Sequences again.

Recall) Let X be a top sp. A sequence of elts in X is

$$a : \mathbb{N} \to X$$
$$n \mapsto a_n$$

The sequence $n \mapsto a_n$ converges to $\ell \in X$ pt
$\forall$ open $U \subseteq X$ st $\ell \in U$ $\exists N \in \mathbb{N}$ st $\forall m > N$ $a_m \in U$.

Eg. $a_n = \dfrac{29n}{31n^2+57}$ in $\mathbb{R}$. Claim $\ell = 0$

Pf) Let $\varepsilon > 0$ be given. Task: find $N \in \mathbb{N}$ st $\forall n \geq N$

$$\left| \frac{29n}{31n^2+57} - 0 \right| < \varepsilon$$

$$= \frac{29n}{31n^2+57} < \varepsilon$$

Choose $N \in \mathbb{N}$ st $\dfrac{1}{N} < \varepsilon$ $\forall n \geq N$

$$|a_n - 0| = \frac{29n}{31n^2+57} < \frac{29n}{31n^2} \leq \frac{29}{31N} = \frac{29}{31} \cdot \varepsilon < \varepsilon$$

$\square$

Defn) Let X be a top sp and let $n \mapsto a_n$ be a seq.

A subsequence of $n \mapsto a_n$ is any function $\mathbb{N} \to X$ of the form $a \circ j$ where

$$j : \mathbb{N} \to \mathbb{N} \quad \text{is strictly increasing!}$$
$$k \mapsto j_k$$

Notation: $a \circ j : \mathbb{N} \to X$

$$k \mapsto a_{j(k)}$$

$$k \mapsto a_{j_k}$$

Lemma If a sequence $n \mapsto x_n$ converges in X, then every subsequence $k \to x_{n_k}$ converges in X.

Moreover, if $n \mapsto x_n$ converges to $\ell \in X$ so does the subsequence.

Pf Fix our subsequence $k \mapsto x_{n_k}$

Fix $U \subseteq X$ to be open and $\ell \in U$

Since $n \mapsto x_n$ converges to ℓ $\exists N' \in \mathbb{N}$ st $\forall n > N'$ $x_n \in U$.

Choose $N = N'$

Then $\forall k > N$, we have $n_k \geq k > N$ so $x_{n_k} \in U$. $\square$

Corollary let $n \mapsto x_n$ be a seq in top sp. X. Then $n \mapsto x_n$ converges to $\ell \in X$ iff every subsequence converges to ℓ

Pf $\Rightarrow$ previous lemma

$\Leftarrow$ immediate since $n \mapsto x_n$ is a subsequence of itself!

Focus on seq in $\mathbb{R}$

Defn A sequence of reals is monotonic p.t it is either increasing or decreasing (not strict).

- monotonic $\uparrow$ p.t $n > m \Rightarrow t_n \geq t_m$
- monotonic $\downarrow$ p.t $n > m \Rightarrow t_n \leq t_m$

Thm Every bounded monotonic function converges.

Pf Wlog suppose $n \mapsto s_n$ is mono $\uparrow$ and bounded.

We want to show this converges so we need a candidate limit.

Let $S = \{s_1, s_2, s_3, \ldots\}$

note: S is non-empty and bdd. Let $\ell = \sup(S)$

let $\varepsilon > 0$ be given $\exists N \in \mathbb{N}$ s.t $S_n \in (\ell-\varepsilon, \ell]$ by characterization of sup.

Since $n \mapsto s_n$ is mono $\uparrow$ we have that $\forall n > N$

$s_n \in (\ell-\varepsilon, \ell]$

$\therefore \forall n > N \quad |s_n - \ell| < \varepsilon$ $\square$.

Big thm - Bolzano - Weierstrass Thm

Every bounded sequence of reals has a convergent sub-sequence.

Pf) let $n \mapsto s_n$ be a bounded sequence

notation: (s_n) is alternate notation

Goal: find a mono $\uparrow$ or $\downarrow$ sequence

Call $n \in \mathbb{N}$ a 'valley' number if $\forall m \geq n$ we have $S_m \geq S_n$ $\rightarrow$

We will use this

I) Spose infinitely many valley nums.

$m_1 \leq m_2 \cdots$

Then, $S_{m_1} \leq S_{m_2} \leq \cdots$

So define the sub seq $k \mapsto S_{m_k}$

This is bdd and mono $\uparrow$ so it converges!

II) There are finitely many valley numbers (may be none!)

Then $\exists N \in \mathbb{N}$ s.t $\forall n > N$, n isn't a valley no

Set $P_1 = N+1$. As P_1 isn't valley $\exists P_2 > P_1$ s.t $S_{P_1} > S_{P_2}$

Similarly $\exists P_3 > P_2$ s.t $S_{P_3} < S_{P_2}$. We continue...

The subseq $k \mapsto S_{P_k}$ is bdd and mono $\downarrow$ $\square$.

Lemma) Squeeze

let $n \mapsto s_n$, $n \mapsto t_n$, $n \mapsto u_n$ be seq of reals
let $\ell \in \mathbb{R}$ if

- $\lim_{n \to \infty} s_n = \ell$
- $\lim_{n \to \infty} t_n = \ell$
- $\exists M$ s.t $\forall n > M$ $s_m \leq u_m \leq t_m$

Then

- $\lim_{m \to \infty} u_m = \ell$

Pf) Show (u_n) converges to ℓ

let $\varepsilon > 0$ be given.

Since $\lim_{n \to \infty} s_n = \lim_{n \to \infty} t_n = \ell$

$\exists N_s, N_t$ s.t $\forall n > N_s, |s_n - \ell| < \varepsilon \iff -\varepsilon + \ell < s_n < \varepsilon + \ell$

$\forall n > N_t$ $|t_n - \ell| < \varepsilon \iff -\varepsilon + \ell < t_n < \varepsilon + \ell$

Then $\forall$ $n > \max(N_s, N_t, M)$

$-\varepsilon + \ell < s_n \leq u_n \leq t_n < \varepsilon + \ell$

$\iff |u_n - \ell| < \varepsilon$ $\square$

Cauchy Sequence

Defn) let (X, d) be a metric sp. The seq $n \mapsto x_n$ is a Cauchy seq pt

$\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ st $\forall n, m > N$ we have

$d(x_n, x_m) < \varepsilon$

Q: What does this have to do with convergence?

Lemma Let (X, ∂) be a metric sp. Every convergent seq is Cauchy.

Pf Spse $n \mapsto x_n$ is a convergent seq to $\ell \in X$.

Let $\varepsilon > 0$ be given. We know $\exists N \in \mathbb{N}$ st $\forall n > N$

$\partial(x_n, \ell) < \frac{\varepsilon}{57}$ → ((we are in metric sp))

$\forall n, m > N$, by the triangle inequality,

$$\partial(x_n, x_m) \leq \partial(x_n, \ell) + \partial(x_m, \ell) < \frac{2\varepsilon}{57} < \varepsilon \qquad \square.$$

Question: Are there sequences that are Cauchy but don't converge?

Ans Yes! $X = (0,1)$ with ∂_{euc} and $n \mapsto \frac{1}{n}$ $(0 \notin X)$

Q: Consider compact metric sp?

Defn The metric sp (X, ∂) is Cauchy complete pt every Cauchy seq converges

non eg $\mathbb{Q}$ consider the sequence

$a_1 = 3 \quad a_2 = 3.14 \quad a_3 = 3.1415 \quad \ldots$

it wants to converge to π but $\pi \notin \mathbb{Q}$

FACT $(\mathbb{R}, \partial_{euc})$ is Cauchy complete

Upshot: to prove a seq of reals converges, show it's Cauchy!

i.e in $\mathbb{R}$ a sequence $n \mapsto x_n$ converges $\Leftrightarrow$ Cauchy

→

lemma) is $[0,1]$ Cauchy complete? **Yes**

Pf) let $n \mapsto x_n$ be a Cauchy seq in $[0,1] \subseteq \mathbb{R}$.
Since $\mathbb{R}$ is C.C, this sequence converges to some $\ell \in \mathbb{R}$

Goal : show $\ell \in [0,1]$

If ℓ is a limit point of $[0,1]$ then, by hw11, $\ell \in [0,1]$
since $[0,1]$ is closed! lets show this

<u>Issue</u>: if $n \mapsto y_n$ is a seq that converges to m,
m might not be an accumulation point of
$S := \{y_n \mid n \in \mathbb{N}\}$

eg $\to$ a constant sequence means $S = \{m\}$
So no accumulation point!

So we consider 2 cases

(I) S is finite (i.e the sequence is eventually const)

$S = \{x_1, \dots, x_n\}$, $n \mapsto x_n$ is Cauchy $\Rightarrow$ convergent
<u>of reals</u>

if S is finite, then $n \mapsto x_n$ is eventually const.
$\exists N \in \mathbb{N}$ s.t $\forall n, m > N$ $x_n = x_m = \ell$

So, $\ell \in S \subseteq [0,1]$ $\square$.

(II) S is infinite

In this case we show ℓ is an accumulation p.t of S.
Then ℓ is an acc pt of $[0,1] \Rightarrow \ell \in [0,1]$ as $[0,1]$
is closed!

Spose S is infinte. Show ℓ is an acc pt of S.

let $U \subseteq [0,1]$ be an open set containing ℓ. We
must show $S \cap (U \setminus \{\ell\}) \neq \emptyset$.

Since $\lim_{n \to \infty} x_n = \ell$ $\exists N \in \mathbb{N}$ s.t $\forall n > N$ $x_n \in U$

Moreover, $\exists m > n$ s.t $x_m \neq \ell$. If not,
$S = \{x_1, x_2 \dots x_N\} \cup \{\ell\} \Rightarrow$ finite!

$\Rightarrow S \cap (U \setminus \{\ell\}) \neq \emptyset$ $\square$.

Can we use the same ideas to show $[a,b]$ $a,b \in \mathbb{R}$ is Cauchy complete. How about an arbitrary closed set?

↳ Yes → it was important that the set was closed!

Lemma Let (X, δ) be a metric sp. Let $n \mapsto x_n$ be Cauchy. Then

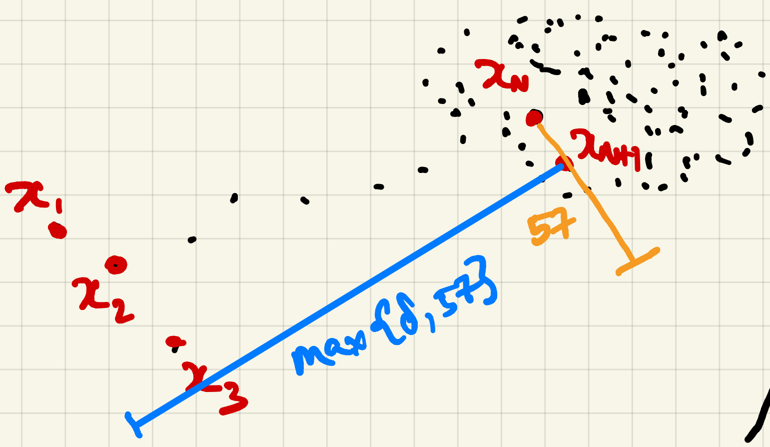
$S = \{x_1, x_2 \dots\}$ is bdd in X

↳ open ball that contains S

Pf Let $\varepsilon = 57$ $\exists N \in \mathbb{N}$ st $\forall n, m > N$ $\delta(x_n, x_m) < 57$

Now take $\{\delta(x_i, x_{N+1}) \mid 1 \le i \le N\}$ → distance of everything before N to $N+1$

↳ finite!



set $\delta := \max(\quad)$

Then $\forall n \in \mathbb{N}$,

$\delta(x_n, x_{N+1}) < \max\{\delta, 57\}$

$\Rightarrow S \subseteq B_{\max\{\delta, 57\}}(x_{N+1})$

Corollary All convergent sequences are bounded!

Pf Convergence ⟹ Cauchy ⟹ bdd ◻.

$\mathbb{R} \longrightarrow$

Back to $\mathbb{R}$

Thm Every Cauchy seq of reals converges (in $\mathbb{R}$)

Pf let $n \mapsto x_n$ be a cauchy seq of reals.
By the previous lemma, this is bounded!
By Bolzano–Weierstrass $\exists$ a convergent sub-seq
$k \mapsto x_{n_k}$ which converges to ℓ.

exclusive to $\mathbb{R}$ here are the sup and inf

Lets show: (x_n) converges to ℓ!
let $\varepsilon > 0$ be given $\exists N \in \mathbb{N}$ st $\forall n,m > N$ we have
$\delta(x_n, x_m) < \frac{\varepsilon}{57}$ ↳ cauchy

Since $k \to x_{n_k}$ converges to ℓ $\exists M \in \mathbb{N}$ st $\forall m > M$
$\delta(x_{n_k}, \ell) < \frac{\varepsilon}{57}$

fix $k > \max(N, M)$ $\Rightarrow n_k \geq k > \max(N, M)$
$\forall n > \max(M, N)$
$\delta(x_n, \ell) \leq \delta(x_n, x_{n_k}) + \delta(x_{n_k}, \ell)$ ↳ triangle
since $x_{n_k}, x_n > N$ & $x_{n_k} > M$
$< \frac{\varepsilon}{57} + \frac{\varepsilon}{57} < \varepsilon$ ◻

$\Rightarrow \mathbb{R}$ is Cauchy complete.

Q: Is being C.C a top invariant (preserved by homeo)?
Ans: No. $\mathbb{R}$ is homeo to $(0,1)$ but $(0,1)$ isn't C.C
↳ Consider isometry!
Q: Are compact metric sp always C.C?
hint: f is cts at $x \in X$ iff $\forall$ seq $n \mapsto x_n$ conv to x
↳ $f: X \to Y$ $\Rightarrow$ $n \mapsto f(x_n)$ conv to $f(x)$

Technical Lemmas for Cauchy seq in $\mathbb{R}$.

Goal: We want Cauchy sq to behave well under arithmetic ops.

→ Cauchy sq of $\mathbb{R}$s closed wrt $+, \cdot, \div$, scalar mult $\times$.

Lemma Spose $n \mapsto a_n$ and $n \mapsto b_n$ are Cauchy seq of reals. let $c \in \mathbb{R}$. Then

(1) $n \mapsto c \cdot a_n$ is Cauchy &

$$\lim_{n \to \infty} c \cdot a_n = c \cdot \lim_{n\to\infty} a_n$$

(2) $n \mapsto (a_n + b_n)$ is Cauchy &

$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

(3) $n \mapsto (a_n \cdot b_n)$ is Cauchy &

$$\lim_{n\to\infty} (a_n \cdot b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

(4) if $\lim_{n\to\infty} b_n \neq 0$ & if $b_n \neq 0 \ \forall n \in \mathbb{N}$ then

$n \mapsto \frac{a_n}{b_n}$ is Cauchy &

$$\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$$

We only need to show these converge as conv $\Rightarrow$ Cauchy

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Pf (1) $\lim_{n\to\infty} c \cdot a_n = c \cdot \lim_{n\to\infty} a_n$

let $\varepsilon > 0$ be given $\exists N \in \mathbb{N}$ st $\forall n > N$ $|a_n - a| < \frac{\varepsilon}{|c|+1}$

needed for 0

$\forall n > N$

$$|c \cdot a_n - c \cdot a| = |c|\,|a_n - a| \leq \frac{|c|\,\varepsilon}{|c|+1} < \varepsilon \qquad \square$$

(2) $\lim_{n\to\infty}(a_n+b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$
$= a \quad = b$

let $\varepsilon>0$ be given

$\exists N_a, N_b$ s.t. $\forall n > N_a \quad |a_n-a| < \frac{\varepsilon}{57}$

$\forall n > N_b \quad |b_n-b| < \frac{\varepsilon}{57}$

$\forall n > \max(N_a, N_b)$

$|(a_n+b_n)-(a+b)| \le |a_n-a| + |b_n-b| < \frac{2\varepsilon}{57} < \varepsilon$ $\square$

(3) $\lim_{n\to\infty}(a_n\cdot b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n = a\cdot b$

let $\varepsilon>0$ be given, since $n\mapsto a_n$ $n\mapsto b_n$ are converg
by the previous lemma they are bounded

$\exists M>0$ st $|a_n|, |b_n| < M \quad \forall n\in\mathbb{N}$

Also, $\exists N_a, N_b$ st $\forall n>N_a \quad |a_n-a| < \frac{\varepsilon}{57M}$

$\forall n > N_b \quad |b_n-b| < \frac{\varepsilon}{57M}$

$\forall n > \max(N_a, N_b)$

0

$|a_nb_n - ab| = |a_nb_n \overbrace{- a_nb + b\cdot a_n}^{} - a\cdot b|$

$\le |a_n|\cdot|b_n-b| + |b|\,|a_n-a| < M\frac{\varepsilon}{57M} + M\frac{\varepsilon}{57M} < \varepsilon$ $\square$

bdd

bdd ?

Is $|b| \le M$?? note $b = \lim_{n\to\infty} b_n$ & $|b_n| < M \quad \forall n\in\mathbb{N}$

squeeze $n\mapsto |b_n|$ & $n \mapsto M \to$ const
$\hookrightarrow$ converges to $|b|$

by squeeze

$\lim_{n\to\infty} b_n \le \lim_{n\to\infty} M \implies |b| \le |M|$ $\square$

(4) Quotients for us !