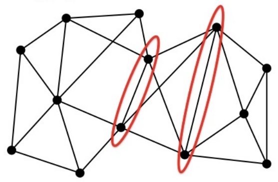
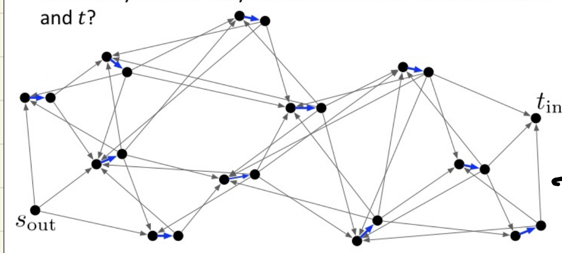
Randomized Vertex Conn

Vertex Connectivity

- Given connected unweighted graph $G = (V, E)$, how many vertices need to be removed to disconnect it?
  - G is called "k-connected" if removing any $k-1$ vertices leaves the remaining graph connected.



- How many vertices do you need to remove to disconnect s and t?



→ transformed in k out graphs.

- **Theorem.** We can test whether G is k-connected in $O(k^2 mn)$ time, $m = |E|$.
- **Proof.** Pick k sources $s_1, \ldots, s_k$. For each s_i and every t, compute an $(s_i)_{out}$-t_{in} flow in G_0 with value $\le k$.

So,

Randomized Vertex Connectivity

- **Theorem.** [Forster, Nanongkai, Saranurak, Yang, and Yingchareonthawornchai 2020] k-connectivity can be tested in $\tilde{O}(k^2 m)$ time, **w.h.p.** → with high prob.
- Uses a few basic facts about ***random sampling***:
  - Suppose an urn contains N balls, ϵN are green and the rest are purple.
    - If you pick $O(\epsilon^{-1} \log N)$ balls, at least one will be green with probability $1 - 1/N^{10}$.

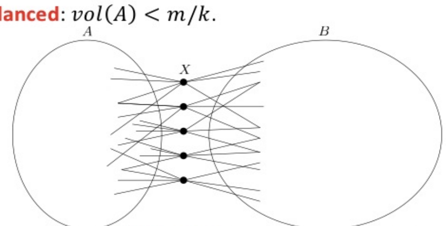
$$1 - (1-\epsilon)^{\epsilon^{-1} \cdot 10 \ln N} \ge 1 - 1/N^{10}.$$

    - If you pick ϵ^{-1} balls, all of them are purple with constant probability.

$$(1-\epsilon)^{\epsilon^{-1}} \approx e^{-1}.$$

    - …and if you do this experiment $O(\log N)$ times, at least one will find only purple balls with probability $1 - 1/N^{10}$.

- If A is a set of vertices, $vol(A) = \sum_{u \in A} \deg(u)$. — 2x both internal & external edges!
- Suppose G is not $(k+1)$-connected and has a k-cut X separating A from B.
- Two possibilities:
  - **Balanced**: $vol(A), vol(B) \ge m/k$.
  - **Unbalanced**: $vol(A) < m/k$.



→ B is larger half (half volume)

- **Claim:** if there is a **Balanced** k-cut, we can find a k-cut in $\tilde{O}(k^2 m)$ time.

sample prop to degree.

Pick ℓ pairs $((s_1, t_1) \ldots (s_\ell, t_\ell))$ — $10K \ln(N)$

Since $K < \sqrt{n}$ $\Rightarrow vol(X) \lesssim vol(A) + vol(B)$

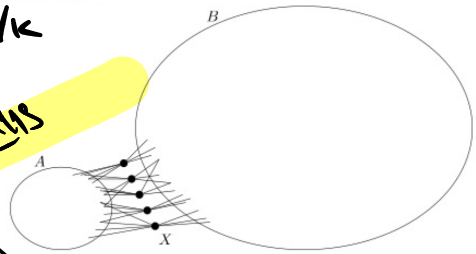
$\Pr(s_1 \in A, t_1 \in B) \ge \frac{1}{K} \cdot \frac{1}{2}$

do max flow using Ford Fulkerson K paths

$O(km) \cdot O(k \ln n)$

Unbalanced Cuts

- Suppose X is an **unbalanced** cut separating (A, B) and $vol(A) = L$. $< m/k$



$\frac{1}{\varepsilon}$ from green nails

α

- Sample $E' \subset E$, $|E'| = O\left(\frac{m}{L} \log n\right)$, $S =$ the endpoints of E'. Then $S \cap A \neq \emptyset$ with high probability $1 - 1/n^{10}$.
- **New problem:** given $s \in A$, if $vol(A) \approx L$, find a k-cut separating s in $\tilde{O}(k^2 L)$ time.

$|S| = O\left(\frac{m}{L} \log n\right)$

trick randomly let L – $2^1, 2^2, \ldots, \frac{m}{k}$

$\to$ proceed as if we have correct L.

Sample E, $|S| = O\left(\frac{m}{L} \log n\right)$

Unbalanced Cuts

- Given s, L, s.t. $\exists$ a k-cut X separating (A, B) with $s \in A$, $vol(A) \leq L$, find a k-cut in $\tilde{O}(k^2 L)$ time.
- Repeat $k + 1$ times:
  - $f = 0$ (start off with zero flow)
  - $G_f =$ residual network of directed graph w.r.t. f.
  - Run DFS from s in G_f, stop after scanning $k \cdot L$ edges.
    - $T =$ the DFS tree
    - $F =$ the non-tree edges scanned
  - Pick an edge $(u, v) \in T \cup F$ uniformly at random.
  - $f' =$ one unit of flow from s to u in T.
  - $f = f + f'$; update G_f.

$\to$ outer loops for $L = 1, 2^1, 2^2, \ldots \frac{m}{k}$

$\frac{m}{L} \lg n$ samples

Try $10 \ln n$ times,

in $(k+1)^{st}$ iter

(*) if can't explore kL edges

$\iff$ found k-cut!

Runtime

$O((k+1)(kL))$ $\to$ single $= O(k^2 L \lg n)$

$\to$ $\boxed{O(nk^2 \log^3 n)}$