



# Transformada de Fourier

Unidad 4

BME423 · Procesamiento de imágenes médicas

Alejandro Veloz

Ingeniería Biomédica

# La transformada de Fourier

# Introducción.

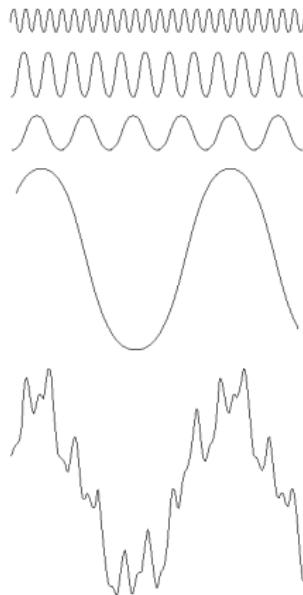
- Dominio espacial.
  - Espacio de la imagen.
  - Se procesa directamente el arreglo de píxeles que constituyen la imagen.
  - Cambios en la posición de los píxeles corresponden a cambios en la escena.
  - Distancias en este dominio corresponden a distancias reales.
- Dominio de la frecuencia.
  - Transforma la imagen en su representación en frecuencia.
  - Cambios en la posición corresponden a cambios en frecuencia.
  - Representa la tasa en la cual las intensidades de la imagen varían en el dominio espacial.
  - Es posible recuperar la información espacial mediante la transformación inversa.

# Introducción.

- Cualquier señal temporal o espacial tiene una representación equivalente en el dominio de la frecuencia.
- En procesamiento de imágenes, las frecuencias significan:
  - Altas frecuencias corresponden a valores de intensidad que cambian rápidamente a lo largo de la imagen (por ejemplo, texto, texturas, etc.).
  - Componentes bajos en frecuencia corresponden a los grandes rasgos de la imagen (por ejemplo, objetos homogéneos que prevalecen en la imagen).

## Series de Fourier.

- Funciones periódicas pueden ser expresadas como suma de componentes seno y/o coseno de diferentes frecuencias, cada uno ponderado por diferentes coeficientes.



**FIGURE 4.1** The function at the bottom is the sum of the four functions above it. Fourier's idea in 1807 that periodic functions could be represented as a weighted sum of sines and cosines was met with skepticism.

# Transformada de Fourier 1-D.

- Funciones **no periódicas**, pero con **área bajo la curva finita** pueden ser expresadas mediante una integral de componentes seno y/o coseno ponderados por una función.
- La transformada de Fourier:

$$F(u) = \int_{-\infty}^{\infty} f(x) \exp(-j2\pi ux) dx, \text{ donde } j = \sqrt{-1}$$

- La transformada inversa:

$$f(t) = \int_{-\infty}^{\infty} F(u) \exp(-j2\pi ux) du$$

## Transformada discreta de Fourier 1-D (DFT).

- La transformada discreta de Fourier:

$$F(u) = \frac{1}{M} \sum_{x=0}^{M-1} f(x) \exp\left(-\frac{j2\pi ux}{M}\right), \text{ para } u = 0, \dots, M-1$$

- Transformada discreta inversa:

$$f(x) = \sum_{u=0}^{M-1} F(u) \exp\left(\frac{j2\pi ux}{M}\right), \text{ para } x = 0, \dots, M-1$$

- La transformada de Fourier de  $f(x)$  existe si y sólo si:
  1.  $f(x)$  es continua por tramos en un intervalo finito.
  2.  $f(x)$  es absolutamente integrable.

## Transformada discreta de Fourier 1-D (DFT).

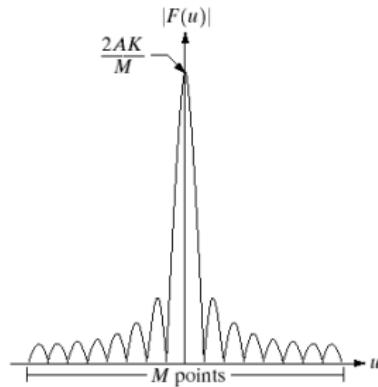
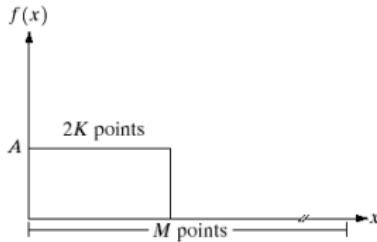
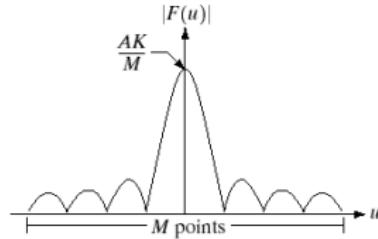
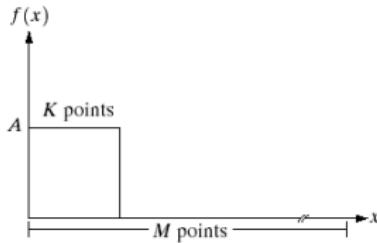
$$F(u) = |F(u)| \exp(-j\phi(u))$$

$$|F(u)| = \left[ \operatorname{Re}(F(u))^2 + \operatorname{Im}(F(u))^2 \right]^{1/2} \text{ (magnitud o espectro de la Transf. de Fourier)}$$

$$\phi(u) = \tan^{-1} \left[ \frac{\operatorname{Im}(F(u))}{\operatorname{Re}(F(u))} \right] \text{ (ángulo o espectro de fase)}$$

$$P(u,v) = |F(u)|^2 \text{ (espectro de potencia)}$$

# Transformada discreta de Fourier 1-D (DFT).



## Transformada discreta de Fourier 2-D.

$$F(u,v) = \Im\{f(x,y)\} = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) \exp\left(-j2\pi\left(\frac{ux}{M} + \frac{vy}{N}\right)\right),$$

para  $u = 0, \dots, M-1; v = 0, \dots, N-1$ .

$$f(x,y) = \Im^{-1}\{F(u,v)\} = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u,v) \exp\left(j2\pi\left(\frac{ux}{M} + \frac{vy}{N}\right)\right),$$

para  $x = 0, \dots, M-1; y = 0, \dots, N-1$ .

## Transformada discreta de Fourier 2-D.

$$|F(u,v)| = \left[ \operatorname{Re}(F(u,v))^2 + \operatorname{Im}(F(u,v))^2 \right]^{1/2} \text{ (espectro de la Transf. de Fourier)}$$

$$\phi(u,v) = \tan^{-1} \left[ \frac{\operatorname{Im}(F(u,v))}{\operatorname{Re}(F(u,v))} \right] \text{ (ángulo de fase)}$$

$$P(u,v) = |F(u,v)|^2 \text{ (espectro de potencia)}$$

$$\Im \left\{ f(x,y)(-1)^{x+y} \right\} = F(u-M/2, v-N/2)$$

...considere  $u_0 = \frac{M}{2}$  y  $v_0 = \frac{N}{2}$

$$\exp \left( -j2\pi \left( \frac{u_0 x}{M} + \frac{v_0 y}{N} \right) \right) = \exp \left( -j\pi(x+y) \right) = (-1)^{x+y}$$

## Transformada discreta de Fourier 2-D.

- La transformada de Fourier es una transformación lineal.

$$\mathfrak{F}\{af(x,y) + bg(x,y)\} = a\mathfrak{F}\{f(x,y)\} + b\mathfrak{F}\{g(x,y)\}$$

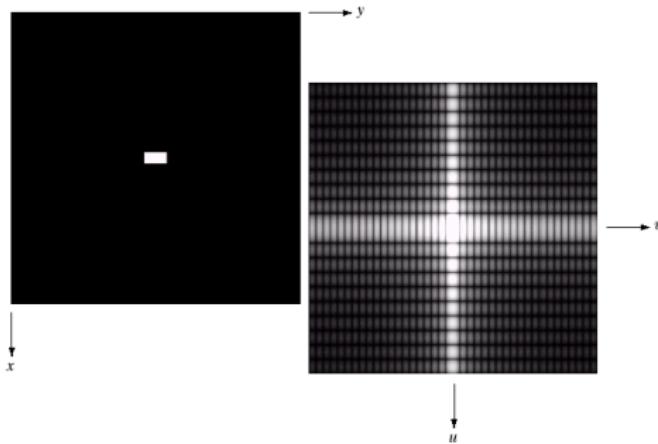
- Algunas propiedades importantes:

$$\text{Real} \quad \longleftrightarrow \quad \text{Symmetric}$$

$$f+g \quad \longleftrightarrow \quad F+G$$

$$f^*g \quad \longleftrightarrow \quad FG$$

## Filtrado en el dominio de la frecuencia.



**Nota:** Para visualizar el espectro de Fourier, aplicar una transformación logarítmica.

# Filtrado en el dominio de la frecuencia.

## Algunos filtros básicos.

- Filtros pasa-bajos.
- Filtros pasa-altos.



$$H(u,v) = \begin{cases} 0 & \text{si } (u,v) = (M/2, N/2) \\ 1 & \text{en otro caso} \end{cases}$$

Filtro notch

# Filtrado en el dominio de la frecuencia.

Algunos filtros básicos.

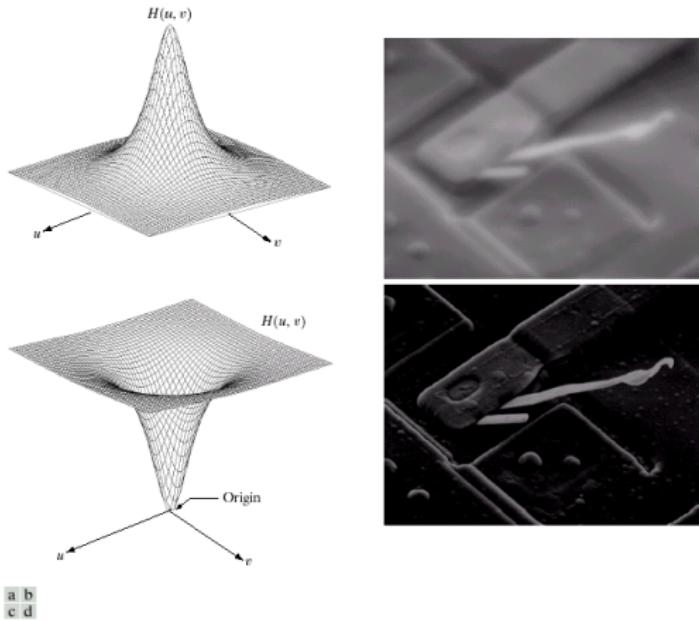


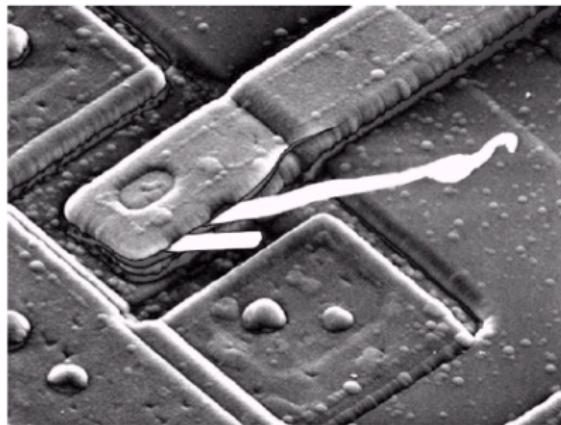
FIGURE 4.7 (a) A two-dimensional lowpass filter function. (b) Result of lowpass filtering the image in Fig. 4.4(a). (c) A two-dimensional highpass filter function. (d) Result of highpass filtering the image in Fig. 4.4(a).

# Filtrado en el dominio de la frecuencia.

## Algunos filtros básicos.

**FIGURE 4.8**

Result of highpass filtering the image in Fig. 4.4(a) with the filter in Fig. 4.7(c), modified by adding a constant of one-half the filter height to the filter function. Compare with Fig. 4.4(a).



# Filtrado en el dominio de la frecuencia.

## Procedimiento básico.

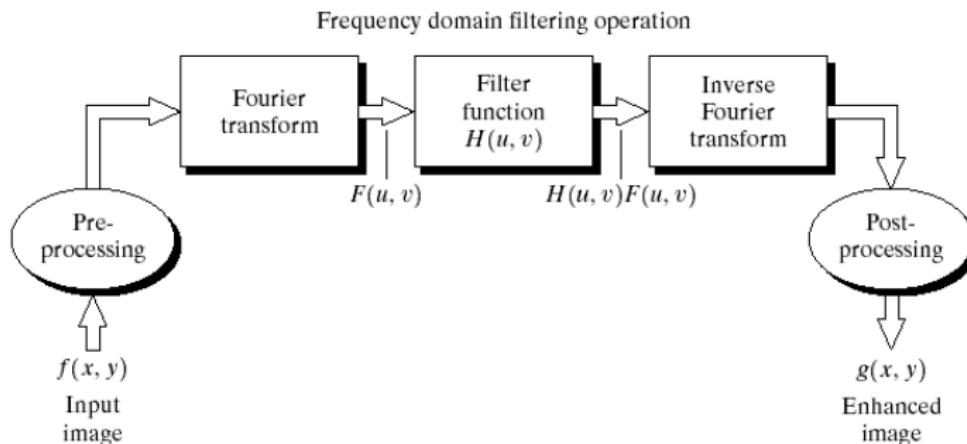
1. Multiplicar la imagen de entrada por  $(-1)^{x+y}$  para centrar la transformada.
2. Computar  $F(u,v)$ .
3. Calcular  $G(u,v) = F(u,v)H(u,v)$ , donde  $H(u,v)$  es la función de transferencia del filtro.
4. Computar la DFT inversa de  $G(u,v)$  obtenida en (3). Es decir:

$$\text{Imagen filtrada} = \mathfrak{F}^{-1}\{G(u,v)\}$$

5. Obtener la parte real de la DFT inversa obtenida en (4).
6. Multiplicar el resultado de (5) por  $(-1)^{x+y}$ .

# Filtrado en el dominio de la frecuencia.

## Procedimiento básico.



**FIGURE 4.5** Basic steps for filtering in the frequency domain.

## Ejemplo.



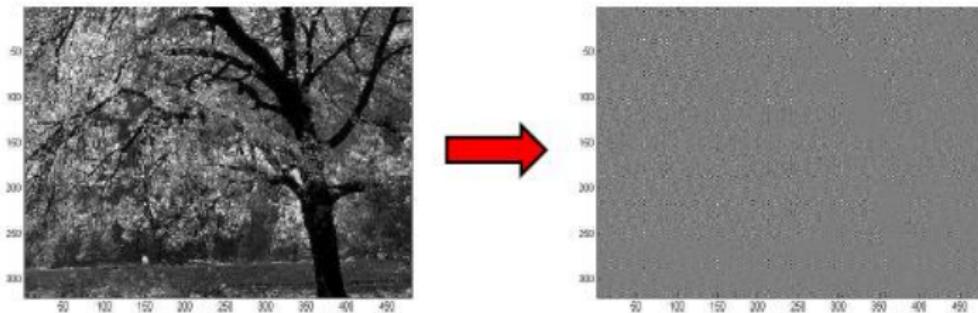
$$\star \begin{bmatrix} 1 & 4 & 6 & 4 & 1 \\ 4 & 16 & 24 & 16 & 4 \\ 6 & 24 & 36 & 24 & 6 \\ 4 & 16 & 24 & 16 & 4 \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix} / 256 =$$



- Suavizado de una imagen con un kernel gaussiano en el dominio espacial.
- De manera alternativa, es posible lograr este efecto en el dominio de la frecuencia.

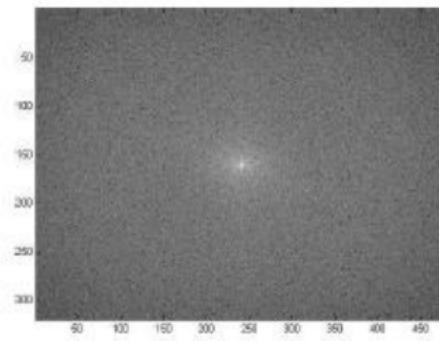
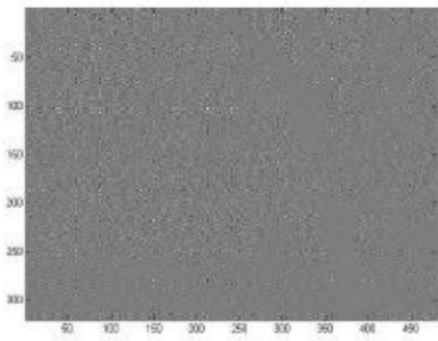
## Ejemplo.

1. Multiplicar la imagen de entrada por  $(-1)^{x+y}$  para centrar la transformada.



## Ejemplo.

2. Computar  $F(u,v)$ .



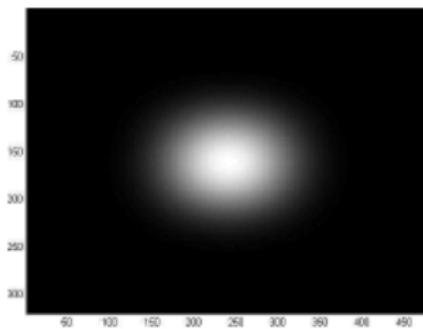
Transformación log

## Ejemplo.

3. Calcular  $G(u,v) = F(u,v)H(u,v)$ , donde  $H(u,v)$  es la función de transferencia del filtro.

$$\begin{bmatrix} 1 & 4 & 6 & 4 & 1 \\ 4 & 16 & 24 & 16 & 4 \\ 6 & 24 & 36 & 24 & 6 \\ 4 & 16 & 24 & 16 & 4 \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix} / 256$$

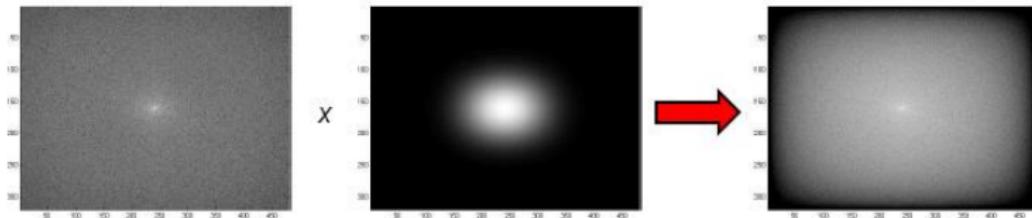
$h(x,y)$



$H(u,v)$

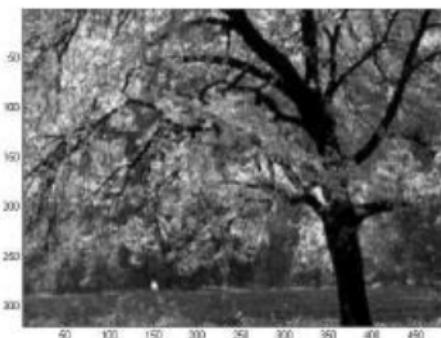
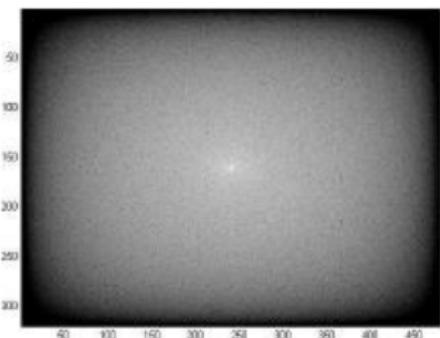
## Ejemplo.

3. Calcular  $G(u,v) = F(u,v)H(u,v)$ , donde  $H(u,v)$  es la función de transferencia del filtro.

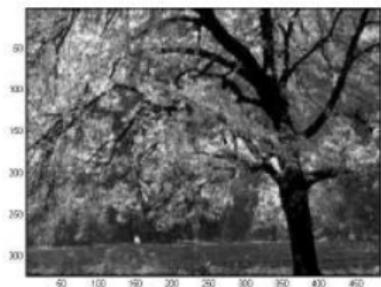


## Ejemplo.

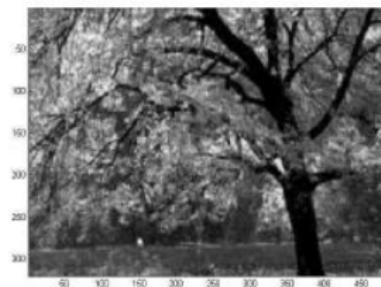
4. Computar la DFT inversa de  $G(u,v)$  obtenida en (3).
5. Obtener la parte real de la DFT inversa obtenida en (4).
6. Multiplicar el resultado de (5) por  $(-1)^{x+y}$ .



## Ejemplo.



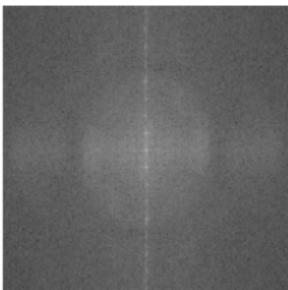
$$f(x,y) * g(x,y)$$



$$F(u,v)G(u,v)$$

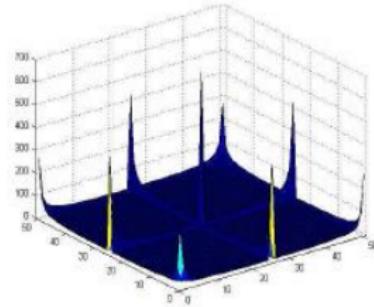
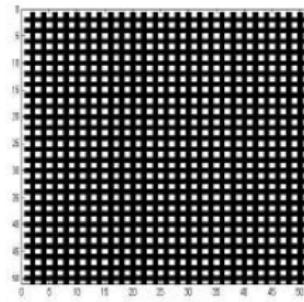
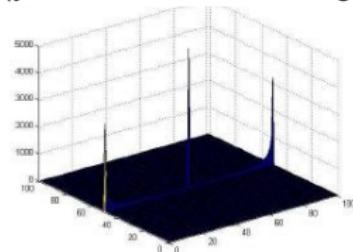
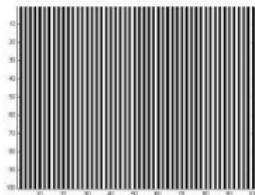
# Ejemplo.

- En ciertas circunstancias es posible inferir información geométrica a partir de la DFT (y su transformación log).



## Ejemplo.

- En ciertas circunstancias es posible inferir información geométrica a partir de la DFT (y su transformación log).



# Filtros en el dominio de la frecuencia

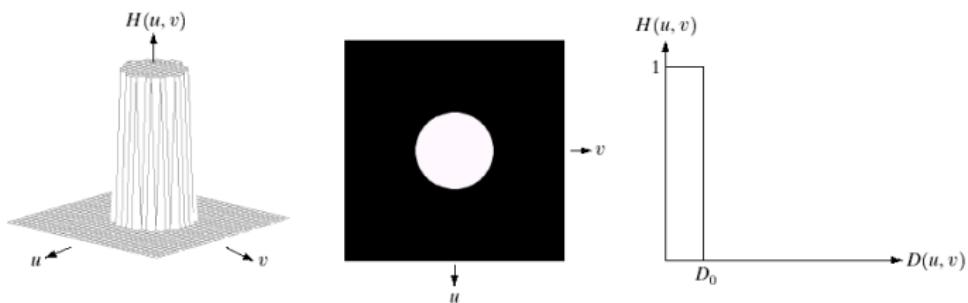
# Filtros

- Filtros de suavizado.
- Filtros de agudizado.

$$G(u,v) = F(u,v)H(u,v)$$
$$g(x,y) = \mathfrak{I}^{-1}\{G(u,v)\}$$

## Filtro de suavizado ideal (ILPF).

$$H(u,v) = \begin{cases} 1 & \text{si } D(u,v) \leq D_0 \\ 0 & \text{si } D(u,v) > D_0 \end{cases}$$
$$D(u,v) = \left[ (u - M/2)^2 + (v - N/2)^2 \right]^{\frac{1}{2}}$$

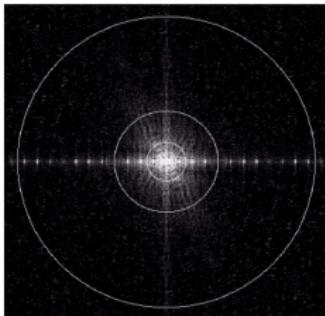
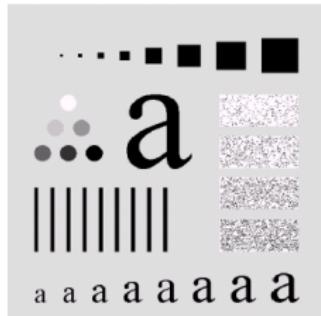


a b c

**FIGURE 4.10** (a) Perspective plot of an ideal lowpass filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross section.

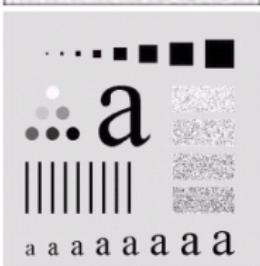
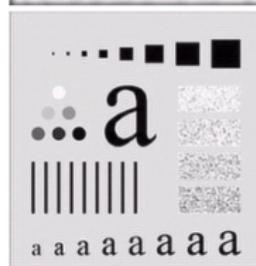
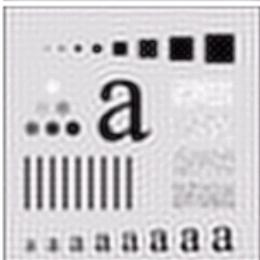
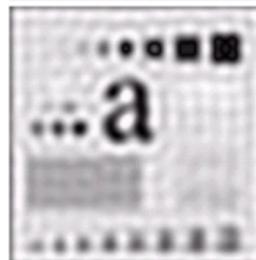
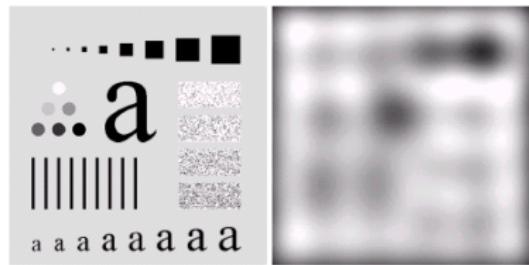
$$P_T = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} P(u,v)$$

$$\alpha = 100 \left[ \sum_u \sum_v P(u,v) / P_T \right]$$



a b

**FIGURE 4.11** (a) An image of size  $500 \times 500$  pixels and (b) its Fourier spectrum. The superimposed circles have radii values of 5, 15, 30, 80, and 230, which enclose 92.0, 94.6, 96.4, 98.0, and 99.5% of the image power, respectively.



a f

**FIGURE 4.12** (a) Original image. (b)–(f) Results of ideal lowpass filtering with cutoff frequencies set at radii values of 5, 15, 30, 80, and 230, as shown in Fig. 4.11(b). The power removed by these filters was 8, 5.4, 3.6, 2, and 0.5% of the total, respectively.

## Filtro de suavizado ideal (ILPF).

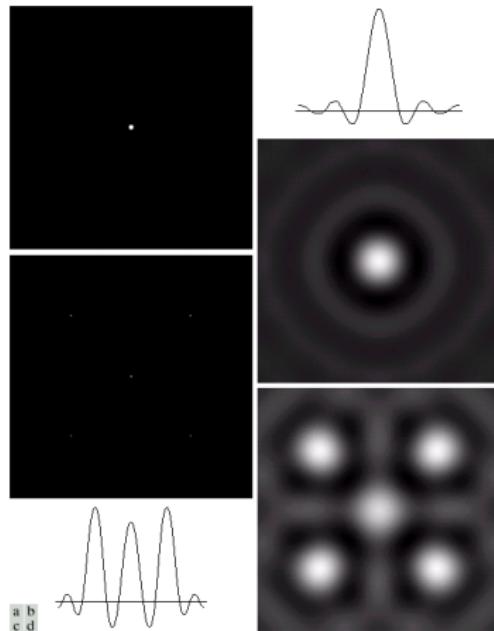
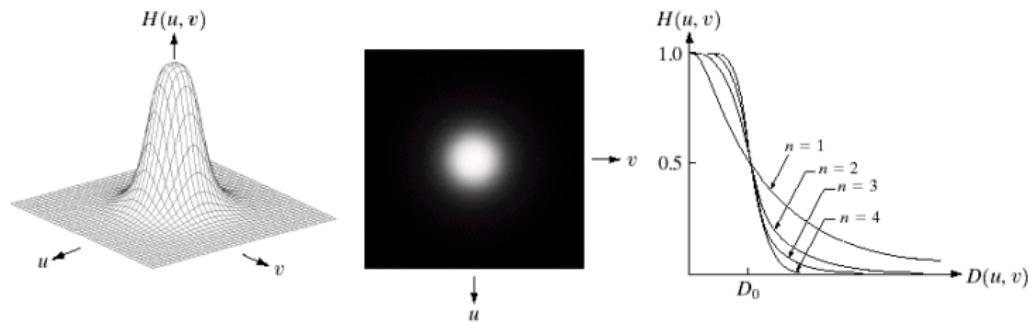


FIGURE 4.13 (a) A frequency-domain ILPF of radius 5. (b) Corresponding spatial filter (note the ringing). (c) Five impulses in the spatial domain, simulating the values of five pixels. (d) Convolution of (b) and (c) in the spatial domain.

## Filtro de suavizado Butterworth de orden n (BLPF).

$$H(u,v) = \frac{1}{1 + [D(u,v)/D_0]^{2n}}$$



a b c

**FIGURE 4.14** (a) Perspective plot of a Butterworth lowpass filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections of orders 1 through 4.

- Filtro de suavizado Butterworth de orden n (BLPF).

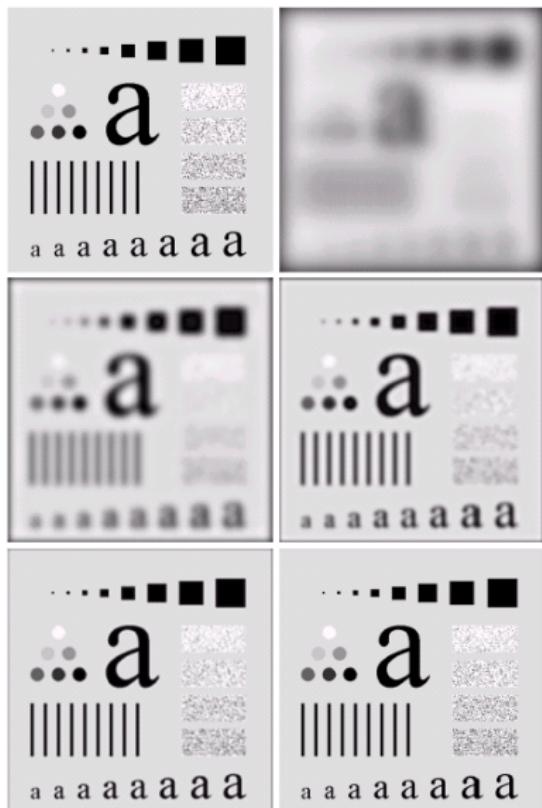
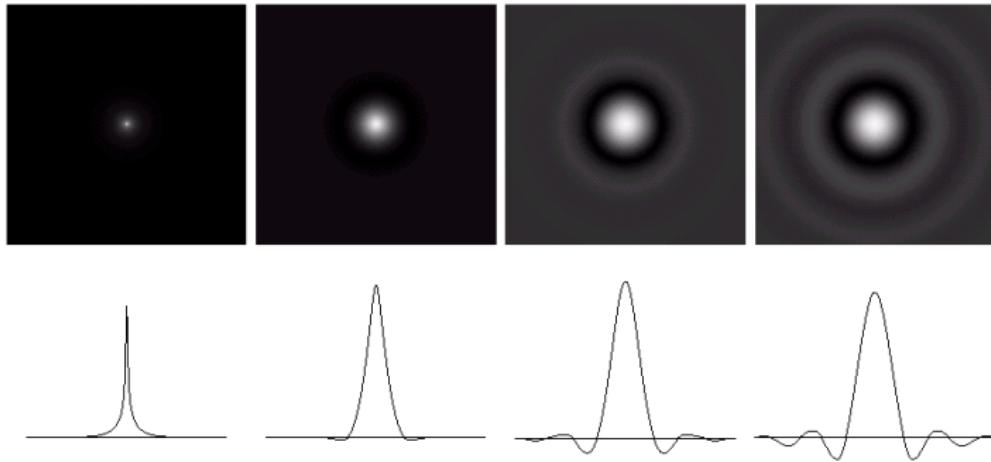


FIGURE 4.15 (a) Original image. (b)-(f) Results of filtering with BLPFs of order 2, with cutoff frequencies at radii of 5, 15, 30, 80, and 230, as shown in Fig. 4.11(b). Compare with Fig. 4.12.

## Filtro de suavizado Butterworth de orden n (BLPF).

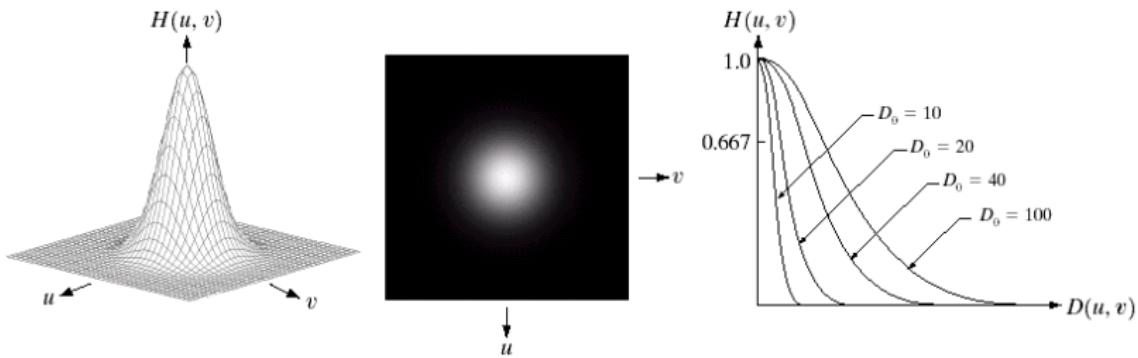


a b c d

**FIGURE 4.16** (a)–(d) Spatial representation of BLPFs of order 1, 2, 5, and 20, and corresponding gray-level profiles through the center of the filters (all filters have a cutoff frequency of 5). Note that ringing increases as a function of filter order.

## Filtro de suavizado Gaussiano (GLPF).

$$H(u,v) = \exp\left(-\frac{D^2(u,v)}{2\sigma^2}\right)$$



a b c

**FIGURE 4.17** (a) Perspective plot of a GLPF transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections for various values of  $D_0$ .

## Filtro de suavizado Gaussiano (GLPF).

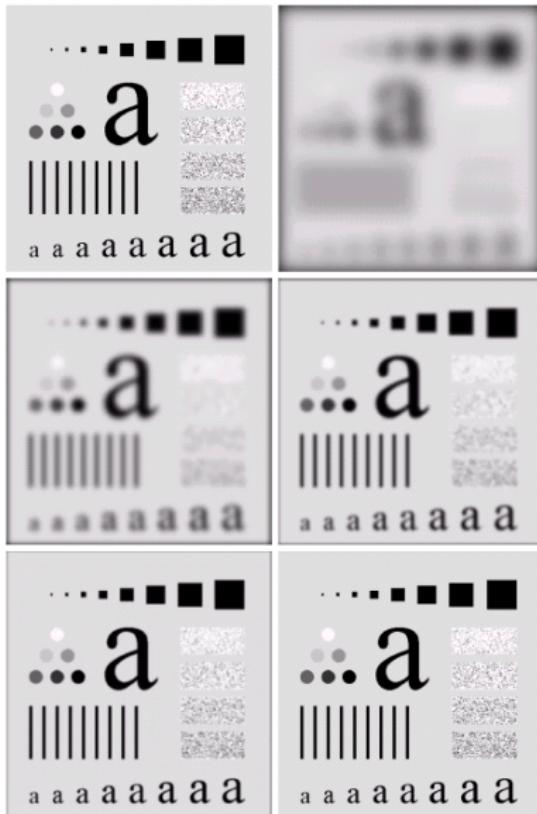


FIGURE 4.18 (a) Original image. (b)-(f) Results of filtering with Gaussian lowpass filters with cutoff frequencies set at radii values of 5, 15, 30, 80, and 230, as shown in Fig. 4.11(b). Compare with Figs. 4.12 and 4.15.

a  
b  
c  
d  
e  
f

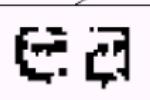
# Ejemplos

a b

**FIGURE 4.19**

- (a) Sample text of poor resolution (note broken characters in magnified view).  
(b) Result of filtering with a GLPF (broken character segments were joined).

Historically, certain computer programs were written using only two digits rather than four to define the applicable year. Accordingly, the company's software may recognize a date using "00" as 1900 rather than the year 2000.



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# Ejemplos



a b c

**FIGURE 4.20** (a) Original image ( $1028 \times 732$  pixels). (b) Result of filtering with a GLPF with  $D_0 = 100$ . (c) Result of filtering with a GLPF with  $D_0 = 80$ . Note reduction in skin fine lines in the magnified sections of (b) and (c).

## Ejemplos



a b c

**FIGURE 4.21** (a) Image showing prominent scan lines. (b) Result of using a GLPF with  $D_0 = 30$ . (c) Result of using a GLPF with  $D_0 = 10$ . (Original image courtesy of NOAA.)

## Filtros de agudizado en el dominio de la frecuencia.

La función de transferencia de los filtros de agudizado en el dominio de la frecuencia está dada por:

$$H_{hp}(u,v) = 1 - H_{lp}(u,v)$$

donde  $H_{lp}(u,v)$  es la función de transferencia del correspondiente filtro pasa bajos.

# Filtros de agudizado en el dominio de la frecuencia.

$$H_{hp}(u,v) = 1 - H_{lp}(u,v)$$

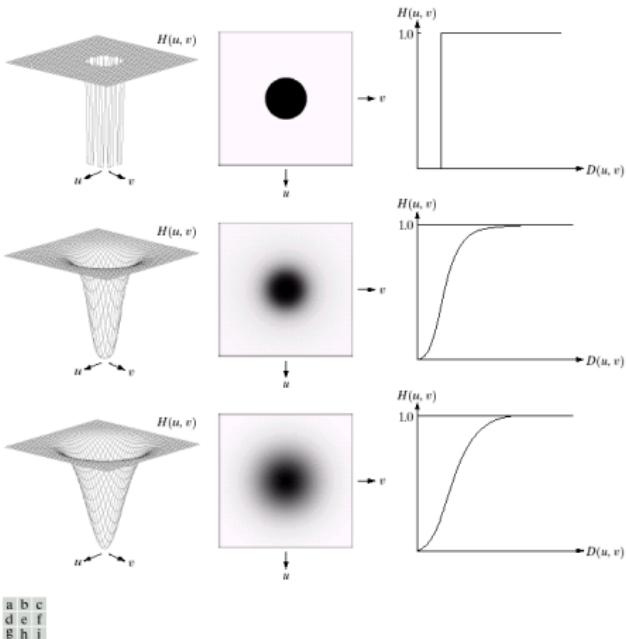
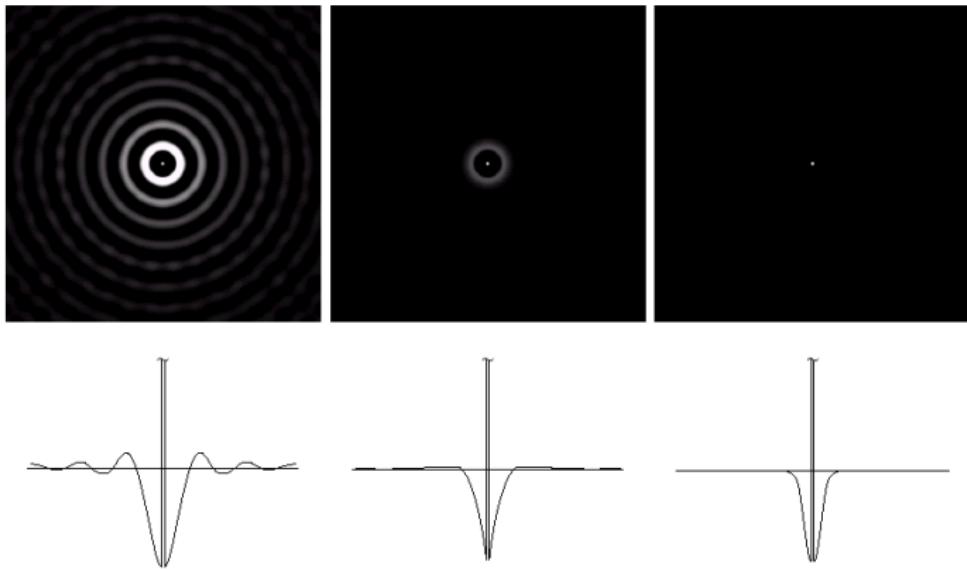


FIGURE 4.22 Top row: Perspective plot, image representation, and cross section of a typical ideal highpass filter. Middle and bottom rows: The same sequence for typical Butterworth and Gaussian highpass filters.

## Filtros de agudizado en el dominio de la frecuencia.



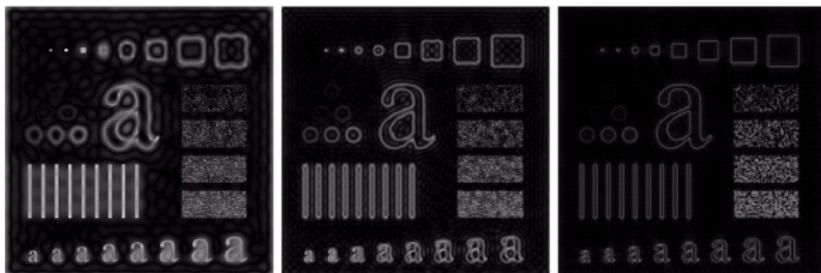
a b c

**FIGURE 4.23** Spatial representations of typical (a) ideal, (b) Butterworth, and (c) Gaussian frequency domain highpass filters, and corresponding gray-level profiles.

## Filtros pasa alto ideal (IHPF).

$$H(u,v) = \begin{cases} 0 & \text{si } D(u,v) \leq D_0 \\ 1 & \text{si } D(u,v) > D_0 \end{cases}$$

$$D(u,v) = \left[ (u - M/2)^2 + (v - N/2)^2 \right]^{\frac{1}{2}}$$

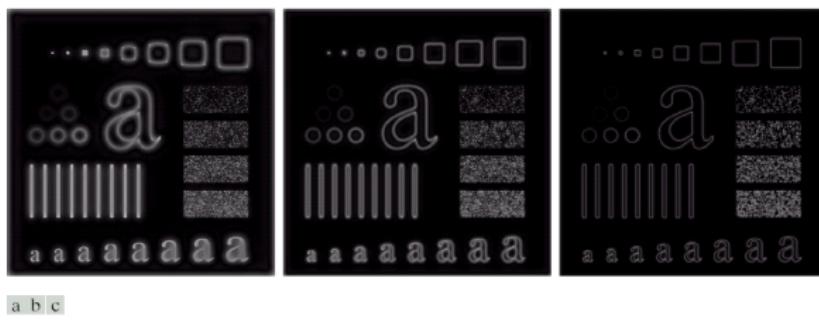


a b c

**FIGURE 4.24** Results of ideal highpass filtering the image in Fig. 4.11(a) with  $D_0 = 15, 30$ , and  $80$ , respectively. Problems with ringing are quite evident in (a) and (b).

# Filtros pasa alto Butterworth (BHPF).

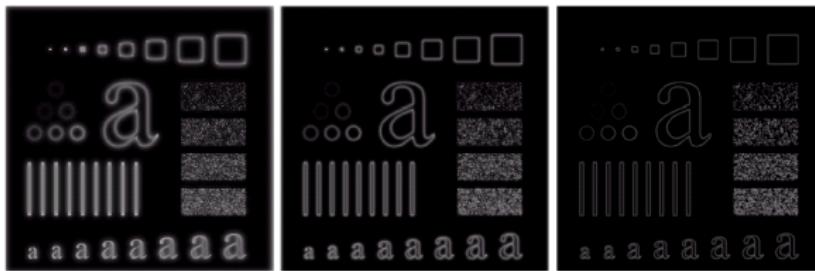
$$H(u,v) = \frac{1}{1 + [D_0/D(u,v)]^{2n}}$$



**FIGURE 4.25** Results of highpass filtering the image in Fig. 4.11(a) using a BHPF of order 2 with  $D_0 = 15$ , 30, and 80, respectively. These results are much smoother than those obtained with an ILPF.

## Filtros pasa alto Gaussiano (GHPF).

$$H(u,v) = 1 - \exp\left(-\frac{D^2(u,v)}{2D_0^2}\right)$$



a b c

**FIGURE 4.26** Results of highpass filtering the image of Fig. 4.11(a) using a GHPF of order 2 with  $D_0 = 15$ , 30, and 80, respectively. Compare with Figs. 4.24 and 4.25.

## El Laplaciano en el dominio de la frecuencia.

$$\Im \left[ \frac{d^n f(x)}{dx^n} \right] = (ju)^n F(u)$$

Entonces :

$$\begin{aligned}\Im \left[ \frac{\partial^2 f(x,y)}{\partial x^2} + \frac{\partial^2 f(x,y)}{\partial y^2} \right] &= (ju)^2 F(u,v) + (jv)^2 F(u,v) \\ \Im [\nabla^2 f(x,y)] &= -(u^2 + v^2) F(u,v)\end{aligned}$$

## Uso del Laplaciano en el dominio de la frecuencia para refuerzo de contraste.

En el dominio del espacio:

$$g(x,y) = f(x,y) - \nabla^2 f(x,y)$$

En el dominio de la frecuencia:

$$G(u,v) = \Im[f(x,y)] - \Im[\nabla^2 f(x,y)]$$

$$G(u,v) = F(u,v) - (u^2 + v^2)F(u,v)$$

$$G(u,v) = [1 - (u^2 + v^2)]F(u,v) \quad \text{por tanto } H(u,v) = 1 - (u^2 + v^2)$$

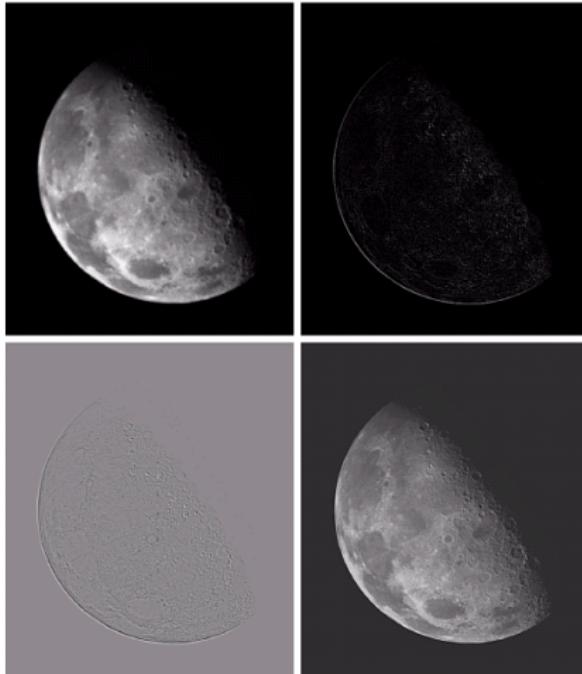
$$g(x,y) = \Im^{-1}[[1 - (u^2 + v^2)]F(u,v)]$$

# Uso del Laplaciano en el dominio de la frecuencia para refuerzo de contraste.

a  
b  
c  
d

FIGURE 4.28

- (a) Image of the North Pole of the moon.  
(b) Laplacian filtered image.  
(c) Laplacian image scaled.  
(d) Image enhanced by using Eq. (4.4-12).  
(Original image courtesy of NASA.)



## Unsharp Masking, High-Boost Filtering y High-Frequency Emphasis Filtering.

Unsharp Masking:

$$f_{hp}(x,y) = f(x,y) - f_{lp}(x,y)$$
$$\therefore H_{hp}(u,v) = 1 - H_{lp}(u,v)$$

High-Boost Filtering:

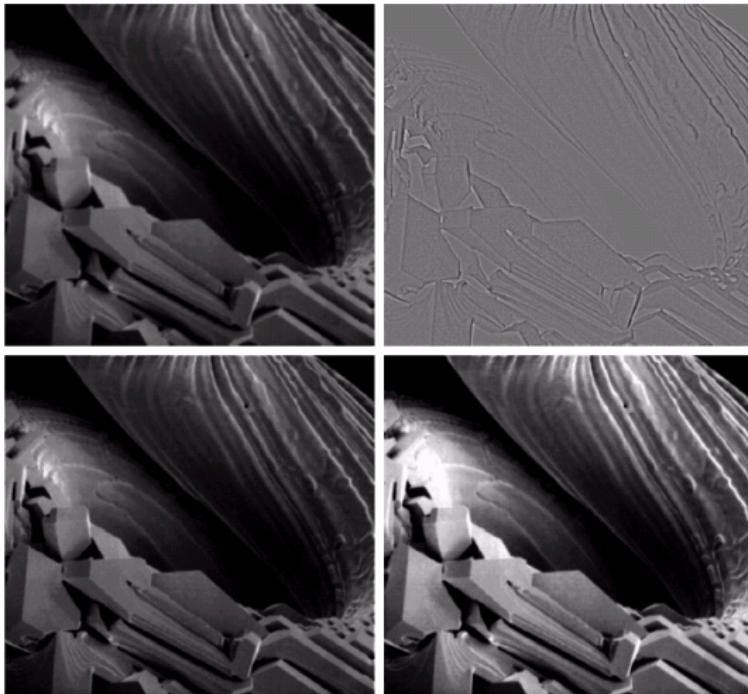
$$f_{hb}(x,y) = Af(x,y) - f_{lp}(x,y); A \geq 1$$
$$f_{hb}(x,y) = (A-1)f(x,y) + f(x,y) - f_{lp}(x,y)$$
$$f_{hb}(x,y) = (A-1)f(x,y) + f_{hp}(x,y)$$
$$\therefore H_{hb}(u,v) = (A-1) + H_{hp}(u,v)$$

# Unsharp Masking, High-Boost Filtering y High-Frequency Emphasis Filtering.

a  
b  
c  
d

**FIGURE 4.29**

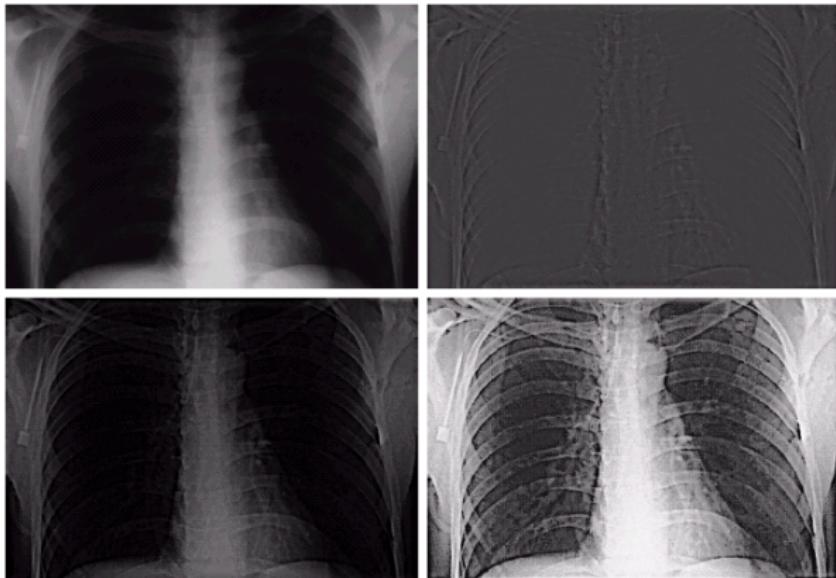
Same as Fig. 3.43, but using frequency domain filtering. (a) Input image. (b) Laplacian of (a). (c) Image obtained using Eq. (4.4-17) with  $A = 2$ . (d) Same as (c), but with  $A = 2.7$ . (Original image courtesy of Mr. Michael Shaffer, Department of Geological Sciences, University of Oregon, Eugene.)



# Unsharp Masking, High-Boost Filtering y High-Frequency Emphasis Filtering.

High-Frequency Emphasis Filtering:

$$H_{hfe}(u,v) = a + bH_{hp}(u,v); a \geq 0, b > a$$



a  
b  
c  
d

**FIGURE 4.30**  
(a) A chest X-ray image. (b) Result of Butterworth highpass filtering.  
(c) Result of high-frequency emphasis filtering.  
(d) Result of performing histogram equalization on (c). (Original image courtesy Dr. Thomas R. Gest, Division of Anatomical Sciences, University of Michigan Medical School.)