Series

Defn) Let (a_n) be a series of reals. Define (s_m),

$$m \longmapsto \sum_{i=1}^{m} a_i$$

This is a sequence of partial sum.

Consider $\lim_{m\to\infty} s_m$

- If this limit exists, we say

$$\sum_{n=1}^{\infty} a_n := \lim_{n\to\infty} s_m$$

and we say the 'infinite series' is summable/converges

- if $\lim_{m\to\infty} s_m = \infty$

Then we say $\sum_{n=1}^{\infty} a_n$ diverges to ∞

$\forall N \in \mathbb{N}\ \exists M \in \mathbb{N}$ s.t. $\forall m > M$ we have $s_m \geq N$

- if $\lim_{m\to\infty} s_m = -\infty$

Then we say $\sum_{n=1}^{\infty} a_n$ diverges to $-\infty$

- Otherwise,

$\sum_{n=1}^{\infty} a_n$ diverges

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eg Consider $a_n = \left(\frac{1}{2}\right)^n$ and calculate $\sum_{n=1}^{\infty} a_n$

$$= \lim_{m\to\infty} \underbrace{\sum_{n=1}^{m} \frac{1}{2^n}}_{s_m} = \lim_{m\to\infty}\left(1 - \frac{1}{2^m}\right) = 1$$

note

$$s_m = \frac{1}{2} + \dots \frac{1}{2^m}$$

$$- \quad 2s_m = 1 + \dots \frac{1}{2^{m-1}}$$

$$-s_m = \frac{1}{2^m} - 1$$

eg 2 $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

$$S_m = \sum_{n=1}^{m} \frac{1}{n(n+1)} \to \text{telescopes}$$

$$\lim_{n\to\infty} S_m = \lim_{m\to\infty} 1 - \frac{1}{m+1} = 1$$

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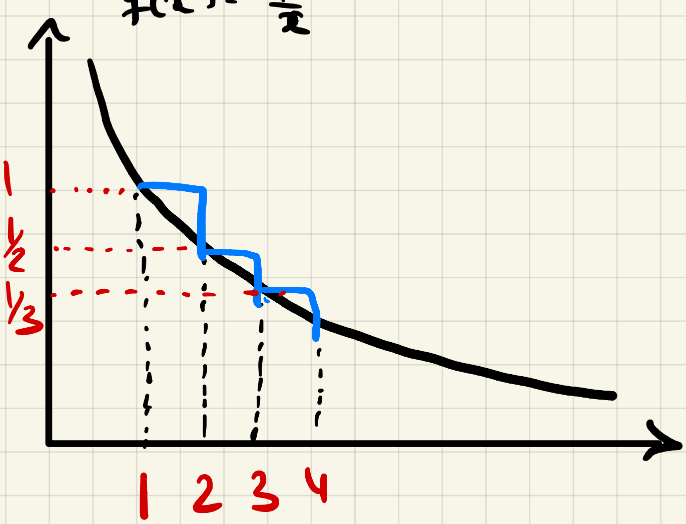
non. eg <u>Harmonic Series</u>

$$\sum_{n=1}^{\infty} \frac{1}{n} \longrightarrow$$



Consider the partition,

$P = \{1, \dots m\}$ of $[1, m]$

We have for $f : \mathbb{R}\setminus\{0\} \to \mathbb{R}$, $x \mapsto \frac{1}{x}$

$$U(f, P) = \sum_{i=1}^{m-1} \frac{1}{i} = S_{m-1} \geq \int_1^m \frac{1}{x} = \ln(m)$$

but also, $S_m = S_{m-1} + \frac{1}{m} \implies S_m > S_{m-1}$

so, $S_m > S_{m-1} \geq \int_1^m \frac{1}{x} = \ln(m)$

So, $\forall n \in \mathbb{N}$ $S_m > \ln(m)$ but $\lim_{m\to\infty} \ln(m) = \infty$

let $N \in \mathbb{N}$ be given $\exists M \in \mathbb{N}$ st $\forall m \geq M$ $\ln(m) > N$
so

$\forall n \geq M$ $S_n > \ln(n) > N$ ▷

So $\displaystyle\sum_{i=1}^{\infty} \frac{1}{i}$ div to ∞

Consecutive term difference can be arbitrarily small !

this satisfies it

note

$m \mapsto S_m$ div to ∞

$\therefore$ (S_m) is not cauchy

This means cauchy is not the same as

$\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t. $\forall n > N$

$|x_n - x_{n+1}| < \varepsilon$

Remark

- $\sum_1^\infty a_n = l$ & $\sum_1^\infty a_n = m \implies l = m$ by unique lim in Hausdorff
- $\sum_1^\infty a_n$ div to $\pm\infty$ Then $\sum_1 a_n$ doesn't converge!
- The convergence behaviour of $\sum_1 a_n$ doesn't change by fiddling with finitely many terms (b/c we only care about tail)

Lemma let $(a_n), (b_n)$ be seq of reals such that

$$\sum_1^\infty a_n = l, \quad \sum_1^\infty b_n = m, \quad m, l \in \mathbb{R}$$

and take $c \in \mathbb{R}$

Then

$$\sum_1^\infty (a_n + c \cdot b_n) = l + cm$$

Pf Immediate from linearity of sequences!

Cauchy Criteria for Convergence

$\sum_{t=1}^\infty a_t$ conv $\iff \lim_{k\to\infty} \sum_{i=1}^k a_i$ exists $\iff k \to \sum_{i=1}^k a_i$ is Cauchy

$\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ s.t $\forall n > m \geq N$ we have

$$\left| \sum_{i=1}^n a_i - \sum_{i=1}^m a_i \right| < \varepsilon$$

$$\iff \boxed{\left| \sum_{i=m+1}^n a_i \right| < \varepsilon}$$

Lemma Divergence Thm

If $\sum a_n$ conv then $\lim_{n\to\infty} a_n = 0$

Pf Spose $\sum_{n=1}^{\infty} a_n$ conv. Let $\varepsilon > 0$ be given

We want to show $\exists N \in \mathbb{N}$, st $\forall n > N$ $|a_n| < \varepsilon$

By Cauchy Criteria of Conv,

$\exists N \in \mathbb{N}$ s.t $\forall n > m > N$ $\left| \sum_{t=m}^{n} a_t \right| < \varepsilon$

set $m = n-1$ $\Rightarrow$ $m+1 = n$

$$\left| \sum_{t=m+1}^{n} a_t \right| = |a_n| < \varepsilon \quad \forall n > N \qquad \square$$

Caution This is not an iff consider $\sum_{1}^{\infty} \frac{1}{n}$

Contrapositive Take $\sum_{1}^{\infty} a_n$ if $\lim_{n\to\infty} a_n \neq 0$ then $\sum a_n$ doesn't converge

Lemma Spose (a_n) is a seq s.t $\exists N \in \mathbb{N}$ st $\forall n > N$ a_n has the same sign (pos or neg).

Then

The seq of partial terms is bdd $\iff$ $\sum a_n$ conv

Pf By the hypothesis, the seq of partial terms is eventually monotone.

By 295 conv $\iff$ bdd !

Corollary Comparison Test

Spose (a_n) & (b_n) st $0 \le a_n \le b_n$ $\forall$ n suff large

Then,

$\sum b_n$ conv $\Rightarrow$ $\sum a_n$ conv

$\sum a_n$ div to ∞ $\Rightarrow$ $\sum b_n$ div to ∞

Pf Wlog $0 \leq a_n \leq b_n \quad \forall n \in \mathbb{N}$. Then $\forall m \in \mathbb{N}$

$$\sum_{n=1}^{m} a_n \leq \sum_{n=1}^{m} b_n \leq \lim_{m\to\infty} \sum_{n=1}^{m} b_n$$ due to monotonicity of partial sums

① If $\sum_{1}^{\infty} b_n$ conv to some $x \in \mathbb{R}$ ((conv div to ∞))

Then $n \mapsto \sum_{n=1}^{m} a_n$ is bdd by x $\forall m \in \mathbb{N}$

Since $\nearrow$ is mono $\uparrow$ it converges too!

So $\sum_{n=1}^{\infty} a_n$ converges!

② If $\sum_{n=1}^{\infty} a_n$ div to ∞ then $\sum_{n=1}^{\infty} b_n$ must div to ∞

as, we have shown, if $\sum b_n$ conv then $\sum a_n$ conv

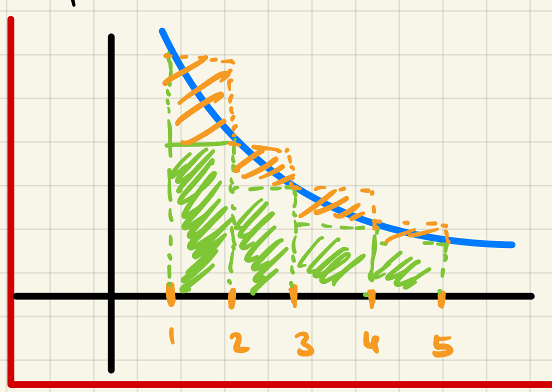
Lemma Integral Test

Spose $g: [1, \infty) \to \mathbb{R}_{\geq 0}$ s.t g is integrable ($\forall a \geq 1 \int_1^a g$ exists) & non-increasing

Then

$$\sum_{n=1}^{\infty} g(n) \text{ conv} \iff \int_1^{\infty} g := \lim_{A\to\infty} \int_1^A g \text{ exists!}$$

Pf



by considering the trivial partition of $[1, \infty)$ and using the lower & upper darboux sum (noting that g is non increasing)

$$g(2) \leq \int_1^2 g \leq g(1)$$

$$g(2)+g(3) \leq \int_1^3 g \leq g(1)+g(2)$$

$$\vdots$$

$$\forall n \in \mathbb{N}_{\geq 2} \implies \sum_{j=2}^{n} g(j) \leq \int_1^n g \leq \sum_{j=1}^{n-1} g(j)$$

Let $S_n = \sum_{j=2}^{n} g(j) \quad I_n = \int_1^n g \quad T_n = \sum_{j=1}^{n-1} g(j)$

$\forall\ n \geq 2$ we have $S_n, I_n, T_n \geq 0$ & $\leftarrow$ by hypothesis

$S_n \leq I_n \leq T_n$

we note $(S_n), (T_n)$ (I_n) are mono $\uparrow$

$\Rightarrow$

if $\int_1^\infty g$ exists $\Rightarrow \lim_{n\to\infty} \int_1^n g$ exists

$\therefore\ n \mapsto S_n$ is mono $\uparrow$ & bdd $\uparrow$ so it converges!

$\Rightarrow \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{j=2}^{n} g(j) \Rightarrow \lim_{n\to\infty} \sum_{j=1}^{n} g(j)$ exists! D

$\Leftarrow$

If $\sum_{j=1}^{\infty} g(j)$ conv $\Rightarrow \lim_{n\to\infty} \sum_{j=1}^{n} g(j)$ conv $\Rightarrow \lim_{n\to\infty} \sum_{j=1}^{n-1} g(j)$ conv!

$\Rightarrow \lim_{n\to\infty} T_n$ conv.

$\therefore\ I_n$ is mono $\uparrow$ & bdd above $\Rightarrow$ it conv

$\therefore\ \lim_{n\to\infty} \int_1^n g$ exists

D

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eg. ① $\int_1^\infty \frac{1}{t}$? $\quad g(t) = \frac{1}{t}$ & $\sum_{n=1}^{\infty} \frac{1}{n}$ ... therefore the integral diverges

② for $s>1$ define

$\zeta : (1,\infty) \to \mathbb{R}$ ——— Riemann - Zeta function

$s \longmapsto \sum_{n=1}^{\infty} \frac{1}{n^s}$

Use integral test to prove that this is well defined!

E.g $\lim_{A\to\infty} \int_1^\infty \frac{1}{x^3}$ exist ?

Lemma) Limit Comparison Test

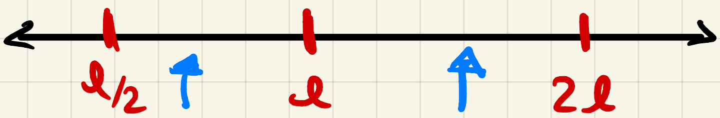
Let $(a_n), (b_n)$ be seq of positives & $\lim_{n\to\infty} \frac{a_n}{b_n} = \ell$

If $\ell > 0$ then

$$\sum a_n \text{ conv} \iff \sum b_n \text{ conv}$$

Pf) Since $\ell > 0$ $\exists N \in \mathbb{N}$ s.t. $\forall n > N$

$$\frac{\ell}{n} < \frac{a_n}{b_n} < 2\ell \quad \to \text{take } \varepsilon = \frac{\ell}{2}$$



$\forall n > N$, $\frac{b_n}{2} \cdot \ell < a_n < 2\ell b_n$

by comparison test, we are done! $\square$

Lemma) Limit Ratio Test

Spose $n \mapsto a_n$ is a seq of positives. Spose.

$\ell = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ exists

Then,
1. $\ell < 1 \implies \sum a_n$ conv
2. $\ell = 1 \implies$ inconclusive
3. $\ell > 1 \implies \sum a_n$ diverges

Lemma) (geometric series)

$\sum_{n=1}^{\infty} a^n$ conv to $\frac{1}{1-a}$ for $|a| < 1$ & div otherwise!

Pf) By hw $\implies \sum_{n=0}^{m} a^n = \frac{1-a^m}{1-a} = S_m$

if $|a| < 1$

$$\lim_{n\to\infty} S_m = \lim_{m\to\infty} \frac{1-a^m}{1-a} = \frac{1}{1-a}$$

if $|a| \geq 1$

$\lim_{m\to\infty} \frac{1-a^m}{1-a}$ dne by div test!

Pf] Ratio

1. $\ell < 1$ choose c s.t $\ell < c < 1$

since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \ell$ $\exists N \in \mathbb{N}$ st $\forall n > N$

$\frac{a_{n+1}}{a_n} < c$ ((taking $\varepsilon = c - \ell$))

$\Rightarrow a_{n+1} < c \cdot a_n \quad \forall n > N$

So, $\forall m \in \mathbb{N}$

$a_{N+1+m} < c \cdot a_{N+1(m-1)}$ repeat process

$< c^2 \cdot a_{N+1(m-2)} < \cdots c^m a_{N+1}$

$\forall k > N$

$$\sum_1^k a_n = \sum_1^N a_n + \sum_{N+1}^k a_n < \sum_1^N a_n + a_{N+1} \sum_{j=0}^{k-(N+1)} c^j$$

$$= \sum_1^N a_n + a_{N+1}\left(\frac{1 - c^{k-N}}{1-c}\right) \quad \text{as } 0 < c < 1$$

$$< \sum_1^N a_n + a_{N+1}\left(\frac{1}{1-c}\right)$$

$\forall k \in \mathbb{N}$

$S_k = \sum_{n=1}^k a_n < \text{constant}$

since (a_n) are positive, $k \mapsto S_k$ is mono $\uparrow$

$\therefore \sum_1^\infty a_n$ conv !

$\square$

$\longrightarrow$

② $l > 1$

Choose δ s.t $1 < \delta < l$ $\exists N \in \mathbb{N}$ s.t

$\forall n > N$ $\delta < \frac{a_{n+1}}{a_n} \Rightarrow \delta a_n < a_{n+1}$

$\forall m \in \mathbb{N}$

$a_{N+1+m} > \delta^m \cdot a_{N+1} > a_{N+1} > 0$ → lower bound for limit

$\Rightarrow \lim_{n \to \infty} a_n \neq 0$

So by div test we have $\sum a_n$ div to ∞ as all $a_n > 0$

$\triangleright$

Lemma) Alternating Series Test!

Spse $n \mapsto a_n$ is non-reals

① $\lim_{n \to \infty} a_n = 0$

② $a_1 \geq a_2 \geq a_3 \ldots\ldots$ mono ↓

Then $\sum (-1)^{n+1} a_n$ converges!

Pf) Consider the sequence of partial sums!

$S_1 = a_1$

$S_2 = a_1 - a_2$

$S_3 = a_1 - a_2 + a_3$

we not by monotonicity of (a_n)

$l \mapsto S_{2l-1}$ (odds)

$k \mapsto S_{2k}$ (evens)

$S_1 \geq S_3 \geq S_5 \ldots$ odds are mono ↓

$S_2 \leq S_4 \leq S_6 \ldots$ even are mono ↑

we also note

$s_1 \geq s_2$, $s_3 = s_2 + a_3 \geq s_2$

$s_5 = s_4 + a_5 \geq s_4 \geq s_2 \ldots\ldots$

$\therefore (\ell_n)$ is mono $\downarrow$ & bdd below by s_2

(k_n) is mono $\uparrow$ & bdd above by s_1

So,

$$\xi := \lim_{k\to\infty} S_{2k} \qquad \theta := \lim_{\ell\to\infty} S_{2\ell-1} \qquad \text{well def!}$$

Claim $\sum(-1)^{n+1}$ to conv, we need to show $\lim_{n\to\infty} s_n$ exists & we are done if $\xi = \theta$

note: $\xi \leq \theta$

Why are we done if $\xi = \theta$ ((must show $\lim_{n\to\infty} s_n$ exist)

Set $\varepsilon > 0$

Spose $\ell = \xi = \theta$

Since $\lim_{k\to\infty} S_{2k} = \xi = \ell \;\exists N_1 \in \mathbb{N}$ st $\forall n > N_1$

$$|s_{2n} - \ell| < \frac{\varepsilon}{57}$$

Similarly $\exists N_2$ st $\forall n > N_2$

$$|s_{2n-1} - \ell| < \frac{\varepsilon}{57}$$

indexing purposes

Set $N = \max(2N_1, 2N_2 - 1)$

$\forall n > N$ we have if n is even $n > 2N_1 \ldots\ldots$

$|s_n - \ell| < \varepsilon/57 \Rightarrow \lim_{n\to\infty} s_n = \ell$

$\Rightarrow \sum(-1)^{n+1}(a_n) = \ell$ $\square$

Now show $\xi = \Theta$

We know, $\Theta \geq \xi$ so

$$0 \leq \Theta - \xi = \lim_{n\to\infty} (S_{2n+1} - S_{2n}) = \lim_{n\to\infty} a_{2n+1} = 0$$

$\square$

Defn A series $\sum a_n$ is absolutely conv iff $\sum |a_n|$ converges

if $\sum a_n$ conv but $\sum |a_n|$ doesn't then $\sum a_n$ is conditionally convergent!

lemma If $\sum a_n$ is absolutely convergent, it is conditionally convergent

That is $\sum |a_n|$ conv $\Rightarrow \sum a_n$ conv.

Pf Let $\varepsilon > 0$ be given $\exists N \in \mathbb{N}$ st $\forall\ n > m > N$

$$\left| \sum_{i=m+1}^{n} |a_i| \right| < \varepsilon \quad \text{by } \underline{CCC}$$

$$\Rightarrow \sum_{i=m+1}^{n} |a_i| < \varepsilon \quad \text{since } \sum_{i=m+1}^{n} |a_i| \text{ is non neg}$$

$$\Rightarrow \left| \sum_{i=m+1}^{n} a_i \right| \leq \sum_{i=m+1}^{n} |a_i| < \varepsilon \quad \text{by triangle !}$$

$\therefore \sum_{i=1}^{\infty} a_i$ converges by CCC

Defn A rearrangement of a series $\sum a_n$ is a new series $\sum b_n$ where

$$b_n := a_{\varrho(n)}$$

for bijection $\varrho : \mathbb{N} \to \mathbb{N}$

Thm Riemann Rearrangement Result.

If $\sum a_n$ is conditionaly convergent then $\forall \alpha \in \mathbb{R} \cup \{\pm\infty\}$ $\exists g : \mathbb{N} \to \mathbb{N}$ such that

$\sum a_{g(n)}$ conv to α!

Pf This result follows from Criteria of conv (below) and thm 7 ch 23 of spivak $\square$

Criteria of Convergence

Spose $\sum a_n$ converges.

Let (p_m) be the seq of positive terms (reindexed)

Let (n_m) be the seq of negative terms (reindexed)

Then:

$\sum a_m$ is conditionally convergent

$\iff$

$\sum p_m = \infty$ OR $\sum n_m = \infty$ → can be made AND

Pf We will show

$\sum a_m$ is abs conv iff $\sum p_m$ AND $\sum n_m$ conv

$\Rightarrow$ Spose $\sum a_m$ conv absolutely

Define the associated seq

$$\left.\begin{aligned} a_m^+ &:= \max\{a_m, 0\} \geq 0 \\ a_m^- &:= \min\{a_m, 0\} \geq 0 \end{aligned}\right]$$

note: $\sum a_m^+ = \sum p_m$, $\sum a_m^- = \sum n_m$

$\sum p_m = \sum a_m \leq \sum |a_m|$ by comparison done

$0 \leq \sum (-1) \cdot n_m = \sum (-1) a_m^- \leq |a_m|$

by comparison test $\sum (-1) \cdot n_m$ conv. By linearity done!

$\square$

$\Leftarrow$

We assume $\sum p_m$ and $\sum n_m$ converge.

$a_m^+ = \max\{a_m, 0\} \geq 0$

$a_m^- = \min\{a_m, 0\} \leq 0$

$a_m = a_m^+ + a_m^-$ (one is always 0)

$|a_m| = a_m^+ - a_m^-$

$$\sum |a_m| = \sum a_m^+ - a_m^-$$

$$= \sum a_m^+ - \sum a_m^-$$

$$= \text{converge}$$

linearity? yes!

$\square$

By contrapositive,

$\sum a_m$ is conditionally conv iff

$\sum p_m = \infty$ (or) $\sum n_m = -\infty$ (by monotonicity)

$\hookrightarrow$ can be promoted to AND

Corollary) Spose $\sum a_n$ is absolutely convergent. Let (b_n) be a rearrangement of (a_n). Then $\sum b_n = \sum a_n$

Pf) Let $\varepsilon > 0$ be given since $\sum a_n$ is abs conv. $\exists\, \ell, \hat{\ell}$ st

$\sum a_n = \ell$, $\sum |a_n| = \hat{\ell}$

We want to show $\sum b_n = \ell$

We can choose $M \in \mathbb{N}$ s.t. $\forall\ m > M$

$$\left| \sum_{n=1}^{m} a_n - \ell \right| < \frac{\varepsilon}{57}$$

take $m = M+1$ & unpack

$$\left| \sum_{n=1}^{m} |a_n| - \hat{\ell} \right| < \frac{\varepsilon}{57} \implies \sum_{n=M+1}^{\infty} |a_n| < \frac{\varepsilon}{57}$$

$\exists\ M' > M+1$ such that

$\{a_1 \ldots a_M\} \subseteq \{b_1 \ldots b_{M'}\}$ They need to be contained somewhere

So $\forall\ q > M'$ we have

$$\left| \sum_{i=1}^{q} b_i - \ell \right|$$

$$= \left| \sum_{i=1}^{q} b_i - \sum_{i=1}^{q} a_i + \sum_{i=1}^{q} a_i - \ell \right|$$

$$\leq \left| \sum_{i=1}^{q} b_i - \sum_{i=1}^{q} a_i \right| + \left| \sum_{i=1}^{q} a_i - \ell \right|$$

all of the a_i for $i \leq M$ cancel out and we have 2 of the tail parts. In the worst case

$$\leq 2 \sum_{n=M+1}^{\infty} |a_n| + \frac{\varepsilon}{57} < \frac{3\varepsilon}{57} \qquad \square$$

Q: Does $\sum b_n$ conv absolutely.

Yes! Since $\sum |b_n|$ is a rearrangement of $\sum |a_n|$