Determinant

We define determinants as

$$\text{Det} : \text{Mat}_{n\times n}(F) \longrightarrow F$$

$$A \longrightarrow \sum_{\sigma\in S_n} \text{sgn}(\sigma) \prod_{i=1}^{n} A_{i\sigma(i)}$$

Can verify that this is the same as our usual if $\underline{n=2,3}$

Eventually, we will show A is inv $\iff \text{Det}(A) \neq 0$

Lemma) let $A \in \text{Mat}_{n\times n}(F)$

$$\text{Det}(A) = \text{Det}(A^t)$$

Pf) By def & $A_{ij} = A^t_{ji}$

$$\text{Det}(A^t) = \sum_{\sigma\in S_n} \text{sgn}(\sigma) \prod_{i=1}^{n} A_{\sigma(i)\, i}$$

we note $\sigma^{-1} : \mathbb{N}_n \to \mathbb{N}_n$ is a bij $\exists$ grp act

$$= \sum_{\sigma\in S_n} \text{sgn}(\sigma) \prod_{i=1}^{n} A_{\sigma^{-1}(\sigma(i))\ \sigma^{-1}(i)}$$

$$= \sum_{\sigma\in S_n} \text{sgn}(\sigma) \prod_{i=1}^{n} A_{i\,\sigma^{-1}(i)}$$

By reindexing

$$= \sum_{\sigma\in S_n} \text{sgn}(\sigma^{-1}) \prod_{i=1}^{n} A_{i\,\sigma(i)}$$

But $\text{sgn}(\sigma^{-1}) = \text{sgn}(\sigma)$ as $\sigma\sigma^{-1} = e$

$$= \sum_{\sigma\in S_n} \text{sgn}(\sigma) \prod_{i=1}^{n} A_{i\,\sigma(i)} = \text{Det}(A)$$

For a 2×2 matrix, we can think of

$\text{Det} \in \text{Alt}^2(F^2)$ where,

$\lambda \in \text{Alt}^2(F^2)$ st it is linear in each spot & $\lambda(\bar{v},\bar{w}) = -\lambda(\bar{w},\bar{v})$ & $F^2 \times F^2 \to F$

So, thinking like this

$$\begin{aligned} F^2 \times F^2 &\longrightarrow F \\ (\bar{v},\bar{w}) &\longmapsto \text{Det}\begin{pmatrix} | & | \\ v & w \\ | & | \end{pmatrix} \end{aligned}$$

Note that it is alternating since

$\text{Det}\begin{pmatrix} a & c \\ b & d \end{pmatrix} = ad - bc$ & $\text{Det}\begin{pmatrix} c & a \\ d & b \end{pmatrix} = cb - ad$

Check,

$\left.\begin{aligned} &\text{Det}(\bar{v}+\bar{x}, \bar{w}) = \text{Det}(\bar{v},\bar{w}) + \text{Det}(\bar{x},\bar{w}) \\ &\text{Det}(\alpha\bar{v}, \bar{w}) = \alpha\, \text{Det}(\bar{v},\bar{w}) \end{aligned}\right\}$ Multilin

This holds generally & we think of

$\text{Det} \in \text{Alt}^n(F^n)$ (note $\dim(\text{Alt}^n(F^n)) = \binom{n}{n} = 1$)

Det is the unique normalized element of this space

That is

$$\boxed{\text{Det}\begin{pmatrix} | & | & & | \\ e_1 & e_2 & \cdots & e_n \\ | & | & & | \end{pmatrix} = 1}$$

Let us prove that it is, in fact, multilinear!

Lemma 1

① $\operatorname{Det}\begin{pmatrix} \cdots & \vert & \cdots & \vert & \cdots \\ & v_1 & & v_2 & \\ & \vert & & \vert & \end{pmatrix} = -\operatorname{Det}\begin{pmatrix} \cdots & \vert & \cdots & \vert & \cdots \\ & v_2 & & v_1 & \\ & \vert & & \vert & \end{pmatrix}$

② $\operatorname{Det}\begin{pmatrix} \cdots & \vert & \cdots \\ & a\vec{v}+b\vec{w} & \\ & \vert & \end{pmatrix} = a\operatorname{Det}\begin{pmatrix} \cdots & \vert & \cdots \\ & \vec{v} & \\ & \vert & \end{pmatrix} + b\operatorname{Det}\begin{pmatrix} \cdots & \vert & \cdots \\ & \vec{w} & \\ & \vert & \end{pmatrix}$

③ $\operatorname{Det}\begin{pmatrix} \cdots & \vert & \cdots & \vert \\ & v & & v \\ & \vert & & \vert \end{pmatrix} = 0$

Pf) Plan: ② + ③ $\Longrightarrow$ ① as

lets show this,

$$\operatorname{Det}\begin{pmatrix} \cdots & \vert & \cdots & \vert & \cdots \\ & v_1+v_2 & & v_1+v_2 & \\ & \vert & & \vert & \end{pmatrix} \overset{③}{=} 0$$

$$\overset{②}{=} \underbrace{\operatorname{Det}\begin{pmatrix} \cdots & \vert & \cdots & \vert & \cdots \\ & v_1 & & v_2 & \\ & \vert & & \vert & \end{pmatrix}} + \underbrace{\operatorname{Det}\begin{pmatrix} \cdots & \vert & \cdots & \vert \\ & v_1 & & v_1 \\ & \vert & & \vert \end{pmatrix}}_{\to\, 0 \ ③} +$$

$$\underbrace{\operatorname{Det}\begin{pmatrix} \cdots & \vert & \cdots & \vert \\ & v_2 & & v_2 \\ & \vert & & \vert \end{pmatrix}}_{\to\, 0} + \operatorname{Det}\begin{pmatrix} \cdots & \vert & \cdots & \vert & \cdots \\ & v_2 & & v_1 & \\ & \vert & & \vert & \end{pmatrix} = 0$$

$$\Longrightarrow \operatorname{Det}\begin{pmatrix} \cdots & \vert & \cdots & \vert \\ & v_1 & & v_2 \\ & \vert & & \vert \end{pmatrix} = -\operatorname{Det}\begin{pmatrix} \cdots & \vert & \cdots & \vert \\ & v_2 & & v_1 \\ & \vert & & \vert \end{pmatrix} \quad \triangleright ①$$

Pf) ② Consider

$$A := \begin{pmatrix} \cdots & \vert & \cdots \\ & a\vec{v}+b\vec{w} & \\ & \vert & \end{pmatrix}$$

↳ col k_0

for $\sigma \in S_n$, set $i_0 = \sigma^{-1}(k_0) \iff \sigma(i_0) = k_0$

So,

$$\operatorname{Det}(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \overbrace{\prod_{i \in \mathbb{N}_n} A_{i\sigma(i)}}$$

$$= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, A_{i_0 k_0} \prod_{i \in \mathbb{N}_n \setminus \{i_0\}} A_{i\sigma(i)}$$

$$= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) [a\bar{v} + b\bar{w}]_{i_0} \prod_{i \in \mathbb{N}_n \setminus \{i_0\}} A_{i\sigma(i)}$$

$$= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) [a\bar{v}_{i_0} + b\bar{w}_{i_0}] \prod_{i \in \mathbb{N}_n \setminus \{i_0\}} A_{i\sigma(i)}$$

$$= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \left[a v_{i_0} \prod_{i \in \mathbb{N}_n \setminus \{i_0\}} A_{i\sigma(i)} + b w_{i_0} \prod_{i \in \mathbb{N}_n \setminus \{i_0\}} A_{i\sigma(i)} \right]$$

and the result follows by pulling it back $\square$

Pf) ③

$$\operatorname{Det}\begin{pmatrix} \cdots & | & \cdots & | & \cdots \\ & \bar{v} & & \bar{v} & \\ & | & & | & \end{pmatrix} = 0$$

$\hookrightarrow$ col j_0 $\quad$ $\hookrightarrow$ col j_1

$$= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} A_{i\sigma(i)} \quad \text{as } S_n = A_n \sqcup B_n$$

$$= \sum_{\sigma \in A_n} \cancel{\operatorname{sgn}(\sigma)}^{1} \prod_{i=1}^{n} A_{i\sigma(i)} + \sum_{\sigma \in B_n} \cancel{\operatorname{sgn}(\sigma)}^{-1} \prod_{i=1}^{n} A_{i\sigma(i)}$$

Since $B_n = (j_0\ j_1) A_n$ as $j_0 \neq j_1$

$$= \sum_{\sigma \in A_n} \prod_{i=1}^{n} A_{i\sigma(i)} - \sum_{\sigma \in S_n} \prod_{i=1}^{n} A_{i((j_0 j_1)\sigma)(i)}$$

$$= \sum_{\sigma \in A_n} \prod_{i=1}^{n} A_{i\sigma(i)} - \prod_{i=1}^{n} A_{i((j_0 j_1)\sigma)(i)}$$

for each $\sigma \in A_n$ $\quad i_0 := \sigma^{-1}(j_0)$

$i_1 := \sigma^{-1}(j_1)$

So

$$\prod_{i=1}^{n} A_{i\sigma(i)} = A_{i_0 j_0} A_{i_1 j_1} \prod_{i \in \mathbb{N}_n \setminus \{i_0, i_1\}} A_{i\sigma(i)}$$

and

$$\prod_{i=1}^{n} A_{i((j_0 j_1)\sigma)(i)} = \underbrace{A_{i_0 j_1} \cdot A_{i_1 j_0}} \prod_{\substack{i \in \mathbb{N}_n \setminus \\ \{i_0, i_1\}}} A_{i((j_0 j_1)\sigma)(i)}$$

as we apply $\underline{(j_0 \ j_1) \text{ after } \sigma}$

We note that, as $i \notin \{i_0, i_1\}$

$$\prod_{\substack{i \in \mathbb{N}_n \setminus \\ \{i_0, i_1\}}} A_{i((j_0 j_1)\sigma)(i)} = \prod_{i \in \mathbb{N}_n \setminus \{i_0, i_1\}} A_{i\sigma(i)} = \alpha(\sigma)$$

So,

$$\mathrm{Det}(A) = \sum_{\sigma \in A_n} \alpha(\sigma) A_{i_0 j_0} A_{i_1 j_1} - \alpha(\sigma) A_{i_0 j_1} A_{i_1 j_0}$$

But $A_{i_0 j_0} = A_{i_0 j_1}$ & $A_{i_1 j_1} = A_{j_1 j_1}$

$\Rightarrow \mathrm{Det}(A) = 0$

$\square$

Can we define the det of a linear transf ?

$\hookrightarrow T : V \to V$

Choose a basis $\mathbb{V}$ of V

① construct ${}_{\mathbb{V}}[T]_{\mathbb{V}}$

② $\mathrm{Det}(T) := \mathrm{Det}({}_{\mathbb{V}}[T]_{\mathbb{V}})$

$\hookrightarrow$ but is this well defined, say we choose $\mathbb{V}'$ as a basis is

$$\mathrm{Det}({}_{\mathbb{V}'}[T]_{\mathbb{V}'}) = \mathrm{Det}({}_{\mathbb{V}}[T]_{\mathbb{V}})$$

Follows from multiplicity so stay tuned

Let us show some stuff we glossed over

$\mathrm{Alt}^n(F^n)$

Is $\mathrm{Det} \in \mathrm{Alt}^n(F^n)$ the zero func?

No as $\mathrm{Det}(\mathrm{Id}_{n \times n}) = 1$ (technically $\mathrm{Det}(e_1 \cdots e_n) = 1$)

$\therefore \dim(\mathrm{Alt}^n(F^n)) \geq 1$

let us show equality. We can cook up inj lin trans $\mathrm{Alt}^n(F^n) \to F$

Pf] Show: $\dim(\mathrm{Alt}^n(F^n)) = 1$

Consider, $\varphi : \mathrm{Alt}^n(F^n) \to F$

$$\lambda \longmapsto \lambda(\bar{e}_1 \cdots \bar{e}_n)$$

is this linear? $\to$ yes by defn of func addition

lets show injectivity. lets examine $\ker(\varphi)$

Fix $\lambda \in \ker(\varphi) \Rightarrow \lambda(\bar{e}_1 \cdots \bar{e}_n) = 0$

To show λ is zero in $\mathrm{Alt}^n(F^n)$, we must show it is the zero function.

$\longrightarrow$

$\forall\, w_1 \cdots w_n \in F^n,\ \lambda(w_1 \cdots w_n) = 0.$

Fix, $w_1 \cdots w_n \in F^n$

$\forall\, k \in \mathbb{N}_n \quad \exists\, b_{1k} \cdots b_{nk} \in F$ s.t

$$w_k = \sum_{i=1}^{n} b_{ik} \vec{e_i}$$

$\lambda(w_1 \ \cdots \ w_k) \to$ unpack with mult

$\lambda\left(\sum_{i=1}^{n} b_{i1} e_i, \ \cdots, \sum_{i=1}^{n} b_{in} e_i\right)$

gets messy with n^n terms

each term will be

$\lambda($ some shuffle of basis vect $)$

some will have repeats $\to \boxed{0}$ by alt

$\lambda($ some shuffle with no repeat$)$

$= 0$ by alt too (unshuffling perm gives -ive signs)

$\Rightarrow$ giant sum $= 0$

$\Rightarrow \varphi$ is inj $\Rightarrow \dim(\mathrm{Alt}^n(F^n)) \leq 1$ $\triangleright$

Corr) Det is a basis of $\mathrm{Alt}^n(F^n)$

Corr) $\mathrm{Alt}^n(F^n) = \mathrm{span}(\mathrm{Det})$

$\hookrightarrow$ any $\lambda \in \mathrm{Alt}^n(F^n)$ is a scalar mult of Det

$\hookrightarrow$ so we get the alternative def from earlier ?

$\hookrightarrow$ unique normalized elt in $\mathrm{Alt}^n(F^n)$

Det are multiplicative!

Lemma) Let $A, B \in \mathrm{Mat}_{n\times n}(F)$

$$\det(AB) = \det(A)\det(B)$$

Pf) Let $A, B \in \mathrm{Mat}_{n\times n}(F)$

$$\lambda_B : \underbrace{F^n \times \cdots \times F^n}_{n \text{ copies}} \to F$$

$$(w_1 \cdots w_n) \longmapsto \det(B \cdot (w_1 \cdots w_2))$$

Claim $\lambda_B \in \mathrm{Alt}^n(F^n)$ (then, we're done!)

$$\lambda_B(w_1 \cdots \bar{w_i} + \bar{v}, \cdots w_n)$$

$$= \mathrm{Det}(B \cdot (w_1 \cdots w_i + v \cdots w_n))$$

$$= \mathrm{Det}(Bw_1 \cdots B(w_i + v) \cdots Bw_n)$$

as Det is lin

$$= \mathrm{Det}(Bw_1 \cdots Bw_i \cdots Bw_n) + \mathrm{Det}(Bw_1 \cdots Bv \cdots Bw_n)$$

$$= \lambda_B(w_1 \cdots w_n) + \lambda_B(w_1 \cdots v \cdots w_n)$$

Similarly we get multiplicity.

Alt?

$$\lambda_B(\bar{w_1} \cdots v, \cdots, \bar{v}, \cdots w_n)$$

$$= \mathrm{Det}(Bw_1 \cdots Bv \cdots Bv \cdots Bw_n) = 0$$

$\therefore \lambda_B \in \mathrm{Alt}^n(F^n)$

$\therefore \exists\, c_B \in F$ s.t. $\lambda_B = c_B \cdot \mathrm{Det}$ but we see,

$$\lambda(e_1 \cdots e_n) = \mathrm{Det}(B \cdot I_{n\times n}) = \mathrm{Det}(B)$$

$\Rightarrow c_B = 1$ as $\det(e_1 \cdots e_n) = 1$

So, $\lambda_B = (\mathrm{Det}(B)) \cdot \mathrm{Det}$

Let $A = \begin{pmatrix} | & & | \\ a_1 & \cdots & a_n \\ | & & | \end{pmatrix}$

$\lambda_B(a_1 \cdots a_n) \overset{\det}{=} \mathrm{Det}(BA)$.

But also $\lambda(a_1 \cdots a_n) = \mathrm{Det}(B) \det(a_1 \cdots a_n)$

$= \mathrm{Det}(B) \det(A)$

$\therefore \mathrm{Det}(B \cdot A) = \mathrm{Det}(B) \mathrm{Det}(A)$ $\triangleright$

$\Rightarrow \mathrm{Det}(BA) = \mathrm{Det}(AB)$

But $BA \neq AB$ always

Back to Det for lin trans!

idea) let $T \in \mathrm{hom}(V, V)$ with V fgvs /F

Choose basis $\mathcal{V}$

Set $\mathrm{Det}(T) = \mathrm{Det}[{}_{\mathcal{V}}[T]_{\mathcal{V}}]$

Choose another basis $\mathcal{V}'$ of V and consider $\mathrm{Det}[{}_{\mathcal{V}'}[T]_{\mathcal{V}'}]$

${}_{\mathcal{V}'}[T]_{\mathcal{V}'} = {}_{\mathcal{V}'}[\mathrm{id}]_{\mathcal{V}} \; {}_{\mathcal{V}}[T]_{\mathcal{V}} \; {}_{\mathcal{V}}[\mathrm{id}]_{\mathcal{V}'}$

$\Rightarrow \mathrm{Det}({}_{\mathcal{V}'}[T]_{\mathcal{V}'}) = \mathrm{Det}(\quad)$

$= \mathrm{Det}({}_{\mathcal{V}}[T]_{\mathcal{V}}) \; \mathrm{Det}({}_{\mathcal{V}}[\mathrm{id}]_{\mathcal{V}'}) \; \mathrm{Det}({}_{\mathcal{V}'}[\mathrm{id}]_{\mathcal{V}})$

$= \mathrm{Det}({}_{\mathcal{V}}[T]_{\mathcal{V}}) \; \mathrm{Det}(\mathrm{id})$

$= \mathrm{Det}({}_{\mathcal{V}}[T]_{\mathcal{V}})$

So this is welldef!

Lemma) if $A \in \mathrm{Mat}_{n \times n}(F)$ is inv

$\mathrm{Det}(A)^{-1} = \mathrm{Det}(A^{-1})$

As $AA^{-1} = \mathrm{id}$

Corr) $\mathrm{Det}(A) \neq 0$ if A is inv!

Defn Let V be fin gen /F & $\dim(V) \geq 1$
Let $T \in \hom(V,V)$. we def

$$\operatorname{Det}(T) := \operatorname{Det}({}_{w}[T]_{w})$$

For any basis w.

Corr if $A \in \operatorname{Mat}_{n \times n}(F)$ is inv, $\det(A) \neq 0$.

Observ fix $a \in F$

1. $\operatorname{Det}(a \cdot T) = a^n \cdot \operatorname{Det}(T)$, $n = \dim(V)$
2. $\operatorname{Det}(T^*) = \operatorname{Det}(T)$

as $\forall$ basis w of V

$${}_{w^*}[T^*]_{w^*} = ({}_{w}[T]_{w})^t$$

3. If $V \neq W$ $S \in \hom(V,W)$ does $\det(S)$ exist?
   1. if $\dim(V) \neq \dim(W)$ we lose
   2. even if they are, the change of basis doesn't work the same! → not inverses
      ↳ we lose

Fun Fact Let V be a VS /F.

$$GL(V) := \{T \in \hom(V,V) \mid T \text{ is an isom}\}$$

↳ General linear grp wrt func comp

If V is finite dim & non trivial.

$$GL(V) \xrightarrow{\operatorname{Det}} F \setminus \{0\} \to \text{wrt } \underline{\underline{\text{mult}}}$$

Is a homomorphism as det is multiplicative.

Goal) $\mathrm{Det}(A) \neq 0 \iff A$ is inv

Task) If $\mathrm{Det}(A) \neq 0 \Rightarrow A$ is inv

idea) Contra positive.

A is not inv $\iff \exists$ non pivotal col

If $\vec{v}$ is a col vec of A, & $\vec{v} \in \mathrm{span}$ (other col)

we use multilin to show $\mathrm{Det}(A) = 0$!

Pf) Suppose A is not inv

$\tilde{A} := \mathrm{rref}(A)$

$\tilde{A}$ not identity!

$\therefore \tilde{A}$ has at least 1 row of all zeros!

$$\tilde{A} = \begin{pmatrix} 1 & \cdots & \cdots \\ & \ddots & \\ 0\,0\,0\,0 & \cdots & 0\,0 \end{pmatrix}$$

$\Rightarrow (\tilde{A})^t$ has a col of all 0s

$\therefore \det(\tilde{A}) = \det((\tilde{A})^t) = 0$!

Recall, exist (inv) elem matrices s.t

$\tilde{A} = E_k \cdots E_1 A$

$\Rightarrow \det(\tilde{A}) = \mathrm{Det}(E_k) \cdots \mathrm{Det}(A) = 0$

$\Rightarrow \mathrm{Det}(A) = 0$ as $\mathrm{Det}(E_n) \neq 0$

Expansion for Calculations.

Can expand along row or col

lemma) Fix $1 \le i_0 \le j_0 \le n$

$$\mathrm{Det}(A) = \sum_{i=1}^{n} (-1)^{i+j_0} A_{i j_0} \cdot \mathrm{Det}(A^{i j_0})$$

mat obtained by removing row i col j

Pf) Laplace expand handout