MML Assignment - 4

Due Date: 25-11-2025

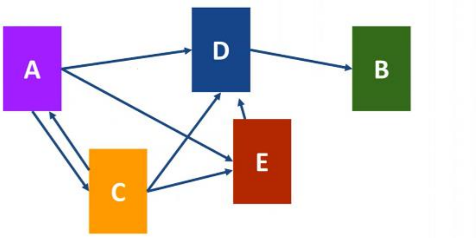
1. Consider a random walk on the integers starting at $X_0 = 0$. At each step, the walker moves +1 with probability p and −1 with probability q = 1 − p. The walk continues until it hits either −r or +m, where r,m are positive integers. (2+3+1=6M)
   a) Write the probability mass function (PMF) of X_n, the position after n steps.
   b) Derive an expression for the probability h that the walker reaches +m before −r, for p ≠q.
   c) Simplify your result for the symmetric case p = q = 1/2 and interpret it.
2. A biased random walk starts at $X_0 = 0$. At each step, (3+1=4M)

$$X_{n+1} = \begin{cases} X_n + 1, & \text{with probability } p, \\ X_n - 1, & \text{with probability } 1 - p. \end{cases}$$

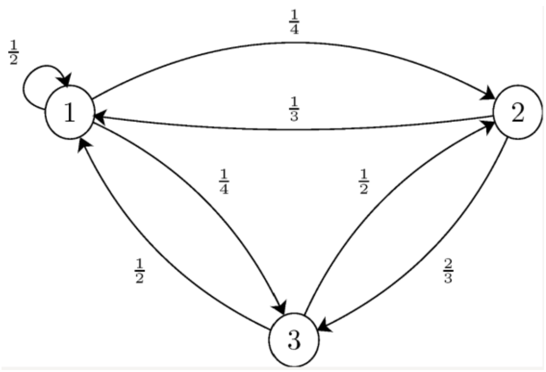
   (a) Derive expressions for the expected value $E[X_n]$ and variance $Var(X_n)$ in terms of n and p.
   (b) Evaluate these numerically for p = 0.6 and n = 10.
3. A Markov chain has states {1, 2, 3} with transition matrix (3+0.5+0.5=5M)

$$P = \begin{bmatrix} 0.2 & 0.5 & 0.3 \\ 0.4 & 0.4 & 0.2 \\ 0.1 & 0.3 & 0.6 \end{bmatrix}.$$

   (a) Find the stationary distribution π = [π1, π2, π3] satisfying π = πP and π1 + π2 + π3 = 1.
   (b) Which state is most likely to be visited in the long run? Briefly justify using your stationary distribution.
   (c) Verify your result by checking that πP = π (show the substituted expressions or a short numeric check).
4. Find the final page rank values for the following pages A, B, C, D and E. Assuming damping factor d=0.85. Make an inference of the page rank obtained for each of the web pages. (10M)



5. Consider the Markov chain shown below



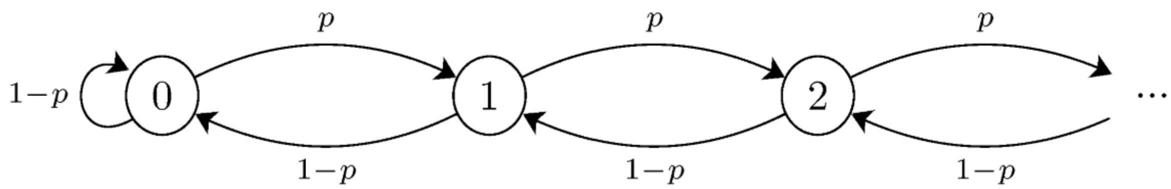
   a. Is this chain irreducible? (1M)

b. Is this chain aperiodic? (1M)

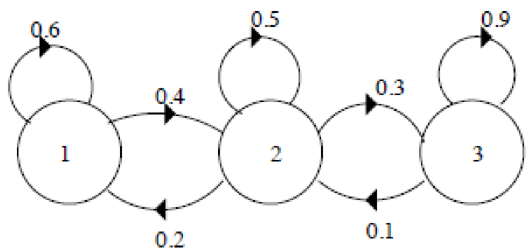
c. Find the stationary distribution for this chain. (3M)

d. Is the stationary distribution a limiting distribution for the chain? (1M)

6. Consider the Markov chain shown in Figure. Assume that 1/2<p<1. Does this chain have a limiting distribution? For all i,j∈{0,1,2,⋯}, find $\lim_{n\to\infty} P(X_n = j|X_0 = i)$ (6M)



7. Consider a Markov chain {Xn; n = 0, 1, . . .}, specified by the following transition diagram.



a. Given that the chain starts with X0 = 1, find the probability that X2 = 2. (2M)

b. Find the steady-state probabilities π1, π2, π3 of the different states. (3M)

c. Let Yn = Xn − Xn−1. Thus, Yn = 1 indicates that the nth transition was to the right, Yn = 0 indicates it was a self-transition, and Yn = −1 indicates it was a transition to the left. Find $\lim_{n\to\infty} P(Y_n = 1)$ (3M)

8. Oscar goes for a run each morning. When he leaves his house for his run, he is equally likely to go out either the front or back door; and similarly, when he returns, he is equally likely to go to either the front or back door. Oscar owns only five pairs of running shoes which he takes off immediately after the run at whichever door he happens to be. If there are no shoes at the door from which he leaves to go running, he runs barefooted. We are interested in determining the long-term proportion of time that he runs barefooted.

(a) Set the scenario up as a Markov chain, specifying the states and transition probabilities. (4M)

(b) Determine the long-run proportion of time Oscar runs barefooted. (4M)

9. An absent-minded professor has two umbrellas that she uses when commuting from home to office and back. If it rains and an umbrella is available in her location, she takes it. If it is not raining, she always forgets to take an umbrella. Suppose that it rains with probability p each time she commutes, independently of other times. What is the steady-state probability that she gets wet on a given day? (8M)

10. Let $(X_0, X_1, . . .)$ be a Markov chain with transition matrix P. (3+2=5M)

(a) Define $(Y_0, Y_1, . . .)$ by setting $Y_n = X_{2n}$ for each n. Show that $(Y_0, Y_1, . . .)$ is a Markov chain with transition matrix P^2.

(b) Find an appropriate generalization of the result in (a) to the situation where we sample every kth (rather than every second) value of $(X_0, X_1, . . .)$.

11. Consider the Markov chain with three states, S={1,2,3}, that has the following transition matrix

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{3} & 0 & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

a. Draw the state transition diagram for this chain. (2M)

b. If we know $P(X_1=1)=P(X_1=2)=1/4$, find $P(X_1=3,X_2=2,X_3=1)$. (4M)

12. Consider the graph shown in the Figure. Using the Metropolis–Hastings algorithm, assign transition probabilities so that the stationary probability distribution of a random walk is $p(u)=3/8, p(v)=1/4, p(w)=1/4, p(x)=1/8$. Assume the maximum degree of any vertex in the graph is 3. (4+2+2 =8M)

(a) Determine the transition probabilities for all possible edges so that the stationary distribution is $p(\cdot)$

(b) Verify that the detailed balance condition $p_i P_{ij}=p_j P_{ji}$ holds for each pair of connected vertices.

(c) Compute the stationary probability of node u from the transition probabilities and verify that it equals 3/8 .