Exploring the dual basis

Let V be a fgips $/\mathbb{R}$

Let $e = (e_1, \dots, e_n)$ be an onb of V.

$$V \xrightarrow{\varphi} V^*$$

$$\bar{v} \longrightarrow \lambda_{\bar{v}} : V \to \mathbb{R}$$
$$y \longmapsto \langle \bar{v}, \bar{y} \rangle$$

We note that $\varphi(e)$ is a basis of V.

So is e^*. How do they relate?

Well, we see that for e_1

$\lambda_{e_1}(e_2) = \langle e_1, e_2 \rangle = 0$

$\lambda_{e_1}(e_3) = 0$

$\vdots$

$\lambda_{e_1}(e_i) = 0 \quad \forall i \in \mathbb{N}_n \quad i \neq 1$ (note $\lambda_{e_1}(e_1) = \langle e_1, e_1 \rangle = \sqrt{\langle e_1, e_1 \rangle^2} = \sqrt{1} = 1$

in general,

$$\lambda_{e_k}(e_\ell) = \begin{cases} 0 & \text{if } \ell \neq k \\ 1 & \text{if } k = \ell \end{cases}$$

So, $\varphi(e) = (\lambda_{e_1} \dots \lambda_{e_n}) = e^*$!

Now, for a given $T \in \hom(V, V)$,

What can we say abt $_e[T^*]_e$? Well...

$$_e[T^*]_e = {}_e[\varphi^{-1} \circ T^* \circ \varphi]_e$$
$$= {}_e[\varphi^{-1}]_{e^*} \cdot {}_{e^*}[T^*]_{e^*} \cdot {}_{e^*}[\varphi]_e$$
$$= {}_e[T]_e^t$$

These are identity!

$T(e) = e^*$

If e is onb of V, $_e[T^*]_e = {}_e[T]_e^t$

Caution: ppl use T^* as notation for T^t

Cor) for char poly, we have .

$$\chi_T(\lambda) = \chi_{T^+}(\lambda)$$

$\hookrightarrow$ The char poly of T is the same as T^+

More exciting

Def) A lin trans T is self adj st (obviously with setup)

$$T = T^+$$

Rem) note $T: V \to V$ is self adj iff $\forall$ onb e of V

$$_e[T]_e = {_e[T^+]_e}$$

But,

$$_e[T^+]_e = ({_e[T]_e})^t$$

$\Rightarrow$

$$_e[T]_e = {_e[T]_e}^t \rightarrow \text{symmetric}$$

So, T is self adj $\Leftrightarrow$

$\forall$ e onb of V $_e[T]_e = {_e[T]_e}^t$

This can be promoted to a single onb

Def) let $A \in Mat_{n \times n}(F)$

A is symmetric st $A^t = A$

eg

① let V be a fdips $\mathbb{R}$.

Let $W \leq V$ be a subsp.

$Proj_W : V \to V$ is self-adj

Pf) let $x, y \in V$

$\exists! \ (\bar{w}, \bar{z}) \in W \oplus W^\perp$

and

$(w', z') \in W \oplus W^\perp$

s.t $x = w + z$, $y = w' + z'$

$\langle \bar{x}, \text{Proj}(\bar{y}) \rangle = \langle \text{Proj}(x), y \rangle$ $\longrightarrow$ Showing this is enough by uniqueness of T^t

$\langle w+z, w' \rangle = \langle w, w'+z' \rangle$

$\Rightarrow$ LHS $\qquad\qquad\qquad\qquad$ RHS

$\langle w, w' \rangle + \cancel{\langle z, w' \rangle}^{\,0} \qquad \langle w, w' \rangle + \cancel{\langle w, z' \rangle}^{\,0}$

$\square$

So, if $\mathcal{C}$ is onb of V then

$_{\mathcal{C}}[\text{Proj}_W]_{\mathcal{C}}$ is symm

② let $A \in \text{Mat}_{n \times n}(\mathbb{R})$ be a symm mat.

Then $L_A : \mathbb{R}_n \to \mathbb{R}_n$ is self adj

Pf) $_{\text{stand basis}}[L_A]_{\text{stand basis}} = A = A^t$ $\qquad$ $\square$

$\hookrightarrow$ Uses that Strong version

Observ) self adj lin trans

① Form a STP wrt +

② Not mult $\Rightarrow$ self adj $\not\Rightarrow$ invertible

eg $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Q) Take sym matrices in $GL_n(\mathbb{R})$. Is this a subgp wrt mult?

ex) Check $(AB)^t = AB$

$\Leftarrow B^t A^t = AB$

Spectral Theorem

Let V be f.d ips $\mathbb{R}$.
let $T \in \hom(V,V)$. Then
T is self adj

$\Longleftrightarrow$

$\exists$ ONB $\mathcal{W}$ of V s.t $_{\mathcal{W}}[T]_{\mathcal{W}}$ is diagonal
$\hookrightarrow$ $\exists$ on eigenbasis!

Corr | let $A \in \mathrm{Mat}_{n \times n}(\mathbb{R})$
If A is symmetric, it is diagonalizable

Pf outline for spectral Thm |

Hard lemma | If $T: V \to V$ is self adj, it has an eigenvalue
$\hookrightarrow$ will prove this later $\to$ ext.

Not so hard | lets say we have the above.
if $T \in \hom(V,V)$ is self adj. Then we have $\lambda_1 \to$ eigenval

- $T((V_{\lambda_1})^\perp) \subseteq (V_{\lambda_1})^\perp$

$\Downarrow$

$(V_{\lambda_1})^\perp$ is an inv subsp.

- $T|_{(V_{\lambda_1})^\perp} : (V_{\lambda_1})^\perp \to (V_{\lambda_1})^\perp$

is also self adjoint

Pf | Fix $\bar{w} \in (V_{\lambda_1})^\perp$, show $T(\bar{w}) \in (V_{\lambda_1})^\perp$
$\forall v \in V_{\lambda_1}$

$\langle \bar{v}, T(\bar{w}) \rangle = \langle T(\bar{v}), \bar{w} \rangle = \langle \lambda_1 \bar{v}, \bar{w} \rangle$
$= \lambda_1 \langle \bar{v}, \bar{w} \rangle = \lambda_1 \cdot 0 = 0$ $\square$

Now, it makes sense to consider the res

$$T|_{(V_{\lambda_1})^\perp} : (V_{\lambda_1})^\perp \to (V_{\lambda_1})^\perp$$

Lets show its self adj. Fix $\bar{w}, \bar{w}' \in (V_{\lambda_1})^\perp \subseteq V$

$\langle \bar{w}, T(\bar{w}') \rangle = \langle T^+(\bar{w}), \bar{w}' \rangle = \langle T(\bar{w}), w' \rangle$ $\square$

Pf plan $\Leftarrow$

Spose w is onb of V consisting of e-vect of $T \in \hom(V,V)$

${}_w[T]_w$ is diagonal

$\Rightarrow {}_w[T]_w = {}_w[T]_w{}^t$

Since w is onb

$({}_w[T^+]_w) = ({}_w[T]_w)^t$

So, ${}_w[T]_w = {}_w[T^+]_w \Rightarrow T = T'$

So, T is self adj!

$\Rightarrow$ Spose T is self adj

by hard lemma,

$\exists \lambda_1$ eval of T. We can write

$V \cong V_{\lambda_1} \oplus (V_{\lambda_1})^\perp$ (by the φ function (addition))

we showed

$T|_{(V_{\lambda_1})^t} : (V_{\lambda_1})^t \to (V_{\lambda_1})^t$ is self adj.

So, $\exists$ an eval of this. called λ_2

Q1 is λ_2 an eval of T?

A: Well obvi

$(V_{\lambda_2}) \subseteq (V_{\lambda_1})^+ \subseteq V$

Continue to get $T|_{(V_{\lambda_2})^\perp} : (V_{\lambda_2})^\perp \to (V_{\lambda_2})^\perp$

continue this process to get eval λ_3.

By induction:

$V \cong V_{\lambda_1} \oplus V_{\lambda_2} \oplus \cdots \oplus V_{\lambda_\ell}$

where $\lambda_1, \lambda_2, \ldots, \lambda_\ell \in \mathbb{R}$ are dist eigen val.

combine the basis for the eigen sp to get ONB!

But, we must show that for distinct eval

λ_i, λ_j, $i \neq j$ $\quad V_{\lambda_i} \subseteq (V_{\lambda_j})^\perp$

fix $v_i \in V_{\lambda_i}$, $v_j \in V_{\lambda_j}$

$$\lambda_i \langle \bar{v}_i, v_j \rangle = \langle \lambda_i v_i, v_j \rangle = \langle T(v_i), v_j \rangle$$

$$\overset{\text{self adj}}{=} \langle v_i, T(v_j) \rangle$$

$$= \lambda_j \langle v_i, v_j \rangle$$

$\Rightarrow \langle v_i, v_j \rangle = 0$ as $\lambda_j \neq \lambda_i$

So, if we appy GS to a basis of V_{λ_i} and take its union ($V_{\lambda_1} \cdots V_{\lambda_n}$) we have ONB of V.

Hard Lemma let V be fdips w $T \in \mathrm{hom}(V, V)$ be self-adj

It has a eigenval!

Pf Step I: Consider

$$f : V \to \mathbb{R}$$
$$v \longmapsto \langle \bar{v}, T(v) \rangle$$

Claim: This is cts (we have a topology) (use seq limit)

Fix $v \in V$. Consider a seq $(v_n) \in V$ that conv to $\bar{v}$

we must show $n \mapsto f(v_n) = \langle v_n, T(v_n) \rangle$

conv to $f(\bar{v})$

let $\varepsilon > 0$ be given, we wanna make

$|f(v_n) - f(v)|$ small

$= |\langle \bar{v_n}, T(v_n)\rangle - \langle \bar{v}, T(v)\rangle|$

$= |\langle v_n, T(v_n)\rangle \underbrace{- \langle v_n, T(v)\rangle + \langle v_n, T(v)\rangle}_{0} - \langle v, T(v)\rangle|$

$= |\langle v_n, T(v_n - v)\rangle + \langle v_n - v, T(v)\rangle|$

$\leq |\langle v_n, T(v_n - v)\rangle| + |\langle v_n - v, T(\bar{v})\rangle|$ Cauchy Schwartz

$\leq \|v_n\| \cdot \|T(v_n - v)\| + \|v_n - v\| \cdot \|T(v)\|$

Recall that, given $T: V \to V$ $\exists\, c \in \mathbb{R}$ s.t $\forall\, w \in V$

$\|T(\bar{w})\| \leq c \cdot \|\bar{w}\|$

as $n \mapsto v_n$ conv, it is bdd.

$\exists\, M \in \mathbb{R}_{>0}$ s.t $\forall n \in \mathbb{N}$ $\|v_n\| < M$

as (v_n) conv to v $\exists\, N \in \mathbb{N}$ s.t $\forall\, n > N$

$$\|v_n - v\| < \frac{\varepsilon}{57\,(c+1)(M+1)(\|v\|+1)} \to K$$

$\forall\, n > N$

$\|v_n\| \cdot \|T(v_n - v)\| + \|T(\bar{v})\| \cdot \|v_n - v\|$

$\leq M \cdot c\|v_n - v\| + c\|v\| \cdot \|v_n - v\|$

$$< M \cdot c \cdot \frac{\varepsilon}{K} + c\|v\| \cdot \frac{\varepsilon}{K} < \varepsilon$$

$\therefore f: V \to \mathbb{R}$ is cts (note, we didn't use self adj)

Step II:

Consider the following compact subset of V

$$S_1(\bar{0}) := \{\bar{v} \in V \mid \|v\| = 1\}$$

↳ "unit sphere"

Claim: $S_1(\bar{0})$ is cmpct.

↳ yes (V) is iso to $\mathbb{R}^n$ & closed & bdd |

Step III:

Consider

$$f|_{S_1(\bar{0})} : S_1(\bar{0}) \to \mathbb{R}$$

By EVT $\exists\, x_0 \in S_1(\bar{0})$ st

$\forall\, x \in S_1(0)$ $\quad f(x_0) \geq f(x)$

Set $\lambda_T := f(\bar{x}_0)$

note $f(\bar{x}_0) = \langle x_0, T(\bar{x}_0)\rangle \neq 0$ as $x_0 \neq 0$

λ_T is cand eval!

cand evect x_0

↳ nonzero so we good

Step IV:

Spose not (classic)

Spose $\lambda_T \bar{x}_0 - T(x_0) \neq 0$

Since it is non zero, we can get unit vect

$$x_1 := \frac{\lambda_T(\bar{x}_0) - T(x_0)}{\|\lambda_T(\bar{x}_0) - T(x_0)\|}$$

$x_1 \in S_1(0)$

Claim| $\bar{x}_0 \perp \bar{x}_1$

$$\langle x_0, x_1\rangle = \langle x_0, \frac{\lambda_T x_0 - T(x_0)}{\| \text{junk} \|}\rangle$$

$$= \frac{1}{\|\text{junk}\|}\langle x_0, \lambda_T x_0 - T(x_0)\rangle$$

$$= \frac{1}{\|j\|}\left(\langle x_0, \lambda_T x_0\rangle - \langle x_0, T(x_0)\rangle\right)$$

$$= \frac{1}{\|j\|}\left(\lambda_T \cancelto{1}{\langle x_0, x_0\rangle} - f(x_0)\right)$$

$$= \frac{1}{\|j\|}(\lambda_T - \lambda_T) = 0$$

$\therefore (x_0, x_1)$ is a orthonormal fam.

In particular, it's an ONB of $W := \mathrm{span}(x_0, x_1)$

Is $T(x_0) \in W$ → yes (rearrange from def of x_1)

$T(x_0) = \langle x_0, T(x_0)\rangle \bar{x}_0 + \langle x_1, T(x_0)\rangle x_1$ → (ONB)

We know, by assumption, This is nonzero so x_1 is nonzero and $\langle x_1, T(x_0)\rangle$ is nonzero

Plan| Show $x_1 \langle x_1, T(x_0)\rangle = 0$

I.p. $\langle x_1, T(x_0)\rangle = 0$

Step 1| Take

$$W \cap S_1(\bar{0}) = \{a x_0 + b x_1 \mid a^2 + b^2 = 1\}$$

Use sine & cos to prove this.

We can write every elt of it as,

$c(t)\bar{x}_0 + s(t)\cdot x_1$ for some $t \in \mathbb{R}$

$$\mathbb{R} \xrightarrow{n} W \cap S_1(\bar{0})$$

$$t \longmapsto c(t)x_0 + s(t)x_1$$

Consider the comp

$g: \mathbb{R} \to \mathbb{R} := f \circ h$

$t \mapsto \langle c(t)\bar{x}_0 + s(t)x_1 , T(c(t)x_0 + s(t)x_1)\rangle$

$$g(t) = (c(t))^2 \langle x_0, T(x_0)\rangle$$
$$+ 2\, s(t)c(t)\langle x_1\, T(x_0)\rangle$$
$$+ s^2(t)\langle x_1, T(x_1)$$

Q] Max val of g. as it is res of $f|_{S_1(0)}$

g is maximised at all t s.t

$h(t) = \bar{x_0}$

note $t=0$ is one such value

$g(0) = \langle x_0, T(x_0)\rangle = \lambda_T$

We wanna show $\langle T(x_0), x_1\rangle = 0$

$\mathbb{R} \xrightarrow{g} \mathbb{R}$

$t=0 \to \max$

Since $g: \mathbb{R} \to \mathbb{R}$ is diffeble & has a extreme val at $t=0$, we must have $g'(0) = 0$

lets see where this leads

$$g'(t) = -2\, c(t)s(t)\langle x_0, T(x_0)\rangle$$
$$+ 2(c^2(t) - s^2(t))\langle x_1, T(x_0)\rangle$$
$$+ 2s(t)c(t)\langle x_1, T(x_1)\rangle$$

$\Rightarrow g'(0) = 2\langle x_1, T(x_0)\rangle = 0$ $\blacksquare$