Lemma let $A \subseteq \mathbb{R}$ be an iwimp spose $f, g : A \to \mathbb{R}$ is diff'ble at $a \in A$ then

$(f \cdot g) : A \to \mathbb{R}$ is diff'ble at a &

$$(f \cdot g)'(a) = f'(a) \cdot g(a) + g'(a) f(a)$$

Pf Since f, g are diff'ble at a, they are cts at a!

we need

$$\lim_{h \to 0} \frac{(f \cdot g)(a+h) - (f \cdot g)(a)}{h}$$

$$= \lim_{h \to 0} \frac{f(a+h) \cdot g(a+h) - f(a) g(a+h) + g(a+h) f(a) - f(a) g(a)}{h}$$

? → since the limits below exist, these exist as sum of limit

$$= \lim_{h \to 0} \frac{f(a+h) g(a+h) - f(a) g(a+h)}{h} + \lim_{h \to 0} \frac{g(a+h) f(a) - f(a) g(a)}{h}$$

? → true since we are cts and prod of limit
cts + diff'ble

$$= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \cdot \lim_{h \to 0} g(a+h) + \lim_{h \to 0} f(a) \cdot \lim_{h \to 0} \frac{g(a+h) - g(a)}{h}$$

$$= f'(a) \cdot g(a) + f(a) g'(a) \quad \square$$

Quotient Rule

$\frac{f}{g} = f \cdot \frac{1}{g}$ → it suffices to show $\frac{1}{g}$ is diff'ble

In fact, it suffices to show

$$h : \mathbb{R} \setminus \{0\} \to \mathbb{R}$$
$$x \mapsto \frac{1}{x}$$ is diff'ble $\forall a \in \mathbb{R} \setminus \{0\}$

→ then we can compose any nonzero fnc chain rule!

→

Lemma $\forall a \in \mathbb{R}\setminus\{0\}$ $h'(a)$ exists (by h given earlier)

Pf $\lim_{h\to 0} \frac{h(a+h)-h(a)}{h} = \lim_{h\to 0} \frac{\left(\frac{1}{a+h} - \frac{1}{a}\right)}{h}$

$= \lim_{h\to 0} \frac{\left(\frac{a-(a+h)}{a\cdot(a+h)}\right)}{h} = \lim_{h\to 0} \frac{-\cancel{h}}{a\cdot\cancel{h}\cdot(a+h)}$

$= \lim_{h\to 0} \frac{-1}{a(a+h)}$ $\longrightarrow$ the fnc $h \mapsto \frac{-1}{a(a+h)}$ is cts at $h=0$

$= \boxed{-\frac{1}{a^2}}$

Corollary Quotient Rule

Let $I \subseteq \mathbb{R}$ be an ivl, Let $f,g : I \to \mathbb{R}$.
Spose f,g are diff'ble at $a \in I$ & $g(a) \neq 0$.

$\left(\frac{f}{g}\right)'(a)$ exists &

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - g'(a)f(a)}{(g(a))^2}$$

Pf Define $h : \mathbb{R}\setminus\{0\} \to \mathbb{R}$
$x \longmapsto \frac{1}{x}$

Then $h \circ g(x) = \frac{1}{g(x)}$

$\therefore$ at $a \in I$

$\frac{f(x)}{g(x)} = f(x) \cdot h \circ g(x)$

$=$ Done

$\square$

Optimization

Defn) Spse $I \subseteq \mathbb{R}$ is an intvl & $f: I \to \mathbb{R}$ is a func. The point $a \in I$ is said to be a critical pt p.t either

① $f'(a) = 0$

② f is not diff'ble at a

if a is a critical point then $f(a)$ is a crit value!

Recall) If f has a local min/max at a & if f is diff'ble at a, then $f'(a) = 0$

So, x-value where we have a local min/max are special $\to$ crit pts

Q) If f has a crit pt at $x=a$, does f have a local min/max?

Ans) No. Consider $f: \mathbb{R} \to \mathbb{R}$
$x \mapsto x^3$

Lemma) If $I \subseteq \mathbb{R}$ is an open intvl. Spse $f: I \to \mathbb{R}$ is diff'ble. If $f': I \to \mathbb{R}$ is positive on I, then f is strictly increasing $\to$ negative $\Rightarrow$ decreasing

Pf) Wlog $f': I \to \mathbb{R}$ positive. Goal: Show f is str ↑

Fix $a, b \in I$ w/ $a < b$ we have

$f|_{[a,b]}\ [a,b] \to \mathbb{R}$ is cts

$f|_{(a,b)}\ (a,b) \to \mathbb{R}$ is diff'ble

by MVT $\exists\ c \in (a,b)$ s.t

$$f'(c) = \frac{f(b) - f(a)}{b-a} \to \text{positive}$$

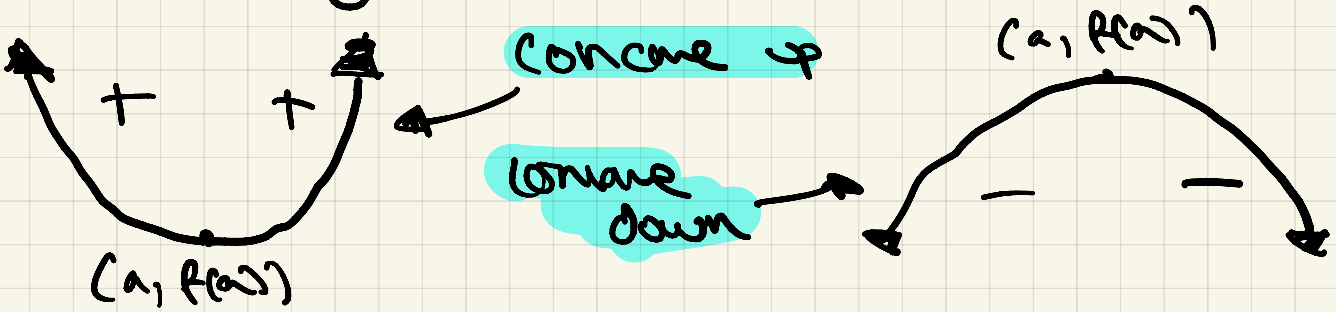
Positive

$\therefore\ f(b) - f(a) > 0 \Rightarrow f(b) > f(a)$ $\square$

Also, we may consider the second derivative!

Lemma Suppose $I \subseteq \mathbb{R}$ is an int. Let $a \in I$. Let $f: I \to \mathbb{R}$ be twice differentiable. If $f'(a)=0$ & $f''(a)>0$ then f attains local min at a.

Corresponding statement holds if $f'(a)=0$ & $f''(a)<0$



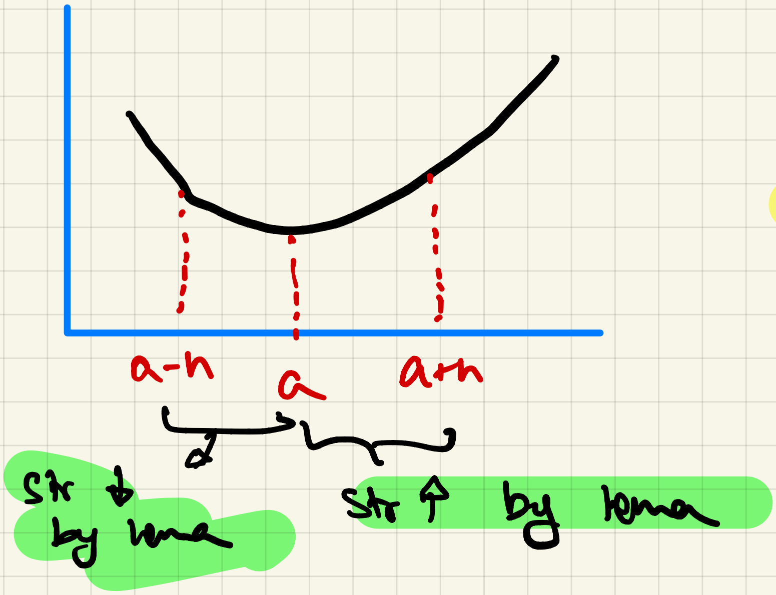
Pf Since $f'(a)=0$, we have,

$$\underbrace{f''(a)}_{>0} = \lim_{h\to 0} \frac{f'(a+h)-f'(a)}{h} = \lim_{h\to 0} \frac{f'(a+h)}{h}$$

$\therefore \lim_{h\to 0} \frac{f'(a+h)}{h} > 0 \to$ consider 1 sided lim

$\Rightarrow f'(a+h)>0$ for $h>0$ sufficiently small!

$\Rightarrow f'(a+h)<0$ for $h<0$ & sufficiently small!



So, f has a local min at a!

FTC II

Recall) FTC I : let $f: [a,b] \to \mathbb{R}$ be cts & define

$$F: [a,b] \to \mathbb{R}$$
$$x \longmapsto \int_a^x f$$

$\forall$ $c \in [a,b]$, F is diffble at c &

$F'(c) = f(c)$

FTC II) If f is integrable on $[a,b]$ & if $f = g'$ for some $g: [a,b] \to \mathbb{R}$ then

$$\int_a^b f = g(b) - g(a)$$

FTC II.2) Same statement as above but make f cts.

Pf) FTC II.2 $\to$ Corollary of FTC I

We have

$$F: [a,b] \to \mathbb{R} \quad \text{is diffble \& } F'(x) = f(x)$$
$$x \mapsto \int_a^x f$$

since $g'(x) = f(x)$ $\exists$ const c st

$F(x) = g(x) + c$

$$F(b) - F(a) = \int_a^b f - \cancel{\int_a^a f}^{\,0} = \int_a^b f$$

$$g(b) - g(a) = (F(b) - c) - (F(a) - c))$$
$$= F(b) - F(a) \qquad \square$$

FTC II) If f is integrable on $[a,b]$ & if $f = g'$ for some $g:[a,b] \to \mathbb{R}$ then

$$\int_a^b f = g(b) - g(a)$$

Pf) let $P = \{a = t_0 < t_1 \cdots < t_n = b\}$ be a partition of $[a,b]$

$g : [t_{i-1}, t_i] \to \mathbb{R}$ is diff'ble (restriction)

$\Rightarrow$ continuous

$g : (t_{i-1}, t_i) \to \mathbb{R}$ is diff'ble

By MVT

$\exists\, x_i^* \in (t_{i-1}, t_i)$ s.t

$$g(t_i) - g(t_{i-1}) = g'(x_i^*)(t_i - t_{i-1})$$
$$= f(x_i^*)(t_i - t_{i-1})$$

$\forall\ 1 \le i \le n$ we note $m_i \le f(x_i^*) \le M_i$

so

$$L(f,P) \le \sum_i f(x_i^*)(t_i - t_{i-1}) \le U(f,P)$$

$$L(f,P) \le \sum_i (g(t_i) - g(t_{i-1})) \le U(f,P)$$

$\hookrightarrow$ telescopes

$$L(f,P) \le g(b) - g(a) \le U(f,P)$$

Holds $\forall$ partitions P of $[a,b]$ $\therefore$ by sup/inf

$$L(f) \le g(b) - g(a) \le U(f)$$

Since f is integrable $U(f) = L(f)$

$$\therefore \int_a^b f = L(f) = U(f) = g(b) - g(a)$$

$\square$

Use ful since we need to find antiderivative to evaluate integrals

D

Extending Polynomials

What is x^r for $r \in \mathbb{R}$ $x > 0$. Well if

$r \in \mathbb{N}$ $\quad x^r := \underbrace{x \cdot x \cdots x}_{r \text{ times}}$

$r \in \mathbb{Z}$ $\quad r > 0 \quad x^r :=$

$r < 0$ then $-r > 0$ so $x^r := \dfrac{1}{x^{-r}}$

$r = 0 \quad x^r := 1$

$r \in \mathbb{Q}$ $\quad x^r = x^{a/b} \quad a \in \mathbb{Z} \quad b \in \mathbb{N}$

consider the meaning of b^{th} root being

$T_{a,n} := \{ x \in \mathbb{R} \mid x^n \leq a \} \to$ nth root of A.

$\hookrightarrow$ we must show $\sup(T_{a,n})$ exists &

$(\sup(T_{a,n}))^n = a$

note: inv functions allow us to discuss roots!

$\therefore \quad x^{a/b} := (\sqrt[b]{x})^a$

Defn let $r \in \mathbb{R}$ & $x > 0$. We define

$x^r := \exp(r \cdot \ln(x))$

so, the fnc $\mathbb{R}_{>0} \to \mathbb{R}$ is diff'ble
$\quad x \mapsto x^r$ $\hookrightarrow$ composition of diff'ble + linearity

$$\frac{\partial}{\partial x} x^r = \frac{\partial}{\partial x} \exp(r \cdot \ln(x))$$

$$= \exp(r \cdot \ln(x)) \cdot \frac{r}{x}$$

$$= \frac{x^r \cdot r}{x} = r \cdot x^{r-1}$$

as $x \neq 0$