Homeomorphisms

Generally, mathematicians are interested in classifying objects

To a set theorist, $\{0,1\} = \{\ddot{\cup}, \ddot{\cap}\}$

Defn 2 sets A, B are said to have the same cardinality iff

$\exists$ a bijection $f: A \to B$

What about the topologist? When are topologies alike?

recall: A top space is a set along with its topology.

Take $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$

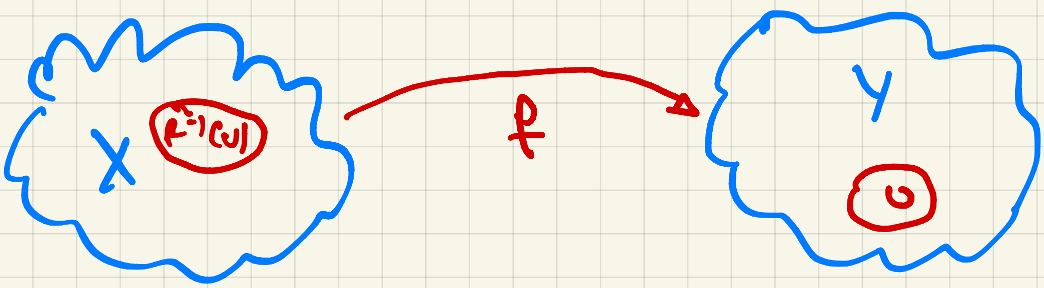
At the very least, we require the underlying sets to be the same.

So, at the very least we require that there exists a bijection

$$f: X \to Y$$

We also insist that it is continuous. What else?

Say $U \in \mathcal{T}_Y$,
$V \in \mathcal{T}_X$



$$\mathcal{T}_X \xrightarrow{f} \mathcal{T}_Y$$

$f^{-1}(U)$ $\qquad$ U

we also need,

$$\mathcal{T}_X \xrightarrow{f} \mathcal{T}_Y$$

V $\qquad$ $f(V)$ to be open

So f^{-1} is cts!

$(f^{-1})^{-1} = f$

Defn let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be top spaces. The bijection:

$$f: X \to Y$$

is a homeomorphism p.t

1. It is continuous
2. $f^{-1}: Y \to X$ is continuous

$f^{-1}: Y \to X$ is continuous means.

$\forall$ open $U \subseteq X$, $(f^{-1})^{-1}(U)$ is open $\iff$ $f(U)$ is open in Y.

That is: f maps open sets to open sets.

Defn 2 The bijection $f: X \to Y$ is a homeomorphism p.t:

1. f is cts, and
2. f sends all open subsets of X to open subsets in Y.

Defn let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be top spaces.

The function $f: X \to Y$ is an open map p.t

$\forall$ open $U \subseteq X$, $f(U) \in \mathcal{T}_Y$.

So, homeomorphisms induce a bijection between topologies.

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eg: Take $X = \mathbb{R}$ with $\mathcal{T}_{disc}$ and $Y = \mathbb{R}$ with $\mathcal{T}_{ind}$

Spose $\exists$ $h: Y \to X$ is a homeomorphism.

h is bijective, cts, h^{-1} is cts.

Take $a, b \in X$ $a \neq b$ consider $\{a\}, \{b\}$ as open subsets in X.

Since h is cts, we know $h^{-1}(\{a\}), h^{-1}(\{b\}) \in \mathcal{T}_Y$

Since h is surjective $h^{-1}(\{a\}), h^{-1}(\{b\}) \neq \emptyset$.

Since h is injective $h^{-1}(\{b\}) \cap h^{-1}(\{a\}) = \emptyset$

Also $\exists\ c, d \in Y$ s.t

$h^{-1}(\{b\}) = \{d\}$ and $h^{-1}(\{a\}) = c$, $c \neq d$.

But $\{c\}$ isn't open in Y! oops!

No homeomorphism from $Y \to X$!

Observe if $h : Y \to X$ is a homeomorphism, so is $h^{-1} : X \to Y$.

Remark Being homeomorphic defines an equivalence relation on top spaces!

Questions:

X homeomorphic? Y

Envision a function that stretches the ○ to the □

$X = \mathbb{R}$ Jeez

$A = [0, 1]$, $B = (0, 1)$

Q: is A and B homeomorphic? The same?

Set theorist: Yes! $\exists$ a bijection $[0,1] \to (0,1)$

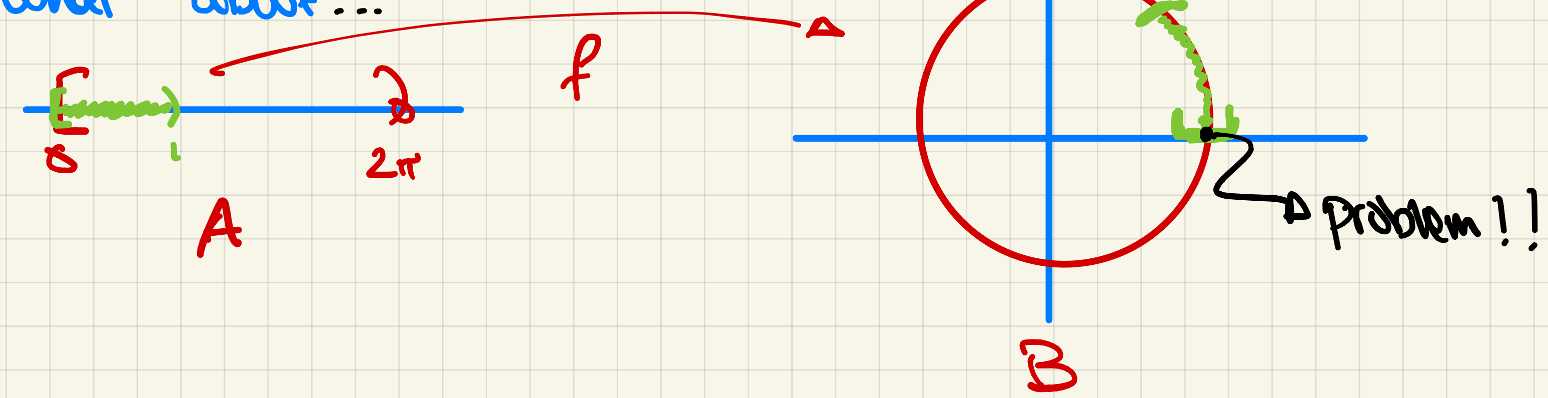
Topologist: No! As $(0,1)$ isn't compact and continuous functions conserve compactness. $\to [0,1]$ is compact

What about ... $C = [0,1)$ $D = (0,1)$?

Both aren't compact as they aren't closed!

Look at Heine-Borel!

What about ...



$f: A \to B$

$x \mapsto (\cos(x), \sin(x))$

1. Bijective, yes
2. Continuous, yes
3. is f^{-1} cts? No.

$[0,1)$ is open in A. $f([0,1))$ is not open.

No ball around 0 is a subset of $f([0,1))$

In $\mathbb{R}^2$!

Scaling is a homeomorphism!

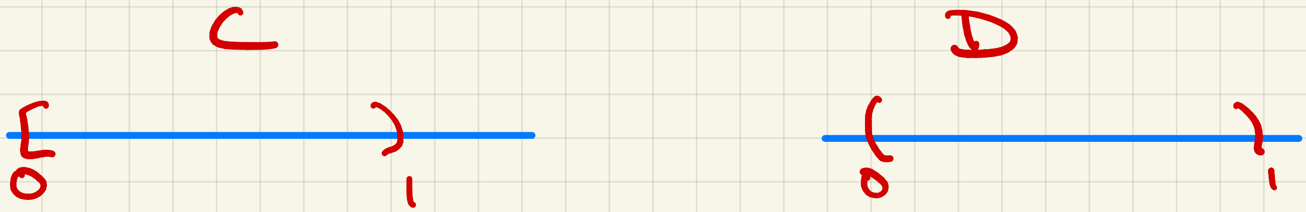
Stretching ○ to ▭ is a homeomorphism!

In fact, all polygons are homeomorphic!

Back to. $C = [0,1)$ $D = (0,1)$

Are these homeomorphic, Consider $X = \mathbb{R}, \mathcal{T}_{euc}$ $Y = \mathbb{R}\setminus\{0\}, \mathcal{T}_{euc}$.

Are X & Y homeomorphic. No! Y is disconnect and X is connected!



$C \setminus \{0\}$ is connected!,

$\exists$ a point $m \in C$ for which $C \setminus \{pt\}$ is connected!

There is no point like that here!

$\forall x \in D$, $D \setminus \{x\}$ is discon.

Defn) Let $(X, \mathcal{T}_X)$ be a connected top space.

The point $x \in X$ is a cut point pt $X \setminus \{x\}$ is disc

With our example, C has 1 non-cut point!

All point in D are cut points!

Show) if $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are connected top spaces and homeomorphic, then

The homeomorphism sends cut points to cut points!

$$h : X \to Y$$

$$h|_{cut(X)} : cut(X) \to cut(Y)$$

is a bijection!

$cut(X) = \{x \in X \mid x \text{ is a cut point of } X\}$

$(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are top spaces. Spose h is a homeomorphism.

Fix $A \subseteq X$

$$h|_A : A \to h(A)$$

is also a homeomorphism!

Consider: $[0,1)$, $(0,1)$

$h|_{[0,1)} : [0,1) \setminus \{0\} \to h([0,1) \setminus \{0\})$

$$\downarrow$$

$$\underbrace{h([0,1)) \setminus h(\{0\})}$$

$\downarrow$ disconnected as $h(0) \neq 0$ as $0 \notin (0,1)$

So, h isn't continuous!

Use these ideas to prove that

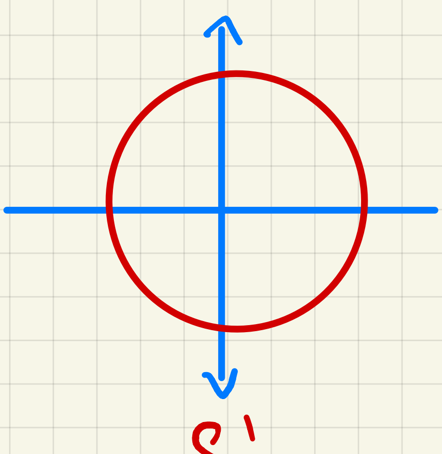
- $\mathbb{R}^2$ with euc top isn't homeomorphic to $\mathbb{R}$

↳ $\mathbb{R} \setminus \{(x_0, y_0)\}$ is path connected so it is connected

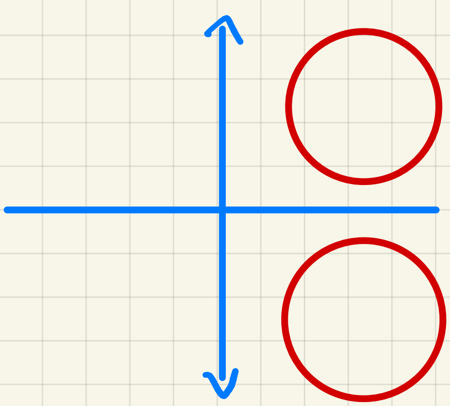
↳ so not cut points!

But $\mathbb{R} \setminus \{pt\}$ is disconnected so infinite cut points

Consider:



S'

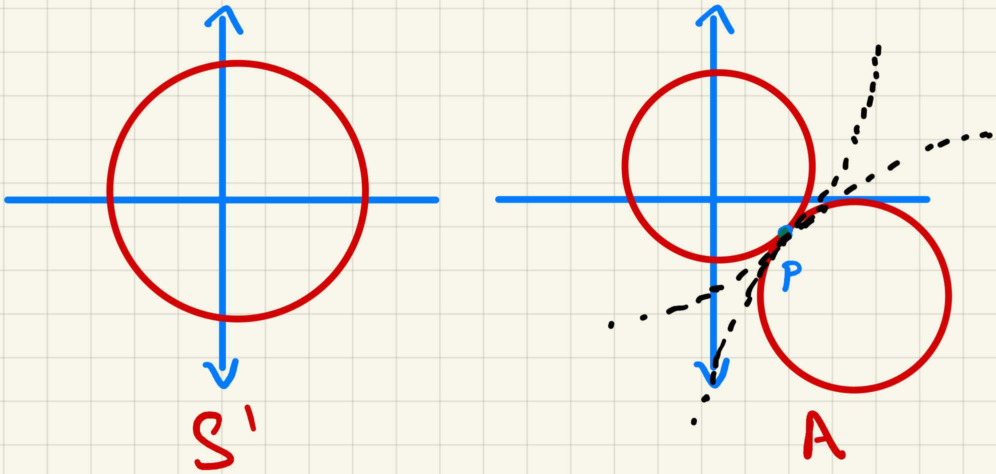


A'

These can't be homeomorphic as B is disconnected and A is connected.

↳ path connected $\Rightarrow$ connected

How about:

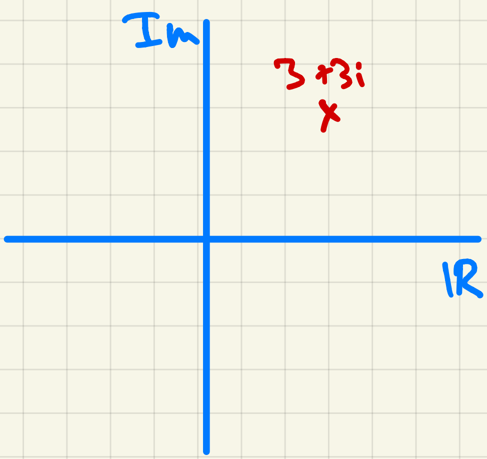


Both are connected, but A has a cut point!

↳ consider the two black circles intersect $A \setminus \{p\}$

∴ Since A has no cut points, they aren't homeomorphic!

$\mathbb{C}$



$$\mathbb{C} := \{a+bi \mid a, b \in \mathbb{R}\}$$

$$i^2 = -1$$

$\mathbb{C}$ vs $\mathbb{R}^2$

$$B_1(-5+3i) \simeq B_1((-5,3))$$

$$= \{(x,y) \in \mathbb{R}^2 \mid \delta((x,y),(-5,3)) < 1\}$$

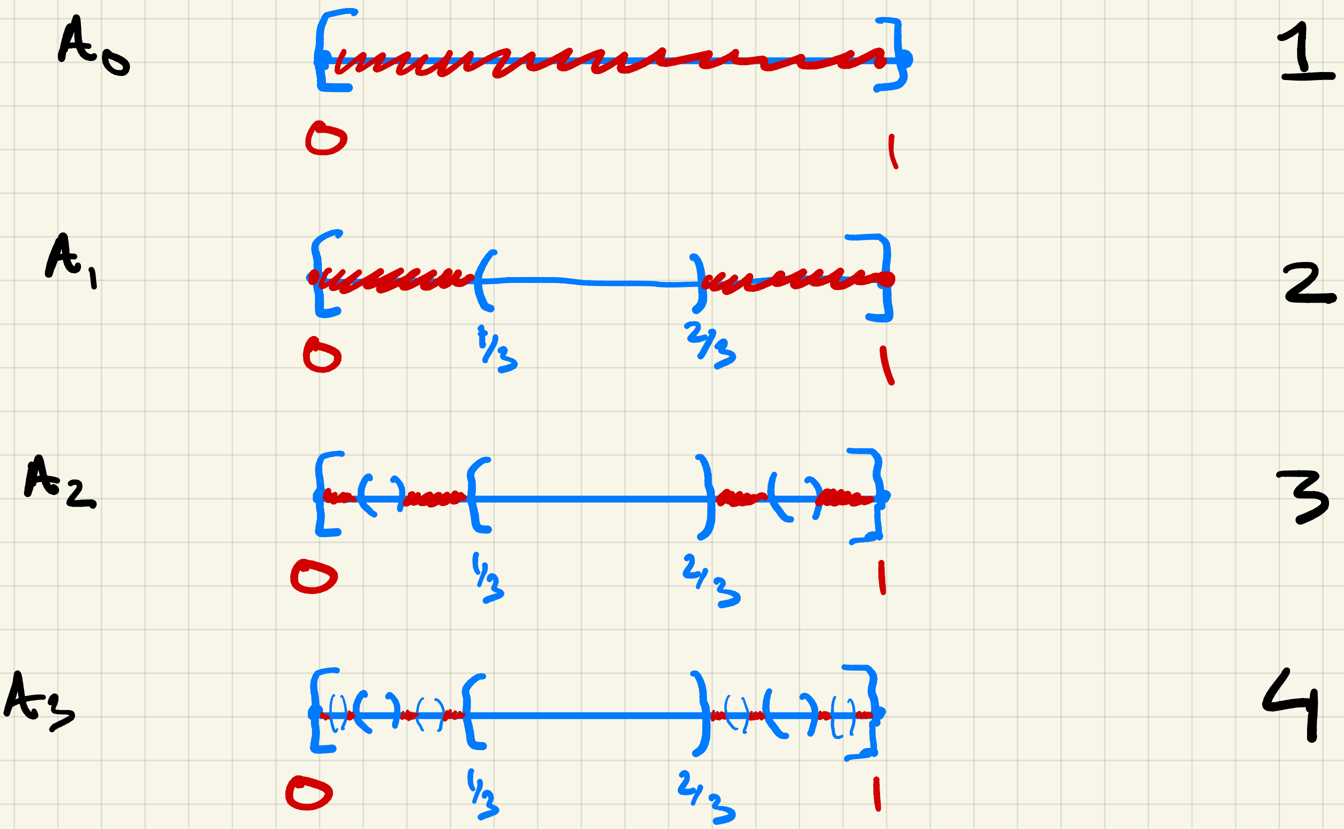
The topology on $\mathbb{C}$ comes from the euclidean metric

→

Cantor Set

Cantor Set

No of closed sets



Consider → subspace topology

$\bigcap_{i\in\mathbb{N}\cup\{0\}} A_i = ?$ we know $0, 1, 1/3, 2/3 \in$ of it

So it is non empty!

It is also bounded

Claim/ $\bigcap_{i\in\mathbb{N}\cup\{0\}} A_i$ is the intersection of infinitely many closed subsets of $\mathbb{R}$

→ We always remove open sets so each A_i is closed!

Can prove: By demorgan's laws

→ Intersection of an arbitrary collection of closed subsets is closed in a top sp!

Defn) The set $C := \bigcap_{i \in \mathbb{N} \cup \{0\}} A_i$ is the middle-third Cantor set!

note: ① It is a metric space (from $\mathbb{R}$)

② It is closed + bounded as a subset of $\mathbb{R}$ with euc top!

③ By Heine-Borel → It is a compact metric sp!

↳ $A \subseteq \mathbb{R}$ closed + bdd $\Rightarrow$ compact (so we wrap it)

④ C is totally disconnected!

Defn) A top sp $(X, \mathcal{T}_X)$ is totally disconnected pt the only connected subsets of X are the singletons and $\emptyset$.

Note → $(\mathbb{R}, \mathcal{T}_{disc})$ is totally disconnected

→ $\mathbb{Z}$ with euc top is totally disconnected

→ $\mathbb{N}$

→ $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are also totally disconnected!

⑤ C is perfect

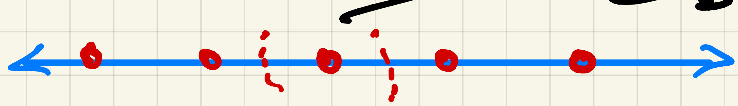
Defn) A subspace A of $(X, \mathcal{T}_X)$ is perfect p.t

① It is a closed subset of X

② A has no isolated points

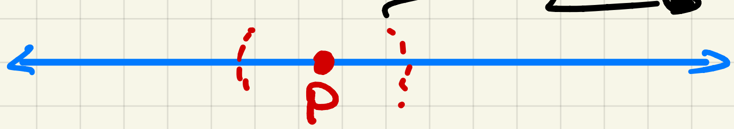
Iso-points ?

$A = \mathbb{N}$



→ open set isolating it

$A = \mathbb{Q}$



→ $\mathbb{Q}$ is dense so no point is isolated!

Defn) Let $(X, \mathcal{T})$ be a topological sp. Let $A \subseteq X$

The element $a \in A$ is a isolated point of A pt $\exists$ open $U \subseteq X$ that contains a and it contains no other elements of A.

or $U \cap (A \setminus \{a\}) = \emptyset \to$ implies $\{a\}$ is open in $\mathcal{T}_A$

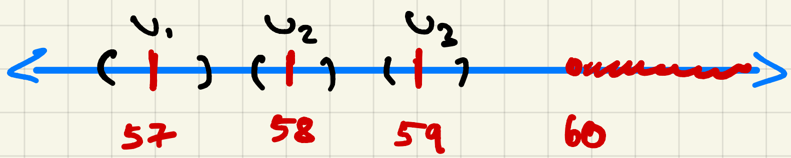
Eg $X = \mathbb{R}$ $A = \{57\} \to 57$ is an iso pt.

$X = \mathbb{R}$ $A = [0, 1]$

there are none due to the nature of $U \in \mathcal{T}_{Eucl}$

$X = \mathbb{R}$ $\{57, 58, 59\} \cup (60, \infty)$

there are 3



Q: Spose $a \in A$ is not an isolated pt. What does it mean?

It means $\forall U \in \mathcal{T}$ s.t $a \in U$ $\exists a' \in A \setminus \{a\}$ s.t $a' \in U$

or $\forall U \in \mathcal{T}$ we have

$$U \cap (A \setminus \{a\}) \neq \emptyset$$

In this case we call $a \in A$ an accumulation (limit) pt of A.

Eg: $X = \mathbb{R}$ $A = \{57\}$ no accumulation pts

$X = \mathbb{R}$ $A = [0, 1]$ every pt is an acc pt

$X = \mathbb{R}$ $A = \{57, 58, 59\} \cup (60, \infty)$

All $a \in (60, \infty)$ are acc pts.

Note: We can consider acc pts of a set A that don't belong to A.

Consider: $X = \mathbb{R}$ $A = (0, 1)$ $a = 1, 0$

Defn let $(X, \mathcal{T})$ be a top sp. let $A \subseteq X$

Thus the elt $x_0 \in X$ is an acc pt of A pt $\forall U \in \mathcal{T}$ st $x_0 \in U$ we have

$$U \cap (A \setminus \{x_0\}) \neq \emptyset$$

Q: What does this do for us?

That is, given a set $A \subseteq X$, we can study the set of acc pts of A & the set of isolated pts of A.

These objects tell us abt A!!

eg: $X = \mathbb{R}$ $A = (0,1)$ accumulation pt of $A \to [0,1]$

$X = \mathbb{R}$ $A = [0,1]$ acc pt of $A \to [0,1]$

$X = \mathbb{R}$ $A = \{57\}$ iso pt of $A \to \{57\}$

Q: Does a closed set contain all of its acc pts? Iff?

Defn The top sp $(X, \mathcal{T})$ is said to be Hausdorff pt $\forall$ distinct $x, y \in X$ $\exists$ disjoint $U, V \in \mathcal{T}$ st

$$x \in U, \quad y \in V$$

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tagline: in a hausdorff sp we can always separate distinct points with open sets

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eg: $X = \mathbb{R}$ $\mathcal{T}_{ind} = \{\mathbb{R}, \emptyset\}$

$x = 57$ $y = 0$ can't be seperated so,

$(\mathbb{R}, \mathcal{T}_{ind})$ is not Hausdorff.

Q: Does this have anything with being metrizable?

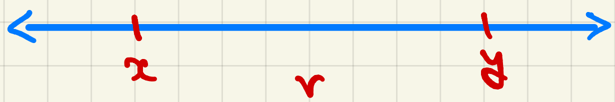
Q: Are metric sp Hausdorff? Yes!

Pf) let (X,d) be a metric sp. lets show this is Hausdorff !

let $x,y \in X$ $\quad y \neq x$. Task: find $U,V \in \mathcal{T}_d$ st

① $x \in U$, $y \in V$

② $U \cap V = \emptyset$



let $r = d(y,x)$ → positive!

Consider $U := B_{r/57}(x)$

$V := B_{r/57}(y)$ } open as they are balls

Don't intersect due to triangle ineq

Sp $z \in U \cap V$

$$\left.\begin{aligned} d(z,x) &< \frac{r}{57} \\ d(z,y) &< \frac{r}{57} \end{aligned}\right\} \quad d(y,x) \le d(x,z) + d(y,z) = \frac{2r}{57} < r \quad \text{oops...}$$

So Metrizable → Hausdorff !