Row Reduction

if we want to determine if a matrix $A \in Mat(n \times n)$ is inv, we must check if cols are spanning

$\Leftrightarrow$ cols basis (as we have n cols)

$\Leftrightarrow$ cols are lin ind

To check the latter, we use row reduction!

Defn) A matrix is said to be in echlon form pt

1. in every row, the first non zero entry is 1 (called a pivot)
2. The pivot 1 of a lower row is to the right of a pivot 1 of a higher row.
3. A col with pivot 1, all other entries are 0
4. Any all zero rows are at the bottom!

e.g

$$\begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Defn) Elementary row ops. They include

1. Swap 2 rows
2. Multiply row by non-zero field elt
3. add a non zero multiple of one row to another

This is justified by hubbard

Thm) 2.17 Hubbard

Given a matrix A, $\exists$ a matrix $\tilde{A}$ in echlon form that can be obtained from A by elt row ops

$\hookrightarrow$ to get this we may use Gaussian Elim!

e.g

$$A = \begin{pmatrix} 1 & 2 & 3 & 1 \\ -1 & 1 & 0 & 2 \\ 1 & 0 & 1 & 2 \end{pmatrix} \begin{matrix} \\ +R_1 \\ -R_1 \end{matrix} \rightarrow \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & 3 & 3 & 3 \\ 0 & -2 & -2 & 1 \end{pmatrix} \times \tfrac{1}{3}$$

$$\rightarrow \begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & -2 & 2 & 1 \end{pmatrix} \begin{matrix} -2R_2 \\ \\ +2R_2 \end{matrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix} \times \tfrac{1}{3}$$

$$\rightarrow \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{matrix} +R_3 \\ -R_3 \\ \\ \end{matrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Defn) Col of A is said to be pivotal pt the corresponding col of $\tilde{A}$ has a pivotal 1.
((highlighted above))

To be proved!

Recall, given matrix A, $\operatorname{im}(L_A) = \operatorname{span}(\text{cols}(A))$ ($F^n \rightarrow F^m$)

But, a basis of the image is the pivotal cols!

&

non pivotal cols give info abt kernel!

($\rightarrow$ note: They can't be basis of ker as col $\in F^m$, $\ker(L_A) \subseteq F^n$!

Recall:

The motivation behind this was to find out whether a square matrix was invertible!

$\Longleftarrow$ L_A is bij / surj / inj

$\hookrightarrow$ which use the kernel / image.

Defn) let $A \in Mat_{m \times n}(F)$. Note $L_A : F^n \to F^m$.

$ker(A) := ker(L_A)$ → info abt inj

$im(A) := im(L_A)$ → info abt surj!

Claim) let $B \in Mat_{m \times n}(F)$. let $\tilde{B}$ be row red of B.

$$ker(B) = ker(\tilde{B})$$

Pf) let $\vec{x} \in ker(B)$ so,

$Bx = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \in F^m$ → put this in aug mat

$$\left(B \;\middle|\; \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right) \xrightarrow{rref} \left(\tilde{B} \;\middle|\; \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right)$$

So, $\tilde{B}\vec{x} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \Rightarrow x \in ker(\tilde{B})$

So, $ker(\tilde{B}) \subseteq ker(B)$

The same argument yields $ker(B) \subseteq ker(\tilde{B})$ $\square$

Q) How do the images compare?

Well, if the statement earlier is true.

$dim(im(B)) = dim(im(\tilde{B}))$ → they might not be

As B & $\tilde{B}$ have the same corresponding pivotal cols

How does row reducing now find the im?

let $B = \begin{pmatrix} 1 & -1 & -2 \\ 2 & -4 & -8 \\ 2 & -3 & -6 \end{pmatrix}$

we want to know what pops out. So,

$B \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

So lets throw it in an aug mat →

$$\left(\begin{array}{ccc|c} 1 & -1 & -2 & y_1 \\ 2 & -4 & -8 & y_2 \\ 2 & -3 & -6 & y_3 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3y_1 - y_3 \\ 0 & 1 & 2 & 2y_1 - y_3 \\ 0 & 0 & 0 & 2y_1 + y_2 - 2y_3 \end{array}\right)$$

So, $x_1 = 3y_1 - y_3$

$x_2 + 2x_3 = 2y_1 - y_3$

$\boxed{0 = 2y_1 + y_2 - 2y_3}$ → this is the restriction

So, Im(B) $= \left\{ \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \middle| \; 2y_1 + y_2 - 2y_3 = 0 \right\}$

↳ 1 restriction gives us 2D plane (look at annihilators)

+ also 2 pivotal cols ⟹ Im(B) $= \text{span}\left[\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ -4 \\ -3 \end{pmatrix} \right]$

Thm] The image of a matrix is the span of its cols ((we have this from 04))

A basis of the image is the family obtained from the pivotal cols!

Pf] Hubbard.

Take this further

Given a family of vectors $(\vec{v}_1 \dots \vec{v}_k)$ in F^n.

How do we extract a linearly independent subfam that has the same span?

$$\begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_k \\ | & | & & | \end{pmatrix}$$ → and row reduce to find pivotal cols!

Back to Kernels

We showed that for $B \in Mat_{m \times n}(F)$,

$Ker(B) = Ker(\tilde{B}) \to$ info given through non-pivotal cols

$$B = \begin{pmatrix} 1 & -1 & -2 \\ 2 & -4 & -8 \\ 2 & -3 & -6 \end{pmatrix} \quad \tilde{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

So,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} x_1 = 0 \\ x_2 + 2x_3 = 0 \end{matrix}$$

$$Ker(B) = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{matrix} x_1 = 0 \\ x_2 + 2x_3 = 0 \end{matrix} \right\}$$

$\hookrightarrow$ should be 1D

So, we pivot with x_3 here.

$x_1 = 0x_3$, $x_2 = -2x_3$, $x_3 = x_3$ free parameter

$\hookrightarrow$ label of non piv col 1

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0x_3 \\ -2x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$$

$$\therefore Ker(B) = Span \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$$

Thm Rank-Nullity

Let $B \in Mat_{m \times n}(F)$. Then

$$\dim(ker(B)) + \dim(im(B)) = n$$

Pf By husband,

$$\left.\begin{matrix} \dim(ker(B)) = \text{no of non piv col} \\ \dim(im(B)) = \text{no of piv col} \end{matrix}\right\} \text{all col} = n$$

$\square$

Corr) let V, W be fgvs $/\mathbb{F}$.
let $T \in \hom(V, W)$

$$\dim(\ker(T)) + \dim(\operatorname{im}(T)) = \dim(V)$$

Pf) Choose $\mathcal{V}, \mathcal{W}$ and work with matrices!

Q) If $T \in \hom(V, W)$, how do we find a basis for
- $\operatorname{im}(T)$?
- $\ker(T)$?

A) Choose basis $\mathcal{V}, \mathcal{W}$.
Compute basis of ker, im of ${}_{\mathcal{W}}[T]_{\mathcal{V}}$
and use isomorphisms to send to V, W respectively!

But, why does row reduction work?

Elementary Matrices

The matrix $E_{rs} \in \mathrm{Mat}_{m \times m}(\mathbb{F})$ is obtained by swapping row r & s of $\mathrm{id}_{m \times m}$

$$E_{rs} = \begin{array}{c} \\ r \\ \\ s \\ \end{array} \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 0 & & 1 & \\ & & & \ddots & & \\ & & 1 & & 0 & \\ & & & & & \ddots \end{pmatrix}$$

(column labels r, s above the columns of the 0/1 entries)

For $A \in \mathrm{Mat}_{m \times n}(\mathbb{F})$, $E_{rs} \cdot A$ is the matrix obtained by swapping rows r, s of A

Fix $c \in \mathbb{F} \setminus \{0\}$

The matrix $E_T(c) \in \mathrm{Mat}_{m \times m}(\mathbb{F})$ is obt by multiplying row T by c

$$E_T(c) = \begin{array}{c} \\ \\ T \\ \\ \end{array} \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & c & \\ & & & & \ddots \end{pmatrix}$$

Again, for $A \in \mathrm{Mat}_{m \times n}(\mathbb{F})$ $E_T(c) \cdot A \in \mathrm{Mat}_{m \times n}(\mathbb{F})$ reflects the obvious

The matrix $E_{qp}(\partial) \in Mat_{m\times n}(F)$ is adding ∂ row q to row p.

$$E_{qp}(\partial) = \begin{array}{c} \\ p \\ \\ q \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccc} & p & & q & \end{array} \\ \begin{pmatrix} \ddots & & & & \\ & 1 & & \partial & \\ & & \ddots & & \\ & & & 1 & \\ & & & & \ddots \end{pmatrix} \end{array}$$

Again for $A \in Mat_{m\times n}(F)$ $E_{qp}(\partial)\cdot A$ represents the obvious

Note: All these elementary matrices are inv!

- $E_{rs}^{-1} = E_{rs}$
- $E_r(c)^{-1} = E_r(c^{-1})$
- $E_{qp}(\partial)^{-1} = E_{qp}(-\partial)$

So, we see that for $A \in Mat_{m\times n}(F)$

$\tilde{A} = E_k \cdots E_1 E_2 A$ where E_i are elementary!

This makes it super clear why $\ker(A) = \ker(\tilde{A})$

The invertibility of $E_i \Rightarrow E_i\vec{v} = \vec{0} \iff \vec{v} = 0$!

Here is the upshot!

This helps us find out inverses when they exist. $A \in Mat_{n\times n}(F)$

$$\left(A \;\middle|\; Id_{n\times n} \right) \xrightarrow{\text{row ops}} \left(\tilde{A} \;\middle|\; B \right)$$

B should be A^{-1}. But only if $\tilde{A} = Id_{n\times n}$.

If A^{-1} exists $\Rightarrow L_A$ is bij $\Rightarrow \exists\, n$ pivotal cols $\Rightarrow \tilde{A} = Id_{n\times n}$

Why?

$$ID_{n\times n} = E_k \cdots E_1 A$$
$$B = E_k \cdots E_1$$

sub $\Rightarrow BA = ID_{n\times n} \longrightarrow$ but this is only 1 sided!

Lemma) let V be fin gen IF.

let $T \in \hom(V,V)$. If $\exists\, S \in \hom(V,V)$ s.t $T \circ S = id_V$ or $S \circ T = Id_V$

1. S is !
2. $S \circ T = T \circ S = id_V$

Pf) 2 => 1 by associativity of composition!

2) $T \circ S = id_V \Rightarrow T \circ S$ is surj

$\Rightarrow T$ is surj $\Rightarrow$ T is an isomorphism

$\Rightarrow T$ is inv so $\exists!\, L : V \to V$ s.t

$T \circ L = L \circ T = id_V$

lets show $L = S$

$L = L \circ Id_V = L \circ (T \circ S) = (L \circ T) \circ S$

$= S$ D

Corr) The mat $A \in Mat_{n \times n}(F)$ is inv

$\iff$

$\exists\, B \in Mat_{n \times n}$ s.t AB or $BA = id_{n \times n}$

Upshot) The prior process gives inverse!