Logarithms

Note: $\int_a^b f$ if $a > b$ then $\int_a^b f := -\int_b^a f$

Consider

$$\gamma : (0,\infty) \to \mathbb{R}$$
$$t \longmapsto \frac{1}{t}$$

We have that γ is cts. By the earlier theorem, $\forall\, a,b \in (0,\infty)$

$\int_a^b \gamma$ is defined!

Defn)

$$\ln : (0,\infty) \to \mathbb{R}$$
$$x \longmapsto \int_1^x \frac{1}{t}$$

Remark: if $0 < x < 1$ then $\int_1^x \frac{1}{t} = -\int_x^1 t$

we use this to understand $\ln(x)$ in this case!

note: $\ln(1) = \int_1^1 \gamma = 0$

Lemma) Spose $a,b,c \in \mathbb{R}$ with $a < b < c$ and $f : [a,c] \to \mathbb{R}$ is a bounded function. Then

f is int on $[a,c]$ $\Longleftrightarrow$ $\mathrm{res}_{[a,b]} f$ is integrable on $[a,b]$; $\mathrm{res}_{[b,c]} f$ is integrable on $[b,c]$

Moreover, when the above is the case

$$\int_a^c f = \int_a^b f + \int_b^c f$$

Pf] $\Longleftarrow$ Use Darboux Criterion

Let $\varepsilon > 0$ be given,

$\exists\ P_{ab}$ of $[a,b]$ s.t

$$U(f, P_{ab}) - L(f, P_{ab}) < \frac{\varepsilon}{57}$$

$\exists\ P_{bc}$ of $[b,c]$ s.t

$$U(f, P_{bc}) - L(f, P_{bc}) < \frac{\varepsilon}{57}$$

Take $P := P_{ab} \cup P_{bc}$, we have,

$$U(f,P) - L(f,P)$$

$$= U(f, P_{ab}) + U(f, P_{bc}) - \left(L(f, P_{ab}) - L(f, P_{bc})\right)$$

$$= \frac{\varepsilon}{57} + \frac{\varepsilon}{57} < \varepsilon \quad \square$$

$\Longrightarrow$ Use Darboux Definition

Let $\varepsilon > 0$ be given. Since f is integrable on $[a,c]$

$\exists$ partition P' of $[a,c]$ s.t

$$U(f,P') - L(f,P') < \varepsilon$$

Let P be a refinement of P' by

$$P := P' \cup \{b\} = \{a = t_0 < t_1 < \dots < t_{j-1} < b < t_{j+1} \dots < t_n = c\}$$

$$\left.\begin{array}{l} P_{ab} = \{a = t_0 < \dots < t_j = b\} \\ P_{bc} = \{b = \underset{= t_j}{s_0} < \dots < \underset{= t_n}{s_m} = c\} \end{array}\right\} P$$

So, $\left[U(f, P_{ab}) - L(f, P_{ab})\right] + \left[U(f, P_{bc}) - L(f, P_{bc})\right]$

$$= U(f,P) - L(f,P) < \varepsilon$$

Since and are non negative, it must be that

$$U(f, P_{ab}) - L(f, P_{ab}) < \varepsilon \ \text{ AND } \ U(f, P_{bc}) - L(f, P_{bc}) < \varepsilon$$

$\square$

finally: Show equivalences in integral values

$\forall$ partitions P' of $[a,c]$. let $P = P' \cup \{b\}$

we have

$$L(f,P') \le L(f,P) = L(f,P_{ab}) + L(f,P_{bc})$$
$$\le \int_a^b f + \int_b^c f \le$$
$$U(f,P_{ab}) + U(f,P_{bc}) = U(f,P) \le U(f,P')$$

we know $\int_a^c f$ exists, so we can show that it is the unique real number that is sandwiched as such

$$L(f,P') \le \int_a^c f \le U(f,P')$$

$\forall$ partitions P' of $[a,c]$

$$\therefore \int_a^c f = \int_a^b f + \int_b^c f \qquad \square$$

Consequence] let $I \subseteq \mathbb{R}$ be a closed & bdd interval.

Spse $f : I \to \mathbb{R}$ is integrable. If $u, v, w \in I$ then,

$$\int_u^w f = \int_u^v f + \int_v^w f.$$

no order restrictions here. Pf: HW9

from this,

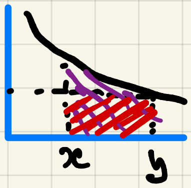
Lemma] The fnc $\ln : (0,\infty) \to \mathbb{R}$ is strictly $\uparrow$

$$x \longmapsto \ln(x)$$

Pf] Spse $0 < x < y$. we want $\ln(x) < \ln(y)$

examine $\ln(y) - \ln(y) = \int_1^y \frac{1}{t} - \int_1^x \frac{1}{t}$

$$= \int_1^y \frac{1}{t} + \int_x^1 \frac{1}{t} = \int_x^y \frac{1}{t} > 0 \; ? \qquad \longrightarrow$$



We have that this integral exists.
Consider $P = \{x, y\}$

$$L\left(\frac{1}{t}, P\right) = \frac{(y-x)}{y} \leq \int_x^y \frac{1}{t}$$

positive!

$$\therefore \int_x^y \frac{1}{t} > 0$$

$$\Rightarrow \ln(y) - \ln(x) > 0$$:) $\square$

Next goal: let's prove it is cts

Lemma) Spose $f : [a,b] \to \mathbb{R}$ is bdd by M.
If f is integrable, then

$$-M(b-a) \leq \int_a^b f \leq M(b-a)$$

Pf) Consider the partition $P = \{a, b\}$
Then $-M(b-a) \leq L(f, P) \leq \int_a^b f$ and

$$\int_a^b f \leq U(f, P) \leq M(b-a)$$

Lemma) if $f : [a,b] \to \mathbb{R}$ is integrable then,
$|f| : [a,b] \to \mathbb{R}$ is integrable &

$$\left| \int_a^b f \right| \leq \int_a^b |f|$$

Pf) HW8

$\longrightarrow$

Lemma) if $f:[a,b]\to\mathbb{R}$ is integrable. For $x\in[a,b]$ define:

$$F(x) := \int_a^x f$$

$F:[a,b]\to\mathbb{R}$ is cts

Pf) Fix anchor point $c\in[a,b]$ → pointwise continuity

let $\varepsilon>0$ be given.

Task: $\forall z\in[a,b]$ if $|z-c|<\delta$ then $|F(z)-F(c)|<\varepsilon$

Since f is integrable, it is bounded. $\therefore \exists M>0$ s.t $\forall x\in[a,b]$ $|f(x)|<M$

set $\delta := \dfrac{\varepsilon}{57\cdot M}$

if $z\in[a,b]$ s.t $|z-c|<\delta$ then,

$$|F(z)-F(c)| = \left|\int_a^z f - \int_a^c f\right| = \left|\int_c^z f\right|$$

$$\leq \int_c^z |f| \leq M\cdot|z-c| < M\cdot\delta = \frac{\varepsilon}{57} < \varepsilon \qquad \square$$

Since f is bdd above by M so is $|f|$

Corollary) Does this mean $\ln:(0,\infty)\to\mathbb{R}$ is cts?

fix anchor pt $y\in(0,\infty)$.

Since $(0,\infty)$ is open $\exists r>0$ s.t $B_r(y)\subseteq(0,\infty)$

$\therefore [y-\frac{r}{2}, y+\frac{r}{2}]\subseteq(0,\infty)$ WLOG suppose $y>0$

$\therefore [1, y+\frac{r}{2}]\subseteq(0,\infty)$

Then $F:[1, y+\frac{r}{2}]\to\mathbb{R}$

$s \longmapsto \int_1^s \frac{1}{t}$ is cts.

As $y\in[1, y+\frac{r}{2}]$, f is cts at y

alternate pf in notes use δ

Note] $\ln : (0,\infty) \to \mathbb{R}$ is cts & strictly increasing

$\therefore$ ln is a continuous bij from $(0,\infty)$ to its image.

By HW3 $\to$ its inverse is continuous

$\Rightarrow$ $\ln : (0,\infty) \to \ln((0,\infty))$ is a homeomorphism!

Lets figure out its image

we know $\ln((0,\infty)) \subseteq \mathbb{R}$ and is an interval since ln is cts and $(0,\infty)$ is connected.

We will show that the inverse isn't bounded above or below...

$\hookrightarrow$ then it is $\underline{\mathbb{R}}$

Claim] $\ln((0,\infty)) = \mathbb{R}$

Pf] enough to show

(1) $\lim_{x\to\infty} \ln(x) = \infty$ $\to$ $\forall N \in \mathbb{N}$ $\exists M \in \mathbb{N}$ s.t. $\forall n > M$ we have $\ln(x) > N$

$\Rightarrow$ height   greater than height

(2) $\lim_{x\to 0^+} \ln(x) = -\infty$ $\to$ $\forall N \in \mathbb{N}$ $\exists \delta > 0$ s.t. if $0 < x < \delta$ then $\ln(x) < -N$

notion of 0^+

(1) implies that the image is unbounded above

to show $\lim_{x\to\infty} \ln(x) = \infty$ consider $\ln(2^n)$

$$= \int_1^{2^n} \frac{1}{t} \geq L\left(\frac{1}{t}, P\right) \quad P = \{1 < 2 < 4 \; \dots \; 2^n\} \text{ of } [1, 2^n]$$

Since $\frac{1}{t}$ is decreasing

$$L\left(\frac{1}{t}, P\right) = \sum_{i=1}^{n} (2^i - 2^{i-1})\left(\frac{1}{2^i}\right) = \frac{n}{2} \Rightarrow \ln(2^n) \geq \frac{n}{2}!$$

$\hookrightarrow \sum_{i=1}^{n} \frac{1}{2}$

Fix $N \in \mathbb{N}$ let $M = 2^{2n+8}$, if $x > M$ since $\ln$ is strictly increasing

$$\ln(x) > \ln(M) = \ln(2^{2n+8}) \geq \frac{2n+8}{2} > N \quad \square$$

$$\Rightarrow \lim_{x \to \infty} \ln(x) = \infty$$

Show $\lim_{x \to 0^+} \ln(x) = -\infty$ ....

These show $\ln((0,\infty))$ is unbounded above & below

So $\ln((0,\infty)) = \mathbb{R}$.

In particular $\ln$ is a homeomorphism from $(0,\infty) \to \mathbb{R}$

★ since $\ln : (0,\infty) \to \mathbb{R}$ is bijective, it has an inverse.

$\exp : \mathbb{R} \to (0,\infty)$ s.t

$$\exp(t) = s \iff t = \ln(s)$$

We know this is cts by earlier!