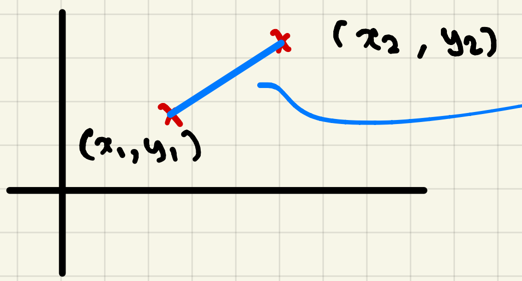
Metric Spaces!

Metrics

$$\mathbb{R}^2 = \mathbb{R}\times\mathbb{R} = \{(x,y) \mid x,y \in \mathbb{R}\}$$



$$d((x_1,y_1),(x_2,y_2))$$
$$= \sqrt{(y_2-y_1)^2 + (x_2-x_1)^2}$$

↳ euclidean distance!

We are going to get a top on $\mathbb{R}^2$ with a metric!

Defn) Let X be a set. A fnc $d: X\times X \to \mathbb{R}_{\geq 0}$ is a metric on X s.t.

1. $d(x,y) = d(y,x) \quad \forall x,y \in X$
2. $d(x,y) = 0 \iff x = y$
3. triangle inequality
$$d(x,y) \leq d(x,z) + d(y,z)$$
$$\forall x,y,z \in X.$$

Defn) The pair (X, d) is a metric space!

Is $\mathbb{R}^2$ w/ euclidean distance a metric space?
Ie is euc distance a metric on $\mathbb{R}^2$

Ans: Yes → check

$$d: \mathbb{R}^2\times\mathbb{R}^2 \longrightarrow \mathbb{R}_{\geq 0}$$

$$((x_1,y_1),(x_2,y_2)) \longmapsto \sqrt{(y_2-y_1)^2 + (x_2-x_1)^2}$$

great! Now we can measure distances in $\mathbb{R}^2$ w this metric!

How does that help with finding a top?

Q: Can we think of $\mathbb{R}$, with a distance f as a metric space?

A: Yes! Abs value!

$(\mathbb{R}, |\,|)$ is a metric space!

Q: We know theres a top on $\mathbb{R}$ (euc). How is this related to the metric?

$$\mathcal{T}_{euc} := \{ A \subseteq \mathbb{R} \mid \forall a \in A \ \exists r > 0 \text{ s.t. } B_r(a) \subseteq A \}$$

$$B_r(a) := \{ x \in \mathbb{R} \mid |x - a| < r \}$$

Bingo!

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lets start with a metric space (X, d)

consider:

$$\mathcal{T}_d := \{ A \subseteq X \mid \forall a \in A \ \exists r > 0 \text{ s.t. } B_r(a) \subseteq A \}$$

$$B_r(a) := \{ x \in X \mid d(x, a) < r \}$$

Defn) let (X, d) be a metric space.

Let $A \subseteq X$ be open $\iff$ $\forall a \in A \ \exists r > 0$ s.t. $B_r(a) \subseteq A$.

$$\mathcal{T}_d := \{ A \subseteq X \mid A \text{ is open} \}$$

$\boxed{\text{Claim:}}$ $\mathcal{T}_d$ is a topology

Pf) ① $\emptyset, X \in \mathcal{T}_d$ ✓

② $\mathcal{T}_d$ closed wrt arbitrary unions!

Let I be an indexing set. let

$$\{ U_i \mid i \in I \} \subseteq \mathcal{T}_d$$

fix an $x \in \bigcup_{i \in I} U_i$ $\longrightarrow$

$\exists\, j \in I$ s.t. $x \in U_j$. Since $U_j \in \mathcal{T}_0$ $\exists\, r > 0$ s.t. $B_r(x) \subseteq U_j$

Since $U_j \subseteq \bigcup_{i \in I} U_i$

we have $B_r(x) \subseteq \bigcup_{i \in I} U_i$

$\therefore$ open under arb unions!

3 Check $\mathcal{T}_0$ is closed wrt finite intersections:

Let $U_1 \dots U_k \in \mathcal{T}_0$

Consider $\bigcap_{i=1}^{k} U_i$ $\to$ show in $\mathcal{T}_0$

fix $x \in \bigcap_{i=1}^{k} U_i$

$x \in U_j \quad \forall\, 1 \le j \le k$

So, $\exists\, r_i > 0$ s.t. $B_{r_i}(x) \subseteq U_i \quad \forall\, 1 \le i \le k$.

Set $r = \min(r_1 \dots r_k)$

Then $B_r(x) \subseteq B_{r_i} \subseteq U_i \quad \forall\, 1 \le i \le k$

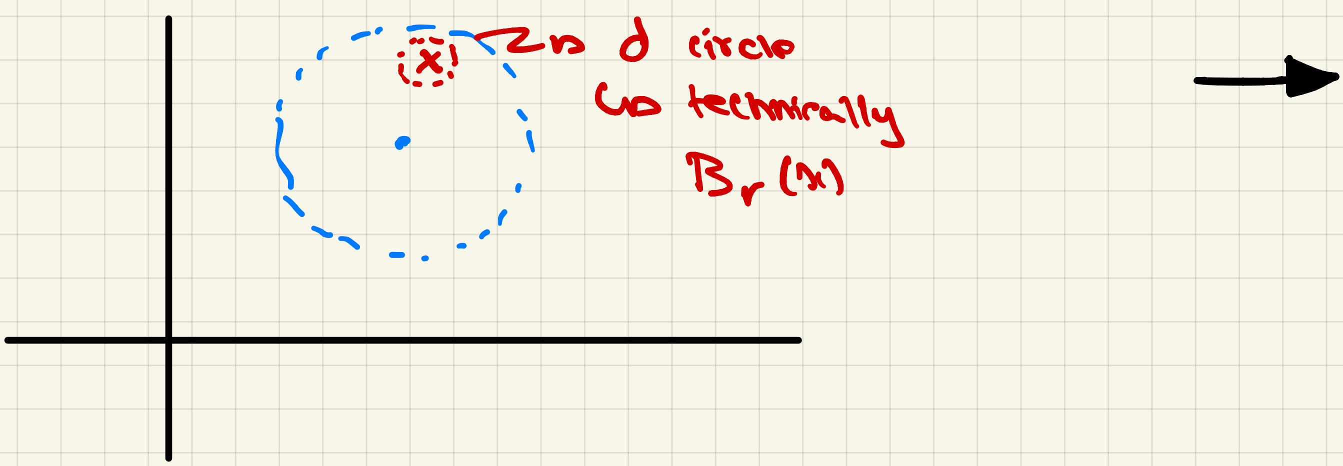
$\Rightarrow B_r(x) \subseteq \bigcap_{i=1}^{k} U_i$

$\square$

Thus, this is a topology on X.

$\mathbb{R}^2$, d_{euc} $\to$ euclidean metric

$\hookrightarrow$ in this distances are circles



Q: given a top space $(X, \mathcal{T})$ is there a metric we can define on X, s.t the topology coming from the metric is equal to $\mathcal{T}$?

Ans eg: $X = \mathbb{R}$ $\qquad \mathcal{T} = \mathcal{T}_{disc} = P(X)$

Does $\mathcal{T}_{disc}$ come from a metric.

In $\mathcal{T}_{disc}$, singletons are open. We would need a metric for which

$\exists r > 0$ s.t $B_r(a) \subseteq \{a\}$

$\{a\} \Leftarrow$

consider. $\delta(x,y) = \begin{cases} 0 & \text{if } x = y \\ 57 & \text{if } x \neq y. \end{cases}$

is it a metric? ....

① ✓

② ✓

③ ✓

What do the balls look like?

take $a = 0$ $\quad r = \frac{1}{2}$

$B_{\frac{1}{2}}(0) = \{x \in \mathbb{R} \mid \delta(x,0) < \frac{1}{2}\}$

$= \{0\}$

$B_{100}(0) = \{x \in \mathbb{R} \mid \delta(x,0) < 100\}$

$= \mathbb{R}$.

Wait) What is $\mathcal{T}_\delta$

$\mathcal{T}_\delta = \{A \subseteq \mathbb{R} \mid \forall a \in A \ \exists r > 0 \text{ s.t } B_r(a) \subseteq A\}$

note $\mathcal{T}_\delta$ contains singletons. By closure under $\cup$

$\mathcal{T}_\delta = P(X) = \mathcal{T}_{disc}$

Defn) The top $(X, \mathcal{T})$ is said to be metrizable iff $\exists$ a metric δ on X s.t.

$$\mathcal{T}_\delta = \mathcal{T}.$$

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guess : $X = \mathbb{R}$ w/ $\mathcal{T}_{ind}$ is not metrizable

$\mathcal{T}_{ind} = \{\emptyset, X\}$

if there is a metric δ on $\mathbb{R}$, s.t $\mathcal{T}_\delta = \mathcal{T}_{ind}$

Then

$\mathcal{T}_\delta$ contains $\mathbb{R}$, $\emptyset$.

Q: Are $B_r(a)$ always in $\mathcal{T}_\delta$? Yes (shown below).

→ take 2 elements.

We know $57 \neq 100$, $r = \delta(57, 100) > 0$

Consider $B_{r/2}(57)$, $B_{r/2}(100)$

They aren't the same as, $57 \in B_{r/2}(57)$ and $57 \notin B_{r/2}(100)$

But they are both open and non-empty. Thus $\mathcal{T}_\delta$ has 2 distinct open nonempty sets.

So, it is impossible for $\mathcal{T}_\delta = \mathcal{T}_{ind}$

Q) Are $B_r(c)$ always in $\mathcal{T}_\delta$? Yes!

Pf) let (X, δ) be a metric space. let $a \in X$.

let $r > 0$. Show $B_r(a)$ is open in X.

That is $\forall\, y \in B_r(a)$ $\exists\, s > 0$ s.t $B_s(y) \subseteq B_r(a)$

fix $y \in B_r(a)$ choose $s = r - \delta(y, a)$

lets check if

$B_s(y) \subseteq B_r(a)$ →

take $z \in B_s(y)$ $\quad \delta(z,y) < s$

lets show $\delta(z,a) < r \to z$ in $B_r(a)$ then

$\delta(z,a) \le \delta(z,y) + \delta(y,a)$ due to triangle inequality

$< s + \delta(y,a)$

$< r - \delta(y,a) + \delta(y,a)$ → from def of s

$\delta(z,a) < r$

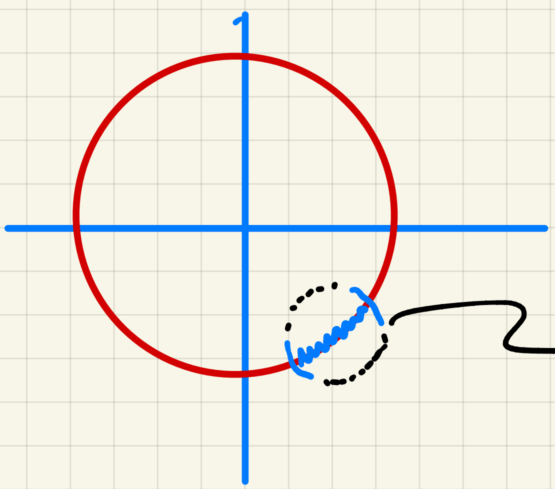
$\therefore$ $z \in B_r(a) \Rightarrow B_s(y) \subseteq B_r(a)$!

Upshot: In a metric space open balls are open sets!

Additional Examples

Consider $\mathbb{R}^2$ with the euclidean metric!

we get a topology $(\mathbb{R}^2, \mathcal{T}_0)$



$S^1 := \{(x,y) \in \mathbb{R}^2 \mid \delta((x,y),(0,0)) = \}$

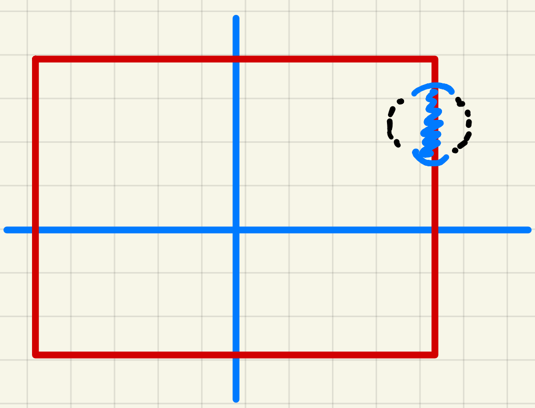
↳ circle of dimension 1

→ These types of 'open' arcs are open as

$\wr = \bigcirc \cap S^1$

where $\bigcirc$ is open in $\mathbb{R}^2$ so $\wr$ open in S^1

↳ open ball



Now we have a square. Again, clearly $\wr$ is open in the square's subspace topology.

How are these 2 topologies related? Look at homeomorphisms!