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CHRONICLE

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V. E. Krivonozhko, O. B. Utkin, and A. V. Antonov (Moscow). *Optimization Models in Performance Analysis of Complex Systems* (September 25, 2000).

To estimate the performance of a large company as a whole viewed as a complex system, we use the technique of Operating Environment Analysis (OEA) [1–3].

Consider a set of n observable production facilities (PF) whose operation is to be assessed. Each PF consumes m input products and manufactures r output products. Thus, let $X_j = (x_{1j}, \ldots, x_{mj}) \geq 0$ be the vector of input parameters (expenditures), and let $Y_j = (y_{1j}, \ldots, y_{rj}) \geq 0$, $j = 1, \ldots, n$, be the vector of output parameters (production). We assume that each PF has at least one positive input and at least one positive output. The set T of production possibilities is defined as $T = \Big\{(X, Y) \mid \textit{the output vector } Y \geq 0 \textit{ can be obtained for the input vector } X \geq 0\Big\}$. On the basis of observed vectors (X_j, Y_j), $j = 1, \ldots, n$, the set T is specified empirically by the following postulates.

Postulate 1 (convexity). If $(X, Y) \in T$ and $(X', Y') \in T$, then

$$(\lambda X + (1-\lambda)X', \lambda Y + (1-\lambda)Y') \in T \qquad \forall \lambda \in [0,1].$$

Postulate 2 (monotonicity). If $(X, Y) \in T$, $X' \geq X$, and $Y' \leq Y$, then $(X', Y') \in T$.

Postulate 3 (the minimal extrapolation). The set T is the intersection of all sets T' satisfying Postulates 1 and 2 and such that $(X_j, Y_j) \in T'$ for all $j = 1, \ldots, n$.

This production model formalism allows one to select effective production possibilities from all pairs (X, Y) of input and output vectors.

Definition 1. A production vector (X^*, Y^*) is said to be *effective* if $(X^*, Y^*) \in T$ and there is no vector $(X, Y) \in T$ different from (X^*, Y^*) and satisfying $X \leq X^*$ and $Y \geq Y^*$.

In the space E^{m+r} of production parameters, effective points form an *effective hypersurface* (the *front*), which is an analog of the production function. There exist various optimization models in the framework of the OEA technology which allow one to find various points of the effective hypersurface and find performance measures. However, an attempt to use the classical OEA in transitional economy results in "strange" situations. For example, the OEA can assign a performance measure of 100% to a low-performance company. The analysis of such situations shows that one needs to modify the standard technology.

Consider a PF $Z_0 = (X_0, Y_0)$ for which some of the input parameters are negative. Let I_0^- be the set of indices of negative components of Y_0. By J^+ we denote the set of PF such that $X \geq 0$ and $Y \geq 0$.

We redefine the set T on the basis of Postulates 1–3 and the set

$$J^+ : T = \left\{(X, Y)/X \geq \sum_{j \in J^+} X_j \lambda_j,\ Y \leq \sum_{j \in J^+} X_j \lambda_j,\ \sum_{j \in J^+} \lambda_j = 1,\ \lambda_j \geq 0\right\}, \qquad T^+ = T \cap R_+^{m+r}.$$

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Thus, to improve the performance of the PF $Z_0 = (X_0, Y_0)$, we must bring it from the current state into the set $T^+(Y_0)$. We introduce the sets

$$T_x(Y_0) = \left\{ X \Big/ \sum_{j\in J^+} x_{kj}\lambda_j \le x_k,\ k = 1,\dots,m,\ \sum_{j\in J^+} y_{ij}\lambda_j \ge y_{i0},\ i \notin I_0^-,\ \sum_{j\in J^+} \lambda_j = 1,\ \lambda_j \ge 0 \right\},$$

and $T^+(Y_0) = \{(X,Y)/X \in T_x(Y_0),\ (X,Y) \in T^+\}$.

Now, to improve the performance of Z_0, we must bring it into the set $T^+(Y_0)$ along a path of minimal length while keeping the positive components of Y_0 at least at the same level. Let us consider two norms, $p = \infty$ and $p = 1$. First, we consider the norm $p = \infty$. Recall that the norm $p = \infty$ is defined as follows: $\|Z\|_\infty = \max_i |z_i|$. To find the minimum distance between Z_0 and $T^+(Y_0)$ in the ∞-norm, we use the following technique.

Algorithm 1.

Step 1. We solve the problem $\min \tau_1$ with the following constraints:

$$\sum_{j\in J^+} X_j\lambda_j \le X_0 + \tau_1 d_1, \qquad d_1 = (1,\dots,1), \qquad \sum_{j\in J^+} Y_j\lambda_j \ge Y_0, \qquad \sum_{j\in J^+} \lambda_j = 1, \qquad \lambda_j \ge 0.$$

Step 2. We solve the problem $\min \tau_2$ with the constraints $Y_0 + \tau_2 d_2 \ge 0$, where

$$d_{i2} = \begin{cases} 0 & \text{if} \quad i \notin I_0^- \\ 1 & \text{if} \quad i \in I_0^-. \end{cases}$$

Step 3. We take $\tau = \max\{\tau_1^*, \tau_2^*\}$ as the minimum distance between Z_0 and $T^+(Y_0)$ in the ∞-norm.

Theorem 1. *The quantity τ provided by Algorithm 1 is the minimum distance between (X_0, Y_0) and $T^+(Y_0)$ in the ∞-norm.*

Let us proceed to the norm $p = 1$ for measuring the distance between Z_0 and $T^+(Y_0)$. Recall that this norm is defined as $\|Z\|_1 = \sum_{i=1}^{m+r} |z_i|$. To find the minimum distance between $Z_0 = (X_0, Y_0)$ and $T^+(Y_0)$ in the 1-norm, we use the following method.

Algorithm 2.

Step 1. We solve the problem $\min \tau_1 = \sum_{k=1}^m w_k^1$, $\sum_{j\in J^+} x_{kj}\lambda_j \le x_{k0} + w_k^1$, $k = 1,\dots,m$, $\sum_{j\in J^+} y_{ij}\lambda_j \ge y_{i0}$, $i \notin I_0^-$, $\sum_{j\in J^+} \lambda_j = 1$, $\lambda_j, w_k^1 \ge 0$.

Step 2. Solve the problem $\min \tau_2 = \sum_{i\in I_0^-} w_i^2$, $y_{i0} + w_i^2 \ge 0$, $i \in I_0^-$.

Step 3. We take $\tau = \tau_1^* + \tau_2^*$ as the minimum distance between Z_0 and the set $T^+(Y_0)$ in the 1-norm.

Theorem 2. *The quantity τ provided by Algorithm 2 is the minimum of the distance between (X_0, Y_0) and $T^+(Y_0)$ in the 1-norm.*

After an application of Algorithm 1 or 2, Z_0 exits to the boundary of the set $T^+(Y_0)$ and becomes at least weakly effective. Then the standard approach can be applied to Z_0.

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A. P. Nosov, V. V. Fomichev, and A. S. Shepit'ko (Moscow). *Methods for Exponential Stabilization of Bilinear Systems* (October 9, 2000).

We consider the problem of stabilizing the system

$$\dot{x} = Ax + \sum_{i=1}^m u_i B_i y, \qquad y = cx, \tag{1}$$

at zero, where $x \in \mathbf{R}^n$, $u \in R^m$, $y \in \mathbf{R}^l$, and A, B_i, and C are constant matrices of appropriate dimensions. We assume that system (1) is in general position, i.e., the pairs $\{A, B_i\}$ are controllable pairs and the pair $\{A, C\}$ is observable ($i = 1, \dots, m$).

We obtain a necessary and sufficient condition for the stabilizability of such a system on the plane with respect to the state in the class of constant controls for a scalar ($m = 1$) control.

Theorem. *System* (1) *is exponentially stable if and only if one of the following alternatives takes place*:

1^0 $-\operatorname{tr} A > 0$ *and* $\det A > 0$;

2^0 $\det B > 0$;

3^0 $\det B < 0$, $\Delta_1 > 0$, *and either* $\mu_1 < 0$ *for* $\operatorname{tr} B > 0$, *or* $\mu_2 > 0$ *for* $\operatorname{tr} B < 0$, *where* μ_1 *and* μ_2 *are the roots of the equation* $\gamma_2(-\operatorname{tr} A/\operatorname{tr} B + \mu) = 0$;

4^0 $\det B < 0$, $\operatorname{tr} B = 0$, $-\operatorname{tr} A > 0$, *and* $\Delta_2 > 0$;

5^0 $\det B = 0$, $\operatorname{tr} B \neq 0$, *and* $[\operatorname{tr} A \operatorname{tr} B - \operatorname{tr}(AB)] \operatorname{tr} B > 0$;

6^0 $\det B = 0$, $\operatorname{tr} B = 0$, $-\operatorname{tr} A > 0$, *and* $\operatorname{tr}(AB) \neq 0$.

In the case of a vector ($m = 2$) control for a system on the plane, a necessary and sufficient condition for stabilizability in the class of constant controls can be stated as follows.

Lemma. *System* (1) *is stabilizable in the class of constant controls if and only if the system of inequalities*

$$-\langle l, u\rangle - \operatorname{tr} A > 0, \qquad \langle Qu, u\rangle + \langle h, u\rangle + \det A > 0$$

is compatible, *where* $\langle\, , \rangle$ *is the inner product and*

$$Q = \begin{bmatrix} \det B_1 & (1/2)\,(\operatorname{tr} B_1 \operatorname{tr} B_2 - \operatorname{tr} B_1 B_2) \\ (1/2)\,(\operatorname{tr} B_1 \operatorname{tr} B_2 - \operatorname{tr} B_1 B_2) & \det B_2 \end{bmatrix},$$

$$h = \begin{bmatrix} \operatorname{tr} A \operatorname{tr} B_1 - \operatorname{tr} AB_1 \\ \operatorname{tr} A \operatorname{tr} B_2 - \operatorname{tr} AB_2 \end{bmatrix}, \qquad l = \begin{bmatrix} \operatorname{tr} B_1 \\ \operatorname{tr} B_2 \end{bmatrix}, \qquad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

On the basis of the lemma, we have obtained a number of simpler sufficient stabilizability conditions. Moreover, for two-dimensional systems in general position with a scalar output y, we have shown that the observation problem (that is, the problem of reconstruction of the phase vector x) can be reduced to the observation problem for a system with a degenerate bilinearity matrix (i.e., with $\operatorname{rank} B = 1$). For such systems, we have constructed asymptotic observers and, on their basis, synthesized algorithms of stabilization of three-dimensional systems with respect to the output.

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V. A. Brusin (Nizhni Novgorod). *Frequency Conditions for the Robust Stability of a Certain Class of Switching Systems* (October 23, 2000).

We consider the system described by the equations

$$\dot{z}(t) = Pz(t) + Qu(t), \qquad u(t) = C_i z(t), \qquad C_i = \begin{cases} C_1 & \text{for} \quad \sigma = 1 \\ C_2 & \text{for} \quad \sigma = 2, \end{cases} \tag{1}$$

where $z \in \mathbf{R}^N$, $u \in \mathbf{R}^N$, $\sigma(t) = 1, 2$, is the "switching" signal [1], whose values (1 or 2) can depend not only on the process but on external (with respect to the system in question) processes as well. We introduce a definition of admissible switching laws such that only finitely many switchings are possible on each finite time interval.

Definition 1. System (1) is said to be *robustly exponentially stable with respect to a family* $\mathscr{U}$ *of admissible switching laws* if, for any $u(t) \in \mathscr{U}$ and for all processes $z(t)$ satisfying (1), there exist positive constants $c_0 > 0$ and $\delta > 0$ such that

$$|z(t)| \leq c_0 e^{-\delta t} |z(0)|, \qquad t \geq 0. \tag{2}$$

Theorem 1. *Let the following conditions be satisfied*:
(1) *one of the matrices* $P + QC_i$, $i = 1, 2$, *is stable*;
(2) *the frequency condition*

$$\operatorname{Re}\Pi(j\omega) = (1/2)\left(\Pi(j\omega) + \Pi^{\mathrm{T}}(-j\omega)\right) \geq \varepsilon\left(I + \left\|(j\omega I - P)^{-1}q\right\|^2\right)$$

is satisfied for some a, b, $a + b < 0$, $\alpha > 0$, $\beta > 0$, *and* $\varepsilon > 0$ *and for all* $\omega > 0$ *such that* $\det(j\omega I - P)$; *here* $j = \sqrt{-1}$, I *is the identity matrix*, T *is the transposition symbol*, *and*

$$-\Pi(jw) = (I - C_1W(-jw))^{\mathrm{T}}\,(C_3W(jw) + aI) + (I - C_1W(-jw))^{\mathrm{T}}\,(C_4W(jw) + bI)\,,$$
$$W(p) = (pI - P)^{-1}Q, \qquad C_3 = \left(\alpha^2 - a\right)C_2 - \alpha^2 C_1, \qquad C_4 = \left(\beta^2 - b\right)C_1 - \beta^2 C_2.$$

Then system (1) *is robustly exponentially stable.*

This result can naturally be generalized to automated control systems consisting of a controller

$$A(p)x(t) = B(p)y(t), \qquad p = d/dt, \tag{3}$$

and switching objects

$$D(p)y(t) = \begin{cases} M_1(p)x(t) & \text{for} \quad \sigma = 1 \\ M_2(p)x(t) & \text{for} \quad \sigma = 2, \end{cases} \tag{4}$$

where $A(p)$, $B(p)$, $D(p)$, and $M_i(p)$ are polynomials in $p = d/dt$ such that the fractions $B(p)M_i(p)/A(p)D(p)$ are proper and $\deg M_i(p) \leq \deg D(p) + \deg A(p) - \deg B(p)$. For systems of the form (3), (4), we prove a theorem on the robust exponential stability.

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S. A. Belov (Nizhni Novgorod). *Symmetry and Sufficient Optimality Conditions in Domain Optimization Problems* (November 27, 2000).

We consider a well-known domain optimization problem with an integral functional and an elliptic boundary value problem as a constraint and obtain sufficient conditions for the disk to be optimal.

The following problem is well known in the theory of domain shape optimization problems: on a family T of connected domains Ω with given area S and with $C^{2,\alpha}$ boundaries, minimize (or maximize) the functional

$$J(\Omega) = \int\limits_{\Omega} g(u(x))dx, \tag{1}$$

where g is a given sufficiently smooth function and u is found from the boundary value problem

$$\Delta u(x) = -1 \qquad (x \in \Omega), \tag{2}$$
$$u(x) = 0 \qquad (x \in \Gamma). \tag{3}$$

Note that if $g(u) = u$, then we obtain the classical problem on the maximum of the torsional rigidity of a homogeneous elastic bar [1]; if $g(u) = u^{1/2}$, then we obtain the problem on ground water reserves in filtration theory [2]. Without loss of generality, we can assume that $g(0) = 0$ and $S = \pi$. Necessary first-order optimality conditions are also well known in problems of the form (1)–(3). Namely, a domain Ω with boundary Γ provides a minimum in the optimization problem (1)–(3) only if there exists a constant λ such that

$$\frac{\partial u}{\partial n}\frac{\partial p}{\partial n} = \lambda \qquad (x \in \Gamma), \tag{4}$$

where the function p is found from the dual problem

$$\Delta p(x) = \partial g(u(x))/\partial u \qquad (x \in \Omega), \tag{5}$$
$$p(x) = 0 \qquad (x \in \Gamma). \tag{6}$$

Now we suppose that Ω is the unit disk and Γ is the unit circle. Then $u = u(r)$, $p = p(r)$, and hence the first-order necessary condition (4) is satisfied for any function g, i.e., the disk is always a stationary point in our problem. To establish conditions for the disk to be a solution of the optimization problem, we consider increments of functionals up to second-order infinitesimals. To this end, we consider domains Ω_h of the class $C^{2,\alpha}$ whose boundaries are described by an equation of the form $r = 1 + h(\phi)$ in the polar coordinates. Then

$$J(\Omega_h) - J(\Omega) = \int\limits_\Gamma \frac{\partial g}{\partial u} F^2 ds - 4g(1/4) \int\limits_\Gamma F \frac{\partial F}{\partial n} ds + \frac{1}{2} \int\limits_\Omega \frac{\partial^2 g}{\partial u^2} F^2 dx + \alpha(h) \int\limits_\Gamma h^2 ds, \tag{7}$$

where $\alpha(h) \to 0$ as $\|h\|_{C^{2,\alpha}} \to 0$ and the function F is found from the problem

$$\Delta F = 0 \qquad (x \in \Omega), \tag{8}$$

$$F = h/2 \qquad (x \in \Gamma). \tag{9}$$

Representing the perturbation h by a Fourier series and performing some manipulations, we obtain

$$J(\Omega_h) - J(\Omega) = 2\pi \sum_{n=2}^{\infty} n \left(a_n^2 + b_n^2\right) \left(\int\limits_0^1 \frac{\partial g}{\partial u}(u(r)) r^{2n-1} dr - 2g(1/4) \right) + \alpha(h) \int\limits_\Gamma h^2 ds. \tag{10}$$

It is important that the summation starts from $n = 2$. This is explained by the presence of a constraint imposed on the area and the translation invariance. Now, using the method of introduction of the nearest circle [3], we obtain sufficient optimality conditions.

Theorem 1. *Suppose that there exists a constant* $\beta > 0$ (*respectively*, < 0) *such that*

$$\int\limits_0^1 \frac{\partial g}{\partial u} r^{2n-1} dr - 2g(1/4) > \beta/n \qquad (< \beta/n) \tag{11}$$

for $n \geq 2$. *Then the unit disk is a local minimum* (*respectively*, *maximum*) *in the optimization problem* (1)–(3).

The following assertion is an interesting corollary to Theorem 1.

Theorem 2. *Let* g *be monotonically increasing* (*respectively*, *decreasing*) *on the closed interval* $[0, 1/4]$. *Then the unit disk is the maximum* (*respectively*, *the minimum*) *in the optimization problem* (1)–(3).

It follows from Theorem 2 that the unit disk is optimal in the above-considered problems on the torsional rigidity and ground water reserves.

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A. V. Pukhlikov (Moscow). *The Dynamics of a Discontinuous System near a Singular Point of the Sliding Field* (December 25, 2000).

Near a smooth break hypersurface, a discontinuous dynamical system has the local form of a pair (v_+, v_-) of smooth dynamical systems on $U \cong \mathbf{R}^m_{(x_1,\ldots,x_m)}$, where $Q = \{x_1 = 0\} \subset U$ is the break hypersurface. On the "halves" $U_\pm = \{\pm x_1 \geq 0\}$, the motion is determined by the dynamical systems $v_\pm$, respectively; moreover, the vectors $v_\pm(p)$ are transversal to Q for almost all points $p \in Q$ and are codirected, i.e., satisfy $v_+(x_1)\, v_-(x_1) > 0$. At transversality points, the discontinuous system has the same behavior as an ordinary smooth system at a nonsingular point. However, even the simplest failure of the transversality condition changes the geometric properties of trajectories drastically. In general position, there are three types of singularities: hyperbolic, parabolic, and elliptic [1, 2]. In the hyperbolic and parabolic cases, the local behavior of the system near Q can be described in a very simple manner. But in the elliptic case, the dynamics is not trivial: the system rapidly oscillates around the manifold $E \subset Q$ of elliptic singular points determined by the condition $v_\pm \in T_x Q$. In this case, on the hyperplane $E \subset Q$ of elliptic singular points, the system $v = (v_\pm)$ induces a *sliding field* $\xi \in \Gamma(TE)$ [2]. At points $p \in E$ at which the sliding field does not vanish, $\xi(p) \neq 0$, there is only one behavior type of the system v (the normal form exists and is unique, see [1]). In particular, the system v is structurally stable at such a point.

Here we investigate the geometric properties of a discontinuous dynamical system v in general position near an (isolated) hyperbolic singular point $o \in E$ of the sliding field. Let $z = (s, u) \in \mathbf{R}_+ \times \mathbf{R}^n$ be a coordinate system on Q_+ such that $s \in \mathbf{R}_+$ is half the first return time of the system to the break hypersurface Q. Let $Q_\pm$ be halves in which E locally divides Q, $Q = Q_+ \cup Q_-$, $Q_+ \cap Q_- = E$, and let $s = 0$ be the equation of the elliptic hypersurface E. The total return mapping is the composition of two return mappings: $R = R_- \circ R_+ : Q_+ \to Q_+$ and similarly for Q_-. It was shown in [2] that, in the coordinates (s, u), the total return mapping $R : Q_+ \to Q_+$ has the form

$$R : (s, u) \mapsto \left(\frac{s}{1 + s g(s, u)},\ \exp(s\xi)u + s^2 G(s, u) \right), \tag{1}$$

where $g(s, u) = g_+(s, u)$ is some smooth function on Q_+ and $G : Q_+ \to \mathbf{R}^n_u$ is some smooth mapping; moreover, $g(o) \neq 0$.

The dynamics of the discontinuous system v near the point $o \in E$ is completely determined by the dynamics of the discrete flow $\{R^k,\ k \in \mathbf{Z}_+\}$ of mappings. Therefore, to describe the geometric properties of the system v, it is necessary and sufficient to reduce the mapping R to a convenient form near the singular point of the sliding field.

We assume that $g(o) > 0$, i.e., trajectories of the system v approach the elliptic surface E. If $g(o) < 0$, then we replace v by $(-v)$ and reduce the problem to the case $g(o) > 0$.

Theorem. *There exists a homeomorphism $T : U \cap Q_+ \to U^\sharp \ni o$, where $U^\sharp \subset Q_+$ is a small neighborhood of the point o in Q_+, such that $T \circ R \circ T^{-1} = R^\sharp$, i.e.,*

$$\begin{array}{ccc} U \cap Q_+ & \xrightarrow{R} & U \cap Q_+ \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle T} \\ U^\sharp & \xrightarrow{R^\sharp} & U^\sharp, \end{array} \tag{2}$$

where

$$R^\sharp : (s, u) \mapsto \left(\frac{s}{1 + s\bar{g}(u)},\ \exp(s\xi)u \right); \tag{3}$$

moreover, $\bar{g}(u) = g(0, u) = g|_E$ and T is the identity mapping id_E on $U \cap E$.

The theorem provides a complete description of geometric properties of the flow $\{R^\bullet\}$ and the system v near the point o. The proof of the theorem is a (rather nontrivial) generalization of the classical proof of the Grobman–Hartman theorem. To obtain the precise analog of the Grobman–Hartman theorem for the problem in question, the technique must be further strengthened.

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