Two Dimensional Guillotine Strip Packing

immediate

Abstract

To-Do [8]

1 Introduction

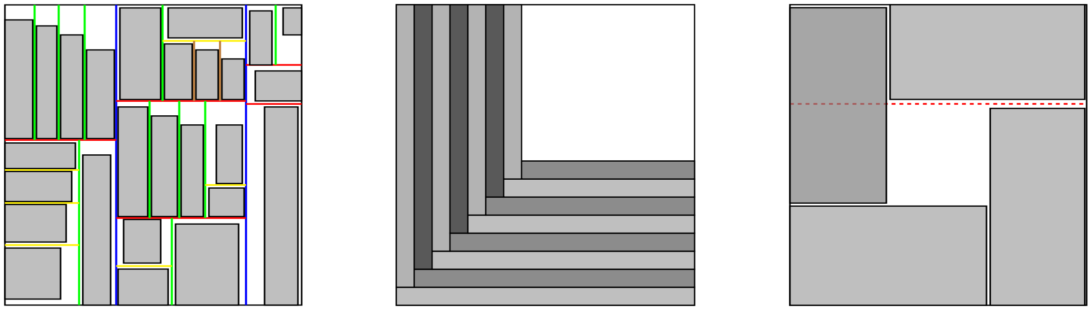


Figure 1: The first packing is a 5-stage guillotine separable packing and the second packing is a $n-1$-stage guillotine separable packing. The third packing is not guillotine separable as any end-to-end cut in the strip intersects a rectangle in the packing

1.1 Prior Work

1.2 Our Contribution

In this paper, we obtain tight algorithms for the 2-dimensional guillotine strip packing in both the pseudo-polynomial time and the polynomial time domains. Specifically, our first result is a $(1+\varepsilon)$ approximation algorithm with pseudo-polynomial running time. Note that this result shows a significant gap in the pseudo-polynomial running time domain between general strip packing and guillotine strip packing since general strip packing has a $\frac{5}{4}$ approximation hardness for any pseudo-polynomial time algorithm [6]. We also obtain algorithms with approximation guarantees $\frac{3}{2}+\varepsilon$ and $\frac{5}{3}+\varepsilon$ with polynomial running time. The $\frac{3}{2}+\varepsilon$ approximation algorithm is tight due to the $\frac{3}{2}$ approximation hardness of 2-dimensional guillotine strip packing which is due to a reduction from the Partition problem (see Theorem 36). Note that this is the first time that the barrier of approximation $\frac{5}{3}$ has been broken for a variant of strip packing (In [5], the authors obtained a $\frac{5}{3}+\varepsilon$ algorithm for general strip packing and in [2, 3] the authors achieved a $\frac{5}{3}+\varepsilon$ algorithm for Sliced strip packing).

In all three results, we exploit a container based packing using guillotine cuts. For the PPTAS, we make use of the crucial property of guillotine cuts to obtain a solution such that all items are packed in constant number of L-compartments (just as in [8]). Within these L-compartments, we ensure that vertical items cannot be packed in the horizontal arms of the L and the horizontal items cannot be packed in the vertical arms of the Ls. Further, using additional $O(\varepsilon)$ fraction of the total height, we convert the L compartments into constant box compartments. Then using resource augmentation we get a nice packing for the horizontal box compartments and for the vertical compartments, we first round the vertical rectangle heights to a multiple of δ^2 and using guillotine cuts show that the vertical rectangles are nicely packed in $O_\delta(1)$ containers. Finally we guess the constant containers and pack the items using an algorithm for the Generalized Assignment problem (GAP) which runs in pseudo-polynomial time.

For the $\frac{5}{3}+\varepsilon$ approximation, we have from the PPTAS that there exists a nice packing of all items in constant containers. But since GAP cannot be solved optimally in polynomial time, we pack the tall items of height greater than $\frac{1}{2}OPT$ separately and the rest in containers. Hence, we require to pack a strip S_1 of width $\varepsilon_3 W$ and height $\frac{1}{2}OPT$ which accounts for the unpacked items in the polynomial time algorithm for GAP and the rounding of containers. First, we show using guillotine cuts that there exists a packing in which all the tall containers can be packed touching the bottom of the strip. We then push all the items/containers completely lying in the upper half of our strip vertically upwards by $\frac{1}{2}OPT$ and then construct constant number of strips in our packing by extending vertical lines from the x co-ordinates of both corners of the containers in our packing and make two cases based on whether or not there is a wide enough strip whose bottom lies below $\frac{2}{3}OPT$. In the first case when indeed such a strip exists we show that by pushing the items which were packed above OPT further upwards by $\frac{1}{3}OPT$, we can pack S_1 in one of the strips constructed and the overall height is at most $(\frac{5}{3} + O(\varepsilon))OPT$. In the other case, we have that all wide strips have their bottoms lying above $\frac{2}{3}OPT$ and hence the area of items which are packed above OPT is at most $(\frac{1}{3} + O(\varepsilon))OPT \cdot W$. We then show that by virtue of guillotine cuts, items with width at least $\frac{1}{2}W$ which were intially packed above OPT can be packed within OPT such that the tall items are sorted within themselves. This is a novel packing in which the non-tall pseudo items partition the tall items but the tall items are still packed in a sorted order. For the rest of the items above OPT, we pack them along with S_1 using Steinberg's algorithm to again get a final packing of height at most $(\frac{5}{3} + O(\varepsilon))OPT$.

For the $\frac{3}{2} + \varepsilon$ approximation, we show that we can find a wide strip which has the lowest bottom height even after rearranging the tall items. Then using a non-uniform shifting of items, we show that we finally get a packing in which the tall items are packed in a sorted order, we pack a box of sufficient width and height $\frac{1}{2}OPT$ at the guessed lowest bottom strip height in which we again have a container based nice packing of items along with the strip S_1.

1.3 Temp:Macros

Symbol	Macro
μ	$\backslash meu$
δ	$\backslash dlta$
Input item set I	$\backslash R$
Optimal height OPT	$\backslash opt$
Bin Width W	$\backslash W$
Item Set I_{large}, I_{medium}, I_{ver}, I_{small}, I_{hor} I_{skew}	$\backslash Rla$, $\backslash Rme$, $\backslash Rve$, $\backslash Rsm$, $\backslash Rho$, $\backslash Rsk$
(Box) **B**-, (L) **L**-, (Vertical) **V**-, (Horizontal) **H**- Compartment ($\mathcal{C}$)	$\backslash Bc$, $\backslash Lc$, $\backslash Vc$, $\backslash Hc$, $\backslash C$
Bin $\mathcal{B}$	$\backslash B$

• Changed the notation to Large instead of Big!!!!!

2 A Pseudo-polynomial time approximation scheme

2.1 Preliminaries

2.1.1 Classification Of Items

We classify the input items according to their heights and widths. For two constants $1 \geq \delta > \mu > 0$ to be defined later, we classify each item $i \in I$ as:

- *Large*: $w_i > \delta W$ and $h_i > \delta OPT$;
- *Horizontal*: $w_i > \delta W$ and $h_i \leq \mu OPT$;
- *Vertical*: $w_i \leq \delta W$ and $h_i > \delta OPT$;
- *Medium:*
  - Either $\delta OPT \geq h_i > \mu OPT$;
  - Or $\delta W \geq w_i > \mu W$ and $h_i \leq \mu OPT$.
- *Small*: $w_i \leq \mu W$ and $h_i \leq \mu OPT$;

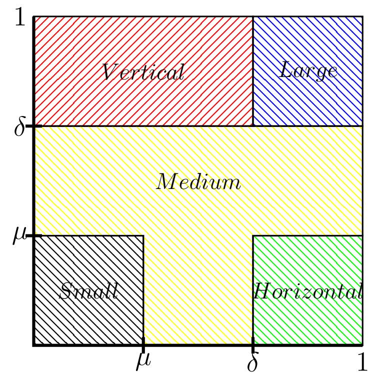


Figure 2: Item Classification. x-axis represents width scaled to 1 (by normalizing w.r.t. W). y- axis represents height scaled to 1 (by normalizing by OPT)

Let $I_{large},\ I_{hor},\ I_{ver},\ I_{medium},\ I_{small}$ be the set of large, horizontal, vertical, medium and small rectangles, respectively. We also define the set of skewed items as $I_{skew} = \ I_{hor} \cup I_{ver}$.

• redefined skew items according to the given classification

Using standard shifting arguments, one can show that medium items occupy marginal area, allowing us to ignore them initally and packing them in the end.

Lemma 1. *[9] Let $\varepsilon > 0$ and $f(.)$ be any positive increasing function such that $f(x) < x\ \forall x \in (0, 1]$. Then we can efficiently find $\delta, \mu \in \Omega_\varepsilon(1)$, with $\varepsilon \geq f(\varepsilon) \geq \delta \geq f(\delta) \geq \mu$ so that the total area of medium rectangles is at most $\varepsilon(OPT \cdot W)$.*

• can we directly refer to [4] for this??

2.1.2 Compartments

Our goal is obtain a bin that can be partitioned into compartments, such that the bin corresponds to an $(1+\varepsilon)$-approximate solution with all the items are placed in a structured way inside these compartments. We will use four types of compartments: *box (**B**)-compartments*, **L**-*compartments*, $Horizontal$(**H**)-*compartments* and *Vertical (**V**)-compartments*.

Definition 1 (**B**-compartment)**.** *A* **B**-compartment *B is an axis-aligned rectangle that satisfies $B \subseteq K$.*

Definition 2. *[**L**-compartment] An* **L**-*compartment L is a subregion of K bounded by a simple rectilinear polygon with six edges $e_0, e_1, \ldots, e_5$ such that for each pair of horizontal (resp. vertical) edges e_i, e_{6-i} with $i \in \{1, 2\}$ there exists a vertical (resp. horizontal) line segment ℓ_i of length less than $\delta\frac{W}{2}$ such that both e_i and e_{6-i} intersect ℓ_i but no other edges intersect ℓ_i.*

• why was it $\delta W/2$ in SoCG?

Definition 3 (**H**-compartment)**.** *An* **H**-*compartment is a box compartment with height less than $\delta\frac{OPT}{2}$. In other words, it is a degenerate case of* **L**-*compartment with the horizontal line segment between pair of vertical edges e_i, e_{6-i} with $i \in \{1, 2\}$; has length zero.*

Definition 4 (**V**-compartment)**.** *An* **V**-*compartment is a box compartment with width less than $\delta\frac{W}{2}$. In other words, is a degenerate case of* **L**-*compartment with the vertical line segment between pair of horizontal edges e_i, e_{6-i} with $i \in \{1, 2\}$; has length zero.*

Since the length of the line segments ℓ_i is less than $\delta W/2$, this implies that inside an **L**-compartment L we cannot place large items, inside the horizontal part of L we cannot place vertical items, and inside the vertical part of L we cannot place horizontal items. Similarly, a **H**- (resp. **V**-) compartment cannot contain large and vertical (resp. horizontal) items.

We seek for a structured packing inside of these compartments according to the following definitions. Inside box-compartments, we want only one type of items and we want that the skewed items are placed in a very simple way, see Figure 3.

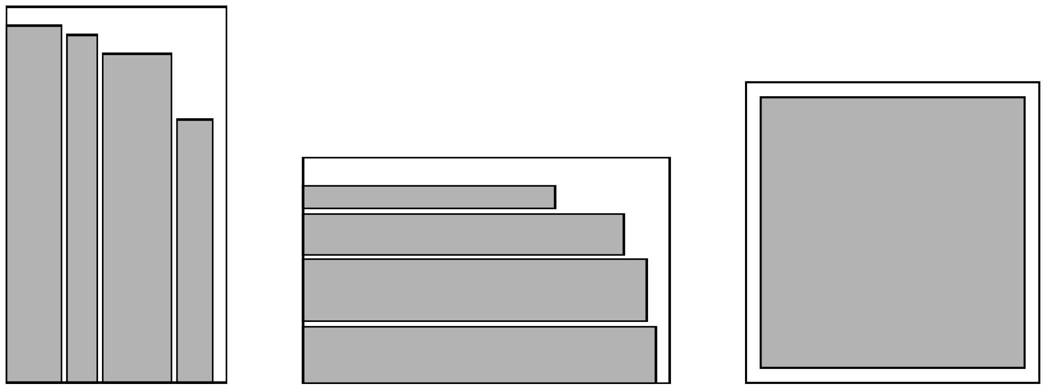


Figure 3: Nice packing of items in vertical, horizontal and large containers respectively

Definition 5 (Nice Packing: **B**). *Let B be a* **B***-compartment and let $I_B \subseteq I$ be a set of items that are placed non-overlappingly inside B. We say that the placement of I_B is* nice *if the items in I_B are guillotine separable and additionally*

- *I_B contains only one item, or*
- *$I_B \subseteq I_{hor}$ and the items in I_B are stacked on top of each other inside B, or*
- *$I_B \subseteq I_{ver}$ and the items in I_B are placed side by side inside B, or*
- *$I_B \subseteq I_{small}$ and for each item $i \in I_B$ it holds that $w_i \leq \varepsilon \cdot w(B)$ and $h_i \leq \varepsilon \cdot h(B)$*

Inside **L**-compartments we allow only horizontal and vertical items and we want them to be placed in a similar way as in the boxes, see Figure 4 a.

Definition 6 (Nice Packing: **L**). *Let L be an* **L***-compartment and let $I_L \subseteq I$ be a set of items that are placed non-overlappingly inside L. We say that the placement of I_L is* nice *if*

- *$I_L \subseteq I_{hor} \cup I_{ver}$, and*
- *the items in $I_L \cap I_{hor}$ are stacked on top of each other inside L, and*
- *the items in $I_L \cap I_{ver}$ are stacked side by side inside L.*

Similar definition follows for **V** and **H** since they are degenerate cases of **L**.

Definition 7 (Nice Packing: **V** and **H**). *Let V, H be two* **V***- and* **H***-compartments respectively and let $I_V, I_H \subseteq I$ be a set of items that are placed non-overlappingly inside V and H respetively. We say that the placement of I_V and I_H is* nice *if*

- *$I_L \subseteq I_{hor} \cup I_{ver}$,*
- *the items in $I_V \cap I_{ver}$ are stacked side by side inside V and*
- *the items in $I_H \cap I_{hor}$ are stacked on top of each other inside H.*

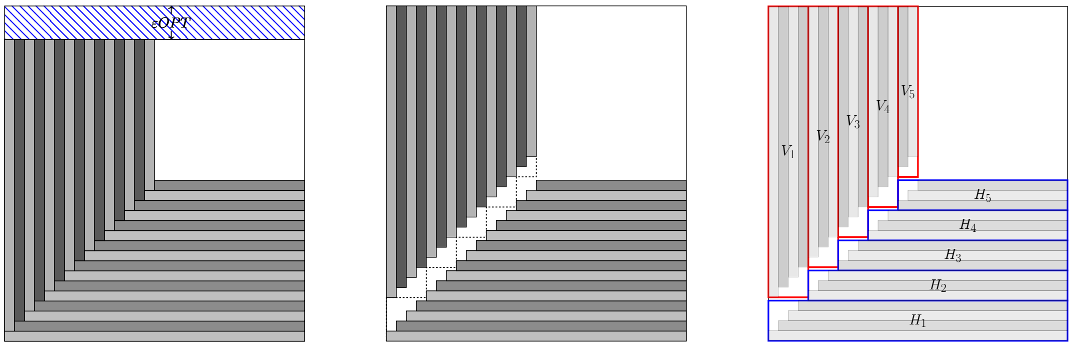
2.2 Structural Lemma

Theorem 2. *Given a set of items I packed in a $W \times OPT$ bin, there exists a bin of $\mathcal{B}'$ of size $W \times (1 + O(\varepsilon))OPT$ packing all items from I into a set of $O_\varepsilon(1)$ compartments $\mathcal{C}$ such that:*

- *the compartments $\mathcal{C}$ admit a pseudo-guillotine cutting sequence,*
- *the items in I can be placed nicely inside the compartments $\mathcal{C}$ in bin $\mathcal{B}'$,*
- *each compartment is either* **V**, **H** *or* **B**.

• naming it as a theorem

• should we mention that we only have rectangular i.e. vertical, horizontal or box compartments only?



(a) (b) (c)

Figure 4: Using an extra εOPT height, we convert a packing of items I in an **L** compartment into another packing such that the items in I are packed in compartments $\mathcal{C}' = \mathbf{V} \cup \mathbf{H}$ which are guillotine separable and $|\mathcal{C}'| = O_\varepsilon(1)$, where $\mathbf{V} = \cup_{i=1}^{i=5} V_i$ and $\mathbf{H} = \cup_{i=1}^{i=5} H_i$

To achieve results in Theorem 2, we consider the given set of times in optimal height bin and then apply a set of procedures to pack the items into the bin $\mathcal{B}'$ of size $W \times (1 + O(\varepsilon))OPT$ ensuring all the required properties. We begin by considering only the set of skewed items and large items $I_{skew} \cup I_{large}$ removing the rest of the small and medium items. We will pack the items removed in later steps in Subsection 2.5.

Lemma 3. *[8] Given a set of skew items I_{skew} packed in a $W \times OPT$ bin $\mathcal{B}$, there exists a packing all items from I_{skew} into bin $\mathcal{B}$ which can be divided into set of $O_\varepsilon(1)$ compartments $\mathcal{C}$ such that:*

- *the compartments $\mathcal{C}$ admit a pseudo-guillotine cutting sequence,*
- *each compartment is either* **L** *or a* **B**.

Once we obtain the packing of skewed items in boxes and **L** containers, we convert the packing to a nice packing of only vertical, horizontal and box containers using an additional $O(\varepsilon)OPT$ height.

Lemma 4. *Given a set of skew items I_{skew} packed in a $W \times OPT$ bin $\mathcal{B}$ which can be divided into set $\mathcal{C}$ of $O_\varepsilon(1)$* **L** *and* **B** *compartments, then there exists a packing all items from I_{skew} into bin $\mathcal{B}'$ of size $W \times (1 + O(\varepsilon))OPT$, which can be divided into set of $O_\varepsilon(1)$ compartments $\mathcal{C}'$ such that:*

- *all the items in $\mathcal{B}'$ are guillotine separable,*
- *the items in I_{skew} can be placed inside the compartments $\mathcal{C}'$ in bin $\mathcal{B}$.*
- *each compartment is either* **V**, **H** *or a* **B**.
- *compartments are pseudo guillotine separable.*

Proof. The proof consists of two steps. The first step is to process the individual **L** compartments to obtain $O_\varepsilon(1)$ **V** compartments. Intuitively we do this by shifting the vertical leg of the **L** by $O(\varepsilon)OPT$ amount vertically and allowing division into a constant number of new **V** compartments. In the second step, we obtain packing of items in $O_\varepsilon(1)$ **H**-compartments.

We now explain the process in detail by considering an **L**-compartment L defined by a simple rectilinear polygon with six edges $e_0, e_1, \ldots, e_5$ as given in Definition 2. Let the vertical and horizontal legs be L_V and L_H respectively. W.l.o.g we can assume that the horizontal leg lies on the bottom right of the vertical (other cases can be handled analogously). Let p_1 be the bottommost point where L_V and L_H intersect. Let e_5 (resp. e_4) be the shorter vertical (resp. horizontal) edge of L_V (resp. L_H). Let the height and width of L compartment be $h(L),\ w(L)$ respectively.

We first shift each item in L_V vertically by an $\varepsilon h(L)$ amount, see Figure 4. We then create the boundary curve with $\frac{2}{\varepsilon}$ bends. We begin at the p_1, i.e., the leftmost bottommost point of the L compartment.

Using a ray shooting argument, continue drawing in the horizontal direction till the ray hits an horizontal item. Then we bend the ray and move it upwards till it hits one of the vertical items. Recursively repeating this process till the ray hits one of the bounding edges e_5 or e_4. Let this point be p_2. The trace of the ray defines the boundary curve $C_{H,V}$ between L_H and L_V, starting at p_1 and ending at p_2. Now we argue that the number of bends in the given boundary curve $C_{H,V}$ is $\frac{4}{\varepsilon}$. It is clear that the number of bends in $C_{H,V}$ will be twice as much as the number of distinct vertical paths in the curve $C_{H,V}$. Since each vertical item is shifted vertically by an $\varepsilon h(L)$ height initially, each such vertical path will be at least $\varepsilon h(L)$ in length. This establishes that the number of such distinct vertical paths can be at most $\frac{1}{\varepsilon}$ which bounds the number of bends by at most $2\frac{1}{\varepsilon} + 2 \leq \frac{4}{\varepsilon}$ bends.

We then further create the vertical compartments by first extending the projections from bends of boundary curve $C_{H,V}$ in vertical direction. If the vertical projection does not intersect with any of the vertical items, this can be considered as a guillotine cut and used to separate the items. If any such vertical projection intersects any item in L_V, we cannot divide the vertical leg using this line. Instead, we exploit the fact that vertical leg has first stage of guillotine cuts separating them since the **L** considered is pseudo-guillotine separable. Now, we instead consider the two nearest guillotine cuts, one on the immediate left and other on the immediate right of the vertical projection. This would divide the vertical leg into 3 vertical boxes. Thus at the end of this procedure, we will have at most $\frac{24}{\varepsilon}$ new **V**-compartments.

Similarly we extend the horizontal projections from bends of boundary curve $C_{H,V}$ dividing the horizontal leg L_H into $O_\varepsilon(1)$ **H**-compartments.

We do the procedure for all the **L**-compartments. The total height added considering $\varepsilon.h(L)$ vertical shifts corresponding to each L which is a **L**-compartment, can be at most $\varepsilon.OPT$ since the cumulative height of **L** containers on the top of each other can at most be OPT. This gives us $O_\varepsilon(1)$ new **H**- and **V**-compartments.

□

Obtaining Nice packing

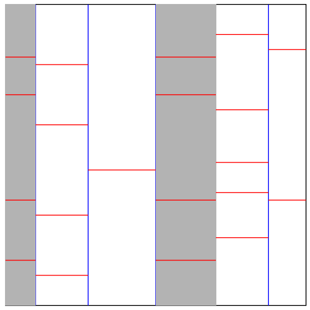
Once we have obtained $O_\varepsilon(1)$ number of vertical compartments, we proceed to ensure that the considered compartments are nice. To ensure the same, we exploit the property of guillotine packing. Intuitively, we consider stages of guillotine cuts and ensure constant number of vertical configurations created by these cuts. We now formally proceed with the proof by defining a configuration for a box-compartment.

Definition 8. *Consider a box B $a \times b$ containing a set of rectangles $\mathcal{I}$ such that all the rectangles are either vertical or large and for each $i \in \mathcal{I}$, $h_i = c_i\delta^2 b$ where $c_i \in [\frac{1}{\delta^2}]$ (Assume that $\frac{1}{\delta^2} \in \mathbb{Z}$ for sake of simplicity). Then given that the next guillotine cut stage is horizontal and considering those horizontal guillotine cuts , the configuration for box B is defined as $(j_1, j_2, ..., j_k)$ if the box B is cut horizontally at $y = j_l\delta^2 b$ for each $l \in [k]$, where $j_1, ..., j_k \in \mathbb{Z}$ and $k \leq \frac{1}{\delta^2} - 1$.*

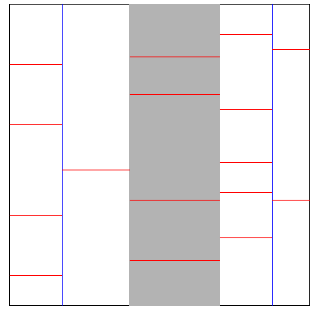
Lemma 5. *Consider a box B $a \times b$ containing a set of rectangles $\mathcal{I}$ such that all the rectangles are either vertical or large. Then we can obtain a nice packing of $\mathcal{I}$ in a box B' $a \times b(1+\delta)$ such that:*

1. *number of compartments inside the box B' is $O_\delta(1)$*
2. *compartments are pseudo guillotine separable.*

Proof. Let the first stage cuts be vertical otherwise we can divide the box into at most $\frac{1}{\delta}$ smaller boxes using the horizontal cuts and repeat the following process on the smaller boxes. Now consider the first stage of vertical cuts. This divides B into vertical strips. Now consider the next stage of horizontal cuts. This divides the strips into smaller regions. Now we shift these regions vertically so that the bottom edge of each such region is at a height (w.r.t. bottom edge of B) which is an integral multiple of $\delta^2 b$. Since a strip can contain at most $\frac{1}{\delta}$ regions, we need an extra space of δb at the top. Number of different possible configurations is at most $\frac{1}{\delta} \cdot (\frac{1}{\delta^2})^{\frac{1}{\delta}}$. Now we rearrange the strips so that the strips with same configuration are placed side by side. Next for each configuration, we merge the regions at the same height to form a box compartment. Now repeat the same procedure on each box compartment until you



(a)



(b)

Figure 5: (a) 2 stages of guillotine cuts for a guillotine compartment containing vertical rectangles. (b) Since rounded heights of vertical rectangles are integral multiples of δ^2, merge configurations with same set of horizontal cuts to get $O_\delta(1)$ configurations .

get a box compartment in which the rectangles are placed side by side only. Let the number of stages in B be $2k+1$ (odd number because the first and last stage cuts are vertical). Now we claim that the total number of compartments created in the box B is at most $(\frac{1}{\delta} \cdot (\frac{1}{\delta^2})^{\frac{1}{\delta}})^k$. We prove this by induction on k. For $k = 0$, it holds as there is only box compartment as all the items are placed side by side. Now let us assume that the hypothesis is true for $k = i$. Now we show that the hypothesis will be true for $k = i+1$. When the procedure is applied on the box with $2(i+1)+1$ stages, it creates $\frac{1}{\delta} \cdot (\frac{1}{\delta^2})^{\frac{1}{\delta}}$ compartments each consisting of at most $2i+1$. From inductive hypothesis we get that each of these compartment contain at most $(\frac{1}{\delta} \cdot (\frac{1}{\delta^2})^{\frac{1}{\delta}})^i$ compartments in which the items are nice packed. Hence in the box with $2(i+1)+1$ we have $(\frac{1}{\delta} \cdot (\frac{1}{\delta^2})^{\frac{1}{\delta}})^{i+1}$ compartments in which the items are nice packed. This proves our hypothesis. Now as $k = O(\frac{1}{\delta})$, the total number of compartments is $O_\delta(1)$. This concludes the proof of this lemma. $\square$

For horizontal container, we can apply resource augmentation (Lemma 33) and obtain a nice packing. For vertical containers, we apply Lemma 5

2.3 Packing Small Items

Now consider a vertical container C in which the items are nicely packed. Now we rearrange items such that items are placed side by side in a non-increasing order from left to right with no gap between the adjacent items and items touch the bottom edge of C. Now group the items in C based on their heights. For all $i \in [(1-\frac{1}{\delta})/\delta^2]$, group i contains the items with heights in the range $(\frac{OPT}{\delta} + (i-1) \cdot OPT \cdot \delta^2, \frac{OPT}{\delta} + i \cdot OPT \cdot \delta^2]$. Now for each group i, consider the minimal rectangular region containing all the items in group i and make it a container. Repeat a similar process for the horizontal containers too. Now the ratio of the area of the rectangles in a container and the area of the container is at least $1-\delta$. Now we form a non-uniform grid by extending the edges of the containers until it hits a container. Now the empty grid cells are guillotine separable as the containers when considered as pseudo items are guillotine separable and the guillotine cuts coincide with one of the edges of the containers. Now we can choose μ, δ appropriately such that the total area of empty grid cells with height less than $\varepsilon' OPT$ or width less than $\varepsilon' W$ is $O(\delta^2 \cdot OPT \cdot W)$ where $\varepsilon' = \mu/\varepsilon$. Note that the total area of the grid cells which have height more than $\varepsilon' OPT$ and width more than $\varepsilon' W$ is $[(1-\delta^2)OPT \cdot W -$area of compartments$]$ which is greater than $[(1-\delta^2)OPT \cdot W -$area of skewed items$]$ which is greater than $(1-\delta)a(I_{small})$. Hence using $NFDH$ we can pack small rectangles of area $O(1-O(\varepsilon))a(I_{small})$ in the grid cells which have height more than $\varepsilon' OPT$ and width more than $\varepsilon' W$.

2.4 Assigning and Packing Items

In this section, we describe the main algorithm that packs most of the items. From Lemma 4 we have that except for $I''_{small} \subset I_{small}$ and I_{medium} where $a(I''_{small}) \leq O(\varepsilon)OPT \cdot W$ and $a(I_{medium}) \leq O(\varepsilon)OPT \cdot W$, all other items can be packed nicely in a set of containers $\mathcal{C}$ such that $|\mathcal{C}| = O_\varepsilon(1)$.

First we assume that our algorithm is given as an input a value OPT' such that $OPT \leq OPT' \leq (1+\varepsilon)OPT$. This assumption can be removed as follows. We compute a 2-approximation APX by Steinberg's algorithm(which is a guillotine separable packing) and then run our algorithm for all the constantly many values $OPT' = (1+\varepsilon)^j \frac{APX}{2(1+\varepsilon)}$ which fit in the range $[\frac{APX}{2(1+\varepsilon)}, APX(1+\varepsilon)]$. One of these values will satisfy the claim. In order to keep the notation light we denote OPT' by OPT. Therefore all the approximation factors would be scaled by a factor of $(1+\varepsilon)$ in order to consider the true value of OPT.

We guess the position of each container C which requires $OPT \cdot W^{O_\varepsilon(1)}$ guesses while also ensuring the guillotine separability of the containers. This adds another multiplicative factor of $O_\varepsilon(1)$ by way of a geometric dynamic programming algorithm (Lemma 32). Further for each container we guess its type; whether it is *empty*, *large*, *vertical container*, *horizontal container* or ε-*area container*. That is we guess if we have an empty container, or a container with a single large item, or a container with only vertical items, or a container with only horizontal items, or a container with only small items. For those containers which we have guessed to be *large containers* , we simply guess this item. This can be done in $O_\delta(1)$ time since we have only constant number of large items.

For the small items, we simply pack them using the Next-Fit-Decreasing-Height (NFDH) algorithm in those containers, which we have guessed to be ε-*area containers*. For the vertical and horizontal items, we assign them by reducing the remaining problem to an instance of Maximum Generalized Assignment problem (GAP). After assigning them, we pack them nicely depending on whether the container is a horizontal or a vertical one.

2.5 Packing Leftout Items

From the previous lemmas, we have that $a(I'_{small}) \leq O(\varepsilon)OPT \cdot W$ and $a(I_{medium}) \leq O(\varepsilon)OPT \cdot W$. By appropriately choosing μ, δ, we can ensure that the total area of horizontal items not packed till now is at most $O(\varepsilon)OPT \cdot W$. We simply use NFDH to pack these items on the top and since NFDH ensures a guillotine packing, we overall have a guillotine packing. Let H_{add} be the additional height on top we require to pack these items. By Lemma 29, we have that

$$\begin{aligned} H_{add} &\leq 2a(I'_{small})/W + 2a(I_{medium})/W + 2a(I'_{hor})/W \\ &\leq O(\varepsilon)OPT \end{aligned}$$

3 A polynomial time $\frac{3}{2} + \varepsilon$ approximation

3.1 Classification of items

First we assume that our algorithm is given as an input a value OPT' such that $OPT \leq OPT' \leq (1+\varepsilon)OPT$. This assumption can be removed as follows. We compute a 2-approximation APX by Steinberg's algorithm(which is a guillotine separable packing) and then run our algorithm for all the constantly many values $OPT' = (1+\varepsilon)^j \frac{APX}{2(1+\varepsilon)}$ which fit in the range $[\frac{APX}{2(1+\varepsilon)}, APX(1+\varepsilon)]$. One of these values will satisfy the claim. In order to keep the notation light we denote OPT' by OPT. Therefore all the approximation factors would be scaled by a factor of $(1+\varepsilon)$ in order to consider the true value of OPT.

We classify the input items according to their heights and widths. For two constants $1 \geq \delta > \mu > 0$ to be defined later, we classify each item $i \in I$ as:

- *Tall*: $h_i > \frac{1}{2}OPT$

- *Large*: $w_i > \delta W$ and $\frac{1}{2}OPT \geq h_i > \delta OPT$;
- *Horizontal*: $w_i > \delta W$ and $h_i \leq \mu OPT$;
- *Intermediate*: $w_i \leq \delta W$ and $\frac{1}{2}OPT \geq h_i > \delta OPT$;
- *Medium:*
  - Either $\delta OPT \geq h_i > \mu OPT$;
  - Or $\delta W \geq w_i > \mu W$ and $h_i \leq \mu OPT$.
- *Small*: $w_i \leq \mu W$ and $h_i \leq \mu OPT$;

Let $I_{tall},\ I_{large},\ I_{hor},\ I_{Intermediate},\ I_{medium},\ I_{small}$ be the set of tall, large, horizontal, intermediate, medium and small rectangles, respectively. We also define the set of skewed items as $I_{skew} = I_{hor} \cup I_{Intermediate}$.

Using standard shifting arguments, one can show that the medium items occupy marginal area, allowing us to ignore them initally and packing them in the end.

Lemma 6. *[9] Let $\varepsilon > 0$ and $f(.)$ be any positive increasing function such that $f(x) < x\ \forall x \in (0, 1]$. Then we can efficiently find $\delta, \mu \in \Omega_\varepsilon(1)$, with $\varepsilon \geq f(\varepsilon) \geq \delta \geq f(\delta) \geq \mu$ so that the total area of medium rectangles is at most $\varepsilon(OPT \cdot W)$.*

3.2 Structural lemma

From the PPTAS we have that $I_{tall} \cup I_{Intermediate}$ have a nice packing in $O_\delta(1)$ containers. In all such containers, since we have a nice packing, we sort pack the items in an order of non-increasing heights from left to right. We then apply a guillotine cut that separates I_{tall} from $I_{Intermediate}$ in the container and we get tall containers and intermediate containers which are both $O_\delta(1)$ many. We define a useful operation called *Mirror* to get a structured packing.

Definition 9 (Mirror)**.** *Consider in a guillotine cutting sequence we have a compartment $\mathcal{C}$ and assume without loss of generality that $\mathcal{C}$ is cut by horizontal guillotine cuts (An equivalent operation holds for vertical cuts). In the next level of the guillotine cutting tree, for some $t \in \mathbb{N}$ it gets cut into t compartments $\mathcal{C}_1, \mathcal{C}_2, ..., \mathcal{C}_t$ where $\mathcal{C}_i \subseteq \mathcal{C}\ \forall i \in [t]$ in that order from top to bottom. For any fixed $k \in [t]$, Mirror operation changes the sequence of the guillotine compartments to $\mathcal{C}_1, ..., \mathcal{C}_{k-1}, \mathcal{C}_{k+1}, ..., \mathcal{C}_t, \mathcal{C}_k$(from top to bottom).*

Claim 1. *If the packing of a set of items I in rectangular strip R was guillotine separable, then the Mirror operation does not change the guillotine separability of I.*

Proof. Consider the guillotine cutting tree for the packing of I in R. *Mirror* operation for any compartment $\mathcal{C}$ just changes the ordering of the children of $\mathcal{C}$. The subtrees of the children of the node representing $\mathcal{C}$ stay unchanged because of the *Mirror* operation on $\mathcal{C}$. Hence, guillotine separability is not affected. □

We restructure the packing of Tall items by considering the guillotine sequence for the packing and then applying the *Mirror* operation to the guillotine compartment containing at least one tall container at each stage of horizontal cuts. Consider an optimum packing of height OPT of the items in I. We show that it is possible to have another packing of I which is guillotine separable such that all the tall items touch the base of the strip.

Lemma 7. *For a given set of items I which can be packed in a strip of height OPT and width W such that they are guillotine separable, there exists a guillotine separable packing of items in I such that for any tall item $i \in I_{tall}$, $bottom(i) = 0$ and the tall items are nicely packed in $O_\delta(1)$ containers.*

Proof. Consider an optimal guillotine packing of height OPT of the items in I and any tall item $i \in I_{tall}$. Consider now a guillotine cutting sequence for i in this packing. We employ the following algorithm: If there is a vertical cut, we do nothing. Else if its a horizontal cut, we apply the *Mirror*

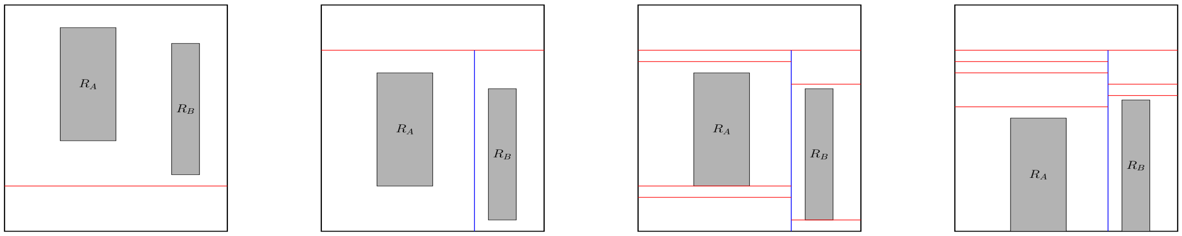


Figure 6: R_A and R_B are tall containers and by the *Mirror* operation on their respective guillotine compartments they can be packed such that the bottoms of both containers intersect the bottom of the strip

operation for the compartment which contains i. We have then that at each step of the cutting sequence, i lies in the compartment which touches the base of the strip, that is, it lies in the bottom-most compartment, see Fig 6. Since two tall items cannot be packed such that one is on top of the other, that is they cannot be packed such that a vertical line passes through both of them, at the end of the cutting sequence we have that the final compartment containing i is the item itself and hence, $bottom(i) = 0$. Since, we have that they can be nicely packed in $O_\delta(1)$ containers and the containers are themselves guillotine separable, we have a nice packing of tall items in $O_\delta(1)$ tall containers such that the tall containers touch the base of the strip. $\square$

Now, we have a packing of height OPT such that there are $O_\delta(1)$ tall containers and they touch the base of the strip and we have the other items (intermediate, horizontal and large) nicely packed in $O_\varepsilon(1)$ containers and these containers are guillotine separable. Since we have a hardness of $\frac{3}{2}$ approximation for guillotine strip packing in polynomial time (see Theorem 36), we are going to need at least $\frac{3}{2}OPT$ height for our packing. Hence we can afford to utilize a region $[0, W] \times [OPT, \frac{3}{2}OPT]$ for our packing.

Thus, assume that the packing P is a container based packing and we separate tall items from the intermediate items as before to obtain a nice packing in $O_\delta(1)$ containers where the tall items are packed in tall containers.

We already have from the PPTAS that the intermediate, large, horizontal and small items can be nicely packed in $O_\varepsilon(1)$ containers that are guillotine separable. Since we want to compute our approximate solution in polynomial time, we use resource augmentation [7] to round the heights of the containers so that their sizes (and hence their positions) belong to a set of cardinality $n^{O_\varepsilon(1)}$ and this set can be computed in polynomial time.

Lemma 8. *Given a set of items I that can be packed nicely in a set of containers $\mathcal{C}$ such that $|\mathcal{C}| = O_\varepsilon(1)$, the packing is guillotine separable and the height of the packing is OPT, there exists another nice packing of $I' \subseteq I$ such that $a(I') \geq (1 - O(\varepsilon))a(I)$, in a set of containers $|\mathcal{C}'| = O_\varepsilon(1)$ such that their sizes belong to a set A such that $|A| = n^{O_\varepsilon(1)}$ and A can be computed in polynomial time. Additionally this new container packing is guillotine separable and has a height of at most $OPT(1 + O(\varepsilon))$.*

We will attempt to pack tall items by just sorting their heights since we cannot employ a container packing for the tall items. For large items, a nice packing entails having a singular large item in a container. Hence, the cardinality of the set of sizes of containers for large items is $O_\delta(1)$ since there are only $O(1/\delta^2)$ many large items. We use NFDH to pack the small items nicely in ε-area containers . We reduce the problem of packing horizontal and intermediate items nicely in the containers to a GAP instance.

Note that GAP cannot be solved optimally in polynomial time due to the inherent hardness of the k-Partition problem (see Lemma 35) for constant $k \geq 2$. In other words, we show a reduction from the k-Partition problem to the problem of packing rectangles in k containers (nicely). We then show that to circumvent this, by using two rectangular strips S_1 and S_2, such that $w_{S_1} = \varepsilon_1 W$, $h_{S_1} = \frac{1}{2}OPT$ and $w_{S_2} = W$, $h_{S_2} = \varepsilon_2 OPT$ where ε_1 and ε_2 are constants to be determined later, we can pack all horizontal and intermediate items in polynomial time in the containers and S_1 and S_2.

Lemma 9. *Given a set of items I' and a set of $O_\delta(1)$ horizontal and intermediate containers C such that the items in I' can be nicely packed in the containers, we can pack the items in I' in $C \cup S_1 \cup S_2$ in $n^{O_\varepsilon(1)}$ time.*

Proof. First we guess in $O_\delta(1)$ time for each container if it is an intermediate container or a horizontal container. Now consider first the packing of the intermediate items in the guessed intermediate containers. Any intermediate container can have at most $O(1/\varepsilon_3)$ intermediate items packed in it which have width at least $\varepsilon_3 W$. Since we have $O_\delta(1)$ containers we consider all the possibilities of packing of $\varepsilon_3 W$-wide items in the containers. Then we reduce the remaining problem to an instance of GAP with profits of the items as their area but item i has profit 0 in container C_j if it cannot be packed in it. We know by Lemma 28 that if we augment the sizes of the containers by a factor of ε_3 we can pack all items in $O_\varepsilon(n^{O_\varepsilon(1)})$ time in the containers. We know that since all the remaining items have width at most $\varepsilon_3 W$, there can be at most one item per container which crosses the true boundary of the respective container when packed in the augmented container. Hence, we have at most $O_\delta(1)$ items of width at most $\varepsilon_3 W$ to be packed which occupy a total width of $\varepsilon_3 W \cdot O_\delta(1) \leq \varepsilon_1 W$. Since they are at most $\frac{1}{2}OPT$-tall, we can pack them in S_1. Similar arguments work for the horizontal items and the unpacked items in the containers can be packed in S_2. □

Consider the packing P again. An important assumption here is that in the container packing P, in any tall container, the tall item with the largest height touches the top of the tall container. If this is not so, we can reduce the height of the container so that it is the case and the packing P is still feasible. The strip S_2 can be easily packed on top with only an $O(\varepsilon)$ increase in the height of the packing. For the packing of S_1, we construct rectangular strips as follows: extend vertical lines upwards till $y = OPT$ from the left and right edges of those containers for which these lines do not intersect any rectangle. For every two consecutive such lines, we extend a horizontal line from either the line $y = \frac{1}{2}OPT$ or the minimum among the upper endpoints of the containers from which we extend these lines, whichever is higher. We now have $O_\delta(1)$ regions and by an averaging argument we can show that there is at least one such region which has width at least $W/O_\delta(1)$. And since we have $W/O_\delta(1) \geq \varepsilon_3 W$, we have sufficient width for packing S_1 . Now, consider such a strip of width at least $\varepsilon_3 W$ which has the lowest bottom height. Since there are only at most $f(\varepsilon)$ number of strips and $\varepsilon_3 f(\varepsilon) = 1$, we know that such a strip exists. Lets say the height of the bottom of this strip is h. First, we push all the containers which lie completely above $y = h$ vertically upwards by $\frac{1}{2}OPT$. Call this packing P_1. The height of the packing now is at most $(\frac{3}{2} + O(\varepsilon))OPT$.

Our strategy now is to be able to guess this height h (which we will later show is possible and which is ensured because of our assumption that in any tall container, the tall item with the largest height touches the top of the tall container) after sorting and packing the tall items in non-increasing order of their heights from left to right. Now, in P_1, if we had pushed all containers lying completely above $y = \frac{1}{2}OPT$ vertically upwards by $\frac{1}{2}OPT$, we get a new packing (lets call it P'). Now using this structured packing, if we sort and pack the tall items in non-increasing order of their heights, we will have a strip of width at least $\varepsilon_3 W$ and whose bottom lies at $y = h$ which can be guessed in polynomial time(to be proved later). Then we can just pack our S_1 in this strip and pack all the containers in polynomial time and employ our GAP approach to pack the items within the containers. But the problem with this is among the containers pushed vertically upwards in P', there can be some who do not completely lie above $h + \frac{1}{2}OPT$ due to them having width at most $\varepsilon_3 W$ and might intersect our strip S_1.

To tackle this problem, consider the line segment l joining the points $(0, h)$ and (W, h) and let the tall containers which it intersects be $\mathcal{C}_1, ..., \mathcal{C}_t$ in that order from left to right. Clearly, all these tall containers have their tops above $y = h$. Then we have the line segment l broken up into further smaller line segments $l_1, l_2, ..., l_{t+1}$ removing the part of the line containing any tall containers which were above $y = h$. That is, l_i is a line segment joining the points $(right(\mathcal{C}_{i-1}), h)$ and $(left(\mathcal{C}_i), h)$ for all $i \in \{2, ..., t\}$. And, l_1 is a line segment joining the points $(0, h)$ and $(left(\mathcal{C}_1), h)$ and l_{t+1} is a line segment joining the points $(right(\mathcal{C}_t), h)$ and (W, h). Now, consider a region between $y = 0$ and $y = OPT$ spanning the line segment l_i horizontally for any such $i \in [t + 1]$. The idea now is to shift all the non-tall containers which intersect l_i in a non-uniform way such that we get a packing in which in this horizontal region spanning l_i, the tall containers are packed entirely below $y = h$ and all the non-tall containers are either completely below $y = h$ or completely between $y = h$ and $y = h + \frac{1}{2}OPT$ or completely above $y = h + \frac{1}{2}OPT$. We do such a shifting procedure for all the line segments $l_1, ..., l_{t+1}$ and then have a packing in which there are tall containers $\mathcal{C}_1, ..., \mathcal{C}_t$ which lie completely above $y = h$ and then we have a strip of lowest bottom height and sufficient width on top of which we pack S_1. And finally the rest of the tall containers and non-tall pseudo items lie below this line $y = h$ and we have regions from $y = h$ and $y = h + \frac{1}{2}OPT$ spanning the line segments $l_1, ..., l_{t+1}$ where in we pack the rest of the non-tall containers. We now first show how this packing (call it P_2) is obtained, in particular the shifting procedure and after that we show it is a feasible, guillotine packing of height at most $(\frac{3}{2}+O(\varepsilon))OPT$. Finally, we show P_2 can be converted to another packing P_3 with tall items sorted and packed in order of non-increasing heights from left to right. Further, we guess the height $y = h$ and show that all non-tall pseudo items lie below $y = h$ and we extend $y = h$ from the right end of the strip which is $x = W$ till we intersect a tall item. Let this point be (x_0, h). We then show that the packing P_3 is a guillotine separable packing of height at most $(\frac{3}{2} + O(\varepsilon))OPT$ and the non tall containers are packed either to the right hand side of the tall items, but they lie completely below $y = h$ or they are packed in the region $[x_0, W] \times [h, h + \frac{1}{2}OPT]$ or they are packed completely above $y = h + \frac{1}{2}OPT$.

First, we show how to guess the height $y = h$ in polynomial time.

Claim 2. *Given a guillotine packing P of rectangular items I of optimal height OPT, we can, in polynomial time find the height of the strip with the lowest bottom height and which has width at least $\varepsilon_3 W$.*

Proof. Because of the assumption that the height of a tall container is the same as the height of the greatest height tall item packed in it, and if the strip with the lowest bottom height and with width at least $\varepsilon_3 W$ was a tall strip, then the height h is the same as the height of the greatest tall item in that tall container which resulted in that strip being constructed. Thus we can just iterate over heights of all the tall items and guess h. And if the strip with the lowest bottom height and with width at least $\varepsilon_3 W$ was a non-tall strip, that is, one constructed because of the two corners of a non-tall container, then after guessing the sizes and the packing of non-tall containers which are supposed to be packed below $y = h$, we iterate over heights of the non-tall containers which are at most $O_\delta(1)$ many. Thus, we can guess the height of the strip with the lowest bottom height and with width at least $\varepsilon_3 W$ with the help of $O(n)$ guesses. $\square$

Lemma 10. *Given a guillotine packing P of rectangular items I of optimal height OPT, there exists another guillotine, feasible packing P_2 of height at most $(\frac{3}{2} + O(\varepsilon))OPT$ such that it is a container based nice packing and*

1. *Any non-tall container either completely lies below $y = h$ or in one of the regions spanned by l_i from some $i \in [t + 1]$ and and in between $y = h$ and $y = h + \frac{1}{2}OPT$ or lies completely above $y = h + \frac{1}{2}OPT$.*
2. *In the packing P_2, the tall containers are packed at the same positiions as those in P. The Strip S_1 is packed in one of the regions spanned by l_i from some $i \in [t + 1]$ and and in between $y = h$ and $y = h + \frac{1}{2}OPT$. The Strip S_2 is packed on top of the packing.*

Proof. In the packing P_1, consider the regions spanned by the lines $l_1, ..., l_{t+1}$. In these regions all tall containers only lie below $y = h$. But there might be non-tall containers which intersect $y = h$. So employ a non-uniform shifting strategy as follows: Among the non-tall containers in R_1 whose bottoms lie above $\frac{1}{2}OPT$ (Type 1) we push them vertically upwards by $h - \frac{1}{2}OPT$. For those containers whose bottoms lie below $\frac{1}{2}OPT$ and whose tops lie above $y = h$ (Type 2), we push them vertically upwards till their bottoms are at $y = h$. We pack S_1 in the strip with the lowest bottom height and which has width at least $\varepsilon_3 W$.

Now observe that after this shifting procedure, no containers intersect either the line $y = h$ or the line $y = h + \frac{1}{2}OPT$. This is because the only containers which were previously intersecting $y = h$ were

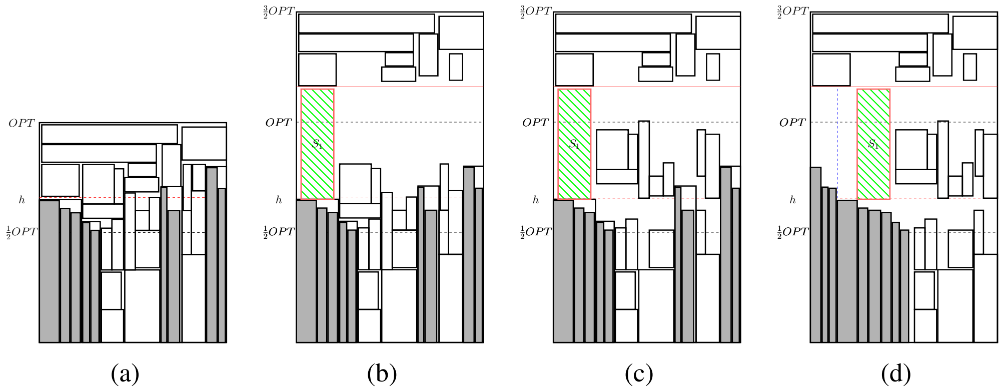


Figure 7: (a) A guillotine separable container packing P with items nicely packed in containers. The dashed red line indicates the height h. (b) The packing P_1 where items completely packed in $[h, OPT]$ are shifted by $\frac{1}{2}OPT$ vertically upward. The thick red line indicates $y = h + \frac{1}{2}OPT$ which separates the items shifted up from the items below. S_1 is packed in the strip of sufficient width and lowest height h. (c) Packing P_2 where containers of Type 1 and 2 are moved accordingly so they do not intersect $y = h$. (d) Final packing P_3

either Type 1 or Type 2 containers. For Type 2 containers, their height is at most $\frac{1}{2}OPT$ and we are ensuring that their bottoms touch $y = h$. For Type 1 containers as well, the height is at most $\frac{1}{2}OPT$ and since we are shifting them vertically upwards by $h - \frac{1}{2}OPT$, their bottoms now lie above $y = h$. And before shifting, their tops were at a height of at most OPT. Hence, after shifting their tops lie within $h + \frac{1}{2}OPT$. To prove guillotine separability of this packing, just showing guillotine separability of the containers suffices since the packing is a nice packing. As no item crosses $y = h + \frac{1}{2}OPT$, we can have a guillotine cut at $y = h + \frac{1}{2}OPT$. Further we can separate the containers lying above this line separately by guillotine cuts since they were guillotine separable in the initial packing, and for the containers in R_1, we separate the tall containers which have height more than $y = h$ by two guillotine cuts at both of their corners. Then in between such tall containers, we apply the guillotine cut $y = h$ and separate the tall containers and non-tall containers since they were guillotine separable in the initial packing. For the containers in these regions above $y = h$, note that the Type 2 containers can be separated via guillotine cuts at both of their left and right corners and since such containers crossed $y = h$ to begin with, there was nothing that was packed above them in P_1. For the Type 1 containers, they are guillotine separable as they were guillotine separable in the initial packing and they have been pushed by an equal height vertically upwards. The height of the packing is the same as the height of P_1 which is at most $(\frac{3}{2} + O(\varepsilon))OPT$. $\square$

3.3 Packing the items

The packing P_2 can be converted to a packing P_3 where tall items are packed from left to right in a sorted order of non-increasing heights. In the packing P_2, after the first guillotine cut at $y = h + \frac{1}{2}OPT$, we consider the guillotine cuts separating the tall containers having height more than h. Consider these guillotine compartments to be $C_1, ..., C_t$. Let the rest of the guillotine compartments which were between two of C_i and C_{i+1} be B_{i+1} for $i \in \{1, ..., t-1\}$. B_1 is the guillotine compartment from $[0, left(C_1)] \times [0, OPT]$ and B_{t+1} is the guillotine compartment from $[right(C_t), W] \times [0, OPT]$. We then sort these tall containers which have height greater than h in order of non-increasing heights and pack their respective guillotine compartments in that order from the left of the strip S. The rest of the guillotine compartments are packed to the right of them and this packing is guillotine separable because of the *Mirror* operation. That is, the order of the packing of the guillotine compartments from left to right is $C_1, C_2, ..., C_t, B_1, ..., B_{t+1}$. Let $right(C_t) = x_1$ in this packing.

Observe now that all the containers which were packed between $y = h$ and $y = h + \frac{1}{2}OPT$ in P_2 along with S_1 are packed in $[x_1, W] \times [h, h + \frac{1}{2}OPT]$ in this packing. Further, transform the current packing into one where all tall items are in a sorted order of non-increasing heights from left to right. This can be easily done by a series of swap operations between the tall items and the non-tall pseudo items. Observe that since the total width of the tall containers having height greater than h was x_1, the tall items having height greater than h can have total width at most x_1(lets say this width is x_2). Thus, if we sort and pack tall items in an order of non-increasing heights. Further, we guess the height h in polynomial time by Claim 2 and consider the line $y = h$ from the right end of the strip S until it intersects a tall item(which happens at $x = x_2$). Then, we can pack all the containers which were packed between $y = h$ and $y = h + \frac{1}{2}OPT$ in P_2, in $[x_2, W] \times [h, h + \frac{1}{2}OPT]$ since $x_2 \leq x_1$. Hence, we have a packing P_3 of height at most $(\frac{3}{2} + O(\varepsilon))OPT$ which is guillotine separable and we show now that it can be guessed in polynomial time.

Lemma 11. *Given a guillotine separable packing P of I of height OPT , there exists a guillotine packing P_3 of $I \cup S_1 \cup S_2$ of height at most $(\frac{3}{2} + O(\varepsilon))OPT$.*

Lemma 12. *There exists an algorithm for guillotine strip packing running in $n^{O_\varepsilon(1)}$ time which returns a packing of height at most $(\frac{3}{2} + O(\varepsilon))OPT$, If OPT is the height of the optimal guillotine packing.*

Proof. The algorithm we employ guesses the packing P_3 in polynomial time. As mentioned before, we guess the heights $OPT($ to within an error of $O(\varepsilon OPT))$ and the height h of the strip with the lowest bottom height and with width at least $\varepsilon_3 W$ in polynomial time. Further, we sort the tall items in non-increasing order of their heights and pack them in that order from the left part of the strip with no gap between them so that for each item i in I_{tall}, $bottom(i) = 0$. We then guess the sizes of the $O_\varepsilon(1)$ many non-tall containers in $n^{O_\varepsilon(1)}$ time.

Then we construct a container packing (with $O_\varepsilon(1)$ containers) of the remaining items with the property that the containers themselves are guillotine separable and items are packed nicely in each container. We first guess whether each container is an intermediate, horizontal, ε-area or large container in $O_\delta(1)$ time. Since there are only $O(1/\delta^2)$ large items, we know their container sizes. Then, we guess the container sizes for each of the set of intermediate, horizontal and ε-area containers from a set of $n^{O_\varepsilon(1)}$ sizes by Lemma 8. This set is computed in polynomial time. Note that by resource augmentation for the small, intermediate and horizontal containers we might not be able to pack such items with area at most $O(\varepsilon)OPT \cdot W$ in the guessed containers . But we know that we can pack them either inside S_1, S_2 or by using at most $O(\varepsilon)OPT$ height of top of the packing by Lemma 9.

Now, for packing the containers we partition the set of containers into 3 subsets C_1, C_2 and C_3 which can be done in at most $3^{O_\varepsilon(1)}$ ways. C_1 is the set of containers which are packed below $y = h$ adjacent to the tall items, C_2 the set of the containers which are packed between $y = h$ and $y = h + \frac{1}{2}OPT$, C_3 the set of containers which are packed above $h + \frac{1}{2}OPT$ and below $(\frac{3}{2} + O(\varepsilon))OPT$. Then we construct the line segment $y = h$(first by guessing the height h in polynomial time by Claim 2), starting from the right end of the strip till the point x_2 where it intersects a tall item. Now, based on our guesses for the non-tall containers, we try all possible packings in $O_\varepsilon(1)$ time. For the containers to be packed between $y = h$ and $y = h + \frac{1}{2}OPT$, we pack them only to the right of $x = x_2$. Also note that since there are $O(\frac{1}{\delta^2})$ large items, we can easily pack them nicely by brute force.

Finally we just need to pack the remaining items nicely in the containers for which we convert the instance to an instance of the Maximum Generalized Assignment problem with one bin per container and the size of j-th container C_j is $a(C_j) = w(C_j) \times h(C_j)$ (We do not include the S_1 and S_2 in this reduction to GAP). We build an instance of GAP as follows. There is one item R per rectangle $R \in I$, with profit $a(R)$. For each horizontal container C_j , we create a knapsack j of size $S_j := h(C_j)$. Furthermore, we define the size $s(R, j)$ of rectangle R w.r.t. knapsack j as $h(R)$ if $h(R) \leq h(C_j)$ and $w(R) \leq w(C_j)$. Otherwise $s(R, j) = \infty$ (meaning that R does not fit in C_j). The construction for intermediate containers is symmetric. For each area container C_j we create a knapsack j of size

$S_j = a(C_j)$ and define the size $s(R, j)$ of rectangle R w.r.t. knapsack j as $a(R)$ if $h(R) \leq \varepsilon' h(C_j)$ and $w(R) \leq \varepsilon' w(C_j)$, setting $b(R, j) = \infty$ otherwise (meaning that the rectangle is not small enough with respect to the dimensions of the container). We now use the approach mentioned in Lemma 9 and finish the packing of all the intermediate and horizontal items in the appropriate containers and pack the small items in the in ε-area containers, all in $n^{O_\varepsilon(1)}$ time. For the leftover items, we need only $O(\varepsilon)$ extra height which we pack in the next subsection using NFDH. For each of the guesses of the algorithm for the entire packing, we finally check if the container packing is guillotine separable by Lemma 32 and if the containers are guillotine separable then the entire packing is (since NFDH for the leftover items still maintains the guillotine separability of the packing). $\square$

3.4 Packing leftout items

From the Lemma **??** and Lemma **??**, we have that $a(I'_{small}) \leq O(\varepsilon)OPT \cdot W$ and $a(I_{medium}) \leq O(\varepsilon)OPT \cdot W$. By appropriately choosing μ, δ, we can ensure that the total area of horizontal items not packed till now is at most $O(\varepsilon)OPT \cdot W$. We simply use NFDH to pack these items on the top and since NFDH ensures a guillotine packing, we overall have a guillotine packing. Let H_{add} be the additional height on top we require to pack these items. By Lemma 29, we have that

$$\begin{aligned} H_{add} &\leq 2a(I'_{small})/W + 2a(I_{medium})/W + 2a(I'_{hor})/W \\ &\leq O(\varepsilon)OPT \end{aligned}$$

4 Conclusion

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A A polynomial time $\frac{7}{4}+\varepsilon$ approximation

A.1 Classification of items

First we assume that our algorithm is given as an input a value OPT' such that $OPT \leq OPT' \leq (1+\varepsilon)OPT$. This assumption can be removed as follows. We compute a 2-approximation APX by Steinberg's algorithm(which is a guillotine separable packing) and then run our algorithm for all the constantly many values $OPT' = (1+\varepsilon)^j \frac{APX}{2(1+\varepsilon)}$ which fit in the range $[\frac{APX}{2(1+\varepsilon)}, APX(1+\varepsilon)]$. One of these values will satisfy the claim. In order to keep the notation light we denote OPT' by OPT. Therefore all the approximation factors would be scaled by a factor of $(1+\varepsilon)$ in order to consider the true value of OPT.

We classify the input items according to their heights and widths. For two constants $1 \geq \delta > \mu > 0$ to be defined later, we classify each item $i \in I$ as:

- *Tall*: $h_i > \frac{1}{2}OPT$
- *Large*: $w_i > \delta W$ and $\frac{1}{2}OPT \geq h_i > \delta OPT$;
- *Horizontal*: $w_i > \delta W$ and $h_i \leq \mu OPT$;
- *Intermediate*: $w_i \leq \delta W$ and $\frac{1}{2}OPT \geq h_i > \delta OPT$;
- *Medium:*
  - Either $\delta OPT \geq h_i > \mu OPT$;
  - Or $\delta W \geq w_i > \mu W$ and $h_i \leq \mu OPT$.
- *Small*: $w_i \leq \mu W$ and $h_i \leq \mu OPT$;

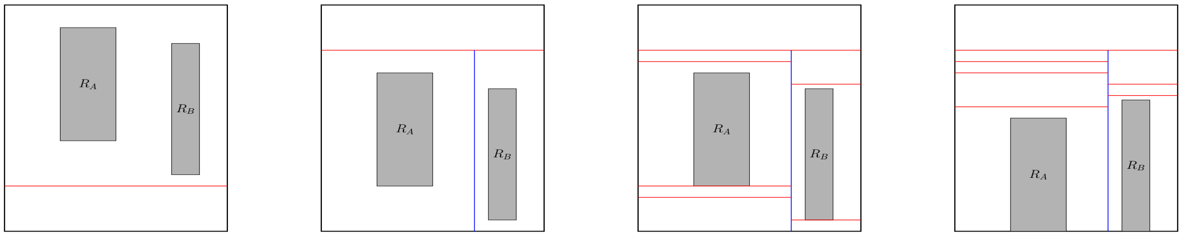


Figure 8: R_A and R_B are tall containers and by the *Mirror* operation on their respective guillotine compartments they can be packed such that the bottoms of both containers intersect the bottom of the strip

Let $I_{tall},\ I_{large},\ I_{hor},\ I_{Intermediate},\ I_{medium},\ I_{small}$ be the set of tall, large, horizontal, intermediate, medium and small rectangles, respectively. We also define the set of skewed items as $I_{skew} = I_{hor} \cup I_{Intermediate}$.

Using standard shifting arguments, one can show that the medium items occupy marginal area, allowing us to ignore them initally and packing them in the end.

Lemma 13. *[9] Let $\varepsilon > 0$ and $f(.)$ be any positive increasing function such that $f(x) < x\ \forall x \in (0, 1]$. Then we can efficiently find $\delta, \mu \in \Omega_\varepsilon(1)$, with $\varepsilon \geq f(\varepsilon) \geq \delta \geq f(\delta) \geq \mu$ so that the total area of medium rectangles is at most $\varepsilon(OPT \cdot W)$.*

A.2 Structural lemma

From the PPTAS we have that $I_{tall} \cup I_{Intermediate}$ have a nice packing in $O_\delta(1)$ containers. In all such containers, since we have a nice packing, we sort pack the items in an order of non-increasing heights from left to right. We then apply a guillotine cut that separates I_{tall} from $I_{Intermediate}$ in the container and we get tall containers and intermediate containers which are both $O_\delta(1)$ many. We define a useful operation called *Mirror* to get a structured packing.

Definition 10 (Mirror)**.** *Consider in a guillotine cutting sequence we have a compartment $\mathcal{C}$ and assume without loss of generality that $\mathcal{C}$ is cut by horizontal guillotine cuts (An equivalent operation holds for vertical cuts). In the next level of the guillotine cutting tree, for some $t \in \mathbb{N}$ it gets cut into t compartments $\mathcal{C}_1, \mathcal{C}_2, ..., \mathcal{C}_t$ where $\mathcal{C}_i \subseteq \mathcal{C}\ \forall i \in [t]$ in that order from top to bottom. For any fixed $k \in [t]$, Mirror operation changes the sequence of the guillotine compartments to $\mathcal{C}_1, ..., \mathcal{C}_{k-1}, \mathcal{C}_{k+1}, ..., \mathcal{C}_t, \mathcal{C}_k$(from top to bottom).*

Claim 3. *If the packing of a set of items I in rectangular strip R was guillotine separable, then the Mirror operation does not change the guillotine separability of I.*

Proof. Consider the guillotine cutting tree for the packing of I in R. *Mirror* operation for any compartment $\mathcal{C}$ just changes the ordering of the children of $\mathcal{C}$. The subtrees of the children of the node representing $\mathcal{C}$ stay unchanged because of the *Mirror* operation on $\mathcal{C}$. Hence, guillotine separability is not affected. □

We restructure the packing of Tall items by considering the guillotine sequence for the packing and then applying the *Mirror* operation to the guillotine compartment containing at least one tall container at each stage of horizontal cuts. Consider an optimum packing of height OPT of the items in I. We show that it is possible to have another packing of I which is guillotine separable such that all the tall items touch the base of the strip.

Lemma 14. *For a given set of items I which can be packed in a strip of height OPT and width W such that they are guillotine separable, there exists a guillotine separable packing of items in I such that for any tall item $i \in I_{tall}$, $bottom(i) = 0$ and the tall items are nicely packed in $O_\delta(1)$ containers.*

Proof. Consider an optimal guillotine packing of height OPT of the items in I and any tall item $i \in I_{tall}$. Consider now a guillotine cutting sequence for i in this packing. We employ the following algorithm: If there is a vertical cut, we do nothing. Else if its a horizontal cut, we apply the *Mirror* operation for the compartment which contains i. We have then that at each step of the cutting sequence, i lies in the compartment which touches the base of the strip, that is, it lies in the bottom-most compartment, see Fig 8. Since two tall items cannot be packed such that one is on top of the other, that is they cannot be packed such that a vertical line passes through both of them, at the end of the cutting sequence we have that the final compartment containing i is the item itself and hence, $bottom(i) = 0$. Since, we have that they can be nicely packed in $O_\delta(1)$ containers and the containers are themselves guillotine separable, we have a nice packing of tall items in $O_\delta(1)$ tall containers such that the tall containers touch the base of the strip. □

Now, we have a packing of height OPT such that there are $O_\delta(1)$ tall containers and they touch the base of the strip and we have the other items (intermediate, horizontal and large) nicely packed in $O_\varepsilon(1)$ containers and these containers are guillotine separable. Since we have a hardness of $\frac{3}{2}$ approximation for guillotine strip packing in polynomial time (see Theorem 36), we are going to need at least $\frac{3}{2}OPT$ height for our packing. Hence we can afford to utilize a region $[0, W] \times [OPT, \frac{3}{2}OPT]$ for our packing. So for those items which are completely packed in $[0, W] \times [\frac{1}{2}, OPT]$ we shift them by $\frac{1}{2}OPT$ vertically upward. This new packing is still guillotine separable and the only items which intersect the line segment joining the points $(0, \frac{1}{2}OPT)$ and $(W, \frac{1}{2}OPT)$ are the tall items (or the tall containers) or possibly some horizontal or intermediate containers. Lets call this packing P_1. We now restructure P_1 to have all tall items in a sorted order (with respect to their heights) at the bottom of the strip.

Claim 4. *The packing P_1 can be restructured so that we have all items in I_{tall} packed consecutively in a sorted order of non-increasing heights such that all of them touch the bottom of the strip, the new packing is still guillotine separable and the height of the packing is at most $\frac{3}{2}OPT$.*

Proof. Consider the packing P_1 as described before. Observe that since there are no items which lie exclusively in $[0, W] \times [\frac{1}{2}OPT, OPT]$ the first guillotine cut can be the horizontal cut at $y = OPT$. Now we can consider the guillotine cutting sequence of items in $[0, W] \times [OPT, \frac{3}{2}OPT]$ separately from the guillotine cutting sequence of $[0, W] \times [0, OPT]$. So we consider vertical cuts separating all the tall containers from the other containers. That is for each tall container C_i which is packed at $[left(C_i), right(C_i)] \times [0, top(C_i)]$ we consider the cuts at $x = left(C_i)$ and $x = right(C_i)$. These cuts do not intersect any items as there are no items in the region $[0, W] \times [\frac{1}{2}OPT, OPT]$. So now we just use the *Mirror* operation to pack the tall containers consecutively in sorted orde of non-increasing heights from the left side of the strip and move all the non-tall containers to the right of the tall containers. By the property of the *Mirror* operation, we still have a guillotine packing and because the items in $[0, W] \times [OPT, \frac{3}{2}OPT]$ are guillotine separable. □

We already have from the PPTAS that the intermediate, large, horizontal and small items can be nicely packed in $O_\varepsilon(1)$ containers that are guillotine separable. Since we want to compute our approximate solution in polynomial time, we use resource augmentation [7] to round the heights of the containers so that their sizes (and hence their positions) belong to a set of cardinality $n^{O_\varepsilon(1)}$ and this set can be computed in polynomial time.

Lemma 15. *Given a set of items I that can be packed nicely in a set of containers $\mathcal{C}$ such that $|\mathcal{C}| = O_\varepsilon(1)$, the packing is guillotine separable and the height of the packing is OPT, there exists another nice packing of $I' \subseteq I$ such that $a(I') \geq (1 - O(\varepsilon))a(I)$, in a set of containers $|\mathcal{C}'| = O_\varepsilon(1)$ such that their sizes belong to a set A such that $|A| = n^{O_\varepsilon(1)}$ and A can be computed in polynomial time. Additionally this new container packing is guillotine separable and has a height of at most $OPT(1 + O(\varepsilon))$.*

Proof. Proof follows from applying resource augmentation (Lemma 33). □

Now, we have a direct packing for tall items by just sorting their heights. For large items, a nice packing entails having a singular large item in a container. Hence, the cardinality of the set of sizes of containers for large items is $O_\delta(1)$ since there are only $O(1/\delta^2)$ many large items. We use NFDH to pack the small items nicely in ε-area containers . We reduce the problem of packing horizontal and intermediate items nicely in the containers to a GAP instance.

Note that GAP cannot be solved optimally in polynomial time due to the inherent hardness of the $k - Partition$ problem (see Lemma 35) for constant $k \geq 2$. In other words, we show a reduction from the $k - Partition$ problem to the problem of packing rectangles in k containers (nicely). We then show that to circumvent this, by using two rectangular strips S_1 and S_2, such that $w_{S_1} = \varepsilon_1 W$, $h_{S_1} = \frac{1}{2}OPT$ and $w_{S_2} = W$, $h_{S_2} = \varepsilon_2 OPT$ where ε_1 and ε_2 are constants to be determined later, we can pack all horizontal and intermediate items in polynomial time in the containers and S_1 and S_2.

Lemma 16. *Given a set of items I' and a set of $O_\delta(1)$ horizontal and intermediate containers C such that the items in I' can be nicely packed in the containers, we can pack the items in I' in $C \cup S_1 \cup S_2$ in $n^{O_\varepsilon(1)}$ time.*

Proof. First we guess in $O_\delta(1)$ time for each container if it is an intermediate container or a horizontal container. Now consider first the packing of the intermediate items in the guessed intermediate containers. Any intermediate container can have at most $O(1/\varepsilon_3)$ intermediate items packed in it which have width at least $\varepsilon_3 W$. Since we have $O_\delta(1)$ containers we consider all the possibilities of packing of $\varepsilon_3 W$-wide items in the containers. Then we reduce the remaining problem to an instance of GAP with profits of the items as their area but item i has profit 0 in container C_j if it cannot be packed in it. We know by Lemma 28 that if we augment the sizes of the containers by a factor of ε_3 we can pack all items in $O_\varepsilon(n^{O_\varepsilon(1)})$ time in the containers. We know that since all the remaining items have width at most $\varepsilon_3 W$, there can be at most one item per container which crosses the true boundary of the respective container when packed in the augmented container. Hence, we have at most $O_\delta(1)$ items of width at most $\varepsilon_3 W$ to be packed which occupy a total width of $\varepsilon_3 W \cdot O_\delta(1) \leq \varepsilon_1 W$. Since they are at most $\frac{1}{2}OPT$-tall, we can pack them in S_1. Similar arguments work for the horizontal items. ☐

Consider the packing P_1 again. The strip S_2 can be easily packed on top with only an $O(\varepsilon)$ increase in the height of the packing. For the packing of S_1, we show that since we have $O_\delta(1)$ containers and since in P_1 we do not have any item (contained exclusively) in $[0, W] \times [\frac{1}{2}OPT, OPT]$, we always have a sufficiently wide and empty rectangular region such that we can pack S_1 with an additional $\frac{1}{4}OPT$ increase in the height of the packing. Let's say R_1 is the region $[0, W] \times [0, OPT]$ and R_2 is the region $[0, W] \times [OPT, \frac{3}{2}OPT]$. Consider the construction: extend vertical lines upwards till $y = OPT$ from the left and right edges of those containers for which these lines do not intersect any rectangle. For every two consecutive such lines, we extend a horizontal line from either the line $y = \frac{1}{2}OPT$ or the minimum among the upper endpoints of the containers from which we extend these lines, whichever is higher. We now have $O_\delta(1)$ regions and by an averaging argument we can show that there is at least one such region which has width at least $W/O_\delta(1)$. And since we have $W/O_\delta(1) \geq \varepsilon_3 W$, we have sufficient width for packing S_1. Lets consider one such rectangular region of sufficient width (call it R_3). Now we have two cases.

1. If $bottom(R_3) \leq \frac{3}{4}OPT$ we can push R_2 above by $\frac{1}{4}OPT$ and pack S_1 in the region $[left(R_3), right(R_3)] \times [bottom(R_3), \frac{5}{4}OPT]$, see Figure 9 c .
2. Else if $bottom(R_3) \geq \frac{3}{4}OPT$, we again push R_2 above by $\frac{1}{4}OPT$ and this time pack S_1 in the region $[left(R_3), right(R_3)] \times [OPT, bottom(R_3) + \frac{3}{4}OPT]$, see Figure 10 c .

Call this packing P_2(irrespective of the case).

Claim 5. *P_2 is a feasible, guillotine separable packing and has a height of at most $(\frac{7}{4} + O(\varepsilon))OPT$.*

Proof. Since we know that P_1 is a feasible, guillotine separable packing, by moving all items in the region R_2 equally by $\frac{1}{4}OPT$ and then packing S_2 on top, neither guillotine separability is affected

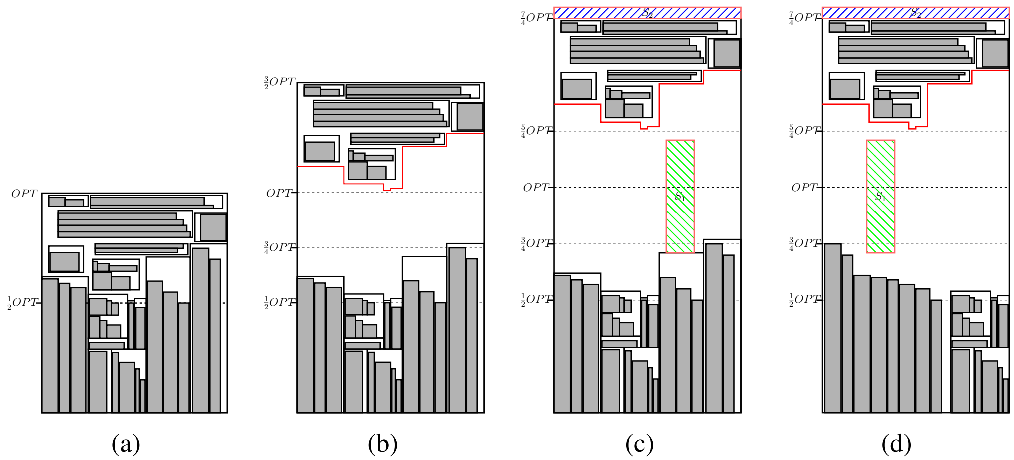


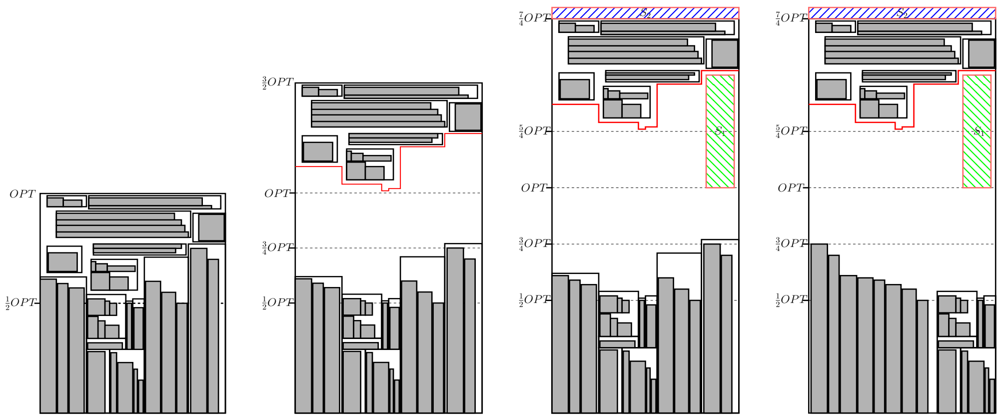
Figure 9: (a) A guillotine separable packing P with items nicely packed in containers . (b) The packing P_1 where items completely packed in $[\frac{1}{2}OPT, OPT]$ are shifted by $\frac{1}{2}OPT$ vertically upward. The red line is obtained by shifting the upper boundary of the packing in R_1 by $\frac{1}{2}OPT$. (c) Case 1 of P_2. (d)Case 1 of P_3

nor feasibility. The height of the packing now is at most $(\frac{7}{4}+O(\varepsilon))OPT$(within an additive error of $O(\varepsilon)OPT$). We will consider both the cases separately. Consider case 1 now. Since we are in Case 1, there exists a rectangular strip whose left and right end edges (lets say are $x=l_1$ and $x=l_2$) are upward extensions of guillotine cuts for containers in our packing. When extended upward they do not cut any other rectangle because of our construction and since no rectangles/containers are now present in R_2 nor do they intersect any horizontal guillotine cut. Hence such a rectangular strip is guillotine separable along with our existing guillotine cutting sequence for all the rectangles and since the bottom of this strip lies below $\frac{3}{4}OPT$ and its width is at least $\varepsilon_3 W$, we have enough space to pack S_1 in this region which is upper bounded in the y co-ordinate by $\frac{5}{4}OPT$.

In Case 2, we have a rectangular strip of with at least $\varepsilon_3 W$ with its bottom point at a height of at least $\frac{3}{4}OPT$. We move all the items in R_2 upward by the same height of $\frac{1}{4}OPT$. Observe then that packing S_1 in that rectangular strip region but with $bottom(S_1)=OPT$ leaves enough space for it to be packed as any item in that x interval from R_2 has height at least $\frac{3}{2}OPT$ after moving those items upward. This settles the feasibilty of the packing. For guillotine separability, we use the fact that the initial packing was guillotine separable. First we include the horizontal cut $x=OPT$ and then we can see that items in R_1 are guillotine separable from the property of the initial packing and items in $[0,W]\times[OPT,(\frac{7}{4}+O(\varepsilon))OPT]$ are guillotine separable if S_1 is not included. But consider the fact that we have a rectangular region of sufficient width formed by two vertical lines $x=l_1$ and $x=l_2$ is on top of a container(or possibly empty space) and hence in the guillotine cutting sequence for the original packing we had the guillotine cuts at $x=l_1$ and $x=l_2$ which may have been bounded on the lower side by a previous horizontal cut(but which is irrelevant). Hence when considering the packing of S_1 we know that along with the other items which were in R_2 we can have 2 vertical guillotine cuts at $x=l_1$ and $x=l_2$ and there are no horizontal cuts crossing the region $[l_1,l_2]\times[OPT,\frac{3}{2}OPT]$ as we can ignore the cuts which were present in the original packing in R_1 through the container below our rectangular strip. Hence we have a guillotine separable packing overall.

□

We modify the packing P_2 to a packing P_3 as we did for P_1 to pack the tall items in a non-increasing order of their heights such that all the tall items touch the bottom of the strip S and the tallest item touches the left edge of the Strip S. The non-tall containers which were in R_1 are packed to the right of the tall items in the same fashion as before, just that they are shifted by the *Mirror* operation to the right. By similar arguments as used in the packing P_2, we show that S_1 can be packed such that the height of



(a) (b) (c) (d)

Figure 10: (a) A guillotine separable packing P with items nicely packed in containers . (b) The packing P_1 where items completely packed in $[\frac{1}{2}OPT, OPT]$ are shifted by $\frac{1}{2}OPT$ vertically upward. The red line is obtained by shifting the upper boundary of the packing in R_1 by $\frac{1}{2}OPT$. (c) Case 2 of P_2. (d)Case 2 of P_3

P_3 is at most $(\frac{7}{4} + O(\varepsilon))OPT$.

Claim 6. *P_3 is a feasible, guillotine separable packing and has a height of at most $(\frac{7}{4} + O(\varepsilon))OPT$.*

Proof. The proof follows from Claim 5 and Claim 4 which we use to pack all the tall items in a non-increasing order of their heights starting from $x = 0$ and leaving no empty space between, such that all the tall items are touching the bottom of the strip S and rearrange the non-tall containers to the right of the packing of the tall items. If we are in Case 2, the same packing of all the items as was in Claim 5 will be feasible with height at most $\frac{7}{4}OPT$ and is a guillotine separable packing, see Figure 10 d. If we are in Case 1, there still exists a sufficiently wide rectangular strip the bottom of which is at a height of at most $\frac{3}{4}OPT$ and hence we can pack S_1 on top of it, , see Figure 9 d. And guillotine separability, feasibility and height of the packing being at most $(\frac{7}{4} + O(\varepsilon))OPT$ will follow exactly from Claim 5. $\square$

A.3 Assigning and packing items

Lemma 17. *There exists an algorithm for guillotine strip packing running in $n^{O_\varepsilon(1)}$ time which returns a packing of height at most $(\frac{7}{4} + O(\varepsilon))OPT$, If OPT is the height of the optimal guillotine packing.*

Proof. By Claim 6 we know that there exists a packing P_3 of height at most $(\frac{7}{4} + O(\varepsilon))OPT$. As mentioned before, we guess the value of OPT to within an additive error of $O(\varepsilon OPT)$ in polynomial time. We then sort the tall items in non-increasing order of their heights and pack them in that order from the left part of the strip with no gap between them so that for each item i in I_{tall}, $bottom(i) = 0$. Then we construct a container packing (with $O_\varepsilon(1)$ containers) of the remaining items with the property that the containers themselves are guillotine separable and items are packed nicely in each container. Since there are only $O(1/\delta^2)$ large items, we know their container sizes. Then, we guess the container sizes for each of the set of intermediate, horizontal and ε-area containers from a set of $n^{O_\varepsilon(1)}$ sizes by Lemma 15. This set is computed in polynomial time. Note that by resource augmentation for the small, intermediate and horizontal containers we might not be able to pack such items with area at most $O(\varepsilon)OPT \cdot W$ in the guessed containers . But we show that we can pack them either inside S_1, S_2 or by using at most $O(\varepsilon)OPT$ height of top of the packing. Now, for packing the containers we partition the set of containers into 2 subsets C_1 and C_2 which can be done in at most $2^{O_\varepsilon(1)}$ ways. C_1 is the set of containers which we pack in R_1 adjacent to the tall items. Then we make the rectangular strip construction with the top of the rectangular strips being $y = OPT$ and check if we are in Case 1 or Case 2. If we are in Case 1 we pack S_1 in the appropriate rectangular strip. We then pack the containers in C_2 in all possible ways in the region $[0, W] \times [\frac{5}{4}OPT, \frac{7}{4}OPT]$. If we are in Case 2, we pack the containers in C_2 in all possible ways in the region $[0, W] \times [OPT, \frac{7}{4}OPT]$ along with S_1 .We pack S_2 on top of the highest container. Since there are $O(\frac{1}{\delta^2})$ large items, we can easily pack them nicely by brute force.

Finally we just need to pack the remaining items nicely in the containers for which we convert the instance to an instance of the Maximum Generalized Assignment problem with one bin per container and the size of j-th container C_j is $a(C_j) = w(C_j) \times h(C_j)$ (We do not include the S_1 and S_2 in this reduction to GAP). We then make a guess for each container whether it is a large, intermediate, horizontal or ε-area container in $O_\varepsilon(1)$ time. We build an instance of GAP as follows. There is one item R per rectangle $R \in I$, with profit $a(R)$. For each horizontal container C_j , we create a knapsack j of size $S_j := h(C_j)$. Furthermore, we define the size $s(R, j)$ of rectangle R w.r.t. knapsack j as $h(R)$ if $h(R) \leq h(C_j)$ and $w(R) \leq w(C_j)$. Otherwise $s(R, j) = \infty$ (meaning that R does not fit in C_j). The construction for intermediate containers is symmetric. For each area container C_j we create a knapsack j of size $S_j = a(C_j)$ and define the size $s(R, j)$ of rectangle R w.r.t. knapsack j as $a(R)$ if $h(R) \leq \varepsilon' h(C_j)$ and $w(R) \leq \varepsilon' w(C_j)$, setting $b(R, j) = \infty$ otherwise (meaning that the rectangle is not small enough with respect to the dimensions of the container). We now use the approach mentioned in Lemma 16 and finish the packing of all the intermediate and horizontal items in the appropriate containers and pack the small items in the in ε-area containers, all in $n^{O_\varepsilon(1)}$ time. For the leftover items, we need only $O(\varepsilon)$ extra height which we pack in the next subsection using NFDH. For each of the guesses of the algorithm for the entire packing, we finally check if the container packing is guillotine separable by Lemma 32 and if the containers are guillotine separable then the entire packing is (since NFDH for the leftover items still maintains the guillotine separability of the packing). $\square$

A.4 Packing leftout items

From the previous lemmas, we have that $a(I'_{small}) \leq O(\varepsilon)OPT \cdot W$ and $a(I_{medium}) \leq O(\varepsilon)OPT \cdot W$. By appropriately choosing μ, δ, we can ensure that the total area of horizontal items not packed till now is at most $O(\varepsilon)OPT \cdot W$. We simply use NFDH to pack these items on the top and since NFDH ensures a guillotine packing, we overall have a guillotine packing. Let H_{add} be the additional height on top we require to pack these items. By Lemma 29, we have that

$$\begin{aligned} H_{add} &\leq 2a(I'_{small})/W + 2a(I_{medium})/W + 2a(I'_{hor})/W \\ &\leq O(\varepsilon)OPT \end{aligned}$$

B A polynomial time $\frac{5}{3} + \varepsilon$ approximation

B.1 Classification of items

First we assume that our algorithm is given as an input a value OPT' such that $OPT \leq OPT' \leq (1+\varepsilon)OPT$. This assumption can be removed as follows. We compute a 2-approximation APX by Steinberg's algorithm(which is a guillotine separable packing) and then run our algorithm for all the constantly many values $OPT' = (1+\varepsilon)^j \frac{APX}{2(1+\varepsilon)}$ which fit in the range $[\frac{APX}{2(1+\varepsilon)}, APX(1+\varepsilon)]$. One of these values will satisfy the claim. In order to keep the notation light we denote OPT' by OPT. Therefore all the approximation factors would be scaled by a factor of $(1+\varepsilon)$ in order to consider the true value of OPT.

We classify the input items according to their heights and widths. For two constants $1 \geq \delta > \mu > 0$ to be defined later, we classify each item $i \in I$ as:

- *Tall*: $h_i > \frac{1}{2}OPT$

- *Large*: $w_i > \delta W$ and $\frac{1}{2}OPT \geq h_i > \delta OPT$;
- *Horizontal*: $w_i > \delta W$ and $h_i \leq \mu OPT$;
- *Intermediate*: $w_i \leq \delta W$ and $\frac{1}{2}OPT \geq h_i > \delta OPT$;
- *Medium:*
  - Either $\delta OPT \geq h_i > \mu OPT$;
  - Or $\delta W \geq w_i > \mu W$ and $h_i \leq \mu OPT$.
- *Small*: $w_i \leq \mu W$ and $h_i \leq \mu OPT$;

Let $I_{tall},\ I_{large},\ I_{hor},\ I_{Intermediate},\ I_{medium},\ I_{small}$ be the set of tall, large, horizontal, intermediate, medium and small rectangles, respectively. We also define the set of skew items as $I_{skew} = I_{hor} \cup I_{Intermediate}$.

Using standard shifting arguments, one can show that the medium items occupy marginal area, allowing us to ignore them initally and packing them in the end.

Lemma 18. *[9] Let $\varepsilon > 0$ and $f(.)$ be any positive increasing function such that $f(x) < x\ \forall x \in (0,1]$. Then we can efficiently find $\delta, \mu \in \Omega_\varepsilon(1)$, with $\varepsilon \geq f(\varepsilon) \geq \delta \geq f(\delta) \geq \mu$ so that the total area of medium rectangles is at most $\varepsilon(OPT \cdot W)$.*

B.2 Structural lemma

We make use of a lot of ideas from Subsection A.2 in the packing of tall items and then packing S_1 and S_2. However we make the bifurcation of cases in a slightly different manner and make use of Steinberg's algorithm [10] in one of the cases. In the last section, for getting the $\frac{7}{4}+O(\varepsilon)$ approximation we showed that there exists an empty rectangular strip of sufficient width and depending on its height in the packing, we made our bifurcation. We do it in a similar manner in this section and show that either there exists a rectangular strip of sufficient width such that bottom of this strip lies below $\frac{2}{3}OPT$ and then we can pack it like Case 1 in Section A such that the height of the packing is at most $(\frac{5}{3}+O(\varepsilon))OPT$. If not, then the area of the items in R_2 (as defined in Section A) in packing P_2 is at most $(\frac{2}{3}+O(\varepsilon))OPT \cdot W$. Then we show that by packing some of the wide items in R_1 and using Steinberg's algorithm, we can pack all the rest of the items from R_2 and S_1 and S_2 in a box of size $W \times (\frac{2}{3}+O(\varepsilon))OPT$ which again implies a guillotine packing of height at most $(\frac{5}{3}+O(\varepsilon))OPT$.

Lemma 19. *For a given set of items I which can be packed in a strip of height OPT and width W such that they are guillotine separable, there exists a guillotine separable packing of items in I such that for any tall item $i \in I_{tall}$, $bottom(i) = 0$ and the tall items are nicely packed in $O_\delta(1)$ containers.*

We now define P_1 exactly as it was defined in the last section by shifting of the items which are packed completely in $[0,W] \times [\frac{1}{2}, OPT]$ by $\frac{1}{2}OPT$ vertically upward. We restructure P_1 to have all tall items in a sorted order (with respect to their heights) at the bottom of the strip S.

Claim 7. *The packing P_1 can be restructured so that we have all items in I_{tall} packed consecutively in a sorted order of non-increasing heights such that all of them touch the bottom of the strip, the new packing is still guillotine separable and the height of the packing is at most $\frac{3}{2}OPT$.*

We already have from the PPTAS that the intermediate, large, horizontal and small items can be nicely packed in $O_\varepsilon(1)$ containers that are guillotine separable. Since we want to compute our approximate solution in polynomial time, we use resource augmentation [7] to round the heights of the containers so that their sizes (and hence their positions) belong to a set of cardinality $n^{O_\varepsilon(1)}$ and this set can be computed in polynomial time.

Lemma 20. *Given a set of items I that can be packed nicely in a set of containers $\mathcal{C}$ such that $|\mathcal{C}| = O_\varepsilon(1)$, the packing is guillotine separable and the height of the packing is OPT, there exists another nice packing of $I' \subseteq I$ such that $a(I') \geq (1-O(\varepsilon))a(I)$, in a set of containers $|\mathcal{C}'| = O_\varepsilon(1)$ such that their sizes belong to a set A such that $|A| = n^{O_\varepsilon(1)}$ and A can be computed in polynomial time. Additionally this new container packing is guillotine separable and has a height of at most $OPT(1+O(\varepsilon))$.*

Lemma 21. *Given a set of items I' and a set of $O_\delta(1)$ horizontal and intermediate containers C such that the items in I' can be nicely packed in the containers, we can pack the items in I' in $C \cup S_1 \cup S_2$ in $n^{O_\varepsilon(1)}$ time.*

Consider the packing P_1 again. The strip S_2 can be easily packed on top with only an $O(\varepsilon)$ increase in the height of the packing. For the packing of S_1, we construct rectangular strips as in Subsection A.2 and show that either S_1 can be packed just like Case 1 of packing P_2 in Section A (with a different additional height region requirement) or the total area of the rectangles which are packed in R_2 in the packing P_1 is at most $(\frac{1}{3} + O(\varepsilon))OPT \cdot W$. We thus have the following lemma.

Lemma 22. *In the packing P_1, we have either of the two Cases.*

1. *There exists a rectangular strip such that the bottom of this strip lies below $\frac{2}{3}OPT$ and it has width at least $\varepsilon_3 W$ or*
2. *The area of the region in R_2 occupied by rectangles packed therein is at most $(\frac{1}{3} + O(\varepsilon))OPT \cdot W$ for appropriately chosen constant ε_3.*

Proof. Consider $\varepsilon_3 = \varepsilon / f(\varepsilon)$ where $f(\varepsilon)$ represents the upper bound on the number of containers in our packing in R_1. Hence, either we do have a rectangular strip whose bottom lies below $\frac{2}{3}OPT$ and whose width is at least $\varepsilon_3 W$ or all the rectangular strips whose bottom is at a height of at most $\frac{2}{3}OPT$ have width at most $\varepsilon_3 W$. Hence the total width occupied by such strips is at most $\varepsilon_3 W \cdot f(\varepsilon) \leq \varepsilon W$. Thus, all the rest of the strips have a width of at least $(1 - \varepsilon)W$ altogether and their bottom lies above $\frac{2}{3}OPT$. This implies that the area of the region in R_2 in packing P_1 occupied by rectangles packed therein is at most $(\frac{1}{3} + O(\varepsilon))OPT \cdot W$ □

B.2.1 Some strip(tall/non-tall) has height at most $\frac{2}{3}OPT$ and width at least $\varepsilon_3 W$

Just like in Case 1 of packing P_2 in Section A, we consider the packing P_1 and shift the rectangles in R_2 by $\frac{1}{6}OPT$. Now, we pack S_1 in one of the rectangular strips whose bottom lies below $\frac{2}{3}OPT$ and whose width is more than $\varepsilon_3 W$ and this packing is feasible since all the rectangles which were packed in R_2 in P_1 now lie in $[0, W] \times [\frac{7}{6}OPT, \frac{5}{3}OPT]$. Call this packing P_2'. We have the following claims to formalize the above ideas for the packing P_2'.

Claim 8. *P_2' is a feasible, guillotine separable packing and has a height of at most $(\frac{5}{3} + O(\varepsilon))OPT$.*

We modify the packing P_2' to a packing P_3' as we did for P_1 to pack the tall items in a non-increasing order of their heights such that all the tall items touch the bottom of the strip S and the tallest item touches the left edge of the Strip S. The intermediate, large, small and horizontal containers which were in R_1 are packed to the right of the tall items in the same fashion as before, just that they are shifted by the *Mirror* operation to the right. By similar arguments as used in the packing P_2', we show that S_1 can be packed such that the height of P_3' is at most $(\frac{5}{3} + O(\varepsilon))OPT$.

Claim 9. *P_3' is a feasible, guillotine separable packing and has a height of at most $(\frac{5}{3} + O(\varepsilon))OPT$.*

B.2.2 No strip (tall/non-tall) has height at most $\frac{2}{3}OPT$ and width at least $\varepsilon_3 W$

Claim 10. *If $w_{max} \leq \frac{W}{2}$ and $h_{max} \leq 2x$ then we can pack rectangles of area $(W \cdot x)$ into a box of width W and height $2x$.*

Proof. We make use of Steinberg's algorithm to prove the claim. For a set of rectangles I', if

$$2a(I') \leq wh - (2w_{max} - w)_+(2h_{max} - h)_+$$

then I' can be packed into a box of size $w \times h$. Setting the values for $h, w, w_{max}, a(I')$ in the above inequality, we have that $w_{max} \leq W$, $h_{max} \leq h$ and also $L.H.S = 2W \cdot x$, $R.H.S = 2W \cdot x$ since $2w_{max} - w \leq 0$. Hence, $L.H.S \leq R.H.S$ and the claim holds. □

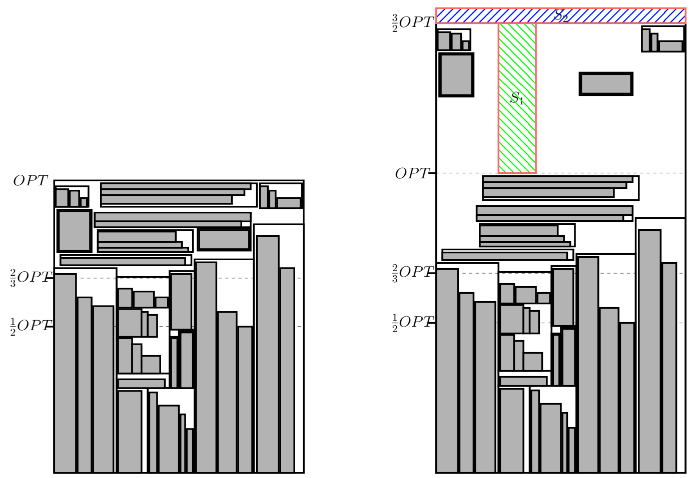
By Lemma 22 we have that the area of the region in R_2 occupied by rectangles packed therein is at most $(\frac{1}{3}+O(\varepsilon))OPT \cdot W$. The idea here is to pack all the rectangles i such that $w_i \geq \frac{W}{2}$ in R_1 along with the other items already packed according to P_1. Once we are able to achieve that, then by Claim 10 we can pack the remaining rectangles along with S_1 in $[0,W] \times [OPT, (\frac{5}{3}+O(\varepsilon))OPT]$ and since Steinberg's algorithm produces a guillotine packing, the overall packing would be guillotine separable.

Now, we consider the packing P_1 but with the addition of the horizontal containers of width at least $\frac{W}{2}$(call them 'wide' containers) with their height again reduced by $\frac{1}{2}OPT$. Call this packing P_4. Now, we want to restructure this packing so that the tall items are packed in a sorted order of non-increasing heights. They need not be touching in this restructured packing. If we restructure the packing such that the in the packing restricted to tall items, they are packed in a sorted order of non-increasing heights, then packing the other containers is not a problem which we do by guessing their positions since we have a $O_\varepsilon(1)$ number of containers and by Lemma 21 we can efficiently pack the items in these containers. Since we know that the containers are guillotine separable, we now consider the guillotine cutting sequence for P_4 till we get guillotine compartments which either do not have any tall container or do not have a container with width at least $\frac{1}{2}W$. We make use of the fact that at any vertical cut stage of the guillotine sequence, there can be at most one such guillotine compartment among its children which has a container with width at least $\frac{1}{2}W$ and the fact that in P_1 all tall containers are such that their bottom intersects the bottom of S and with these observations we rearrange the guillotine compartments and the items within the compartments such that the tall items are in a sorted order of non-increasing heights. Once we ensure this, as stated before we can guess the positions of the other containers and pack items in them and S_1, S_2 by reducing the problem to an instance of GAP. And we pack S_1 and the rest of the items in R_2 by Steinberg's algorithm in a box of height at most $(\frac{2}{3}+O(\varepsilon))OPT$ on top of R_1 so that the total height of the packing is at most $(\frac{5}{3}+O(\varepsilon))OPT$.

Lemma 23. *Given a guillotine separable packing P of I of height OPT such that there is no strip whose bottom lies below $\frac{2}{3}OPT$ and width at least $\varepsilon_3 W$, there exists a guillotine packing P_5 of $I \cup S_1 \cup S_2$ of height at most $(\frac{5}{3}+O(\varepsilon))OPT$.*

Proof. We show that all items in P_1 in R_1 along with the wide containers can be packed in the region R_1 and this packing is guillotine separable. Then by Claim 10, we get a packing of height at most $(\frac{5}{3}+O(\varepsilon))OPT$. By considering the cut $y = OPT$ and then since the packing of items in $[0,W] \times [OPT, (\frac{5}{3}+O(\varepsilon))OPT]$ is guillotine separable since its a Steinberg packing, the entire packing is guillotine separable.

So consider the packing P_4 as defined before. Since we know from the PPTAS that the containers are guillotine separable, we consider their guillotine cutting sequence until the respective guillotine compartment either does not have any tall container or does not have a wide container, see Figure 12 a. With this termination condition in mind, at the horizontal cut stage we let the guillotine compartments be as they are. At the vertical cut stage, by the *Mirror* operation we bring those compartments to the left which do not have any wide containers, see Figure 12 c . Now, in the final step of restructuring we ensure that the tall items are in a sorted order of non-increasing heights starting from the left side, by a swap operation. Lets say $i_1, i_2, ..., i_t$ for some integer t are the tall items in this packing such that $left(i_1) < left(i_2) < ... < left(i_t)$. Now if there are 2 tall items i_j and i_k such that $j < k$ and $h_{i_j} < h_{i_k}$ then we show another packing P_4' with tall items $i_1', i_2', ..., i_t'$ in that order from left to right, such that $i_l' = i_l$ for $l > k$, $i_k' = i_j$. The ordering of tall items in this new packing to the left of i_k' does not matter as by the above step we first bring the shortest tall item to the rightmost position among the tall items, then the second shortest tall item and so on. In this way we can ensure all tall items are sorted in non-increasing order of their heights (they need not be consecutively packed and can have other containers in between). Note that the tall items packed in the tall containers can be separated using at most one more stage of vertical cuts and then in these guillotine compartments for the tall containers we additionally have that no other item is present because of the structure of P_1. Hence, we consider the respective guillotine compartments of i_j and i_k. We then simply keep on swapping the guillotine compartment of i_j with the next guillotine compartment according to our guillotine cutting sequence



(a) (b)

Figure 11: (a) A guillotine separable nice packing P. (b) The guillotine packing P_4 of height at most $\frac{3}{2}OPT$. Here the rectangular strips S_1 and S_2 are shown packed only for presenting that they need to be packed in the region above $y = OPT$. It might be possible we cannot pack them within R_2 along with the other items

until we finally swap with the guillotine compartment of i_k, see Figure 13 . Notice that after each such swap the feasibility of the packing is maintained as height of any such guillotine compartment which we swap is at least the height of the guillotine compartment of i_k according to our structured packing and the packing remains feasible as we are just employing the *Mirror* operation successively. This proves the lemma. □

B.3 Assigning and packing items

Lemma 24. *There exists an algorithm for guillotine strip packing running in $n^{O_\varepsilon(1)}$ time which returns a packing of height at most $(\frac{5}{3} + O(\varepsilon))OPT$, If OPT is the height of the optimal guillotine packing.*

Proof. By Lemma 23 and Claim 8 we know that there exists a guillotine separable packing P_5 of height at most $(\frac{5}{3} + O(\varepsilon))OPT$. As mentioned before, we guess the value of OPT to within an additive error of $O(\varepsilon OPT)$ in polynomial time. We proceed with assuming that it is Case 1 of the packing and if that turns out to be incorrect we construct the packing according to Case 2. Hence first pack the tall items from the left edge of the strip such that their bottoms intersect the bottom of the strip and with no space in between any of them. Since we know that the items can be packed nicely in $O_\varepsilon(1)$ containers and the sizes of the containers belong to a set that can be computed in $n^{O_\varepsilon(1)}$ time, we guess the sizes of all the $O_\varepsilon(1)$ containers along with their respective types in $n^{O_\varepsilon(1)}$ time . Now, for packing the containers we partition the set of containers into 2 subsets C_1 and C_2 which can be done in at most $2^{O_\varepsilon(1)}$ ways. C_1 is the set of containers which we pack in R_1 adjacent to the tall items while C_2 is the set of containers which we pack in $[0, W] \times [\frac{7}{6}OPT, (\frac{5}{3} + O(\varepsilon))OPT]$. Then we make the rectangular strip construction with the top of the rectangular strips being $y = OPT$ and check if there exists a rectangular strip of width at least $\varepsilon_3 W$ and whose bottom lies below $\frac{2}{3}OPT$. If there indeed exists such a strip, we proceed exactly as per Case 1 in Lemma 17 just that S_1 is packed such that its bottom lies at a height of at most $\frac{2}{3}OPT$. We first pack all the $O(\frac{1}{\delta^2})$ large items and then reduce the problem to an instance of GAP just like in Lemma 17 and by Lemma 21 we know that all the items can be packed in polynomial time in $C_1 \cup C_2 \cup S_1 \cup S_2$.

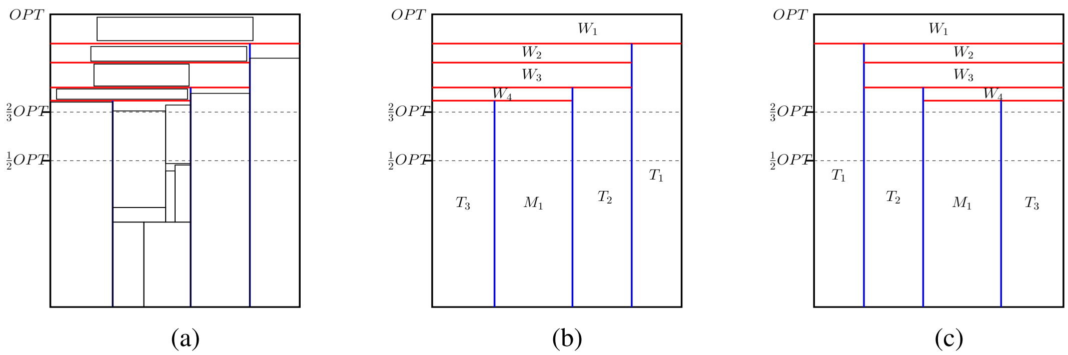


Figure 12: (a) The containers from the packing of Figure 11 a with the corresponding guillotine cuts until any guillotine compartment either does not have a wide container or does not have a tall container. (b) W_1, W_2, W_3, W_4 are pseudo-items that contain wide containers and T_1, T_2, T_3 are pseudo-items that contain tall containers. M_1 contains netiher type of containers . (c) The rearrangement of the tall pseudo-items and M_1 so that we have them in a non-increasing order of heights from left to right.

If there exists no rectangular strip from the previous construction such that its bottom lies below $\frac{2}{3}OPT$ and it also has a width of at least $\varepsilon_3 W$, then by Lemma 22 we have Case 2 of the packing. Then we guess the sizes of all the $O_\varepsilon(1)$ containers along with their respective types in $n^{O_\varepsilon(1)}$ time . Now, for packing the containers we partition the set of containers into 2 subsets C_1 and C_2 which can be done in at most $2^{O_\varepsilon(1)}$ ways but in this case we ensure that C_1 contains all of the wide containers also. C_1 is the set of containers which we pack in R_1 along with the tall items and we ignore C_2. For the containers in C_1, we guess their positions in between the tall items or otherwise on top of some other container, and the tall items are themselves packed in sorted order of non-increasing heights. There can be at most $n^{O_\varepsilon(1)}$ many guesses for such a packing in R_1. Then, considering S_1 and S_2 we convert this problem to an instance of GAP just like in Lemma 17. Note that we have not packed S_1 and S_2 yet. By our reduction to a GAP instance and the corresponding algorithm we ensure that items with at least $(\frac{2}{3} + O(\varepsilon))OPT \cdot W$ get packed in $C_1 \cup S_1 \cup S_2$. Now once we pack items as according to the GAP reduction, we consider all the items which have not been packed and along with S_1 and S_2 pack them by Steinberg's algorithm in $[0, W] \times [OPT, (\frac{5}{3} + O(\varepsilon))OPT]$, see Figure 13 c.

We then check the guillotine separability of the packing using Lemma 32. Finally the leftout items are packed according to NFDH on top which take up only $O(\varepsilon)OPT$ extra height and guillotine separability is ensured since NFDH guarantees a guillotine separable packing. Correctness of the algorithm is guaranteed by Lemma 23 and Claim 8. □

B.4 Packing leftout items

From the previous lemmas, we have that $a(I'_{small}) \leq O(\varepsilon)OPT \cdot W$ and $a(I_{medium}) \leq O(\varepsilon)OPT \cdot W$. By appropriately choosing μ, δ, we can ensure that the total area of horizontal items not packed till now is at most $O(\varepsilon)OPT \cdot W$. We simply use NFDH to pack these items on the top and since NFDH ensures a guillotine packing, we overall have a guillotine packing. Let H_{add} be the additional height on top we require to pack these items. By Lemma 29, we have that

$$\begin{aligned} H_{add} &\leq 2a(I'_{small})/W + 2a(I_{medium})/W + 2a(I'_{hor})/W \\ &\leq O(\varepsilon)OPT + \delta OPT \end{aligned}$$

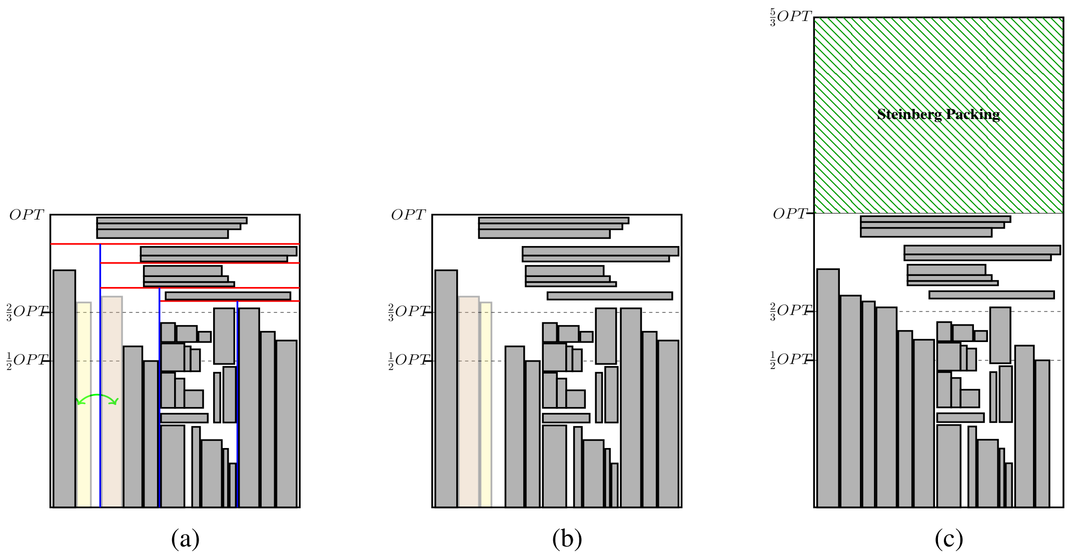


Figure 13: (a) The part of packing P_4 within $y = OPT$ along with corresponding guillotine cuts in red and blue colors. The yellow and brown rectangles are not packed in a non-increasing order of heights. (b) The yellow and brown rectangles are swapped without violating the guillotine cut property for the overall packing. (c)The packing P_5 obtained by repeated swapping of tall items along with potentially non-tall pseudo-items such tall items are packed in a sorted order of non-increasing heights (not necessarily consecutively). The rest of the items along with S_1 and S_2 shown in Figure 11 b above $y = OPT$ are packed by Steinberg's algorithm above $y = OPT$.

C A polynomial time $\frac{13}{8} + \varepsilon$ approximation

We make use of the techniques from the previous Sections to get an algorithm for guillotine strip packing with a better approximation ratio. Classification of items same as in Section B. Medium items packed as before.

C.1 Structural Lemma

We first show like before that there exists a packing of the items such that the tall containers/items touch the bottom of the strip S by the *Mirror* operation. Then we use ideas from both Section B and A to get our improvement. First, we show that there either exists a rectangular strip of sufficient width whose bottom lies below $\frac{5}{8}OPT$ or we have that the area of rectangles in R_2 in packing P_1 (as defined before) is at most $(\frac{3}{8} + \varepsilon)OPT \cdot W$. In the first case we get a $(\frac{13}{8} + O(\varepsilon))OPT$ approximation by identical proofs for Case 1 of Section B. We divide Case 2 into 2 more cases and show that either there exists a strip whose bottom lies above $\frac{7}{8}OPT$ or all strips of sufficient width have their bottom below $\frac{7}{8}OPT$ in which case apart from the wide items/containers, we pack the remaining items along with S_1 and S_2 by Steinberg's algorithm on top of $\frac{7}{8}OPT$. Finally we pack the wide items on top and show that the overall height of this packing is at most $(\frac{13}{8} + O(\varepsilon))OPT$. In this case there might be thin strips whose bottom may lie above $\frac{7}{8}OPT$ but if they are tall strips we rearrange them and pack them in the left end of the strip S. If they are non-tall strips then their width is at most εW and they are formed because of a non-tall container having height at most $\frac{1}{2}OPT$. Hence we can pack these thin items on top of our base of $y = \frac{7}{8}OPT$ along with the other items to be packed using Steinberg's algorithm with at most $O(\varepsilon)OPT$ increase in the overall height.

Lemma 25. *Given a guillotine separable packing P of I of height OPT such that there is no strip whose bottom lies below $\frac{5}{8}OPT$ or lies above $\frac{7}{8}OPT$ and has width at least $\varepsilon_3 W$, there exists a guillotine packing P_5 of $I \cup S_1 \cup S_2$ of height at most $(\frac{13}{8} + O(\varepsilon))OPT$.*

Lemma 26. *There exists an algorithm for guillotine strip packing running in $n^{O_\varepsilon(1)}$ time which returns a packing of height at most $(\frac{13}{8} + O(\varepsilon))OPT$, If OPT is the height of the optimal guillotine packing.*

D Tools

D.1 Maximum Generalized Assignment Problem

In this section we show that there is an pseudo polynomial time algorithm for the Maximum Generalized Assignment Problem (GAP) if the number of bins is constant. In GAP, we are given a set of k bins with capacity constraints and a set of n items that have a possibly different size and profit for each bin and the goal is to pack a maximum-profit subset of items into the bins. Let us assume that if item i is packed in bin j, then it requires size $s_{ij} \in \mathbb{Z}$ and profit $p_{ij} \in \mathbb{Z}$.

Let C_j be the capacity of bin j for $j \in [k]$. Let $p(OPT)$ be the cost of the optimal assignment.

Lemma 27 ([4])**.** *There is a $O(n \prod_{j=1}^{k} C_j)$ time algorithm for the maximum generalized assignment problem with k bins and returns a solution with maxiumum profit $p(OPT)$.*

Proof. For each $i \in [n]$ and $c_j \in [C_j]$ and $j \in [k]$, let $S_{i,c_1,...,c_k}$ denote a subset of the set of items $\{1, 2, ..., i\}$ packed into the bins such that the profit is maximized and the capacity of bin j is at most c_j. Let $P[i, c_1, c_2, ..., c_k]$ denote the profit of $S_{i,c_1,...,c_k}$. Clearly $P[i, c_1, c_2, ..., c_k]$ is known for all $c_j \in [C_j]$ for $j \in [k]$. Moreover we define $P[i, c_1, c_2, ..., c_k] = 0$ if $c_j < 0$ for any $j \in [k]$. We can compute the value of $P[i, c_1, c_2, ..., c_k]$ by a dynamic program that exploits the following recurrence:

$$P[i, c_1, c_2, ..., c_k] = \max\{P[i-1, c_1, c_2, ..., c_k], \\ \max_j\{p_{ij} + P[i-1, c_1, ..., c_j - s_{ij}, ..., c_k]\}\}$$

This dynamic program clearly runs in $O(n \prod_{j=1}^{k} C_j)$ corresponding to the entries in the DP table. □

We also have the following result for GAP with augmentation of the bins by a factor of $(1 + \varepsilon)$

Lemma 28 ([4]). *There is a $O((\frac{1+\varepsilon}{\varepsilon})^k n^{k+1})$ time algorithm for the maximum generalized assignment problem with k bins which returns a solution with profit at least $p(OPT)$ if we are allowed to augment the bin capacities by a $(1 + \varepsilon)$-factor for any fixed $\varepsilon > 0$.*

D.2 Next Fit Decreasing Height

One of the most recurring tools used as a subroutine in countless results on geometric problems is the Next Fit Decreasing Height(NFDH) algorithm which was originally analyzed in [1] in the context of strip packing. We will use two standard results related to NFDH for our requirements. We will provide the proofs of the both these results for sake of completeness.

Suppose we have a set of rectangles I'. NFDH computes in polynomial time a packing (without rotations) of I' as follows. It sorts the items $i \in I'$ in nonincreasing order of their heights h_i (corresponding widths w_i) and considers items in that order $i_1, ..., i_n$ (lets call this list L). Let width of the strip be W and define $A(L) = \sum_i h_i \cdot w_i$. Then the algorithm works in rounds $j \geq 1$. At the beginning of round j, it is given an index $n(j)$ and a horizontal segment $L(j)$(level j) going from the left to the right of C. Initially $n(1) = 1$ and $L(1)$ is the bottom side of C which is the first level. In round j, the algorithm packs a maximal set of items $i_{n(j)}, ..., i_{n(j+1)-1}$, with the bottom side touching $L(j)$ one next to the other from left to right. The segment $L(j+1)$ is the horizontal segment containing the top of $i_{n(j)}$ and extending from the left to the right of C. The space between two consecutive levels will be called a block. The algorithm continues in this manner till all items in L are packed. Hence we have a sequence of blocks $B_1, ..., B_k$ where the index increases from the bottom to the top of the packing and B_k is the last block in the packing of rectangles in L. Let A_i denote the total area of rectangles in block B_i and let H_i denote the height of block B_i. By way of our algorithm we have that $H_1 \geq H_2 \geq ... \geq H_k$. We state the result regarding the height of this packing through the following lemma.

Lemma 29 ([1]). *For a list L of rectangles ordered by nonincreasing height,*

$$\text{NFDH}(L) \leq 2A(L)/W + H_1$$

,

Proof. For each i, let x_i denote the width of the first rectangle in B_i and y_i be the total width of the rectangles in B_i. For each $i < k$, the first rectangle in B_{i+1} does not fit in B_i. Therefore $y_i + x_{i+1} > W$, $1 \leq i < k$. Since each rectangle in B_i has height at least H_{i+1}, and the first rectangle in B_{i+1} has height H_{i+1}, $A_i + A_{i+1} \geq H_{i+1}(y_i + x_{i+1}) > H_{i+1}W$. Therefore,

$$\begin{aligned} \text{NFDH}(L) &= \sum_{i=1}^{i=k} H_i \leq H_1 + \sum_{i=1}^{i=k-1} A_i/W + \sum_{i=2}^{i=k} A_i/W \\ &\leq H_1 + 2A(L)/W \end{aligned}$$

□

The second result is in the context of using NFDH to pack items inside a box. Suppose you are given a box C of size $w \times h$ and a set of items I' each one fitting in the box (without rotations). NFDH computes in polynomial time a packing (without rotations) of $I'' \subset I'$ as mentioned before. But unlike the previous lemma the process halts at round r when either all items are packed or $i_{n(r+1)}$ cannot be packed in the box. The following lemma describes the result regarding this packing.

Lemma 30. *Assume that for some parameter $\epsilon \in (0, 1)$, for each $i \in I'$ one has $w_i \leq \varepsilon w$ and $h_i \leq \varepsilon h$. Then NFDH is able to pack in C a subset $I'' \subset I'$ of area at least $a(I'') \geq \min\{a(I'), (1 - 2\varepsilon)w \cdot h\}$. In particular, if $a(I') \leq (1 - 2\varepsilon)w \cdot h$, all items are packed.*

Proof. The claim trivially holds if all items are packed. Thus suppose that is not the case. Observe that $\sum_{j=1}^{r+1} h(i_{n(j)}) > h$, otherwise item $i_{n(r+1)}$ would fit in the next shelf above $i_{n(r)}$: hence $\sum_{j=2}^{r+1} h(i_{n(j)}) > h - h(i_{n(1)}) \geq (1 - \varepsilon)h$. Observe that the total width of items packed in each round j is at least $w - \varepsilon w$, since $i_{n(j+1)}$, of width at least εw does not fit to the right of $i_{n(j+1)-1}$. It follows that the total area of items packed in round j is at least $(w - \varepsilon w)h_{n(j+1)-1}$, and thus

$$a(I'') \geq \sum_{j=1}^{r}(w - \varepsilon w)h_{n(j+1)-1} \geq w \sum_{j=2}^{r+1}(1 - \varepsilon)h_{n(j)} \geq (1 - \varepsilon)^2 w \cdot h \geq (1 - 2\varepsilon)w \cdot h$$

□

We will make use of Steinberg's algorithm [10] as a subroutine in our polynomial time algorithms.

Theorem 31 (Steinberg [10])**.** *We are given a set of rectangles I' and box Q of size $w \times h$. Let $w_{max} \leq w$ and $h_{max} \leq h$ be the maximum width and the maximum height among the items in I' respectively. Also we denote $x_+ := \max\{x, 0\}$. If,*

$$2a(I') \leq wh - (2w_{max} - w)_+(2h_{max} - h)_+$$

then I' can be packed into Q

We present an algorithm which checks whether a set of axis-aligned packed rectangles are guillotine separable.

Lemma 32. *Given a set of packed rectangles I' specified by their positions $(left(i), right(i)) \times (bottom(i), top(i))$ for each $i \in I'$ with $|I'| = n$, we can check in $O(n^3)$ time whether they are guillotine separable.*

Proof. Using standard shifting arguments, we can pack all the rectangles in a box of $[0, 2n-1] \times [0, 2n-1]$ where all the rectangles have integer co-ordinates for all of their four corners. To demonstrate this, we show how to do this in the x-direction, i.e., we consider the projections of the rectangles on the x-axis and consider both of the endpoints for each rectangle and then we assign them integer co-ordinates from $[0, 2n - 1]$ in the same order. We do this similarly for the y co-ordinates. Now, we specify a recursive procedure where for a rectangle compartment $\mathcal{C}$ in which we have a subset of rectangles $I'' \subseteq I'$ packed, we check all of the horizontal cuts and the vertical cuts at integral points which are feasible, incorporate such feasible cuts and recurse on the resulting smaller rectangle compartments. That is, we check all cuts which are line segments joining $(i, 0)$ and $(i, 2n - 1)$ for $i \in [2n - 1]$ and line segments joining $(0, j)$ and $(2n - 1, j)$ for $j \in [2n - 1]$ and incorporate either all feasible horizontal cuts or all feasible vertical cuts and recurse further(with alternating cuts). We only check cuts at integral co-ordinates since all rectangles are packed at locations which have integer co-ordinates by our preprocessing.

It is easy to see that in each level of the guillotine cutting sequence, at least 1 rectangle is separated from a rectangular compartment or else we can declare that they are not guillotine separable. And at each level of the guillotine cutting sequence we spend at most $O(n^2)$ time overall checking all the possible feasible cuts for each respective compartment. Since previous argument implies we can have at most $O(n)$ levels for the guillotine cutting sequence, the algorithm runs in $O(n^3)$ time. □

D.3 Resource Augmentation

In this section we show that the proof techniques used in [4] for packing rectangles with resource augmentation maintain the guillotine separability of the rectangles.

Lemma 33. *(Resource Augmentation Packing Lemma [4]) Let I' be a collection of rectangles that can be packed into a box of size $a \times b$, and $\varepsilon_{ra} > 0$ be a given constant. Here a denotes the height of the box and b denotes the width. Then there exists a container packing of $I'' \subseteq I'$ inside a box of size $a \times (1+\varepsilon_{ra})b$ (resp. $(1+\varepsilon_{ra})a \times b$) such that:*

1. *$p(I'') \geq (1 - O(\varepsilon_{ra}))p(I')$;*
2. *the number of containers is $O_{\varepsilon_{ra}}(1)$ and their sizes belong to a set of cardinality $n^{O_{\varepsilon_{ra}}(1)}$ that can be computed in polynomial time;*
3. *the total area of the the containers is at most $a(I') + \varepsilon_{ra}ab$;*

E Hardness

Theorem 34. *The k-Partition problem is NP-hard for any constant $k \geq 2$.*

Lemma 35. *Given a set of rectangles $\mathcal{R}$ and a set of rectangular containers $\mathcal{C}$ such that $|\mathcal{C}| = k$, where $k \geq 2$ is a positive integer, it is NP-hard to decide whether rectangles in $\mathcal{R}$ can be packed in containers in $\mathcal{C}$ such that in any container, they can be packed only in 1 level horizontally. That is, there can be no items on top of the other in a container packing.*

Proof. We reduce any instance of the k-Partition problem P to an instance I of the above problem. Given positive integers $i_1, ..., i_n$, $k \geq 2$ where k is a constant integer and $T = \sum_{j=1}^{j=n}$ in the k-Partition problem, we have to decide if the numbers can be partitioned into k sets $S_1, ..., S_k$ such that for each set S_l, $l \in [k]$, $\sum_{i_j \in S_l} i_j = T/k$. For each number i_j, in instance I we have a corresponding rectangle R_j such that $h(i_j) = 1$ and width $w_j = i_j$. We also have the set of containers $\mathcal{C} = \{C_1, ..., C_k\}$ where for each $p \in [k]$, $h(C_p) = 1$ and $w(C_p) = T/k$.

Now, we prove the equivalence of the reduction. If indeed P is a Yes instance, then for $j \in [n]$ if $i_j \in S_l$ we can pack rectangle R_j in C_l. Clearly this packing is feasible and all rectangles are packed side by side since height of each container is 1. If P is a No instance, then assume for contradiction that we can pack all the rectangles in the containers. Since the height of each container is same as the height of each rectangle, we must have a packing where rectangles are stacked side by side and no rectangle can be packed above the other in any container. Also, since $\sum_{j=1}^{j=n} w(R_j) = T$, it cannot be the case that any container has any empty area and thus, all containers should be packed completely. Now in this packing, for $j \in [n]$ if R_j is packed in C_l we choose $i_j \in S_l$ and we then have that each set S_l, $l \in [k]$, $\sum_{i_j \in S_l} i_j = T/k$. This proves the NP hardness. $\square$

Theorem 36. *There exists no polynomial time algorithm for the 2-dimensional guillotine strip packing problem with an approximation ratio $(\frac{3}{2} - \varepsilon)$ for any $\varepsilon > 0$ unless **P = NP**.*

Proof. Consider the following reduction from the 2-Partition problem. For an instance of the Partition problem P where we are given positive integers $i_1, ..., i_n$ such that $T = \sum_{j=1}^{j=n} i_j$ and where we have to check if we can partition the given numbers into two sets S_1 and S_2 such that $\sum_{i_j \in S_1} i_j = \sum_{i_k \in S_2} i_k = T/2$, we construct the following instance I of the 2-dimensional guillotine strip packing problem: Rectangles $\mathcal{R} = \{R_1, ..., R_n\}$ such that $h(R_k) = 1$ for any $k \in [n]$ and $w(R_k) = i_k$ and we want to check if there exists a guillotine separable packing of the rectangles in $\mathcal{R}$ in a half strip of width $T/2$ such that height of this packing is at most 2.

We now show that for the above reduction, If the answer to the 2-Partition instance P is Yes, there exists a guillotine separable packing of height exactly 2 for instance I. And if the answer to the instance P is No, the optimal guillotine separable packing has height at least 3. Note that any optimal packing can have only have an integral height as all rectangles have a height of exactly 1. Now, if the answer to P is Yes, we have 2 sets S_1 and S_2 such that $S_1 \cup S_2 = \{i_1, ..., i_n\}$ and $\sum_{i_j \in S_1} i_j = \sum_{i_k \in S_2} i_k = T/2$ where $T = \sum_{j=1}^{j=n} i_j$. Thus, we first pack all rectangles corresponding to numbers in S_1 from left to right at the bottom of the half-strip starting from $x = 0$ and without leaving any gap. Since the height of each rectangle is 1, we pack all rectangles similarly as before corresponding to numbers in S_2 from left to right on top of this packing. This results in a packing of height 2. It is a 2-stage guillotine separable packing because we first consider the horizontal cut $y = 1$ and then we separate all the rectangles in the 2 individual guillotine compartments by way of vertical cuts.

We show that if there exists a guillotine separable packing of rectangles in $\mathcal{R}$ of height at most 2, then the answer to the instance P would be Yes. Observe that any packing of the rectangles has to have a height of at least 2 since the area of the rectangles in $\mathcal{R}$ is T and the width of the half strip is T. If we have a guillotine separable packing of height 2, then by the area lower bound and the fact that all the rectangles have height 1, we have a guillotine cut at $y = 1$ and both the resulting compartments are completely filled by rectangles packed side by side without any space in between. Hence we consider all items corresponding to rectangles packed in 1 compartment as S_1 and the others as S_2. Hence, we have a positive 2-Partition instance. Taking the contrapositive of this statement proves our first claim of equivalence of the reduction.

If we have a polynomial time algorithm A for the 2-dimensional guillotine strip packing problem with an approximation ratio $(\frac{3}{2} - \varepsilon)$ for $\varepsilon > 0$, then

1. If the instance I is a Yes instance, we have a guillotine separable packing of height 2 and by applying the algorithm, we get a guillotine separable packing of height at most $2(\frac{3}{2} - \varepsilon) < 3$. And since only integral height packings are possible, we get a packing of height 2.
2. If the instance I is a No instance, from our reduction we have a guillotine packing of height at least 3.

Consider the following polynomial time algorithm for the 2-Partition problem. For an instance P of 2-Partition, we reduce the problem to an instance I of 2-dimensional guillotine strip packing as described. Then we apply the approximation algorithm A on this instance. If we get a packing of height 2, then by our previous claim for the reduction, P is a Yes instance. Else if we get a packing of height at least 3 we have a No instance.

This proves the theorem. $\square$