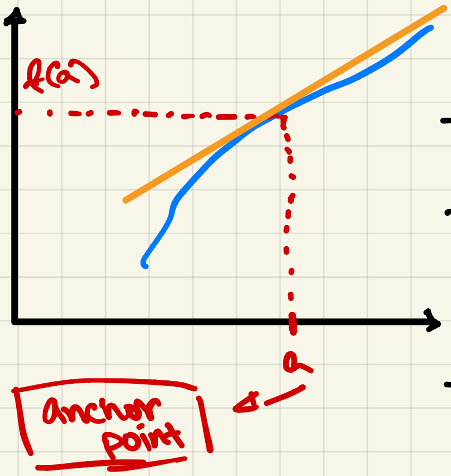
# Taylor Polynomials

**Goal:** To approximate a function with polynomials as they are easy to deal with → easy diffate




Best Approx - At anchor pt with polynomial

→ deg 0 → $P_{0,a}(x) = f(a)$

→ deg 1 → $P_{1,a}(x) = f(a) + f'(a)(x-a)$

↳ equation of tangent!

→ deg 2 → $P_{2,a}(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!}$

where does this come from?

Lets back up for a second.

The idea is, we want the best polynomial approximation of a function $f$ to have the following properties

$P_{n,a}(x)$ → polynomial of degree $n$ anchored at $a$!

$P_{n,a}(a) = f(a)$

$P'_{n,a}(a) = f'(a)$

$\vdots$

$P^{(k)}_{n,a}(a) = f^{(k)}(a)$

$\vdots$

$P^{(n)}_{n,a}(a) = f^{(n)}(a)$ → of course we imply $f$ is $n$ times diff'able!

lets say

$P_{n,a}(x) = a_0 + a_1(x-a) + a_2(x-a)^2 \ldots a_n(x-a)^n$

What can we say about these coefficients?

$\boxed{P_{n,a}(a) = f(a)}$ so $P_{n,a}(a) = \boxed{a_0 = f(a)}$

Similarly

$P'_{n,a}(x) = a_1 + 2a_2(x-a) \ldots n \cdot a_n(x-a)^{n-1} \Rightarrow \boxed{a_1 = f'(a)}$

We continue... in general

$$\boxed{a_n = \frac{f^{(n)}(a)}{n!}}$$

Show this,

$$P_{n,a}(x) = \sum_{k=0}^{n} a_k (x-a)^k$$

$$P_{n,a}^{(j)}(x) = \sum_{k=j}^{n} a_k \cdot k(k-1)\cdots(k-j+1)(x-a)^{k-j}$$

only the first survives!

So, $P_{n,a}^{(j)}(a) = a_j \cdot j \cdot (j-1) \cdots\cdots (2)\cdot 1 = a_j\, j!$

$\Rightarrow \quad a_j = \dfrac{f^{(j)}(a)}{j!} \quad \to$ from wishlist

Wait!

We have solved for the coefficients of $P_{n,a}(x)$ so that $\forall\ 0 \le i \le n$ we have

$$P_{n,a}^{(i)}(a) = f^{(i)}(a)$$

So a polynomial that satisfies the wish list exists and is unique

Defn) Spose $I \subseteq \mathbb{R}$ is an ivint and let $a \in I$ be given. Spose $f: I \to \mathbb{R}$ is $n$ times diffble at $a$. Then,

$$P_{n,a}(x) := \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Is the taylor polynomial of deg $n$, centered at $a$ associated with $f$.

Defn) We define the error function of the Taylor Polynomial as

$\to$ remainder

$$R_{n,a}(x) = f(x) - P_{n,a}(x)$$

**Thm** Let $I \subseteq \mathbb{R}$ be an iwimp & let $a \in I$, let $n \in \mathbb{N}$ and $f: I \to \mathbb{R}$ be a fnc.

(I) **Lagrange** Spose $f$ is $(n+1)$ times diff'ble on $I$. Then $\forall x \in I$ $\exists t$ between $a, x$ s.t

$$R_{n,a}(x) = \frac{f^{(n+1)}(t)}{(n+1)!}(x-a)^{n+1}$$

→ explicit form for remainder

(II) **Cauchy** Spose $f$ is $(n+1)$ times diff'ble on $I$. Then $\forall x \in I$ $\exists t$ btw $a$ & $x$ s.t

$$R_{n,a}(x) = \frac{f^{(n+1)}(t)}{n!}(x-t)^n(x-a)$$

(III) Spose $f$ is $(n+1)$ times diff'ble at $a$ & $f^{(n+1)}$ is cts on $I$ then

$$R_{n,a}(x) = \int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n \, dt$$

**Pf** fix $x \in I$. Wlog $x > a$

**Define** $F: I \to \mathbb{R}$

$$F(y) = f(y) + f'(y)\cdot(x-y) + \frac{f''(y)}{2!}(x-y)^2 + \dots$$

$$F(y) = \sum_{k=0}^{n} \frac{f^{(k)}(y)}{k!}(x-y)^k$$

→ Poly centered at $y$

→ diff'ble $\Rightarrow$ cts

**Observe**

- $F(a) = P_{n,a}(x)$
- $F(x) = f(x)$
- $F'(y) = f'(y) + (f'(y)(-1) + f''(y)(x-y)) \dots\dots$

→ telescopes to $\frac{f^{(n+1)}(y)}{n!}(x-y)^n$

**Define** $G: [a,x] \to \mathbb{R}$

$$G(y) = (x-y)^{n+1}$$

→

Apply CMVT to.

$G : [a, x] \to \mathbb{R}$ $\qquad$ $F\big|_{[a,x]} : [a,x] \to \mathbb{R}$

Both are cts and diff'ble as they are polynomials.

$\exists\, t \in (a, x)$ s.t

$G'(t)\,(F(x) - F(a)) = F'(t)\,(G(x) - G(a))$

$$\Rightarrow (n+1)(x-t)^n(-1)\underbrace{(f(x) - P_{n,a}(x))}_{R_{n,a}(x)} = \frac{f^{(n+1)}(t)}{n!}(x-t)^n\,(0 - (x-a)^{n+1})$$

$$\Rightarrow R_{n,a}(x) = \frac{f^{(n+1)}(t)}{(n+1)!}(x-a)^{n+1}$$

So we have I D.

For II apply MVT to $F : [a,b] \to \mathbb{R}$

$\exists\, t \in (a,b)$ st

$F'(t)\,(x-a) = F(x) - F(a)$

$$(x-a)\frac{f^{(n+1)}(t)}{n!}(x-t)^n = f(x) - P_{n,a}(x) = R_{n,a}(x)$$

D

for III

Apply FTC to

$$\int_a^x F'(t)\,dt = F(x) - F(a) = R_{n,a}(x)$$

D

Done with Calc 🎉