C-type Functions

Defn) let $I \subseteq \mathbb{R}$ be an iwimp. A fuction $f: I \to \mathbb{R}$ is of

- C^0-type p.t f is cts
- C^1-type p.t
  1. f is diff'ble on I (including endpts)
  2. $f': I \to \mathbb{R}$ is of type C^0

More generally for some $n \in \mathbb{N}$, we say $f \in C^n(I)$ p.t

1. f is diff'ble on I
2. $f': I \to \mathbb{R} \in C^{n-1}(I)$

Defn) for $I \subseteq \mathbb{R}$ an iwimp & $n \in \mathbb{N} \cup \{0\}$ define (used above)

$$C^n(I) := \{ f: I \to \mathbb{R} \mid f \text{ is of } C^n\text{-type} \}$$

Note) As diff'ble $\Rightarrow$ cts we have

$$C^{58}(I) \subseteq C^{57}(I) \subseteq \dots\dots C^0(I)$$

Observe) $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto |x|$, $f \in C^0(\mathbb{R})$ but $f \notin C^1(\mathbb{R})$ so,
$C^1(\mathbb{R}) \neq C^0(\mathbb{R})$ and $C^1(\mathbb{R}) \subsetneq C^0(\mathbb{R})$

Similarly $f: \mathbb{R} \to \mathbb{R}$

$$x \mapsto \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$f \in C^2(\mathbb{R}) \setminus C^1(\mathbb{R})$
$\hookrightarrow$ $\boxed{\text{check}}$
so
$C^2(\mathbb{R}) \subsetneq C^1(\mathbb{R})$

can generalise to $x^n \sin(1/x)$ to show $C^n(\mathbb{R}) \setminus C^{n-1}(\mathbb{R}) \neq \emptyset$

Defn) let $I \subseteq \mathbb{R}$ be iwimp. We define

$$C^\infty(I) := \bigcap_{n \in \mathbb{N} \cup \{0\}} C^n(I)$$

e.g. poly fnc, $x \mapsto \sin(x)$, $x \mapsto \exp(x)$ $\in C^\infty(\mathbb{R})$

Claim: $f \in C^\infty(I)$ is 'infinitely diff'ble' that is

$\forall n \in \mathbb{N}$ $\exists f^{(n)} : I \to \mathbb{R}$! → immediate from definition!

→ we call these func smooth

→ eg $f : \mathbb{R} \to \mathbb{R}$ | from 295 |

$$x \longmapsto \begin{cases} \exp(-\frac{1}{x^2}) & , x \neq 0 \\ 0 & , x = 0 \end{cases}$$

$f \in C^\infty(\mathbb{R})$

Lemma) let $I \subseteq \mathbb{R}$ be an iwimp. Then

(A) $\forall n \in \mathbb{N} \cup \{0\}$ $C^n(I)$ is closed wrt

- addition
- multiplication
- division by 'non vanishing' fnc!

(B) Spose $f : I \to \mathbb{R}$ is a fnc & $f(I) \subseteq J$ where J is an iwimp. If $f \in C^n(I)$ & $h \in C^n(J)$ then

$h \circ f \in C^n(I)$

Pf) Consider $n=0$

(A) Follows from 295!

(B) follows from 295! → we know this in arbitrary topological space

Consider $n=1$

(A) Show $C^1(I)$ is closed wrt $+, \cdot, \div$

fix $f, g \in C^1(I)$ & we must show

① $(f+g): I \to \mathbb{R} \in C^1(I)$ → show diffble & cts derivative

by linearity rule, we know derivative exists &

$(f+g)' = f' + g'$ but $f', g' \in C^0(I)$ so $(f+g)' \in C^0(I)$

done.

② $(f \cdot g): I \to \mathbb{R} \in C^1(I)$

by product rule,

$(fg)' = fg' + f'g \in C^0$ as $f, f', g, g' \in C^0(I)$

∎

③ suffices to show $\frac{1}{g}: I \to \mathbb{R} \in C^1(I)$ → straightforward!

B let $f \in C^1(I)$ and $f(I) \subseteq J$ → is imp

let $h \in C^1(J)$. we must show $h \circ f \in C^1(I)$

diffble & by chain rule

$(h \circ f)' = f'(h' \circ f)$ since $f', f \in C^0(I)$ & $h' \in C^0(J)$

it follows that $(h \circ f)' \in C^0(I)$

→

Use induction to show property in generality

$\mathcal{X} := \{ n \in \mathbb{N} \mid (A) \text{ \& } (B) \text{ hold for } C^n(I) \}$

we showed $1 \in \mathcal{X}$

let $m \in \mathcal{X}$

(A) Fix $f, g \in C^{m+1}(I)$

we know $(f+g)' = f' + g' \in C^m(I)$ as $m \in \mathcal{X}$ and $f', g' \in C^m(I)$

Similarly $(f \cdot g) \in C^{m+1}(I)$

(B) Use similar induction

$\square$

Question Can we say the same about $C^\infty(I)$?

YES!

Corollary $C^\infty(I)$ is follows (A) & (B) from earlier

Pf fix $f, g \in C^\infty(I)$, show $(f+g) \in C^n(I)$ $\forall n \in \mathbb{N} \cup \{0\}$

fix $n \in \mathbb{N} \cup \{0\}$

$f, g \in C^n(I) \Rightarrow f+g \in C^n(I)$ by lemma above

Argue similarly for rest

$\square$

Remark Smooth functions are nice. But we will discuss nicer functions → $\mathbb{R}$-analytic.

↳ Requires taylor polynomials → series