Fast Fourier Transf

Recall: $A(x) = a_0 + a_1 x + \dots + a_3 x^3$, $A_{even} = a_0 + a_2 x$ $A_{odd} = a_1 + a_3 x$

↳ we get $A(x) = A_{even}(x^2) + x \cdot A_{odd}(x^2)$ upshot get $A(-x)$ too

↳ motivates using complex roots of unity.

e.g. $3 \times 3 \to 0011_2 \cdot 0011_2 \to A(x) = 1 + x + 0x^2 + 0x^3$ → use trailing zeros

evaluate at 4 pts to get pt value $\text{FFT}(1,1,0,0,i) = \begin{pmatrix} 1, & i, & -1, & -i \\ 2, & 1+i, & 0, & 1-i \end{pmatrix}$

recursive calls are identical → $\text{FFT}(1,0,-1) = (1,1)$ using

$\Rightarrow \text{FFT}(1,1,0,0,i) = (2, 1+i, 0, 1-i)$

↳ evaluate poly at $[1,1]$

use w^{-1}

$C = A^2$ has repr $(4, 2i, 0, -2i)$

recall! FFT is mat mult → $\text{IFFT}(4,2i,0,-2i,i) = \frac{1}{n}\text{FFT}(4,2i,0,-2i,\frac{1}{i})$

$\text{FFT}(4,2i,0,-2i,-i) = (4, 8, 4, 0)$ (evaluated at $1, -i, -1, i$)

even & rec

$\text{FFT}(4,0,-1) = (4,4)$

odd $\text{FFT}(2i,-2i,-1) = (0,4i)$

$\therefore C \to (1,2,1,0) \to 1 + 2x + x^2 \to C(2) = 9$

Q) $A \subseteq \{1, \dots, 100n\}$ $|A| = n$ $A+A = \{x \mid \exists y,z \in A \text{ so } y+z = x\}$

$A+A \subseteq \{1, \dots, 200 \cdot n\}$ treat A as a bit vector.

so $\{3,4,5\} \to P_A(x) = x^3 + x^4 + x^5$

↳ $(0,0,0,1,1,1,\dots)$

$$(P_A(x))^2 = \sum_{i=0}^{200n} c_i x^i$$

→ number of ways to make i

in fact $0 \to i$ not in $A+A$

$$T(n) = 2T\left(\frac{n}{2}\right) + n = O(n \log n)$$

Linear Time Selection

→ How many matches required to find k^{th} best team? 1^{st} is standard group

→ we wanna do it in linear time.

- Problem:
  - Given n totally ordered elements and $1 \le k \le n$, Find the k^{th} smallest element → rank k
  - Only allowed to *compare* two elements
- $k = 1$
  - $n-1$ comparisons necessary and sufficient
- $k = 2$? ... good exercise $n + \log n$
- median ($k = n/2$)? arbitrary k?

→ Rank(e, S) → gives rank of e in S

→ Quickselect (k, S)
- → pick random elt e
- → r = Rank(e, S) → $|S| - 1$ comp!
- → $S_<$ = elt in S less than e
- → $S_>$ = " " " greater " "
- → if $r = k$ return e
- → if $k < r$ quickselect $(k, S_<)$
  else quickselect $(k-r, S_>)$

Random Selection

- $Q(k,n)$ = Expected # comparisons made by QuickSelect
- Leads to an complicated recurrence relation:

$$Q(k,n) = n - 1 + \frac{1}{n-1}\left(\sum_{r=k+1}^{n-1} Q(k,r) + \sum_{r=1}^{k-1} Q(k-r, n-r)\right)$$

rank k → when $r \neq k$

upper bd. Forget about k & recurse on bigger half

$$Q(n) \le n-1 + \frac{1}{n}\sum_{r=1}^{n-1} \max\{Q(r), Q(n-r)\}$$

$$= n-1 + \frac{2}{n}\sum_{r=n/2}^{n-1} Q(r)$$

Random Selection

- Want to show that $Q(n)$ is linear in n
- A correct proof: assume $Q(n') \le cn'$ for ***some*** fixed c and ***all*** $n' < n$.
- $Q(n) = n - 1 + \frac{2}{n}\sum_{r=n/2}^{n-1} Q(r)$

$$\le n - 1 + \frac{2}{n}\sum_{r=n/2}^{n-1} cr$$
$$= n - 1 + \frac{2c}{n}\left(\frac{n}{2} + \left(\frac{n}{2}+1\right) + \cdots + (n-1)\right)$$
$$= n - 1 + \frac{2c}{n} \cdot \frac{n}{4}\left(\frac{3n}{2} - 1\right)$$
$$= n - 1 + \frac{3cn}{4} - \frac{c}{2}$$
$$< cn \qquad (c \ge 4)$$

Horrible in worst case

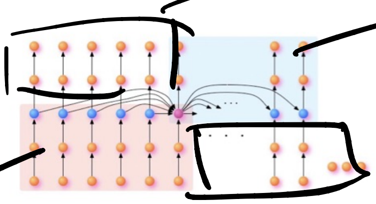
→ $(n-1) + (n-2) + \cdots + 1 = O(n^2)$
→ pick extremal points!

Median of Medians

→ find an elt e that's guaranteed to split list in almost half
→ replace quickselect random step with this.

Median of Medians Algorithm

- Select(k,S)
  - Divide S into $|S|/5$ groups of 5 (+ at most 4 leftover)
  - S′ contains median of each group
  - Find median e of S′ recursively
  - r = Rank(e,S)



→ can't say anything

→ $> e$

$< e$

Median of Medians Algorithm

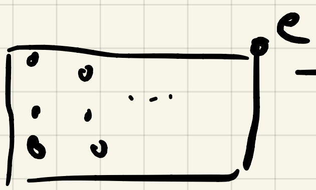
- Select(k,S)
  - Divide S into $|S|/5$ groups of 5 (+ at most 4 leftover)
  - S′ contains median of each group
  - Find median e of S′ recursively
  - r = Rank(e,S)
  - If r=k return e
  - If r>k return Select(k,$S_<$)
  - If r<k return Select(k-r,$S_>$)

$T(n)$ = # comparisons for any value of k → 8 comp to sort list of 5. $\frac{n}{5}$ list

$$T(n) = T\left(\frac{n}{5}\right) + T\left(\frac{7n}{10}+4\right) + \left(\frac{8n}{5}\right) + n$$

largest size of $S_>, S_<$

$O(n)$



→ at least $\frac{3n}{10} - 4$ less than

symmetrically $\frac{3n}{10} - 4$ greater than

→ max $S_>, S_<$ $\quad n - \left(\frac{3n}{10} - 4\right)$

simplify

$$T(n) = T\left(\frac{n}{5}\right) + T\left(\frac{7n}{10}\right) + 3n$$

Claim: $T(n) \le c \cdot n$ assume for $\forall n' < n$

$$\le c\frac{n}{5} + c\cdot\frac{7n}{10} + 3n$$

$$= c(.9)\cdot n + 3n \le cn$$

$$= (3.9)\, cn$$

$c \ge 30$