Assignment 3: Probability Theory

Math for ML (AIT-512)

Submission Deadline: 2-11-2025

Instructions: Answer all questions. Submit in a single PDF file in LMS (handwritten answers with scan and applying grayscale filter, if necessary). If plagiarism is detected, it will be given -20 marks.

Problem 1: A coin is tossed three times with each 3-tuple outcome being equally likely. Find the probability of obtaining (H, T, H) if it is known that the outcome has 2 heads. Do this by using conditional probability. (5)

Problem 2: A digital communication system transmits one of three values: -1, 0, 1. A channel adds noise to cause the decoder to sometimes make an error. The error rates are 12.5% if -1 is transmitted, 75% if a 0 is transmitted, and 12.5% if a 1 is transmitted. If the probability for the various symbols being transmitted are P(-1) = P(1) = 0.25 and P(0) = 0.5, find the probability of error. (5)

Problem 3: The cumulative distribution function of the continuous random variable V is

$$F_V(v) = \begin{pmatrix} 0, v \leq -5 \\ c(v+5)^2, -5 \leq v < 7 \\ 1, v \geq 7 \end{pmatrix}$$

(a) What is c? (3)
(b) What is P(V > 4)? (2)
(c) What is P(-3 < V < 0)? (2)
(d) What is the value of *a* such that P(V > *a*) = 2/3? (3)

Problem 4: The discrete random variable X has the following CDF

$$F_X(x) = \begin{pmatrix} 0, x < -3 \\ 0.4, -3 \leq x < 5 \\ 0.8, 5 \leq x < 7 \\ 1, x \geq 7 \end{pmatrix}$$

(a) Plot the CDF. (3)
(b) Draw the PMF of X**.** (2)
(c) Find $E[X]$**.** (2)
(d) Find $Var[X]$**.** (3)

Problem 5. (5*2=10)

Random variables X and Y have joint pdf:

$$f_{X,Y}(x,y) = \begin{pmatrix} 4xy, 0 \le x \le 1, 0 \le y \le 1 \\ 0, otherwise \end{pmatrix}$$

(a) What are $E[X]$ and $Var[X]$?
(b) What are $E[Y]$ and $Var[Y]$?
(c) What is $Cov[X,Y]$?
(d) What is $E[X+Y]$?
(e) What is $Var[X+Y]$

Problem 6: (5+5)

Let X, Y be i.i.d. random variables with distribution N(0,α). Show that PDF of $\mathrm{Z} = \mathrm{X}^2 + \mathrm{Y}^2$ follows exponential distribution.

What will be the PDF for sum of squares of N such Random Variables where each follows standard Gaussian distribution. Show your steps to arrive at the answer.

Problem 7: (5)

Suppose $X_1, X_2, \ldots, X_n$ are jointly Gaussian random variables with $\mathrm{COV}(X_i, X_j) = 0$ for $i \neq j$. Show that $X_1, X_2, \ldots, X_n$ are independent random variables.

Problem 8: (5)

Let $W = \max(X_1, X_2, \ldots, X_n)$ and $Z = \min(X_1, X_2, \ldots, X_n)$, where the X_i are independent random variables with the same distribution. Find $F_W(w)$ and $F_Z(z)$.

Problem 9: (10)

Let $Y = X + N$ where X (the "signal") and N (the "noise') are independent zero-mean Gaussian random variables with different variances. Find the correlation coefficient between the observed signal Y and the desired signal X. Find the value of x that maximizes $f_X(x \mid y)$.

Problem 10: (3+5+2)

Given a random vector $\mathbf{X}$, find the joint pdf of the following transformation:

$$\begin{aligned} Z_1 &= g_1(X_1) = a_1 X_1 + b_1, \\ Z_2 &= g_2(X_2) = a_2 X_2 + b_2, \\ &\vdots \\ Z_n &= g_n(X_n) = a_n X_n + b_n. \end{aligned}$$

Let X_1 and X_2 be independent exponential variables, parameter λ. Find the joint density function of

$$Y_1 = X_1 + X_2, \qquad Y_2 = X_1 / X_2,$$

and show that they are independent.

Problem 11 (2+4+4)

Let X and Y have the joint density $f(x, y) = cx(y - x)e^{-y}$, $0 \le x \le y < \infty$.

(a) Find c.

(b) Show that:

$$f_{X|Y}(x \mid y) = 6x(y - x)y^{-3}, \qquad 0 \le x \le y,$$
$$f_{Y|X}(y \mid x) = (y - x)e^{x-y}, \qquad 0 \le x \le y < \infty.$$

(c) Deduce that $\mathbb{E}(X \mid Y) = \frac{1}{2}Y$ and $\mathbb{E}(Y \mid X) = X + 2$.

Problem 12 (3+7)

(I) (3 marks)

MATLAB. Let X be a constant, scalar random variable taking the value m. It is easy to see that $F_X(x) = u(x - m)$, where u is the unit step function. It then follows that $f_X(x) = \delta(x - m)$. Use the following MATLAB code to plot the $N(0, 1/n^2)$ density for $n = 1, 2, 3, 4$ to demonstrate that as the variance of a Gaussian goes to zero, the density approaches an impulse; in other words, a constant random variable can be viewed as the limiting case of the ordinary Gaussian.

```
x=linspace(-3.5,3.5,200);
s = 1;    y1 = exp(-x.*x/(2*s))/sqrt(2*pi*s);
s = 1/4;  y2 = exp(-x.*x/(2*s))/sqrt(2*pi*s);
s = 1/9;  y3 = exp(-x.*x/(2*s))/sqrt(2*pi*s);
s = 1/16; y4 = exp(-x.*x/(2*s))/sqrt(2*pi*s);
plot(x,y1,x,y2,x,y3,x,y4)
```

(II) 2+2+3 Marks

Let $X \sim N(0, 1)$ and put $Y := 3X$.

(a) Show that X and Y are jointly Gaussian.

(b) Find their covariance matrix, $\text{cov}([X, Y]')$.

(c) Show that they are not jointly continuous. *Hint:* Show that the conditional cdf of Y given $X = x$ is a unit-step function, and hence, the conditional density is an impulse.

Hint:

A random vector $X = [X_1, \ldots, X_n]'$ is said to be Gaussian or normal if every linear combination of the components of X, e.g.,

$$\sum_{i=1}^{n} c_i X_i,$$

is a scalar Gaussian random variable. Equivalent terminology is that $X_1, \ldots, X_n$ are **jointly Gaussian** or **jointly normal**.