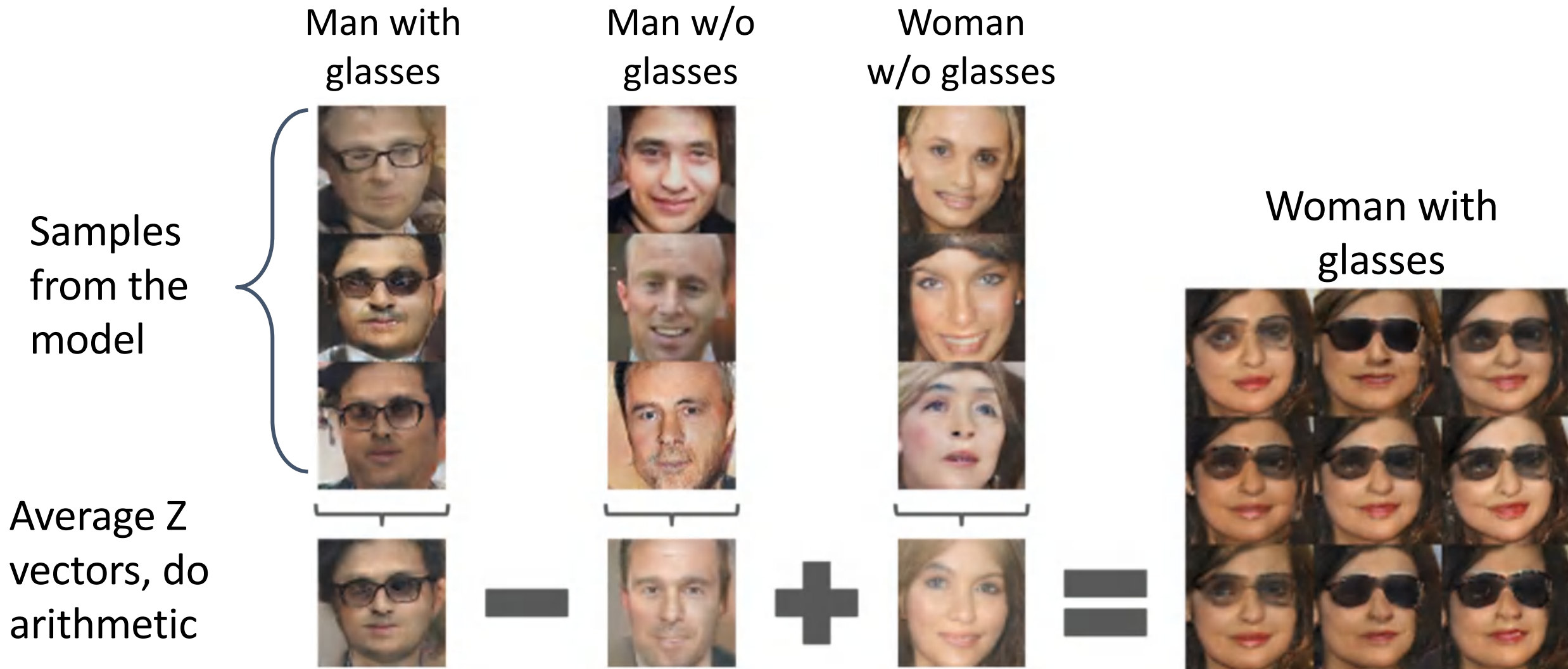


Average Z  
vectors, do  
arithmetic



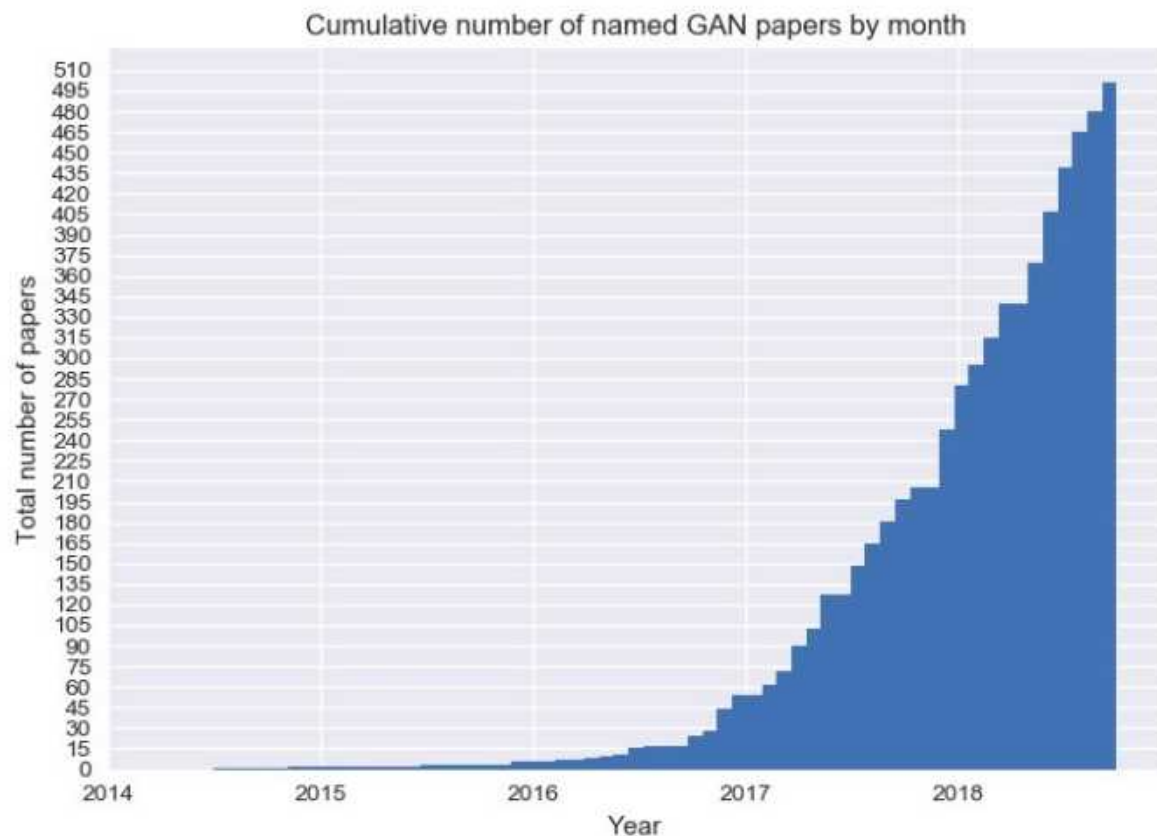
Radford et al, ICLR 2016

# Generative Adversarial Networks: Vector Math



Radford et al, ICLR 2016

## 2017 to present: Explosion of GANs



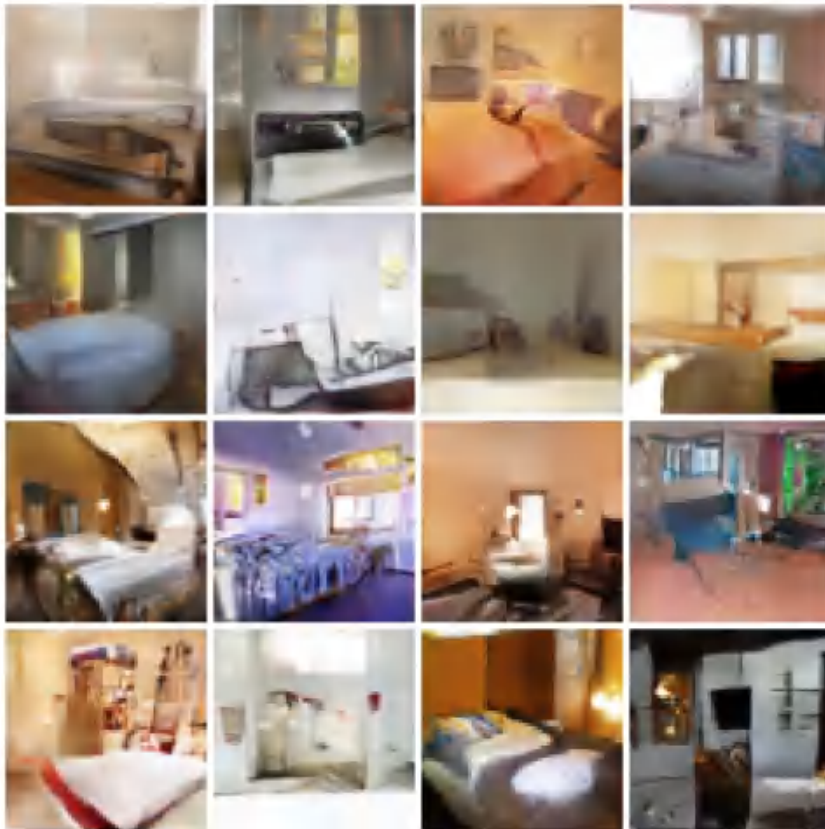
<https://github.com/hindupuravinash/the-gan-zoo>

- [illegible]



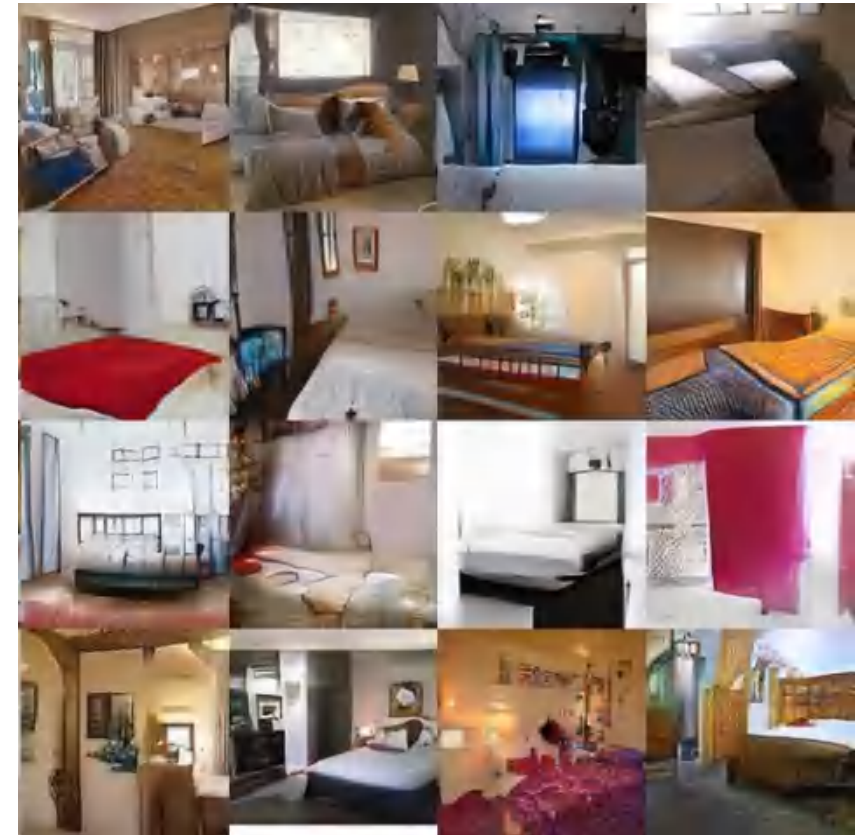
# GAN Improvements: Improved Loss Functions

## Wasserstein GAN (WGAN)



Arjovsky, Chintala, and Bottou, "Wasserstein GAN", 2017

## WGAN with Gradient Penalty (WGAN-GP)



Gulrajani et al, "Improved Training of Wasserstein GANs", NeurIPS 2017

# GAN Improvements: Higher Resolution

256 x 256 bedrooms



1024 x 1024 faces



Karras et al, "Progressive Growing of GANs for Improved Quality, Stability, and Variation", ICLR 2018



# GAN Improvements: Higher Resolution

512 x 384 cars

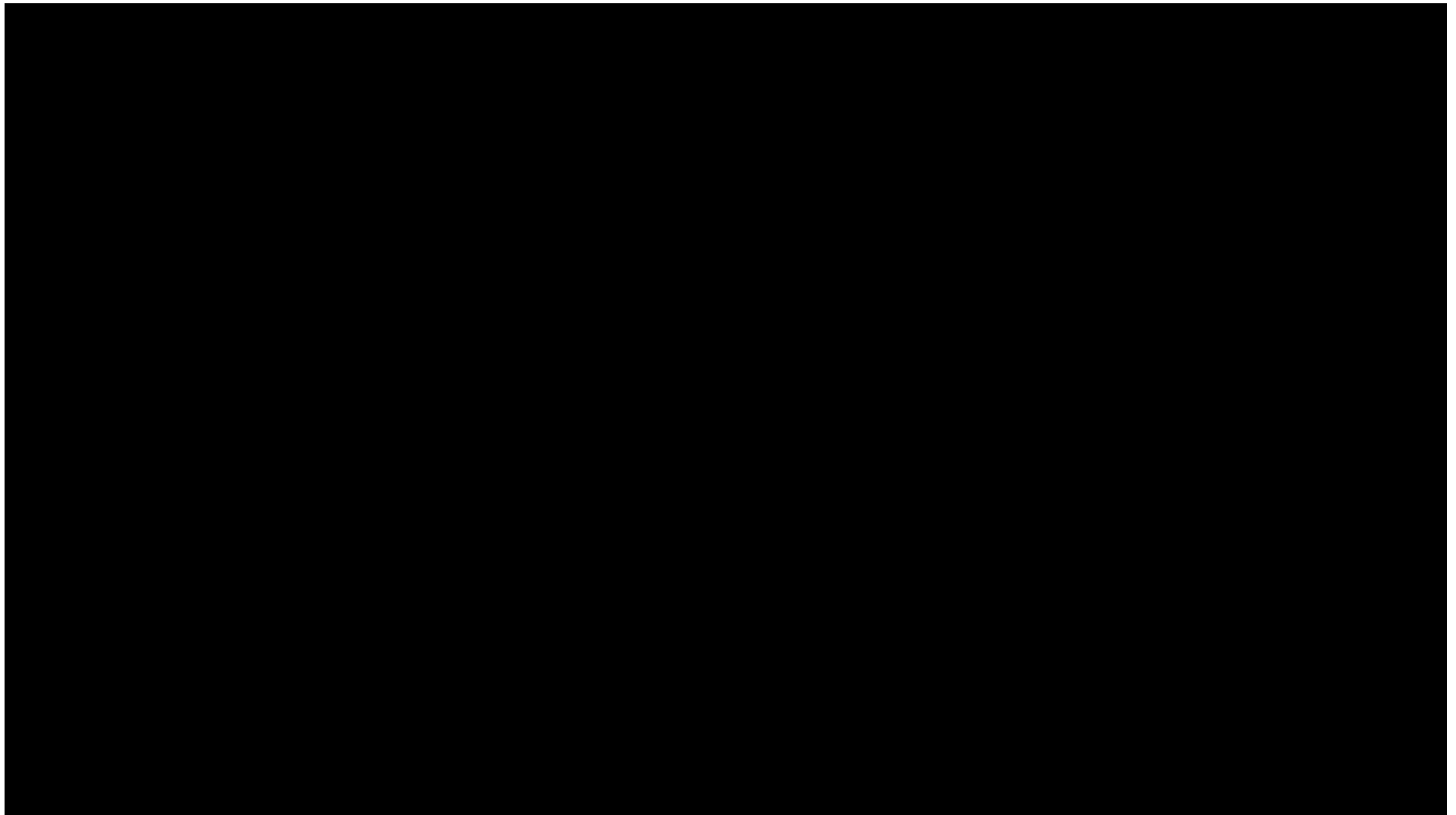


1024 x 1024 faces



Karras et al, "A Style-Based Generator Architecture for Generative Adversarial Networks", CVPR 2019

[Images](#) are licensed under [CC BY-NC 4.0](#)



Karras et al, "A Style-Based Generator Architecture for Generative Adversarial Networks", CVPR 2019

Video is licensed under [CC BY-NC 4.0](#).  
Source: [https://drive.google.com/drive/folders/1NFO7\\_vH0t98J13ckJYFd7kuaTkYeRJ86](https://drive.google.com/drive/folders/1NFO7_vH0t98J13ckJYFd7kuaTkYeRJ86)

# StyleGAN2



Karras et al, "Analyzing and Improving the Image Quality of StyleGAN", CVPR 2020



# Conditional GANs

**Recall:** Conditional Generative Models learn  $p(x|y)$  instead of  $p(x)$   
Make generator and discriminator both take label  $y$  as an additional input!

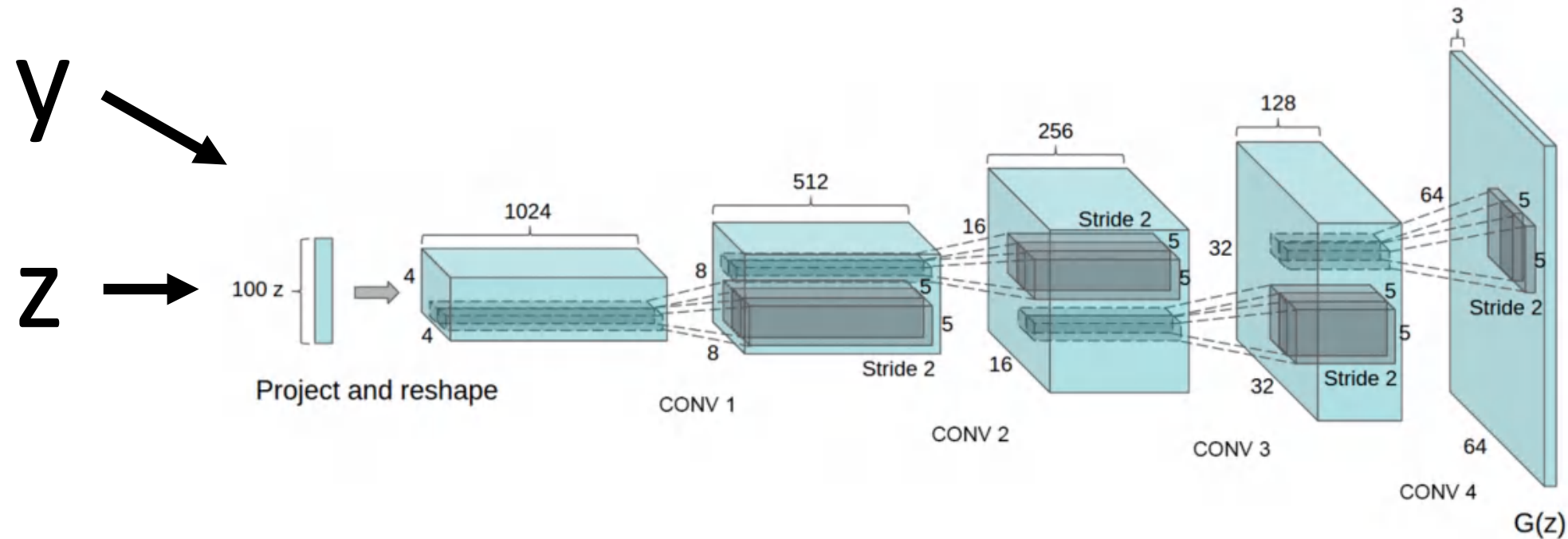


Figure credit: Radford et al, "Unsupervised Representation Learning with Deep Convolutional Generative Adversarial Networks", ICLR 2016

# Conditional GANs: Conditional Batch Normalization

## Batch Normalization

$$\begin{aligned}\mu_j &= \frac{1}{N} \sum_{i=1}^N x_{i,j} \\ \sigma_j^2 &= \frac{1}{N} \sum_{i=1}^N (x_{i,j} - \mu_j)^2 \\ \hat{x}_{i,j} &= \frac{x_{i,j} - \mu_j}{\sqrt{\sigma_j^2 + \epsilon}} \\ y_{i,j} &= \gamma_j \hat{x}_{i,j} + \beta_j\end{aligned}$$



Learn a separate  
scale and shift  
for each  
different label  $y$

## Conditional Batch Normalization

$$\begin{aligned}\mu_j &= \frac{1}{N} \sum_{i=1}^N x_{i,j} \\ \sigma_j^2 &= \frac{1}{N} \sum_{i=1}^N (x_{i,j} - \mu_j)^2 \\ \hat{x}_{i,j} &= \frac{x_{i,j} - \mu_j}{\sqrt{\sigma_j^2 + \epsilon}} \\ y_{i,j} &= \gamma_j^y \hat{x}_{i,j} + \beta_j^y\end{aligned}$$

# Conditional GANs: Spectral Normalization

Welsh springer spaniel



Fire truck



Daisy



Miyato et al, "Spectral Normalization for Generative Adversarial Networks", ICLR 2018

128x128 images on ImageNet



# Conditional GANs: Self-Attention

Goldfish



Indigo bunting



Redshank



Saint Bernard



Zhang et al, "Self-Attention Generative Adversarial Networks", ICML 2019

128x128 images on ImageNet



# Conditional GANs: BigGAN



Brock et al, "Large Scale GAN Training for High Fidelity Natural Image Synthesis", ICLR 2019

512x512 images on ImageNet

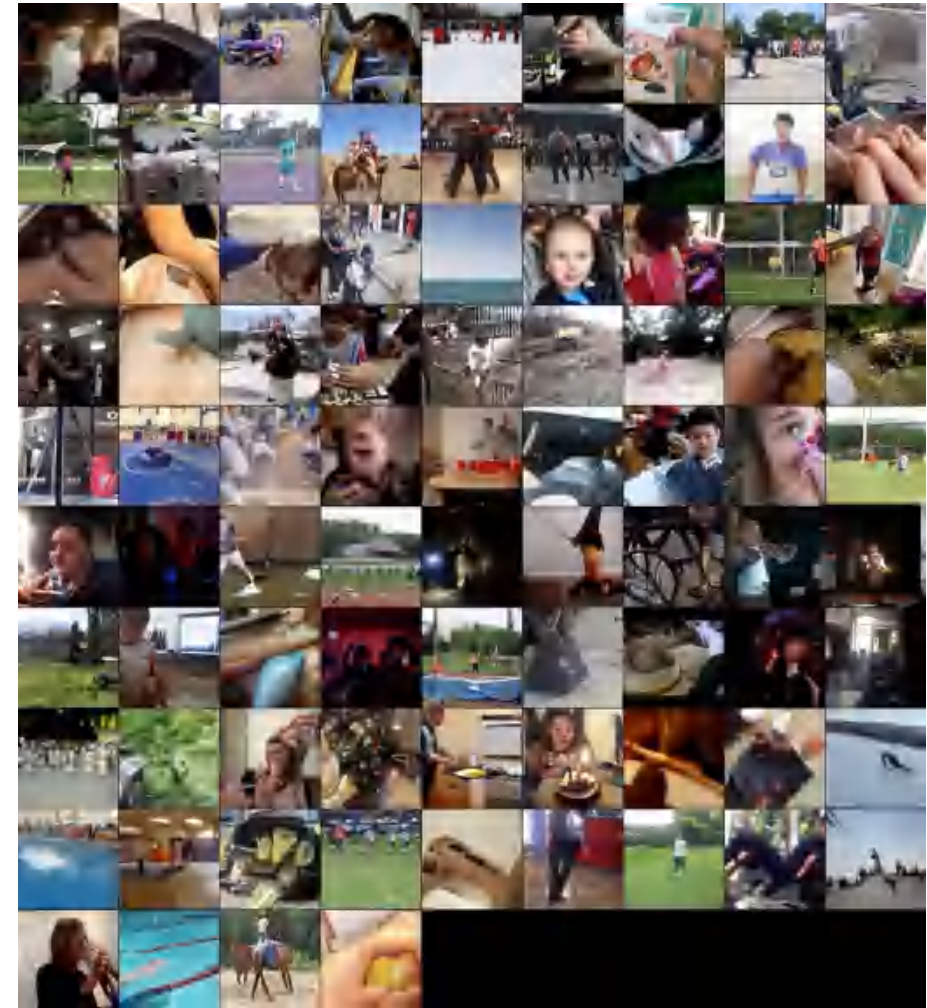


# Generating Videos with GANs



64x64 images, 48 frames

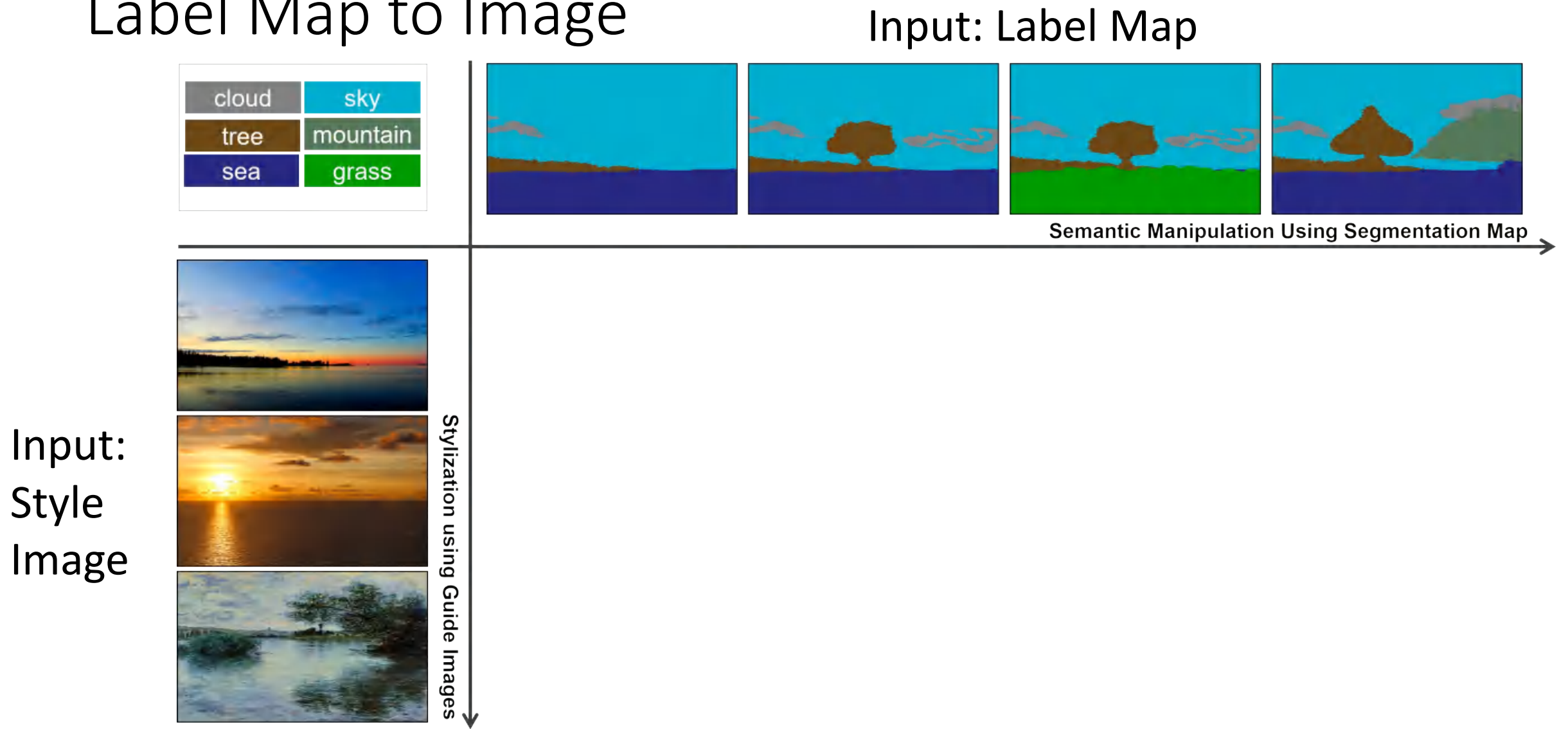
<https://drive.google.com/file/d/1FjOQYdUuxPXvS8yeOhXdPQMapUQakLi/view>



128x128 images, 12 frames

[https://drive.google.com/file/d/165Yxuvvu3viOy-39LhhSDGtczbWphj\\_i/view](https://drive.google.com/file/d/165Yxuvvu3viOy-39LhhSDGtczbWphj_i/view)

# Label Map to Image



Park et al, "Semantic Image Synthesis with Spatially-Adaptive Normalization", CVPR 2019



# Label Map to Image

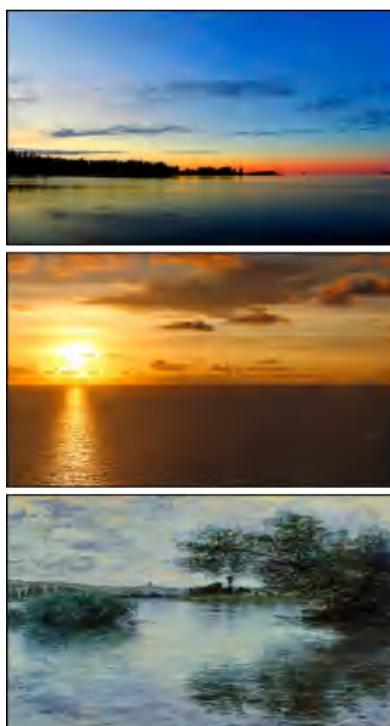
Input: Label Map

cloud	sky
tree	mountain
sea	grass

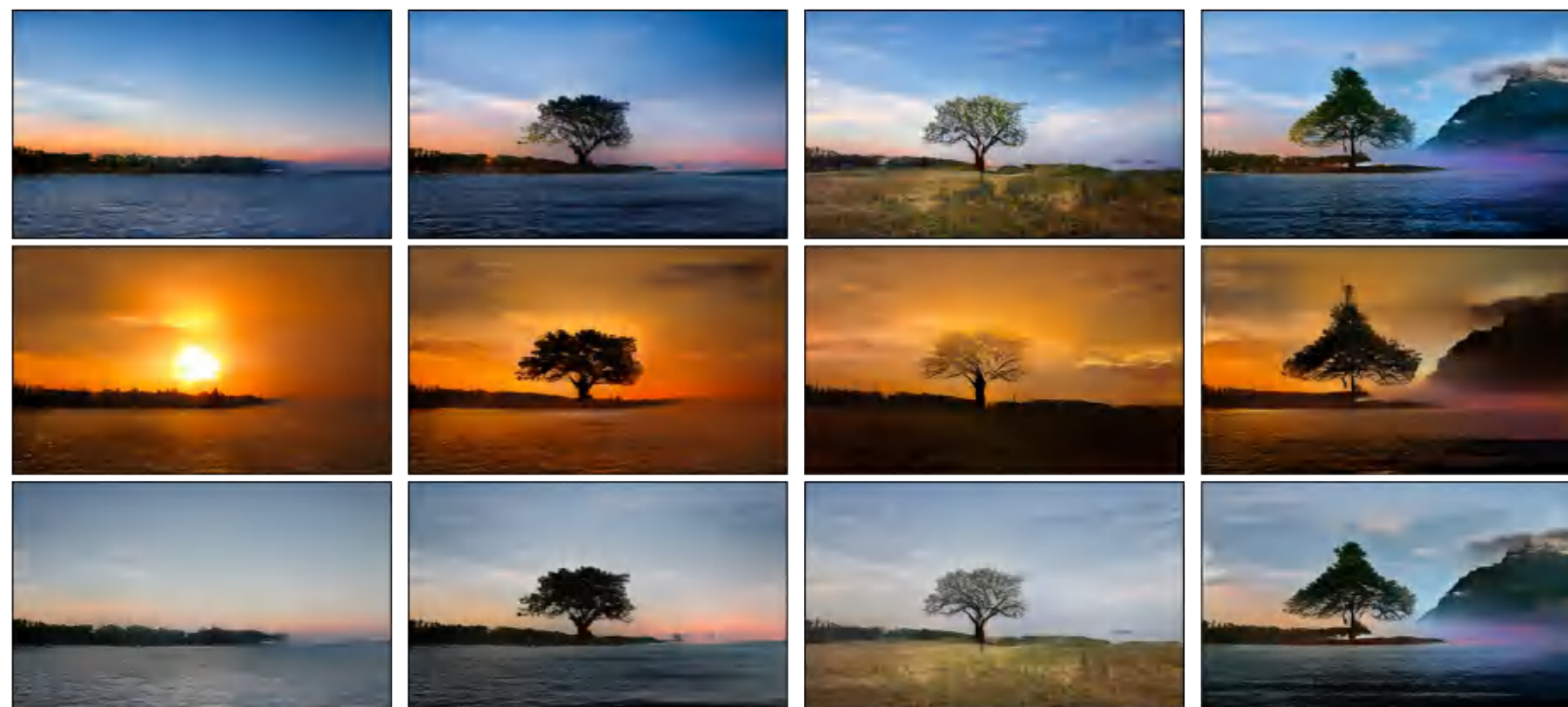


Semantic Manipulation Using Segmentation Map →

Input:  
Style  
Image



Stylization using Guide Images ↓

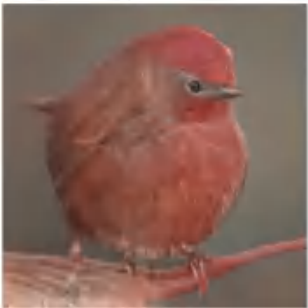


Park et al, "Semantic Image Synthesis with Spatially-Adaptive Normalization", CVPR 2019



# Conditioning on more than labels! Text to Image

This bird is red and brown in color, with a stubby beak



The bird is short and stubby with yellow on its body



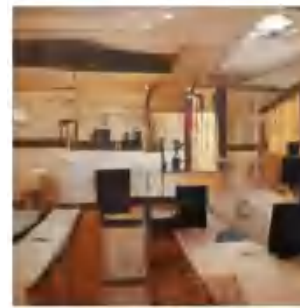
A bird with a medium orange bill white body gray wings and webbed feet



This small black bird has a short, slightly curved bill and long legs



A picture of a very clean living room



A group of people on skis stand in the snow



Eggs fruit candy nuts and meat served on white dish



A street sign on a stoplight pole in the middle of a day



Zhang et al, "StackGAN++: Realistic Image Synthesis with Stacked Generative Adversarial Networks.", TPAMI 2018

Zhang et al, "StackGAN: Text to Photo-realistic Image Synthesis with Stacked Generative Adversarial Networks.", ICCV 2017

Reed et al, "Generative Adversarial Text-to-Image Synthesis", ICML 2016

# Text to Image: DALL-E

**Step 1:** Train VQ-VAE (discrete grid of latent codes)

**Step 2:** Train autoregressive Transformer model to predict sequence of latent codes  
(Giant model on 250M image/text pairs)

**Step 3:** Given text prompt, sample new image codes; pass through VQ-VAE decoder to generate images

Ramesh et al, "Zero-Shot Text-to-Image Generation", ICML 2021

# Text to Image: DALL-E

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*an illustration of a baby hedgehog in a christmas sweater walking a dog*

# Text to Image: DALL-E

**Step 1:** Train VQ-VAE (discrete grid of latent codes)

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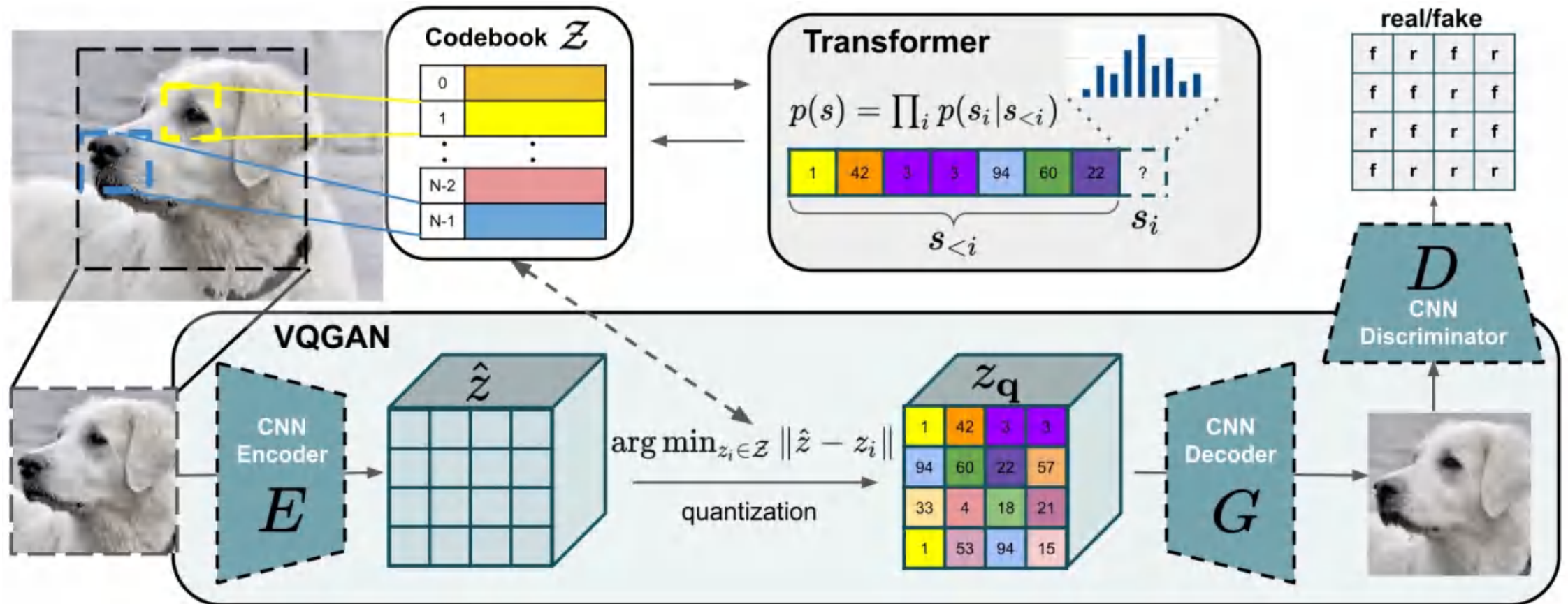
**Step 3:** Given text prompt, sample new image codes; pass through VQ-VAE decoder to generate images



*a neon sign that reads “backprop”. a neon sign that reads “backprop”. backprop neon sign*

Ramesh et al, “Zero-Shot Text-to-Image Generation”, ICML 2021

# VQ-GAN



Esser et al, "Taming Transformers for High-Resolution Image Synthesis", CVPR 2021



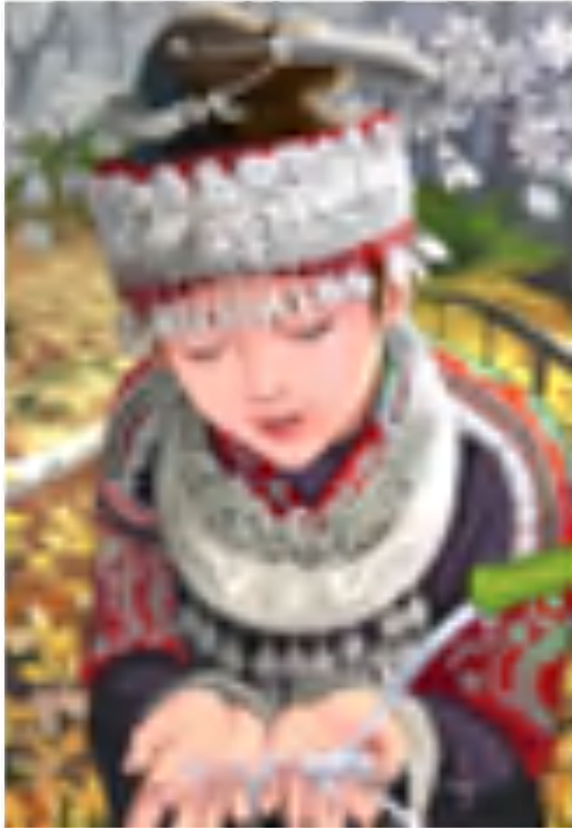
# VQ-GAN (Semantic Segmentation to Image)



Esser et al, "Taming Transformers for High-Resolution Image Synthesis", CVPR 2021

# Image Super-Resolution: Low-Res to High-Res

bicubic  
(21.59dB/0.6423)



SRResNet  
(23.53dB/0.7832)



SRGAN  
(21.15dB/0.6868)



original

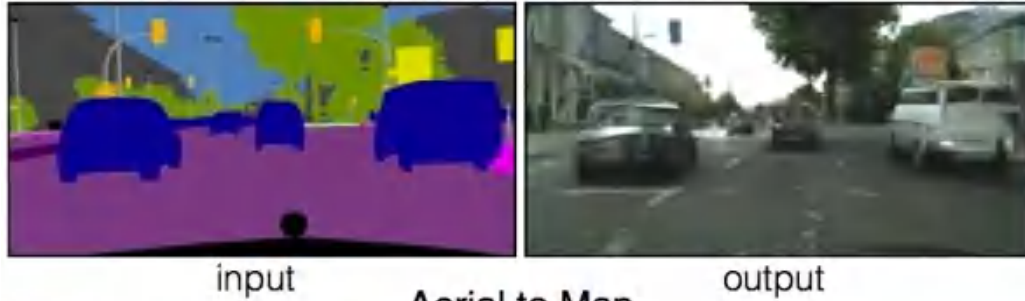


Ledig et al, "Photo-Realistic Single Image Super-Resolution Using a Generative Adversarial Network", CVPR 2017

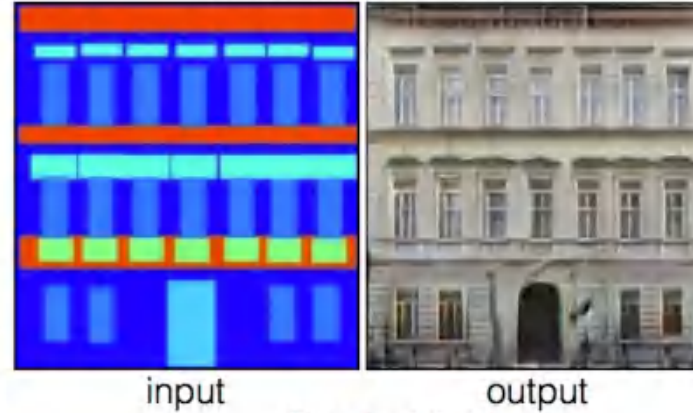


# Image-to-Image Translation: Pix2Pix

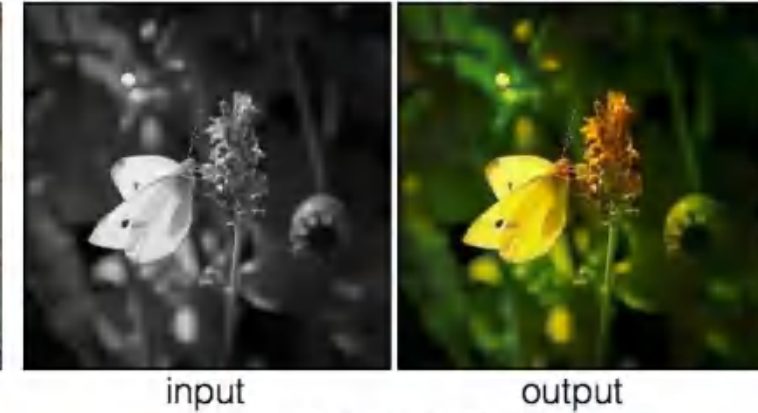
Labels to Street Scene



Labels to Facade



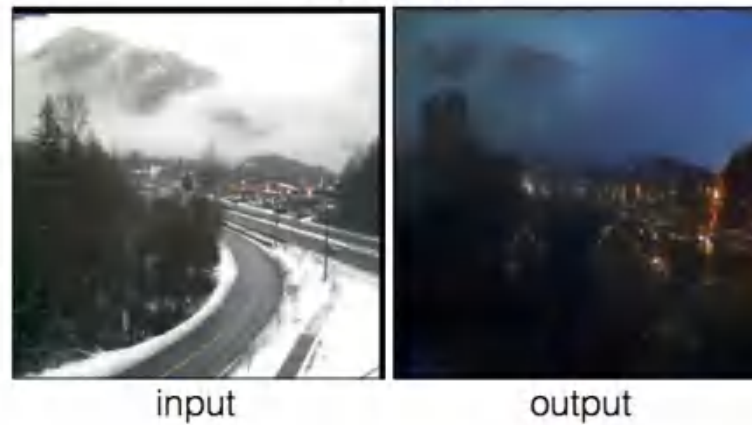
BW to Color



Aerial to Map



Day to Night



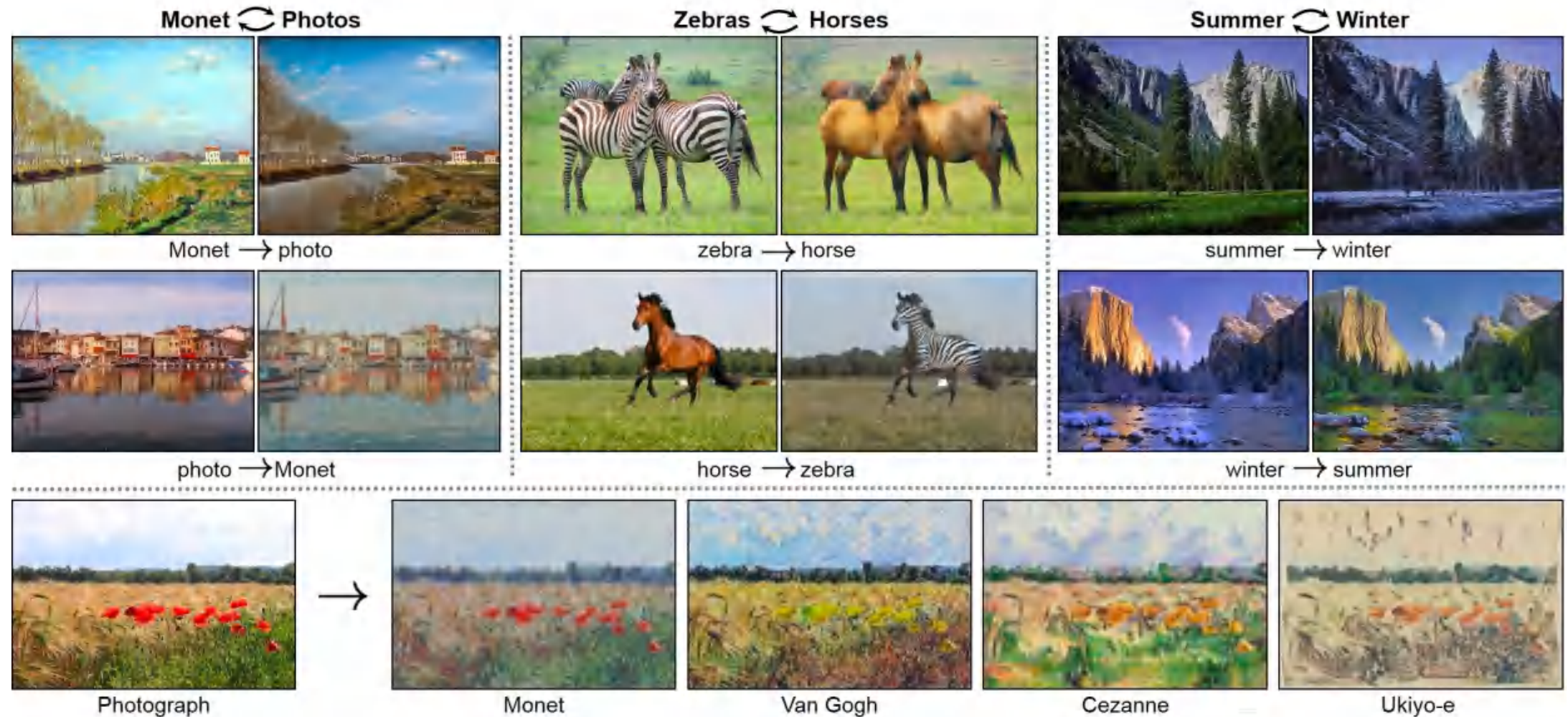
Edges to Photo



Isola et al, "Image-to-Image Translation with Conditional Adversarial Nets", CVPR 2017



# Unpaired Image-to-Image Translation: CycleGAN



Zhu et al, "Unpaired Image-to-Image Translation using Cycle-Consistent Adversarial Networks", ICCV 2017

# Unpaired Image-to-Image Translation: CycleGAN

Input Video: Horse

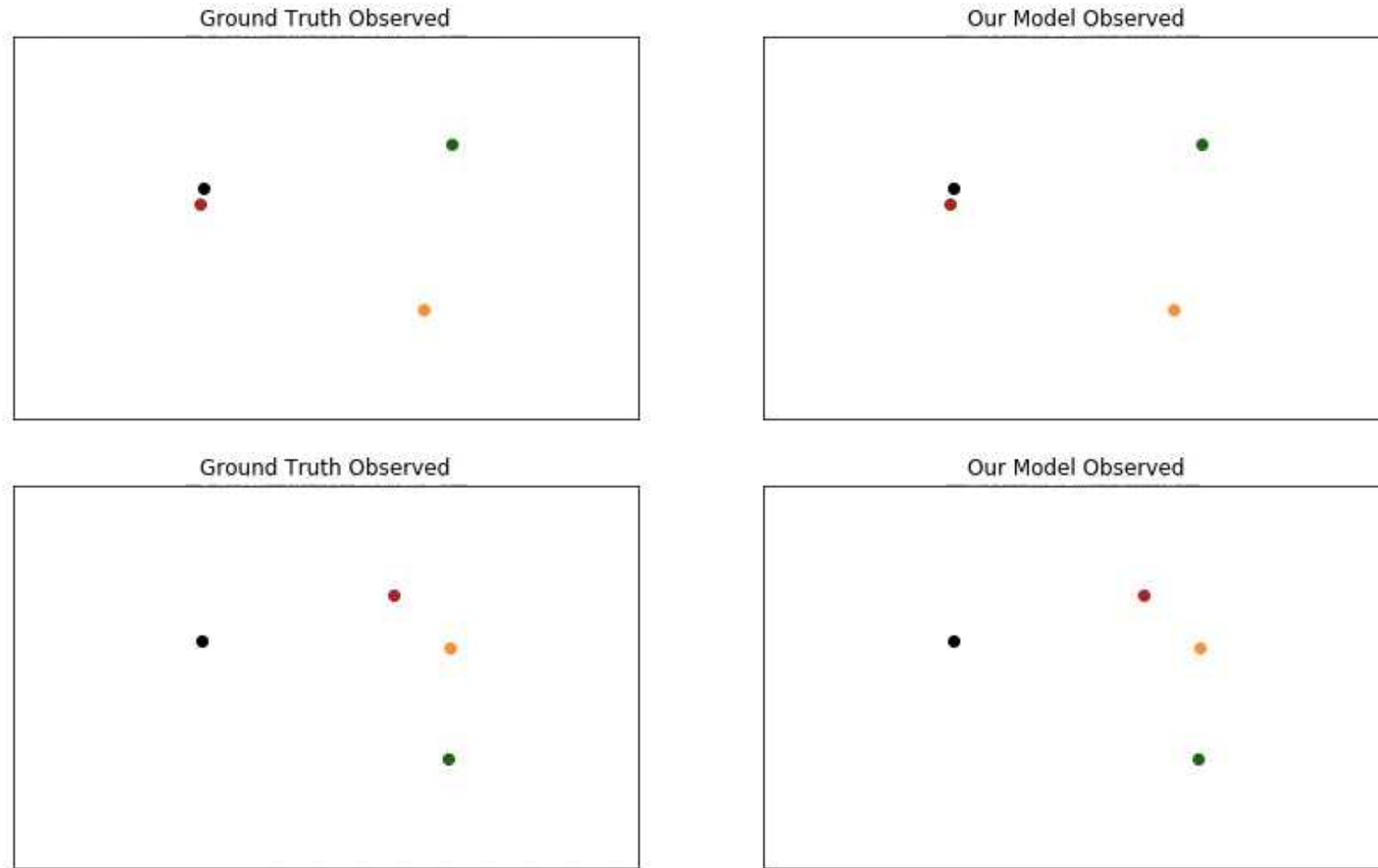
Output Video: Zebra



<https://www.youtube.com/watch?v=9reHvktowLY>

Zhu et al, "Unpaired Image-to-Image Translation using Cycle-Consistent Adversarial Networks", ICCV 2017

# GANs: Not just for images! Trajectory Prediction



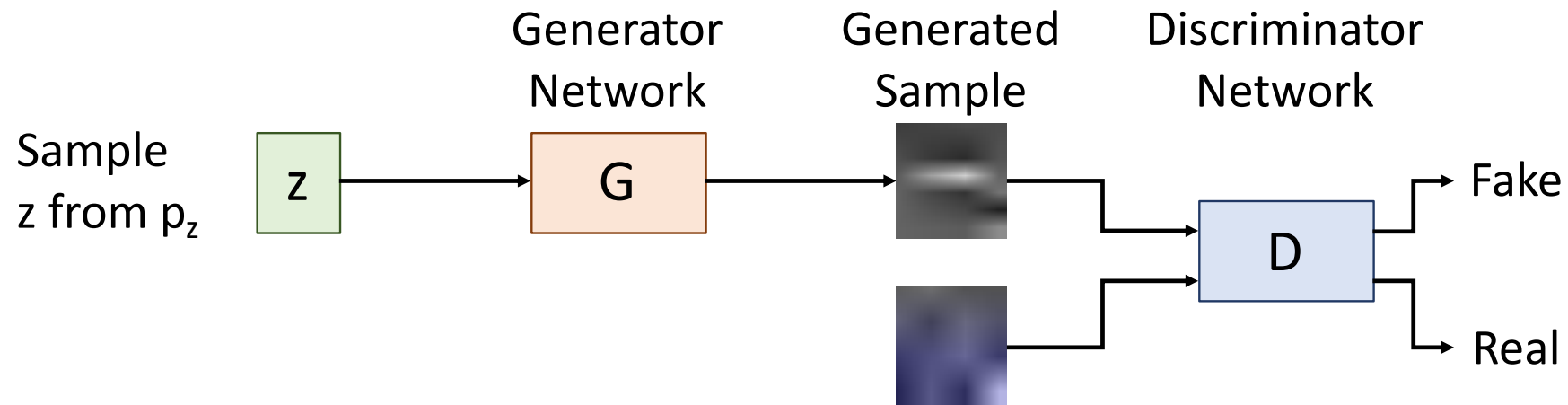
Gupta, **Johnson**, Li, Savarese, Alahi, "Social GAN: Socially Acceptable Trajectories with Generative Adversarial Networks", CVPR 2018

# GAN Summary

Jointly train two networks:

**Discriminator:** Classify data as real or fake

**Generator:** Generate data that fools the discriminator



Under some assumptions, generator converges to true data distribution  
Many applications! Very active area of research!

# Taxonomy of Generative Models

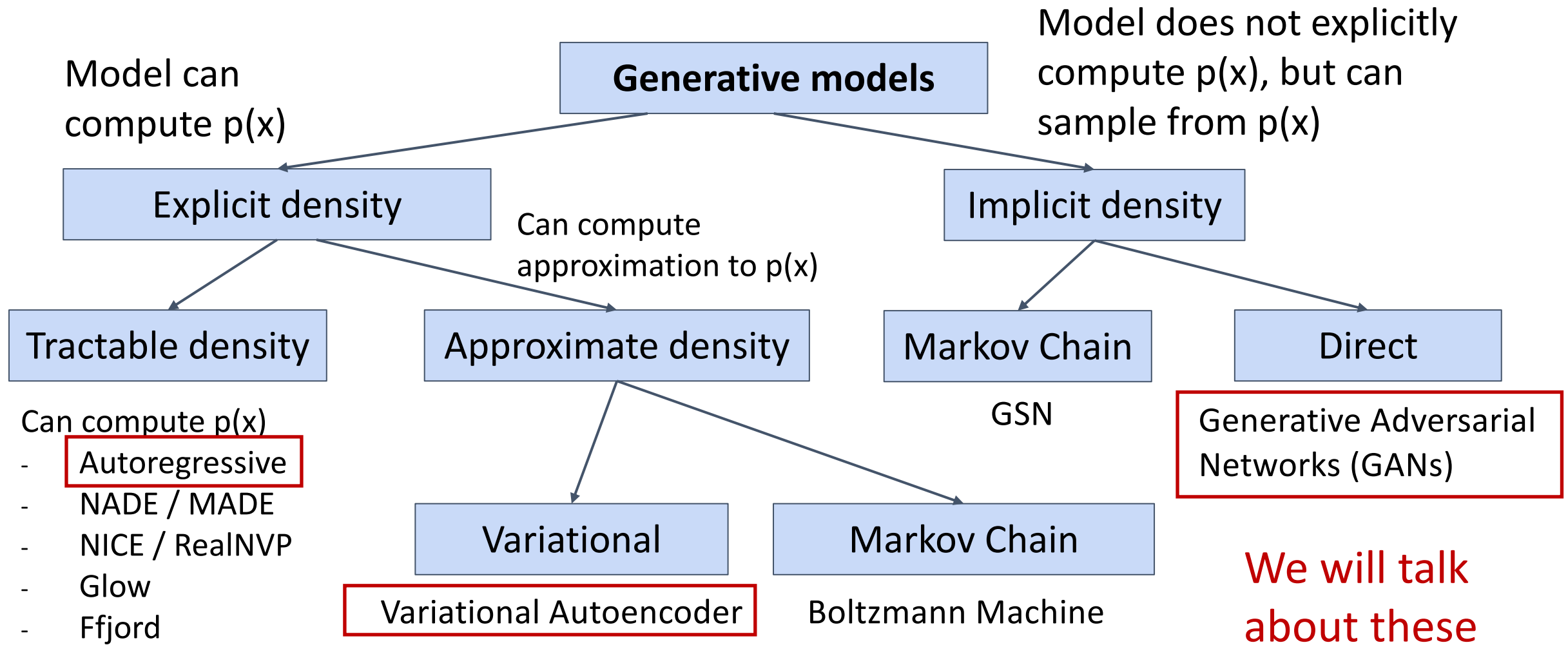


Figure adapted from Ian Goodfellow, Tutorial on Generative Adversarial Networks, 2017.



# Generative Models Summary

**Autoregressive Models** directly maximize likelihood of training data:

$$p_{\theta}(x) = \prod_{i=1}^N p_{\theta}(x_i | x_1, \dots, x_{i-1})$$

Good image quality, can evaluate with perplexity. Slow to generate data, needs tricks to scale up.

**Variational Autoencoders** introduce a latent  $z$ , and maximize a lower bound:

$$p_{\theta}(x) = \int_Z p_{\theta}(x|z)p(z)dz \geq E_{z \sim q_{\phi}(z|x)}[\log p_{\theta}(x|z)] - D_{KL}(q_{\phi}(z|x), p(z))$$

Latent  $z$  allows for powerful interpolation and editing applications.

**Generative Adversarial Networks** give up on modeling  $p(x)$ , but allow us to draw samples from  $p(x)$ . Difficult to evaluate, but best qualitative results today

# Next Time: Visualizing Models and Generating Images