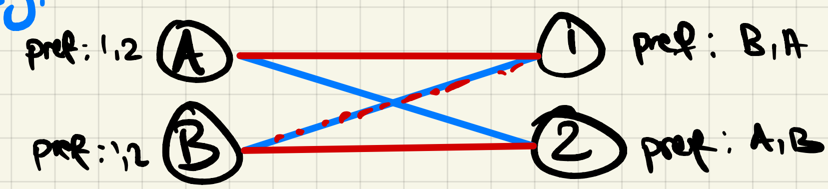
Lec 1

Stable Matching

- Match n men to n women given a ranked preference list
  - ↳ both men & women rank!
- stable matching is what we want!
- e.g. med students & hospitals
- → when does cheating occur?

e.g.



Stable

Unstable

↳ (B,1) cheats

blocking pair

M = set of men   W = set of women

A matching $F \subseteq M \times W$

↳ let mate(x) = y   if $(x,y) \in F$ or $\perp$ otherwise

↳ blocking pair (m,w)   m prefers w to mate(m) & same for w

Is there always stable matching? Can there be many

How fast?

→ trivial → n! matchings

Gale Shapley Alg

Idea: men will propose, women choose

Single man proposes according to pref list

woman can accept & dump or

reject & keeps date

- Let date(a) := $\perp$ $\forall$ $m \in M$, $w \in W$

Repeat:

1) let m be single. m proposes in pref order to w

→ who he hasn't proposed to yet

2) if w pref m to date(m) or date(m) = $\perp$

   set date(date(w)) := $\perp$

   date(w) = m

   date(m) = w

Repeat until all men taken!

We observe

→ since no man proposes to a woman twice (n^2 proposals max)
→ once a woman is dating she's never single
→ men get worse, women get better situations

Correctness – contradiction

Let us say we get a matching F with blocking pair (m, w) →

1. $w <_m$ mate(m)
2. $m <_w$ mate(w)

!

3. ① ⇒ m proposed to w before mate(m)
4. at some point w rejected m in favor of m' ⇒ $m' <_w m$
5. As w's dates improve, mate (w) $\leq_w$ m'
6. mate(w) $<_w$ m' $\leq_w$ m but also $m <_w$ mate(w) oops!

no total order

Notation

$<_m$ "total order for m"

$<_w$ → for w

e.g. $w <_m w'$

m prefers w to w'

→ might seem odd

Thm) let us say that F is the matching from GS. Let F' be any other stable matching.

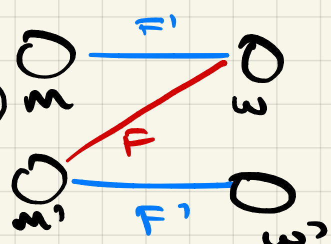
(A) proposer optimality: every man prefers F to F'

(B) chooser pessimality: every woman prefers F' to F

Pf (A) Contradiction

let us say (m,w) is 'eligible' if (m,w) ∈ F' for ^some stable F'

1. let (m,w) ∈ F' be the first eligible pair rejected in F by GS
2. w rejects m for m' in F
3. So $m' <_w m$, $w' <_{m'} w$ as F' stable ($w <_{m'} w'$) leads to instability
4. Thus m' proposed to w' eventually rejected & proposed to w



5. Since (m,w) is the 1st eligible pair rejected we see (m', w') was eligible & rejected earlier.

oops!

Pf (B) ⌐ case) (same img as earlier)

1. $m <_w m'$ &

2. $w <_m w'$ by part (A)

3. $w' <_m w$ (F) is stable)

4. contradiction