Lets show some cts functions!

- If X is a top sp we know

  $id_X : X \to X$ is cts

- If X & Y are top sp then any constant func

  $f: X \to Y$
  $x \mapsto y_0$

  is cts

What else? Consider $(\mathbb{R}, \mathcal{T}_{euc})$

Lemma) $\forall n \in \mathbb{N}$, the func $\mathbb{R} \to \mathbb{R}$ given by $x \mapsto x^n$ is cts

Pf) let $S := \{ n \in \mathbb{N} \mid f: \mathbb{R} \to \mathbb{R}$ by $x \mapsto x^n$ is cts $\}$

- $1 \in S$? yes $id_{\mathbb{R}}$
- Spose $k \in S$.

consider $k+1$ $\quad f: \mathbb{R} \to \mathbb{R}$
$\qquad x \mapsto x^{k+1}$

$f(x) = \underbrace{x^k}_{\text{cts}} \cdot \underbrace{x}_{\text{cts}}$ by hw 5 f is cts!

S is inductive!

Defn) A polynomial func of degree n, $p: \mathbb{R} \to \mathbb{R}$ is a function of the form

$$p(x) = a_n x^n + a_{n-1} x^{n-1} \dots a_0 x^0$$

with $a_0, a_1, \dots a_n \in \mathbb{R}$ and $a_n \neq 0$

eg
$$\left.\begin{array}{l} p(x) = 3x^5 + 5x \\ p(x) = 17 \end{array}\right\} \text{Polynomials dg } \begin{array}{l} 5 \\ 0 \end{array}$$

Lemma Every polynomial func is cts!

Pf let $T := \{k \in \mathbb{N} \mid$ all polynomial functions of deg $< k$ are cts$\}$

- $1 \in T$? Yes, all constant functions are cts
- spose $l \in T$ $\to$ show $l+1 \in T$

spose $g: \mathbb{R} \to \mathbb{R}$ is a poly func of degree l

$$g(x) = \underbrace{a_l x^l}_{\text{cts}} + \underbrace{\cdots\cdots a_1 x + a_0}_{\text{poly fnc of deg} < l}$$

$\therefore$ $g(x)$ is the sum of 2 cts functions. By hw5 g is cts.

$\Rightarrow$ $l+1 \in T$ so $T = \mathbb{N}$

Inverse functions:

Recall: if $f: A \to B$ is injective then

$f: A \to f(A)$ is bijective so,

$f^{-1}: f(A) \to A$ is a function

eg $f: \mathbb{R} \setminus \{0\} \to \mathbb{R}$ $\to$ $f(\mathbb{R} \setminus \{0\}) = \mathbb{R} \setminus \{0\}$

$x \mapsto \frac{1}{x}$

so, $f: \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$ is bijective

$f^{-1}: \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$

$x \mapsto \frac{1}{x}$ $\to$ involution lol

Defn let $A \subseteq \mathbb{R}$. The fnc $f: A \to \mathbb{R}$ is strictly $\uparrow$ p.t

if $x, y \in Y$ with $x < y$, then $f(x) < f(y)$

Defn let $A \subseteq \mathbb{R}$. The fnc $f: A \to \mathbb{R}$ is strictly $\downarrow$ p.t

if $x, y \in Y$ with $x < y$, then $f(x) > f(y)$

Lemma Let $I \subseteq \mathbb{R}$ be an iwimp (interval w inf many pts)

if $f: I \to \mathbb{R}$ is injective & cts then f is strictly $\uparrow$ or $\downarrow$

Pf Spse $a, b, c \in I$ with $a < b < c$

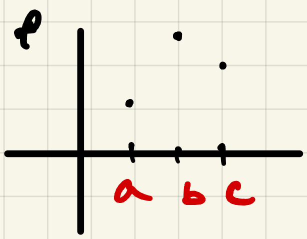
Lets compare

$$f(a) \;\begin{matrix} < \\ < \\ > \\ > \end{matrix}\; f(b) \;\begin{matrix} < \\ > \\ < \\ > \end{matrix}\; f(c)$$

Goal: Exclude the bad ones!

① If $f(a) < f(b)$ & $f(b) > f(c)$

(1.1) If $f(a) < f(c)$



In this case $f(c)$ is between $f(a)$ & $f(b)$. So $f(c)$ is the intermediate value!

By IVT $\exists\, d \in [a, b]$ s.t $f(d) = f(c)$ but $d \neq c$ as $c > b$ oops...

(1.2) If $f(c) < f(a)$

by ivt $\exists\, d \in [b, c]$ st $f(d) = f(a)$ but $d \neq a$ oops.

(1.3) If $f(c) = f(a)$ oops .... since $c \neq a$

② Similar as above but $f(a) > f(b)$ and $f(b) < f(c)$

So, we only consider

$f(a) < f(b) < f(c)$

$f(a) > f(b) > f(c)$

Then it follows that if $a, b, c, d \in I$ with $a < b < c < d$ either

$$f(a) < f(b) < f(c) < f(d) \quad \text{or}$$

$$f(a) > f(b) > f(c) > f(d)$$

We need to show that f is either strictly increasing or decreasing

Fix $a, b \in I$ w $a < b$

① If $f(a) < f(b)$ we will show f is strictly increasing

spse $x, y \in I$ with $x < y$

$$\left.\begin{array}{l} x < y < a < b \\ x < a < y < b \\ x < a < b < y \\ a < x < y < b \\ a < x < b < y \\ a < b < x < y \end{array}\right\} \text{in all cases, } f(x) < f(y) \text{ so, } f \text{ is strictly increasing}$$

② If $f(a) > f(b)$ argue similarly to get that f is strictly decreasing

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Lemma (version of inverse function thm)

Spse $I \subseteq \mathbb{R}$ is an interval. If $f : I \to \mathbb{R}$ is cts & injective then $f^{-1} : f(I) \to \mathbb{R}$ is cts

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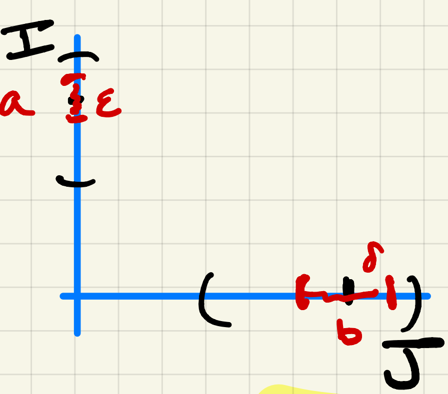
By the prev lemma either f is str $\uparrow$ or $\downarrow$ wlog

Now fix $a \in I$, $b = f(a) \in f(I) = J$

so $f^{-1} : J \to I$



Goal: Show f is cts at b!

let $\varepsilon > 0$ be given

★ spse I is open, we can shrink ε if necessary s.t. $B_\varepsilon(a) \subseteq I$

if δ works for ε it will $\forall \varepsilon' > \varepsilon$

task: find a good δ s.t.

if $y \in B_\delta(b) \cap J$ then $f^{-1}(y) \in B_\varepsilon(a)$

note: since $B_\delta(I) \subseteq I$ the points $a-\frac{\varepsilon}{2}, a+\frac{\varepsilon}{2} \in I$

compare

$$f(a-\tfrac{\varepsilon}{2}), \quad f(a), \quad f(a+\tfrac{\varepsilon}{2})$$

Since f is strictly $\uparrow$

$$f(a-\tfrac{\varepsilon}{2}) < b < f(a+\tfrac{\varepsilon}{2})$$

In J as $J = f(I)$

Set $\delta = \min\{f(a+\frac{\varepsilon}{2})-b, \; b-f(a-\frac{\varepsilon}{2})\}$

$\hookrightarrow$ both >0 by inequality.

Check that this works:

for $y \in B_\delta(b)$, we have $y \in [f(a-\frac{\varepsilon}{2}), f(a+\frac{\varepsilon}{2})]$

Since $B_\delta(b) \subseteq [f(a-\frac{\varepsilon}{2}), f(a+\frac{\varepsilon}{2})]$

By IVT applied to f,

$\exists\, x \in [a-\frac{\varepsilon}{2}, a+\frac{\varepsilon}{2}]$ s.t $y = f(x)$. Moreover

this x is unique as f is injective,

so, $f^{-1}(y) = x \implies x \in [a-\frac{\varepsilon}{2}, a+\frac{\varepsilon}{2}] \subseteq B_\varepsilon(a)$

So f^{-1} is cts!

If I is not open & a is not an end pt of I we're okay!

If a is an endpoint of I we may add on stuff on either side of the interval and use $\mathrm{res}_I f$.

This reduces to the previous case!

Defn Spose $A \subseteq \mathbb{R}$ a function $f: A \to \mathbb{R}$ is said to be uniformly cts on A pt $\forall \varepsilon > 0$ $\exists \delta > 0$ st if $x, y \in A$ w $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$

How do we compare this with regular continuity?

recall: The fnc $f: A \to \mathbb{R}$ is cts pt $\forall a \in A$ $\forall \varepsilon > 0$ $\exists \delta > 0$ if $x \in A$ & $|x - a| < \delta$ then $|f(x) - f(a)| < \varepsilon$.

Observe: If $f: A \to \mathbb{R}$ is uniformly cts on A then it is cts on A!

Q:

→ if $f: A \to \mathbb{R}$ is cts is it unif cts?

→ do these functions need to bdd? ✗

eg: $f: \mathbb{R} \to \mathbb{R}$ $\quad x \mapsto x$

let $\varepsilon > 0$ and take $\delta = \varepsilon$

→ uniformly cts and not bounded!

$f: \mathbb{R} \to \mathbb{R}$ $\quad x \mapsto 57$

let $\varepsilon > 0$ and take $\delta = 6$

→ unif cts

non eg: Strategy → find func for which δ is dependent on 'a'

$f: (0, \infty) \to \mathbb{R}$ $\quad x \mapsto \frac{1}{x}$

f is cts by the 'quotient' rule of continuity!

fix $\varepsilon = 1/2$, δs are shrinking depending on a.
Show that no matter what $\delta > 0$ we choose

$\exists\ x, y \in (0, \infty)$ st $|x - y| < \delta$ & $|f(x) - f(y)| \geq \varepsilon$

fix $\delta > 0$ by arch $\exists N \in \mathbb{N}$ st $\frac{1}{N} < \delta$

So, $\frac{1}{N}, \frac{1}{N+1} \in (0, \infty)$ and $\left|\frac{1}{N} - \frac{1}{N+1}\right| = \frac{1}{N(N+1)} < \delta$

but $\left|f\left(\frac{1}{N}\right) - f\left(\frac{1}{N+1}\right)\right| = |N - (N+1)| = 1 \geq \varepsilon$

So f is not unif cts

Principle of Unif Continuity

Let $a, b \in A$ with $a < b$ if $f : [a,b] \to \mathbb{R}$ is cts then f is unif cts on $[a,b]$.

Pf Let $\varepsilon > 0$ be given. Since f is cts at each $x \in [a,b]$,

$\exists\ \delta > 0$ s.t if $y \in [a,b]$ & $|x - y| < \delta$ then $|f(x) - f(y)| < \frac{\varepsilon}{57}$

Consider open cover of $[a,b]$

$$\mathcal{E} := \{ B_{\frac{\delta_x}{2}}(x) \mid x \in [a,b] \}$$

Since $[a,b]$ is cmpct $\exists$ a finite subcover $\mathcal{E}' \subseteq \mathcal{E}$

$$\mathcal{E}' = \{ B_{\frac{\delta_{x_1}}{2}}(x_1),\ B_{\frac{\delta_{x_2}}{2}},\ \ldots\ldots\ B_{\frac{\delta_{x_k}}{2}} \}$$

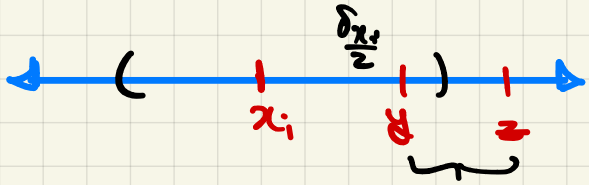
Choose $\delta = \min\{ \frac{\delta_{x_1}}{2}, \ldots \frac{\delta_{x_k}}{2} \}$

Claim: this works!

Choose $y, z \in [a,b]$ s.t $|y - z| < \delta$

Since $y, z \in [a,b]$ $\exists\ i$ with $1 \leq i \leq k$ s.t

$y \in B_{\frac{\delta_{x_i}}{2}}(x_i)$



$\delta < \frac{\delta_{x_i}}{2}$ so $z \in B_{\delta_{x_i}}(x_i)$

What about z

$$\begin{aligned}
|z - x_i| &= |z - y + y - x_i| \\
&\leq |z - y| + |y - x_i| \\
&< \delta + \frac{\delta_{x_i}}{2} \\
&< \delta_{x_i}
\end{aligned}$$

$\therefore\ z \in B_{\delta_{x_i}}(x_i)$ $\quad y \in B_{\frac{\delta_{x_i}}{2}}(x_i)$

$\longrightarrow$

$$|g(z) - g(y)| = |g(z) - g(x_i) + g(x_i) - g(y)|$$

$$\leq |g(z) - g(x_i)| + |g(x_i) - g(y_i)$$

$z \in B_{\delta_{x_i}}(x_i) \qquad y \in B_{\frac{\delta_{x_i}}{2}}(x_i) \subseteq B_{\delta_{x_i}}(x_i)$

So, since f is cts at x_i

$$< \frac{2\varepsilon}{5\pi} < \varepsilon \qquad \square$$

Doesn't work for non-compact Domains!