Global Min Cut

→ Graph $G = (V, E, c)$ $c: E \to \mathbb{N}$ capacity

→ X a cut & $c(X) = \sum_{\substack{v \in X \\ u \notin X}} c(v,u)$

↳ called a s,t cut if $s \in X$ & $t \notin X$

Note Any two nodes s-t on opp side of min global cut have same min s-t cut

Global min cut → given a graph & find a cut that is minimal so $X \neq \emptyset$ or V

↳ it is the capacity of some s-t cut

↳ so some max flow

→ trivial soln → iterate over all choices of s,t & calculate min cut.

Better fix s and move t around (for some choice of t they lie on diff sides of partition)

↳ so $O(n \cdot \text{min s-t})$ ~ $n \cdot O(\min\{mn, mn^2\})$

If unit cap $\boxed{n \cdot O(\min\{m^{3/2}, mn^{2/3}\})}$

↳ focus on this now!

Karger's Contraction Alg $\xrightarrow{\text{begins}}$ $O(n^4)$, $O(mn^2 \log n)$

↓ trick

$O(\min(n^2 \log^2 n, m \log^4 n))$

Claim The number of edges that cross the cut $(X, V \setminus X)$ is at most $2m/n$

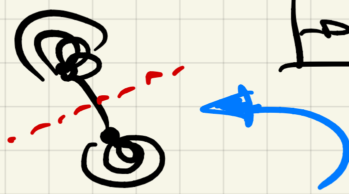
Pf) min cut $\leq \min_{v \in V} \deg(v)$ $\boxed{\text{isolate } v}$

$\leq$ average degree $= 2m/n$

Karger's Algorithm

→ never pick a self loop → arises from new!

- pick a random edge
- contract it (fuse endpoints) → new vertex inherits all edges of both end
  → may result in a multigraph
- repeat!
  → until 2 vertices!



→ the unique cut of this graph corresponds to the min cut iff we never contracted an edge in the min-cut

What happens if we contract wrong edge? → All bets are off

① time taken for a sequence of $n-2$ edge contr?

$O(n^2)$ or $O(m \log n)$

② Probability that we never contract an edge crossing a min cut?

claim: $P(\text{min cut ret}) \geq \frac{2}{n(n-1)}$

→ so repeat karger a few times (will address) and ret best!

→ what if we wanted const succ prob $= O(1)$

$\left(1 - \frac{2}{n^2}\right)^k$ → recall from prev lec

→ fail k times

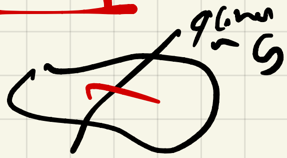
$k = n^2$ gives const

$k = cn^2 \log n$ gives high prob → $1 - \frac{1}{\text{poly } n}$

⇒ overall run time $n^2 \log n \cdot O(n^2)$

or $n^2 \log n \cdot O(m \log n)$

Pf of Claim Pr (Kargers ret min cut)



$$\Pr(\text{pick bad edge}) \le \frac{2m/n}{m} = \frac{2}{n}$$

$2m/n$ → max size of min cut!

m → total edges!

let A_i → i^{th} contr. doesn't mess up!

$P(A_1) \ge 1 - \frac{2}{n}$ ... $P(A_j \mid A_1 \cap \dots \cap A_{j-1}) = 1 - \frac{2}{n-(j-1)}$

$P(A_2 \mid A_1) \ge 1 - \frac{2}{n-1}$

given size of graph

want $P(A_1 \cap \dots \cap A_{n-2})$ = Pr(min cut survives to end)

$$= \prod_{i=1}^{n}\left(1 - \frac{2}{n-i+1}\right)$$

→ telescopes 2 terms apart!

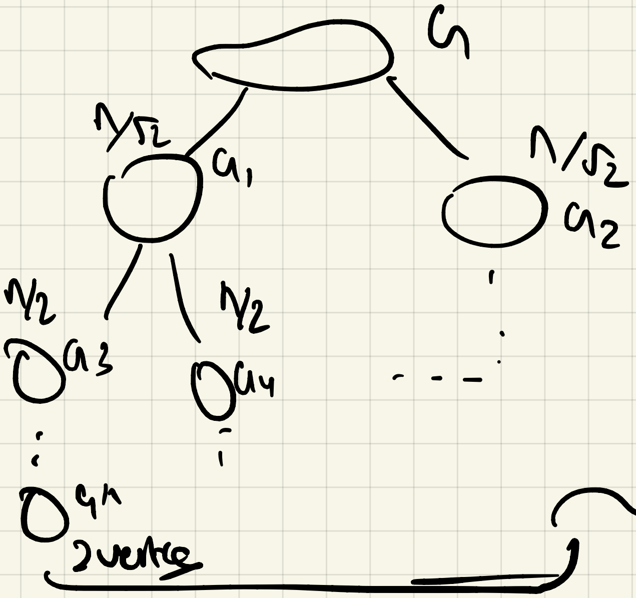
$$= \frac{2}{n(n-1)}$$

$P(A \cap B) = P(A)\,P(B \mid A)$

$P(B_1 \cap \dots \cap B_n)$
$= P(B_1)P(B_2 \mid B_1)\,P(B_3 \mid B_1 \cap B_2) \cdots P(B_n \mid B_1 \cap \dots \cap B_{n-1})$

Speed up Algorithm!

Start with G graph → apply contr until $\frac{n}{\sqrt{2}}$ vertices!



shrink by factor of $\sqrt{2}$

→ take min of this! level

Punchline:

runtime of tree

$n^2 \log n$ or $m \log^2 n$

success $\ge \frac{1}{\log n}$

∴ to get high prob $\ge 2 \log n$ trees

→ $O(n^2 \log^3 n)$ or

$O(m \log^4 n)$

Pf) Runtime is $n^2 \log n$!

[to combine]

↪ recursive struct! $T(n) = 2T\left(\frac{n}{\sqrt{2}}\right) + O(n^2)$

↪ master thm $T(n) \in n^2 \log n$

Success Prob!

$\geq \Omega\left(\frac{1}{\log n}\right)$

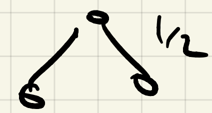
Chance min wt survives $n - \frac{n}{\sqrt{2}}$ contr

↪ same telescope!

[same!]

$\Rightarrow \frac{\cdots}{n(n-1)} = \boxed{\frac{1}{2}}$

Bacteria



1/2 prob of splitting in 2

↪ or dies!

K-levels of iter

$\Rightarrow$ prob bac surviving after n-levels!

Prob of surv at level k ≥

$P_k = 1 - \big(\underbrace{1 - \frac{P_{k-1}}{2}}_{\text{doesn't surv}}\big)^2$