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1 Background and History

1.1 Omega Function

Let $\Omega(n)$ denote the **number of prime factors** of n (when counted with multiplicities). For example, $\Omega(1) = 0$, $\Omega(p) = 1$, $\Omega(p.q) = \Omega(p^2) = 2$, $\Omega(p_1^{e_1} \cdots p_k^{e_k}) = e_1 + \cdots + e_k$.

A central question that we have in number theory is: What is the distribution of the values of $\Omega(n)$. If we look at the values of the function for the first two-hundred positive integers, we find that $\Omega(n)$ is growing. In fact, it grows like $\log(\log(n))$, but besides the average growth, it behaves very randomly. We shouldn't forget here that there are loads of primes within the first two-hundred integers. So, it's understood that:

- The distribution of the values of $\Omega(n)$ follows no notable pattern. It appears to be random.
- Knowing $\Omega(n-1)$, $\Omega(n-2)$, \cdots , $\Omega(n-m)$ does not allow us to predict $\Omega(n)$.

1.2 Some Classical Results in Multiplicative Number Theory

The study of the distribution of the values of $\Omega(n)$ has a long and rich history and is closely related to fundamental questions about the prime numbers.

The **natural density** of a set $A \subset \mathbb{N}$ is defined as $d(A) = \lim_{n \rightarrow \infty} |\{1 \leq n \leq N : n \in A\}|/N$. The following statement is a well-known equivalent form of the Prime Number Theorem (von Mangoldt 1897, Landau 1911):

Theorem 1. *The sets $\{n \in \mathbb{N} : \Omega(n) \text{ is even}\}$ and $\{n \in \mathbb{N} : \Omega(n) \text{ is odd}\}$ have natural density of $1/2$.*

This means that, asymptotically, there are as many prime numbers with even numbers of prime factors as there are prime numbers with odd numbers of prime factors. It is natural to ask whether this can be generalized as this function is evenly distributing between even's and odd's. And we can see this in the Pillai-Selberg Theorem (Pillai 1940, Selberg 1939):

Theorem 2. *For all $m \in \mathbb{N}$ and $r \in \{0, \dots, m-1\}$ the set $\{n \in \mathbb{N} : \Omega(n) \equiv r \pmod{m}\}$ has natural density $1/m$.*

A sequence of real numbers $(x_n)_{n \in \mathbb{N}}$ is said to be **uniformly distributed mod 1** if for any continuous $f : [0, 1) \rightarrow \mathbb{C}$ we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(x_n) = \int_0^1 f(x) dx.$$

Theorem 3. *For all irrational α the sequence $\Omega(n)\alpha, n \in \mathbb{N}$, is uniformly distributed mod 1.*

The above is the Erdos-Delange Theorem (Erdos 1946, Delange 1958). The paper rides on these classical results mostly. Let us set the scene for the main characters in the following section.

2 A Dynamic Generalization of the Prime Number Theorem

2.1 1st Main Result

Let X be a compact metric space and $T : X \rightarrow X$ a continuous map. Since

$$T^m \cdot T^n = T^{m+n}, \quad \forall m, n \in \mathbb{N},$$

the transformation T naturally induces an action of $(\mathbb{N}, +)$ on X . We call (X, T) an **additive topological dynamical system**. Every additive topological dynamical system possesses at least one T -invariant Borel probability measure. If (X, T) admits only one such measure, then the system is called **uniquely ergodic**.

Theorem 4. *Let X, μ, T be uniquely ergodic. Then*

$$\lim_{n \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(T^{\Omega(n)}x) = \int f d\mu$$

for every $x \in X$ and $f \in C(X)$.

We can interpret 4(Bergelson-R. 2020) as saying that for any uniquely ergodic system (X, T) and any point $x \in X$ the orbit $T^{\Omega(n)}x$ is uniformly distributed in the space X . Let us apply this theorem to a simple situation like **rotation on two points**. Theorem 4 applied on this recovers the Prime Number Theorem. It also recovers the Pillai-Selberg Theorem and Erdos-Delange Theorem which I was too lazy to prove (it's quite similar to the one I write below).

Proof. Let $X = \{0, 1\}$ and $T : x \mapsto x + 1 \pmod{2}$. This system is uniquely ergodic, with unique invariant measure μ given by $\mu(\{0\}) = \mu(\{1\}) = 1/2$. Let $f : \{0, 1\} \rightarrow \mathbb{R}$ be defined as $f(0) = 1$ and $f(1) = 0$, and take $x = 0$. Then

$$\lim_{n \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(T^{\Omega(n)}x) = d(\{n : \Omega(n) \text{ is even}\}).$$

Since $\int f d\mu = 1/2$, it follows from Theorem 4 that $d(\{n : \Omega(n) \text{ is even}\}) = 1/2$.

QED

3 To. Be. Continued.

This paper is quite tough to understand. I did get through some heavy math but I'll come back to it later.