Contents

1 Background and History

1.1 Omega Function

Let $\Omega(n)$ denote the **number of prime factors** of n (when counted with multiplicities). For example, $\Omega(1) = 0$, $\Omega(p) = 1$, $\Omega(p.q) = \Omega(p^2) = 2$, $\Omega(p_1^{e_1} \cdots p_k^{e_k}) = e_1 + \cdots + e_k$.

A central question that we have in number theory is: What is the distribution of the values of $\Omega(n)$. If we look at the values of the function for the first two-hundred positive integers, we find that $\Omega(n)$ is growing. In fact, it grows like $\log(\log(n))$, but besides the average growth, it behaves very randomly. We shouldn't forget here that there are loads of primes within the first two-hundred integers. So, it's understood that:

- The distribution of the values of $\Omega(n)$ follows no notable pattern. It appears to be random.
- Knowing $\Omega(n-1)$, $\Omega(n-2), \cdots, \Omega(n-m)$ does not allow us to predict $\Omega(n)$.

1.2 Some Classical Results in Multiplicative Number Theory

The study of the distribution of the values of $\Omega(n)$ has a long and rich history and is closely related to fundamental questions about the prime numbers.

The **natural density** of a set $A \subset \mathbb{N}$ is defined as $d(A) = \lim_{n\to\infty} |\{1 \leq n \leq N : n \in A\}|/N$. The following statement is a well-known equivalent form of the Prime Number Theorem (von Mangoldt 1897, Landau 1911):

Theorem 1. *The sets* $\{n \in \mathbb{N} : \Omega(n)$ *is even*$\}$ *and* $\{n \in \mathbb{N} : \Omega(n)$ *is odd*$\}$ *have natural density of* $1/2$.

This means that, asymptotically, there are as many prime numbers with even numbers of prime factors as there are prime numbers with odd numbers of prime factors. It is natural to ask whether this can be generalized as this function is evenly distributing between even's and odd's. And we can see this in the Pillai-Selberg Theorem (Pillai 1940, Selberg 1939):

Theorem 2. *For all* $m \in \mathbb{N}$ *and* $r \in \{0, \dots, m-1\}$ *the set* $\{n \in \mathbb{N} : \Omega(n) \equiv r \mod m\}$ *has natural density* $1/m$.

A sequence of real numbers $(x_n)_{n\in\mathbb{N}}$ is said to be **uniformly distributed mod 1** if for any continuous $f : [0,1) \to \mathbb{C}$ we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x_n) = \int_0^1 f(x)dx.$$

Theorem 3. *For all irrational* α *the sequence* $\Omega(n)\alpha, n \in \mathbb{N}$, *is uniformly distributed mod 1.*

The above is the Erdos-Delange Theorem (Erdos 1946, Delange 1958). The paper rides on these classical results mostly. Let us set the scene for the main characters in the following section.

2 A Dynamic Generalization of the Prime Number Theorem

2.1 1^{st} Main Result

Let X be a compact metric space and $T : X \to X$ a continuous map. Since

$$T^m \cdot T^n = T^{m+n}, \quad \forall m, n \in \mathbb{N},$$

the transformation T naturally induces an action of $(\mathbb{N}, +)$ on X. We call (X, T) an **additive topological dynamical system**. Every additive topological dynamical system possesses at least one T-invariant Borel probability measure. If (X, T) admits only one such measure, then the system is called **uniquely ergodic**.

Theorem 4. *Let X, μ, T be uniquely ergodic. Then*

$$\lim_{n\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(T^{\Omega(n)}x) = \int f d\mu$$

for every $x \in X$ and $f \in C(X)$.

We can interpret 4(Bergelson-R. 2020) as saying that for any uniquely ergodic system (X, T) and any point $x \in X$ the orbit $T^{\Omega(n)}x$ is uniformly distributed in the space X. Let us apply this theorem to a simple situation like **rotation on two points**. Theorem 4 applied on this recovers the Prime Number Theorem. It also recovers the Pillai-Selberg Theorem and Erdos-Delange Theorem which I was too lazy to prove (it's quite similar to the one I write below).

Proof. Let $X = \{0, 1\}$ and $T : x \longmapsto x + 1 \mod 2$. This system is uniquely ergodic, with unique invariant measure μ given by $\mu(\{0\}) = \mu(\{1\}) = 1/2$. Let $f : \{0, 1\} \to \mathbb{R}$ be defined as $f(0) = 1$ and $f(1) = 0$, and take $x = 0$. Then

$$\lim_{n\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(T^{\Omega(n)}x) = d(\{n : \Omega(n) \text{ is even}\}).$$

Since $\int f d\mu = 1/2$, it follows from Theorem 4 that $d(\{n : \Omega(n) \text{ is even}\}) = 1/2$. QED.

Theorem 4 applied to **unipotent affine transformations** on tori yields the following polynomial extensions of the Erdos-Delange Theorem, which includes an extension of the DeKonick-Katai Theorem to all irrational α as a special case. And when applied to certain **constant length substitution systems** gives an analogue of a classicla result of Gelfond, provinding new insight into the digital expansion of $\Omega(n)$ is base q.

Corollary 4.1. *Let $Q(n) = c_k n^k + \cdots + c_1 n + c_0$. Then $Q(\Omega(n)), n \in \mathbb{N}$, is uniformly distributed mod 1 iff at least one of the coefficients is irrational.*

Corollary 4.2. *Let $s_q(n)$ denote the sum of digits of n in base q. If m and $q-1$ are coprime then for all $r \in \{0, 1, \ldots, m-1\}$ the set of n for which $s_q(\Omega(n)) \equiv r \mod m$ has asymptotic density $1/m$.*

Note that our proofs are elementary and self-contained. In particular, we don't use any tools or results from analytic number theory.

3 Multiplicative Systems

A **multiplicative topological dynamical system** is a pair (Y, S) where Y is a compact metric space and $S = (S_n)_{n \in \mathbb{N}}$ is an action of $(\mathbb{N}, \cdot)$ by continuous maps on Y, i.e.,

$$S_{nm} = S_n \cdot S_m, \quad \forall n, m \in \mathbb{N}.$$

3.1 2^{nd} Main Result

The main question here is whether Theorem 4 remains true if (X, T^{Ω}) is replaced by more general multiplicative systems (Y, S)? Let us call a multiplicative topological dynamical system (Y, S) **finitely generated** if $\{S_p : p \text{ prime}\}$ is finite.

Theorem 5. *Let Y, ν, S be finitely generated and strongly uniquely ergodic. Then*

$$\lim_{n \to \infty} \frac{1}{N} \sum_{n=1}^{N} g(S_n y) = \int g d\nu$$

for all $y \in Y$ and $g \in C(Y)$.

4 Disjointness of additive and multiplicative Systems

4.1 Sarnak's Liouville disjointness conjecture

Recall that the Liouville function is $\lambda(n) = (-1)^{\Omega(n)}$. The Liouville disjointness conjecture is as follows,

Conjecture 1. *For any zero entropy additive topological dynamical system* (X, T) *we have*

$$\lim_{n\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(T^n x)\lambda(n) = 0$$

for all $x \in X$ *and* $f \in C(X)$.

We have learnt from Theorems 4 and 5 that a natural generalization of $\lambda(n)$ are sequences of the form $g(S_n y)$ coming from a multiplicative topological dynamical system (Y, S).

4.2 Disjointness

We call two bounded arithmetic functions $a, b : \mathbb{N} \to \mathbb{C}$ **asymptotically independent** if

$$\lim_{n\to\infty} \left[\frac{1}{N} \sum_{n=1}^{N} a(n)\overline{b(n)} - \left(\frac{1}{N} \sum_{n=1}^{N} a(n) \right) \cdot \left(\frac{1}{N} \sum_{n=1}^{N} \overline{b(n)} \right) \right] = 0.$$

Disjointness is defined when $a(n)$ and $b(n)$ are asymptotically independent.

When we consider multiplicative rotation on two points, Liouville disjointness conjecture can be reformulated to state that multiplicative rotation on two points in disjoint from every zero entropy additive topological dynamical system.

Conjecture 2. *If* $(X, T$ *is a* ***zero entropy*** *additive topological dynamical system and* (Y, S) *a* ***finitely generated*** *multiplicative topological dynamical system and either is aperiodic, then,* $(X, T$ *and* (Y, S) *are disjoint.*

Theorem 6. *Conjecture 2 holds when* (X, T) *is a nilsystem.*

Theorem 7. *Conjecture 2 holds when* (X, T) *is a horocycle flow.*