LP Duality and Optimum Matching

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EECS 477

Aaron Bernstein: Negative-Weight Single-Source Shortest Paths in Near-linear Time

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Rutgers University

WHERE: 3725 Beyster Building

MAP

WHEN: Friday, November 4, 2022 @ 3:00 pm - 4:00 pm

This event is free and open to the public

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WEB: Event Website

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Zoom link

LPs and Duality.

- Primal LP: $x \in \mathbb{R}^n, A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$
  - ***Maximize*** $c^T x \quad (\sum_{i=1}^{n} c_i x_i)$
  - Subject to: $Ax \leq b, x \geq 0.$
- Dual LP: $y \in \mathbb{R}^m$
  - ***Minimize*** $y^T b$
  - Subject to: $c^T \leq y^T A, y \geq 0.$

feasible dual

opt

feasible primal

- Assume Primal is feasible and bounded (objective not ∞).
- <u>**Weak Duality Theorem**</u>. If x, y are feasible, $c^T x \leq y^T b$.

- <u>**Strong Duality Theorem**</u>. If x^*, y^* are opt., $c^T x^* = {y^*}^T b$

Duality Cheatsheet

Primal	**Dual**
Maximize	***Minimize***
ith constraint ≤	$y_i \geq 0$
ith constraint ≥	$y_i \leq 0$
ith constraint =	y_i unconstrained
$x_j \geq 0$	jth constraint ≥
$x_j \leq 0$	jth constraint ≤
x_j unconstrained	jth constraint =

If the primal is in "standard" format…

… the dual is also in "standard" format…

LPs and Duality.

- **<u>Strong Duality Theorem</u>**. If x^*, y^* are opt., $c^T x^* = {y^*}^T b$

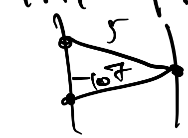
- An implication of Strong Duality.
  - Suppose j^{th} constraint is "loose" $\left(\sum_{i=1}^{n} a_{i,j} x_i^*\right) < b_j$.
  - What does this tell us about y_j^*? It must be 0.
- **<u>Complementary Slackness Conditions:</u>**
  - If $y_j \neq 0$ then the j^{th} constraint in the primal is *tight* (=).
  - If the j^{th} constraint in the primal is *loose* then $y_j = 0$.
  - If $x_i \neq 0$ then the i^{th} constraint in the dual is *tight* (=).
  - If the i^{th} constraint in the dual is *loose* then $x_i = 0$.

Max Weight Perfect Matching

- Given ***bipartite*** graph $G = (A \cup B, E)$, $E = A \times B$, and ***weight function*** $w: E \to \mathbb{R}$, find the maximum weight perfect matching.
  - A ***matching*** is a set of edges that share no endpoints.
  - A ***perfect matching*** is one in which all vertices are adjacent to a matching edge.

bipartite graph

$|A| = |B|$



$Ax \le b$

LP max

$$x(u,v) = \begin{cases} 0 & (u,v) \in \text{matching} \\ 1 & \notin \text{matching} \end{cases}$$

$$\max \sum_{(u,v)} x(u,v) \cdot w(u,v)$$

$$\forall v \quad \sum_u x(u,v) = 1 \quad \text{(only 1 edge selected)}$$

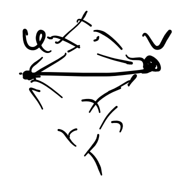
$$x \ge 0$$

Dual

$$\min \sum_v 1 \cdot y(v)$$

$\longrightarrow$ $y(v)$ unconstrained

$$y(u) + y(v) \ge w(u,v)$$



(coeff of $x(u,v)$)

Complementary Slackness

$$X(u,v) > 0 \Rightarrow y(u)+y(v) = w(u,v)$$

Max Weight Perfect Matching

- Given ***bipartite*** graph $G = (A \cup B, E)$, $E = A \times B$, and ***weight function*** $w: E \to \mathbb{R}$, find the maximum weight perfect matching.
- LP:
  - **Maximize** $\sum_{e \in E} w(e)x(e)$ subject to:
  - $\forall v \in V:\ \sum_{u \in V} x(v,u) = 1$
  - $x(u,v) \geq 0$
- Dual LP:
  - **Minimize** $\sum_{v \in V} y(v)$ subject to:
  - $\forall (u,v) \in E:\ y(u) + y(v) \geq w(u,v)$
  - y unconstrained.

The Hungarian Algorithm

THE HUNGARIAN METHOD FOR THE ASSIGNMENT PROBLEM[1]

H. W. Kuhn
Bryn Mawr College

1955

Kőnig, Dénes (1931), "Gráfok és mátrixok", Matematikai és Fizikai Lapok, ***38****: 116–119*.
Egerváry, Jenő (1931), "Matrixok kombinatorius tulajdonságairól", Matematikai és Fizikai Lapok, **38**: 16–28

DE AEQUATIONUM DIFFERENTIALIUM SYSTEMATE NON NORMALI AD FORMAM NORMALEM REVOCANDO

AUCTORE

C. G. J. JACOBI,
PROF. ORD. MATH. REGIOM.

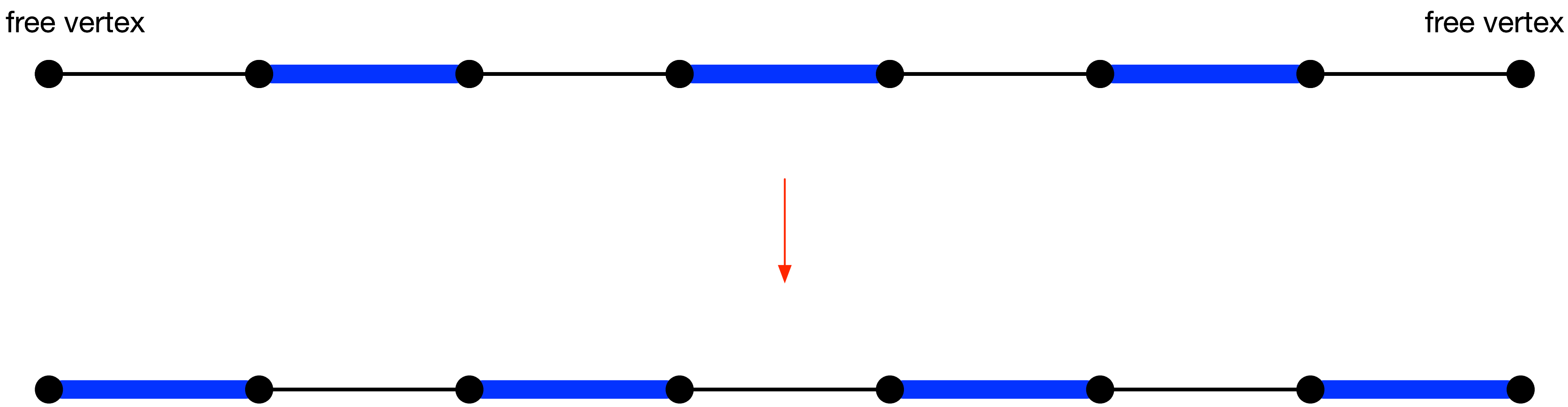
Written 1836, left on a shelf, published posthumously in 1890, discovered & translated in 2006.

The Hungarian Algorithm

- A *primal-dual* algorithm:
  - Maintain a ***feasible dual solution*** at all times.
  - Maintain a primal solution, that is ***usually infeasible***.
  - In each step do one of two things:
    - Do a "**dual adjustment**" — reduce the dual objective.
    - Do a "**primal adjustment**" — make the primal *less infeasible*.
  - Halt when you have a feasible dual & feasible primal with the same objective. By ***Strong Duality***, they are both optimal.

Augmenting Paths

- An ***augmenting path*** is one that
  - Starts and ends at free (unmatched) vertices
  - Alternates between unmatched & matched edges



- **<u>Observation.</u>** If M is a matching and P an augmenting path wrt M, then $M' = (M \cup P) - (M \cap P)$ is also a matching and $|M'| = |M| + 1$.

M matching $\quad e \in M \Leftrightarrow x(e) = 1$

Invariant(★)

Maintain(M, y)

matching dual

1) $(u,v) \in M \Rightarrow y(u) + y(v) = w(u,v)$

2. $(u,v) \notin M \Rightarrow y(u) + y(v) \geq w(u,v)$

If invariant ★ holds and M is perfect, then M is a max weight perfect matching.

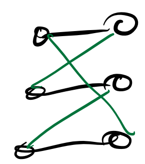
$$w(M) = \sum_{e \in M} w(e)$$

(count every y value exactly once)

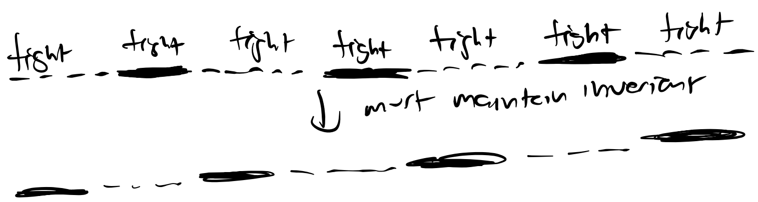
$$= \sum_{u \in V} y(u)$$

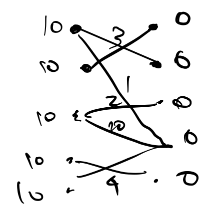
("tightness" from ①)

$$\geq \sum_{e \in M'} w(e) = w(M')$$



M' ← some other matching





Hungarian(G, w):

Initialize $M = \emptyset$

$u \in A \quad y(u) = \max_z w(z)$

$u \in B \quad y(u) = 0$

Repeat ______ ← $O(n^2 \cdot m)$ (n^2 → n)

F_A, F_B unmatched vertices in A, B. → 1 Dijkstra's

$E_= = \{(u,v) \mid y(u) + y(v) = w(u,v)\}$

$V_=$ = all vertices reachable from F_A by alt. path in

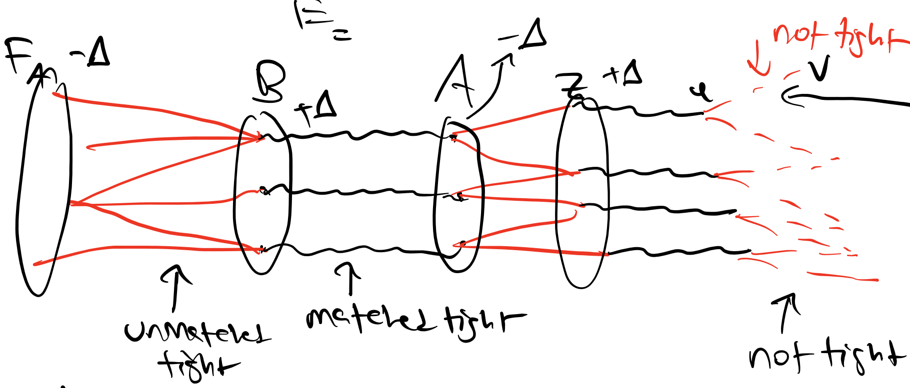
"primal only" → if $(V_= \cap F_B) \neq \emptyset \Rightarrow$ aug path P. Set $M = (M \cup P) / (M \cap P)$

"dual only" else

$3n^2$ times

$\leq |A|$ times ↑

$y(u)+y(v) \geq w(u,v)$



decrease y(u) until becomes tight

to fix

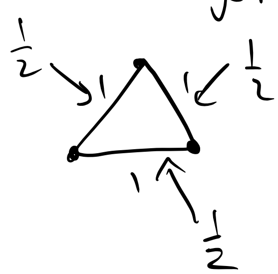
- everything in A, $-\Delta$ } all y values.
- " B, $+\Delta$

$$\Delta = \min_{\substack{u \in (V_= \cap A) \\ v \notin V_=}} y(u)+y(v)-w(u,v)$$

$y(u) \mathrel{-}= \Delta$ if $u \in A \cap V_=$

$y(v) \mathrel{+}= \Delta$ if $v \in B \cap V_=$

Bipartite graph assumption means this algorithm finishes. Not true for a general graph.



fractional soln for general graph.

$\boxed{\text{Extra}}$

Edmond's LP: max ______

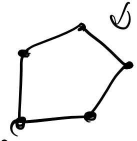
Still polynomial time

$\sum x(u,v) = 1$

$\forall S$

$|S|$ odd, $\sum_{(u,v) \in S} x(u,v) \leq \left\lfloor \frac{|S|}{2} \right\rfloor$

2^n constraints



not allowed odd cycle