Set 1 : IVT, EVT, Sz and the coolness of $\mathbb{R}$

LEC1

LEC2

Let $f:\mathbb{R} \to \mathbb{R}$ be c/s

let $a, b \in \mathbb{R}$, $a \leq b$

IVT says,

I $f(a) < 0 < b$ then

$\exists\, c \in]a,b[$ s.t $f(c) = 0$

Eg. $f:\mathbb{R} \to \mathbb{R}$ $\quad f(x) = x^2 - 2$

$f:\mathbb{R} \to \mathbb{R}$ $\quad f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$

Non eg. $f:\mathbb{R} \to \mathbb{R}$ $\quad f(x) = \begin{cases} \frac{x}{|x|} & x \neq 0 \\ -1 & x = 0 \end{cases}$

IVT asserts the existence of $\sqrt{2}$ in $\mathbb{R}$ through eg 1

What is $\sqrt{2}$?

Lim of fraction?

Infinite Decimal?

OLD SCHOOL $\sqrt{2}$ PROOF

$\exists\, m, n \in \mathbb{Z}$, $n \neq 0$ st $\sqrt{2} = \frac{m}{n}$

$\boxed{\text{Contradiction}}$

<u>WLOG</u> m, n are coprime $\rightarrow \gcd(m,n) = 1$

$2 = \frac{m^2}{n^2} \rightarrow 2n^2 = m^2$

m^2 is even $\Leftrightarrow$ $\boxed{m \text{ is even}}$

lets say $m = 2k$, $k \in \mathbb{Z}$

$2n^2 = 4k^2$

$n^2 = 2k^2 \Rightarrow \boxed{n \text{ is even}}$

$\rightarrow$ contradicts $\gcd(m,n) = 1$

$\therefore \sqrt{2}$ can't be expressed as a fraction

$\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$

New, STRONG Proof

$\forall \; a, b \in \mathbb{N}$

$$\left|\sqrt{2} - \frac{a}{b}\right| \geq \begin{cases} 1 & \text{if } \frac{a}{b} \geq 57 - \sqrt{2} \\ \frac{1}{57b^2} & \text{if } \frac{a}{b} < 57 - \sqrt{2} \end{cases}$$

We see that for any fraction,

$\sqrt{2} - \frac{a}{b} \neq 0 \quad \therefore \quad \frac{m}{n} \neq \sqrt{2}$

Why can we use $\mathbb{N}$? By IVT we know $\sqrt{2}$ is +ive

Case 1 $\quad \frac{a}{b} \geq 57 - \sqrt{2}$

fix $a, b \in \mathbb{N}$ and assume the above

we know $\frac{a}{b} - \sqrt{2}$ is positive $\therefore$

$$\boxed{\left|\sqrt{2} - \frac{a}{b}\right| = \frac{a}{b} - \sqrt{2}}$$

$$\frac{a}{b} - \sqrt{2} \geq (57 - \sqrt{2}) - \sqrt{2}$$

$$\frac{a}{b} - \sqrt{2} \geq 57 - 2\sqrt{2} > 53$$

$$\therefore \left|\sqrt{2} - \frac{a}{b}\right| > 53 > 1$$

Case 2 $\quad \frac{a}{b} < 57 - \sqrt{2}$

fix $a, b \in \mathbb{N}$ and assume the above

$$\left|\sqrt{2} - \frac{a}{b}\right| = \left|\frac{2 - \frac{a^2}{b^2}}{\sqrt{2} + \frac{a}{b}}\right| = \frac{|2b^2 - a^2|}{b^2\left(\sqrt{2} + \frac{a}{b}\right)}$$

$$57 > \sqrt{2} + \frac{a}{b} \implies \frac{1}{57} < \frac{1}{\sqrt{2} + \frac{a}{b}}$$

$\therefore$ as $b^2 \in \mathbb{N}$

$$\frac{1}{57 b^2} < \frac{1}{b^2\left(\sqrt{2} + \frac{a}{b}\right)}$$

$\therefore$ all we need to show now is

$$\underbrace{|2b^2 - a^2|}_{} \geq 1$$

→ non neg integer. So we must simply refute the value 0

$$|2b^2 - a^2| = 0 \implies 2b^2 - a^2 = 0$$

$$a^2 = 2b^2$$

Fundamental Theorem of Arithmetic

Any $\mathbb{Z}$ greater than 1, can be written uniquely as a product of its primes! to powers

Using this fact we can rule out

$$a^2 = 2b^2$$

Write out both a^2 and b^2 as a product of primes. Thus, a^2 & b^2 are a product of an even number of primes

But

$$\underbrace{\underbrace{a^2}_{\text{even \# of primes}} = \underbrace{b^2}_{\text{odd \# of primes}}}$$

Using the uniqueness stated in FTA $a^2 \neq 2b^2$

but FTA doesn't hold for a or $b = 1$

so we can tackle this case by case!

Sub case 1 $\quad b=1$

$2b^2 = 2$ } product of odd # of primes

$\therefore 2 \neq a^2, \; a \in \mathbb{N}$

Sub case 2 $\quad a=1$

$2b^2 = 1$ } but 1 isn't even so $b^2 \notin \mathbb{N}$

$\therefore a^2 \neq 2b^2 \quad a,b \in \mathbb{N}$ and

if $\frac{a}{b} < 5\sqrt{3} - \sqrt{2} \quad \left| \sqrt{2} - \frac{a}{b} \right| \geq \frac{1}{5\sqrt{3}b^2}$

What about IVT over rationals ?

eg: $f : \mathbb{Q} \to \mathbb{Q}$, $x \mapsto x^2 - 2$

IVT says $\exists ?\, c \in (0,2) \cap \mathbb{Q}$ s.t. $f(c)=0$

since $f(0) = -2$
$f(2) = 2$

Is f cts ? What is continuity in $\mathbb{Q}$?

↳ lets say yes

But $\nexists\, c \in (0,2) \cap \mathbb{Q}$ s.t. $f(c)=0$

So IVT doesn't hold for $\mathbb{Q}$

<u>IVT FAILS over the Rationals</u> !

Extreme Value Theorem

$f: \mathbb{R} \to \mathbb{R}$, let $a, b \in \mathbb{R}$, $a < b$

let $f : [a,b] \to \mathbb{R}$ be cts

$\exists\, m, M \in [a,b]$ s.t.

$\forall\, x \in [a,b]$

$$f(m) \leq f(x) \leq f(M)$$

① The function is bounded

② The function achieves extreme values

Def) Bounded

$f : [a,b] \to \mathbb{R}$ is bounded if

$\{ f(x) \mid x \in [a,b] \}$ is a bounded subset of $\mathbb{R}$

Generally, a subset of $\mathbb{R}$ is bounded & below p.t if is bounded above & below

$A \subset \mathbb{R}$ is bounded above p.t

$\exists\, \alpha \in A$ s.t $\forall\, a \in A$, $\alpha \geq a$

$\to$ bounded below is similar

Does EVT hold for $\mathbb{Q}$??

fix $a, b \in \mathbb{Q}$ w $a < b$

$f : ([a,b] \cap \mathbb{Q}) \to \mathbb{Q}$ cts

is it true that

① F is bounded ?

② f achieves min & max ?

No. counter eg $F : ([0,2] \cap \mathbb{Q}) \to \mathbb{Q}$

$$f(x) = \frac{1}{x^2 - 2}$$

lets claim f is cts (note asymptote isn't in the domain)

$f(0) = -\frac{1}{2}$, $f(2) = \frac{1}{2}$

but we have an asymptote

$\therefore$ EVT FAILS

What is so special about $\mathbb{R}$ so that IVT and EVT holds for it but not $\mathbb{Q}$?