- Ordered fields
- Absolute value
→ Archimedian and Density
→ Inductive subsets ⇒ $\mathbb{N}$
→ Suprema, Infima, Maximals and Minimals.

Lec 4 - 6

Ordered Fields and Results in Ordered Fields

Notation

$a \in P \iff a > 0$

$a = 0 \iff a = 0$

$-a \in P \iff a < 0$

Convention

$a > b \iff a - b \in P$

$a < b \iff -(a-b) \in P$

$a \geq b \iff a - b \in P \cup \{0\}$

Absolute Value

function $\to f : F \to P \cup \{0\}$

$$|a| := \begin{cases} a & \text{if } a > 0 \\ 0 & \text{if } a = 0 \\ -a & \text{if } a < 0 \end{cases}$$

Results

In what follows, (F, P) is an ordered field

Lemma $x \in F \setminus \{0\}$, then $x^2 \in P$

PF fix $x \in F \setminus \{0\}$ $\to$ note $F \setminus \{0\}$ is non empty

Since $x \neq 0$, there are 2 cases, by trichotomy.

1. $x \in P \Rightarrow x^2 \in P$ due to closure
2. $-x \in P \Rightarrow (-x)(-x) \in P \Rightarrow x^2 \in P$ due to closure

Corollary we have $1 > 0$

PF Well, $1 = 1 \cdot 1 \neq 0$

So, $1 = 1^2$. By previous lemma, $1 \in P$

Remark Any field that contains $i \in F$ s.t $i^2 = -1$ can't be ordered as $-1 \notin P$.

$\therefore \mathbb{C}$ can't be ordered.

Corollary $2 := 1+1 \in P$ due to closure.

Lemma $a, b, c \in F$. If $a > b$, $b > c$, then $a > c$.

Since, $a > b$, $a - b \in P$

$b > c$, $b - c \in P$

$\therefore (a-b) + (b-c) \in P$

$a - c \in P$

$a > c$

$$\begin{aligned} a - c &= a + 0 + (-c) \\ &= a + ((-b) + b) + (-c) \\ &= (a + (-b)) + (b + (-c)) \\ &= (a - b) + (b - c) \end{aligned}$$

$\in P$ by closure

$\therefore a > c$

Lemma $\forall x \in F$ we have

a) $|x|^2 = x^2$ and

b) $x \leq |x|$

Pf Use trichotomy. Fix $x \in F$.

i) Assume $x \in P$

a) $|x|^2 = x^2 = x^2$ ✓

b) $x = |x|$ ✓

ii) Assume $x = 0$

a) $|x|^2 = 0^2 = 0^2 = x^2$ ✓

b) $x = 0 = |0| = |x|$ ✓

iii) Assume $-x \in P$

a) $|x|^2 = (-x)^2 = (-x)(-x) = x^2$ ✓

b) Since $|x|$ and $-x \in P$. ✓

$|x| + (-x) \in P$

$|x| + (-x) > 0$

$|x| > x$

Theorem) $\forall a,b \in F$ we have

① $|a\cdot b| = |a|\cdot|b|$

② $|a+b| \leq |a|+|b|$ → triangle inequality

Pf) From hw1, Exercise 2(c), if $x,y \in F$ and $x,y \geq 0_F$, then $x^2 \leq y^2$ if and only if $x \leq y$. Since both sides in ① and ② are non-negative, we can check both assertions after squaring each side.

(1') $|ab|^2 = (|a||b|)^2$

(2') $|a+b|^2 \leq (|a|+|b|)^2$

since $|x|^2 = x^2 \ \forall x \in F \Rightarrow |ab|^2 = (ab)^2 = a^2b^2$
$(|a||b|)^2 = |a|^2|b|^2 = a^2b^2$. Thus we proved (1').

For 2',

$$|a+b|^2 = (a+b)^2 = a^2 + 2_F ab + b^2$$

and

$$(|a|+|b|)^2 = |a| + 2_F |a||b| + |b|^2$$
$$= a^2 + 2_F |ab| + a^2 \rightarrow \text{using } (1')$$

Thus 2' reduces to

i — $a^2 + 2_F ab + b^2 \leq a^2 + 2_F |ab| + a^2$

since $x \leq |x| \ \forall x \in F$, we know $ab \leq |ab|$. But $2_F > 0_F$, so $2_F ab \leq 2_F |ab|$. From this i follows.

Comparing $\mathbb{R}$ and $\mathbb{Q}$.

$(\mathbb{R}, (0,\infty))$
$(\mathbb{Q}, (0,\infty) \cap \mathbb{Q})$ } both are ordered fields

We know $\mathbb{R} \neq \mathbb{Q}$ b/c $\sqrt{2} \in \mathbb{R}$, $\sqrt{2} \notin \mathbb{Q}$

Let (F, P) be an ordered field

Defn) Let $A \subseteq F$ & $u \in F$ if $\forall a \in A$, $u \geq a$, then u is an upper bound for A.

Defn) $B \subseteq F$ is bounded above p.t. B has an upper bound

eg)
- $F_{\leq 2} := \{x \in F \mid x \leq 2\}$
  bounded above by 2
- $\mathbb{R}_{<0} := \{r \in \mathbb{R} \mid r < 0\}$
  bounded above by 57
- $\emptyset \subseteq F$ is bounded above by any $u \in F$.

non eg) $\mathbb{Q} \subseteq \mathbb{R}$ is not bounded above in $\mathbb{R}$

$\mathbb{R}$ isn't bounded above $\to$ rather dubious statement.

Defn) Let $A \subseteq F$.

Spose A is bounded above by $\alpha \in F$. α is a least upper bound or supremum for A s.t

① α is an upper bound for A

② α is the least such upper bound; if $u \in F$ is an upper bound for A $\alpha \leq u$.

eg)

- $\emptyset \subseteq F$ has no lub. why? everything is an upper bound. So we need the least element of F. Is there one? ((give $s \in F$, $s-1 \in F$)) → **prove**

- $\{57\} \subseteq \mathbb{R}$
  $\text{lub}(\{57\}) = 57$

- $\text{Lub}(\mathbb{R}_{<0}) = 0$

Remark)

① If a l.u.b exists for $A \subseteq F$. It is !

<u>why</u>

Spose $\alpha, \beta \in F$ are l.ub for A

$\alpha \leq \beta$ and $\beta \leq \alpha$ $\iff$

$(\beta - \alpha) \in P \cup \{0\}$

$(\alpha - \beta) \in P \cup \{0\}$ $\iff$

$\alpha - \beta \in \{0\}$

$\alpha = \beta$

Forward) We will show $\mathbb{N}$ isn't bounded in $\mathbb{Q}$ or $\mathbb{R}$.

$\sup\{x \in \mathbb{Q}_{>0} \mid x^2 < 2\}$ DNE if $F = \mathbb{Q}$

↳ manifestation of holes in $\mathbb{Q}$

L.U.B property)

Spose (F, P) is an ordered field.
If every non-empty bounded above subset of F has a least upper bound, the F is said to have the "lub" property.

Defn) An ordered field is said to be complete p.t it has the L.U.B property.

Claim) (Proved in Dec)
$\exists!$ complete ordered field

Defn) We call this $\mathbb{R}$ $\rightarrow$ assume true for now

Defn) $A \subseteq F$ is bounded below p.t $\exists\, w \in F$ s.t $\forall a \in A$, $w \leq a$.

We say that $\beta \in F$ is the greatest lower bound for $A \subseteq F$ p.t

1. β is a lower bound for A
2. β is the greatest such. If $w \in F$ is a lower bound for A $w \geq \beta$

FACT: if $A \subseteq F$ has a g.l.b it is unique
$\rightarrow$ prove like ! lub

Defn) let β be the glb of A. We call β the infimum of A.

Completeness has a similar meaning. Look at HW3 (maybe).

Recall) $\mathbb{R}$ is the unique complete field

What is $\mathbb{N}$? (set, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$)

Let (F, P) be an ordered field.

Defn) A subset $X \subseteq F$ is inductive p.t.

- $1 \in X$ and
- $x \in X \Rightarrow x+1 \in X$

- F is inductive
- P is inductive $\to$ closure and $1 \in P$.

Lemma) F contains a ! smallest inductive subset. (called $\mathbb{N}_F$)

here: smallest means if $A \subseteq F$ is inductive, then $\mathbb{N}_F \subseteq A$

Pf) Define $\mathbb{N}_F$

$$N_F := \{x \in F \mid x \text{ belongs to every inductive subset } Z \subseteq F\}$$

$$= \bigcap_{\substack{Z \subseteq F \\ Z \text{ is inductive}}} Z$$

intersection of all inductive subsets

Claim 1 $\mathbb{N}_F$ is inductive

① $1 \in \mathbb{N}_F$? Yes as $\forall Z \subseteq F$, Z is inductive
$1 \in Z$.

② spose $x \in \mathbb{N}_F$. Is it true that $x+1 \in \mathbb{N}_F$

$x \in \mathbb{N}_F \Rightarrow \forall Z \subseteq F$, s.t Z is inductive, we have
$x \in Z$

Fix arbitrary inductive subset $Y \subseteq F$. Since
$x \in \mathbb{N}$, we have $x \in Y$. We know $\mathbb{N}_F \subseteq Y$.
Since Y is inductive $x+1 \in Y$.
Since Y is arbitrary, we know $x+1$ belongs to every inductive subset of F.
$\therefore$ $x+1 \in \mathbb{N}_F$

We have shown $\mathbb{N}_F$ is inductive

Claim 2 $\mathbb{N}_F$ is the smallest inductive subset of F.

Pf take arbitrary inductive subset $X \subseteq F$.

Then $\mathbb{N}_F := \bigcap_{\substack{Z \subseteq F \\ Z \text{ is inductive}}} Z \subseteq X$

Claim 3 $\mathbb{N}_F$ is unique

Spose $\mathbb{N}_F'$ is another smallest inductive subset of F.
Since $\mathbb{N}_F'$ is the smallest we have
$\mathbb{N}_F' \subseteq \mathbb{N}_F$
But, by Claim 2, we have
$\mathbb{N}_F \subseteq \mathbb{N}_F'$ since $\mathbb{N}_F'$ is inductive
$\therefore \mathbb{N}_F = \mathbb{N}_F'$ (two way containment)

Remark We can define the following.

$$\mathbb{Z}_F := \{ t \in F \mid |t| \in \mathbb{N}_F \cup \{0\} \}$$

$$\mathbb{Q}_F := \{ r \in F \mid \exists a \in \mathbb{Z}_F \ \exists b \in \mathbb{N}_F \text{ s.t. } r = a b^{-1} \}$$

Question? why is this better than our last def of $\mathbb{Q}$? (ie $\mathbb{Q} := \{ \frac{m}{n} \mid m \in \mathbb{Z}, n \in \mathbb{N})$

This time we do have $\frac{1}{2}, \frac{2}{4}$ in $\mathbb{Q}$, but we, using our arithmetic skills, can realise they are the same. Ie $r = (ca)(cb)^{-1}$

If $F = \mathbb{R}$, we denote,

$\mathbb{N}_F$ by $\mathbb{N}$

$\mathbb{Z}_F$ by $\mathbb{Z}$

$\mathbb{Q}_F$ by $\mathbb{Q}$

Claim $\forall n \in \mathbb{N}$

$$1 + 2 + 3 \ldots\ldots n = \frac{n(n+1)}{2}$$

Pf lets define $S := \{ m \in \mathbb{N} \mid \text{claim holds for } m \}$

by definition $S \subseteq \mathbb{N}$

it is enough to prove S is inductive $\Rightarrow \mathbb{N} \subseteq S$

lets do it

- $1 \in S$? yes $1 = \frac{1(1+1)}{2}$
- $\gamma \in S \Rightarrow \gamma + 1 \in S$? show

Fix $\gamma \in S$. Then

$$\left. \sum_{i=1}^{\gamma} = \frac{\gamma(\gamma+1)}{2} \ \right\} \text{ as } \gamma \in S \text{ by defintion}$$

We need to show $\gamma+1 \in S$, or show
$$\sum_{j=1}^{\gamma+1} j = \frac{(\gamma+1)((\gamma+1)+1)}{2} \Rightarrow \text{easy to do.}$$

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so $\gamma+1 \in S$
$\Rightarrow S$ is inductive $\Rightarrow \mathbb{N} \subseteq S \Rightarrow \mathbb{N} = S$

Lemma $\mathbb{N}$ isn't bounded in $\mathbb{R}$

Pf Proceed by contradiction

Assume $\mathbb{N}$ is bounded above in $\mathbb{R}$. Since $1 \in \mathbb{N}$, $\mathbb{N} \neq \emptyset$. As $\mathbb{R}$ is complete,

$\alpha := \sup(\mathbb{N})$ exists

So $\alpha \geq n \; \forall n \in \mathbb{N}$

$\mathbb{N}$ is inductive, so

$\alpha \geq n+1 \; \forall n \in \mathbb{N}$

$\Rightarrow \alpha - 1 \geq n \; \forall n \in \mathbb{N}$

$\Rightarrow \alpha - 1$ is a upper bound for $\mathbb{N}$ but

$\alpha - 1 < \alpha \rightarrow$ true since $\Rightarrow 1 \in P$

$\hookrightarrow$ contradiction, This would make $\alpha - 1$ $\sup(\mathbb{N})$.

$\therefore \mathbb{N}$ isn't bounded in $\mathbb{R}$.

Remark $\mathbb{N}$ isn't bounded above in $\mathbb{Q}$ as $\mathbb{Q} \subseteq \mathbb{R}$. If $\mathbb{N}$ is bounded above in $\mathbb{Q} \Rightarrow \mathbb{N}$ is bounded above in $\mathbb{R}$.

Archemedian Property

Lemma) Let (F,P) be an ordered field.
let $a,b \in P$. Then

$$(a > b) \iff (a^{-1} < b^{-1})$$

Pf) We will use this from HW 1

$\Leftrightarrow$ $d \in P \iff d^{-1} \in P$.

fix $a,b \in P$

we know $a^{-1}, b^{-1} \in P$. So $a^{-1} \cdot b^{-1} \in P$ by closure.

Thus

$$a < b \iff a \cdot (a^{-1} \cdot b^{-1}) < b \cdot (a^{-1} \cdot b^{-1})$$
$$b^{-1} < a^{-1}$$

Corollary 1 Archemedian Property for $\mathbb{R}$.

$$\forall \varepsilon > 0 \ \exists n \in \mathbb{N} \ \text{s.t.} \ n^{-1} < \varepsilon$$

PF) Proceed by contradiction. Spose false.

$$\exists \varepsilon > 0 \ \text{s.t.} \ \forall n \in \mathbb{N} \quad \varepsilon \leq n^{-1}$$

Then by lemma,

$$(n^{-1})^{-1} \leq (\varepsilon)^{-1} \qquad \text{but } (n^{-1})^{-1} = n$$

So, we have

$$\forall n \in \mathbb{N}, \ n \leq (\varepsilon)^{-1}$$

$\Rightarrow$ $\mathbb{N}$ is bounded in $\mathbb{R}$. oops.

Corollary $\mathbb{Q}$ is dense in $\mathbb{R}$

That is $\forall\, \alpha, \beta \in \mathbb{R}$ w $\alpha < \beta$
$\exists\, q \in \mathbb{Q}$ s.t $\alpha < q < \beta$.

PF Let $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$. Then $\beta - \alpha > 0$.
By archemedian property. $\exists\, n \in \mathbb{N}$ with $\frac{1}{n} < \beta - \alpha$.
$\Rightarrow 1 < n\beta - n\alpha$
Look at handout " int in $\mathbb{R}$"
$\exists\, m \in \mathbb{Z}$ s.t

$$n\alpha < m < n\beta$$

$\longrightarrow$ fairly intuitive as we have the distance between 2 numbers being > 1.

So,

$$\alpha < \frac{m}{n} < \beta$$

More sups & infs!
In what follows (F, P) is an ordered field.

① Let $A \subseteq F$. If $u \in F$ is an upper bound for A. $u' \in F$, s.t $u' < u$, then u' is also an upper bound for A.

Defn Set $B \subseteq F$ is said to be bounded p.t it is bounded above and bounded below

Eg for boundedness

A) fix $a \in \mathbb{R}$
$X_a := \{x \in \mathbb{R} \mid x < a\}$
note $X_a \neq \emptyset$ as $a - 1 \in X_a$
X_a is bounded above by $a+1$, so, by l.u.b property $\sup(X_a)$ exists.

Claim $\boxed{\sup(X_a) = a}$ $\longrightarrow$ 2 assertions
① observe $a \notin X_a$

B) $A := \{ 1 - \frac{1}{n} \mid n \in \mathbb{N} \}$

Note: $1 \notin A$ as $\forall n \in \mathbb{N}$ $1 \neq 1 - \frac{1}{n}$

① $A \neq \emptyset$ as $0 \in A$

② A is bounded above by 57. So sup(A) exists

$\sup(A) = 1$

note $1 \in A$

↳ Proof is an exercise

c) fix $a \in \mathbb{R}_{>0}$

$A := \{ x \in \mathbb{R} \mid x^2 < a \}$

① $A \neq \emptyset$ as $0 \in A$

② A is bounded above:

if $x \in A$, then $x^2 < a < a+1 < (a+1)^2$

↳ as $y > 1 \Rightarrow y^2 > y$

$\Rightarrow x^2 < (a+1)^2$

$\Rightarrow |x| < |a+1|$ and $x \leq |x|$

$\Rightarrow x \leq |x| < |a+1|$

A is bounded above by $|a+1|$. So sup(A) exists.

We call it $\sqrt{a}$.

Caution! Is it true that $(\sqrt{a})^2 = a$

Hope so! We need to prove it, but once we do we would have defined the square root.

In all the examples above, A has no maximal elmt.

Defn Spose (F,P) is an ordered field with $A \subseteq F$. The element $m \in F$ is a maximal element for A p.t

1. $m \in A$
2. m is an upper bound for A.

Observation

If A has a maximal element it is unique.
Can be shown similarly to unique Lub.
A minimal element is defined similarly.

Useful Result Let $A \subseteq \mathbb{R}$. Then

(A has a maximal element)

iff

($\sup(A)$ exists and $\sup(A) \in A$)

Pf Spose A has a maximal element

let $m := \max(A)$

(note $A \neq \emptyset$ as $m \in A$)

and A has an upper bound (m).

Since $\mathbb{R}$ is complete $\sup(A)$ exists.

Since $m \in A$, $m \leq \sup(A)$. As m is an upper bound for A, $\sup(A) \leq m$.

By trichotomy $m = \sup(A)$

To prove the other direction we need to show $\sup(A)$ is a maximal element (which it is by definition in the statement).

Theorem let $A \subseteq \mathbb{R}$, $A \neq \emptyset$.
let $\alpha \in \mathbb{R}$ be an upper bound of A.
Then,

$$(\alpha = \sup(A))$$
iff
$$\left(\forall\ \varepsilon > 0\ \ \exists\ a \in A \text{ s.t } \alpha - \varepsilon < a \leq \alpha\right)$$

2. This is what being a sup means.

Pf $\Longrightarrow$

Spose $\alpha = \sup(A)$
Fix an arbitrary $\varepsilon > 0$.
if no such $a \in A$ exists → that is $\nexists\, a \in (\alpha - \varepsilon, \alpha]$
Then $\alpha - \varepsilon$ is the upper bound for a as
$\forall\, a \in A$ $a < \alpha - \varepsilon$ (as $a \not> \alpha$). Oops $\square$

$\Longleftarrow$

lets assume α is an upper bound for a and
$\forall \varepsilon > 0$ $\exists\, a \in A$ s.t $\alpha - \varepsilon < a \leq \alpha$
We want to show $\alpha = \sup(A)$

Let α' be an upper bound for A, lets show
$\alpha' \geq \alpha$.
By trichotomy, we must simply show $\alpha' \not< \alpha$.
Assuming $\alpha_1 < \alpha$, let $\varepsilon = \alpha - \alpha'$ $(\alpha - \alpha' \in P)$
then $\exists\, a \in A$ s.t

$\alpha' = \alpha - \varepsilon < a \leq \alpha$

Then α' isn't an upper bound. Oops $\square$
by trichotomy $\alpha' \geq \alpha$ so $\sup(A) = \alpha$

Corollary Theorem

Spose $B \subseteq \mathbb{R}$, $B \neq \emptyset$, and B is bounded below. Let β be a lower bound for B.

$$\beta = \inf(B) \iff \forall \varepsilon < 0 \ \exists\, b \in B \text{ st } b \in [\beta, \beta + \varepsilon)$$

Lemma Let $A \subseteq \mathbb{R}$ be non empty,

(i) If A is bounded above then, $\forall t \in \mathbb{R}$

$t \geq \sup(A) \iff t \geq a \ \forall a \in A$.

(ii) If A is bounded below then, $\forall s \in \mathbb{R}$

$s \leq \inf(A) \iff s \leq a \ \forall a \in A$

Remark (i) and (ii) are similar, so we'll only do the proof of (ii).

Pf (ii) $A \subseteq \mathbb{R}$ is non-empty and bounded below.

As $A \neq \emptyset$, $\exists \inf(A)$

fix arbitrary $s \in \mathbb{R}$

$\Rightarrow$ Spose $s \leq \inf(A)$

Then, $s \leq \inf(A) \leq a \ \forall a \in A$. Thus, s is a lower bound for A.

$\Leftarrow$ Spose $s \leq a \ \forall a \in A$

Then s is a lower bound for A.
So, by definition, $s \leq \inf(A)$

Defn Let $A \subseteq \mathbb{R}$ and $b \in \mathbb{R}$
we define

- $-A := \{-x \mid x \in A\}$ ⇝ additive inverses
- $b + A := \{x+b \mid x \in A\}$ ⇝ translation

Thm let $A \subseteq \mathbb{R}$ be nonempty and bounded below. $\Rightarrow \exists \inf(A)$

Consider $-A$

Claim: $-A \neq \emptyset$ and $-A$ is bounded above
↳ check

So, $\exists \sup(-A)$.
how to compute $\inf(A)$
candidate is $-\sup(-A)$.
Show this is correct by showing 2 things:
1. $-\sup(-A)$ is a lower bound for A
2. $-\sup(-A)$ is the g.l.b.

Details: HW3?

Lemma Let $A \subseteq \mathbb{R}$ & $b \in \mathbb{R}$ then,

(I) A bounded above $\iff$ $-A$ bdd below

(II) $A \neq \emptyset$, A bdd above $\Rightarrow$ $-\sup(A) = \inf(-A)$

(III) A bdd above $\iff$ $b+A$ bdd above

(IV) A bdd below $\iff$ $b+A$ bdd below

(V) $A \neq \emptyset$, A bdd above $\Rightarrow$ $\sup(b+A) = \sup(A)+b$

(VI) $A \neq \emptyset$, A bdd below $\Rightarrow$ $\inf(b+A) = \inf(A)+b$

Pf of (I) A bounded above $\iff$ $\exists\, v \in \mathbb{R}$ st $\forall\, a \in A$ $v \leq a$

$\iff$ $\exists\, v \in \mathbb{R}$ s.t $\forall\, a \in A$ $-v \geq -a$

$\iff$ $\exists\, v \in \mathbb{R}$ s.t $\forall\, x \in -A$ $-v \geq x$

$\iff$ $-A$ is bounded below

Pf of II HW3

Pf of III If v is an upper bound for A, the $v+b$ is an upper bound of $b+A$. Conversely if w is an upper bound for $b+A$, $w-b$ is an upper bound for A.

Pf of IV Similar to (III).

Pf of (V) $A \neq \emptyset$ and bounded above so $\exists \sup(A)$

Thus, $b+A = \emptyset$ and $b+A$ is bounded above (by III) so $\sup(b+A)$ exists.

So we have:

- $b+\sup(A) \geq b+a$ $\forall\, a \in A$ $\leftarrow$ by def of $\sup(A)$
- so $b+\sup(A)$ is an upper bound for $b+A$

$\Rightarrow b+\sup(A) \geq \sup(b+A)$

We also have

- $-b+\sup(b+A) \geq -b+(b+x)$ $\forall\, x \in A$
- $-b+\sup(b+A) \geq x$ $\forall\, x \in A$

$\Rightarrow b+\sup(b+A) \leq \sup(A)$ $\quad\leftarrow$

So, $\sup(b+A) \geq b + \sup(A)$

by trichotomy,

$$\sup(b+A) = b + \sup(A)$$

Pf (VII) Similar to (V)