Sequences of function,

Ptwise conv,

Uniform converges &

Preservation of

Cts, int, diff

Seq of functions

Let $I \subseteq \mathbb{R}$ be an iwimp.

Defn Let $n \mapsto (f_n : I \to \mathbb{R})$ be a sequence of functions from I to $\mathbb{R}$.

Let $f : I \to \mathbb{R}$ be a function

(a) We say (f_n) conv pt wise to f on I pt

$\forall x_0 \in I$ the seq $n \mapsto f_n(x_0)$ conv to $f(x_0)$

so

$\forall x_0 \in I \quad \forall \varepsilon > 0 \quad \exists N \in \mathbb{N}$ s.t $\forall n > N \quad |f(x_0) - f_n(x_0)| < \varepsilon$

(b) We say (f_n) conv uniformly to f on I p.t

$\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ s.t $\forall x_0 \in I \ \forall n > N \quad |f(x_0) - f_n(x_0)| < \varepsilon$

Note 1: Look at the change of quantifiers

In (a) N is dependent on x_0 & ε !

In (b) N is dependent on ε !

Note 2: Rather obvious but

Uniform conv $\Rightarrow$ ptwise conv

Eg. ptwise

$g_n : [0,1] \to \mathbb{R}$

$$x \mapsto \begin{cases} 0 & x \in [\frac{1}{n}, 1] \\ 1 & x \in [0, \frac{1}{n}) \end{cases}$$

conv to

$g : [0,1] \to \mathbb{R}$

$$x \mapsto \begin{cases} 0 & x \neq 0 \\ 1 & x = 1 \end{cases}$$

Conv ptwise but not unif. Take $\varepsilon < 1$ to validate !

eg unif

$h_n : \mathbb{R} \to \mathbb{R}$

$$x \mapsto \frac{1}{n(1+x^2)}$$

Conv unif to 0 fnc.

Let $\varepsilon > 0$ be given

$\forall n \in \mathbb{N} \quad \forall x_0 \in \mathbb{R}$

$$|h_n(x_0) - h(x_0)| = \frac{1}{n(1+x^2)} \leq \frac{1}{n}$$

Use Arch to validate unif conv !

Thm Uniform Cauchy Criteria of Conv (UCC)

let $m \mapsto (f_m : I \to \mathbb{R})$ be a seq of fnc.

Then

$\exists\ f : I \to \mathbb{R}$ s.t $f_n \xrightarrow{\text{unif}} f$

$\Longleftrightarrow$

$\forall\ \varepsilon > 0\ \exists N \in \mathbb{N}$ s.t $\forall\ n, m > N\ \forall\ x_0 \in I$

$|f_n(x_0) - f_m(x_0)| < \varepsilon$

Pf $\Longrightarrow$

Let $\varepsilon > 0$ be given.

Due to unif conv $\exists N \in \mathbb{N}$ s.t $\forall\ n > N\ \forall x_0 \in I$

$|f_n(x_0) - f(x_0)| < \frac{\varepsilon}{57}$

So $\forall\ n, m > N\ \forall\ x_0 \in I$

$|f_n(x_0) - f_m(x_0)| = |f_n(x_0) - f(x_0) + f(x_0) - f_m(x_0)|$

$\leq |f_n(x_0) - f(x_0)| + |f_m(x_0) - f(x_0)| < \frac{2\varepsilon}{57}$

$\Longleftarrow$

We need a candidate limit function here!

fix $a \in I$ since $n \mapsto f_n(a)$ is Cauchy $\exists\ \ell_a$ to which it conv,

$f(a) := \ell_a = \lim_{n \to \infty} f_n(a)$

By construction, we have ptwise conv. We need unif conv!

Let $\varepsilon > 0$ be given $\exists N \in \mathbb{N}$ s.t $\forall\ n, m > N\ \forall x_0 \in I$

$|f_n(x_0) - f_m(x_0)| < \frac{\varepsilon}{57}$ in particular $|f_n(x_0) - f_{N+1}(x_0)| < \frac{\varepsilon}{57}$

$\longrightarrow$

So, by the lemma below, ①

$$\lim_{n\to\infty} |f_n(x_0) - f_{N+1}(x_0)| \le \lim_{n\to\infty} \frac{\varepsilon}{57}$$

$$\Rightarrow |f(x_0) - f_{N+1}(x_0)| \le \frac{\varepsilon}{57}$$

$\forall x_0 \in I$

So, $\forall n > I$ $\forall x_0 \in I$

$|f_n(x_0) - f(x_0)|$

$\le |f_n(x_0) - f_{N+1}(x_0)| +$

$|f_{N+1}(x_0) - f(x_0)|$

$< \frac{2\varepsilon}{57} < \varepsilon$ □

Lemma

① if (a_n), (b_n) are 2 conv seq of reals s.t (for sufficiently large n)

$a_n \le b_n$ then

$\lim_{n\to\infty} a_n \le \lim_{n\to\infty} b_n$

by 295 hw □

② If (a_n) is a seq of reals that conv then

$\lim_{n\to\infty} |a_n|$ exists and is

$|\lim_{n\to\infty} a_n|$

since abs val is cts □

Lemma "Uniform limit of cts func is cts"

let $I \subset \mathbb{R}$ be an iwimp.

let

$n \to (f_n : I \to \mathbb{R})$ st $f_n \in C^0(I)$

let $f: I \to \mathbb{R}$ st

$(f_n) \xrightarrow{\text{unif}} f$ Then $f \in C^0(I)$

Pf) We must show f is cts. Let $a \in I$ be given.

Let $\varepsilon > 0$ given. **Task:** find $\delta > 0$ s.t $\forall x \in B_\delta(a) \cap I$

$f(x) \in B_\varepsilon(f(a))$

Since $f_n \xrightarrow{\text{unif}} f$

$\exists N \in \mathbb{N}$ s.t $\forall n > N$, $\forall y \in I$

$|f_n(y) - f(y)| < \frac{\varepsilon}{171}$

★

since f_{N+1} is cts, it is cts at a, so

$\exists \delta > 0$ s.t. if $x \in B_\delta(a) \cap I$ $f(x) \in B_{\varepsilon/57}(f_{N+1}(a))$

Thus, for $x \in B_\delta(a) \cap I$

$|f(x) - f(a)|$

$= |f(x) - f_{N+1}(x) + f_{N+1}(x) - f_{N+1}(a) + f_{N+1}(a) - f(a)|$

$\leq |f(x) - f_{N+1}(x)| + |f_{N+1}(x) - f_{N+1}(a)| + |f_{N+1}(a) - f(a)|$

unif conv ... cts ... $a \in B_\delta(a) \cap I$, unif conv

$< \frac{3\varepsilon}{57} < \varepsilon$ $\square$

Note: Converse isn't true $f_n(x) = \frac{x^2}{n}$ $f_n : \mathbb{R} \to \mathbb{R}$

converges ptwise to 0 function

but not unif

Integrals I

Is the pointwise limit of integrable functions integrable?

No. HW2A

We want for $[a,b] \subseteq \mathbb{R}$ i.e. and $n \mapsto (f_n : [a,b] \to \mathbb{R})$

where $\forall n \in \mathbb{N}$ f_n is integrable on $[a,b]$ and

converges ptwise (only) to $f : [a,b] \to \mathbb{R}$.

① $\int_a^b f$ exists AND

② $\int_a^b f = \lim_{n \to \infty} \int_a^b f_n$

But this doesn't hold in general (hw2a has cntr case)

Thm Let $a, b \in \mathbb{R}$ with $a < b$

Let $n \mapsto (f_n : [a,b] \to \mathbb{R})$ be a seq of integrable functions.

If $f : [a,b] \to \mathbb{R}$ s.t $f_n \xrightarrow{\text{unif}} f$ on $[a,b]$ Then.

$$\boxed{\int_a^b \lim_{n\to\infty} f_n = \int_a^b f = \lim_{n\to\infty} \int_a^b f_n}$$

Pf We will use Darboux integrability criterion!

Let $\varepsilon > 0$ be given

By uniform convergence $\exists N \in \mathbb{N}$ s.t $\forall\, n > N \ \forall x_0 \in [a,b]$

$$|f(x_0) - f_n(x_0)| < \frac{\varepsilon}{17 (b-a)}$$

Since f_{N+1} is integrable, $\exists$ P partition of $[a,b]$ s.t

$$U(f_{N+1}, P) - L(f_{N+1}, P) < \frac{\varepsilon}{17}$$

$U(f,P) - L(f,P) = U(f,P) - U(f_{N+1},P) + U(f_{N+1},P) - L(f_{N+1},P) + L(f_{N+1},P) - L(f,P)$

↳ non negative so

$\leq |U(f,P) - U(f_{N+1},P)| + |U(f_{N+1},P) - L(f_{N+1},P)| + |L(f_{N+1},P) - L(f,P)|$

$< \varepsilon$ → Check hw 3 for details here ◻

$\Rightarrow \int_a^b f$ exists.

Now we show $\lim_{n\to\infty} \int_a^b f_n = f$. We note,

$\forall\, n > N$,

$$\left|\int_a^b f - \int_a^b f_n\right| = \left|\int_a^b (f - f_n)\right| < \left|\int_a^b \frac{\varepsilon}{17 (b-a)}\right| = \frac{\varepsilon}{17} < \varepsilon \quad ◻$$

$\therefore\ n \mapsto \int_a^b f_n$ conv to $\int_a^b f$

We see that derivatives don't work quite as well → Q16

Thm] Let $I \subseteq \mathbb{R}$ iwimp. Let $(f_n : I \to \mathbb{R})$ s.t. $f_n \in C^1(I)$ $\forall n \in \mathbb{N}$

Spose (f_n) and (f_n') satisfy UCCC.

Then $\exists$ $f \in C^1(I)$, $g \in C^0(I)$ s.t.

$f_n \xrightarrow{\text{unif}} f$, $f_n' \xrightarrow{\text{unif}} g$, $f' = g$ → limit of derivative is derivative of limit

Pf] By UCC, we already know, as $C^1(I) \subseteq C^0(I)$

$\exists$ $f, g \in C^1(I)$ s.t.

$f_n \xrightarrow{\text{unif}} f$, $f_n' \xrightarrow{\text{unif}} g$

We must show

1. $f \in C^1(I)$
2. $f' = g$

fix $a \in I$ we claim $\forall x \in I$

$f(x) = \int_a^x g + f(a)$ note:: g is integrable as it is cts

If we show this were done B/c by FTC

Claim ⟹ f is diffble with $f' = g$!

Claim] By FTC

$f_n(x) - f_n(a) = \int_a^x f_n'$ $\forall n \in \mathbb{N}$

By our integral theorem & FTC

$\int_a^x g = \lim_{n\to\infty} \int_a^x f_n' = \lim_{n\to\infty} (f_n(x) - f_n(a))$

$= f(x) - f(a)$ D

Corollary

Spose $p \in \mathbb{N} \cup \{0\}$ spose $m \to (f_m : I \to \mathbb{R})$ is a seq of fnc in $C^p(I)$ s.t

$m \to (f^{(j)}(m))$ is a seq that satisfies UCC $\forall\ 0 \le j < p+1$.

Then $\exists f \in C^p(I)$ s.t

$(f_m^j) \xrightarrow{\text{unif}} f^j \quad \forall\ 0 \le j < p+1$