Compactness, Path connectedness and EVT

Motivation : Extreme Value Theorem

Spose $a, b \in \mathbb{R}$ $a < b$

Spose $f [a,b] \to \mathbb{R}$ is cts

Then $\exists x_m \in [a,b]$ s.t

$$f(x_m) \le f(x) \quad \forall x \in [a,b]$$

and $\exists x_M \in [a,b]$ s.t

$$f(x_M) \ge f(x) \quad \forall x \in [a,b]$$

→ we're saying this function achieves min / max values.

Consider $f([a,b]) \subseteq \mathbb{R}$.

We are trying to show that $f([a,b])$ has a max / min value.

What do we know about $f([a,b])$, well since cts functions preserve connectedness, it is connected and therefore an interval.

What kinds of intervals have max and min elts?

1. bound (above & below)
2. Closed → if open we couldn't find a max / min.

We would like to show that $f([a,b])$ is closed & bounded

Now, a topological vacation . . . .

Defn let $(X, \mathcal{T})$ be a top space. let $A \subseteq X$. A collection of open sets $\mathcal{C} \subseteq \mathcal{T}$ is called an open cover of A p.t

$$A \subseteq \bigcup_{C \in \mathcal{C}} \mathcal{C}$$ ⟶ like a blanket

eg: $X = \mathbb{R}$ $\mathcal{T} = \mathcal{T}_{euc}$

⟶ open cover of $\mathbb{Q}$?

$\mathcal{C} = \{\mathbb{R}\}$

$\mathcal{C} = \{ B_{57}(x) \mid x \in \mathbb{Q} \} \subseteq \mathcal{T}$

→ consider $\mathcal{C} = \{(m-2, m+2) \mid m \in \mathbb{Z}\}$

Claim: this is an open cover of $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$.

⟶ $\mathcal{C} = \{ (\frac{1}{n}, 1) \mid n \in \mathbb{N} \}$

open cover of $(0,1)$

⟶ $\mathcal{C} = \{(0,5)\}$

open cover of $[e, \pi]$.

⟶ $\mathcal{C} = \mathbb{R}$ is an open cover of any subset of $\mathbb{R}$.

Defn Spose $\mathcal{C}$ is an open cover of a subset $A \subset X$.

A subcollection $\mathcal{C}' \subseteq \mathcal{C}$ is called an open subcover of A p.t

$$A \subseteq \bigcup_{C \in \mathcal{C}'} C$$

Claim Let $\mathcal{C} = \{ (\frac{1}{n}, 1) \mid n \in \mathbb{N} \}$

Then $\mathcal{C}$ is an open cover of $(0,1)$ but there is no finite subcover of $\mathcal{C}$ that covers $(0,1)$.

Pf 1) We first show that $\mathcal{E}$ is a open subcover of $(0,1)$.

let $x \in (0,1)$ task: find $U \in \mathcal{E}$ s.t $x \in U$.

By archimedian property, $\exists N \in \mathbb{N}$ s.t

$\frac{1}{N} < x$, we see that $x \in (\frac{1}{N}, 1)$,

and $(\frac{1}{N}, 1) \in \mathcal{E}$ so, $(0,1) \subseteq \bigcup_{c \in \mathcal{E}} c$

Pf 2) Show no finite subcover works.

Spose $\mathcal{E}'$ is a finite subcover of $\mathcal{E}$. We can say.

$\mathcal{E}' = \{ (\frac{1}{n_1}, 1), (\frac{1}{n_2}, 1), (\frac{1}{n_3}, 1) \dots (\frac{1}{n_k}, 1) \}$

set $M := \max \{ n_1, n_2 \dots n_k \}$

$$\bigcup_{i=1}^{k} (\frac{1}{n_i}, 1) = (\frac{1}{m}, 1).$$

But then $(0, 1) \not\subseteq (\frac{1}{m}, 1)$

$$\not\subseteq \bigcup_{i=1}^{k} (\frac{1}{n_i}, 1)$$

So, $\mathcal{E}'$ doesn't cover $(0,1)$

$\Rightarrow \nexists$ a finite subcover of $\mathcal{E}$ that covers $(0,1)$.

What kind of subsets $A \subseteq \mathbb{R}$ have the property that every open cover admits a finite subcover?

- $\emptyset$
- $\{57\}$ $\to$ since $\mathcal{E}$ is a cover of A, $\exists U \in \mathcal{E}$ st $57 \in U$ $\mathcal{E}' := \{U\}$. Then $\mathcal{E}'$ is a finite subcover of $\mathcal{E}$ that covers A.
- All finite subsets $\mathbb{R}$.
- What about $\mathbb{R}$?

No. $\mathbb{R}$ doesn't have the property!

Consider $\mathcal{E} = \{B_{57}(x) \mid x \in \mathbb{Q}\}$

note: there is no finite subset of that will cover $\mathbb{R}$ as it isn't bounded!

Defn let $(X, \mathcal{T})$ be a top space, let $A \subseteq X$. The set A is said to be compact i.f.f every open cover of A admits a finite subcover.

Remark: We have shown $(0,1)$ isn't compact.

In fact, we can show (a,b) isn't compact.

How? take $\mathcal{E} := \{(a + \frac{b-a}{n}, b) \mid n \in \mathbb{N}\}$

and show it doesn't admit a finite subcover

What about $(a,b]$ → not compact

consider $\mathcal{E} := \{(a + \frac{b-a}{n}, b+1) \mid n \in \mathbb{N}$

Lemma if $A \subseteq \mathbb{R}$ is compact Then it is closed and bounded.

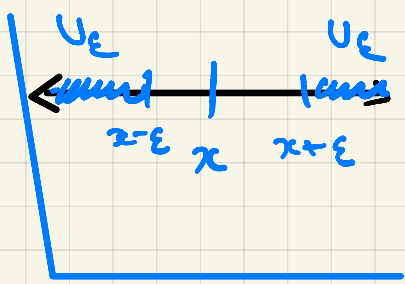
Pf First show A is closed. → show $\mathbb{R} \setminus A$ is open.

note: $\mathbb{R} \setminus A \neq \emptyset$ as $A \subseteq \mathbb{R}$ is compact and $\mathbb{R}$ isn't.

Fix $x \in \mathbb{R} \setminus A$. task find $r > 0$ s.t. $B_r(x) \in \mathbb{R} \setminus A$

$\forall \varepsilon > 0$ define $U_\varepsilon := (-\infty, x - \varepsilon) \cup (x + \varepsilon, \infty)$

note U_ε is an open subset of $\mathbb{R}$.



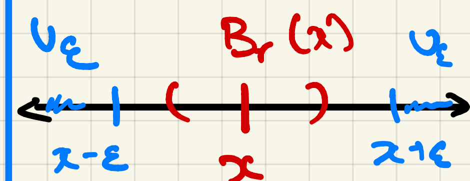
Claim: $\mathcal{E} = \{U_\varepsilon \mid \varepsilon \in \mathbb{R}_{>0}\}$

is an open cover of $\mathbb{R} \setminus \{x\}$.

note: $A \subseteq \mathbb{R} \setminus \{x\}$ as $x \notin A$.

and thus $\mathcal{E}$ is an open cover of A. Since A is cmpct, $\exists$ a finite subcover $\mathcal{E}' \subseteq \mathcal{E}$ that covers A.

$$\mathcal{E}' := \{ U_{\varepsilon_1}, \dots U_{\varepsilon_k} \}$$



Clearly $U_\varepsilon \cap B_r(x)$ is $\emptyset$.

set $\varepsilon = \min(\varepsilon_1 \dots \varepsilon_k)$

Then $\displaystyle\bigcup_{i=1}^{k} U_{\varepsilon_i} = U_\varepsilon$

So, $A \subseteq U_\varepsilon$, choose $r = \frac{\varepsilon}{57}$

So, $B_r(x) \cap U_\varepsilon = \emptyset = B_r(x) \cap A$.

So, $B_r(x) \subseteq \mathbb{R} \setminus A$

$\Rightarrow$ $\mathbb{R} \setminus A$ is open & A is closed!

Pf] Show A is <u>bounded</u>

Idea: cook up a smart choice of an open cover of A to start.

Choose: $\mathcal{E} = \{ B_n(0) \mid n \in \mathbb{N} \}$ an open cover of $\mathbb{R}$ so open cover of A

Since A is cmpct there is a finite subcover $\mathcal{E}' \subseteq \mathcal{E}$ that covers A.

$$\mathcal{E}' = \{ B_{n_1}(0), B_{n_2}(0), \dots B_{n_k}(0) \}$$

$$N := \max(n_1 \dots n_k)$$

Then $A \subseteq \displaystyle\bigcup_{i=1}^{k} B_{n_i}(0) = B_N(0)$

Ie, A is bdd above by N and $-N$!

$\square$.

Alternate) We said let $(X, \mathcal{T})$ be a top space let $A \subseteq X$. A is cmpct iff every open cover admits a finite subcover.

How about: let $(X, \mathcal{T})$ be a top space. Then X is cmpct iff every cover of X admits a finite subcover

note: from the first definition, the subset $A \subseteq X$ is cmpct iff the topology $(A, \mathcal{T}_A)$ is compact according to the definition.
↳ subspace top.

We showed that $(0,1)$ isn't compact, how about

$L = \{ \frac{1}{n} \mid n \in \mathbb{N} \}$

Q: is L cmpct?

To show it isn't compact, we simply need to show that it isn't bounded or open.

This set is bounded, lets show L isn't closed, ie $\mathbb{R} \setminus L$ isn't open.
↳ a ball around 0 will include elements in L.
Thus, $B_r(0) \not\subseteq \mathbb{R} \setminus L$ and $\mathbb{R} \setminus L$ not open.

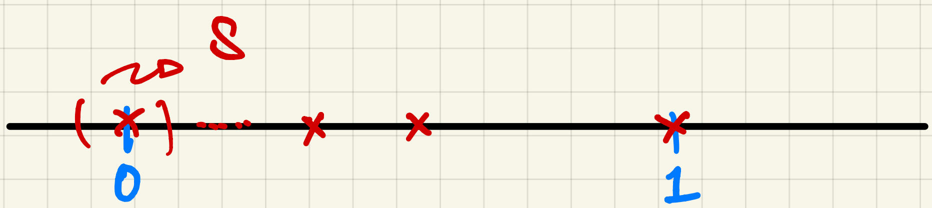
$\therefore$ L isn't compact!

(2) How about $\hat{L} := L \cup \{0\}$

Q: is $\hat{L}$ cmpct?

guess: yes!

Pf) Let $\mathcal{S}$ be an open cover for $\hat{L}$



We have to exhibit a finite subcover.

What's great is that we can cover infinitely many numbers with S!

$\exists S \in \mathcal{E}$ s.t $0 \in S$.

Since S is open $\exists r > 0$ s.t $B_r(0) \subseteq S$.
by archimedian property $\exists N \in \mathbb{N}$ s.t

$$\frac{1}{N} < r$$

So, $\left\{ \frac{1}{n} \mid n \geq N \right\} \subseteq B_r(0) \subseteq S$.

Since $\mathcal{E}$ is an open cover of I, then

$\forall\ 1 \leq k \leq N-1$

$\exists\ V_k \in \mathcal{E}$ s.t $\frac{1}{k} \in V_k$

lets collect these!!

$$\mathcal{E}' := \{ V_k \mid 1 \leq k \leq N-1 \} \cup \{ S \}$$

This is the finite subcover we seek!!

$\therefore I$ is compact!!

Lemma let $\mathcal{E}$ be an open cover of $[a,b]$. Task: extract a finite subcover

Def $\to$ $A := \{ x \in [a,b] \mid \exists$ a finite subcover of $\mathcal{E}$ that covers $[a,x] \}$

We need to show $b \in A$. Specifically

① $b = \sup(A)$, and $\sup(A) \in A$.

What do we know about A?

- $\to$ $A \neq \emptyset$ as $a \in A$ (as $[a,a] = \{a\}$)
- $\to$ b is an upper bound for A!
- $\to$ $\sup(A)$ exists! $\alpha := \sup(A)$?

given that α is the sup and b is an upper bound,
$\alpha \leq b$.

We know $a \leq \alpha \leq b$, so, $\alpha \in [a,b]$

Since $\mathcal{E}$ is an open cover $[a,b]$, $\exists\, U_0 \in \mathcal{E}$ s.t $\alpha \in U_0$.

Since U_0 is open and $\alpha \in U_0$

$\exists\, r > 0$ s.t $B_r(\alpha) \subseteq U_0$

By characteristic of sup $\exists\, a' \in A$ s.t $a' \in (\alpha - r, \alpha]$

$\hookrightarrow$ There is a finite subcover of $\mathcal{E}$ that covers $[a, a']$!

To show $\alpha \in A$, we need to find a finite subcover of $\mathcal{E}$ covers $[a, \alpha]$.

Since $a' \in A$, $\exists\, \mathcal{E}' \subseteq \mathcal{E}$ st.

$$[a, a'] \subseteq \bigcup_{c \in \mathcal{E}'} c \quad \rightarrow \text{finite}$$

we can say,

$$[a, a'] \cup U_0 \subseteq \bigcup_{c \in \mathcal{E}} c \cup U_0$$

Define $\mathcal{E}'' := \mathcal{E}' \cup \{U_0\}$

This is a finite subcover of $\mathcal{E}$ that covers $[a, \alpha]$

$\therefore \alpha \in A$!

We know $\alpha \leq b$, so if $\alpha \neq b$ then $\alpha < b$.

note: $\mathcal{E}''$ covers $\left[a, \alpha + \frac{r}{57}\right]$ so, if $\alpha < b$

then we find $t > \alpha$ and $t < b$ st $\mathcal{E}''$ covers $[a, t]$. Oops, as $t \in A$ and $t > \alpha \rightarrow$ not sup!

$\therefore \alpha = b \rightarrow b \in A \rightarrow$ done! $[a,b]$ is compact

What is the point of this?

Recall: To prove EVT, we wanted to show $f([a,b])$ is closed and bounded!

We already know it is an interval (by the fact that functions preserve connectedness).

So, $f([a,b]) = [c,d]$

Show → compactness is conserved by cts functions

$[a,b]$ → $f[a,b]$ → closed + bounded + interval
cmpct → cmpct

↳ $\exists c,d \in \mathbb{R}$ s.t $f([a,b]) = [c,d]$

Theorem let $(X, \mathcal{T}_X)$ & $(Y, \mathcal{T}_Y)$ be top spaces.

Spose X is compact & spose $f: X \to Y$ is cts. Then $f(X)$ is cmpct!

Pf Let $\mathcal{E}$ be an open cover of $f(X)$

task: obtain finite subcover of $\mathcal{E}$ that covers $f(X)$

Consider $\mathcal{E}' := \{ f^{-1}(U) \mid U \in \mathcal{E} \}$.

↳ inverse image of all (open) sets in $\mathcal{E}$.

① Every element of $\mathcal{E}'$ is open in X (as f is cts)

② is $\mathcal{E}'$ an open cover of X?

↳ yes let $x \in X$, then $f(x) \in f(X)$. So $\exists\, V \in \mathcal{E}$ s.t $f(x) \in V$, so $x \in f^{-1}(V)$

$\therefore X \subseteq \bigcup_{C \in \mathcal{E}'} C$

Since X is cmpct, $\exists$ a finite subcover of $\mathcal{E}$ that covers X.

lets say this is

$\mathcal{E}'' = \{ f^{-1}(U_1) \;.....\; f^{-1}(U_n) \}$

guess:

$\mathcal{E}''' = \{U_1 \;....\; U_n\}$ is a finite subcover of $\mathcal{E}$ that covers $f(X)$.

we know U_1 is open.

Q1) Are they open sets $\rightarrow$ yes!

Q2) is it a subcover of $\mathcal{E}$? check

Q3) finite ? ✓

2) show it is a cover of $f(X)$

ie $f(X) \subseteq \bigcup_{i=1}^{n} U_i$

take an element $y \in f(X)$. Then $f^{-1}(\{y\}) \subseteq X$.

So, $f^{-1}(\{y\}) \subseteq \bigcup_{i=1}^{n} f^{-1}(U_i)$

$$f(f^{-1}\{y\}) \subseteq f\left(\bigcup_{i=1}^{n} f^{-1}(U_i) \right)$$

$$\{y\} \subseteq \bigcup_{i=1}^{n} f(f^{-1}(U_i))$$

$$\subseteq \bigcup_{i=1}^{n} U_i$$

$\therefore$ $y \in \bigcup_{i=1}^{n} U_i$

So, $f(X) \subseteq \bigcup_{i=1}^{n} U_i$ $\square$

EVT] let $a, b \in \mathbb{R}$ w $a < b$

let $f : [a,b] \to \mathbb{R}$ be cts.

Then $\exists\ x_m, x_M \in [a,b]$ s.t

$\forall\ x \in [a,b]$ $f(x_m) \leq f(x) \leq f(x_M)$

Pf] The domain $[a,b]$ is a compact subset of $\mathbb{R}$.

Since f is cts, the image $f([a,b])$ is compact.

$\therefore$ $f([a,b])$ is closed & bounded!

Since $[a,b]$ is an interval, it is connected. Thus $f([a,b])$ is an interval.

$\Rightarrow \exists\ c, d \in \mathbb{R}$ s.t $f([a,b]) \in [c,d]$.

$\therefore$ $f([a,b])$ has max'l and min elt!

Defn] let $(X, \mathcal{T})$ be a top space. We say that X is path connected pt

$\forall\ x, y \in X$ $\exists$ a path connecting them

Defn) A path connecting $x, y \in X$ is a cts function

$f : [0,1] \to X$ s.t

$f(0) = x$, $f(1) = y$

Eg: $(\mathbb{R}, \mathcal{T}_{euc})$ is

by $(\mathbb{R} \setminus \{0\}, \mathcal{T}_{euc})$ isn't

is there a cts function $f : [0,1] \to \mathbb{R} \setminus \{0\}$

s.t $f(0) = -57$ and $f(1) = 57$

No, by ev+ $f(c) = 0$ $\longrightarrow$ bad!!

Q: Relationship between connected & path connected?

First: Are constant functions always continuous?

Take: $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ & a point $y_0 \in Y$.

Is $f: X \to Y$, $x \mapsto y_0$ cts?

$\forall U \subseteq Y$ open,

$$f^{-1}(U) = \begin{cases} \emptyset & \text{is } y_0 \notin U \\ X & \text{is } y_0 \in U \end{cases}$$

$\therefore$ the function is continuous!

Claim: if $(X, \mathcal{T})$ is path connected, then $(X, \mathcal{T})$ is connected!

Claim HW7: If the subset $A \subseteq \mathbb{R}$ (2nd euc top) is path connected, it is connected!

↳ in general not true

EX: In $\mathbb{R}^2$ the graph of $\sin(\frac{1}{x})$ and line segment of y axis from -1 to 1

→ this is connected but not path connected!