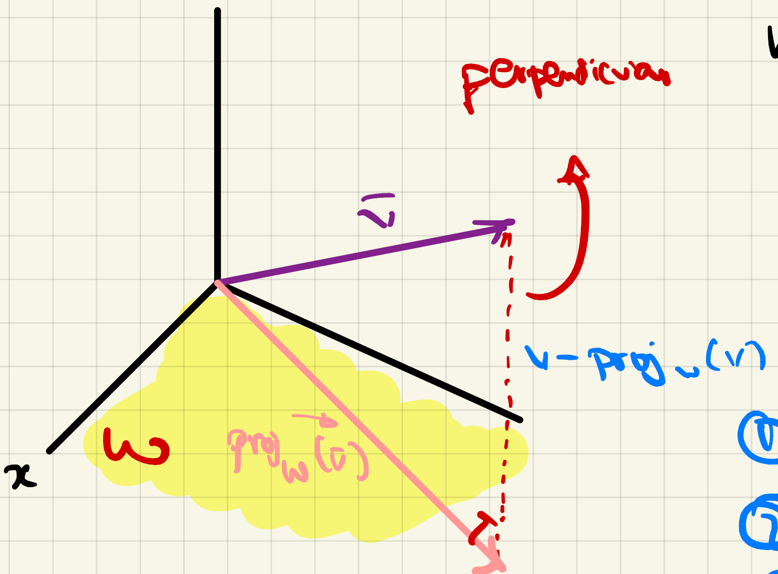
Projection Motivation

We are ready to use orthogonality to cook up a brand new lin trans.

Suppose V is a ips $/\mathbb{R}$. Let $W \leq V$ be a subspace

W is an ips too! By restriction.

E.g.) $V = \mathbb{R}^3$ W = xy plane



We want a function that takes $\vec{v} \in V$ and returns its 'shadow' in W

Wish list for

$\text{proj}_W V \to V$

1. $\text{proj}_W \in \text{hom}(V, V)$
2. $\text{im}(\text{proj}_W) = W$
3. $\text{proj}_W(\vec{w}) = \vec{w} \quad \forall \vec{w} \in W$
4. $\langle \text{proj}_W(\vec{v}), v - \text{proj}_W(v) \rangle = 0$

   ↳ orthogonality

Let V be a fin gen ips. Let $W \leq V$ be a subsp.

Defn) The set

$$W^\perp := \{ \vec{v} \in V \mid \langle \vec{v}, \vec{w} \rangle = 0 \quad \forall w \in W \}$$ "W perp"

Note: $W^\perp$ is a subsp of V by multilinearity

Eg.) Recall: $V^* := \text{hom}(V, \mathbb{R})$ for $W \leq V$

$$W^\circ := \{ \lambda \in V^* \mid \lambda(\vec{w}) = 0 \quad \forall w \in W \}$$

sorta feels like W° is a cousin to $W^\perp$

for V fg ips, $W \leq V \Rightarrow W^\circ \leq V^*$ there exists

$$V \xrightarrow{\psi} V^*$$

$$\bar{v} \longmapsto \lambda_v : V \longrightarrow \mathbb{R}$$

$$\bar{u} \longrightarrow \langle \bar{v}, \bar{u} \rangle$$

$\psi(\bar{v}) = \lambda_{\bar{v}} \in V^*$

Claim/ ψ is linear

Pf/ Basically follows from multilinearity of ip.

Claim/ ψ is injective

Pf/ Show that $\ker(\psi)$ is trivial!

Let $v \in \ker(\psi)$

$\Rightarrow \psi(\bar{v})$ is the zero functional

$\Rightarrow \lambda_v(\bar{u}) = \langle \bar{v}, \bar{u} \rangle = 0 \quad \forall u \in V$

in particular,

$\lambda_{\bar{v}}(\bar{v}) = \langle \bar{v}, \bar{v} \rangle = 0$

$\Rightarrow \|\bar{v}\|^2 = 0$

$\Rightarrow \bar{v} = 0$

$\therefore \ker(\psi)$ is trivial $\Rightarrow \psi$ is inj!

Corr/ If V is a fgips, $\psi : V \to V^*$ is an iso.

Pf/ $\dim(V) = \dim(V^*)$

now .... back to our Q .....

Upshot/ Natural (canonical) iso!

Claim: The image of the restriction

$\varphi|_{W^\perp} : W^\perp \to V^*$

is exactly W°

Pf) Show $\varphi(W^\perp) = W^\circ$ by 2 way containment

let $\bar{v} \in W^\perp$

consider $\varphi(\bar{v}) = \lambda_{\bar{v}}$ $\to$ show $\lambda_{\bar{v}} \in W^\circ$

fix $w \in W$

$\lambda_{\bar{v}}(w) = \langle \bar{w}, \bar{v} \rangle = 0$ by def of $W^\perp$

$\Rightarrow \varphi(W^\perp) \subseteq W^\circ$

For the other direction, let $\lambda \in W^\circ$

$\exists!\ x \in V$ s.t

$\varphi(\bar{x}) = \lambda$ as $\varphi : V \to V^*$ is an iso.

we are done if we show $\bar{x} \in W^\perp$

$\lambda = \lambda_x$ so $\forall w \in W$

$\lambda(w) = \langle \bar{x}, \bar{w} \rangle = 0$ $\Rightarrow \bar{x} \in W^\perp$ $\triangleright$

Corr) $\varphi : W^\perp \to W^\circ$ is a isom (pt $W^\perp$ is fg)

Corr) $\dim(W^\perp) = \dim(W^\circ)$

Corr) let V be fg ips

let $W \subseteq V$ subsp

$\dim(W) + \dim(W^\perp) = \dim(V)$

Pf) Recall

$\dim(W) + \dim(W^\circ) = \dim(V)$

& $\dim(W^\circ) = \dim(W^\perp)$ $\triangleright$

lemma | let V be an ips. let $W \subseteq V$ be a subsp. then:

① $\varphi : W \oplus W^\perp \to V$

$(\bar{w}, \bar{x}) \mapsto \bar{w} + x$

is an injective lin trans

If V is fin gen, φ is isom!

② $W \subseteq (W^\perp)^\perp$

if V is fin gen, $W = (W^\perp)^\perp$

Pf | ① Check linearity (ez)

lets show injectivity!

let $(\bar{w}, \bar{x}) \in \ker(\varphi)$

$\Rightarrow \varphi(w, x) = 0 \Rightarrow w + x = 0$

$\Rightarrow \|x + w\|^2 = 0$

$\langle w+x, w+x \rangle = 0$ 0 as $x \in W^\perp$, $w \in W$

$= \langle w, w \rangle + \cancel{2\langle x, w \rangle} + \langle x, x \rangle$

$= \|\bar{w}\|^2 + \|x\|^2 = 0$

$\Rightarrow \|w\| = 0$, $\|x\| = 0 \Rightarrow \boxed{(w, x = 0}$

So, we're injective!

Spose V is fg!

$\dim(W \oplus W^\perp) = \dim(W) + \dim(W^\perp) = \dim(V)$ fin gen!

$\therefore \varphi$ is bijective

$\square$

$\square$

② lets show $W \subseteq (W^\perp)^\perp$

let $w \in W$.

$\forall x \in W^\perp$, $\langle \bar{w}, \bar{x} \rangle = 0 \Rightarrow x \in (W^\perp)^\perp$

$\Rightarrow W \subseteq (W^\perp)^\perp$

Spose V is a fin gen ips. we can show equality

we note:

$\dim(W) + \dim(W^\perp) = \dim(V)$

$\dim(W^\perp) + \dim((W^\perp)^\perp) = \dim(V)$

$\Rightarrow \dim(W) = \dim((W^\perp)^\perp)$

So, $W = (W^\perp)^\perp$ $\square$

Now: we finally get back to proj

let V be a fin gen ips

$W \subseteq V$ is a subsp

Def, $\mathrm{proj}_W : V \to V$ → really $V \to W$

$v \to \underset{\omega}{?}$

By prev result. $\exists!$ $(w, x) \in W \oplus W^\perp$

st $\bar{w} + \bar{x} = v$

def $\mathrm{proj}_W(\bar{v}) = \bar{w}$

Dis this linear?

well proj_W is realy

$$V \xrightarrow{\varphi^{-1}} W \oplus W^\perp \longrightarrow W$$

$$v \longmapsto (w, x) \longrightarrow w$$

φ^{-1} is $\in \mathrm{hom}(V, W \oplus W^\perp)$ as φ is an isom

Check that the second is linear $\Rightarrow$ done by composition!

② Show $im(proj_W) = W$

well $\subseteq$ is trivial

Fix $\bar{w} \in W$

$proj_W(\bar{w}) = \bar{w}$ as $w = w + \bar{0}$ $\in [W^\perp]$

uniqueness!

$\square$

③ $\forall w \in W \quad proj_W(\bar{w}) = \bar{w}$

Done above!

④ $\forall v \in V, \quad \langle \bar{v} - proj_W(\bar{v}), proj_W(\bar{v}) \rangle = 0$

well, if we can show

$\forall v \in V \quad v - proj_W(\bar{v}) \in W^\perp$ we're done!

why is $v - proj_W(\bar{v}) \in W^\perp$?

fix $\bar{v} \in V$ $\exists!$ $(\bar{w}, \bar{x}) \in W \oplus W^\perp$ s.t. $\bar{w} + \bar{x} = \bar{v}$

$proj_V = \bar{w}$

$\Rightarrow v - proj_W(v) = (w + x) - \bar{w}$

$= x \in W^\perp$

$\square$

So..... we have a new lin trans given V a fdips $W \subseteq V$ subsp.

Def $Proj_W : V \to V$ as above

$proj_W : V \to V$ is idempotent

That is $proj_W \circ proj_W = proj_W$

$\rightarrow$

* consider the geometry.

for $\bar{v} \in V$, $proj_W(\bar{v})$ is the "closest" vec to v in W

lets show that the above is true & unique:

fix $\bar{v} \in V$

$\bar{v} = \bar{w} + \bar{x}$ where $(\bar{w}, \bar{x}) \in W \oplus W^\perp$

$proj_W(\bar{v}) = \bar{w}$

fix arbitrary $w' \in W$ and consider

$d(\bar{v}, \bar{w}')$ we wanna show this is min when $\bar{w}' = w$

$d(\bar{v}, \bar{w}')$

$= \|(\bar{x} + \underbrace{\bar{w}) - \bar{w}'}_{\text{in } W \text{. lets call it } w''}\|$

$= \| x + w'' \|$ is smallest when the following is min!

$\| x + w'' \|^2 = \langle x + w'', x + w'' \rangle$

$= \langle x, x \rangle + 2 \cancel{\langle x, w'' \rangle}^{0} + \langle w'', w'' \rangle$

$= \|x\|^2 + \|w''\|^2$

is min when $\|w''\|^2 = 0 \Rightarrow \boxed{w' = w}$ D

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in 'real life' how do I compute $proj_W$?

Plan Take a on basis of W to be $(e_1 \ldots e_m)$

→ extend to onb of V

↳ steinitz exchange to basis of V

$(e_1 \ldots e_m, v_1 \ldots v_n)$

↳ use GS to get onb of V. But note $e_1 \ldots e_m$ are uneffected

$(e_1 \ldots e_m, e_{m+1} \ldots e_{m+n})$ is onb of V.

now, to compute. Fix $v \in V$.

$$\bar{v} = \sum_{i=1}^{m+n} \langle v, e_i \rangle e_i$$

$$= \underbrace{\sum_{i=1}^{m} \langle v, e_i \rangle e_i}_{\text{in } W} + \underbrace{\sum_{j=m+1}^{m+n} \langle v, e_j \rangle e_j}_{\text{in } W^\perp}$$

$$\boxed{\mathrm{proj}_W(\bar{v})}$$

Context for GS $\to$ Recap

V ips with $(v_1, \dots, v_m)$ lin ind

$\exists!$ on form $(u_1, \dots, u_m)$

① $\mathrm{span}(v_1, \dots, v_\ell) = \mathrm{span}(u_1, \dots, u_\ell)$ $\forall \ \ell \in \mathbb{N}_m$

② $\langle u_j, v_j \rangle \Rightarrow 0$ $\forall \ j \in \mathbb{N}_m$

recipe

$$\bar{u_j} = \frac{\bar{v_j} - \left(\sum_{i=1}^{j-1} \langle v_j, u_i \rangle u_i \right)}{\| \text{top} \|}$$

$\to$ $\mathrm{proj}_W(v_j)$ $\to$ $W = \mathrm{span}(v_1, \dots, v_{j-1})$

$v_j - \mathrm{proj}_W(v_j) \in W^\perp$ by design!

$\exists$ a geometric recipe for $\boxed{GS}$

((in particular pt ④ of proj with list))

Adjoints!

lemma) Let V be a fgips.

$\forall\, T \in \hom(V,V)$

$\exists!\, T^+ \in \hom(V,V)$ st

$\langle T(x), y\rangle = \langle x, T^+(y)\rangle$

$\forall\, x,y \in V$

wut.... how do we begin???

So, $T^+ \in \hom(V,V)$

recall $\psi : V \to V^*$ as the natural iso!

$v \mapsto \lambda_v : V \to \mathbb{R}$

$$V \xrightarrow{\psi} V^* \xrightarrow{T^*} V^* \xrightarrow{\psi^{-1}} V$$

$$v \mapsto \lambda_v \mapsto T^*(\lambda_v) \to \psi^{-1}(T^*(\lambda_v))$$

$T^+ := \psi^{-1} \circ T^* \circ \psi$

is a candidate!

Show, $\forall\, x,y \in V$ $\langle x, T(y)\rangle = \langle T^+(x), y\rangle$

$$\begin{tikzcd}
\lambda_y \arrow[rr, mapsto] & & T^*(\lambda_y) := \lambda_y \circ T \arrow[d, equal] \\
V^* \arrow[r, "T^*"] & V^* & \lambda_{T^+(y)} \\
V \arrow[u, "\psi"] \arrow[r, "\psi^{-1} \circ T^* \circ \psi := T^+"'] & V \arrow[u, "\psi"'] & \\
y \arrow[uuu, mapsto] \arrow[rr, mapsto] & & T^+(y) \arrow[uu, mapsto]
\end{tikzcd}$$

since the diagram commutes.

$\lambda_y \circ T = \lambda_{T^+(y)}$ as fns in V^*

now lets show the stuff (another exp on pg 29 of week 15)

Fix $y \in V$, $x \in V$

$\lambda_y \cdot T(x) = \lambda_{T^+(y)}(x)$

$\downarrow$ def $\qquad\qquad \downarrow$ def

$\langle y, T(x) \rangle = \langle T^+(y), x \rangle$ $\square$

Pf) uniqueness

Spose $S \in \hom(V, V)$ st

$\langle x, T(y) \rangle = \langle S(x), y \rangle \quad \forall x, y \in V$

$\Rightarrow$

$\langle T^+(y), x \rangle = \langle S(y), x \rangle \quad \forall x, y \in V$

by multilin

$\langle T^+(y) - S(y), x \rangle = 0 \quad \forall x, y \in V$

fix $y \in V$. Take $x = T^+(y) - S(y)$

$\Rightarrow \langle T^+(y) - S(y), T^+(y) - S(y) \rangle = 0$

$\Rightarrow T^+(y) = S(y) \quad \forall y \in V$ $\square$

Wait: in the pf of uniqueness, did we use the fact that V is fin gen?

No. But we did in $\exists$ (ψ) !

Rem) in the inf dim setting we can def the adj of a lin trans as

Defn) let V be a ips. let $T \in \hom(V, V)$

if $\exists\, T^+ \in \hom(V, V)$ st $\forall x, y \in V$

$\langle T(x), y \rangle = \langle x, T^+(y) \rangle$

Then T^+ is unique and the adj of T

Note: we are guaranteed $\exists$ if V is fg

Lemma) let V be f.d.i.p.s.

let $T \in \hom(V,V)$

Then $(T^+)^+ = T$

Pf) for $x, y \in V$ we have

$\langle T(x), \bar{y} \rangle = \langle x, T^+(y) \rangle$

and

$\langle T^+(x), \bar{y} \rangle = \langle x, (T^+)^+(y) \rangle$

but we may swap x & y to get

$\langle T(x), \bar{y} \rangle = \langle \bar{y}, (T^+)^+(x) \rangle$

$\Rightarrow T = (T^+)^+ \longrightarrow$ same idea as above