Functions, Balls, intervals, open-ness, limits

→ $\varepsilon - \delta$

More fun with Functions

<u>Recall</u> : $f : A \to B$ is $f \subseteq A \times B$ s.t

$\forall a \in A \quad \exists!\ b \in B$ s.t $(a,b) \in f$

we write $b = f(a)$

Defn Spose $f : A \to B$ is a function.

we define $f^{-1} \subseteq (B \times A)$ by

$f^{-1} = \{ (b,a) \in B \times A \mid (a,b) \in f \}$

Warning : no claim that f^{-1} is a function.

Natural Q : when is it a function?

when it satisfies,

$\forall b \in B \quad \exists!\ a \in A$ st. $(b,a) \in f^{-1} \iff (a,b) \in f$.

Defn A function $f : A \to B$ is said to be surjective (onto) p.t

$\forall b \in B \ \exists a \in A$ st. $f(a) = b$

Defn A function $f : A \to B$ is injective (one-to-one) pt

whenever, $f(a) = f(a')$ we have $a = a'$.

non eg :- $f : \mathbb{R} \to \mathbb{R}$
$x \to x^2$

eg : $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$
$x \mapsto x^2$

Defn A function is bijective p.t it is both injective and surjective

↳ we call a bijective function a bijection.

Defn] The function $f: A \to B$ is called invertible if f^{-1} is also a function

When is $f^{-1}: B \to A$ a function?

$\underbrace{\forall b \in B}_{\text{surjective}} \; \underbrace{\exists!}_{\text{injective}} \; a \in A$ s.t $(b, a) \in f^{-1} \Rightarrow (a, b) \in f$

Defn] Let $f: A \to B$, the subset $f(A) \subseteq B$ is the image of f when

$$f(A) := \{ f(a) \in B \mid a \in A \}$$

$f(A) = B$ iff f is surjective

Lemma] Let $f: A \to B$ be a function

f is invertible $\Longleftrightarrow$ f is bijective

Pf] $\Rightarrow$

since f is invertible. f^{-1} is a function. Thus,

$\forall b \in B \; \exists! a \in A$ s.t $(a, b) \in f \Rightarrow$ injective + surjective

$\Leftarrow$

f is bijective. This means,

① $\forall b \in B \; \exists a \in A$ st $f(a) = b$

② if $f(a) = f(a')$ then $a = a'$

putting ① and ② together,

$\forall b \in B \; \exists! a \in A$ s.t $(a, b) \in f$

$\therefore f^{-1}$ is a function.

Defn Spose $f: A \to B$ is invertible, we call $f^{-1}: B \to A$ the inverse of f. This means:

$$A \xrightarrow{f} B \xrightarrow{f^{-1}} A$$

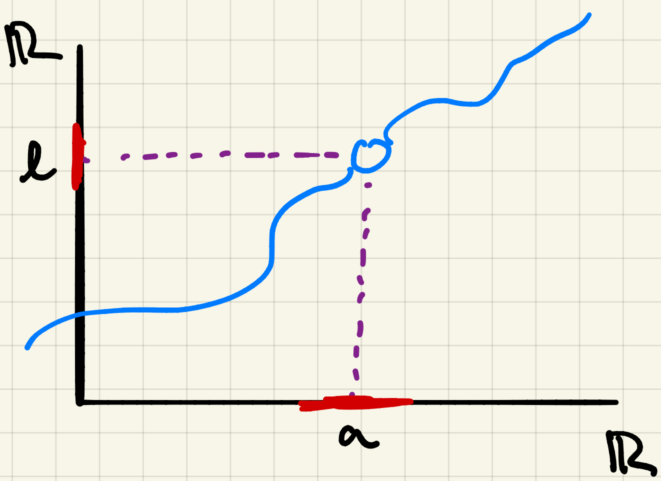
$$a \mapsto b \mapsto a$$

So, $f^{-1} \circ f: A \to A$ is the identity function on A

Similarly $f \circ f^{-1}: B \to B$ is the identity function on B.

Limits

take $f: \mathbb{R} \to \mathbb{R}$



What does

$\lim_{x \to a} f(a) = l$ mean?

Ideal

for every small neighbornood around l, there exists a small neighborhood around (but not including) a such that $\forall$ values of x in the latter neighbornood, $f(x)$ belongs to the former neighborhood.

We start with l !!!

We are now going to work to unpack this

Defn A subset $A \subseteq \mathbb{R}$ is called an interval p.t $\forall x, y \in A$, if $z \in \mathbb{R}$ if $x < z < y$, then $z \in A$.

Singletons and $\emptyset$ are vacuously intervals

$\mathbb{R}$ is an interval

$\mathbb{Q}$ is not an interval (irrationals)

Fact ∃ exactly 9 types of intervals in $\mathbb{R}$!

Let $a, b \in \mathbb{R}$ with $a \leq b$

- $(a,b) := \{x \in \mathbb{R} \mid a < x < b\}$
- $(a,b] := \{x \in \mathbb{R} \mid a < x \leq b\}$ ↳ $\emptyset$ is a part of this
- $[a,b) := \{x \in \mathbb{R} \mid a \leq x < b\}$
- $[a,b] := \{x \in \mathbb{R} \mid a \leq x \leq b\}$
- $(a,\infty) := \{x \in \mathbb{R} \mid a < x\}$
- $[a,\infty) := \{x \in \mathbb{R} \mid a \leq x\}$
- $(-\infty,b) := \{x \in \mathbb{R} \mid x < b\}$
- $(-\infty,b] := \{x \in \mathbb{R} \mid x \leq b\}$
- $(-\infty,\infty) := \mathbb{R}$

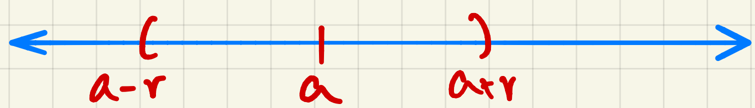
These are the <u>open</u> intervals! What does that mean?

Defn A is a neighborhood of $p \in \mathbb{R}$ provided that A is an open interval containing p.

Defn Let $a \in \mathbb{R}$ and $r > 0$. The set:

$$B_r(a) := \{x \in \mathbb{R} \mid a - r < x < a + r\}$$

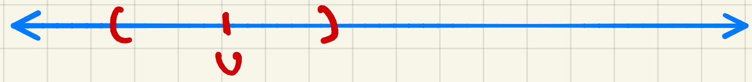
is called an open ball centered at a with a radius r.



note: $B_r(a) := (a-r, a+r)$

↳ consists of all real numbers x s.t. that the distance between x and the center (a) is less than the radius r.

Defn Let $U \subseteq \mathbb{R}$, then U is **open** p.t $\forall u \in U$, $\exists r > 0$ s.t $B_r(u) \subseteq U$.



$\exists ?$ $r > 0$ such that $B_r(u) \subseteq U$?

it should be the case that open ball is an open subset of $\mathbb{R}$.

Similarly, we should check ALL "open" intervals are open subsets of $\mathbb{R}$.

Claim: let $a, b \in \mathbb{R}$, then (a,b) is an open interval

(I) Show it is an interval
(II) Show it is an open subset of $\mathbb{R}$

(I) **Pf)** if $a \geq b$, then $(a,b) = \emptyset$ so it is vacuously true.

if $a < b$,

Fix $x, y \in (a,b)$ with $x < y$

Fix $z \in \mathbb{R}$ with $x < z < y$ → show $z \in (a,b)$

Since $x \in (a,b)$, $x > a$
Since $y \in (a,b)$, $y < b$

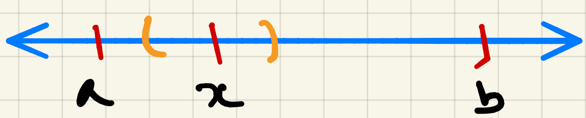
Thus, $a < x < z < y < b \implies z \in (a,b)$

(II) **Pf)** now show (a,b) is an open subset of $\mathbb{R}$.

if $a \geq b$ → $(a,b) = \emptyset$ and we are done

if $a < b$, $(a,b) \neq \emptyset$

Fix $x \in (a,b)$ task: find $r > 0$ s.t $B_r(x) \subseteq (a,b)$



Chose, $r = \min(x-a, b-x)$

then $B_r(x) \subseteq (a,b)$ → lets check

Check: Let $y \in B_r(x)$

$$a = x - (x-a) \leq x - r < y < x + r \leq x + (b - x) = b$$

$\therefore \quad a < y < b$ so $y \in (a,b)$

So $B_r(x) \subseteq (a,b)$

Corollary) Fix $a \in \mathbb{R}$ & $r > 0$. Then,

$B_r(a) = (a-r, a+r)$ is an open interval

Defn) The set $C \subseteq \mathbb{R}$ is closed p.t $\mathbb{R} \setminus C$ is open

Ex

$\left.\begin{array}{l} [a, \infty) \\ (-\infty, b] \\ [a,b] \end{array}\right\}$ Closed

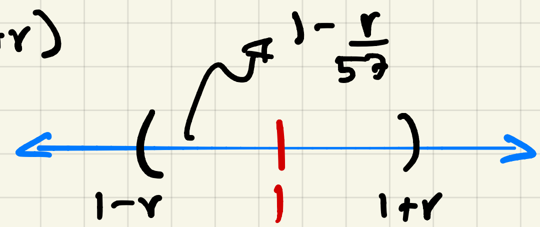
$\left.\begin{array}{l} \emptyset \\ \mathbb{R} \end{array}\right\}$ Clopen

What about $[1,2)$? focusing on 1 it can't be open or closed!

Claim) $\exists r > 0$ st $B_r(1) \subseteq A$

Pf) fix $r > 0$ $\quad B_r(1) = (1-r, 1+r)$

$\exists y \in B_r(1)$ s.t $y < 1$



So, $1 - \frac{r}{57} \in B_r(1)$

but $1 - \frac{r}{57} \notin A \quad \therefore B_r(1) \not\subseteq A$ and so A isn't open

A similar argument can be used for $\mathbb{R} \setminus A$.

Union of Open Sets

Is the union of open sets open? Yes! Let's prove

Let Δ be an indexing set (we're labeling open sets)

$$f: \Delta \to P(\mathbb{R})$$
$$\lambda \to U_\lambda$$

Spse $\forall \lambda \in \Delta$, U_λ is an open subset of $\mathbb{R}$.

Claim:

$$\bigcup_{\lambda \in \Delta} U_\lambda \text{ is an open subset of } \mathbb{R}$$

Pf Let $x \in \bigcup_{\lambda \in \Delta} U_\lambda$

$\exists \lambda' \in \Delta$ s.t $x \in U_{\lambda'}$ (by union)

Since $U_{\lambda'}$ is open,

$\exists r > 0$ st $B_r(x) \subseteq U_{\lambda'}$

So, $B_r(x) \subseteq \bigcup_{\lambda \in \Delta} U_\lambda$ Done!

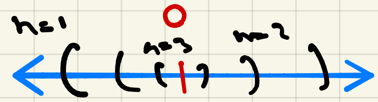
"an arbitrary union of open sets is open!"

Eg: $(1,2) \cup (3,4)$

$\forall n \in \mathbb{N}$ define $U_n := \left(-\frac{1}{n}, \frac{57}{n} + n\right)$ $\bigcup_{n \in \mathbb{N}} U_n$ is open

Non eg: $\mathbb{Q} \subseteq \mathbb{R}$ isn't open.

Intersection of open sets

take the example $U_n := (-\frac{1}{n}, \frac{1}{n})$ 

$\bigcap_{n \in \mathbb{N}} U_n$ is not open. Claim: it is $\{0\}$ which isn't open.

"it is not true that an arbitrary $\cap$ of open sets is open"

Claim: Let $V_1, V_2 \ldots V_n$ be open subsets of $\mathbb{R}$.

$\bigcap_{i=1}^{n} V_n$ is open

"intersection of finitely many open sets is open"

Pf: $x \in \bigcap_{i=1}^{n} V_i$

task; find $r > 0$ s.t $B_r(x) \subseteq \bigcap_{i=1}^{n} V_i$

Since x is in the intersection, $x \in V_i \ \forall\, 1 \le i \le n$

Each V_i is open so...

$\forall i \ \exists r_i > 0$ s.t $B_{r_i}(x) \subseteq V_i$

Since true: set $r = \min(r_1, \ldots\ldots r_n)$ note: $r > 0$

Lets show this r works, that is $B_r(x) \subseteq \bigcap_{i=1}^{n} V_i$

We get this immediately since

$\forall\, 1 \le i \le n$ $\quad r \le r_i$

$B_r(x) \subseteq B_{r_i}(x) \subseteq V_i$

$\Rightarrow B_r(x) \subseteq \bigcap_{i=1}^{n} V_i \qquad \square$

What about closed sets?

Which subsets of $\mathbb{R}$ are both closed and open?

$\mathbb{R}$, $\emptyset$, others.....

The collection of all open sets of $\mathbb{R}$.

$\mathcal{T} = \{ U \subseteq \mathbb{R} \mid U \text{ is open} \}$

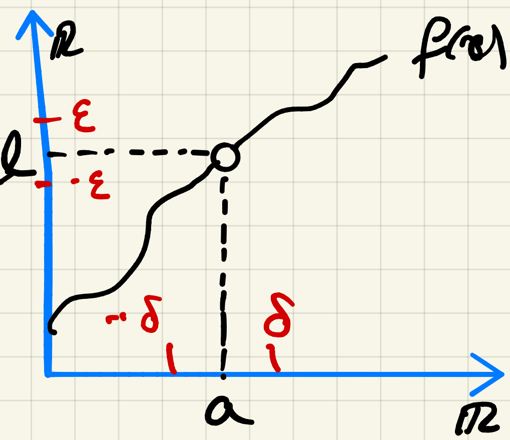
is called the topology on $\mathbb{R}$ (euclidean topology).

note! we have shown the 3 criteria for topology

1. $\mathbb{R}, \emptyset \in \mathcal{T}$
2. $\mathcal{T}$ is "closed" wrt taking arbitrary unions
3. $\mathcal{T}$ is "closed" wrt to finite intersections.

Back to the main attraction

We went down a rabbit hole. Limit → neighborhood → interval → ball → open.



$$\lim_{x \to a} f(x) = \ell$$

Defn) let $f: \mathbb{R} \to \mathbb{R}$, let $a, \ell \in \mathbb{R}$. We say f approaches ℓ as x approaches a" p.t one of the following holds.

1. $\forall \varepsilon > 0 \; \exists \delta > 0$ st if $x \in B_\delta(a) \setminus \{a\}$, then $f(x) \in B_\varepsilon(\ell)$

2. $\forall \varepsilon > 0 \; \exists \delta > 0$ s.t $\forall x \in \mathbb{R}$ with $0 < |x-a| < \varepsilon$ we have $|f(x) - \ell| < \varepsilon$

3. $\forall$ open subsets $V \subseteq \mathbb{R}$ that contain ℓ, $\exists$ an open subset $U \subseteq \mathbb{R}$ that contains a s.t $\forall \in U \setminus \{a\}$ we have $f(x) \in V$.

Examples for ε, δ.

(A) $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto 57x - 2$. Show $\lim_{x \to 1} f(x) = 55$

Let $\varepsilon > 0$ be given

task: find $\delta > 0$ s.t

$\forall\, x \in B_\delta(1) \setminus \{1\}$, we have $f(x) \in B_\varepsilon(55)$

$0 < |x - 1| < \delta$ $\qquad$ $|f(x) - 55| < \varepsilon$

$\delta = \frac{\varepsilon}{57}$

Fix $x \in \mathbb{R}$ s.t

$0 < |x - 1| < \delta$ (x is in ball)

Then $|f(x) - 55| = 57\,|x - 1|$

$< 57 \cdot \delta$

$= 57 \cdot \frac{\varepsilon}{57}$

$= \varepsilon$

$|f(x) - 55| < \varepsilon$

we proved it !!!

Scratch

$= |f(x) - 55|$

$= |57x - 2 - 55|$

$= |57x - 57|$

$= 57\,|x - 1|$ → happens to be $|x - a|$

choose $\delta = \frac{\varepsilon}{57}$

(B) $g : \mathbb{R} \to \mathbb{R}$ $\quad g(x) = 2x^2 + 2x - 4$

Show $\lim_{x \to -3} g(x) = 8$

Pf: Show $\forall\, \varepsilon > 0\ \exists\, \delta > 0$ s.t if $0 < |x - (-3)| < \delta$ then $|g(x) - 8| < \varepsilon$

Choose $\delta = \min\left(1, \frac{\varepsilon}{12}\right)$

if $0 < |x+3| < \delta$

Then

$$|g(x)-8| = |x+3||2x-4| < \delta\, |2x-4| < \delta \cdot 12 \le \frac{\varepsilon}{12} \cdot 12 = \varepsilon$$

Scratch

$|g(x) - 8|$

$= |2x^2 + 2x - 12| = |x+3||2x-4|$

its easy to get $|x+3|$ but what about $|2x-4|$. what do we know about if $|x+3| < \delta$

$-\delta < x+3 < \delta$

$-2\delta - 6 < 2x < 2\delta - 6$

$-2\delta - 10 < 2x - 4 < 2\delta - 10$

if $\delta \le 1$, then

$-12 < 2x - 4 < -8$

$8 < |2x-4| < 12$

Lets prove limits are unique.

Lemma Let $f:\mathbb{R}\to\mathbb{R}$ and spose

$$\lim_{x\to a} f(x) = l \quad \text{and} \quad \lim_{x\to a} f(x) = m$$

Then $l=m$

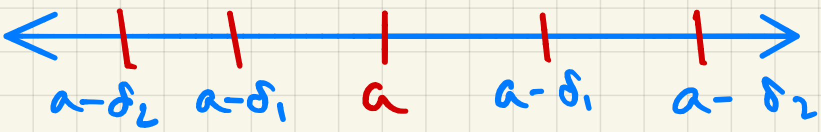
spose $l\neq m$ and set $\varepsilon = \frac{|l-m|}{57}$ note: $\varepsilon>0$

by the 2 limits

$\exists\, \delta_1>0$ s.t $x\in B_{\delta_1}(a)\setminus\{a\}$, then $f(x)\in B_\varepsilon(l)$

$\exists\, \delta_2>0$ s.t $x\in B_{\delta_2}(a)\setminus\{a\}$, then $f(x)\in B_\varepsilon(m)$

Then, $\forall\, x\in\left(B_{\delta_1}(a)\setminus\{a\}\right)\cap\left(B_{\delta_2}(a)\setminus\{a\}\right)$



chose $\delta=\min(\delta_1,\delta_2)$

$\forall\, x\in B_\delta(a)\setminus\{a\}$ $\square$ holds

$$\begin{aligned}
|l-m| &= |l-f(x)+f(x)-m| \\
&\le |l-f(x)| + |f(x)-m| \\
&< \varepsilon + \varepsilon \\
&= 2\varepsilon = \frac{2|l-m|}{57} < |l-m| \quad \text{oops}
\end{aligned}$$

Caution

open sets (thus topology) underlie alot of what is going on here.

In some topologies, limits are not unique.

Exercise: Let $f: \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} 57, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Show $\lim_{x \to 0} f(x)$ doesn't exist. Negate for all $\ell \in \mathbb{R}$

$\forall \ell \in \mathbb{R} \ \exists \varepsilon > 0$ s.t $\forall \ \delta > 0$

$\exists x \in B_\delta(a) \setminus \{a\}$ st $f(x) \notin B_\varepsilon(\ell)$

Let $\ell \in \mathbb{R}$. Task: find ε such that

Choose $\varepsilon = \frac{1}{2}$, fix $\delta \in \mathbb{R}$, $\delta > 0$

let $x_1, x_2 \in B_\delta(0) \setminus \{0\}$ with $x_2 < 0$ and $x_1 > 0$

$\hookrightarrow$ certainly exist as $\delta \neq 0$

Claim we can't have,

$$f(x_1) \in B_\varepsilon(\ell) \quad \text{and} \quad f(x_2) \in B_\varepsilon(\ell)$$

If we did then,

$$57 = |f(x_1) - f(x_2)| \to \text{by def of } f(x)$$
$$= |f(x_1) - \ell + \ell - f(x_2)|$$
$$\leq |f(x_1) - \ell| + |\ell - f(x_2)|$$
$$< \varepsilon + \varepsilon$$

$57 < 1$ oops....

we proved $\lim_{x \to 0} f(x)$ DNE

Fake definition: We can show that the last limit exists with the fake definition.

What else can we say about limits?

Lemma Let $c \in \mathbb{R}$
let $f, g : \mathbb{R} \to \mathbb{R}$ be functions
Let $m, n, a \in \mathbb{R}$ s.t

$$\lim_{x \to a} f(x) = m \quad \text{and} \quad \lim_{x \to a} g(x) = n$$

We have

① $\lim_{x \to a} (c \cdot f)(x) = c \cdot m$

② $\lim_{x \to a} (f + g)(x) = m + n$

③ $\lim_{x \to a} (f \cdot g)(x) = mn$

Pf ①. Let $\varepsilon > 0$ be given

Find $\delta > 0$ s.t if $0 < |x-a| < \delta$, then $|f(x) - m| < \dfrac{\varepsilon}{|c|+57}$

then

$$|c|\,|f(x) - m| < \frac{(|c|+1)\,\varepsilon}{|c|+57}$$

for $0 < |x-a| < \delta$ we have,

$$|c \cdot f(x) - cm| < \frac{(|c|+1) \cdot \varepsilon}{|c|+57} < \varepsilon$$

Pf ② Let $\varepsilon > 0$ be given

$\lim_{x \to a} f(x) = m$ so we can choose $\delta_1 > 0$ s.t

if $0 < |x-a| < \delta_1$ then $|f(x) - m| < \frac{\varepsilon}{57}$

$\lim_{x \to a} g(x) = n$ so, we can choose $\delta_2 > 0$ s.t

if $0 < |x-a| < \delta_2$ then $|f(x) - n| < \frac{\varepsilon}{57}$

Then, $\forall x$ satisfying

$0 < |x-a| < \delta \to \delta = \min(\delta_1, \delta_2)$

we have,

$$|f(x) + g(x) - (m+n)| \leq |f(x) - m| + |g(x) - n|$$
$$< \frac{\varepsilon}{57} + \frac{\varepsilon}{57} < \varepsilon$$

Pf ③ Let $\varepsilon > 0$ be given

Choose $\delta_1 > 0$ s.t, if $x \in B_{\delta_1}(a) \setminus \{a\}$ then,

$|f(x) - m| < \frac{\varepsilon}{57(|n|+1)}$

Similarly, choose $\delta_2 > 0$ such that if $x \in B_{\delta_2}(a) \setminus \{a\}$ then,

$|g(x) - n| < \min\left(1, \frac{\varepsilon}{57(|m|+1)}\right)$

for $x \in B\delta_2(a) \setminus \{a\}$ we have

$-1 \leq g(x) - n \leq 1$

$-1-n \leq g(x) \leq 1+n$

$-(1+|n|) \leq g(x) \leq 1+|n|$

$|g(x)| \leq |n+1|$ for $x \in B_{\delta_2}(a) \setminus \{a\}$

Now chose $\delta = \min(\delta_1, \delta_2)$. Then for $x \in B_\delta(a) \setminus \{a\}$

$|f(x) \cdot g(x) - m \cdot n| = |f(x) \cdot g(x) - m \cdot g(x) + m \cdot g(x) - m \cdot n|$

$= |(f(x) - m)\, g(x) + m\,(g(x) - n)|$

$\leq |g(x)|\,|f(x) - m| + |m|\,|g(x) - n|$

$< \frac{\varepsilon}{(|n|+1)\,57}(|n|+1) + |m| \cdot \frac{\varepsilon}{57\,(|m|+1)}$

$< \frac{\varepsilon}{57} + \frac{\varepsilon}{57} < \varepsilon$ $\square$

Now, we also want to divide (and be mindful of 0).

Lemma Spose $h: \mathbb{R} \to \mathbb{R}$ let $a, \ell \in \mathbb{R}$ w $\ell \neq 0$

If $\lim_{x \to a} h(x) = \ell$, then

$\exists\, \delta > 0$ s.t. $\forall\, x$ satisfying $0 < |x - a| < \delta$, we have

$|h(x)| > \frac{|\ell|}{2}$

Pf Choose $\varepsilon = \frac{|\ell|}{2}$

Then $\exists\, \delta > 0$ s.t. if $0 < |x - a| < \delta$ then

$|h(x) - \ell| < \varepsilon = \frac{|\ell|}{2}$

$\Rightarrow -\frac{|\ell|}{2} + \ell < h(x) < \frac{|\ell|}{2} + \ell$

note: if $\ell > 0$, then we have $\forall\, x \in B_\delta(a) \setminus \{a\}$

$-\frac{|\ell|}{2} + \ell < h(x)$ since $\frac{|\ell|}{2} = \frac{\ell}{2}$

$\Rightarrow \frac{\ell}{2} < h(x)$ so

$|h(x)| > \frac{|\ell|}{2}$

if $l<0$, then $\forall x \in B_\delta(a)\setminus\{a\}$

$$h(x) < \frac{|l|}{2} + l \Rightarrow h(x) < \frac{l}{2} < 0$$

$$\Rightarrow |h(x)| > \left|\frac{l}{2}\right|$$

Why do we need this? To talk about division !!

Lemma $f:\mathbb{R}\to\mathbb{R}$, $g:\mathbb{R}\to\mathbb{R}$, $a\in\mathbb{R}$

$$\lim_{x\to a} f(x) = m$$

$$\lim_{x\to a} g(x) = n \qquad \text{spose } n\neq 0$$

Then: $\displaystyle\lim_{x\to a}\left(\frac{f}{g}\right)(x) = \frac{m}{n}$

Pf It is sufficient to show $\displaystyle\lim_{x\to a}\left(\frac{1}{g}\right)(x) = \frac{1}{n}$

notes about 0: $g:\mathbb{R}\to\mathbb{R}$, $\frac{1}{g}: A\to\mathbb{R}$

$A := \{x\in\mathbb{R} \mid g(x)\neq 0\}$ but there's more

Let $\varepsilon>0$ be given. Since $\lim_{x\to a} g(x)=n$

$\exists\, \delta>0$ s.t if $0<|x-a|<\delta$,

then, $|g(x)-n| < \frac{\varepsilon\cdot|n|^2}{2}$

and

$|g(x)| > \frac{|n|}{2} > 0$ by lemma

$\therefore$ $|g(x)|$ is non zero

Scratch

we want to make the following small.

$$\left|\frac{1}{g(x)} - \frac{1}{n}\right|$$

$$= \left|\frac{n-g(x)}{g(x)\cdot n}\right|$$

Thus, $\forall x\in B_\delta(a)\setminus\{a\}$ we have

$$\left|\frac{1}{g(x)} - \frac{1}{n}\right| = \left|\frac{n-g(x)}{g(x)\cdot n}\right| < \frac{\varepsilon\cdot|n|^2}{2}\cdot\frac{2}{|n|^2} < \varepsilon$$

What about composition?

want: lim to behave well under composition.

spse: $\lim_{x \to a} f(x) = \ell \qquad \lim_{y \to \ell} (g(y)) = m$

$$\lim_{x \to a} g(f(x)) = m$$

Does this actualy work?

eg1 $f, g : \mathbb{R} \to \mathbb{R}$

$f : s \mapsto 1$

$$f : t \mapsto \begin{cases} 3, & t \neq 1 \\ -2, & t = 1 \end{cases}$$

$g \circ f : \mathbb{R} \to \mathbb{R}$

$x \mapsto -2$

So, $\lim_{x \to 0} g \circ f(x) = -2$ oops we wanted 3

We have found a problem we are forced to fix.

patch, we need $\lim_{y \to \ell} g(y) = g(\ell) = m$

We are forced to meet continuity.

Patched lemma discussed later:

if $\lim_{x \to a} f(x) = \ell$ and

$\lim_{y \to \ell} g(x) = m$ and $m = g(\ell)$

then $\lim_{x \to a} g(f(x)) = g(\ell) = m$