Linear Prog

→ Zingermans deli's scones & muffins

↳ Resource req:

↳ 12 scones → 5 cups of flour, 2 eggs, 1 cup sugar

" muf → 4 , 4 , 2

→ Scones are \$10 / dozen

Muf " \$12 / doz

→ resource constraint

50 flour, 30 eggs, 20 cups of sugar

So, use 2 var. x_1 → doz of scones

x_2 → " " muf

max obj → $10x_1 + 12x_2$

↳ constr:

$5x_1 + 4x_2 \leq 50$ flour

$2x_1 + 4x_2 \leq 30$ eggs

$x_1 + 2x_2 \leq 20$ sugar

$x_1 \geq 0$

$x_2 \geq 0$

} no anti muffins/scones

The LP language

↳ real variables $x_1, \dots, x_n$

↳ max → $c_1x_1 + c_2x_2 + \dots + c_nx_n$ → to min negate c_i and max

↳ constr → m of em

- $a_{1,1}x_1 + \dots + a_{1,n}x_n \leq b_1$
- $a_{2,1}x_1 + \dots + a_{2,n}x_n \leq b_2$
- ⋮
- $a_{m,1}x_1 + \dots + a_{m,n}x_n \leq b_m$

→ negate if you want $\geq$ on some constr.

and $x_1, \dots, x_n \geq 0$

→ if you wanna drop $x_i \geq 0$, duplicate it to x_i', x_i'' and replace x_i with $(x_i' - x_i'')$ and stick to $x_i', x_i'' \geq 0$

→ can also have equality constr with duplicating inequality having $\geq b_i$, $\leq b_i$.

3 possibilities
- no feasible soln
- obj is unbounded
- feasible max soln, ∃ on bdry

lower bd, → find soln in feasible set & compare the objective for this

upper bd, → do work like

$3 \times ①$

$10x_1 + 12x_2 \le 15x_1 + 12x_2 \le \boxed{150}$

max obj → $10x_1 + 12x_2$

constr: $5x_1 + 4x_2 \le 50$ → ①

$2x_1 + 4x_2 \le 30$ → ②

$x_1 + 2x_2 \le 20$ → ③

$x_1 \ge 0$

$x_2 \ge 0$

$2 \times ① + 1 \times ②$

$10x_1 + 12x_2 \le 12x_1 + 12x_2 \le \boxed{130}$

want linear comb of constr so that coeff exceed the objective func

y_1 ① y_2 ② y_3 ③

$10x_1 + 12x_2 \le (5y_1 + 2y_2 + y_3)x_1 + (4y_1 + 4y_2 + 2y_3)x_2 \le y_1 \cdot 50 + y_2 \cdot 30 + y_3 \cdot 20$

for best upper bd

min → $50y_1 + 30y_2 + 20y_3$

with const $5y_1 + 2y_2 + y_3 \ge 10$

$4y_1 + 4y_2 + 2y_3 \ge 12$

also need

$y_1, y_2, y_3 \ge 0$

Dual LP

otherwise the signs flip & comb not possible

Changes in the primal → Changes in dual

1

– Objective: **maximize** $10x_1 + 12x_2$

– Subject to:

- $5x_1 + 4x_2 \geq 50$
- $2x_1 + 4x_2 \leq 30$
- $x_1 + 2x_2 \leq 20$
- $x_1 \geq 0, x_2 \geq 0$

What if changed this to a "≥" ? How would this affect the dual?

– Objective: **minimize** $50y_1 + 30y_2 + 20y_3$

– Subject to:

- $5y_1 + 2y_2 + y_3 \geq 10$
- $4y_1 + 4y_2 + 2y_3 \geq 12$
- $y_2, y_3 \geq 0, y_1 \leq 0$

"≤" constraints ↔ non-negative duals
"≥" constraints ↔ non-positive duals

account for sign flip when adding inequalities ⇒ ≤

2

– Objective: **maximize** $10x_1 + 12x_2$

– Subject to:

- $5x_1 + 4x_2 = 50$
- $2x_1 + 4x_2 \leq 30$
- $x_1 + 2x_2 \leq 20$
- $x_1 \geq 0, x_2 \geq 0$

What if changed this to a "=" ? How would this affect the dual?

– Objective: **minimize** $50y_1 + 30y_2 + 20y_3$

– Subject to:

- $5y_1 + 2y_2 + y_3 \geq 10$
- $4y_1 + 4y_2 + 2y_3 \geq 12$
- $y_2, y_3 \geq 0$, y_1 unconstrained.

"≤" constraints ↔ non-negative duals
"≥" constraints ↔ non-positive duals
"=" constraints ↔ unconstrained duals

you can multiply by anything and ≤ ineq satisfied

– Objective: **maximize** $10x_1 + 12x_2$

– Subject to:

- $5x_1 + 4x_2 \leq 50$
- $2x_1 + 4x_2 \leq 30$
- $x_1 + 2x_2 \leq 20$
- $x_1 \geq 0, x_2 \leq 0$

What if we changed "$x_2 \geq 0$" to "$x_2 \leq 0$". How would this affect the dual?

– Objective: **minimize** $50y_1 + 30y_2 + 20y_3$

– Subject to:

- $5y_1 + 2y_2 + y_3 \geq 10$
- $4y_1 + 4y_2 + 2y_3 \leq 12$
- $y_1, y_2, y_3 \geq 0$

When "$x_2 \leq 0$", x_2's constraint in the dual becomes an "≤" inequality.

– Objective: **maximize** $10x_1 + 12x_2$

– Subject to:

- $5x_1 + 4x_2 \leq 50$
- $2x_1 + 4x_2 \leq 30$
- $x_1 + 2x_2 \leq 20$
- $x_1 \geq 0, x_2 \leq 0$

What if we changed "$x_2 \geq 0$" to "$x_2 \leq 0$". How would this affect the dual?

– Objective: **minimize** $50y_1 + 30y_2 + 20y_3$

– Subject to:

- $5y_1 + 2y_2 + y_3 \geq 10$
- $4y_1 + 4y_2 + 2y_3 \leq 12$
- $y_1, y_2, y_3 \geq 0$

When "$x_2 \leq 0$", x_2's constraint in the dual becomes an "≤" inequality.

PRIMAL LP (in Standard Form)
Maximize: $c^T x$
Subject to: $Ax \leq b,\ x \geq 0$
$A \in \mathbb{R}^{m \times n};\ c, x \in \mathbb{R}^n;\ b \in \mathbb{R}^m$

DUAL LP (in Standard Form)
Minimize: $b^T y$
Subject to: $A^T y \geq c,\ \ y \geq 0$

- **Theorem.** The dual of the dual LP is the primal LP.
- **Strong Duality Theorem.** If Opt(Primal) exists, Opt(Dual) exists, and Opt(Primal) = Opt(Dual).
  – If Primal is infeasible/unbounded, Dual is infeasible/unbounded.

PRIMAL LP (in Standard Form)
Maximize: $c^T x$
Subject to: $Ax \leq b,\ x \geq 0$
$A \in \mathbb{R}^{m \times n};\ c, x \in \mathbb{R}^n;\ b \in \mathbb{R}^m$

DUAL LP (in Standard Form)
Minimize: $b^T y$
Subject to: $A^T y \geq c,\ \ y \geq 0$

- **Theorem.** The dual of the dual LP is the primal LP.
- ***Weak* Duality Theorem.** If Opt(Primal) exists,
  Opt(Primal) ≤ Opt(Dual).

$$\sum_{j=1}^{n} c_j x_j \leq \sum_{j=1}^{n}\left(\sum_{i=1}^{m} a_{ij} y_i\right) x_j \qquad j^{\text{th}} \text{ constraint in Dual LP}$$

The primal LP objective

$$= \sum_{i=1}^{m}\left(\sum_{j=1}^{n} a_{ij} x_j\right) y_i \qquad \text{Rearrange terms}$$

$$\leq \sum_{i=1}^{m} b_i y_i \qquad i^{\text{th}} \text{ constraint in Primal LP}$$

Complementary Slackness

- (Strong Duality) implies $\sum_j c_j x_j^* = \sum_i b_i y_i^*$.
- What does this imply?
  – $c_j x_j^* = \left(\sum_{i=1}^{m} a_{ij} y_i^*\right) x_j^*$
    - Must be that $x_j^* = 0$
      OR $c_j = \sum_{i=1}^{m} a_{ij} y_i^*$.
  – $\left(\sum_{j=1}^{n} a_{ij} x_j^*\right) y_i^* = b_i y_i^*$
    - Must be that $y_i^* = 0$
      OR $b_i = \sum_{j=1}^{n} a_{ij} x_j^*$.

$$\sum_{j=1}^{n} c_j x_j \leq \sum_{j=1}^{n}\left(\sum_{i=1}^{m} a_{ij} y_i\right) x_j$$

$$= \sum_{i=1}^{m}\left(\sum_{j=1}^{n} a_{ij} x_j\right) y_i$$

$$\leq \sum_{i=1}^{m} b_i y_i$$

constr holds with equality

when we have that these are optimal!

Cheat Sheet

Duality Cheatsheet

Primal	Dual
Maximize	***Minimize***
ith constraint ≤	$y_i \geq 0$
ith constraint ≥	$y_i \leq 0$
ith constraint =	y_i unconstrained
$x_j \geq 0$	jth constraint ≥
$x_j \leq 0$	jth constraint ≤
x_j unconstrained	jth constraint =
If $x_j \neq 0$ $\Rightarrow$	Then ith constraint holds with equality
Then jth constraint holds with equality $\Leftarrow$	If $y_j \neq 0$

If the primal is in "standard" format…

… the dual is also in "standard" format…

"Complementary Slackness" conditions

- Primal:
  - max $10x_1 + 12x_2$
  - Subject to:

$$5x_1 + 4x_2 \leq 50$$
$$2x_1 + 4x_2 \leq 30$$
$$x_1 + 2x_2 \leq 20$$
$$x_1 \geq 0, x_2 \geq 0$$

- Dual:
  - min $50y_1 + 30y_2 + 20y_3$
  - Subject to:

$$5y_1 + 2y_2 + y_3 \geq 10$$
$$4y_1 + 4y_2 + 2y_3 \geq 12$$
$$y_1, y_2, y_3 \geq 0$$

- Primal achieves optimum at $(x_1, x_2) = \left(\frac{20}{3}, \frac{25}{6}\right)$.
  - Objective value is $10x_1 + 12x_2 = 350/3$.
  - ***We know what "x_1" and "x_2" represent.***
- Dual achieves optimum at $(y_1, y_2, y_3) = \left(\frac{4}{3}, \frac{5}{3}, 0\right)$ Shadow prices
  - Objective value is $50y_1 + 30y_2 + 20y_3 = 350/3$.
  - ***What is the proper interpretation of the duals y_1, y_2, y_3 ?***