Setting: V is a vs $/\mathbb{R}$

Defn) An inner product on V is a func

$$V \times V \longrightarrow \mathbb{R}$$
$$(v, w) \longmapsto \langle v, w\rangle$$

That satisfies

1. It is bilinear
2. $\forall v \in V$ $\langle v, v\rangle \geq 0$ and $\langle v, v\rangle = 0 \Rightarrow v = 0$
3. $\forall v, w \in V$, $\langle \bar{v}, \bar{w}\rangle = \langle \bar{w}, \bar{v}\rangle$

Observ) (1) By prop 1, if we fix $\bar{w}$ the func

$$V \longrightarrow \mathbb{R}$$
$$v \longmapsto \langle \bar{v}, \bar{w}\rangle$$

is linear. So it is in the dual!

(2) $\langle \bar{v}, \bar{0}\rangle = \langle \bar{v}, \bar{0} + \bar{0}\rangle = \langle \bar{v}, 0\rangle + \langle \bar{v}, 0\rangle$

$\Rightarrow \langle \bar{v}, \bar{0}\rangle = 0$!

E.g (1) Standard dot product $V = \mathbb{R}^n$

$$\left\langle \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \right\rangle = \sum_{i=1}^{n} x_i y_i$$

(2) $V = C^0([0,1])$

$$\langle f, g\rangle = \int_0^1 f(t)\, g(t)\, dt$$

A inner product gives us a norm!

Defn) let V be a ips $/\mathbb{R}$

We define, for $v \in V$, $\|\bar{v}\| := \sqrt{\langle \bar{v}, \bar{v} \rangle}$

↳ gives us a norm!

↳ hw 10

A norm gives a metric where

$d(\bar{v}, \bar{w}) := \|\bar{v} - \bar{w}\| = \sqrt{\langle \bar{v}-\bar{w}, \bar{v}-\bar{w} \rangle}$

→ is it a norm

① $\|\bar{v}\| = 0 \iff \bar{v} = 0$

② $\|c\bar{v}\| = |c| \cdot \|\bar{v}\|$

③ $\|\bar{v} + \bar{w}\| \leq \|v\| + \|w\|$

①, ② are immediate

③ requires justification

Inner products tell us abt angles!

Defn) let V be ips $/\mathbb{R}$.

let $\bar{v}, \bar{w} \in V$

They are said to be orthogonal p.t $\langle \bar{v}, \bar{w} \rangle = 0$

Rem: $\bar{0}$ is orthogonal to everything.

Defn) let V be ips $/\mathbb{R}$.

The vector $\bar{v} \in V$ is a unit vec p.t

$\|\bar{v}\| = \sqrt{\langle v, v \rangle} = 1$

Note: if $\bar{w} \in V$ is nonzero, $\dfrac{\bar{w}}{\|\bar{w}\|}$ is a unit vec

Q) Does every vs $/\mathbb{R}$ have an inner product?

A) Yes! (ish)

If V is f.g, let $W = (v_1 \dots v_n)$ be a basis

if $\bar{v}, \bar{w} \in V$.

$$v = \sum a_i v_i \qquad w = \sum b_i v_i$$

The following is a inner product

$$\langle v, w \rangle = \sum a_i b_i$$

Q) we said any ip gives a norm.

Does every norm come from an ip?

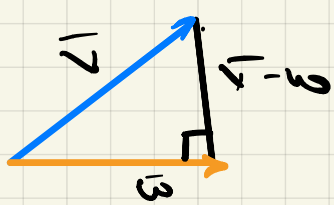
A) No. The p-norm, if $p \neq 2$, doesn't

e.g $p = 57$

Motivating stuff

The standard inner product on $\mathbb{R}^2$ gives us len, orthogonality, and a notion of angle

- Assuming we have law of cosine. We got the following from nonzero vec $\bar{v}, \bar{w} \in \mathbb{R}$



Cos rule gives

$$\|v-w\|^2 = \|v\|^2 + \|w\|^2 - 2\|\bar{v}\|\|\bar{w}\| \cos(\theta)$$

$$= \langle v-w, v-w \rangle - \langle v, v \rangle + \langle w, w \rangle - 2\|\bar{v}\|\|\bar{w}\| \cos\theta$$

Using multilinearity

$$= \langle \bar{v}, \bar{v} \rangle - \langle v, -w \rangle - \langle w, -v \rangle + \langle w, w \rangle$$

$$\Rightarrow -2\langle \bar{v}, w \rangle = -2\|\bar{v}\|\|\bar{w}\| \cos\theta$$

$$\Rightarrow \cos\theta = \frac{\langle \bar{v}, \bar{w} \rangle}{\|\bar{v}\|\|\bar{w}\|} \Rightarrow \left| \frac{\langle \bar{v}, \bar{w} \rangle}{\|\bar{v}\|\|\bar{w}\|} \right| \leq 1$$

insist non zero vec

Cauchy Schwarz

Thm Cauchy Schwartz

Let V be an ips $/\mathbb{R}$ for $\bar{v}, \bar{w} \in V$

$$|\langle \bar{v}, \bar{w} \rangle| \leq \|\bar{v}\| \cdot \|\bar{w}\|$$

with equality iff $\bar{v}, \bar{w}$ are lin dep!

Pf Fix $\bar{v}, \bar{w} \in V$. $\longrightarrow$ warm up

If $\bar{v}, \bar{w}$ are lin dep $\bar{v} = c\bar{w}$ for some $c \in \mathbb{R}$

LHS: $|\langle v, w \rangle| = |\langle cw, w \rangle| = |c\langle w, w \rangle|$

$= |c||\langle w, w \rangle|$

RHS: $\|\bar{v}\| \|\bar{w}\| = \|cw\| \|w\| = |c| \|w\|^2$

$= |c| |\langle w, w \rangle|$

Now, fix $\bar{v}, \bar{w} \in V$.

We want $|\langle v, w \rangle| \leq \|\bar{v}\| \cdot \|\bar{w}\|$.

If $\bar{w} = \bar{0}$, we're done. So sps this isn't the case

If we can show that the statement holds for a unit vector w, we are done for all non-zero cases

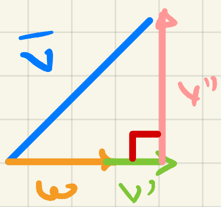
↳ follows from scale invariance

↳ if it holds for $\bar{w}$, it will $c\bar{w}$

Spose $\|\bar{w}\| = 1$

Show $|\langle v, w \rangle| \leq \|\bar{v}\|$

Motivation



sorta $\bar{v}$'s length in $\bar{w}$'s direct

$v' := \langle \bar{v}, \bar{w} \rangle \cdot \bar{w}$

$v'' := \bar{v} - \bar{v}'$

We want to show a right angle.

$\Rightarrow \|v\|$ is the largest

↳ show v', v'' are orth!

Show, $\langle v', v'' \rangle = 0$

So, $\langle v', v'' \rangle = \langle v', v - v' \rangle = \langle v', \bar{v} \rangle - \langle v', v' \rangle$

$= \langle \langle v,w \rangle w, v \rangle - \langle \langle v,w \rangle \bar{w}, \langle v,w \rangle \bar{w} \rangle$

$= \langle v,w \rangle \langle w,v \rangle - \langle v,w \rangle^2 \underbrace{\langle \bar{w}, \bar{w} \rangle}_{\to 1}$

$= \langle v,w \rangle^2 - \langle v,w \rangle^2 = 0$

Recall, $\bar{v} = \bar{v}'' + \bar{v}'$

$\langle \bar{v}, \bar{v} \rangle = \langle v'' + v', v'' + v' \rangle$

$= \langle v'', v'' \rangle + 2 \underbrace{\langle v'', v' \rangle}_{\to 0} + \langle v', v' \rangle$

$= \|v''\|^2 + \langle v', v' \rangle \to$ by above

$= \|v''\|^2 + \langle v,w \rangle^2$

$\Rightarrow \|v\| = \sqrt{\langle \bar{v}, \bar{v} \rangle} \geq \langle v,w \rangle$ D.

Almost there ... need the other direction of equality

to get this, we must show, if

$\langle v, v \rangle = \langle \bar{v}, w \rangle^2 \Rightarrow \vec{v}, \vec{w}$ are lin dep.

Spose $\langle \bar{v}, w \rangle^2 = \langle v, v \rangle$

So, $\langle v,w \rangle^2 - \|\bar{v}\|^2 = 0$

We saw that,

$\|v''\|^2 = \|v\|^2 - \langle v,w \rangle^2 = 0$

$\Rightarrow v'' = 0 \Rightarrow 0 = v - v'$

$0 = v - \langle v,w \rangle \bar{w}$

So, $\therefore \bar{v} = \langle v,w \rangle \bar{w}$ D.

Hubbard's pf page 57 week 13!

With CS we can now define the angle between 2 vecs!
for $v, w \in V$

$$\cos(\theta) = \frac{\langle \bar{v}, \bar{w} \rangle}{\|\bar{v}\| \, \|\bar{w}\|}$$

θ is the unique angle in $[0, \pi]$ satisfying

CS guarantees the above is in $[-1, 1]$
With this, we get law of cos for free!
↳ see convad for sine law!

yet to show triangle inequality.

Lemma) let V be an IPS /$\mathbb{R}$
$\forall v, w \in V$ $\quad \|v+w\| \leq \|v\| + \|w\|$
moreover equality holds iff
$\bar{v} = 0$ or $w \in \mathbb{R}_{\geq 0} \bar{v}$ → condition blue

Pf) Fix $v, w \in V$.
Claim: it is enough to show

$$\langle v+w, v+w \rangle \leq \langle v, v \rangle + 2\|v\| \, \|w\| + \langle w, w \rangle$$

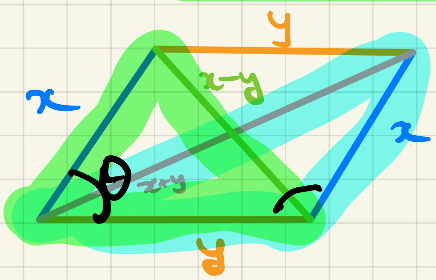
with equality iff blue
→ this is simply squaring both sides.
lets unpack the left side!

$\langle v+w, v+w \rangle = \langle v, v \rangle + 2\langle v, w \rangle + \langle w, w \rangle$ → done!
↳ from CS, we have $|\langle v, w \rangle| \leq \|v\| \, \|w\|$ □
with equality iff v, w are lin dep.
But $\langle v, w \rangle \leq |\langle v, w \rangle|$ with equality iff $\langle v, w \rangle \geq 0$
So, $\langle v, \alpha \bar{v} \rangle \geq 0 \Rightarrow \alpha \|v\|^2 \geq 0 \Rightarrow v = 0$ or $\alpha \geq 0$

Corr) inner products give norms!

Note: We mentioned that there are norms that don't come from inner products

Fun F) A norm $\|\cdot\|$ comes from a ip $\iff$ it satisfies the parallelogram law



$$\|x-y\|^2 = \|x\|^2 + \|y\|^2 - 2\|x\|\|y\| \cos\theta$$

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\|x\|\|y\| \cos(\theta)$$

$$= -\cos(180-\theta)$$

by adding the two, $\forall\, x, y$

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

Defn) let V be a ips $/\mathbb{R}$.
An indexed fam $(v_i \mid i \in I)$ is orthonormal st

$$\langle v_i, v_j \rangle = \begin{cases} 0 & \text{if } i \neq j \longrightarrow \text{ortho} \\ 1 & \text{if } i = j \longrightarrow \text{normal} \Rightarrow \|v_i\| = 1 \end{cases}$$

notes) if $(v_i \mid i \in I)$ is on, calculations are easy.

lemma) Spse $\mathcal{B} = (e_i \mid i \in I)$ is an onb of V.
$\forall\, v \in V$ $\quad v = \sum_{i \in I} \langle e_i, v \rangle e_i$

Pf) fix $v \in V$, $\exists\, a_i$ st

$$v = \sum_{i \in I} a_i e_i$$

$\forall\, j \in I$ $\quad \langle v, e_j \rangle = \langle \sum_{i \in I} a_i e_i, e_j \rangle$

$$= \sum_{i \in I} a_i \langle e_i, e_i \rangle = a_i \langle e_i, e_i \rangle$$

$$= a_i \qquad \square$$

also, we get.

If $v = \sum_{i \in I} a_i e_i \Rightarrow a_i = \langle v, e_i \rangle$

$w = \sum_{i \in I} b_i e_i \Rightarrow b_i = \langle w, e_i \rangle$

we have, $\langle v, w \rangle = \sum_{i \in I} a_i b_i$ → again comes from multilin & on

Lemma) Spose V is fgips /$\mathbb{R}$.

And $e = (e_1, \dots, e_n)$ is a onb.

fix $T \in \hom(V, V)$.

set $c := \left(\sum_{i=1}^{n} \|T(e_i)\|^2 \right)^{1/2}$

$\forall v \in V$,

$\|T(v)\| \le c \cdot \|v\|$

Pf) Fix $v \in V$.

$v = \sum_{i \in \mathbb{N}_n} a_i e_i$

$\|v\| = \sqrt{\langle v, v \rangle} = \sqrt{\sum_{i \in \mathbb{N}_n} a_i^2}$

$\|T(v)\| = \|T(\sum_{i \in \mathbb{N}_n} a_i e_i)\|$

$= \| \sum_{i \in \mathbb{N}_n} T(a_i e_i) \| \le \sum_{i \in \mathbb{N}_n} \|T(a_i e_i)\|$ → triangle

$= \sum_{i \in \mathbb{N}_n} |a_i| \, \|T(e_i)\|$

But, we see that, this is can be represented as

$u = \sum_{i \in \mathbb{N}_n} |a_i| e_i \qquad u' = \sum_{i \in \mathbb{N}_n} \|T(e_i)\| e_i$

$\|T(v)\| \le \langle u, u' \rangle \le |\langle u, u' \rangle|$ ↩ Cauchy Schwarz

$\le \|u\| \cdot \|u'\| = \|v\| \cdot c$

Lemma) Let V be ips $/\mathbb{R}$.

Let $(v_i \mid i \in I)$ be orthogonal.

Then: $(v_i \mid i \in I)$ is ~~Automatically~~ lin ind!

Pf) Spose $\sum_{i \in I} c_i v_i = 0$

$\forall j \in I$

$$0 = \langle 0, v_j \rangle = \langle \sum_{i \in I} c_i v_i , v_j \rangle$$
$$= \sum_{i \in I} c_i \langle v_i , v_j \rangle = c_j \langle v_j , v_j \rangle$$
$$= c_j$$

D

Thm | Gram-Schmidt Orth Process

Let V be an ips $/\mathbb{R}$.

Spose $(v_1, \dots v_m)$ are lin ind.

$\exists!$ orn fam $(u_1, \dots u_m)$ s.t

① $\mathrm{span}(v_1 \dots v_\ell) = \mathrm{span}(u_1, \dots u_\ell) \quad \forall \ell \in \mathbb{N}_m$

② $\langle v_i , u_i > 0 \quad \forall\ 1 \le i \le m$

Pf) The existence pf is constructive

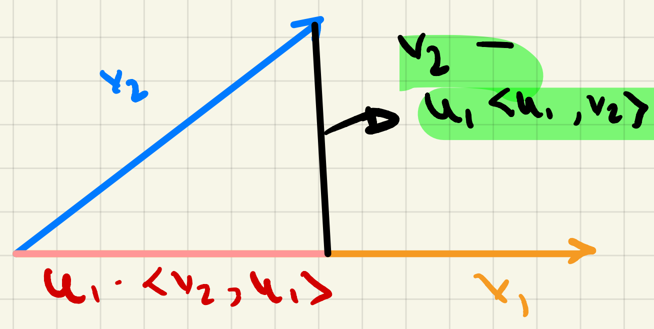
The uniqueness pf is in a Conrad Handout!

m=1| $u_1 = \frac{v_1}{\|v_1\|} \rightarrow \ > 0$ as $v_1 \neq 0$

$\Rightarrow \langle u_1 , v_1 \rangle = \langle \frac{v_1}{\|v_1\|} , v_1 \rangle > 0$

m=2| (v_1, v_2)

take $u_1 = \frac{v_1}{\|v_1\|}$



is $v_2 - u_1 \langle u_1 , v_2 \rangle$ orn to u_1?

$\langle v_2 - \langle u_1, v_2\rangle u_1 , u_1 \rangle$

$= \langle v_2 , u_1 \rangle - \langle u_1 , v_2 \rangle \cancelto{1}{\langle u_1 , u_1 \rangle}$

$= \boxed{0}$ so yes.

Take, $u_2 = \dfrac{v_2 - \langle u_1 , v_2 \rangle \cdot \overline{u}_1}{\| v_2 - \langle u_1 , v_2 \rangle u_1 \|}$

$\hookrightarrow$ not zero as it would imply u_1 is a scalar mult of v_2.

$\Rightarrow$ v_2 is scalar mult of v_1 oops

<u>Check</u> 1 span

2 $\langle u_1 , v_1 \rangle > 0$, $\langle u_2 , v_2 \rangle > 0$ ✓

<u>m=3</u> Same for u_1 , u_2

$$u_3 = \frac{v_3 - (\langle u_1 , v_3 \rangle u_1 , \langle u_2 , v_3 \rangle u_2)}{\| v_3 - (\langle u_1 , v_3 \rangle u_1 , \langle u_2 , v_3 \rangle u_2) \|}$$

D

rinse & repeat !