Continuity, Topology, Connectedness, IVT!

Patched Lemma

if $\lim_{x \to a} f(x) = \ell$, $\lim_{y \to \ell} g(y) = m$ and $g(\ell) = m$

$\Rightarrow$ essentially g is cts at $y = '\ell'$

then $\lim_{x \to a} g(f(x)) = m$

Pf) Let $\varepsilon > 0$

Since $\lim_{y \to \ell} g(y) = m$ $\exists$ $\delta_g > 0$ s.t $\forall y$ satisfying $0 < |y - \ell| < \delta_g$ we have $g(y) \in B_\varepsilon(m)$.

But since $g(\ell) = m$, we have

$\forall y$ satisfying $|y - \ell| < \delta_g$, $g(y) \in B_\varepsilon(m)$.

We know if $y = \ell$, then $g(y) = g(\ell) = m \in B_\varepsilon(m)$

Since $\lim_{x \to a} f(x) = \ell$, $\exists$ $\delta > 0$ s.t if x satisfies $0 < |x - a| < \delta$ then $f(x) \in B_{\delta_g}(\ell)$ $\to$ take $\varepsilon = \delta_g > 0$

Then we ultimately have,

$\forall x$ satisfying $0 < |x - a| < \delta$, $|f(x) - \ell| < \delta_g$, so,

$|g(f(x)) - m| < \varepsilon$

$\therefore \lim_{x \to a} g(f(x)) = m$

A very important idea emerged.

Defn) Spose $f: \mathbb{R} \to \mathbb{R}$ is a function and $a \in \mathbb{R}$. We say that f is cts at a p.t.

$\lim_{x \to a} f(x) = f(a)$. $\Rightarrow$ 2 assertions here!!

$\longrightarrow$

We can unpack this using ε and δ

" f is cts at a pt $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t $\forall x$ satisfying $|x - a| < \delta$, then $|f(x) - f(a)| < \varepsilon$ "

Defn The function $f: \mathbb{R} \to \mathbb{R}$ is said to be cts p.t $\forall a \in \mathbb{R}$, f is cts at a.

This definition is a little clunky. Lets use topology to get a better one.

Let A, B be sets. Let $g: A \to B$ be a fnc.

for $S \subseteq A$, we define the image of S under g as:

$$g(S) := \{ b \in B \mid \exists s \in S \text{ with } g(s) = b \}$$

note: if g is surjective $g(A) = B$.

the image of g is just $g(A)$

for $T \subseteq B$, define the inverse image of T under g.

$$g^{-1}(T) := \{ a \in A \mid g(a) \in T \}$$

Eg let $f: \mathbb{R} \to \mathbb{R}$ be $f: x \mapsto x^2$

$$f([1,4]) = [1, 16]$$

$$f^{-1}((0,1)) = (-1, 0) \cup (0, 1)$$

Lets discuss properties of these objects.

Claim Let $g: A \to B$ be a fnc

Let $S_1 \subseteq S_2 \subseteq A$ then, $g(S_1) \subseteq g(S_2)$

Pf Fix $y \in g(S_1)$. $\exists\, x \in S_1$ s.t. $g(x) = y$.

Note, $x \in S_2$, so $g(x) \in S_2 \iff y \in S_2$.

so, $g(S_1) \subseteq g(S_2)$

Claim Let $T_1 \subseteq T_2 \subseteq B$

Then $g^{-1}(T_1) \subseteq g^{-1}(T_2)$

Let $S_1, S_2 \subseteq A$ and $T_1, T_2 \subseteq B$

$$g(S_1 \cup S_2) = g(S_1) \cup g(S_2)$$

$$g(S_1 \cap S_2) \subseteq g(S_1) \cap g(S_2)$$

$$g^{-1}(T_1 \cap T_2) = g^{-1}(T_1) \cap g^{-1}(T_2)$$

$$g^{-1}(g(S)) \supseteq S$$

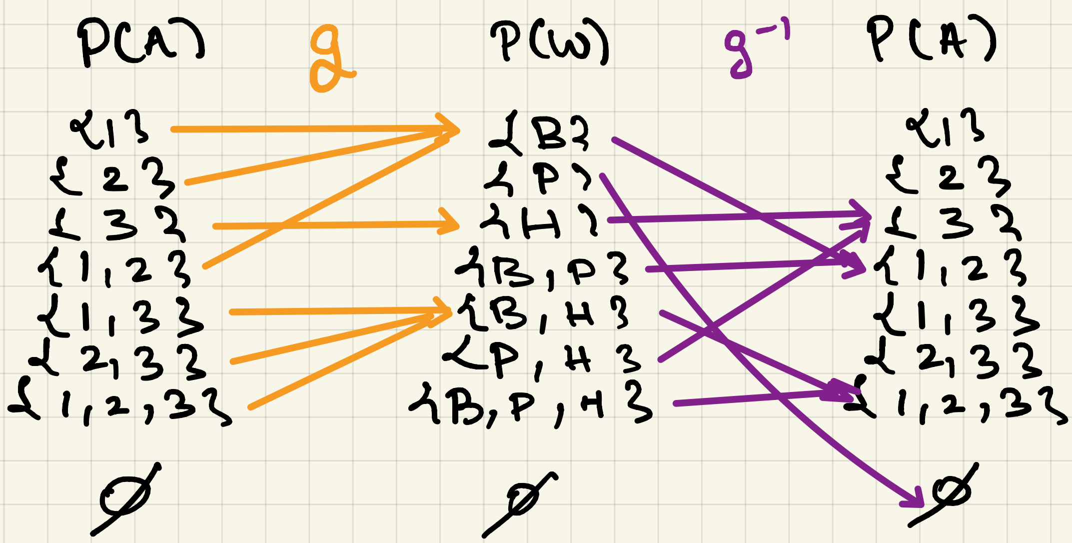
$$g(g^{-1}(T)) \subseteq T$$

consider
$g: \mathbb{R} \to \mathbb{R}$
$x \mapsto x^2$
$T = \{-1, 4\}$

WARMUP: $W = \{B, P, H\}$ $A = \{1, 2, 3\}$

$g: A \to W$

$1 \to B$
$2 \to H$
$3 \to P$



Theorem

The fnc $f:\mathbb{R}\to\mathbb{R}$ is cts $\iff$ $\forall\, U \subseteq \mathbb{R}$ open, the set $f^{-1}(U)$ is open in $\mathbb{R}$

$\mathbb{R}$ (of $U\subseteq\mathbb{R}$) $\to$ target

$\mathbb{R}$ (of "open in $\mathbb{R}$") $\to$ domain.

Eg $f:\mathbb{R}\to\mathbb{R}$, $x \mapsto x^2$

$$f^{-1}(\emptyset) = \emptyset$$

$$f^{-1}(\mathbb{R}) = \mathbb{R}$$

$$f^{-1}(0,57) = (-\sqrt{57},\sqrt{57})\setminus\{0\}$$

$$f^{-1}(0,\infty) = \mathbb{R}\setminus\{0\}$$

$$f^{-1}(-3,4) = (-2,2) \text{ or } (-2,0] \cup [0,2)$$

Non eg $f:\mathbb{R}\to\mathbb{R}$

$$x \mapsto \begin{cases} 57 & x\geq 0 \\ 0 & x<0 \end{cases}$$

$$f^{-1}(56,58) = [0,\infty)$$

$\to$ not open!

Pf $\Leftarrow$

Let $a\in\mathbb{R}$. Let $\varepsilon>0$ be given.

Task: find a $\delta>0$ st $\forall\, x\in B_\delta(a)$, $f(x)\in B_\varepsilon(f(a))$

Take $U = B_\varepsilon(f(a))$. This is open.

Consider $f^{-1}(U)$. By hypothesis this is open.

Note $a\in f^{-1}(U)$. So we can find a radius δ.

$B_\delta(a) \subseteq f^{-1}(U)$.

Want: this to be the δ we seek.

$$B_\delta(a) \subseteq f^{-1}(U)$$

$$f(B_\delta(a)) \subseteq f(f^{-1}(U)) \subseteq U.$$

In particular

$$f(B_\delta(a)) \subseteq B_\varepsilon(f(a))$$

done.

Pf) $\Rightarrow$ let $U \subseteq \mathbb{R}$ be open.

Task: $f^{-1}(U)$ is open.

note: if $f^{-1}(U)$ is $\emptyset$ then we are done

Spose $f^{-1}(U) \neq \emptyset$.

fix $x \in f^{-1}(U)$. Find $r>0$ s.t $B_r(x) \subseteq f^{-1}(U)$

Consider $f(x) \in U$.

Because U is open $\exists\ \varepsilon>0$ s.t $B_\varepsilon(f(x)) \subseteq U$.

we f is cts at x. So, $\exists\ \delta>0$ s.t

$\forall\ y \in B_\delta(x)$ we have $f(y) \in B_\varepsilon(f(x)) \subseteq U$

So, $\forall\ y \in B_\delta(x) \to f(y) \in U$.

$\Rightarrow \forall\ y \in B_\delta(x)$ $y \in f^{-1}(U)$.

$\therefore B_\delta(x) \subseteq f^{-1}(U)$.

So, $f^{-1}(U)$ is open.

Defn) The function $f: \mathbb{R} \to \mathbb{R}$ is cts p.t $\forall$ open $U \subseteq \mathbb{R}$ we have $f^{-1}(U)$ is open in $\mathbb{R}$.

Defn) A topology on X is $\mathcal{T}_x \subseteq P(x)$ that satisfies

1. $X, \emptyset \in \mathcal{T}_x$
2. $\mathcal{T}_x$ is closed under arbitrary unions
3. $\mathcal{T}_x$ is closed under finite intersections.

Defn) The pair $(X, \mathcal{T}_x)$ is called a topological space.

Defn) Let $(X, \mathcal{T}_x)$ be a top space. The elements of $\mathcal{T}_x$ are the "open subsets" of X.

Eg: Let $X = \mathbb{R}$

$$\mathcal{T}_{eucl} = \left\{ A \subseteq \mathbb{R} \,\middle|\, \begin{matrix} \forall a \in A & \exists r > 0 \\ \text{st} & B_r(a) \subseteq A \end{matrix} \right\}$$

we have checked the validity of this topology on $\mathbb{R}$

Defn Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be top spaces.
The fnc $f: X \to Y$ is cts p.t.
$\forall\, U \in \mathcal{T}_Y$, we have $f^{-1}(U) \in \mathcal{T}_X$

Explore topologies

$X = \mathbb{R}$ $\quad \mathcal{T}_{ind} = \{\emptyset, \mathbb{R}\} \to$ indiscrete topology.

Guess. Let $g: \mathbb{R} \to \mathbb{R}$ be a fnc. Let $\mathcal{T}$ be the topology on the domain.
Take $\mathcal{T}_{ind}$ on the codomain.
g is always cts as
$g^{-1}(\emptyset) = \emptyset$ $\quad g^{-1}(\mathbb{R}) = \mathbb{R}$
$g^{-1}(\mathbb{R}) := \{x \in \mathbb{R} \mid f(x) \in \mathbb{R}\}$

Swap the topologies $\quad f: \underset{\substack{\uparrow \\ \text{indiscrete}}}{\mathbb{R}} \to \mathbb{R}$ $\to$ anything.

Is $f: x \mapsto x$ cts?

If $\mathcal{T}$ contains any set other than $\emptyset$ or $\mathbb{R}$. This isn't cts.

Spose $V \in \mathcal{T}$, $V \neq \emptyset, \mathbb{R}$. $f^{-1}(V) = V \notin \mathcal{T}_{ind}$

Q: Are there any cts functions of the form above?
Yes! Constant functions. for $U \subseteq \mathbb{R}$, $f^{-1}(U) = \emptyset$ or $\mathbb{R}$.
U doesn't have the constant

Another topology

$X = \mathbb{R}$ $\quad \mathcal{T}_{disc} = P(\mathbb{R}) \longrightarrow$ can be generalized to any set X.

Interesting feature .... every set is clopen!!! $\Rightarrow \{57\}$ is open

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functions with this topology on domain $\mathbb{R}$ are always cts for obvious reasons.

Trivially $\quad f: \underset{disc}{\mathbb{R}} \to \underset{indisc}{\mathbb{R}}$ is cts.

What about $\quad f: \underset{ind}{\mathbb{R}} \to \mathbb{R}_{2\,disc}$?

Constant functions are cts here too!!

Lemma Let X, Y, Z be top spaces and $f: X \to Y$ & $g: Y \to Z$ be cts.

Then the composition $(g \circ f): X \to Z$ is cts.

Pf Consider $U \subseteq Z$ to be open. ie $U \in \mathcal{T}_Z$

Consider $(g \circ f)^{-1}(U) = f^{-1} \circ g^{-1}(U)$

We know $g^{-1}(U)$ is open in Y as g is cts.

Since f is cts $f^{-1}(g^{-1}(U))$ is open in X!

$\square$

On day 1.

$$f : \mathbb{Q} \to \mathbb{Q}$$

$t \longmapsto t^2 - 2$ was said to be cts.

Now: We use topology to understand what it means for $f : A \to \mathbb{R}$, $A \subseteq \mathbb{R}$ to be cts.

This motivates a discussion of subspace topology!!!

Sppse $(X, \mathcal{T}_X)$ is a top space. Let $A \subseteq X$.

Then A inherits the topological structure of X.

$$\mathcal{T}_A := \{ A \cap Y \mid Y \in \mathcal{T}_X \}$$

This will work and give us a top space on A.

We call it the subspace topology on $A \to (A, \mathcal{T}_A)$ is a top space

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eg: $X = \mathbb{R}$ $\quad \mathcal{T}_X = \mathcal{T}_{euc}$

$A = [0, 1)$

$B = [0, 0.5)$ note $B \subseteq A$. $\to$ is B open in A.

B is open in A iff $B = A \cap Y$, Y in $\mathcal{T}_{euc}$.

$B = [0, 1] \cap (-0.5, 0.5)$

B is open in A but not X.

On the next page we prove the validity of the subspace topology!!

Defn if $(X, \mathcal{T})$ is a top space and $A \subseteq X$. Then A inherits the structure of a top space from X.

$$\mathcal{T}_A := \{ A \cap U \mid U \in \mathcal{T} \}$$

Claim: $\mathcal{T}_A$ is a top on A

① $A \in \mathcal{T}_A \rightarrow A \cap X = A$ ✓

$\emptyset \in \mathcal{T}_A \rightarrow \emptyset \cap A = \emptyset$ ✓

② Let I be an indexing set.

Let $\{U_i \mid i \in I\}$ be a collection of elements in $\mathcal{T}_A$.

Show $\bigcup_{i \in I} U_i \in \mathcal{T}_A$. $\xrightarrow{\text{so}}$ $\bigcup_{i \in I} U_i = A \cap V$ for V in $\mathcal{T}$.

Since $U_i \in \mathcal{T}_A$ $\exists V_i \in \mathcal{T}$ st $U_i = A \cap V_i$

$\bigcup_{i \in I} V_i \cap A = A \cap \bigcup_{i \in I} V_i = A \cap V \in \mathcal{T}_A$

certainly in $\mathcal{T}$.

③ Let $U_1, U_2 \dots U_k \in \mathcal{T}_A$

Show: $\bigcap_{i=1}^{k} U_i \in \mathcal{T}_A$

Since $U_i \in \mathcal{T}_A$ $\exists V_i \in \mathcal{T}$ s.t. $U_i = V_i \cap A$ $\forall i$ $1 \leq i \leq k$.

So, $\bigcap_{i=1}^{k} U_i = \bigcap_{i=1}^{k} (A \cap V_i)$

$= A \cap \left(\bigcap_{i=1}^{k} V_i \right)$ → closed in $\mathcal{T}$

$= A \cap V \in \mathcal{T}_A$.

We now have many examples of Top spaces!

$X = \mathbb{R}$ $\quad \mathcal{T} = \mathcal{T}_{Eucl}$ $\quad A = \mathbb{Q}$

Consider $\mathcal{T}_A$ as a top on $\mathbb{Q}$

$B \subseteq \mathbb{Q}$ is open p.t. $\exists\, V \in \mathcal{T}_{Eucl}$ s.t $B = V \cap \mathbb{Q}$.

Q: is $\{57\}$ open in $\mathbb{Q}$?

$\{57\} = \underset{\text{open in } \mathbb{R}}{V} \cap \mathbb{Q}$?

No, because of density!!

V must contain 57 $\exists\, r > 0$ s.t $B_r(57) \subseteq V$.

So, many elements in $\mathbb{Q}$!

Q: Is $\{57\}$ closed?

This is the same as "is $\mathbb{Q} \setminus \{57\}$ open?".

$\mathbb{Q} \setminus \{57\} = V \cap \mathbb{Q}$, take $V = \mathbb{R} \setminus \{57\}$

thus, it is closed!!

More examples!

- $A = \mathbb{Z}$ $\quad \mathbb{Z} \subseteq \mathbb{R}$.

So, $B \subseteq \mathbb{Z}$ is open p.t. $\exists\, V \in \mathcal{T}$ s.t $B = \mathbb{Z} \cap V$.

So, singletons are open!! $\{57\} = \mathbb{Z} \cap (56.8, 57.8)$

This is essentially the discrete topology!! Wild!

$\{57\}$ is also closed as $\mathbb{Z} \setminus \{57\} = \mathbb{Z} \cap [\mathbb{R} \setminus \{57\}]$.

$\mathbb{N}$ is clopen in $\mathbb{Z}$! $\mathbb{N} = \mathbb{Z} \cap (0, \infty)$

Q: Is $\mathbb{N}$ open in $\mathbb{Q}$?

No. Because of density you get other rationals!

Defn) Spose $A \subseteq \mathbb{R}$.

Let $f: A \to \mathbb{R}$ be a function. Let $a \in A$.

$\lim_{x \to a} f(x) = l$ means,

$\forall\, \varepsilon > 0\ \exists\, \delta > 0$ s.t. if $x \in (B_\delta(a) \setminus \{a\}) \cap A$

then $f(x) \in B_\varepsilon(l)$

Similarly, f is cts at a means

$\forall\, \varepsilon > 0\ \exists\, \delta > 0$ s.t. if $x \in B_\delta(a) \cap A$ then

$f(x) \in B_\varepsilon(f(a))$

Defn) A top space $(X, \mathcal{T})$ is said to be connected iff it isn't disconnected.

Defn) A top space $(X, \mathcal{T})$ is disconnected iff $\exists$ nonempty, open subsets $A, B \subseteq X$ s.t

$A \cup B = X$ $\to$ whole space is seperated.

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eg: $\mathbb{Z}$ inheriting $\mathcal{T}_{eue}$

$A = \{m \in \mathbb{Z} \mid m \geq 0\}$

$B = \{m \in \mathbb{Z} \mid m < 0\}$

nonempty? ✓ disjoint? ✓ open in $\mathbb{Z}$? ✓

$\mathbb{Z} = A \cup B$

So, $\mathbb{Z}$ is disconnected

$\longrightarrow$

eg: How about $\mathbb{Q}$?

consider

$A = (\sqrt{2}, \infty) \cap \mathbb{Q}$

$B = (-\infty, \sqrt{2}) \cap \mathbb{Q}$

Are A and B open in $\mathbb{Q}$? Yes

They are non-empty and also disjoint!

And, $A \cup B = \mathbb{Q}$!

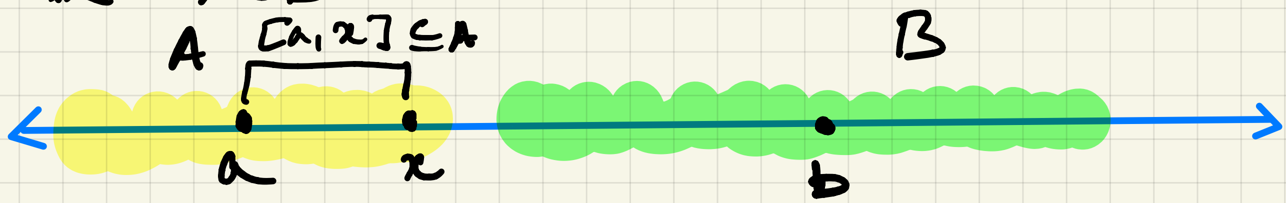
Thus, $\mathbb{Q}$ is disconnected.

if X has the indiscrete top it is connected

Lemma) $\mathbb{R}$ w $\mathcal{T}_{eucl}$ is connected

Pf) Spose $\mathbb{R}$ is disconnected. Then $\exists$ non-empty disjoint open sets $A, B \subseteq \mathbb{R}$ st

$\mathbb{R} = A \cup B$



Fix $a \in A$ and $b \in B$

WLOG, spose $a < b$

Consider $C := \{x \in \mathbb{R} \mid x \geq a$ and $[a, x] \subseteq A\}$

Observe:

→ $C \subseteq A$

→ C is bounded above by b.

↳ if not, $\exists\, c \in C$ s.t $c > b$, $[a, c] \subseteq A$, but $b \in [a, c]$ → oops disjoint

→ $C \neq \emptyset$ as $a \in C$

So, $\alpha := \sup(C)$ exists!!

α must exist in either A or B → disconnected.

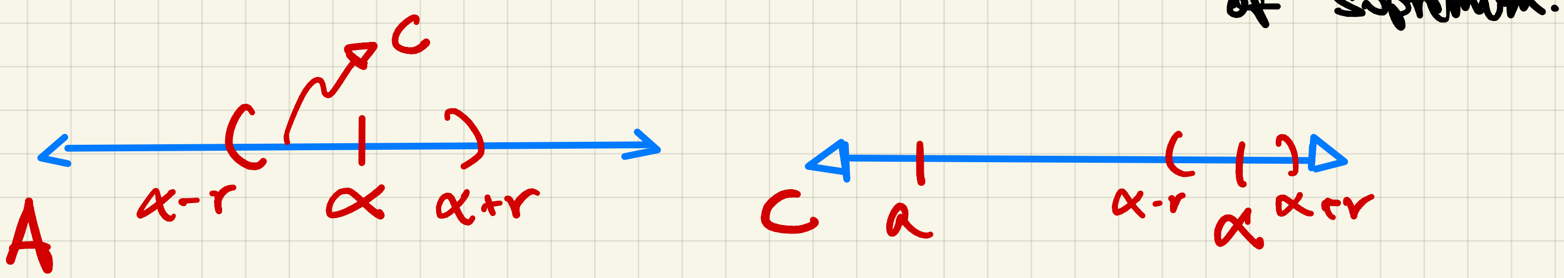
We will rule out both

(1) Spose $\alpha \in A$

Since A is open $\exists\, r > 0$ s.t. $B_r(\alpha) \subseteq A$.

Since $\alpha = \sup(C)$

$\exists\, c \in C$ st $c \in (\alpha - r, \alpha]$ → characterization of supremum.



Since $c \in C$ we know,

$[a, c] \subseteq A$. Furthermore, $[c, \alpha + \frac{r}{57}] \subseteq A$

as $B_r(\alpha) \subseteq A$ and

$[c, \alpha + \frac{r}{57}] \subseteq B_r(\alpha)$

We then have,

$$[a, \alpha + \tfrac{r}{57}] = [a, c] \cup [c, \alpha + \tfrac{r}{57}]$$

both of which $\subseteq A$

$$[a, \alpha + \tfrac{r}{57}] \subseteq A$$

$\therefore\ \alpha + \frac{r}{57} \in C$ → oops, $\alpha := \sup(C)$.

(II) Spose $\alpha \in B$

Since B is open

$\exists\, r > 0$ s.t $B_r(\alpha) \subseteq B$

Also, $\exists\, c \in C$ s.t $c \in (\alpha - r, \alpha] \subseteq B_r(\alpha)$

But then $c \in B \cap C$. Oops,

$B \cap C = \emptyset$ as $C \subseteq A$ and $A \cap C = \emptyset$. oops

Thus, $\mathbb{R}$ isn't disconnected ! $\to$ we heavily used completeness.

On hw5: all intervals in $\mathbb{R}$ are connected.

$\hookrightarrow$ only connected subsets of $\mathbb{R}$.

Top theorem "connectedness is conserved by cts functions":

Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ be top spaces.

Spose X is connected and $f: X \to Y$ is cts.

Then $f(X)$ is connected !

Pf Spose $f(X) \subseteq Y$ is disconnected. Then,

$\exists\, A, B \subseteq f(X)$ that are open in $f(X)$ st

$f(X) = A \cup B$. A, B are nonempty and disjoint.

By subspace top on $f(X) \subseteq Y$

$\exists$ open sets $U_A, U_B \subseteq Y$ s.t

$A = f(X) \cap U_A$ $\qquad$ $B = f(X) \cap U_B$

Then, $f^{-1}(A) = f^{-1}(f(X) \cap U_A)$

$$= f^{-1}(f(X)) \cap f^{-1}(U_A)$$

$$= X \cap f^{-1}(U_A)$$

$$= f^{-1}(U_A)$$

Similarly $f^{-1}(B) = f^{-1}(U_B)$

Since f is cts, both $f^{-1}(U_B)$ and $f^{-1}(U_A)$ are open in X. Since $A \cap B = \emptyset$,

$f^{-1}(U_A) \cap f^{-1}(U_B) = \emptyset$

A and B are non-empty, so the same applys for the sets above.

Claim $X = f^{-1}(U_A) \cup f^{-1}(U_B)$ oops, then X is disconnected.

two way containment

$\supseteq$ is free as both are subsets of X

$\subseteq$ is also true.

We reach a contradiction and thus, $f(X)$ is conneted in Y.

Corollary IVT. Let $a, b \in \mathbb{R}$. with $a < b$

Spose $f : [a, b] \to \mathbb{R}$ is cts.

Let y be btw $f(a)$ and $f(b)$

$\exists \; c \in [a, b]$ st $y = f(c)$.

Pf we have $[a, b]$ is an interval, thus it is a connected subset of $\mathbb{R}$.

Since f is cts $f([a, b])$ is connected subset of $\mathbb{R}$. So, by nw, also an interval!

furthermore $f(a), f(b) \in f([a, b])$ so, if y between $f(a)$ and $f(b)$ then

$y \in f[a, b]$.

So, $\exists c \in [a, b]$ s.t $y = f(c)$.