Building $\mathbb{R}$

Plan: We build $\mathbb{N} \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{R}$

Resources → Conrad Handouts
Naive Set Theory – Halmos
Math 481 / 582

Peano Axioms

$\exists$ triple $(\mathbb{N}, \sigma, 1)$ s.t

- $\mathbb{N}$ is a set
- $\sigma : \mathbb{N} \to \mathbb{N}$ is injective (secretly $+1$) } we haven't defined $+$
- $1 \in \mathbb{N} \Rightarrow \mathbb{N} \neq \emptyset$

So that $\sigma(n) \neq 1 \quad \forall n \in \mathbb{N}$ and

$\forall S \subseteq \mathbb{N}$ if $1 \in S$ and if $n \in S \Rightarrow n+1 \in S$ then $S = \mathbb{N}$

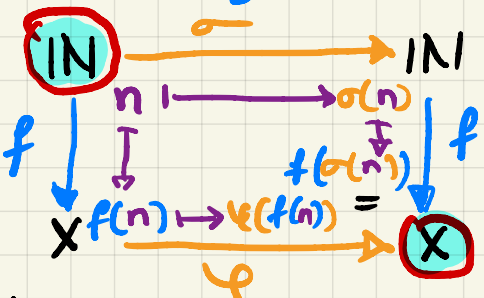
To be stern & earnest, we need to now define $+$, $\times$, order & prime factorization to show $\mathbb{N}$ behaves as expected

Principle of Recursion

Let X be a set & let $\varphi : X \to X$ & let $a \in X$. Then $\exists!\ f : \mathbb{N} \to X$ s.t

- $f(1) = a$
- $\forall n \in \mathbb{N} \quad f(\sigma(n)) = \varphi(f(n))$

Visualize This!



→ This diagram commutes as, no matter which path we choose from $\mathbb{N}$, we end up with the same value by the above

Using the secret definition of σ:

$f(1) = a \Rightarrow f(\sigma(1)) = f(2) = \varphi(a) \Rightarrow f(\sigma(2)) = f(3) = \varphi \circ \varphi(a)$

f allows us to compose φ with itself!

↳ we can define addition & multiplication

Addition

fix $m \in \mathbb{N}$

Take $X = \mathbb{N}$, set $a = \sigma(m)$, set $\varphi = \sigma$

By principle of recursion

$\exists!\ f_m : \mathbb{N} \to \mathbb{N}$ s.t. • $f_m(1) = \sigma(m) = a$

• $f_m(\sigma(n)) = \sigma(f_m(n))\ \forall n \in \mathbb{N}$

↳ sorta $f_m(n+1) = f_m(n) + 1$

In particular,

$f_m(1) = \sigma(m)$ `=' $m+1$

$f_m(2) = \sigma(f_m(1))$ `=' $\sigma(m+1)$ `=' $m+2$

$f_m(\sigma(2))$ `=' $f_m(3) = \sigma(f_m(2))$ `=' $\sigma(m+2)$ `=' $m+3$

So, $f_m : \mathbb{N} \to \mathbb{N}$ is addition on the left by m

Defn $\forall n \in \mathbb{N}$ define, (for fixed m giving $f_m : \mathbb{N} \to \mathbb{N}$)

$m +_{\mathbb{N}} n := f_m(n)$

Multiplication — repeated addition

fix $m \in \mathbb{N}$ $X = \mathbb{N}$ $\varphi = f_m$ $a = m$

By principle of recursion

$\exists N_m : \mathbb{N} \to \mathbb{N}$ s.t.

- $N_m(1) = m$
- $N_m(\sigma(n)) = f_m(N_m(n))$

$N_m(1) = m$

$N_m(2)$ `=' $N_m(\sigma(1)) = f_m(N_m(1)) = f_m(m) = m +_{\mathbb{N}} m$

Defn $\forall n \in \mathbb{N}$ define (having fixed $m \in \mathbb{N}$ to give ν_m)

$$m \cdot n := \nu_m(n)$$

We should check $+$ & $\cdot$ are commutative, associative & distribution holds → Conrad

Lemma $\forall n \in \mathbb{N}$ $f_1(n) = \sigma(n)$ → formalizing that σ is $+1$

Pf let $S = \{n \in \mathbb{N} \mid f_1(n) = \sigma(n)\}$

note: $1 \in S$ as $f_1(1) = \sigma(1)$ by definition of f_1

Spose $n \in S$, lets show $\sigma(n) \in S$, that is

$f_1(\sigma(n)) = \sigma(\sigma(n))$

Since $n \in S$, $f_1(n) = \sigma(n)$ so

$f_1(\sigma(n)) = \sigma(f_1(n)) = \sigma(\sigma(n))$

Principle of rec ✓

So, $\sigma(n) \in S$. By Peano axioms $S = \mathbb{N}$ □

Lemma $\forall n \in \mathbb{N}$ $\nu_1(n) = n$ '$n \cdot 1 = n$'

Pf let $\lambda = \{n \in \mathbb{N} \mid \nu_1(n) = n\}$

by definition $\nu_1(1) = 1$ so $1 \in \lambda$

Check $n \in \lambda \Rightarrow \sigma(n) \in \lambda$ that is $\nu_1(\sigma(n)) = \sigma(n)$

by definition of ν_1 coming from principle of rec

$\nu_1(\sigma(n)) = f_1(\nu_1(n))$ since $n \in \lambda$

$= f_1(n) = \sigma(n)$ by prev lemma

By Peano axioms $\lambda = \mathbb{N}$ □

Establishing Order

Lemma $\forall n, m, q \in \mathbb{N}$ we have

$m = n \iff m + q = n + q$ → we can cancel

Pf up to us!

Lemma $\forall n, m \in \mathbb{N}$, $m \neq n$ exactly one of the following holds

1) $\exists!\ r \in \mathbb{N}$ s.t $m = n + r$ => we say $m >_{\mathbb{N}} n$
2) $\exists!\ s \in \mathbb{N}$ s.t $m + s = n$ => we say $n >_{\mathbb{N}} m$

Pf Conrad

Corollary Trichotomy !

Building $\mathbb{Z}$

Consider $X = \mathbb{N} \times \mathbb{N}$ and define the (equivalence) relation $\sim_{\mathbb{Z}}$ by $(n, m) \sim_{\mathbb{Z}} (l, \theta) \iff n +_{\mathbb{N}} \theta = l +_{\mathbb{N}} m$ (sorta $n - m = l - \theta$)

Let us show this is an equivalence relation!

1) **Reflexive**: $(n, m) \sim_{\mathbb{Z}} (n, m)$ yes✓ as $n + m_{\mathbb{N}} = n +_{\mathbb{N}} m$
2) **Symmetric**: Show $(n, m) \sim_{\mathbb{Z}} (l, \theta) \Rightarrow (l, \theta) \sim_{\mathbb{Z}} (n, m)$

   yes as addition is commutative

$(n, m) \sim_{\mathbb{Z}} (l, \theta) \Rightarrow n +_{\mathbb{N}} \theta = l +_{\mathbb{N}} m \Rightarrow \theta +_{\mathbb{N}} n = m +_{\mathbb{N}} l$ □

3) **Transitive**: $(n, m) \sim_{\mathbb{Z}} (l, \theta)$ & $(l, \theta) \sim_{\mathbb{Z}} (p, q)$

   $\Rightarrow (n, m) \sim_{\mathbb{Z}} (p, q)$ → $n + \theta = l + m$ → $l + q = p + \theta$

adding the above $(n + \theta) + (l + q) = (l + m) + (p + \theta)$

from comm + ass + cancelling

$\Rightarrow n +_{\mathbb{N}} q = p +_{\mathbb{N}} m \iff (n, m) \sim_{\mathbb{Z}} (p, q)$ □

Defn $\mathbb{Z} := \mathbb{N} \times \mathbb{N} / \sim_{\mathbb{Z}}$ } set of equiv classes

Eg. '4' $= [(57, 53)] = [(6, 1)] \dots$

'-3' $= [(6, 9)] = \dots$

Adding in $\mathbb{Z}$

$[(n, m)] +_{\mathbb{Z}} [(l, k)] := [(n +_{\mathbb{N}} l \;,\; m +_{\mathbb{N}} k)]$

Remark: this is a definition on classes that uses elements to define a rule!

To see if this is **well defined**, check

if $[(n, m)] = [(n', m')]$

& $[(l, k)] = [(l', k')]$

$[(n, m)] +_{\mathbb{Z}} [(l, k)] = [(n', m')] +_{\mathbb{Z}} [(l', k')]$

Defn $0_{\mathbb{Z}} : [(1, 1)] = [(57, 57)]$

Does this have the property of being the **identity wrt $+_{\mathbb{Z}}$**!

$0_{\mathbb{Z}} + [(l, k)] = [(1, 1)] +_{\mathbb{Z}} [(l, k)] = [(1 +_{\mathbb{N}} l \;,\; 1 +_{\mathbb{N}} k)]$

immediate from comx cancelling! $\leftarrow = [(l, k)]$ □

Multiplication in $\mathbb{Z}$

$[(n, m)] \cdot_{\mathbb{Z}} [(l, k)] = [(n \cdot_{\mathbb{N}} l +_{\mathbb{N}} m \cdot_{\mathbb{N}} k \;,\; n \cdot_{\mathbb{N}} k + m \cdot_{\mathbb{N}} l)]$

→ where does this come from?

$(n - m)(l - k) = nl - ml - nk + mk \Rightarrow nl + mk - (ml + nk)$

Again → check is this is well defined & if

$1_{\mathbb{Z}} := [(57, 56)]$ is the **identity wrt $\cdot_{\mathbb{Z}}$**