Vector Spaces

let F be a field. A vector space V over F is a set equipped with

$+ : V \times V \to V$ → vector addition

$\cdot : F \times V \to V$ → scalar mult

St $(V, +)$ is an abelian grp (we have a 0 vector)

AND, $\forall c_1, c_2 \in F$, $\forall v_1, v_2 \in V$

- $c(v_1 + v_2) = cv_1 + cv_2$
- $(c_1 + c_2) \cdot v_1 = c_1 v_1 + c_2 v_1$
- $(c_1 \cdot c_2) \cdot v_1 = c_1 (c_2 v_1)$

→ some form of distributivity & associativity

- $1_F v = v \quad \forall v \in V$

e.g) $\mathbb{R}^2$ over $\mathbb{R}$

$\mathbb{R}[x]$ over $\mathbb{R}$

$F[x]_{\leq 57}$ over F

⋮

Sequences of reals over $\mathbb{R}$

↳ bdd

↳ Cauchy

↳ conv to 0 ...

let $I \subseteq \mathbb{R}$ be an int

$C^0(I)$, $C^1(I)$ ... $C^\omega(I)$

set of integrable func

set of bdd func

F^n over F generally

$\mathrm{Mat}_{m \times n}(F)$ over F

Defn) A vector is an element of a vector space!

lemma) let $v \in V$, $c \in F$

$$c \cdot v = \vec{0} \iff c = 0 \text{ or } v = \vec{0}$$

Pf) $\Leftarrow$ spose $c = 0$

$$c \cdot v = 0 \cdot v = (0+0)v = 0v + 0v$$

$$c \cdot v = 0v + 0v \implies 0 = 0 \cdot v$$

since we are an abelian grp

Spose $\boxed{v = \vec{0}}$

$cv = c \cdot \vec{0} = c(0+0) = c\vec{0} + c\vec{0}$

$c \cdot \vec{0} = c\vec{0} + c \cdot \vec{0} \Rightarrow \boxed{c \cdot \vec{0} = 0}$

$\Rightarrow$ spose $c \cdot \vec{v} = \vec{0}$ if $c \neq 0$, $\exists c^{-1}$

So, $c \cdot \vec{v} = \vec{0} \Rightarrow (c^{-1} \cdot c) \cdot \vec{v} = c^{-1}\vec{0} \to 0$ by above

by axiom $\Rightarrow 1\vec{v} = \vec{0} \Rightarrow \boxed{\vec{v} = \vec{0}}$

by axiom

$\square$

Lemma) let V be a vs over F

① $\vec{0}$ is unique and additive inv are unique

② $(-1_F) \cdot \vec{v} + \vec{v} = \vec{0} \quad \forall v \in V$

Pf) ① follows as $(V,+)$ is a grp

② $(-1) \cdot \vec{v} + \vec{v} = (v)(-1+1) = (v)(0) = 0$ by above

$\square$

Vector subsp

Defn) let V be a vs $/F$

A subset $W \subseteq V$ is a vector subspace of V p.t

- $(W,+)$ is a subgrp of V
- W is closed wrt scalar mult

↳ $\mathrm{im}(\cdot|_W) \subseteq W$

Lemma) If V is vs $/F$. If W is a vsubspace of V, it is a vs $/F$.

Pf) ① By def

② Immediate as $W \subseteq V$ $\square$.

Obs) let V be a vs $\mathbb{F}$. If w_1, w_2 are subsp of V

- $W_1 + W_2 := \{\bar{w}_1 + \bar{w}_2 \mid w_1 \in W_1, w_2 \in W_2\}$
- $W_1 \cap W_2$

are vs $\mathbb{F}$ (and subsp of V)

But, $W_1 \cup W_2$ is not. $\Rightarrow$ closure wrt addition doesn't work if they're disjoint!

note: for now, sums are finite (unable to take limits)

So, if J is an indexing set and $v_j \in V \ \forall j \in J$

$\sum_{j \in J} a_j v_j$ is a sum where abfmz!

Defn) Indexing set and indexed families

Let I be a set (possibly empty). An indexed family of vect in VS V $\mathbb{F}$ is a function

$$f : I \to V$$
$$i \mapsto \vec{v}_i$$

$(v_i \mid i \in I) \to$ is the indexed family

Note, I need not be injective as

$(v_1, v_1, v_2) \neq (v_1, v_2)$

by abusing notation

$(v_i \mid i \in I) \subseteq V$

if $J \subseteq I$ $(v_j \mid j \in J) \subseteq (v_i \mid i \in I)$

$\hookrightarrow$ subfamily

note:

① $\sum_{\varnothing} b_k v_k = 0$ ② $\sum_{i \in I} a_i v_i + \sum_{i \in I} b_i v_i = \sum_{i \in I} (a_i + b_i) v_i$ ③ $c \sum_{i \in I} a_i v_i = \sum_{i \in I} (c a_i) v_i$

④ if $I = I' \sqcup I''$ $\sum_{i \in I} a_i v_i = \sum_{i \in I'} a_i v_i + \sum_{i \in I''} a_i v_i$

Defn let V be vs $\mathbb{F}$.

An indexed family $(v_i \mid i \in I)$ is linearly ind iff

$$\sum_{i \in I} a_i \vec{v}_i = \vec{0} \implies a_i = 0, \forall i \in I$$

Eg. Standard Family

singleton fam of nonzero vect.

If a family has $\vec{0}$ it is linearly dep

Any fam with repeats $\implies$ $\Downarrow$ immediate from above

↳ scaled repeats?

Defn An indexed family $(v_i \mid i \in I)$ is linearly dep iff it isn't lin ind

note: Empty family is lin ind (vacuously)

↳ Any subfam of a lin ind fam is lin ind!

lemma let V be a vs $/\mathbb{F}$.

The family $(v_i \mid i \in I)$ is

lin dep $\iff$ $\exists\, k \in I$, scalars $b_i \in \mathbb{F}$ s.t.

$$v_k = \sum_{i \in I \setminus \{k\}} b_i v_i$$

↳ v_k is a linear comb of other vecs!

Pf $\implies$ span $(v_i \mid i \in I)$ is lin dep

So, $\exists$ scalars $a_i \in \mathbb{F}$ not all zero s.t.

$$\sum_{i \in I} a_i v_i = 0 \qquad \exists\, k \in I \text{ s.t. } a_k \neq 0.$$

$$\text{So, } a_k v_k = -\sum_{i \in I \setminus \{k\}} a_i v_i \qquad \text{as } a_k \neq 0\ \exists\, a_k^{-1}$$

$$v_k = \sum_{i \in I \setminus \{k\}} \frac{-a_i}{a_k} \vec{v}_i$$

$\square$

$\Leftarrow$ $\exists k \in I$ s.t $\exists b_j \in F$

$v_k = \sum_{i \in I \setminus \{k\}} b_i \vec{v_i}$ note $-1 \neq 0$ in F. So,

$$0 = \sum_{i \in I \setminus \{k\}} b_i \vec{v_i} + (-1) v_k$$

$\Rightarrow$ $(v_i \mid i \in I)$ is lin dep !

Corr Subfam of lin ind fam is lin ind (not true for lin dep)

Eg look at week 5 pg 43 onwards (but its easy)

Defn Span$(\vec{v_i} \mid i \in I)$

let $(v_i \mid i \in I)$ be a indexed fam in V. The span of the family

$$\text{span}(v_i) := \left\{ \sum_{i \in I} a_i \vec{v_i} \mid a_i \in F, \text{ a.b.f.n.z} \right\}$$

This is the smallest vector subsp that contains $(v_i \mid i \in I)$
($\hookrightarrow$ by containment !

Note: span$(\emptyset) = \vec{0}$

Defn The vs V is said to be fin gen if

$\exists$ a finite indexed family $B \subseteq V$ s.t span$(B) = V$

Eg $\mathbb{R}^3$, $\text{Mat}_{m \times n}(\mathbb{R})$, $\mathbb{C}$ over $\mathbb{R}$, $\mathbb{Q}[\sqrt{2}]$

$\mathbb{R}^3 \to$ standard; $\text{Mat}_{m \times n}(\mathbb{R}) \to$ standard vect

Non eg $\mathbb{R}[x] \to$ always have a max.

$C^\infty(\mathbb{R})$, $C^\omega(\mathbb{R})$

$\mathbb{C}$ over $\mathbb{Q}$, $\mathbb{R}$ over $\mathbb{Q}$

Note: $\forall \alpha \in \mathbb{C}$, $\mathbb{Q}[\alpha]$ is a vs $/\mathbb{Q}$.

Lemma $\mathbb{Q}[\alpha]$ is fin gen $/\mathbb{Q}$

$\Longleftrightarrow$ $\quad \mathbb{Q}[\alpha] := \mathrm{span}_{\mathbb{Q}}(\alpha^i \mid i \in \mathbb{N} \cup \{0\})$

α is algebraic

Pf $\Rightarrow$ let $\alpha \in \mathbb{C}$

Spse $\mathbb{Q}[\alpha]$ is fin gen $/\mathbb{Q}$

$\exists v_1, v_2 \ldots v_m \in \mathbb{Q}[\alpha]$ s.t

$\mathbb{Q}[\alpha] = \mathrm{span}(v_1, \ldots, v_m)$

for each $1 \le i \le m$ $v_i \in \mathbb{Q}[\alpha]$ so

$v_i = q_{i_0} + q_{i_1}\alpha + \cdots + q_{i_{n_i}}\alpha^{n_i}$ → deg is n_i

with $q_{i_k} \in \mathbb{Q}$ $\forall i \in \mathbb{N}_m$ $\forall k \in \mathbb{N}_{n_i}$

rewrite $v_i = \mathrm{mess}_i$

let $M = \max\{n_1 \ldots n_m\}$

$\alpha^{M+1} \in \mathbb{Q}[\alpha]$ so for $d_i \in \mathbb{Q}$

$\alpha^{M+1} = d_1\vec{v_1} + \cdots + d_m\vec{v_m}$

$= d_1\,\mathrm{mess}_1 + \cdots + d_m\,\mathrm{mess}_m$

expand and write in powers of α

$\alpha^{M+1} = r_0 + r_1\alpha + \cdots r_M\alpha^M$

So, α is a root of the following nonzero poly

$p(x) = x^{M+1} - (r_0 + r_1 x + \cdots r_M x^M)$

nonzero

so $\alpha \in \overline{\mathbb{Q}}$ → it is algebraic

→ to make coeff in $\mathbb{Z}$ just multiply to get $\tilde{p}(x)$

$\Leftarrow$ Spose α is algebraic

$\deg p \geq 1$

$\exists$ nonzero poly $p \in \mathbb{Z}[x]$ st $p(\alpha) = 0$

let $N = \deg(p) \geq 1$

$$p(x) = \sum_{n=0}^{N} a_n x^n \quad \text{with } a_i \in \mathbb{Z} \ \& \ a_N \neq 0$$

$\boxed{\text{Claim:}}$ $\mathbb{Q}[\alpha] = \text{span}(1, \alpha, \ldots \alpha^{N-1})$

we will show this by 2 way containment

$\supseteq$ is trivial as $(1, \alpha, \ldots, \alpha^{N-1}) \subseteq (1, \alpha, \ldots\ldots)$

let $v \in \mathbb{Q}[\alpha]$

$$v = \sum_{i \in \mathbb{N} \cup \{0\}} b_i x^i \qquad b_i \in \mathbb{Q} \text{ abfmz}$$

Def $\deg(\vec{v}) = \max\{i \mid b_i \neq 0\}$

note: this is nonemp as if $v = 0$ we are done immediat

So spose $v \neq 0$

if $\deg(v) < N$ we're done!

spose $\deg(v) \geq N$

let $j := \deg(v) - (N-1)$

we will induct on j.

if $j = 1$, $\deg(\vec{v}) = N$, we are done as α^N can be written in terms of $1, \alpha, \alpha^2 \ldots, \alpha^{N-1}$

$$\text{as } 0 = \sum_{n=0}^{N} a_n \alpha^n \Rightarrow \alpha^N = \sum_{n=0}^{N-1} \frac{a_n}{a_N} \alpha^n$$

So, done with base case as $\vec{v} \in \text{span}(1, \ldots, \alpha^{N-1})$

Spose if $1 \leq j < k$ then $\vec{v} \in$ RHS (strong ind)

Show this holds if $j = k \Rightarrow \deg(v) = k + N - 1$

$$\vec{v} = b_{N-1+k}\, \alpha^{(N-1)+k} + \underbrace{\vec{v}'}_{\text{taking the remaining terms}} \quad \& \quad \deg(v') < N - 1 + k$$

So, $\vec{v} = b_{N-1+k}\alpha^{k-1} \cdot \alpha^N + \vec{v}'$ use previous sheet

$$= b_{N-1+k}\alpha^{k-1}\left(-\frac{a_0}{a_N} - \frac{a_1}{a_N}\alpha \cdots - \frac{a_{N-1}}{a_N}\alpha^{N-1}\right) + v'$$

$$= b_{N-1+k}\left(-\frac{a_0}{a_N}\alpha^{k-1} - \frac{a_1}{a_N}\alpha^k - \cdots \frac{a_{N-1}}{a_N}\alpha^{N+k-2}\right) + v'$$

deg of this $\leq N-1+k-1$ deg of this $< N-1+k$

By inductive hypothesis, done! $v \in RHS$

Lingo) let V be a vs $/F$

We say $(\vec{v_i} \mid i \in I)$ is spanning iff

$V = \text{span}(\vec{v_i} \mid i \in I)$

Defn) let V be a vs $/F$

A basis for V is a linearly independent, spanning indexed fam

eg (e_1, e_2) in $\mathbb{R}^2$

Defn) let V be a vs $/F$.

We say V is finite dimensional iff it has a finite basis.

Note: immediately fin dim $\Rightarrow$ fin gen. The other direction also holds (later)