Setting

Spose V is a vs /F.

Let $T \in \mathrm{hom}(V,V)$

Defn let W subsp of V. we say W is an invariant subsp for T p.t

$$T(W) \subseteq W$$

e.g. Trivial subsp, image, kernel, whole sp.

Q Spose V is a one-d vs over F. what does T do?

$\hookrightarrow$ T simply scales vectors!

Fix basis $\mathcal{V} = (t)$ of V. Consider ${}_{\mathcal{V}}[T]_{\mathcal{V}} \in \mathrm{Mat}_{1\times 1}(F)$

$T(w) = ?$

$\forall w \in V$ $\exists$ scalar α_w s.t $w = \alpha_w t$

${}_{\mathcal{V}}[T]_{\mathcal{V}}(\alpha_w) = \lambda \alpha_w$

$T(w) = \lambda \cdot \alpha_w t = \lambda \overline{w}$

$\therefore$ any lin trans from a 1D vs to itself is just scaling.

$$T : V \to V$$
$$\overline{w} \mapsto \lambda \overline{w}$$

Def let V be a vs /F. let $T \in \mathrm{hom}(V,V)$

The scalar $\lambda \in F$ is an eigenval of T p.t

$\exists\ \overline{v} \in V \setminus \{\overline{0}\}$ s.t $T(\overline{v}) = \lambda \overline{v}$

In this case $\overline{v}$ is the eigenvec assc with λ

Note If $\overline{v}$ is an eigenvec with assc val λ,

$\mathrm{span}(\overline{v})$ is a 1D inv subsp! $T|_{\mathrm{span}(\overline{v})} : \mathrm{span}(\overline{v}) \circlearrowleft$

$$w \mapsto \lambda \overline{w}$$

note: Many eigenvectors can be assc with an e-val.

e.g. If $\bar{v}, \lambda$ are an eigen pair

$\forall w \in \operatorname{span}(\bar{v})$, λ is an eigenval of $\bar{w}$

e.g. $V = \mathbb{R}^2$ $\quad T: V \to V$

$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \end{pmatrix}$

$\to$ 1 is the eigenval for all vectors!

note: We may have no eigenval / vect

e.g. Rotation by $90°$ about origin in $\mathbb{R}^2$

What is the big deal?

Let V be fg $\mathbb{F}$. Let $T \in \hom(V, V)$

Spose $\mathcal{V} = (v_1 \dots v_n)$ is an eigenbasis corr. to $(\lambda_1 \dots \lambda_n)$

We see that,

$$_{\mathcal{V}}[T]_{\mathcal{V}} = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$

$\to$ diagonal

Q) Spose V is a vs / $\mathbb{F}$ and $\dim(V) = n$.

Spose $\mathcal{V}$ is a basis for which $_{\mathcal{V}}[T]_{\mathcal{V}}$ is diagonal.

Is $\mathcal{V}$ an eigenbasis? Yes! follow the diagram!

Defn) The mat A that $_{n \times n}(\mathbb{F})$ is diag pt

$A_{ij} = 0 \quad \forall\ 1 \le i \ne j \le n$

Defn) Let V be a fg vs $\mathbb{F}$. Let $T \in \hom(V, V)$.

We say T is diagable pt

$\exists$ a basis $\mathcal{V}$ so that $_{\mathcal{V}}[T]_{\mathcal{V}}$ is diag

$\iff \exists$ an eigenbasis of V.

Thm) Let V be a vs / F.

Let $T \in \hom(V, V)$.

Spose $(\lambda_1, \ldots, \lambda_m)$ are distinct eval with corr vectors $(v_1, \ldots, v_m)$

Then $(v_1, \ldots, v_m)$ is lin ind

Pf) HW 9a

Tagline: "Evec from distinct eval are lin ind" $\square$

Corr) Spose V has dim n. Then, $T \in \hom(V, V)$ has, at most, n distinct evals

How do we find eval / evec!

Just unpack: Let $T \in \hom(V, V)$

$T(\bar{v}) = \lambda v \implies T(\bar{v}) - \lambda \bar{v} = 0$

So, $\lambda \cdot id_V(\bar{v}) - T(\bar{v}) = 0$

$\iff (\lambda \cdot id_V - T)(\bar{v}) = 0 \implies \bar{v} \in \ker(\lambda \cdot id_V - T)$

insist $\bar{v} \neq 0$. When do we have a nonzero $v \in \ker$

$\implies \dim(\ker(\lambda \cdot id_V - T)) \geq 1$

$\implies (\lambda \cdot id_V - T) : V \to V$ is not injective

So, if V is fg, we know that

$(\lambda \cdot Id_V - T)$ is not inj $\iff \det(\lambda \cdot Id_V - T) = 0$

So, we are interested in $\lambda \in F$ st

$\operatorname{Det}(\lambda \cdot id_V - T) = 0 \to$ These give evals!

Recipe

(I) Write down charpoly

$\chi(\lambda) := \operatorname{Det}(\lambda \cdot Id_V - T)$

& find roots in F.

All roots are eval by design!

(II) For each eigen val λ_i set

$V_{\lambda_i} := \ker(\lambda_i \cdot id_V - T) \longrightarrow$ eigen space

by design $\dim(V_{\lambda_i}) \geq 0$

all nonzero vect in V_{λ_i} are eigenvect!

(III) Is T diagable? Is there an e basis?

Extract basis of

$V_{\lambda_1} \ldots V_{\lambda_n}$ to build basis of V if possible!

wait! if v is a basis of V_{λ_1} & w is of V_{λ_2}

since $\lambda_1 \neq \lambda_2$, $V_{\lambda_1} \cap V_{\lambda_2} = \emptyset$

Remark) We have an e basis iff

$$\sum_{i=1}^{n} \dim(V_{\lambda_i}) = \dim(V)$$

PP) Brian conrad handout on decomposing vs into eigen pieces!

Notation: Direct Sum

Let W_1 & W_2 be vs /F. We define a new vs /F.
$W_1 \oplus W_2$ called the direct sum of W_1 & W_2

- $W_1 \oplus W_2 = W_1 \times W_2$
- $(w_1, w_2) \oplus (w_1', w_2') := (w_1 +_1 w_1', w_2 +_2 w_2')$
- for $c \in F$ $\quad c \cdot (w_1, w_2) := (cw_1, cw_2)$

Can show

- $W_1 \oplus W_2$ is a vs /F
- $\dim(W_1) + \dim(W_2) = \dim(W_1 \oplus W_2)$
- If $W_3, W_4 \leq V$
  $W_3 \oplus W_4 \cong W_3 + W_4 \iff W_3 \cap W_4 = \{\vec{0}\}$

Diagonilizability

Pf $T \in \text{hom}(V,V)$,
T is diagonable iff $\exists$ eigenbasis iff
$V \cong \bigoplus_{i=1}^{m} V_{\lambda_i}$ where $\lambda_1 \dots \lambda_m$ are distinct eval

Lemma) Let V be a vs /F. Let $T \in \text{hom}(V,V)$
Then T is not inj $\iff$ 0 is an eval!

Pf) from def!

Note: By Change of Basis.
If A is diagonable $\exists B$ s.t
$C = BAB^{-1}$
(C is diag!)