# One sided Limits, Derivatives &

# Differentiability, Chain Rule,

# One Sided Limits

Let $f:\mathbb{R}\to\mathbb{R}$ and $a\in\mathbb{R}$

$\lim_{x\to a^+} f(x) = l$ means ...

$\forall \varepsilon>0\ \exists\ \delta>0$ s.t if $0<x-a<\delta$ then

$$|f(x)-l|<\varepsilon$$

Similarly, $\lim_{x\to a^-} f(x) = l$ means

$\forall \varepsilon>0\ \exists\ \delta>0$ s.t if $0<a-x<\delta$ then

$$|f(x)-l|<\varepsilon$$

$\to$ The objects allow us to consider limits at endpoints!

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**Lemma** let $f:\mathbb{R}\to\mathbb{R}$ & $a\in\mathbb{R}$

$$\lim_{x\to a} f(x) = l \iff \lim_{x\to a^+} f(x) = l \text{ and } \lim_{x\to a^-} f(x) = l$$

**Pf** lemma in HW9

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**Ex** let $f:\mathbb{R}\to\mathbb{R}$ be a func. Let $a\in\mathbb{R}$ Show....

$$\lim_{x\to a} f(x) = f(a) \iff \lim_{n\to 0}\left(f(a+n)-f(a)\right)=0$$

**Pf** Use $\varepsilon$ $\delta$

!!!

$\longrightarrow$

# Derivatives

**Motivation:** Let $f:\mathbb{R}\to\mathbb{R}$

find a line that best approximates $f$ at $x=a$
cause lines are nice!
↳ goes thru $(a, f(a))$

Slope of secant line through $(a, f(a))$, $(a+h, f(a+h))$

$$m_{sec} = \frac{f(a+h)-f(a)}{h}$$

$$m_{tan} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

**Defn** Spse $A \subseteq \mathbb{R}$ is an "interval"! Let $a\in A$ and $f: A\to\mathbb{R}$ be a fnc.

We say $f$ is differentiable at a pt.

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \text{ exists}$$

★ if $a$ is an endpoint of $A$ we use 1 sided limits

If the limit exists then we denote it as $f'(a)$ and call it the derivative of $f$ at $a$

If $f: A\to\mathbb{R}$ is diff'ble at all $a\in A$, then we say $f$ is diff'ble on $A$ and we define

$$f': A \to \mathbb{R}$$
$$a \mapsto f'(a)$$

called the derivative of $f$.

**eg** Let $n\in\mathbb{N}$ and define $g:\mathbb{R}\to\mathbb{R}$, $x\mapsto x^n$. Fix $a\in\mathbb{R}$. Is $g$ diff'ble at $a$?

**Compute**

$$\lim_{h\to 0} \frac{g(a+h)-g(a)}{h}$$

$$= \lim_{h\to 0} \frac{(a+h)^n - a^n}{h} \longrightarrow$$

$$= \lim_{h\to 0} \frac{\sum_{k=0}^{n} \binom{n}{k} a^{n-k} \cdot h^k - a^n}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{a^n} + n \cdot a^{n-1} \cdot h + \binom{n}{2} \ldots\ldots - \cancel{a^n}}{h}$$

$$= \lim_{h\to 0} a^{n-1} \cdot n + \binom{n}{2} a^{n-2} \cdot h \ldots\ldots h^{n-1}$$

↳ this is a polynomial in $h$ so we simply sub $h=0$
↳ cts

$$g'(a) = n \cdot a^{n-1}$$

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**Lemma)** let $I \subseteq \mathbb{R}$ be an ivimp. let $f, g : I \to \mathbb{R}$ be diff'ble on $I$

① $(f+g) : I \to \mathbb{R}$ is diffble particularly $(f+g)' = f' + g'$

② $\forall\, c \in \mathbb{R}$ $(c \cdot f) : I \to \mathbb{R}$ is diffble & $(c \cdot f)' = c \cdot f'$

Tagline : linearity of differentiation

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**Corollary)** all polynomial functions are diffble !

PF) Up to us !

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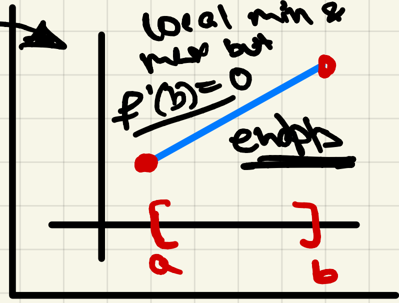
**Defn)** Let $I \subseteq \mathbb{R}$ be an ivimp we say that the fn $f : I \to \mathbb{R}$ has a local max $a \in I$ s.t

$$\exists\, \delta > 0 \text{ st } \forall x \in B_\delta(a) \cap I \quad f(a) \geq f(x)$$

We similarly define local mins.

**Lemma** Spse $I \subseteq \mathbb{R}$ is an open iwimp. Let $c \in I$ & $f : I \to \mathbb{R}$. If $f$ attains a local min or max at $c$ & $f$ is diff'ble at $c$

$$f'(c) = 0$$




**Pf** WLOG suppose $f$ attains a local min at $c$:

Then $f(c) \le f(x)$ $\forall x \in I$ that are near $c$.
That is, $c+h \in I$ & $|h|$ is small enough then,

$$f(c+h) - f(c) \ge 0$$

We are trying to show $f'(c) = 0$

We know $f'(c)$ exists from hypothesis.

**note** $\lim_{h \to 0^+} \dfrac{f(c+h) - f(c)}{h} \begin{matrix} \ge 0 \\ \ge 0 \end{matrix} \quad \ge 0$

**and** $\lim_{h \to 0^-} \dfrac{f(c+h) - f(c)}{h} \begin{matrix} \ge 0 \\ \le 0 \end{matrix} \quad \le 0$

$\therefore$ since $\lim_{h \to 0}$ exists it must be that

$$\lim_{h \to 0} = \lim_{h \to 0^+} = \lim_{h \to 0^-} = 0$$

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lets compare differentiability & continuity!

Clearly continuity $\not\Rightarrow$ diff'bility consider $f : \mathbb{R} \to \mathbb{R}$
$x \mapsto |x|$

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**Lemma** Let $I \subseteq \mathbb{R}$ be an iwimp. If $f : I \to \mathbb{R}$ is diff'ble at $a \in I$ it is cts at $a$.

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**Pf** To show $f$ is cts at $\underline{a}$, from the previous lemma it suffices to show

$$\lim_{h \to 0} f(a+h) - f(a) = 0$$

Well....

$$\lim_{h\to 0} f(a+h) - f(a)$$

$$= \lim_{h\to 0} \underbrace{\frac{f(a+h) - f(a)}{h}} \cdot \underbrace{h}$$

limit of prod is prod of lim

both these limits exist so, → as $f'(a)$ exists

$$= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \cdot \lim_{h\to 0} h$$

$$= f'(a) \cdot 0 = \boxed{0}$$

$\Rightarrow$ $f$ is cts at $a$ $\blacksquare$

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**Chain Rule**

Spose $f, g : \mathbb{R} \to \mathbb{R}$ let $a \in \mathbb{R}$

Spose $f$ is diffble at $g(a)$ and $g$ is diffble at $a$

Then $f \circ g : \mathbb{R} \to \mathbb{R}$ is diffble at $a$. Moreover

$$(f \circ g)'(a) = f'(g(a)) \cdot g'(a).$$

Lemma Take the following auxiliary func.

$\chi : (-\rho, \rho) \to \mathbb{R}$ $\rho > 0$ → we get a neighborhood of 0

$$h \mapsto \begin{cases} \dfrac{f(g(a+h)) - f(g(a))}{g(a+h) - g(a)} & \text{if } g(a+h) - g(a) \neq 0 \\ f'(g(a)) & \text{otherwise} \end{cases}$$

We claim that this is cts at $h = 0$!

Note: these are the $f$ & $g$ from above!

→

**Pf]** Let $\varepsilon > 0$ be given

Task: find $\delta > 0$ s.t. if $h \in (-\delta, \delta)$ then

$$|\chi(h) - \chi(0)| < \varepsilon$$

Since $f$ is diff'ble at $g(a)$, we have

$$\lim_{p \to 0} \frac{f(g(a)+p) - f(g(a))}{p} - f'(g(a)) = 0$$

So $\exists\, \delta' > 0$ s.t. if $p \in (-\delta', \delta')$ then

$$\left| \frac{f(g(a)+p) - f(g(a))}{p} - f'(g(a)) \right| < \varepsilon$$

Since $g$ is diff'ble at $a$, it is cts at $a$ so $\exists\, \delta > 0$ s.t. if $h \in (-\delta, \delta) \setminus \{0\}$, then

$$|g(a+h) - g(a)| < \delta'$$

Fix this $\delta$ and show this works. Spse $h \in (-\delta, \delta)$ we have

$$|g(a+h) - g(a)| < \delta'$$

2 cases now

**C I]** $g(a+h) - g(a) = 0$

Then, $\chi(h) = f'(g(a)) = \chi(0)$

$$|\chi(h) - \chi(0)| = 0 < \varepsilon$$

$$\boxed{\begin{array}{l} \chi : (-\rho, \rho) \to \mathbb{R} \\ h \mapsto \begin{cases} \dfrac{f(g(a+h)) - f(g(a))}{g(a+h) - g(a)} \\ f'(g(a)) \end{cases} \end{array}}$$

**C II]** $g(a+h) - g(a) \neq 0$

Then what is $\chi(h) - \chi(0)$?

$$\left| \frac{f(g(a+h)) - f(g(a))}{g(a+h) - g(a)} - f'(g(a)) \right|$$

we have $|g(a+h) - g(a)| < \delta'$ so, substituting

$< \varepsilon$

$\square$

# Pf for Chain Rule using Auxiliary fnc

$$\lim_{h\to 0} \frac{f(g(a+h)) - f(g(a))}{h}$$ is what we want to evaluate

$$= \lim_{h\to 0} \chi(h) \cdot \frac{g(a+h) - g(a)}{h} \rightarrow$$ Claim

Why?

CI : $g(a+h) - g(a) \neq 0$ is immediate!

CII : $g(a+h) - g(a) = 0$

then $f(g(a+h)) - f(g(a)) = 0$

So lhs and RHS are 0 and :)

$\chi : (-\rho, \rho) \to \mathbb{R}$

$$h \mapsto \begin{cases} \dfrac{f(g(a+h)) - f(g(a))}{g(a+h) - g(a)} \\ f'(g(a)) \end{cases}$$

So, $\lim_{h\to 0} \chi(h) \cdot \frac{g(a+h) - g(a)}{h}$ — both limits exist so.

$$= \lim_{h\to 0} \chi(h) \cdot \lim_{h\to 0} \frac{g(a+h) - g(a)}{h}$$

$$= \chi(0) \cdot g'(a)$$

$$= f'(g(a)) \cdot g'(a)$$

$\Rightarrow f\circ g$ is diff'ble at $a$ and $(f\circ g)'(a)$ $\square$