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Maths for ML (AIT-512)

Assignment - 3 Probability Theory

Q1) A coin is tossed three times with each 3-tuple outcome being equally likely. Find the probability of obtaining (H, T, H) if it is known that the outcome has 2 heads.

Ans: Coin tossed 3 times

$S = \{ (HHH), (TTT), (TTH), (THT), (HTH), (HHT), (THH), (HTT) \}$

Outcome equally likely; Hence for eg: $P(\{THH\}) = 1/8$

Event A: Obtaining $\{(H,T,H)\}$

Event B: outcome has exactly 2 Heads $\{(HHT)\ (THH)\ (HTH)\}$

☆ Conditional Probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(\{HTH\} \cap \{(HHT), (THH)\ (HTH)\})}{P(HHT) + P(THH) + P(HTH)}$$

$$= \frac{P(\{HTH\})}{P(\{HHT\}) + P(\{THH\}) + P(\{HTH\})}$$

$$= \frac{1/8}{3/8}$$

$$= \frac{1}{3}$$

Q2) A digital comm. sys transmits one of three values: -1, 0, +1. A channel adds noise to cause the decoder to sometimes make an error. Error rate: 12.5% if -1 transmitted, 75% if 0 is transmitted and 12.5% if +1 transmitted. Probability for various symbols being transmitted are $P(-1) = P(1) = 0.25$ and $P(0) = 0.5$. Find probab. of error.

Given:

Ans: $P(S_{-1})$: Probability of transmitting symbol $-1 = 0.25$

$P(S_0)$: Probability of transmitting symbol $0 = 0.5$

$P(S_1)$: Probability of transmitting symbol $1 = 0.25$

$P(E)$: Probability of an error has occurred $= ?$

$P(E/S_{-1})$: Probability of an error given $-1 = 0.125$

$P(E/S_0)$: Probability of an error given $0 = 0.75$

$P(E/S_{+1})$: Probability of an error given $1 = 0.125$

$$P(E) = P(E/S_{-1})\,P(S_{-1}) + P(E/S_0)\,P(S_0) + P(E/S_1)\,P(S_1)$$

$$= 0.125 \times 0.25 + 0.75 \times 0.5 + 0.125 \times 0.25$$

$$= 0.03125 + 0.375 + 0.3125$$

$$= \underline{\underline{0.4375}}$$

Q3) The cdf of continuous random variable V

$$F_V(v) = \begin{cases} 0, & v \leq -5 \\ c(v+5)^2, & -5 \leq v < 7 \\ 1, & v \geq 7 \end{cases}$$

(a) What is c?

Ans: PDF $f_V(v)$ derivative of the CDF $F_V(v)$

$$f_V(v) = \frac{d}{dv} F_V(v)$$

for $-5 \leq v < 7$: $\dfrac{d\,(c(v+5)^2)}{dv} = c\,\dfrac{d\,(v+5)^2}{dv} = \underline{2c(v+5)}$

for range $v \leq -5$ & $v \geq 7$, constant values whose differentiation is 0.

$$f_V(v) = \begin{cases} 2c(v+5) & -5 \leq v < 7 \\ 0 & \text{otherwise} \end{cases}$$

$$\int_{-5}^{7} 2c(v+5)\,dv = 1$$

$$2c\int_{-5}^{7}(v+5)\,dv = 1$$

$$2c\left[\frac{(v+5)^2}{2}\right]_{-5}^{7} = 1$$

$$c\left[(7+5)^2 - (-5+5)^2\right] = 1$$

$$c\left[(12)^2 - (0)^2\right] = 1$$

$$144c = 1$$

$$\underline{\underline{c = 1/144}}$$

(b) What is $P(V > 4)$?

Ans: $$P(V>4) = \int_4^7 \frac{1}{144} \times 2(v+5)\,dv$$

$$= \frac{1}{72}\left[\left(\frac{(v+5)^2}{2}\right)\right]_4^7$$

$$= \frac{1}{144}\left[(7+5)^2 - (4+5)^2\right]$$

$$= \frac{1}{144}\left[(12)^2 - (9)^2\right]$$

$$= \frac{1}{144}\left[144 - 81\right] = \frac{63}{144} = \frac{7}{16}$$

(c) What is $P(-3 < V < 0)$?

Ans: $P(-3 < V < 0) = \int_{-3}^{0} \frac{1}{144} \times 2(v+5)\, dv$

$= \frac{1}{72} \left[\frac{(v+5)^2}{2} \right]_{-3}^{0}$

$= \frac{1}{144} \left[(0+5)^2 - (-3+5)^2 \right]$

$= \frac{1}{144} \left[(5)^2 - (2)^2 \right] = \frac{1}{144}(25-4) = \frac{21}{144} = \frac{7}{48}$

(d) Value of 'a' such that $P(V > a) = 2/3$?

Ans $P(V > a) = \int_{a}^{7} \frac{1 \times 2}{144} (v+5)\, dv = \frac{2}{3}$

$\Rightarrow \frac{1}{72} \left[\frac{(v+5)^2}{2} \right]_{a}^{7} = \frac{2}{3}$

$\Rightarrow \frac{1}{144} \left[(7+5)^2 - (a+5)^2 \right] = \frac{2}{3}$

$\Rightarrow \frac{1}{144} \left[(12)^2 - (a+5)^2 \right] = \frac{2}{3}$

$\Rightarrow \frac{1}{144} \left(144 - (a+5)^2 \right] = \frac{2}{3}$

$\Rightarrow 1 - \frac{(a+5)^2}{144} = \frac{2}{3}$

$\Rightarrow \frac{(a+5)^2}{144} = 1 - \frac{2}{3} = \frac{1}{3}$

$(a+5)^2 = \frac{144}{3} = 48$

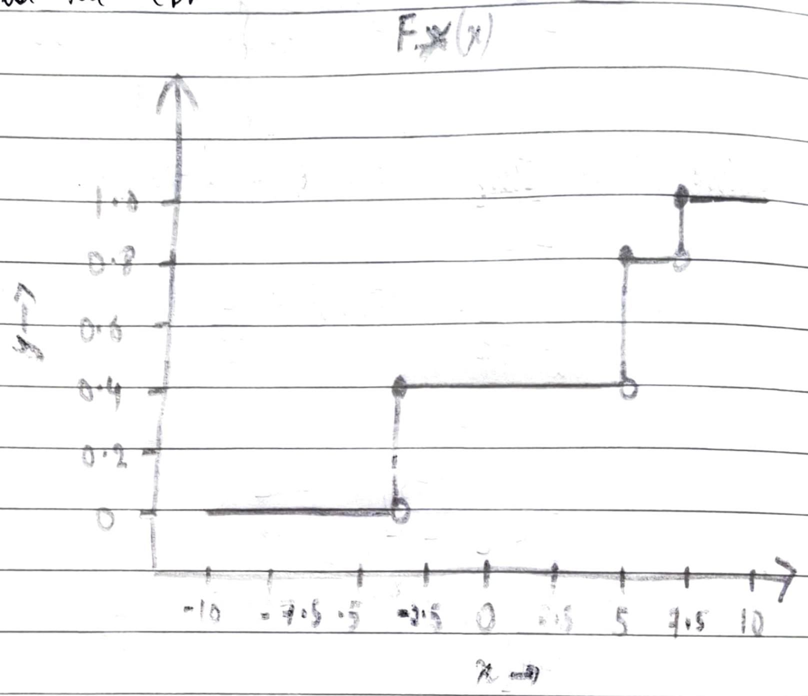
$a = -5 \pm \sqrt{48}$

$a = -5 + 4\sqrt{3}$

Q4) Discrete random variable X has following CDF.

$$F_X(x) = \begin{cases} 0, & x < -3 \\ 0.4, & -3 \le x < 5 \\ 0.8, & 5 \le x < 7 \\ 1, & x \geq 7 \end{cases}$$

(a) Plot the CDF.



(b) Draw the PMF

At $x = -3$: $F_X(-3) - F_X(\text{just before } -3) = 0.4 - 0 = 0.4$

At $x = 5$: $F_X(5) - F_X(\text{just before } 5) = 0.8 - 0.4 = 0.4$

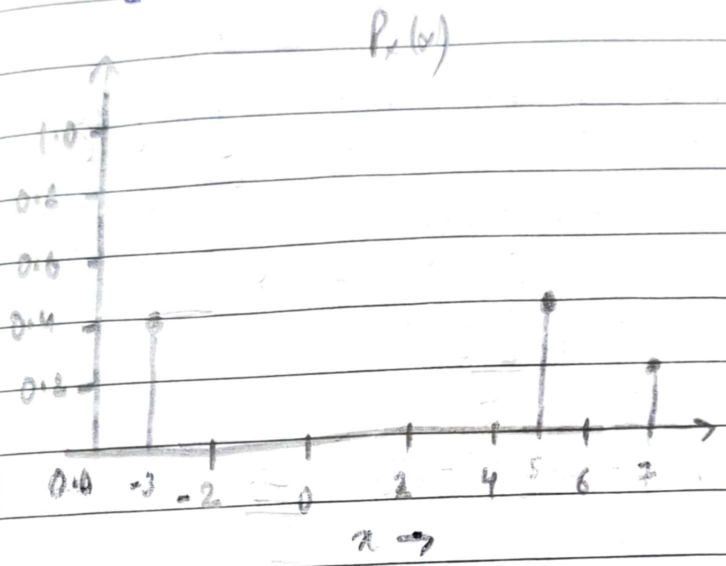
$P(X = 5) = 0.4$

At $x = 7$: $F_X(7) - F_X(\text{just before } 7) = 1 - 0.8 = 0.2$

$P(X = 7) = 0.2$

for other values, probab. is 0.

$$P_X(x) = \begin{cases} 0.4 & x = -3 \\ 0.4 & x = 5 \\ 0.2 & x = 7 \\ 0 & \text{otherwise} \end{cases}$$



(i) Find $E[X]$

Ans: $E[X] = (-3)\,P(X=-3) + (5)\cdot P(X=5) + (7)\,P(X=7)$

$E[X] = (-3)(0.4) + 5(0.4) + 7(0.2)$

$E(X) = -1.2 + 2 + 1.4$

$E[X] = \underline{\underline{2.2}}$

(ii) Find $Var[X]$.

Ans: $Var[X] = E[X^2] - (E[X])^2$

$E[X^2] = (-3)^2(0.4) + (5)^2(0.4) + (7)^2(0.2)$

$= 9(0.4) + 25(0.4) + 49(0.2)$

$= 3.6 + 10 + 9.8$

$E[X^2] = \underline{\underline{23.4}}$

X	X^2	$P(x)$
-3	9	0.4
5	25	0.4
7	49	0.2

$(E[X])^2 = (2.2)^2$ from part i

$Var[X] = 23.4 - (2.2)^2 = 23.4 - 4.84 = \underline{\underline{18.56}}$

Q5) Random variables X and Y have joint pdf:

$$f_{X,Y}(x,y) = \begin{cases} 4xy, & 0 \le x \le 1, \quad 0 \le y \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) What are $E[X]$ and $var[X]$?

Ans: Marginal pdf for X : $f_X(x) = \int_0^1 4xy \, dy$

$$= 4x \int_0^1 y \, dy$$

$$= 4x \left[\frac{y^2}{2}\right]_0^1$$

$$f_X(x) = 2x\left[(1)^2 - (0)^2\right] = \underline{\underline{2x}}$$

Marginal pdf for Y: $f_Y(y) = \int_0^1 4xy \, dx$

$$= 4y \int_0^1 x \, dx$$

$$= 4y \left[\frac{x^2}{2}\right]_0^1$$

$$= 2y\left[(1)^2 - (0)^2\right]$$

$$= \underline{\underline{2y}}.$$

$\therefore$ $E[X] = \int_0^1 x \cdot f_X(x) \, dx = \int_0^1 x(2x) \, dx$

$$= \int_0^1 2x^2 \, dx$$

$$= \frac{2}{3}\left[x^3\right]_0^1$$

$$= \underline{\underline{\frac{2}{3}}} \rightarrow (1)$$

$$Var[X] = E[X^2] - (E[X])^2$$

$$E[X^2] = \int_0^1 x^2 \cdot f_X(x)\, dx = \int_0^1 x^2\, 2x\, dx = \int_0^1 2x^3\, dx$$

$$= \frac{2}{4}\left[x^4\right]_0^1$$

$$= \frac{1}{2} \rightarrow (2)$$

$$Var[X] = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{2} - \frac{4}{9} = \frac{9-8}{18} = \frac{1}{18}$$

(b) What are, $E[Y]$ and $Var[Y]$?

Ans: $E[Y] = \int_0^1 y \cdot f_Y(y)\, dy = \int_0^1 y \cdot 2y\, dy$

$$= \frac{2}{3}\left[y^3\right]_0^1 = \frac{2}{3}$$

$$E[Y^2] = \int_0^1 y^2 \cdot f_Y(y)\, dy = \int_0^1 y^2\, 2y\, dy = \int_0^1 2y^3\, dy = \frac{1}{2}$$

$$Var[Y] = \frac{1}{2} - \left(\frac{2}{3}\right)^2 = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}$$

(c) What is $Cov[X, Y]$?

Ans: $Cov[X,Y] = E[XY] - E[X]\,E[Y]$

$$E[XY] = \int_0^1\int_0^1 (xy)(4xy)\, dx\, dy$$

$$= 4\int_0^1\int_0^1 x^2 y^2\, dx\, dy$$

$$= 4\int_0^1 y^2 \int x^2 dx\, dy$$

$$= 4\int_0^1 y^2 \cdot \frac{1}{3}\left[x^3\right]_0^1 dy$$

$$= \frac{4}{3}\int y^2 dy = \frac{4}{3}\frac{\left[y^3\right]_0^1}{3}$$

$$= \frac{4}{9}.$$

$$Cov(X,Y) = \frac{4}{9} - \left(\frac{2}{3}\right)\left(\frac{2}{3}\right) = \frac{4}{9} - \frac{4}{9} = 0.$$

(d) What is $E[X+Y]$?

Ans: $E[X+Y] = E[X] + E[Y]$

$$E[X+Y] = \frac{2}{3} + \frac{2}{3} = \frac{4}{3}.$$

(e) What is $Var[X+Y]$?

Ans. $Var[X+Y] = Var[X] + Var[Y] + 2\,Cov[X,Y]$

$$= \frac{1}{18} + \frac{1}{18} + 2\cdot(0)$$

$$= \frac{2}{18} = \frac{1}{9}.$$

Q11) Let X and Y have joint density $f(x,y) = cx(y-x)e^{-y}$,
$0 \le x \le y < \infty$.

(a) Find c.

$y = 0$ to ∞, and $\forall y$, $x = 0$ to y.

Ans:

$$\int_0^\infty \left(\int_0^y cx(y-x)e^{-y}dx\right) dy = 1$$

$$\int_0^\infty ce^{-y}\left(\int_0^y (xy - x^2)dx\right) dy = 1$$

$$\int_0^\infty \left(ce^{-y} \left[\frac{x^2 y}{2} - \frac{x^3}{3} \right]_0^y \right) dy = 1$$

$$\int_0^\infty ce^{-y} \left[\left(\frac{y^3}{2} - \frac{y^3}{3} \right) - (0) \right] dy = 1$$

$$\int_0^\infty ce^{-y} \left(\frac{y^3}{6} \right) dy = 1$$

$$\underbrace{\frac{c}{6} \int_0^\infty y^3 e^{-y} \, dy}_{} = 1 \quad \text{— (a)}$$

Gamma func form: $\int_0^\infty x^{a-1} e^{-x} \, dx$

here $a - 1 = 3 \Rightarrow \underline{a = 4}$

$\Gamma(4) = (4-1)! = 3! = \underline{6}$

$\therefore \frac{c}{6} \times 6 = 1 \Rightarrow \boxed{c = 1}$

(b) Show that: $f_{X|Y}(x|y) = 6x(y-x)y^{-3}, \quad 0 \le x \le y,$

$f_{Y|X}(y|x) = (y-x)e^{x-y}, \quad 0 \le x \le y < \infty$

Ans: $c = 1$; $f(x,y) = x(y-x)e^{-y}$

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$$

$$f_Y(y) = \int_0^y x(y-x)e^{-y} \, dx$$

from (a): $f_Y(y) = \frac{1}{6} y^3 e^{-y}$

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{x(y-x)e^{-y}}{\frac{1}{6}y^3 e^{-y}}$$

$$= \frac{6x(y-x)}{y^3}$$

$$= 6x(y-x)y^{-3}$$

$$\therefore \quad f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)}$$

$$f_X(x) = \int_x^\infty x(y-x)e^{-y}\,dy = x\int_x^\infty \underset{①}{(y-x)}\,\underset{②}{e^{-y}}\,dy$$

Integration by parts.

$$\int (y-x)e^{-y}\,dy = -(y-x)e^{-y} - \int -e^{-y}\,dy$$

$$= -(y-x)e^{-y} - e^{-y}$$

$$= \left[-(y-x)e^{-y} - e^{-y}\right]_x^\infty \qquad e^{-\infty} = 0$$

$$= (0) - \left(-(x-x)e^{-x} - e^{-x}\right)$$

$$= 0 - \left(-(0)\cdot e^{-x} - e^{-x}\right)$$

$$= e^{-x}$$

$$f_X(x) = x \cdot e^{-x}$$

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{x(y-x)e^{-y}}{x\cdot e^{-x}}$$

$$= (y-x)\,e^{x-y}$$

(i) Deduce that $E(X|Y) = \frac{1}{2}Y$ and $E(Y|X) = X+2$

Ans: $E[X|Y] = \int_{-\infty}^{\infty} x \cdot f_{X|Y}(x|y)\,dx = \int_0^y x\left[\frac{6x(y-x)}{y^3}\right] dx$

$$= \frac{6}{y^3}\int_0^y x^2 y - x^3\,dx$$

$$= \frac{6}{y^3}\left[\frac{x^3 y}{3} - \frac{x^4}{4}\right]_0^y$$

$$= \frac{6}{y^3}\left[\frac{y^4}{3} - \frac{y^4}{4}\right]$$

$$E[X|Y] = \frac{6}{y^3}\cdot\frac{y^4}{12} = \frac{y}{2}$$

$\therefore$ $E[Y|X] = \int_{-\infty}^{\infty} y \cdot f_{Y|X}(y|x)\,dy = \int_x^{\infty} y\left[(y-x)e^{x-y}\right] dy$

$$= e^{x}\int_x^{\infty}(y^2 - xy)e^{-y}\,dy$$

Integration by parts: $I = \left[-(y^2-xy)e^{-y}\right]_x^{\infty} - \int -(2y-x)e^{-y}\,dy$

$u = 2y - x \rightarrow du = 2\,dy$; $dv = e^{-y}dy \Rightarrow v = -e^{-y}$

$$I = \left[-(2y-x)e^{-y}\right]_x^{\infty} - \int_x^{\infty} -2e^{-y}\,dy$$

$$I = (0 - (-(2x-x)e^{-x})) + 2\int_x^{\infty} e^{-y}\,dy$$

$$I = xe^{-x} + 2\left[-e^{-y}\right]_x^{\infty}$$

$$I = xe^{-x} + 2e^{-x} = (x+2)e^{-x}$$

$$E[Y|X] = e^{x}\cdot I = e^{x}(x+2)e^{-x} = x+2$$

19) Let $Y = X + N$ where X (signal) and N (noise) are independent zero-mean Gaussian random variables with different variances. Find the correlation coefficient b/w the observed signal Y and desired signal X. Find the value of x that maximizes $f_X(x|y)$.

Ans: Given

$X \sim N(0, \sigma_X^2)$ $\quad$ $N \sim N(0, \sigma_N^2)$

Mean of Y: $E[Y] = E[X+N] = E[X] + E[N] = 0 + 0 = 0$

Variance of Y: $Var(Y) = Var(X+N) = Var(X) + Var(N)$

$= \sigma_X^2 + \sigma_N^2$ — ①

$Y \sim N(0, \sigma_X^2 + \sigma_N^2)$

Correlation coefficient $\rho_{XY} = \dfrac{Cov(X,Y)}{\sigma_X \sigma_Y}$

from ① $\sigma_Y = \sqrt{\sigma_X^2 + \sigma_N^2}$

$\therefore Cov(X,Y) = E[XY] - E[X]E[Y]$

$Cov(X,Y) = E[XY] - 0.0 = E[XY]$

$Y = X+N$

$E[XY] = E[X(X+N)] = E[X^2 + XN] = E[X^2] + E[XN]$

$Var(X) = E[X^2] - (E[X])^2 \rightarrow Var(X) = E[X^2] = \sigma_X^2$

$E[XN] = E[X]E[N] = 0.0 = 0$ $\quad$ since X, N independent

$Cov(X,Y) = E[XY] = \sigma_X^2$

$$\rho_{XY} = \frac{\sigma_X^2}{\sigma_X\sqrt{\sigma_X^2 + \sigma_N^2}} = \frac{\sigma_X}{\sqrt{\sigma_X^2 + \sigma_N^2}}$$

Value of x that maximize $f_X(x|y)$

$$F_Z(z) = \int_0^{2\pi}\left(\int_0^{\sqrt{z}} \frac{1}{2\pi\sigma} e^{-r^2/2\sigma} \cdot r\, dr\right) d\theta$$

Let $u = r^2/2\sigma$ $\qquad \dfrac{du}{c} = \dfrac{2r\,dr}{2\sigma} = \dfrac{r\,dr}{\sigma}$

$r\,dr = \sigma\,du$

Limits: $r = 0,\ u = 0 \quad \rightarrow r = \sqrt{z} \quad u = z/2\sigma$

$$= \int_0^{2\pi} \frac{1}{2\pi\sigma} \int_0^{z/2\sigma} e^{-u} (\sigma\,du)\, d\theta$$

$$= \int_0^{2\pi} \frac{1}{2\pi} \left[-e^{-u}\right]_0^{z/2\sigma} d\theta$$

$$= \int_0^{2\pi} \frac{1}{2\pi} \left(1 - e^{-z/2\sigma}\right) d\theta$$

$$F_Z(z) = 2\pi \cdot \left[\frac{1}{2\pi}\left(1 - e^{-z/2\sigma}\right)\right] = 1 - e^{-z/2\sigma}$$

for PDF $f_Z(z) = \dfrac{d}{dz} F_Z(z) = \dfrac{d}{dz}\left(1 - e^{-z/2\sigma}\right)$

$$f_Z(z) = \frac{1}{2\sigma} e^{-z/2\sigma}$$

follows expo distrib : $f_Z(z) = \lambda e^{-\lambda z}$ $\quad \lambda = \frac{1}{2\sigma}$

2) PDF for sum of squares of N such random variables

Let $X_1, \ldots, X_n$ be i.i.d standard normal random variable

$X_i \sim N(0,1)$ $\qquad$ Let $S_N = \sum_{i=1}^{N} x^2$

$$E[X/Y=y] = E[X] + \frac{cov(X,Y)}{Var(Y)}(y - E[Y])$$

$$= 0 + \frac{\sigma_X^2}{\sigma_X^2 + \sigma_N^2}(y-0)$$

$$x = \frac{\sigma_X^2}{\sigma_X^2 + \sigma_N^2} \cdot y$$

Q6) Let X, Y be I.I.D random variables with distrib. $N(0, \sigma)$. Show that the PDF of $Z = X^2 + Y^2$ follows exponential distribution.
What will be the PDF for sum of square of N such random variable where each follows standard Gaussian distrib.

Ans.
$$F_Z(z) = P(Z \leq z)$$
$$= P(X^2 + Y^2 \leq z)$$

$$F_Z(z) = \iint_{x^2+y^2 \leq z} f(x,y)\,dx\,dy$$

$\therefore$ PDF for $X \sim N(0, \sigma)$

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-x^2/2\sigma}$$

PDF for $Y \sim N(0, \sigma)$

$$f_Y(y) = \frac{1}{\sqrt{2\pi\sigma}} e^{-y^2/2\sigma}$$

$$f(x,y) = \frac{1}{2\pi\sigma} e^{-(x^2+y^2)/2\sigma}$$

Substitute x with $r\cos\theta$ and y with $r\sin\theta$

$x^2 + y^2 \leq z \rightarrow r^2 < z \rightarrow 0 \leq r < \sqrt{z}$

angle becomes 0 to 2π

Computing MGF

$$M_{X_i^2}(t) = E[e^{x_i^2 t}] = \int_{-\infty}^{\infty} e^{x_i^2 t} \frac{1}{\sqrt{2\pi}} e^{-x_i^2/2} \, dx_i = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x_i^2(-t+\frac{1}{2})} \, dx_i$$

$$E[e^{X_i^2 t}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x_i^2(-t+1/2)} \, dx$$

$$\gamma(n) = \int_0^{\infty} x^{n-1} e^{-x} \, dx \qquad \gamma(n) = \begin{cases} (n-1)! \, ; & n \in \mathbb{Z}^+ \ (+ve) \\ (n-1)\,\gamma(n-1); & n \text{ is frac} \end{cases}$$

Let $X_i^2\left(-t+\frac{1}{2}\right) = p$

$$dp = 2x_i(-t+1/2)\, dx_i$$

$$dx_i = \frac{dp}{2(-t+1/2)\sqrt{\frac{p}{-t+1/2}}} \qquad x_i = \sqrt{\frac{p}{(-t+1/2)}}$$

$$dx_i = \frac{dp}{2\sqrt{p(-t+1/2)}} \qquad \left[\begin{array}{l} -t+1/2 > 0 \\ \Rightarrow t < \frac{1}{2} \end{array}\right]$$

$$E[e^{x_i^2 t}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-p} \frac{dp}{2\sqrt{p(-t+1/2)}}$$

$$= \frac{2}{\sqrt{2\pi}} \cdot \frac{1}{2\sqrt{t+1/2}} \int_0^{\infty} e^{-p} p^{1/2} \, dp$$

$$= \frac{1}{\sqrt{2\pi\left(\frac{1}{2}-t\right)}} \int_0^{\infty} p^{(1/2-1)} e^{-p} \, dp = \frac{1}{\sqrt{2\pi\left(\frac{1}{2}-t\right)}} \gamma(1/2)$$

$$= \frac{\sqrt{\pi}}{\sqrt{2\pi(1/2-t)}}$$

$$M_{X_i^2}(t) = \frac{1}{\sqrt{1-2t}}$$

for independent X_i's $\quad M_{S_N}(t) = E[e^{S_N t}] = E\left[e^{(x_1^2+x_2^2+\dots+x_n^2)t}\right]$

$$= E[e^{x_1^2 t}]\, E[e^{x_2^2 t}] \dots \dots E[e^{x_n^2 t}]$$

$$= M_{X_1^2}(t) \cdot M_{x_2^2}(t) \dots \dots M_{X_N^2}(t)$$

$$= \left(\frac{1}{\sqrt{1-2t}}\right)^N$$

$$M_{S_N}(t) = (1-2t)^{-N/2}$$ corresponds to MGF of Gamma $\left(\alpha = \frac{N}{2}, \ \theta = 2\right)$

Gamma PDF: $f_{S_N}(z) = \dfrac{1}{\gamma(\alpha)\,\theta^{\alpha}} z^{\alpha-1} e^{-z/\theta}, \quad z \geq 0$

for $\alpha = N/2$ and $\theta = 2$

$$f_{S_N}(z) = \frac{1}{\gamma(N/2)\, 2^{N/2}} z^{N/2-1} e^{-z/2}, \quad z \geq 0$$

Q7) Suppose $X_1, X_2 \ldots X_n$ are jointly Gaussian random variables with $\mathrm{Cov}(X_i, X_j) = 0$ for $i \neq j$. Show that $X_1, X_2 \ldots X_n$ are independent random variables.

Ans: Goal: $f_X(x_1, \ldots x_n) = f_{X_1}(x_1) \cdot f_{X_2}(x_2) \cdots f_{X_n}(x_n)$

Given: $f_X(x) = \dfrac{1}{(2\pi)^{n/2} |\Sigma|^{1/2}} e^{\underbrace{-\frac{1}{2}(x-u)^T \Sigma^{-1} (x-u)}_{\text{solving independently}}}$ — ①

$\Sigma_{ii} = \mathrm{Cov}(X_i, X_i) = \mathrm{Var}(X_i) = \sigma_i^2$

$\Sigma_{ij} = \mathrm{Cov}(X_i, X_j) = 0$ for $(i \neq j)$

$$\Sigma = \begin{bmatrix} \sigma_1^2 & 0 & 0 & \cdots \\ 0 & \sigma_2^2 & & \cdots \\ 0 & & \sigma_3^2 & \\ \vdots & & & \ddots \\ 0 & \cdots & & \sigma_n^2 \end{bmatrix}$$

$|\Sigma| = \sigma_1^2 \cdot \sigma_2^2 \cdots \sigma_n^2 = \prod_{i=1}^{n} \sigma_i^2$

$|\Sigma|^{1/2} = \prod_{i=1}^{n} \sigma_i$

$$\Sigma^{-1} = \begin{bmatrix} 1/\sigma_1^2 & 0 & \cdots & 0 \\ 0 & 1/\sigma_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1/\sigma_n^2 \end{bmatrix}$$

$\therefore = (x-u)^T \Sigma^{-1} (x-u)$

$$= \begin{bmatrix} x_1 - u_1, \ldots & x_n - u_n \end{bmatrix} \begin{bmatrix} 1/\sigma_1^2 & \cdots & 0 \\ \vdots & \ddots & \\ 0 & \cdots & 1/\sigma_n^2 \end{bmatrix} \begin{pmatrix} x_1 - u_1 \\ \vdots \\ x_n - u_n \end{pmatrix}$$

$$= \frac{(x_1 - u_1)^2}{\sigma_1^2} + \frac{(x_2 - u_2)^2}{\sigma_2^2} + \cdots + \frac{(x_n - u_n)^2}{\sigma_n^2} = \sum_{i=1}^{n} \frac{(x_i - u_i)^2}{\sigma_i^2}$$

Putting each values in ①

$$f_X(x) = \frac{1}{(2\pi)^{n/2} \prod_{i=1}^{n} \sigma_i} e^{-\frac{1}{2}\sum_{i=1}^{n} \frac{(x_i - u_i)^2}{\sigma_i^2}}$$

$$f_X(x) = \left(\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\,\sigma_i}\right) \cdot \left(\prod_{i=1}^{n} e^{-\frac{(x_i - \mu_i)^2}{2\sigma_i^2}}\right)$$

$$f_X(x) = \prod_{i=1}^{n} \left[\frac{1}{\sqrt{2\pi}\,\sigma_i} e^{-\left(\frac{x_i - \mu_i}{2\sigma_i^2}\right)^2} \right]$$

Marginal PDF $f_{X_i}(x_i)$ for $X_i \sim N(\mu_i, \sigma_i^2)$

Hence $f_X(x) = \prod_{i=1}^{n} f_{X_i}(x_i)$

Q12) II) Let $X \sim N(0,1)$ put $Y = 3X$

(a) Show that X and Y are jointly gaussian.

Ans: Every linear combination of components must be scalar Gaussian variable.

Let K be general linear combination

$K = c_1 X + c_2 Y$ $\quad c_1, c_2$ constant

$Y = 3X$

$K = c_1 X + c_2 (3X)$

$= c_1 X + 3c_2 X$

$= (c_1 + 3c_2) X$

Let $m = c_1 + 3c_2$ $\quad m$ is constant

$K = mX$

Given $X \sim N(0,1)$ Mean: $E[K] = E[mX] = m\,E[X] = m \cdot 0 = 0$

$Var(K) = Var(mX) = m^2\, Var(X) = m^2 \cdot 1 = \underline{\underline{m^2}}$

Hence $K \sim N(0, m^2)$ K (Any linear combination) of X and Y is scalar Gaussian variable

X and $Y \rightarrow$ jointly ~~distribution~~ Gaussian.

(b) Find covariance matrix $Cov([X,Y]')$

$V = [X,Y]^T$

$$\Sigma = \begin{pmatrix} Var(X) & Cov(X,Y) \\ Cov(Y,X) & Var(Y) \end{pmatrix}$$

$Var(X) = 1$ since $X \sim N(0,1)$

$Var(Y) = 3X$

$Var(Y) = Var(3X) = 3^2 \cdot Var(X) = 9 \cdot 1 = 9$

$Var(Y) = 9$

$Cov(X,Y) = E[XY] - E[X]E[Y]$

$E[X] = 0 \quad E[Y] = E[3X] = 3E[X] = 3 \cdot 0 = 0$

$E[XY] = E[X(3X)] = E[3X^2] = 3 \cdot E[X^2]$

$Var(X) = E[X^2] - (E[X])^2$

$1 = E[X^2] - (0)^2$

$E[X^2] = 1$

$E[XY] = 3 \cdot 1 = 3$

$Cov(X,Y) = E[XY] = 3$

$Cov(Y,X) = Cov(X,Y) = 3$

$$\Sigma = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$$

(c) Show that they are not jointly continuous.

cdf $F_{Y|X}(y|x) = P(Y \le y \mid X = x)$

$Y = 3X$

$F_{Y|X}(y|x) = P(3X \le y \mid X = x)$

$F_{Y|X}(y|x) = P(3x \le y \mid X = x)$

If $y \geq 3x$ then $3x \le y$ is T probability $= 1$

If $y < 3x$ $3x \le y$ is F probability $= 0$

$$F_{Y|X}(y|x) = \begin{cases} 0, & y < 3x \\ 1 & y \geq 3x \end{cases}$$

$\therefore$ conditional density is derivative of cdf w.r.t y:

$$f_{Y|X}(y|x) = \frac{d}{dy} F_{Y|X}(y|x)$$

$$= \delta(y - 3x)$$ ← dirac delta function (impulse)

A) A distribution is "jointly continuous" if it has a joint PDF $f_{X,Y}(x,y)$ has regular func.

joint pdf $f_{X,Y}(x,y) = f_{Y|X}(y|x) f_X(x)$

$f_{Y|X}(y|x)$ is an impulse, the joint PDF $f_{X,Y}(x,y)$ is singular and contains delta function.

Probability mass concentrated on $y=3x$ and not on 2D area.

Therefore, X and Y are not jointly continuous.

This is degenerate jointly Gaussian distribution, occurs when covariance matrix is singular (determinant is 0).

$$\det(\Sigma) = (1)(9) - (3)(3) = 0.$$

I) Use MATLAB code to plot the $N(0, 1/n^2)$ density

PTO for answer

Q8) Let $W = \max(X_1, X_2, \dots X_n)$ and $Z = \min(X_1, X_2, \dots X_n)$ where X_i are independent random variables with the same distrib. Find $F_W(w)$ and $F_Z(z)$

Ans: CDF: $F_X(x) = P(X_i \leq x)$

$F_W(w) = P(W \leq w)$ Since $\max^m \leq w$, then all are $\leq w$

$F_W(w) = P(X_1 \leq w$ and $X_2 \leq w \dots$ and $X_n \leq w)$

Also it is given that X_i are independent

$F_W(w) = P(X_1 \leq w) \cdot P(X_2 \leq w) \dots P(X_n \leq w)$

Since, all have identical distribution

$F_W(w) = [F_X(w)]^n$

CDF: $P(Z \leq z) = 1 - P(Z > z)$

$P(Z > z) = P(\min(X_1 \dots X_n) > z)$

Smallest $> z$ then all are $> z$

$P(Z > z) = P(X_1 > z$ and $X_2 > z \dots X_n > z)$

all X_i are independent

$P(Z > z) = P(X_1 > z) \cdot P(X_2 > z) \dots P(X_n > z)$

and $P(X_i > z) = 1 - P(X_i \leq z) \Rightarrow 1 - F_X(z)$

$P(Z > z) = (1 - F_X(z))(1 - F_X(z)) \dots (1 - F_X(z))$

$= [1 - F_X(z)]^n$

$F_Z(z) = P(Z \leq z) = 1 - P(Z > z)$

$= 1 - [1 - F_X(z)]^n$

Q10) Give random vector X, find the joint pdf

$z_1 = g_1(X_1) = a_1 X_1 + b_1$

$z_2 = g_2(X_2) = a_2 X_2 + b_2$

$\vdots$

$z_n = g_n(X_n) = a_n X_n + b_n$

Let X_1 & X_2 be i. exponential variables, λ.

joint density func of $Y_1 = X_1 + X_2$ $Y_2 = X_1/X_2$ and show independence.

Ans: $x_i = \frac{z_i - b_i}{a_i}$ joint cdf of $z = (z_1, z_2, \ldots, z_n)$

$= F_z(z_1, z_2, \ldots, z_n) = P(Z_1 \le z_1, Z_2 \le z_2, \ldots, Z_n \le z_n)$

If $a_i > 0 \to X_i \le \frac{z_i - b_i}{a_i}$

$$F_z(z_1, z_2, \ldots, z_n) = P\left(X_1 \le \frac{z_1 - b_1}{a_1}, X_2 \le \frac{z_2 - b_2}{a_2}, \ldots, X_n \le \frac{z_n - b_n}{a_n}\right)$$

$$= P(X_1 \le x_1, X_2 \le x_2, \ldots X_n \le x_n)$$

$$= F_X(x_1, x_2, \ldots, x_n)$$

joint pdf of $z - f_z(z_1 \ldots z_n) = \frac{\delta^n F_z}{\delta z_1 \delta z_2 \ldots \delta z_n}$

$$= \frac{\delta F_X}{\delta x_i} \frac{\delta}{\delta z_i}\left(\frac{z_i - b_i}{a_i}\right) = \frac{\delta F_X}{\delta x_i} \frac{1}{a_i}$$

$$f_z(z_1, \ldots, z_n) = \frac{\delta^n F_X}{\delta x_1 \delta x_2 \ldots \delta x_n} \frac{1}{a_1} \frac{1}{a_2} \ldots \frac{1}{a_n}$$

$$= f_X(x_1, x_2, \ldots x_n) \frac{1}{\pi a_i}$$

(b) $Y_1 = X_1 + X_2$, $Y_2 = X_1/X_2$

$$X_1 = \begin{cases} \lambda e^{-\lambda x_1} & ; x_1 \ge 0 \\ 0 & ; x_1 < 0 \end{cases} \qquad X_2 = \begin{cases} \lambda e^{-\lambda x_2} & ; x_2 \ge 0 \\ 0 & x_2 < 0 \end{cases}$$

$$Y_1 = \lambda(e^{-\lambda x_1} + e^{-\lambda x_2}) \qquad Y_2 = \frac{\lambda e^{-\lambda x_1}}{\lambda e^{-\lambda x_2}} = e^{-\lambda(x_1 - x_2)}$$

joint pdf of X_1 & X_2 $= f_{X_1,X_2}(x_1, x_2) = \lambda^2 e^{-\lambda(x_1 + x_2)}$

joint cdf of Y_1 & $Y_2 \Rightarrow F_{Y_1,Y_2}(y_1, y_2) = P\left(X_1 + X_2 \le y_1, \frac{X_1}{X_2} \le y_2\right)$

$$F_{Y_1,Y_2}(y_1, y_2) = \int_{-\infty}^{\infty} \int_{x_1/y_2}^{y_1 - x_1} \lambda^2 e^{-\lambda(x_1 + x_2)}\, dx_1\, dx_2$$

$$= \int_{x_1} \underbrace{\left(\int_{x_1/y_2}^{y_1 - x_1} \lambda^2 e^{-\lambda x_1} e^{-\lambda x_2} \, dx_2 \right)}_{\text{inner}} dx_1$$

Solving inner: $\lambda^2 e^{-\lambda x_1} \left[\frac{e^{-\lambda x_2}}{-\lambda} \right]_{x_1/y_2}^{y_1 - x_1}$

$$= \lambda e^{-\lambda x_1} \left[e^{-\lambda (x_1/y_2)} - e^{-\lambda (y_1 - x_1)} \right]$$

$$= \lambda e^{-\lambda x_1} \left[e^{-\lambda (x_1/y_2)} - e^{-\lambda y_1} e^{\lambda x_1} \right]$$

$$F_{Y_1 Y_2} = \int_0^{\frac{y_1 y_2}{1+y_2}} \lambda \left(e^{-\lambda x_1 \left(\frac{1+y_2}{y_2} \right)} - e^{-\lambda y_1} \right) dx_1$$

$$= \lambda \left[\frac{e^{-\lambda x_1 \frac{1+y_2}{y_2}}}{-\lambda \left(\frac{1+y_2}{y_2} \right)} - x_1 e^{-\lambda y_1} \right]_0^{\frac{y_1 y_2}{1+y_2}}$$

$$F_{Y_1 Y_2} = \frac{y_2}{1+y_2} \left(-e^{-\lambda y_1} - \lambda y_1 e^{-\lambda y_1} + 1 \right)$$

joint pdf of Y_1 & Y_2 $= \frac{\partial^2 F_{Y_1 Y_2}}{\partial y_1 \partial y_2}$

$$\frac{\partial F_Y}{\partial y_1} = \frac{y_2}{1+y_2} \frac{\partial}{\partial y_1} \left(1 - e^{-\lambda y_1} - \lambda y_1 e^{-\lambda y_1} \right)$$

$$= \frac{y_2}{1+y_2} \left(\lambda^2 y_1 e^{-\lambda y_1} \right)$$

$$f_{Y_1 Y_2} = \left(\lambda^2 y_1 e^{-\lambda y_1} \right) \left[\frac{1(1+y_2) - y_2(1)}{(1+y_2)^2} \right]$$

$$= \frac{\lambda^2 y_1 e^{-\lambda y_1}}{(1+y_2)^2}$$

$f_{Y_1 Y_2}$ can be split into $\left(\lambda^2 y_1 e^{-\lambda y_1} \right) \times \left(\frac{1}{(1+y_2)^2} \right)$

1st term depends on y_1 & 2nd on y_2.

These terms represent marginal density.

Q12

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```matlab
%Aayank Singhai

clear all;
close all;
clc;

domain = linspace(-4, 4, 1000);
varList = [1, 0.25, 1/9, 1/16];
numCurves = numel(varList);

pdfData = zeros(numCurves, numel(domain));

gaussianPDF = @(x_vec, v) (1./sqrt(2*pi*v)) .* exp(-(x_vec.^2)./(2*v));

integralAreas = zeros(numCurves, 1);
for i = 1:numCurves
    currentVar = varList(i);
    pdfData(i, :) = gaussianPDF(domain, currentVar);
    integralAreas(i) = trapz(domain, pdfData(i, :));
end

disp('--- Numerical Integration Check ---');
disp('Variance       | Area');
disp('--------------------------------');
for i = 1:numCurves
    fprintf('Var: %-7.4f | Area: %-7.6f\n', varList(i), integralAreas(i));
end

figure(1);
set(gcf, 'Color', 'w', 'Position', [100 100 900 450]);
colorMap = jet(numCurves);

hold on;
for i = 1:numCurves
    plot(domain, pdfData(i, :), 'LineWidth', 2, 'Color', colorMap(i,:));
end
hold off;

legendEntries = arrayfun(@(v) sprintf('\\sigma^2 = %.3f', v), varList, 'UniformOutput', fal
legend(legendEntries, 'Location', 'northeast');
xlabel('x');
ylabel('PDF f(x)');
```

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```
legendEntries = arrayfun(@(v) sprintf('\\sigma^2 = %.3f', v), varList, 'UniformOutput', fal
legend(legendEntries, 'Location', 'northeast');
xlabel('x');
ylabel('PDF f(x)');
title('Gaussian PDF Comparison as Variance Decreases');
grid on;
box on;
set(gca,'FontSize',12);

figure(2);
set(gcf, 'Color', 'w', 'Position', [200 200 700 450]);

animVarList = [linspace(1, 0.1, 15), 0.08, 0.05, 0.02, 0.01, 0.005];

for v = animVarList
    y_anim = gaussianPDF(domain, v);

    plot(domain, y_anim, 'LineWidth', 2.5, 'Color', 'r');

    ylim([0 max(pdfData(:))*1.1]);
    xlim([domain(1) domain(end)]);

    title(sprintf('Approaching the Impulse: \\sigma^2 = %.4f', v), 'FontSize', 14);
    xlabel('x');
    ylabel('PDF f(x)');
    grid on;

    drawnow;
    pause(0.25);
end

figure(3);
set(gcf, 'Color', 'w', 'Position', [300 300 700 450]);

plot(domain, pdfData(1,:), 'LineWidth', 1.5, 'Color', [0.7 0.7 0.7]);
hold on;
for i = 2:numCurves
    plot(domain, pdfData(i,:), 'LineWidth', 2, 'Color', colorMap(i,:));
end
hold off;

xlim([-0.5 0.5]);
```

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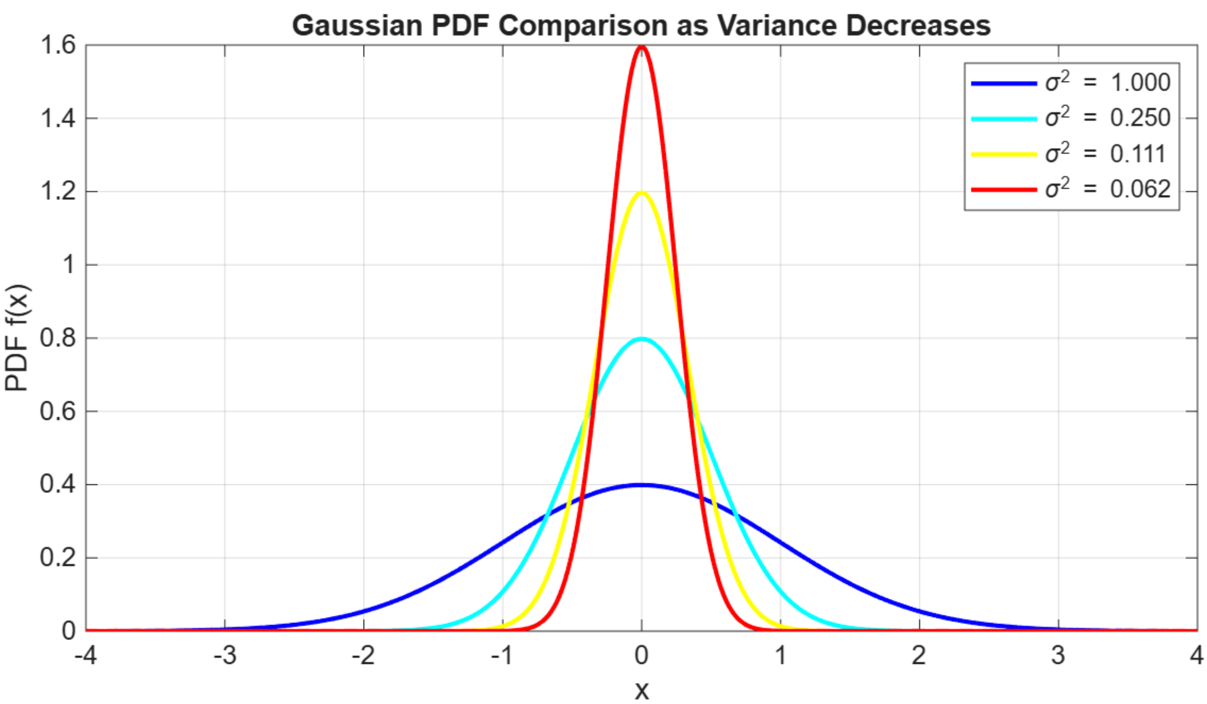
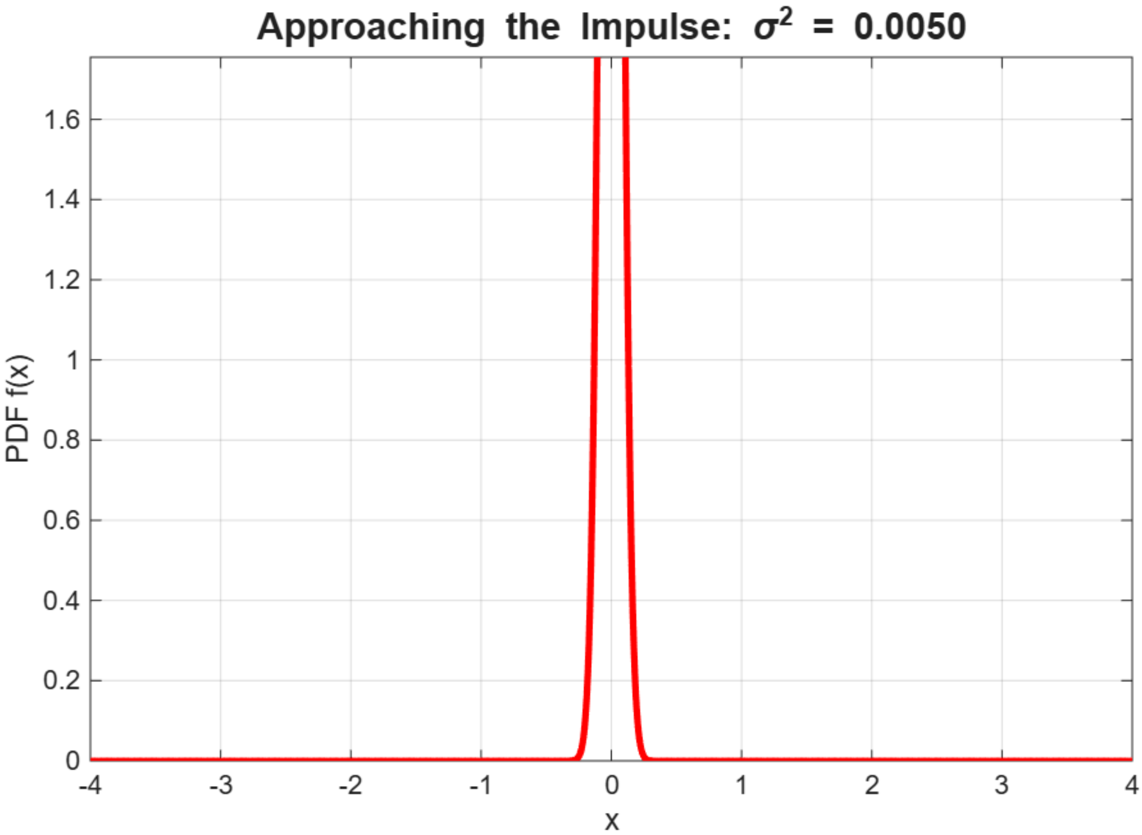
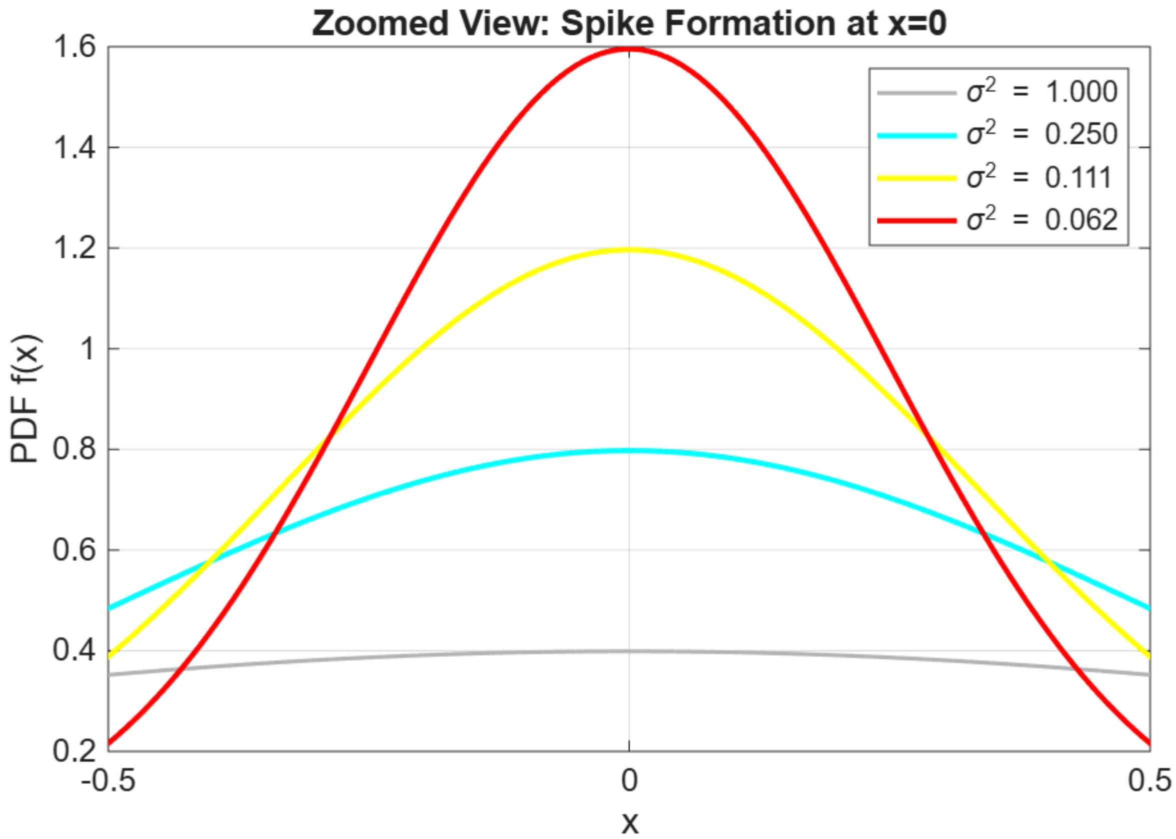
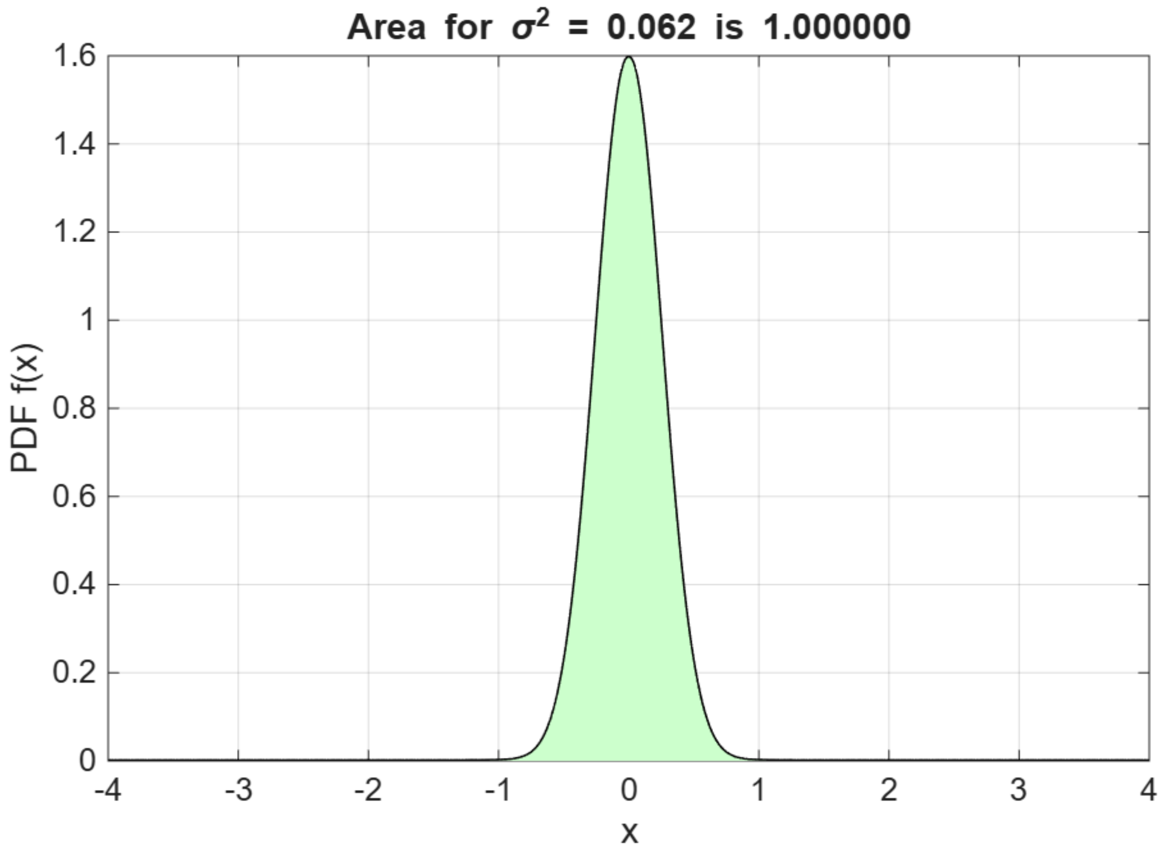
```
for i = 2:numCurves
end
hold off;

xlim([-0.5 0.5]);
legend(legendEntries, 'Location', 'northeast');
xlabel('x');
ylabel('PDF f(x)');
title('Zoomed View: Spike Formation at x=0');
grid on;
box on;
set(gca,'FontSize',12);

figure(4);
set(gcf, 'Color', 'w', 'Position', [400 400 700 450]);

smallestVarIdx = numCurves;
y_smallest = pdfData(smallestVarIdx, :);

plot(domain, y_smallest, 'k', 'LineWidth', 1.5);
hold on;

fill(domain, y_smallest, [0.8 1 0.8], 'EdgeColor', 'none');

xlabel('x');
ylabel('PDF f(x)');
title(sprintf('Area for \\sigma^2 = %.3f is %.6f', ...
    varList(smallestVarIdx), integralAreas(smallestVarIdx)), 'FontSize', 12);
grid on;
set(gca,'FontSize',12);
```




Approaching the Impulse: σ^2 = 0.0050
PDF f(x)
x
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-4
-3
-2
-1
0
1
2
3
4


Zoomed View: Spike Formation at x=0
σ2 = 1.000
σ2 = 0.250
σ2 = 0.111
σ2 = 0.062
PDF f(x)
x
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
-0.5
0
0.5


Area for σ2 = 0.062 is 1.000000
PDF f(x)
x
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-4
-3
-2
-1
0
1
2
3
4