**Theorem** Very useful Thm

Let $V$ be a vs over $\mathbb{F}$. Let $(v_i \mid i \in I)$ be a fam

The following are equiv

1. The family is a basis for $V$
2. $\text{span}(v_i \mid i \in I) = V$ and no proper subfam is spanning
3. $(v_i \mid i \in I)$ is lin ind and is not properly contained in a lin ind fam

tagline: "bases as minimal spanning / maximal lin ind families"

**Pf** $1 \Longleftrightarrow 2$, $3 \Longleftrightarrow 1$

All done by contrapositive!

$1 \Longrightarrow 2$

Spose a proper subfam of $(v_i \mid i \in I)$ spans $V$.

$\exists\, i_0 \in I$ s.t. $V = \text{span}(v_i \mid i \in I \setminus \{i_0\})$

as $v_{i_0} \in I$, $\exists$ scalars $a_i \in \mathbb{F}$ s.t.

$$v_{i_0} = \sum_{i \in I \setminus \{i_0\}} a_i \vec{v_i}$$

So, $(v_i \mid i \in I)$ not lin ind!

$2 \Longrightarrow 1$

Spose we have a minimal spanning fam. We must show its lin ind.

contrapos

Spose the family is not lin ind. By classy lemma

$\exists\, i_0 \in I$ $\exists$ scalars $a_i \in \mathbb{F}$ s.t.

$$v_{i_0} = \sum_{i \in I \setminus \{i_0\}} a_i \vec{v_i}$$

So $\text{span}_{\mathbb{F}}(v_i \mid i \in I \setminus \{i_0\}) = V$ so we don't have minimal spanning fam!

① ⟹ ③

Spose $(v_i \mid i \in I)$ is properly contained in a lin ind fam in V.

$\exists\, \vec{w} \in V$ s.t $(\vec{w}, v_i \mid i \in I)$ is lin ind.

$\therefore\ \vec{w} \notin \operatorname{span}(v_i \mid i \in I) \Rightarrow$ $(v_i \mid i \in I)$ is not spanning

↳ not basis

③ ⟹ ① $(\neg 1 \Rightarrow \neg 3)$

Spose the family is not spanning. → (not lin ind doesn't matter)

Then $\exists\, \vec{w} \in V$ s.t $w \notin \operatorname{span}(v_i \mid i \in I)$

We see, $(\vec{w}, v_i \mid i \in I)$ is lin ind.

$\therefore\ (v_i \mid i \in I)$ is not a maximal lin ind fam $\square$

Corollaries of VUT

Corr) let V be a vs /F and let B be a basis.

No proper subfamily of B is a basis

Corr) let V be a finitely generated vs /F. Then

① Every spanning family contains a finite basis

② Every basis of V is finite

Pf) ① let $S := (v_i \mid i \in I)$ be spanning.

Goal: get a finite basis from $S$

As V is fg,

$\exists\ \vec{w}_1, \ldots, \vec{w}_m$ s.t $V = \operatorname{span}(w_1, \ldots, w_m)$

for each $j \in \mathbb{N}_m$ scalars $a_{ji}$ (ap fin)

$$\vec{w}_j = \sum_{i \in I} a_{ji} \vec{v}_i$$

Collect the indices of nonzero scalars,

$I_0 := \{ i \in I \mid \exists\, j \in \mathbb{N}_m \text{ s.t } a_{ji} \neq 0 \}$ → this is finite

$I_0$ might be empty iff $V$ is trivial
But, we use $I_0$ to get a finite spanning subfam
Claim: $(v_i \mid i \in I_0)$ is spanning
follows as $w_j = \sum_{i \in I} a_{ji} \vec{v}_i = \sum_{i \in I_0} a_{ji} \vec{v}_i$
$\therefore \mathcal{J} = (v_i \mid i \in I_0)$ is finite & spanning.
Let $E$ be the minimal indexed fam wrt containment in
$$\{ \mathcal{F} \subseteq \mathcal{J} \mid \operatorname{span}(\mathcal{F}) = V \}$$
Why does such an $E$ exist?
look at all subfamilies $\mathcal{F} \subseteq \mathcal{J}$ and find one that spans and has no spanning subfam!
$\therefore E$ is a basis of $V$ $\square$

Pf ② Show every basis is finite.
Let $B$ be a basis of $V \Rightarrow B$ is spanning
By ① $\exists$ a finite, spanning subfam $B' \subseteq B$.
By VUT, $B' = B \Rightarrow B$ is finite

Steinitz Exchange Lemma
Let $V$ be a vs $/F$.
let $X = (x_i \mid i \in I)$ be a finite lin ind fam
Let $Y = (y_j \mid j \in J)$ be spanning
$\exists$ an injection $f : I \hookrightarrow J$ st
$V = \operatorname{span}(X \cup (y_j \mid j \in J \setminus f(I)))$
removing / replacing etc

Pf) Induct on $|I|$. Let

$S = \{n \in \mathbb{N} \mid \text{if } |I| < n \text{ then the result holds}\}$

→ $1 \in S$ as $I = \emptyset \Rightarrow f(I) = 0$ so

$(X \cup (y_j \mid j \in J \setminus f(I))) = (y_j \mid j \in J)$

→ Suppose $k \in S$. We must show the statement holds if $|I| = k \Rightarrow$ wlog $I = \mathbb{N}_k$

$\Rightarrow X = (x_1, \dots, x_{k-1}, x_k)$

Let $I' = \mathbb{N}_{k-1}$. We see $X' = (x_1 \dots x_{k-1})$ is lin ind as $X' \subseteq X$.

As $|I'| = k-1$, $\exists$ injection $f' : I' \hookrightarrow J$ s.t

$V = \text{span}(X' \cup (y_j \mid j \in J \setminus f'(I')))$

We aim to get $f : I \hookrightarrow J$ with a similar property.

Let $f = f'$ on $I' \subseteq I$

Since $x_k \in V$ $\exists$ scalars $a_j, c_i \in F$ almost all zero st

$$x_k = \sum_{i \in I'} c_i x_i + \sum_{j \in J \setminus f'(I')} a_j \vec{y}_j$$

We note, not all $a_j = 0$ as $X$ is lin ind

$\exists j_0 \in J \setminus f'(I')$ st $a_{j_0} \neq 0$

Consider $a_{j_0} \vec{y}_{j_0}$. Solve for $\vec{y}_{j_0}$

$$y_{j_0} = \sum_{i \in I'} (-c_i)(a_{j_0})^{-1} \vec{x}_i + \sum_{\substack{j \in J \setminus f'(I') \\ j \neq j_0}} (-a_j)(a_{j_0})^{-1} y_j$$

Lets say $f(k) = j_0 \Rightarrow$ we have completely def $f$.

Is $f$ injective? Yes as $f'$ is and $j_0 \in J \setminus f'(I')$
$= J \setminus f(I')$

Now, let $v \in V$ be given. We know (by inductive hyp)

$\exists$ scalars $b_j, a_i \in F$ st

$$v = \sum_{i \in I'} a_i \bar{x}_i + \sum_{\substack{j \in J \setminus f'(I') \\ j \neq j_0}} b_j y_j + b_{j_0} y_{j_0}$$

we observe, $f'(I') \cup \{j_0\} = f(I)$.

So, $j \in J \setminus f'(I')$, $j \neq j_0 \Rightarrow j \in J \setminus f(I)$

So,

$$v = \sum_{i \in I'} a_i \vec{x}_i + \sum_{j \in J \setminus f(I)} b_j y_j + b_{j_0} \cdot$$

mess from earlier

$$\sum_{i \in I'} (-c_i)(a_{j_0})^{-1} \vec{x}_i + \sum_{j \in J \setminus f(I)} (-a_j)(a_{j_0})^{-1} y_j$$

As $I' \subseteq I$, we are done! $k+1 \in S$ $\square$

**Corr]** let $V$ be a vs /F.

If $V$ admits a basis with $n$ elts, then every basis has $n$ elts!

**Corr]** let $V$ be a vs /F.

If $V$ admits a basis with $m$ elts. Then every linearly ind family has, at most, $m$ elts!

**PF]** let $(v_i \mid i \in I)$ be lin ind fam in $V$.

Sppse $B = (w_1 \dots w_m)$ is a basis.

If $|I| > m$ then we can extract a subfamily of $I$ with $m+1$ elts (lin ind)

By Steinitz $\exists$ injection $\mathbb{N}_{m+1} \hookrightarrow \mathbb{N}_m$ oops $\square$

**Corr)** If $B$ is a basis with $n$ elts and $S$ is spanning with $m$ elts, it is a basis!

**Pf)** if $S$ is not minimal sp family

$\exists\ S' \subsetneq S$ that spans $\Rightarrow |S'| < |S| = m$

By Steinitz, $\exists$ an injection $\mathbb{N}_m \hookrightarrow \mathbb{N}_{|S'|}$

oops $\square$

**Corr)** Let $V$ be fin gen vs over $F$.

If $B$ and $B'$ are bases of $V$ then

$|B| = |B'|$

**Pf)** By VUT corr, $B, B'$ are finite

$B = (v_1 \dots v_m)$, $B' = (w_1 \dots w_n)$

By Steinitz, $\exists$ injections $\mathbb{N}_m \hookrightarrow \mathbb{N}_n \Rightarrow n \geq m$

$\mathbb{N}_n \hookrightarrow \mathbb{N}_m \Rightarrow m \geq n$

So, $n = m$ $\square$

**Defn)** Let $V$ be a vs over $F$.

We define the dimension of $V$ to be

$\infty \to$ if $V$ is not fin gen

$|B| \to$ if $V$ is fin gen & $B$ is a basis $\square$

$\hookrightarrow$ well def by above!

**Lemma)** Spse $V$ is fin gen $/F$. Let $W \subseteq V$ be a subsp

1. $W$ is fin gen
2. $\dim(W) \leq \dim(V)$
3. $\dim(W) = \dim(V) \iff W = V$

Pf) ① + ②

let $L$ be lin ind in $W \Rightarrow$ lin ind in $V$.

So, $|L| \leq \dim(V)$. Consider

$\{L \subseteq W \mid L$ is lin ind fam$\}$

Find maximal elt (by containment)

Let this be $M$. This is a basis of $W$ and $|M| \leq \dim(V)$

Also $|M| = \dim(W)$ ◻

③ $\Leftarrow$ immediate.

Spose $\dim(W) = \dim(V)$

Let $B$ be a basis of $W$. By VUT, it is a max lin ind fam of $W$.

But $B$ is also max lin ind of $V$ as $|B| = \dim(W) = \dim(V)$

So, $B$ is a basis of $V$.

So, $W = \operatorname{span}(B) = V$ ◻.