DP Shortest Path

Have a directed graph $G=(V,E)$ and $\ell : E \to \mathbb{R}$ length for edge. $|V| = n$, $|E| = m$

Want $\text{dist}(u,v) = \min_{\text{paths } u \to v} \ell(p)$ $\qquad \ell(\text{path}) = \sum_{e \in \text{path}} \ell(e)$

If DNE path u,v then let $\text{dist}(u,v) = \infty$ $\longrightarrow$ edge case 1

Edge case 2 $\to$ net negative cycles $\to$ assume DNE

All pair Shortest Path

$\forall u,v$ find $\text{dist}(u,v)$

Single Source Shortest Path

for $s \in V$, $\forall u \in V$ get $\text{dist}(s,u)$

There are simplification if $\ell : E \to \mathbb{R}_{\geq 0}$

Obs 1 every subpath of a shortest path is a subpath

Obs 2 $\exists$ some sortest ST path with at most $n-1$ edges

spose not

$s = v_0 \to v_1 \to v_2 \to \cdots \to t = v_k \qquad k \geq n$

by pidgeonhole, we have that $v_i = v_j$ for $i \neq j$, $i,j \in \mathbb{N}$ nwlg

$\Rightarrow$ it must be the case that $\ell(Q) = 0$ $\quad$ (Q subpath v_i, v_j)

$\Rightarrow$ can remove to ensure path has fewer than $n-1$ edges

Recursion

$$\text{dist}(s,t) = \min_{x \text{ adj to } t} \left(\text{dist}(s,x) + \ell(x,t) \right) \; ?$$

but if x adj to t then $\underline{\text{dist}(s,t) \text{ uses } \text{dist}(s,t) \text{ uses } \text{dist}(s,t)}$. circular dependency.

To break symmetry... if optimal path of s,t has i edges, $\text{dist}(s,x)$ has $i-1$ edges.

$d^{(i)}(s,t)$ = length of shortest s,t path using $\leq i$ paths.

$$d^{(i)}(s,t) = \min_{x : (x,t) \in E} \left(d^{(i-1)}(s,x) + \ell(x,t) \right) \longrightarrow \text{single source}$$

Pseudo

Dy alg 2

$$d^0(s,t) = \begin{cases} 0 & s=t \\ \infty & s \neq t \end{cases}$$

For $i = 1, \dots, n-1$:

For all $t \in V$,

$$d^{(i)}(s,t) = \min_x \left(d^{(i-1)}(s,x) + \ell(x,t) \right)$$

edges x2 — n vertices degree

Runtime → $O(nm)$ → BELLMAN FORD

$$d(s,t) = \begin{cases} 0 & s=t \\ \infty & s \neq t \end{cases}$$

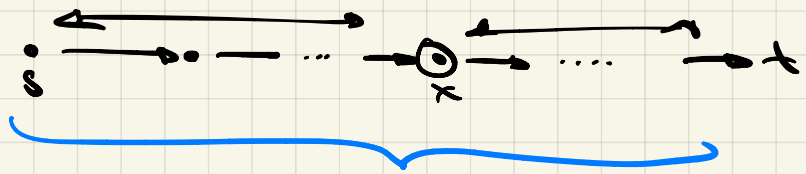
Repeat $n-1$ times

For all $(x,t) \in E$

$$d(s,t) = \min \left(d(s,t),\ d(s,x) + \ell(x,t) \right)$$

Diff formulation

Midpoint instead of pred



say $\leq 2^k$ → each one $\leq 2^{k-1}$

S, not in all as d, covered

$$dist^{(2k)}(s,t) = \min_{x \in V} \left(dist^{(2k-1)}(s,x) + dist^{(2k-1)}(x,t) \right)$$

↳ not as useful for SSSP. But nice for APSP

↳ PATH DOUBLING APSP.

↳ when $K=0$ $\quad d^{(0)}(u,v) = \begin{cases} 0 & u=v \\ \ell(u,v) & (u,v) \in E \\ \infty & (u,v) \notin E \end{cases}$

↳ care about for $k = 1, \dots, \lceil \log(n-1) \rceil$ — $\log n$

For ALL $u \in V$ → n

For ALL $v \in V$ → n

$$d^{(2k)}(u,v) = \min_x \left(d^{(2k-1)}(u,x) + d^{(2k-1)}(x,v) \right)$$ → n

Runtime $n^3 \log n$

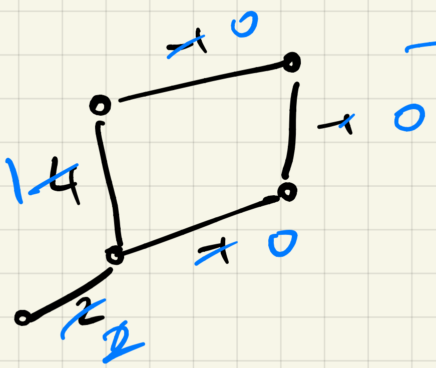
Trick → go from SSSP → SSSP$^+$

Thm| Given $G = (V, E, \ell)$ $\quad \ell : E \to \mathbb{R}$

Output $G' = (V, E, \ell')$ $\quad \ell' : E \to \mathbb{R}_{\geq 0}$

P is a shortest path in G iff shortest in G' too

e.g.|



→ smoothed out negatives along cycle

$p : V \to \mathbb{R}$ ⇝ price func → diff by vs

$$\ell_p(u,v) = \ell(u,v) + p(u) - p(v)$$

Claim: 1) Shortest wrt ℓ equiv to shortest wrt ℓ_p $\forall p$

2) C is a cycle, $\ell(C) = \ell_p(C)$

Pf 2| Sum will telescope

$$\ell_p(C) = \sum_{i=0}^{n-1} \ell_p(v_i, v_{i+1})$$

$v_0 = v_n$

$$= \sum_{i=0}^{n} \ell(v_i, v_{i+1}) + p(v_i) - p(v_{i+1})$$

tele

$$= \sum_{i=0}^{n} \ell(v_i, v_{i+1}) = \ell(C)$$

Pf 1) a path P

$$\underset{v_0}{\bullet} \to \underset{v_1}{\bullet} \to \cdots \to v_k$$

$$\ell_p(P) = \sum_{i=0}^{k-1} \ell_p(v_i, v_{i+1}) = \sum_{i=0}^{k-1} \ell(v_i, v_{i+1}) + p(v_i) - p(v_{i+1})$$

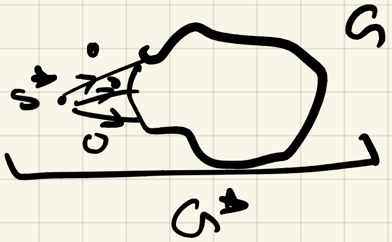
$$= \left(\sum_{i=0}^{k-1} \ell(v_i, v_{i+1}) \right) + \underbrace{p(v_0) + p(v_k)}_{\text{indep of path}}$$

⇒ shortest path from v_0 to v_k in G, ℓ same same as G', ℓ'

Upshot → can recover distances too!

Q: Can we always find a price function like this?

Pf of Thm | Find a good price → $\ell_p(u,v) \geq 0 \ \forall u,v \in V$

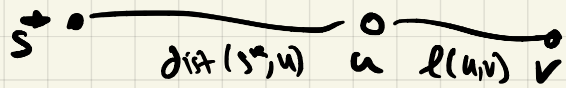


connect s^* to every pt in G with len 0 edge! → one way

obvious $dist(u,v) = dist^*(u,v)$

cx as can't go into s^*!

$$dist(s^*, v) \overset{def}{=} p(v)$$ → SSSP on G^*



Pick $(u,v) \in V$

$$\ell_p(u,v) = \ell(u,v) + p(u) - p(v) = \ell(u,v) + dist(s^*,u) - dist(s^*,v)$$

→ gives $dist(s^*,v) \leq dist(s^*,u) + \ell(u,v)$

$$\Rightarrow \boxed{\ell_p(u,v) \geq 0}$$