Linear Transformations

Defn let V, W be vs $/\mathbb{F}$.
The function $T: V \to W$ is a lin trans p.t

1. T is a grp homomorphism wrt $(V,+), (W,+)$
   $T(v + v') = T(v) + T(v')$
2. $\forall v \in V \quad \forall \alpha \in \mathbb{F}$
   $T(\alpha v) = \alpha T(v)$

Note: immediately $T(0_V) = 0_W$
$T(-v) = -T(v)$

Defn let V, W be vs $/\mathbb{F}$

$\text{Hom}(V,W) := \{T: V \to W \mid T \text{ is a lin trans}\}$

Cool $\text{hom}(V,W)$ is a vs $/\mathbb{F}$ of $\dim(V) \cdot \dim(W)$
(if they're both fin gen)

Basic Results

let $T, T' \in \text{hom}(V,W)$

1. $\forall c \in \mathbb{F} \quad (cT): V \to W \quad \in \text{hom}(V,W)$
   $v \mapsto c \cdot T(v)$
2. $(T+T'): V \to W \quad \in \text{hom}(V,W)$
3. $T: V \to W$ is inj $\iff \ker(T) = \{0_V\}$
4. $\ker(T) \subseteq V$ is a subsp of V
5. $\text{im}(T) \subseteq W$ is a subsp of W
6. If $T'' \in \text{hom}(W,V)$, $T \circ T'' \in \text{hom}(V,W)$

Lemma) let V, W vs $/\mathbb{F}$.

If $T \in \hom(V,W)$ is injective, it takes lin ind fam to lin ind fam.

Pf) let $L = (v_i \mid i \in I)$ be lin ind in V

Consider $T(L) := (T(v_i) \mid i \in I) \Rightarrow$ show it is lin in in W

Spose $\exists a_i \in \mathbb{F}$. where

$$\sum_{i \in I} a_i T(v_i) = 0_W$$

$$\Rightarrow \sum_{i \in I} T(a_i v_i) = 0_W$$

$$\Rightarrow T\left(\sum_{i \in I} a_i v_i\right) = 0_W$$

By injectivity,

$$\sum_{i \in I} a_i v_i = 0_W \Rightarrow a_i = 0 \quad \forall i \in I \qquad \square$$

Lemma) let V, W vs $/\mathbb{F}$

let $T \in \hom(V,W)$ be surj.

T takes spanning fam in V to spanning fam in W.

Pf) let $S = (v_i \mid i \in I)$ span V.

let $w \in W$ be given. $\exists v \in V$ s.t $T(v) = w$

but, $\exists a_i \in \mathbb{F}$ st

$$v = \sum_{i \in I} a_i v_i \Rightarrow w = T\left(\sum_{i \in I} a_i v_i\right)$$

$$= \sum_{i \in I} a_i T(v_i)$$

$\therefore$ $T(V)$ is spanning!

does the conv hold? Probably!

Corr) Spose V, W are fg / F. Let $T \in \hom(V,W)$

(1) T inj $\Rightarrow \dim(V) \le \dim(W)$

(2) T surj $\Rightarrow \dim(W) \le \dim(V)$

Pf) (1) Let B be a basis of V.

We see that $T(B)$ is lin ind in W by lemma

So $|T(B)| \le \dim(W)$

By injectivity $|T(B)| = |B| = \dim(V)$ $\square$

(2) Let B be a basis of V.

$T(B)$ is spanning in W and

$|T(B)| \le |B|$

But, $|B| = \dim(V)$ $|T(B)| \ge \dim(W)$ $\square$

Lemma) Let V, W be fin gen / F.

If $\dim(V) = \dim(W)$

Let $T \in \hom(V,W)$

T is inj $\Leftrightarrow$ T is surj

(obviously getting $\Leftrightarrow$ T is bij) $\square$

Pf) Let $B = (v_1, \dots, v_n)$ be a basis of V.

$\Rightarrow$

T inj $\Rightarrow T(B)$ is lin ind, but $|T(B)| = |B| = \dim(V)$

$\therefore$ $T(B)$ is a max lin ind fam in $W \Rightarrow T(B)$ is a bs

$\Rightarrow T(B)$ is spanning

$\Rightarrow$ T is surj!

$\Leftarrow$ same idea $\square$

Understanding Linear Transforms

→ We only need to what a linear trans does to a basis!

Claim) let V, W be vs $/F$. let $B = (v_i \mid i \in I) \subseteq V$ be a basis

let $(\vec{w_i} \mid i \in I) \subseteq W$ be a family

$\exists!\ T \in \hom(V, W)$ s.t $\forall i \in I\ T(v_i) = w_i$

Pf) set $T(v_i) = \vec{w_i}\ \forall\ i \in I$

let $\vec{v} \in V$.

$\vec{v}$ is uniquely represented by a lin comb of B

$\vec{v} = \sum_{i \in I} a_i v_i$

$\therefore T(\vec{v}) = \sum_{i \in I} a_i T(v_i) = \sum_{i \in I} a_i w_i$

↳ Linear by construction

let $S \in \hom(V, W)$ st $S(v_i) = w_i\ \forall\ i \in I$

$S(\vec{v}) = S\left(\sum_{i \in I} a_i v_i\right) = \sum_{i \in I} a_i w_i$ $\square$

What do we know about 'im'?

if $T \in \hom(V, W)$ and $B = (v_i \mid i \in I)$ is a basis of V.

Then $\operatorname{im}(T) = \operatorname{span}(T(v_i) \mid i \in I)$

$\supseteq$ immediate → just reverse engineer.

let $w \in \operatorname{Im}(T)$ $\exists v \in V$ s.t $T(v) = w$

$v = \sum_{i \in I} a_i v_i \Rightarrow T(v) = \sum_{i \in I} a_i T(v_i) \in \operatorname{span}(T(v_i) \mid i \in I)$ $\square$

Note: also works if you just have a spanning fam over V.

say S spans V.

$\operatorname{im}(T) = \operatorname{span}(T(S))$

Lemma) "Equal finite dimensional vs are isomorphic"

let $n \in \mathbb{N}$. Fix F.

let V, W be vs $/F$ s.t. $n = \dim(V) = \dim(W)$

V, W are isomorphic!

Pf) Since V, W are finite d, $\exists$ basis

$B = (v_1, \dots, v_n) \subseteq V$

$B' = (w_1, \dots, w_n) \subseteq W$

Consider the unique lin trans (by earlier)

$T: V \to W$ $\quad \forall i \in \mathbb{N}_n \; T(v_i) = w_i$

Claim T is bij. As $\dim(V) = \dim(W)$, we only need to check surj or inj.

Surj.

$\operatorname{Im}(T) = \operatorname{span}(T(B)) = \operatorname{span}(B') = W$

$\therefore$ surj $\Rightarrow$ inj $\Rightarrow$ bij $\qquad$ D.

Corr) Every vs of dim n over F is isomorphic to F^n $\qquad$ D

lemma) let V be a vs of dim n over F.

Then the following is a bijection

$$☺ : \operatorname{isom}(F^n, V) \longrightarrow \{\text{bases of } V\}$$

isomorphisms from F^n to V

$$T \longmapsto (T(e_1), T(e_2) \dots T(e_n))$$

Pf) Surj. Fix basis B of V, $B = (v_1, v_2, \dots, v_n)$

Consider $T \in \operatorname{hom}(F^n, V)$, $T(e_i) = v_i \; \forall i \in \mathbb{N}_n$

T is a isomorphism by earlier logic & $☺(T) = B$

$\therefore \operatorname{im}(☺) = \{\text{bases of } V\} \Rightarrow$ surj $\qquad$ D

Inj. Spose ☺(T) = ☺(S)

$T, S \in \text{Isom}(F^n, V)$ so

☺$(T) = (T(e_1) \dots T(e_n))$

$=$ ☺$(S) = (S(e_1) \dots S(e_n))$

$\Rightarrow T(e_i) = S(e_i) \quad \forall i \in \mathbb{N}_n$

So, S & T agree on a basis of F^n.

$\Rightarrow S = T \Rightarrow$ inj $\quad$ D

Since we have a bij, consider the inv.

☺$^{-1}$: {basis of V} $\longrightarrow \text{Isom}(F^n, V)$

$v \longmapsto i_v : F^n \to V$

where if $v = (v_1, \dots v_n)$

$i_v(e_i) = v_i \quad \forall i \in \mathbb{N}$

$i_v : F^n \to V$ gives us "coordinates" in V

Visualizing Linear Transformations

Let V, W be fgvs /F where $\dim(V) = n$, $\dim(W) = m$.

* Choose a basis $v = (v_1, \dots, v_n) \subseteq V$
* Choose a basis $w = (w_1, \dots, w_m) \subseteq W$

We have the following commutative diagram

$$\begin{tikzcd}
\overset{v}{V} \arrow[r, "T"] & \overset{w}{W} \\
F^n \arrow[u, "i_v"] \arrow[r, "{}_w[T]_v"] & F^m \arrow[u, "i_w"']
\end{tikzcd}$$

for the diagram to com

${}_w[T]_v = i_w^{-1} \circ T \circ i_v$

So, ${}_w[T]_v \in \hom(F^n, F^m)$

lets understand ${}_W[T]_V$

for each $v_i \in V$, $T(v_i) = \sum_{j=1}^{m} T_{ji} w_j$, $T_{ji} \in F$

if $v = a_1 v_1 + \cdots + a_n v_n$

$$T(v) = T\left(\sum_{i=1}^{n} a_i v_i\right)$$

$$= \sum_{i=1}^{n} a_i T(v_i) = \sum_{i=1}^{n} \left[a_i \sum_{j=1}^{m} T_{ji} w_j \right]$$

upon expansion,

$$= (a_1 T_{11} + a_2 T_{12} \cdots) w_1 +$$
$$(a_1 T_{21} + a_2 T_{22} \cdots) w_2 +$$
$$\vdots$$
$$(a_1 T_{n1} + a_2 T_{n2} \cdots) w_n$$

so, $${}_W[T]_V \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} a_1 T_{11} + a_2 T_{12} \cdots + a_n T_{1m} \\ a_1 T_{21} + a_2 T_{22} \cdots + a_n T_{2m} \\ \vdots \\ a_1 T_{n1} + a_2 T_{n2} \cdots + a_n T_{nm} \end{pmatrix}$$

→ matrix mult on left

${}_W[T]_V$ is rep by $\begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ \vdots & & & \\ T_{m1} & T_{m2} & \cdots & T_{mn} \end{bmatrix}$

$m \times n$ matrix

Pause.... matrices popped up!

Defn) Matrix mult.

Let $A \in Mat_{m \times n}(F)$ $\quad C \in Mat_{n \times p}(F)$

Then AC is defined by, $j \in \mathbb{N}_m$, $k \in \mathbb{N}_p$

$$(AC)_{jk} = \sum_{r=1}^{n} A_{jr} C_{r.k}$$

$AC \in Mat_{m \times p}(F)$

This allows composition

Spose $A \in Mat_{m \times n}(F)$, $C \in Mat_{n \times p}(F)$

we will show the following are lin tras

$$L_A : F^n \to F^m \qquad L_C : F^m \to F^p$$
$$x \to Ax \qquad\qquad y \mapsto Cy$$

$\boxed{L_C \circ L_A = L_{CA}}$ as $x \mapsto Ax \mapsto BAx$

lemma) let $n, m \in \mathbb{N}$ be given. Let F be a field.

let $A \in Mat_{m \times n}$. Define,

$$L_A : F^n \to F^m$$
$$\bar{v} \mapsto A\bar{v}$$

$L_A \in hom(F^n, F^m)$

Pf) Comes from formula of matrix mult!

Q) We have 2 vec sp staring us in the face

$Mat_{m \times n}(F)$ & $hom(F^n, F^m)$

They should be iso right? Consider

$$\Phi : Mat_{m \times n}(F) \longrightarrow hom(F^n, F^m)$$
$$A \mapsto L_A : F^n \to F^m$$

The fact that $\mathcal{B} \in \hom(\mathrm{Mat}_{m\times n}(F), \hom(F^n, F^m))$ comes from the formula for matrix mult.

$\therefore$ $\mathcal{B}$ is linear!

Claim) $\dim(\mathrm{Mat}_{m\times n}(F)) = m \times n$

Pf) Consider a basis for $i \in \mathbb{N}_m$, $j \in \mathbb{N}_n$

$$E_{ij} = \begin{array}{c} \\ i \end{array}\begin{bmatrix} 0 & \cdots & \cdots & \\ 0 & \cdots & \cdots & \\ \vdots & & 1 & \\ \vdots & & & \end{bmatrix}$$
$$\qquad\qquad j$$

This family is of size $m \times n$.

So, to show bijectivity, injectivity is sufficient.

let $M \in \ker(\mathcal{B})$

$\lambda(M)$ is the lin transf $L_M : F^n \to F^n$, $\vec{v} \mapsto M\vec{v}$ $\equiv 0$

So, in particular

$$\begin{pmatrix} M_{11} & \cdots & M_{1n} \\ \vdots & & \\ M_{m1} & \cdots & M_{mn} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \vec{0} \Rightarrow \begin{pmatrix} M_{11} \\ \vdots \\ M_{m1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

So, $\forall\, 1 \le j \le n$, $M\vec{e}_j$ extracts the j^{th} col which is 0!

$\therefore$ $\forall j \in \mathbb{N}_n$, $\forall i \in \mathbb{N}_m$ $M_{ij} = 0$

$\therefore$ $\ker(\mathcal{B})$ is trivial. $\square$

We can also get surj. fix $T \in \hom(F^n, F^m)$

$$A = \begin{pmatrix} | & | & & | \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ | & | & & | \end{pmatrix}$$ does the trick! $\square$

To construct matrix $_{\mathcal{W}}[T]_{\mathcal{V}}$, simply choose with standard basis in F^n!

↳ gives cols!

Check p 18 week 8 if needed

Change of Basis!

We will work with an eg.

$V = \mathbb{R}[x]_{\leq 2} \qquad W = \mathbb{R}[x]_{\leq 2}$

$$T : V \longrightarrow W$$

$$p \longmapsto \frac{\partial^2}{\partial x^2} p + 3x \frac{\partial}{\partial x} p + \frac{\partial}{\partial x}(2xp)$$

if $\mathbb{V} = \mathbb{W} = (1, x, x^2)$

$${}_{\mathbb{W}}[T]_{\mathbb{V}} = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 7 & 0 \\ 0 & 0 & 12 \end{pmatrix} \rightarrow \text{can check}$$

if $\mathbb{V}' = (1, 1+x, x^2) \qquad \mathbb{W}' = (2, 1+3x, x^2+1)$

$${}_{\mathbb{W}'}[T]_{\mathbb{V}} = \begin{pmatrix} 1 & -1/6 & -5 \\ 0 & 7/3 & 0 \\ 0 & 0 & 12 \end{pmatrix}$$

How are these matrices related? <u>Commutative diag!</u>

$$\begin{tikzcd}
\overset{\mathbb{V}'}{V} \arrow[r, "\mathrm{id}_V"] & \overset{\mathbb{V}}{V} \arrow[r, "T"] & \overset{\mathbb{W}}{W} \arrow[r, "\mathrm{id}_W"] & \overset{\mathbb{W}'}{W} \\
F^n \arrow[u, "i_{\mathbb{V}'}"] \arrow[r, "{}_{\mathbb{V}}[\mathrm{id}]_{\mathbb{V}'}"'] & F^n \arrow[u, "i_{\mathbb{V}}"] \arrow[r, "{}_{\mathbb{W}}[T]_{\mathbb{V}}"'] & F^m \arrow[u, "i_{\mathbb{W}}"] \arrow[r, "{}_{\mathbb{V}}[\mathrm{id}]_{\mathbb{W}'}"'] & F^m \arrow[u, "i_{\mathbb{W}'}"]
\end{tikzcd}$$

$$\underbrace{F^n \to F^n \to F^m \to F^m}_{{}_{\mathbb{W}'}[T]_{\mathbb{V}'}}$$

comp / mult

So, $${}_{\mathbb{W}'}[T]_{\mathbb{V}'} = {}_{\mathbb{W}'}[\mathrm{id}_W]_{\mathbb{W}} \cdot {}_{\mathbb{W}}[T]_{\mathbb{V}} \; ! \; {}_{\mathbb{V}}[\mathrm{id}_V]_{\mathbb{V}'}$$

change of basis matrix!

In the example

$${}_{\mathcal{V}}[id]_{\mathcal{V}'} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$${}_{\mathcal{V}'}[id]_{\mathcal{V}} = \begin{pmatrix} 0.5 & -1/6 & -1/2 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

And things work out

Defn) Let $\mathcal{V}$ and $\mathcal{V}'$ be two bases of fin gen V.
The change of co-ordinate ~~matrix~~ from $\mathcal{V}$ to $\mathcal{V}'$ is

$${}_{\mathcal{V}'}[id]_{\mathcal{V}}$$

Obs) ${}_{\mathcal{V}}[id]_{\mathcal{V}} = id_{n\times n}$ $\quad (n = \dim(V))$

$${}_{\mathcal{V}}[id]_{\mathcal{V}} = \underbrace{{}_{\mathcal{V}}[id]_{\mathcal{V}'}}_{A} \cdot \underbrace{{}_{\mathcal{V}'}[id]_{\mathcal{V}}}_{B} = id_{n\times n}$$

$$B \cdot A = {}_{\mathcal{V}'}[id]_{\mathcal{V}'} = id_{n\times n}$$

$\therefore$ A & B are invertible!

Defn) The sq mat $A \in Mat_{n\times n}(F)$ is inv iff
$\exists$ $B \in Mat_{n\times n}$ s.t.

$$A \cdot B = BA = Id_{n\times n}$$

Lemma) The matrix $A \in Mat_{n\times n}(F)$ is inv iff

① $L_A : F^n \to F^n$ is inv

② L_A is lin isom

③ L_A is inj

④ L_A is surj

⑤ cols of A span F^n

Pf) we already have ① $\iff$ ② ③ ④

we also have ④ $\iff$ ⑤ as

$\text{im}(L_A) = \text{span}(L_A(\text{basis of } F^n))$

(so take $e = (e_1 \ldots e_n)$)

$= \text{span}(\text{first col}, \text{sec col}, \ldots, n \text{ col})$

$\iff$ follows from prev!

Row Reduction

if we want to determine if a matrix $A \in Mat(n\times n)$ is inv, we must check if cols are spanning

$\iff$ cols basis (as we have n cols)

$\iff$ cols are lin ind

To check the latter, we use row reduction!

Defn) A matrix is said to be in echlon form pt

① in every row, the first non zero entry is 1 (called a pivot)