Spivak integrals

Defn) Let P be a partition of $[a,b]$

$$P = \{a = t_0, \ \ldots \ t_n = b\}$$

We define the **norm of P** to be

$$\|P\| := \max\{t_i - t_{i-1} \mid 1 \leq i \leq n\}$$

↳ intuitively the max subinterval length

eg: Let P_n be a regular partition of $[a,b]$.

$$\|P_n\| = \frac{b-a}{n}$$

Take this idea back to $\lim_{n \to \infty} U(f, P_n)$ for **Darboux**

it is the same as $\lim_{\|P\| \to 0} U(f, P_n)$

Defn) Spose $f : [a,b] \to \mathbb{R}$ is bounded.

We say f is Spivak Integrable pt.

$$\lim_{\|P\| \to 0} U(f, P) = \lim_{\|P\| \to 0} L(f, P) = \int_a^b f$$

↳ unpack:

$\forall \ \varepsilon > 0 \ \exists \ \delta > 0$ s.t. $\forall$ partitions P of $[a,b]$ s.t. $\|P\| < \delta$ we have

$$|U(f,P) - \ell| < \varepsilon$$

Consider:

$$U(f) \quad \text{vs} \quad \lim_{\|P\| \to 0} U(f, P)$$

$$L(f) \quad \text{vs} \quad \lim_{\|P\| \to 0} L(f, P)$$

If we show that they are are equivalent it is immediate:

→ $S\text{-int} \iff D\text{-int}$

→ $I_S = I_D$ But for now....

Riemann Integrable

Defn) let $f : [a,b] \to \mathbb{R}$ (not necessarily bdd)

let $[a,b] \supseteq P = \{a = t_0, \ldots t_n = b\}$

let x^* be an n-tuple

$x = (x_1^*, x_2^*, \ldots x_n^*)$ st $x_i^* \in \mathbb{R}$ $\forall\ 1 \leq i \leq n$

Defn) x^* is compatible with the partition P pt

$x_i^* \in [t_{i-1}, t_i]$ $\forall\ 1 \leq i \leq n$

For a partition P & x^* compatible, we define the Riemann sum

$$R(f, P, x^*) = \sum_{i=1}^{n} \underline{f(x_i^*)} (t_i - t_{i-1})$$

↳ we don't use max. so f need not be bdd

Defn) $f : [a,b] \to \mathbb{R}$ is said to be Riemann integrable p.t.

$\lim_{\|P\| \to 0} R(f, P, x^*)$ converges to ℓ

In this case, $I_R = \int_a^b f = \ell$

Unpack

$\forall\ \varepsilon > 0\ \exists\ \delta > 0$ st $\forall$ partitions P of $[a,b]$ with $\|P\| < \delta$ then $\forall\ x^*$ compatible with P we have

$$|R(f, P, x^*) - \ell| < \varepsilon$$

Interesting how R doesn't require boundedness.

lets show a result →

Lemma if $f:[a,b]\to\mathbb{R}$ is Riemann integrable, it is bounded.

Pf Spose $f:[a,b]\to\mathbb{R}$ is R-int & not bdd.

Set $I_R := \lim_{\|P\|\to 0} R(f,P,x^*) \to$ exists

$\forall\, \varepsilon>0\ \exists\, \delta>0$ s.t if P is a partition of $[a,b]$ with $\|P\|<\delta$ if x^* is compatible with P then

$$|R(f,P,x^*) - I_R| < \varepsilon$$

Set $\varepsilon = 57$ and fix $\delta>0$ and partition P such that $\|P\|<\delta$.

x^* & z^* are compatible with P we have

$$|R(f,P,x^*) - R(f,P,z^*)|$$

$$= |R(f,P,x^*) - I_R + I_R - R(f,P,z^*)|$$

$$\le |R(f,P,x^*) - I_R| + |R(f,P,z^*) - I_R| < 57 + 57 = 114$$

lets break this. By AP we can choose $n\in\mathbb{N}$ s.t

$\|P_n\| < \delta$.

Spose x^* is compatible w/ P_n. Since f is unbdd $\exists\, j$ with $1\le j\le n$ s.t

$\text{res}_{[t_{j-1},t_j]} f$ is unbounded

So $\exists\, z_j \in [t_{j-1}, t_j]$ s.t

$$|f(z_j) - f(x_j^*)| > \frac{570\cdot n}{b-a}$$

Define z^* by,

$$z_i^* := \begin{cases} x_i^* & i\neq j \\ z_j^* & i=j \end{cases}$$

$\longrightarrow$

Now, we have P_n, x^*, z^* & x^*, z^* are compatible with P_n.

Since $\|P_n\| < \delta$,

$|R(f,P_n,x^*) - R(f,P_n,z^*)|$

$= \sum_1^n f(x_i^*)(t_i - t_{i-1}) - \sum_1^n f(z_i^*)(t_i - t_{i-1})$

by the way defines z^* most of the stuff cancelles

$= |(f(x_j^*) - f(z_j^*))(t_j - t_{j-1})|$

$= \frac{570 \cdot \cancel{n}}{\cancel{b-a}} \frac{\cancel{(b-a)}}{\cancel{n}} = 570 \nless 57$ ⊠

So it must be the case that f is bdd

Goal: Show equivalences in 3 definitions

Lemma) Let $f:[a,b] \to \mathbb{R}$ be bounded. Spse

$$\lim_{\|P\| \to 0} U(f,P) = \ell$$

Then,

$$\lim_{n \to \infty} U(f, P_n) = \ell$$

Pf) Let $\varepsilon > 0$ be given. Since $\lim_{\|P\| \to 0} U(f,P) = \ell$

$\exists\, \delta > 0$ s.t for partition P of $[a,b]$ with $\|P\| < \delta$

$$|U(f,P) - \ell| < \varepsilon$$

Consider the regular Partition P_n. $\|P_n\| = \frac{b-a}{n}$

By AP we can choose $N \in \mathbb{N}$ s.t $\|P_n\| < \delta$

$\therefore\ \forall n \geq N$ $\quad \|P_n\| < \delta$ and $|U(f,P_n) - \ell| < \varepsilon$

$\therefore \lim_{n \to \infty} U(f,P_n) = \ell$

Note: we sorta played with 2 ideas of limits. 1 is of a sequence and the other is old school.

Side Note A sequence in $\mathbb{R}$ $n \mapsto x_n$ converges to ℓ pt

$\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ st $\forall n \geq N$ we have $x_n \in B_\varepsilon(\ell)$

Equivalences in Ideas of Integrals

We have 3 ideas. In Sarah's handout the following is proved

Lemma Let $f : [a,b] \to \mathbb{R}$ be bdd. Then

$$\lim_{\|P\| \to 0} U(f,P) = U(f)$$

$$\| \qquad\qquad \|$$

$$\lim_{\|P\| \to 0} L(f,P) = L(f)$$

Corollary Let $f : [a,b] \to \mathbb{R}$ be bounded.

Then f is S integrable iff f is D-integrable

And we have $I_D = I_S$

We will now show Ⓢ $\Rightarrow$ Ⓡ and $I_S = I_R$

Another handout, Conrad shows Ⓡ $\Rightarrow$ Ⓢ & $I_R = I_S$

$\hookrightarrow$ This is unnecessary info

Lemma If $f : [a,b]$ is bounded & S-integrable then, f is R integrable.

Moreover, $I_S = I_R$

Pf let P be a partition of $[a,b]$ & let x^* be compat. Then, we consider,

$$L(f,P) \leq R(f,P,x^*) \leq U(f,P)$$

$$\sum_1^n (t_n - t_{n-1})\, m_i \qquad \sum_1^n (t_n - t_{n-1}) f(x^*_i) \qquad \sum_1^n (t_n - t_{n-1})\, M_i$$

$$m_i \leq f(x^*_i) \leq M_i \quad \forall i \quad 1 \leq i \leq n$$

Since f is S-integrable, we know

$$\lim_{\|P\|\to 0} U(f,P) = \lim_{\|Q\|\to 0} L(f,Q) = \ell$$ converges

Since f is S-int

Task: Show R int exists. Candidate limit

Let $\varepsilon > 0$ be given. $\exists \delta_1 > 0$ s.t. $\forall$ partitions P of $[a,b]$ s.t. $\|P\| < \delta_1$

$$|L(f,P) - \ell| < \varepsilon$$

$$\iff -\varepsilon < L(f,P) - \ell < \varepsilon$$

Similarly $\exists \delta_2 > 0$ s.t. $\forall$ partitions Q of $[a,b]$ s.t. $\|Q\| < \delta_2$

$$|U(f,Q) - \ell| < \varepsilon$$

$$\implies -\varepsilon < U(f,Q) - \ell < \varepsilon$$

Set $\delta := \min(\delta_1, \delta_2)$ take a partition of $[a,b]$ with $\|P\| < \delta$ and let x^* be compat.

Then,

$$-\varepsilon < L(f,P) - \ell \le R(f,P,x^*) - \ell \le U(f,P) - \ell < \varepsilon$$

$$\implies -\varepsilon < R(f,P,x^*) - \ell < \varepsilon \implies |R(f,P,x^*) - \ell| < \varepsilon$$

So the limit $\lim_{\|P\|\to 0} R(f,P,x^*)$ exists and is ℓ!

$$\implies I_S = \ell = I_R$$

$\square$

Thm) Let $f:[a,b]\to\mathbb{R}$ be a function. TFAE

1. f is bdd & Darboux int
2. f is bdd & S-int
3. f is R-int

$$\begin{tikzcd} D \arrow[r, leftrightarrow, "SK"] & S \arrow[d, bend right, "us"'] \\ & R \arrow[u, bend right, "cont"'] \end{tikzcd}$$

Moreover $I_S = I_R = I_D = \int_a^b f$

Now, we only need to say f is integrable without any specification

Integration is Linear

Lemma Let $f, g : [a,b] \to \mathbb{R}$ be integrable. Let $c \in \mathbb{R}$.
Then

① $\int_a^b f+g = \int_a^b f + \int_a^b g$

These have 2 assertions

② $\int_a^b cf = c\int_a^b f$ $\longrightarrow$ Done by Sarah in handout

Pf ① We will use Riemann Definition. We must show

$\lim_{\|P\| \to 0} R(f+g, P, x^*)$ exists and is $\int_a^b f + \int_a^b g$.

Let $\varepsilon > 0$ be given

$\exists \delta_f$ s.t $\forall$ partitions P of $[a,b]$ with $\|P\| < \delta_f$
and if x^* is compatible then

$$\left|R(f, P, x^*) - \int_a^b f\right| < \frac{\varepsilon}{57}$$

$\exists \delta_g$ s.t $\forall$ partitions Q of $[a,b]$ with $\|Q\| < \delta_g$
and if y^* is compat with Q then

$$\left|R(g, P, y^*) - \int_a^b g\right| < \frac{\varepsilon}{57}$$

take $\delta := \min\{\delta_f, \delta_g\} > 0$

let P be a partition of $[a,b]$ s.t $\|P\| < \delta$ and
let x^* be compat.

$$\left|R(f+g, P, x^*) - \left(\int_a^b f + \int_a^b g\right)\right|$$

$$= \left|\sum_{i=1}^{n} (t_i - t_{i-1})\, f+g(x_i^*) - \int_a^b f - \int_a^b g\right|$$

$$= \left|\sum_{i=1}^{n} (t_i - t_{i-1})(f(x_i^*) + g(x_i^*)) - \int_a^b f - \int_a^b g\right|$$

$$= \left|\underbrace{\sum_{i=1}^{n} (t_i - t_{i-1})\, f(x_i^*) + \sum_{i=1}^{n} (t_i - t_{i-1})\, g(x_i^*)} - \int_a^b f - \int_a^b g\right|$$

$= \left| R(f, P, x^*) + R(g, P, x^*) - \int_a^b f - \int_a^b g \right|$

$= \left| R(f,P,x^*) - \int_a^b f \right| + \left| R(g, P, x^*) - \int_a^b g \right|$

$< \frac{\varepsilon}{57} + \frac{\varepsilon}{57} < \varepsilon$ $\square$

$\Rightarrow \lim_{\|P\| \to 0} R(f+g, P, x^*) = \int_a^b f + \int_a^b g$

$\Rightarrow \int_a^b f+g = \int_a^b f + \int_a^b g$

Thm if $f : [a,b] \to \mathbb{R}$ is cts, it is integrable.

↳ due to EVT f is bounded

PF by EVT f is bdd

Let $\varepsilon > 0$ be given.

Task: Find a partition P s.t $U(f,P) - L(f,P) < \varepsilon$ → Darboux

Recall: Since $f : [a,b] \to \mathbb{R}$ is cts, it is uniformly cts on $[a,b]$

$\exists \delta > 0$ s.t $\forall x, y \in [a,b]$ if $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$

By AP, choose $n \in \mathbb{N}$ s.t $\frac{b-a}{n} < \delta$

Consider regular partition $P_n = \{a = t_0, \dots, t_n = b\}$

note: $f|_{[t_{i-1}, t_i]} : [t_{i-1}, t_i] \to \mathbb{R}$ is cts.

By EVT $\exists x_{m_i}, x_{M_i} \in [t_{i-1}, t_i]$ s.t

$M_i = f(x_{M_i})$, $m_i = f(x_{m_i})$

→

Since x_{M_i}, x_{m_i} exist in a single subinterval,

$|x_{M_i} - x_{m_i}| < \frac{b-a}{n} < \delta$. So by unif cts.

$|f(x_{M_i}) - f(x_{m_i})| < \frac{\varepsilon}{57(b-a)}$ → positive as $b > a$

$\Rightarrow |M_i - m_i| < \frac{\varepsilon}{57(b-a)} \Rightarrow M_i - m_i < \frac{\varepsilon}{57(b-a)}$

↳ as $M_i \geq m_i$

Consequently

$U(f, P_n) - L(f, P_n)$

$= \sum_{i=1}^{n} (t_i - t_{i-1}) M_i - \sum_{i=1}^{n} (t_i - t_{i-1}) m_i$

$= \sum_{i=1}^{n} (M_i - m_i) \left(\frac{b-a}{n}\right)$

$= \sum_{i=1}^{n} \frac{\varepsilon}{57\cancel{(b-a)}} \left(\frac{\cancel{b-a}}{n}\right) = \cancel{n} \frac{\varepsilon}{57\cancel{n}} < \varepsilon$

$\Rightarrow$ f is integrable