Integration

Standing assumption area of rectangle $= b \cdot h$

look at: Scissors congruence

We will discuss the following integrals:

1. Darboux
2. Spivak
3. Riemann

Throughout this discussion $f: [a,b] \to \mathbb{R}$ is bounded on $f([a,b])$ is bounded!

note: f need not be cts!

Upper Darboux Sums

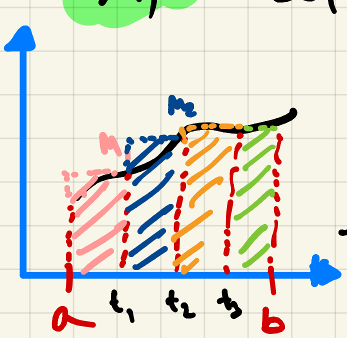
Defn A partition P of $[a,b]$ is a ordered set (finite) $P = \{t_0, \dots t_n\}$ of points in $\mathbb{R}$ where

$$a = t_0 < t_1 \quad \dots \quad < t_n = b$$

Defn if P is a partition of $[a,b]$ then we define,

$$M_i := \sup\{f(x) \mid x \in [t_{i-1}, t_i]\} \quad 1 \le i \le n$$

$\to$ exists as the codomain is odd $*$ $t_{i-1} < t_i$



$$U(f,P) := \sum_{i=1}^{n} (t_i - t_{i-1}) \cdot M_i$$

This is the upper Darboux sum

$\to$ overshooting the area under the curve

eg: $f: [0,1] \to \mathbb{R} \quad x \mapsto x_2$

$$U(f, \{0, 1/3, 1\}) = \frac{19}{27} \quad //$$

Defn Let P, Q be a partitions of $[a,b]$. We say

$P \preceq Q$ iff $P \subseteq Q$ as sets

" Q is a refinement of P "

Lemma if P, Q are partitions of $[a,b]$ with $P \preceq Q$

$U(f,P) \geq U(f,Q)$ → The upper sum of the refinement is more accurate

Pf **Case I:** $\underbrace{Q = P \sqcup \{s\}}_{\text{disjoint union}}$ → $s \notin P$

for $s \in [a,b]$

$P = \{a = t_0 < t_1 < \dots < t_n\}$

$Q = \{a = t_0 < t_1 \dots t_j < s < t_{j+1} \dots t_n = b\}$

Goal: Show $U(f,P) - U(f,Q) \geq 0$

$A_1 := \{f(x) \mid x \in [t_j, s]\}$

$A_2 := \{f(x) \mid x \in [s, t_{j+1}]\}$

$U(f,P) - U(f,Q)$

$= \sum_i^n (t_i - t_{i-1}) \cdot M_i - \Big[\sum_1^j (t_i - t_{i-1}) \cdot M_i + (s - t_j) \sup(A_1)$

$+ (t_{j+1} - s) \sup(A_2) + \sum_{j+2}^n (t_i - t_{i-1}) \Big]$

↓ simplifies to

$= (t_{j+1} - t_j) M_{j+1} - (s - t_j) \sup(A_1) - (t_{j+1} - s) \sup(A_2)$

note $A_1, A_2 \subseteq \{f(x) \mid x \in [t_j, t_{j+1}]\}$

so, $\sup(A_1), \sup(A_2) \leq \sup\{ * \} = \underline{M_{j+1}}$

$\Rightarrow = \underbrace{(t_{j+1} - s)}_{\geq 0} \underbrace{(M_{j+1} - \sup(A_2))}_{\geq 0} + \underbrace{(s - t_j)}_{\geq 0} \underbrace{(M_{j+1} - \sup(A_1))}_{\geq 0}$

≥ 0 □

- Case 2, P and Q differ by > 1 element

try a recursive argument by

$Q \setminus P = \{s_1 \ \dots \ s_m\}$

$P_0 := P$

$P_1 := P_0 \sqcup \{s_1\}$

$P_2 := P_1 \sqcup \{s_2\} \quad = P_0 \sqcup \{s_1, s_2\}$

$\vdots$

$P_n := P_{n-1} \sqcup \{s_m\} \quad = Q$

Apply case 1 to each step

$$U(f,P) = U(f,P_0) \geq U(f,P_1)$$
$$U(f,P_1) \geq U(f,P_2)$$
$$\vdots$$
$$U(f,P_{n-1}) \geq U(f,P_n)$$

$$\Rightarrow U(f,P) \geq U(f,Q) \qquad \square$$

Lower Darboux Sums

Defn) $f : [a,b] \to \mathbb{R}$ is bdd & let P be a partition of $[a,b]$

$$L(f,P) := \sum_{i}^{n} (t_i - t_{i-1}) \cdot m_i \quad \to \text{lower estimates}$$

$m_i := \inf\{f(x) \mid x \in [t_i - t_{i-1}]\}$

Lemma) Let P, Q be a partition of $[a,b]$ if $P \leq Q$

$$L(f,P) \leq L(f,Q)$$

Same idea as earlier see book for proof!

Remark) $\forall$ partitions P of $[a,b]$ we have

$$L(f,P) \leq U(f,P)$$

This comes from def where bound each term in L with the corresponding term in U.

Lemma) If P, Q are two partitions of $[a,b]$ then

$$L(f,P) \leq U(f,Q)$$

Pf) let R be a partition of $[a,b]$ such that it is a common refinement of P and Q.

eg $R = P \cup Q$

Then $L(f,P) \leq L(f,R) \leq U(f,R) \leq U(f,Q)$

Consider

$L(f) := \{ L(f,P) \mid P$ is a partition of $[a,b]\}$

ub exists b/c it is nonempty & bdd above by any upper sum

$U(f): \{ U(f,P) \mid P$ is a partition of $[a,b]\}$

Defn) Suppose $f:[a,b] \to \mathbb{R}$ is bdd. We say that f is darboux integrable if

$U(f) = L(f)$

In this case we define

$$\int_a^b f = U(f) = L(f)$$

Furthermore, if f is non-negative we define the area under the curve is said to be

$$\int_a^b f$$

Eg: Show that if $f:[a,b]\to\mathbb{R}$ is a constant func then it is integrable

$\to \forall i \quad M_i = m_i$

non eg! : $f:[0,1]\to\mathbb{R}$

$$x \mapsto \begin{cases} 1 & x\in\mathbb{Q} \\ 0 & x\notin\mathbb{Q} \end{cases}$$

We can show $L(f)=0$ & $U(f)=1$

Q is $g:[a,b]\to\mathbb{R}$, $x\mapsto x^2$ integrable?

If yes, what is $\int_a^b g$?

We need a better definition of integrability!

Remark | $A:=\{L(f,P) \mid P \text{ is a partition of } [a,b]\}$

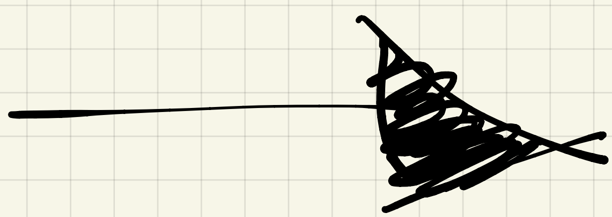
$B:=\{U(f,P) \mid P \text{ is a partition of } [a,b]\}$

$\forall a\in A \quad \forall b\in B$

$a\le b$, so,

$L(f)=\sup(A)\le\inf(B)=U(f)$

That is $L(f)\le U(f)$ always!



Lemma Darboux integrability criterion

f is Darboux integrable

iff

$\forall \varepsilon>0 \;\; \exists P, Q$ partitions of $[a,b]$ s.t. $\underbrace{U(f,P) - L(f,Q)}_{\geq 0 \text{ obviously}} < \varepsilon$

Pf $\Rightarrow$

Let $\varepsilon>0$ be given.

Since $L(f) = \sup(A)$ $\exists$ a partion Q of $[a,b]$ s.t

$L(f) - L(f,Q) < \frac{\varepsilon}{57}$

Since $U(f) = \inf(B)$ $\exists$ a partition P of $[a,b]$ s.t

$U(f,P) - U(f) < \frac{\varepsilon}{57}$

Then $U(f,P) - L(f,Q)$

$= [U(f,P) - U(f)] + [L(f) - L(f,Q)]$

$< \frac{\varepsilon}{57} + \frac{\varepsilon}{57} < \varepsilon$ $\square$

$\Leftarrow$ We need to show $U(f) = L(f)$. We already have that $U(f) \geq L(f)$

We are done if we show $U(f) \leq L(f)$.

From HW we need to show $\forall \varepsilon>0 \;\; U(f) - L(f) < \varepsilon$

Let $\varepsilon>0$ be given by assumption $\exists$ a partitions P, Q s.t

$U(f,P) - L(f,Q) < \varepsilon$

$U(f) - L(f) \leq U(f,P) - L(f,Q) < \varepsilon$ $\square$

Nice! But what about $\int_a^b x^2$??

$\int x^2 \ldots$

assume $b \geq a \geq 0$ $\quad f : [a,b] \to \mathbb{R}^2 \quad x \mapsto x^2$

what is $\int_a^b f$?

We first need to show f is integrable ! Then we compute it .

We will use Darboux int criterion !

There are nice partitions to use

Regular Partition

$P_n := \{ a = t_0 < t_1 \ldots < t_n = b \}$

length of each subinterval is $\frac{b-a}{n}$

$t_i = a + \frac{b-a}{n} \cdot i \qquad 0 \leq i \leq n$

Fix a regular partition P_n

$$U(f, P_n) := \sum_1^n (t_i - t_{i-1}) \cdot M_i$$

$$= \sum_1^n \frac{(b-a)}{n} f(t_i) \to \text{increasing } f$$

$$= \sum_1^n \frac{(b-a)}{n} \left(a + \frac{b-a}{n} \cdot i \right)^2$$

$$L(f, P_n) := \sum_1^n \frac{(b-a)}{n} \left(a + (i-1) \frac{(b-a)}{n} \right)$$

$\Rightarrow U(f, P_n) - L(f, P_n)$ telescopes to

$$= \frac{b-a}{n} \left[(a + (b-a))^2 - a^2 \right]$$

$$= \frac{b-a}{n} \left[b^2 - a^2 \right] = \frac{(b-a)^2 (b+a)}{n}$$

Now we use Darboux int crit $\exists N \in \mathbb{N}$ st

$\frac{1}{N} < \frac{\varepsilon}{(b-a)^2 (b+a)} \to$ if $a = b$ there is only 1 part so sup = inf

So, for P_N $U(f,P_N) - L(f,P_N) = \frac{(b-a)^2(b+a)}{N}$

So, $U(f,P_n) - L(f,P_n) < \varepsilon$

$\boxed{\text{So the integral exists!}}$

Pause: Sequences

What is a sequence? let $(X,\mathcal{T})$ be a top sp
A seq of pts in X is a function

$$\mathbb{N} \to X$$
$$n \mapsto x_n$$

Defn) let $n \mapsto x_n$ be a seq of pts in X. The seq is said to converge to $\ell \in X$ p.t

$\forall\, U \in \mathcal{T}$ that contains ℓ $\exists\, N \in \mathbb{N}$ s.t $\forall\, n \geq N$ we have $x_n \in U$

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eg: take $n \mapsto \frac{1}{n}$ $\ell = 0$

let U be an open set that contains $\ell = 0$.
since U is open, & contains 0 $\exists\, r > 0$ s.t

$$B_r(0) \subseteq U$$

By AP $\exists\, N \in \mathbb{N}$ s.t $\frac{1}{N} < r$ $\forall\, n \geq N$

$\frac{1}{n} \leq \frac{1}{N} < r \Rightarrow \forall\, n \geq N$ $x_n \in U$. $\square$

Fun Fact! $X = \mathbb{R}$ with $\mathcal{T}_{ind}$

Everything converges to every real number!

$X = \mathbb{R}$ with $\mathcal{T}_{Disc}$

Only functions that are eventually const converge!

ex: Define $n \mapsto \frac{c}{n}$ prove this converges to 0.

Ex: if $n \mapsto a_n$ and $n \mapsto b_n$ are 2 seq of reals and if $a_n \le b_n$,

and if both converge then

$$\lim_{n\to\infty} a_n \le \lim_{n\to\infty} b_n$$

if $n \mapsto a_n$ converges to ℓ then we write

$$\lim_{n\to\infty} a_n = \ell .$$

Ex: Spose $n \mapsto a_n$ is a sequence in a hausdorff sp $(X, \mathcal{T})$

Show that if $\lim_{n\to\infty} a_n$ exists, then it is unique

"limits of sequences are unique in Hausdorff spaces"

Back to calculating integral

Let's use a regular partition P_n of $[a,b]$

fix $n \in \mathbb{N}$ & consider $P_n \subseteq [a,b]$

$I_D = \int_a^b x^2$ exists $I_D = U(f) \Rightarrow U(f, P_n) \ge I_D$

$$U(f,P_n) - I_D = U(f,P_n) - L(f) \le U(f,P_n) - L(f,P_n)$$

from earlier $U(f,P_n) - L(f,P_n) = \frac{(b-a)^2 (b+a)}{n}$

Concretely $U(f,P_n) - I_D \le \frac{(b-a)^2 (b+a)}{n}$

These are both sequences $a_n = U(f,P_n) - I_d$

$$b_n = \frac{(b-a)^2 (b+a)}{n}$$

$\forall n \in \mathbb{N}$ $0 \le a_n \le b_n$ by hw since

$\lim_{n\to\infty} b_n = 0$, we have $\lim_{n\to\infty} a_n = 0$

we have $\lim_{n\to\infty} (U(f,P_n) - I_D) = 0$

$\therefore I_D = \lim_{n\to\infty} U(f,P_n)$ $\Rightarrow$ unpack this!

$$U(f,P_n) = \sum_{i=1}^{n} \left(\frac{b-a}{n}\right)\left(a + \frac{i(b-a)}{n}\right)^2$$

$$= \left(\frac{b-a}{n}\right) \sum_{i=1}^{n} \left(a^2 + \frac{2i(b-a)a}{n} + i^2 \frac{(b-a)^2}{n^2}\right)$$

$$= \frac{(b-a)}{n}\left[\sum_{i=1}^{n}(a^2) + \frac{2(b-a)a}{n}\sum_{i=1}^{n} i + \frac{(b-a)^2}{n^2}\sum_{i=1}^{n} i^2\right]$$

$$= \frac{(b-a)}{n}\left[na^2 + \frac{\cancel{2}(b-a)a}{\cancel{n}} \cdot \frac{\cancel{n}(n+1)}{\cancel{2}} + \frac{(b-a)^2}{n^2} \cdot \frac{\cancel{(n)}(n+1)(2n+1)}{6}\right]$$

$$= (b-a)\left[a^2 + a(b-a) + \frac{(b-a)^2 \cdot \cancel{2}}{\cancel{6}\ 3} + \frac{1}{n}[\text{stuff}]\right]$$

$\Rightarrow$

$$\lim_{n\to\infty} U(f,P_n) = (b-a)\left\{(a^2 + ab - a^2) + \frac{1}{3}(b^2 - 2ab - a^2)\right]$$

$$= \frac{b-a}{3}\left[3ab + b^2 - 2ab + a^2\right)$$

$$= \frac{b-a}{3}(a^2 + ab + b^2) = \frac{b^3 - a^3}{3}$$

$$\therefore I_D = \int_a^b x^2 = \frac{b^3 - a^3}{3}$$

Note: once we know the limit existed, we computed it using limits. Specifically

$$\lim_{n\to\infty} U(f,P_n) = I_D = \int_a^b f$$

$\Rightarrow$ intuitively we shrink the subintervals!