March 25, 2019
CS: DS&A
Proof School

Assignment 25: Dijkstra Verification
Due in class Thursday, 3/28

In this short assignment, you'll give a careful proof that Dijkstra's algorithm works. (The next assignment will be a longer coding assignment.) Recall the algorithm:

Start with A, and first relax along edges coming out of A. Then, pick an unprocessed vertex with the smallest current d-value. Call that node processed, and relax along edges coming out of it. Keep going.

Question 1. First, give a quick argument that every vertex v gets relaxed at some point; i.e. no vertex ends up with $d(v) = \infty$. (Hint: Suggested first line: We know G is connected, so there is some path from A to v.)

Question 2. Now, here's the setup for the main part of the argument. Suppose we're in the middle of the algorithm; let P be the set of processed nodes (so $V \setminus P$ is the set of unprocessed nodes.) By induction on stages, we can assume that $d(x) = D(x)$ for all nodes $x \in P$. (When we use d here, we mean the d-value at this particular stage.) We've just chosen $v \in V \setminus P$ to process. By question 1, v has already been relaxed at some point; suppose it was relaxed most recently by $u \in P$ (i.e. suppose the current predecessor of v is u). We claim that at this point $d(v) = D(v)$. We prove this by contradiction: suppose that $D(v) < d(v)$.

Your task for this assignment is to finish the argument. But to make things more clear, *I want you to do this in the style of a two-column argument. Every single statement must be stated separately, and given a reason, unless it's just establishing notation.* For example, "Let w be the weight of the edge from u to v" doesn't need a reason. Something like "$D(v') = d(v')$" definitely needs a reason, though.

You may use the fact, discussed in class, that if $A - v_1 - \ldots - v_k - v$ is a true path from A to v, then $A - v_1 - \ldots - v_i$ is a true path from A to v_i for each $i \leq k$.