Power Series

Defn) Spose $a \in \mathbb{R}$ & $n \mapsto c_n$ is a seq of reals
The expression
$$\sum_{n=0}^{\infty} c_n (x-a)^n \text{ is}$$
A power series centered at A

Note:
- A power series is not a function
- In general, a power series will conv & d/v for some values of x

So, for each $x_0 \in \mathbb{R}$ we consider the series of numbers
$$\sum_{n=0}^{\infty} c_n (x_0 - a)^n$$
For that x_0-value, we can assess the behaviour of the series
↳ we work point-by-point

Defn) Spose $I \subseteq \mathbb{R}$ is an iwimp & let $f: I \to \mathbb{R}$, we say f is $\mathbb{R}$-analytic p.t
f is locally represented by a power series
That is

anchor $\forall a \in I$ $\exists \varepsilon > 0$ (→ local) and a seq $n \to c_n$
s.t $\forall h \in B_\varepsilon(0)$, if $a+h \in I$
$$f(a+h) = \sum_{i}^{\infty} c_n h^n \to \text{represents}$$

or,
$\forall a \in I$ $\exists \varepsilon > 0$ $\exists (c_n)$ st $\forall x \in B_\varepsilon(a) \cap I$
$$f(x) = \sum c_n (x-a)^n \to \boxed{h = x - a}$$

Defn let $I \subseteq \mathbb{R}$ be an interval. The fnc $f : I \to \mathbb{R}$ is of C^ω type p.t it is real analytic and

$$C^\omega(I) := \{ f : I \to \mathbb{R} \mid f \text{ is } \mathbb{R} \text{ analytic} \}$$

TO BE PROVEN

① if $f(a+h) = \sum c_n h^n \quad \forall h \in (-\varepsilon, \varepsilon)$ w/ $a+h \in I$ then, necessarily

$$\boxed{c_n = \frac{f^{(n)}(a)}{n!}} \to \text{taylor coefs}$$

↳ unique
↳ implies $f \in C^\infty(I)$

② $C^\omega(I) \subsetneq C^\infty(I)$

we note, $f : \mathbb{R} \to \mathbb{R}$

$$x \mapsto \begin{cases} \exp\left(-\frac{1}{x^2}\right) & , x \neq 0 \\ 0 & , x = 0 \end{cases}$$

$f \in C^\infty(\mathbb{R})$ but not $C^\omega(\mathbb{R})$

proceed by contradiction.

f is locally represented by a power series at $a = 0$

but, by 295, $\forall n \in \mathbb{N}$ $f^{(n)}(0) = 0$ ← taylor

$\therefore \exists \varepsilon > 0$ s.t $\forall h \in (-\varepsilon, \varepsilon)$, $f(0+h) = \sum c_n h = \sum 0 = 0$

but if $x \neq 0$ $\underline{f(x) = \exp(-1/x^2) \neq 0}$ oops $\square$

Comparing C^∞ vs C^ω

C^ω	C^∞
$f,g \in C^\infty(\mathbb{R})$ let $a \in \mathbb{R}$ $\forall \varepsilon > 0\ \exists\, p \in C^\infty(\mathbb{R})$ s.t $p(x) = f(x)$ on $(-\infty, a-\varepsilon)$ & $p(x) = g(x)$ on $(a+\varepsilon, \infty)$ gluing fnc !	$f,g \in C^\omega(\mathbb{R})$ $a \in \mathbb{R}$ if $(x_n) \in \mathbb{R}\setminus\{a\}$ that conv to to a (limit pt) & $f(x_n) = g(x_n)\ \forall n \in \mathbb{N}$ Then $\boxed{f = g}$

An important example!

Consider $\ell : \mathbb{R} \to \mathbb{R}$, $x \longmapsto \dfrac{1}{1+x^2}$ Claim $\ell \in C^\omega(\mathbb{R})$

At the anchor pt 0, $\ell(0+h) = \dfrac{1}{1+h^2} = \displaystyle\sum_{n=0}^{\infty} (-1)^n (h^{2n})$

↳ this P.S only conv $h \in (-1, 1)$

At $a = 1$ we require another P.series!

In general $a \in \mathbb{R}$

$$\boxed{\ell(a+h) = \sum_{j=0}^{\infty} \frac{\ell^{(j)}(a)}{j!} h^j \quad \forall\, h \in B_{\sqrt{1+a^2}}(0)}$$

This demonstrates

1. Choice of $h \to C_n$ depends on anchor p.t A
2. There may not be a single PS that works
   ↳ need of locally rep by P.S

Recall

let $n \mapsto c_n$ be a seq of $\mathbb{R}$. We say the PS

$\sum c_n y^n$

converges at $y_0 \in \mathbb{R}$ p.t. $\sum c_n (y_0)^n$ converge

Otherwise it diverges!

Spose $n \longmapsto s_n$ is bounded

Define for $m \in \mathbb{N}$ $S_m := \{ s_k \mid k \geq m \}$

$\limsup(s_n) := \lim_{m \to \infty} (\sup(S_m))$ $\Rightarrow$ exists by HW

$\liminf(s_n) := \lim_{m \to \infty} (\inf(S_m))$

Thm) Hadamard

Spose $\sum c_n x^n$ is a power series.

Set $\rho := \dfrac{1}{\limsup(|c_n|^{1/n})}$ $\begin{cases} 0 & \text{if denom is } \infty \\ \infty & \text{if denom is } 0 \end{cases}$

① $\sum c_n x_n$ is abs conv on $(-\rho, \rho)$

② $\sum c_n x_n$ diverges on $\mathbb{R} \setminus [-\rho, \rho]$

③ No info at endps.

Defn) ρ is the radius of conv of the PS!

Pf) fix $x_0 \in \mathbb{R}$ & WLOG $x \neq 0$ (trivial)

CASE 1) $0 < |x_0| < \rho$ $\longrightarrow$ rules out $\rho = 0$ but $\rho = \infty$ allowed!

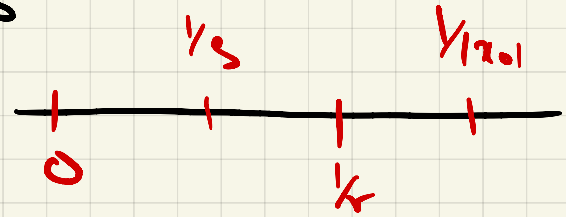
$\rightarrow$

it follows that

$\Rightarrow \frac{1}{|x_0|} > \frac{1}{\rho} = \limsup(|c_n|^{1/n})$

choose γ st $|x_0| < \gamma < \rho$

$\Rightarrow \frac{1}{|x_0|} > \frac{1}{\gamma} > \frac{1}{\rho}$



Since $\frac{1}{\rho} = \limsup(|c_n|^{1/n}) = \lim_{n\to\infty}(\sup\{|c_m|^{1/m} \mid m > n\})$ $\Big] \rightarrow$ is a limit of a mono $\downarrow$ seq

$\exists N \in \mathbb{N}$ s.t $\forall n > N$ we have $|c_n|^{1/n} < \frac{1}{\gamma} < \frac{1}{|x_0|}$

↳ catching the tail

$\Rightarrow |c_n|^{1/n} \cdot |x_0| < \frac{|x_0|}{\gamma} < 1$

$\Rightarrow |c_n| \cdot |x_0|^n < \left(\frac{|x_0|}{\gamma}\right)^n < 1 \rightarrow$ geometric series!

↳ by comparison test (in this case just with the tail) we get

$\sum |c_n||x_0|^n$ converges for $x_0 \in (-\rho, \rho)$

$\therefore$ The power series converges absolutely. $\square$

CASE 2) Take $|x_0| > \rho$ (excluding $\rho = \infty$)

if, $\rho = 0 \Rightarrow \limsup(|c_n|^{1/n}) = \infty$ so,

$\forall N \in \mathbb{N}$ $\exists n > N$ s.t $|c_n|^{1/n} > \frac{1}{|x_0|}$ $\Rightarrow |c_n|^{1/n}|x_0| > 1$
$\Rightarrow |c_n||x_0|^n > 1$

$\therefore \sum c_n x_0^n$ diverges by the div test!

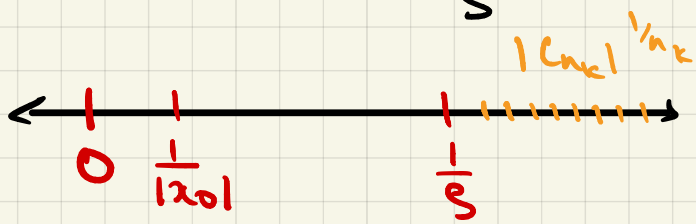
if $\rho \neq 0$ then, $\frac{1}{\rho} \in \mathbb{R}$

$\frac{1}{\rho} = \limsup(|c_n|^{1/n}) = \lim_{n\to\infty}(\sup\{|c_m|^{1/m} \mid m > n\})$

$\exists$ a subsequence of $n \mapsto |c_n|^{1/n}$ that conv to $\frac{1}{\rho}$

Call this $k \mapsto |c_{n_k}|^{1/n_k}$

$\to$ mono decreasing as $n \mapsto |c_n|^{1/n}$ is



$\forall\, k \in \mathbb{N}$ we have

$$|c_{n_k}|^{1/n_k} > \frac{1}{|x_0|} \implies |c_{n_k}| \cdot |x_0|^{n_k} > 1$$

unpack this

$\therefore\ \sum c_n (x_0)^n$ diverges as there is a subsequence of it that doesn't converge to 0 ! $\to$ Div test !

$\to$ Sometimes ratio fails so use Hadamard.

Eg $\sum b_n x^n \qquad b_n = \begin{cases} 2^n & \text{odd } n \\ 3^n & \text{even } n \end{cases}$

ratio test 'flip flops' here. Hadamard gives $\rho = \frac{1}{3}$!

Natural Questions

Let $\sum_{n=0}^{\infty} a_n x_n$ be a ps with radius $\rho > 0$

let $\to$ $f(y) := \sum_{n=0}^{\infty} c_n (y)^n$ for $y \in (-\rho, \rho)$

Is this continuous? Differentiable? How does it behave.

note, by construction, $f : (-\rho, \rho)$ is the ptwise limit of

$$k \longmapsto \sum_{n=0}^{k} c_n x^n = S_k(x)$$

Thm) **Weirstrass M Test**

Let $I \subseteq \mathbb{R}$ be an iwimp. Let $(f_n : I \to \mathbb{R})$ be a seq of func.

Let $n \mapsto M_n$ be a seq of numbers such than

1. $\sum M_n$ converge
2. For all but finitely many naturals $\forall\, x_0 \in I$ $|f_n(x_0)| \leq M_n$

Then $\sum f_n$ converges uniformly ! $k \mapsto S_k : I \to \mathbb{R}$ $\to$ partial sums

Pf] WLOG. Suppose

$\forall n \in \mathbb{N} \quad \forall x_0 \in I \quad |f_n(x_0)| \leq M_n$

We will show Σf_n & $\Sigma |f_n|$ satisfy UCC.

Let $\varepsilon > 0$ be given. Since ΣM_n converges,

$\exists N \in \mathbb{N}$ s.t $\forall n > m > N$

$$\left| \sum_{k=m+1}^{n} M_n \right| = \sum_{k=m+1}^{n} M_n < \varepsilon$$

$\forall a \in I$, we get, by triangle,

$$\left| \sum_{k=m+1}^{n} f_k a \right| \leq \sum_{k=m+1}^{n} |f_k(a)| \leq \sum_{k=m+1}^{n} M_k < \varepsilon$$

$\therefore \exists f : I \to \mathbb{R}, \ g : I \to \mathbb{R}$ s.t

$\Sigma f_n \xrightarrow{\text{unif}} f$

$\Sigma |f_n| \xrightarrow{\text{unif}} g$

$\longrightarrow$ as it is $= \left| \sum_{k=m+1}^{n} |f_k(a)| \right|$

Corollary] let $\Sigma c_\ell x^\ell$ be a ps w roc S

for $x_0 \in (-S, S)$ define

$f(x) := \Sigma c_\ell x_0^\ell$

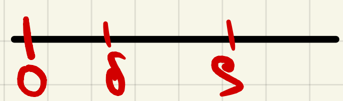
Then $\forall \delta \in (0, S)$, $\Sigma c_\ell x_\ell \xrightarrow{\text{unif}} f$ on $[-\delta, \delta]$

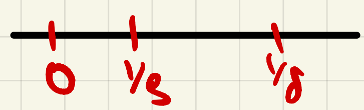
Pf] let $\delta \in (0, S)$ be given.

We must show $k \longmapsto \sum_{n=0}^{k} c_n x^n$ conv unif on $[-\delta, \delta]$

Spse $S \neq 0 \Rightarrow$ this is a useless case

let $\delta \in (0, S) \longrightarrow$





Choose $\gamma \in (\delta, S)$ since $\limsup(|c_n|^{1/n}) = \frac{1}{S} \to$ mono $\downarrow$

$\exists N \in \mathbb{N}$ s.t $\forall n > N \quad |c_n|^{1/n} < \frac{1}{\gamma}$

$\longrightarrow$.

So $\forall n > N$, $\forall x_0 \in [-\delta, \delta]$,

$$|x_0| \cdot |c_n|^{1/n} < \frac{|x_0|}{r} \leq \frac{\delta}{r} < 1$$

$$\Rightarrow |c_n||x_0|^n < \left(\frac{\delta}{r}\right)^n := M_n$$

Since $\sum M_n$ converges (geometric) we apply M test and we're done! $\square$

Corollary 2) if $f(x) = \sum c_\ell x_\ell$, it is cts on $(-S, S)$

Pf) fix $x_0 \in (-S, S)$ & take $\delta \in (|x_0|, S)$

$\therefore$ since ps converge uniformly on $[-\delta, \delta]$ & the fact that each partial sum is cts, we have that f is cts at $x_0 \in [-\delta, \delta]$

So pointwise limits of power series are continuous. And $\forall \delta \in (0, S)$ we have $\sum c_n x_n \xrightarrow{\text{unif}} f$ on $[-\delta, \delta]$

From our Sequences of Functions Discussion

Thm) Spose $\sum c_n x^n$ is a ps with roc $S > 0$.

For $x_0 \in (-S, S)$ set $f(x_0) = \sum c_n x_0^n$

Then

① $k \longmapsto \sum_{n=0}^{k} \frac{c_n x^{n+1}}{n+1}$ converges uniformly on $[-\delta, \delta]$ $\delta \in (0, S)$

and this converges to the antiderivative of f

② $k \longmapsto \sum_{n=0}^{k} n c_n x^{n-1}$ converges unif on $[-\delta, \delta]$, $\delta \in (0, S)$

and this converges to the derivative of f

③ $f \in C^\infty((-S, S))$

PP ① Consider,

$$k \longmapsto \left(S_k = \sum_{n=0}^{k} c_n x^n \right)$$

which conv uniformly on $[-\delta, \delta]$ $\forall \delta \in (0, S)$

Furthermore, $\forall k \in \mathbb{N}$ S_k is integrable on $[a,b] \subseteq [-\delta, \delta]$

so,

$$\lim_{k \to \infty} \left(\int_a^x S_k \right) = \int_a^x \lim_{k \to \infty} f_k = \int_a^x f$$

fix $\delta \in (0, S)$ we note

$$S_k(x) = \sum_{n=0}^{k} \frac{c_n x^{n+1}}{n+1} \text{ conv unif} \iff t_k(x) = \sum_{n=0}^{k} \frac{c_n x^n}{n+1} \text{ conv unif!}$$

Why? As $S_k(x) = x\, t_k(x)$ for $x \in [-\delta, \delta]$

Use UCC and bound $|x| < \delta$ to get this $\rightarrow$ in notes if needed W4 P12

We note that the radius of t_k is

$$\frac{1}{S'} = \limsup \left| \frac{c_n}{n+1} \right|^{1/n} = \limsup \left(\left| \frac{1}{1+n} \right|^{1/n} \cdot |c_n|^{1/n} \right)$$

by hw 2a

$$= \lim_{n \to \infty} \left(\left(\frac{1}{1+n} \right)^{1/n} \right) \cdot \limsup \left(|c_n|^{1/n} \right) = 1 \cdot \frac{1}{S}$$

$\rightarrow$ evaluate by exp & log!

$\Rightarrow S' = S$!

$\therefore t_k$ conv uniformly on $[-\delta, \delta] \Rightarrow S_k$ conv unif on $[-\delta, \delta]$ $\forall \delta \in (0, S)$

(as it is a ps.

② Use same argument.

$$\limsup \left(|n^{\text{th}} \text{ coeff}|^{1/n} \right) = \limsup |c_n|^{1/n} \cdot \lim |n|^{1/n}$$

$$= \frac{1}{S} \; !$$

③ Use induction!

Back to $\mathbb{R}$ analytic functions!

Defn let $I \subseteq \mathbb{R}$ be an int. The function $f \in C^\omega(I)$ pt

$\forall a \in I \ \exists \varepsilon > 0 \ \exists n \mapsto c_n$ st

$\forall h \in B_\varepsilon(0)$ if $a+h \in I$ we have

$$f(a+h) = \sum_{n=0}^{\infty} c_n h^n$$

Remark $\Rightarrow$ it follows from before that $C^\omega(I) \subseteq C^\infty(I)$

Also, $c_n = \frac{f^{(n)}(a)}{n!}$ $\Leftarrow$ term by term division

Important!

An important example!

Consider $\ell : \mathbb{R} \to \mathbb{R}$, $x \mapsto \frac{1}{1+x^2}$ Claim $\ell \in C^\omega(\mathbb{R})$

At the anchor pt 0, $\ell(0+h) = \frac{1}{1+h^2} = \sum_{n=0}^{\infty} (-1)^n (h^{2n})$

$\hookrightarrow$ this P.S only conv $h \in (-1, 1)$

At $a = 1$ we require another P series!

In general $a \in \mathbb{R}$

$$\boxed{\ell(a+h) = \sum_{j=0}^{\infty} \frac{\ell^{(j)}(a)}{j!} h^j \quad \forall h \in B_{\sqrt{1+a^2}}(0)}$$

This demonstrates

① Choice of $n \to c_n$ depends on anchor pt A

② There may not be a single PS that works

$\hookrightarrow$ need to locally rep by P.S

$\Longrightarrow$

Thm Principle of Analytic Continuation

Spose $I \subseteq \mathbb{R}$ is an iwimp.

let $l \in I$ and spose $n \mapsto x_n$ st $x_n \in I \setminus \{l\}$ and

$\lim_{n\to\infty} x_n = l$. $\Rightarrow$ l is an accumulation pt

If $f, g \in C^\omega(I)$ s.t

$f(x_n) = g(x_n)$ $\forall n \in \mathbb{N}$. Then $f = g$ on I!

Warmup Lem

Spose $J \subseteq \mathbb{R}$ is an iwimp. Let $j \in J$.

Let $n \to j_n$ be a seq in $J / \{j\}$ conv to j.

If $h(j_n) = 0$ $\forall n \in \mathbb{N}$, then $h(j) = 0$

Pf $h \in C^\omega(J) \subseteq C^0(J)$

So $h(j) = \lim_{n\to\infty} h(j_n) = 0$ $\square$

Pf let $h = f - g$. We must show $h = 0$ on I.

We claim $h = 0$ on a nbhd of l

note: $h(l) = 0$ as, by above, $h(x_n) = 0$ $\forall$ $n \in \mathbb{N}$

If h is not 0 on a nbhd of l. $\exists k > 0$ st h is represented on a nbhd of l by a ps of the form

eventually a nonzero term.

$$\sum_{\tilde{n} \geq k} a_{\tilde{n}} (x - l)^{\tilde{n}} \quad \text{with } a_k \neq 0$$

We write $h(x) = a_k (x - l)^k \cdot r(x)$ where

$$r(x) = 1 + \sum_{m \geq 1} b_m (x - l)^m \quad \text{w} \quad b_m = \frac{a_{m+k}}{a_k}$$

Since $h(x_n) = 0$ $\forall n$ & $x_n - l \neq 0$ $\forall n$ (with radius of conv -

$\boxed{r(x_n) = 0}$!!

We note $r(\ell) = 1$.

But $r(\ell) = \lim_{n\to\infty} (r(x_n)) = 0$ as r is cts.

Oops!

$\therefore \exists$ nbhd of ℓ for which h is the zero fn!

Step 2

fix $x \in I$. We are done if we show $h(x) = 0$

wlog $x > \ell$. Define

$S := \{ t \in I \mid \ell \le t \le x \text{ and } \mathrm{res}_{[\ell, t]}\, h = 0 \}$

by above $\ell \in S \therefore S \neq \emptyset$

it is also bdd above by x. $\Rightarrow \sup(S)$ exists.

Set $\sigma := \sup(S)$. Certainly $\sigma \le x$.

Plan: Show $h(\sigma) = 0$, $\sigma = x$!

① Why is $h(\sigma) = 0$

We can find a seq $n \mapsto y_n$ in S (by char sup) that conv to σ.

$\therefore$ as $h(y_n) = 0 \ \forall n \in \mathbb{N}$ by lemma $h(\sigma) = 0$

② We proceed by contradiction, $\sigma < x$

we note $\sigma \neq \ell$ by nbhd stuff.

Find a seq $n \to z_n$ contained in $S \setminus \{\sigma\}$ that conv to σ.

If $\sigma \notin S$ this is easy (char sup)

If $\sigma \in S$ then $\forall\ \ell \le k < \sigma$ $k \in S$. Thus, we can still find a seq.

We have a seq (z_n) in $S \setminus \{\sigma\}$ that conv to σ.

$h(z_n) = 0$ as $z_n \in S$.

$\therefore$ by earlier, $\exists$ nbhd of σ st $h = 0$.

In particular $\exists \varepsilon > 0$ s.t $\sigma + \frac{\varepsilon}{57} \in S$. oops $\square$

Why is this Analytic continuation?

Consider extending function.

I $f \in C^\omega(I)$, $F \in C^\omega(J)$, $I \subseteq J$ and

$f(x) = F(x) \quad \forall x \in I$.

Then F is a analytic continuation of f

Our thm says if

$F_1, F_2 \in C^\omega(J)$ and are analytic cont of f.

Then $F_1 = F_2$ on J!

Co e.g in $\mathbb{C}$ analysis is Riemann Zeta Function!

Thm If $f, g \in C^\omega(I)$ for I wimp I then

1. $f + g \in C^\omega(I)$
2. $fg \in C^\omega(I)$
3. if $f(I) \subseteq J$ wimp and $h \in C^\omega(J)$ then $h \circ f \in C^\omega(I)$
4. If $f(I) \subseteq \mathbb{R} \setminus \{0\}$ then $\frac{1}{f} \in C^\omega(I)$

$\Rightarrow$ Behave well!