Motivating Ideas

How do we descr a subsp of a vec space?

→ image of a lin trans?

→ using a basis?

→ using an eqn?

e.g $\mathbb{R}^3$

dim 0 – $\{0\}$

dim 1 – line passing through origin → $\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; ax + by + cz = 0 \right\}$

dim 2 – plane " " "

3 – $\mathbb{R}^3$

↳ 1 eqn describes a plane

2 describes a line

how to we make this rigorous!

Dual spaces

Spose $W = \text{span}(v_1 \dots v_k) \subseteq V \cong F^n$

a) How do we know if $\vec{v} \in V$ is in W?

b) Does there exist a way to present W as a soln space to a system of eqn?

Defn) let V be fgvs $/F$.

let $W \leq V$ be a subsp.

$\text{codim}(W) := \dim(V) - \dim(W)$

→ represents no of eqns we need

eg) if $\text{codim}(W) = 57$ we want

$$W = \bigcap_{j=1}^{57} \ker(\lambda_j) \quad \text{where } \lambda_j \in \text{Hom}(V, F)$$

↳ each one represents a solution space!

Defn The dual space of vs V $/F$ is
$V^* := \hom(V,F)$
$\lambda \in V^*$ is called a linear functional!

Remark if V is fg,
(1) $\dim(V) = \dim(V) \cdot 1$
$= \dim(V) \cdot \dim(F) = \dim(\hom(V,F))$
$= \dim(V^*)$

(2) if $V = \{\vec{0}\}$. $\hom(V,F)$ is just trivial.
$\therefore \hom(V,F) = \{\vec{0}\} \longrightarrow$ 0 functional

(3) if $V \neq \{0\}$, V^* is more than trivial!

Defn / Lemma if $v = (v_1 \dots v_n)$ is a basis of V. Then $v^* = (\lambda_1 \dots \lambda_n)$ is a basis of the dual sp.

$$\lambda_i(v_j) = \begin{cases} 1 & , i=j \\ 0 & , i \neq j \end{cases}$$

This is called the dual basis of v / V.

Pf We note,
$\lambda_i \in V^*$ as its defined on a basis
lets prove v^* is a basis $\iff$ lin ind or span
as $\dim V^* = n$

Spose, for $a_i \in F$
$$\sum_{i=1}^{n} a_i \lambda_i = 0 \longrightarrow 0 \text{ functional!}$$
$\therefore$ let $j \in \mathbb{N}_n$ be given,
$$\sum_{i=1}^{n} a_i \lambda_i(v_j) = 0(v_j) = 0$$
$\hookrightarrow = a_j$

$\therefore \forall j \in \mathbb{N}_n, a_j = 0$! $\longrightarrow \therefore$ lin ind!

Corr) there is an isomorphism from $T: V \to V^*$
Define it as $\forall i \in \mathbb{N}_n$

$$T(v_i) = \lambda_i$$

note:) $(V^*)^*$ is the double dual $(\hom(V^*, F))$

To get $V \cong V^*$ we need to choose a basis.

This is not the case for showing $V \cong (V^*)^*$, There exists a canonical iso

Defn) let V be a vs $/F$. let $W \subseteq V$ (subset)
Then

$$W^0 := \{\lambda \in V^* \mid \lambda(w) = 0, \forall w \in W\}$$

annahilator of W!

ex) Show $W^0 \leq V^*$ is always a subsp!

eg) consider $\mathbb{R}^5$

$V = (e_1 \ldots e_5)$ $\quad V^* = (\lambda_1 \ldots \lambda_5)$

Spose $W = \text{span}(e_1, e_2)$

What is $W^0 \leq V^*$?

we need that, for $\lambda \in W^0$ $\lambda(e_1) = 0, \lambda(e_2) = 0$

$\lambda(e_3, e_4, e_5)$ can be anything!

$\therefore \lambda \in \text{span}(\lambda_3, \lambda_4, \lambda_5) \Rightarrow W^0 \subseteq \text{span}(\lambda_3 .. \lambda_5)$

Conversely, $\lambda \in \text{span}(\lambda_3, \lambda_4, \lambda_5) = a\lambda_3 + b\lambda_4 + c\lambda_5$

let $w \in W \Rightarrow w = a_1 v_1 + a_2 v_2$

linearity $\Rightarrow \lambda(w) = 0 \Rightarrow \lambda \in W^0$

$\therefore W^0 = \text{span}(\lambda_3, \lambda_4, \lambda_5) \to \dim(3)$

we see $\dim(W) + \dim(W^0) = \dim(V)$

$\Rightarrow \dim(W) = \text{codim}(W^0)$

cool!

Thm) let V be fg $\setminus \mathbb{F}$. let $W \subseteq V$ be a subsp.

$$\dim(W^0) + \dim(W) = \dim(V)$$

or

$$\dim(W^0) = \operatorname{codim}(W)$$

Pf) let $(v_1, \dots, v_m)$ be a basis of W.

By Steinitz, we can extend this to a basis of V.

$$(\underbrace{v_1, \dots, v_m}_{W}, \dots v_n)$$

Claim: $\dim(W^0) = n-m$

Basis $(\lambda_{m+1} \dots \lambda_n) \longrightarrow$ these are lin ind as a subfam of the dual basis.

All that is left $\longrightarrow B$

$$\operatorname{span}(\lambda_{m+1}, \dots, \lambda_n) = W^0$$

$\subseteq$ let $\lambda \in \operatorname{span}(B)$.

So $\exists\, a_i \in \mathbb{F}$ s.t

$$\lambda = \sum_{i=m+1}^{n} a_i \lambda_i$$

fix $w \in W = \operatorname{span}(v_1 \dots v_m) \Rightarrow w = \sum_{i \in \mathbb{N}_m} b_i v_i$

$$\therefore \lambda(w) = \sum_{i=m+1}^{n} a_i \lambda_i(w) = \sum_{i=m+1}^{n} a_i \lambda_i\Big(\sum_{j \in \mathbb{N}_m} b_j v_j\Big)$$

$$= \sum_{i=m+1}^{n} a_i \Big[\sum_{j \in \mathbb{N}_m} b_j \lambda_i(v_j)\Big] = \sum_{i=m+1}^{n} a_i \cdot 0 = 0$$

$$1 \leq j \leq m < i \leq n$$

$\supseteq$ let $\lambda \in W^0$ $\exists$ scalars $a_i \in \mathbb{F}$ s.t

$$\lambda = \sum_{i=1}^{n} a_i \lambda_i \longrightarrow \text{we will show } \forall i \in \mathbb{N}_m \; a_i = 0$$

Pick $k \in \mathbb{N}_m$. We have $v_k \in W$.

$0 = \lambda(v_k) = \sum_{i=1}^{n} a_i \lambda_i(v_k) = a_k \lambda_k(v_k) = a_k$

So, $\lambda \in \operatorname{span}(\beta)$

$\therefore W^0 = \operatorname{span}(\beta)$ — a basis of W

So, it follows

$$\operatorname{codim}(W) = \dim(W^0)$$ D

Remark) The subsp W is cut out by $\operatorname{codim}(W)$ eqn.

W is exactly the set of vectors in V that are killed by $\lambda_{m+1}, \dots, \lambda_n$

$$W = \bigcap_{i=m+1}^{n} \ker(\lambda_k)$$ $\lambda_k \in V^*$ as above!

Lemma) If V is a vs $/F$. let W_1 & W_2 be subsp. Then, $W_1 \subseteq W_2$ iff $(W_2)^0 \subseteq (W_1)^0$

Pf) $\Rightarrow$ Say $W_1 \subseteq W_2$.

Let $\lambda \in (W_2)^0$.

fix $w \in W_1 \Rightarrow w \in W_2 \Rightarrow \lambda(w) = 0$

$\therefore \lambda \in (W_1)^0 \Rightarrow (W_2)^0 \subseteq (W_1)^0$

$\Leftarrow$ contrapos. Spose $W_1 \not\subseteq W_2$ (we show $(W_2)^0 \not\subseteq (W_1)^0$)

$\exists v \in W_1$ s.t $v \notin W_2$

find λ s.t $\lambda(v) \neq 0$, $\lambda \in (W_2)^0$

Take basis $W_2 \to (\bar{v}_1, \dots, \bar{v}_k)$

Extend to larger lin ind fam by adding $v \in W_1 \setminus W_2$

$(v_1, \dots, v_k, v)$

Use Steinitz to extend to basis of V.

$(v_1 \ldots v_k, v, \text{other})$

Choose $\nearrow$ dual basis elt. Call it λ.

Clearly $\lambda \in W_2^0$.

But $\lambda \notin (W_1)^0$ as $\lambda(v) = 1$

So $(W_2)^0 \not\subseteq (W_1)^0$ $\square$

Corr) let V be fg. If $W_1, W_2 \leq V$ subsp

$$W_1 = W_2$$
$$\Longleftrightarrow$$
$$(W_1)^0 = (W_2)^0$$

Pf) Prev lemma $\Rightarrow$ 2 way containment

Thm) let V be fg $/F$.

Then V & $(V^*)^*$ are canonically iso.

Pf) Cook up a linear trans:

$$\Phi : V \longrightarrow (V^*)^*$$
$$v \longmapsto e_{\bar{v}} : V^* \to F$$
$$\lambda \mapsto \lambda(v)$$

Is $e_{\bar{v}} \in \hom(V^*, F)$

Show: $e_{\bar{v}}(\lambda_1 + \lambda_2) = e_{\bar{v}}(\lambda_1) + e_{\bar{v}}(\lambda_2)$

yes, this is true as

$(\lambda_1 + \lambda_2)(\bar{v}) := \lambda_1(\bar{v}) + \lambda_2(\bar{v})$ by def of fun +

Scalar mult follows similarly

lets show $\Phi \in \hom(V, V^{**})$ & bijective.

$\longrightarrow$

let $v_1, v_2 \in V$ & $\alpha \in F$.

$\Phi(v_1+v_2) = e_{(v_1+v_2)}$

let $\lambda \in V^*$

$e_{(v_1+v_2)}(\lambda) = \lambda(v_1+v_2) = \lambda(v_1)+\lambda(v_2) = e_{v_1}(\lambda)+e_{v_2}(\lambda)$

So, they are identical as func. Works the same way for scalar mult.

$\therefore\ \Phi \in \hom(V, V^{**})$

As, $\dim(V) = \dim(V^{**})$ injectivity is sufficient.

let v, v' s.t $\Phi(v) = \Phi(v')$

$\Rightarrow e_{\bar{v}} = e_{\bar{v}'}$ as funct

$\forall \lambda \in V^*$ $\quad e_{\bar{v}}(\lambda) = e_{\bar{v}'}(\lambda)$

$\iff \lambda(\bar{v}) = \lambda(\bar{v}') \Rightarrow \lambda(\bar{v} - v') = 0$

Lets show $(\operatorname{span}(\bar{v} - v'))^\circ = V^*$

This would suggest $\dim(\operatorname{span}(v-v')) = 0 \Rightarrow v = v'$

let $x \in \operatorname{span}(\bar{v} - v')$, $\lambda(x) = \lambda(\alpha(v-v')) = 0$ $\forall \lambda \in V^*$

So, $V^* \subseteq (\operatorname{span}(v-v'))^\circ$ Done! $\quad$ D

Dual Maps

We know if V is a vs $/F$ we have V^*.

If W is a vs $/F$ we have W^*

Take $T \in \hom(V, W)$, can we get a map between the dual spaces?

Consider for $\lambda \in W^*$

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[dr, "\lambda \circ T"'] & W \arrow[d, "\lambda"] \\ & F \end{tikzcd}$$

Take $\lambda \in W^*$

$\lambda \circ T \in \text{hom}(V, F) = V^*$ by composition

So, given

$T: V \to W$ we get

$T^*: W^* \to V^*$

$\lambda \mapsto \lambda \circ T$

This is the dual map

Spose V, W, X are vs $\mathbb{F}$.

Take $T \in \text{hom}(V, W)$ $S \in \text{hom}(W, X)$

$\forall \lambda \in X^*$

$$(S \circ T)^*(\lambda) \overset{\text{def}}{=} \lambda \circ (S \circ T)$$
$$= (\lambda \circ S) \circ T$$
$$= T^*(\lambda \circ S)$$
$$= T^*(S^*(\lambda))$$
$$= T^* \circ S^*(\lambda)$$

E.g.) Page 60 week 9

Defn) let $A \in \text{Mat}_{m \times n}(F)$. The transpose of A, A^t, is the matrix in $\text{Mat}_{n \times m}(F)$ s.t.

$\forall i \in \mathbb{N}_n \ \forall j \in \mathbb{N}_m$

$(A^t)_{ij} = A_{ji}$

note: $(A^t)^t = A$!

Commutative Maps

$$\begin{tikzcd} \mathcal{V} & \mathcal{W} \\ V \arrow[r, "T"] & W \\ F^n \arrow[u, "i_{\mathcal{V}}"] \arrow[r, "{}_{\mathcal{W}}[T]_{\mathcal{V}}"'] & F^m \arrow[u, "i_{\mathcal{W}}"'] \end{tikzcd}
\qquad
\begin{tikzcd} \mathcal{W}^* & \mathcal{V}^* \\ W^* \arrow[r, "T^*"] & V^* \\ F^m \arrow[u, "i_{\mathcal{W}^*}"] \arrow[r, "{}_{\mathcal{V}^*}[T^*]_{\mathcal{W}^*}"'] & F^n \arrow[u, "i_{\mathcal{V}^*}"'] \end{tikzcd}$$

${}_{\mathcal{W}}[T]_{\mathcal{V}}$ ↳ $m \times n$ ${}_{\mathcal{V}^*}[T^*]_{\mathcal{W}^*}$. ↳ $n \times m$

lemma) let V, W be fin gen /F s.t

$\left.\begin{aligned} \dim(V) &= n \\ \dim(W) &= m \end{aligned}\right\} \geq 1$

let $T \in \hom(V, W)$

let $\mathcal{V}, \mathcal{W}$ be bases of V, W.

$${}_{\mathcal{V}^*}[T^*]_{\mathcal{W}^*} = \left({}_{\mathcal{W}}[T]_{\mathcal{V}}\right)^t$$

Pf) Suppose $\mathcal{V} = (v_1 \ldots v_n)$
$\mathcal{W} = (w_1 \ldots w_m)$

So, $\mathcal{V}^* = (\lambda_1 \ldots \lambda_n)$
$\mathcal{W}^* = (\mu_1 \ldots \mu_m)$

let

$A = {}_{\mathcal{W}}[T]_{\mathcal{V}}$, $C = {}_{\mathcal{V}^*}[T^*]_{\mathcal{W}^*}$

let $j \in \mathbb{N}_m$ and $k \in \mathbb{N}_n$

Show $A_{jk} = C_{kj}$

By def of ${}_{\mathcal{V}^*}[T^*]_{\mathcal{W}^*}$ & $T^* : W^* \to V^*$
$\mu \mapsto \mu \circ T$

$T^*(\mu_j) \to$ gives values in the j^{th} column of ${}_{\mathcal{V}^*}[T^*]_{\mathcal{W}^*}$
$\in V^*$

$T^*(\mu_j) = \sum_{r=1}^{n} c_{rj}\, \lambda_r$, we care abt c_{kj}

if we evaluate

$$T^*(\mu_j)(\bar{v}_k) = \sum_{r=1}^{n} c_{rj}\, \lambda_r(\bar{v}_k) = c_{rk}$$

$\to$ by dual basis!

But also,

$$T^*(\mu_j)(\bar{v}_k) = \mu_j(T)(\bar{v}_k)$$

$\to$ k^{th} col of A!

So,

$$T(\bar{v}_k) = \sum_{r=1}^{m} A_{rk}(\bar{w}_r)$$

$$\Rightarrow \mu_j(T)(\bar{v}_k) = \mu_j\left(\sum_{r=1}^{m} A_{rk}(\bar{w}_r)\right)$$

$$= \sum_{r=1}^{m} A_{rk}\, \mu_j(w_r) \to \text{dual basis}$$

$$= A_{jk}$$

So, $A_{jk} = c_{kj}$ $\square$.

Remark) let $A \in Mat_{m\times n}(F)$ $B \in Mat_{n\times p}(F)$

$$(AB)^T = B^T A^T$$

follows as

$$(S \circ T)^* = T^* \circ S^*$$

$\to$ pullback contravt

Also) Spose A is inv

$$(A^{-1})^t = (A^t)^{-1}$$

As $\exists\, B \in Mat_{n\times n}(F)$ s.t. $AB = I_{n\times n}$

$$I_{n\times n} = (A\cdot B)^t = B^t A^t$$

$$\Rightarrow (A^t)^{-1} = B^t = (A^{-1})^t$$

Lemma Spose V, W are fin gen $\mathbb{F}$.

let $T \in \hom(V, W)$

(1) $\ker(T^*) = (\mathrm{im}(T))^0$

(2) $\dim(\ker(T^*)) = \dim(\ker(T)) + \dim(W) - \dim(V)$

Pf (2) lets assume 1

$\dim(\ker(T^*)) = \dim((\mathrm{im}(T))^0)$

$= \mathrm{codim}(\mathrm{im}(T))$

rank nullity

$= \dim(W) - \dim(\mathrm{im}(T))$

$= \dim(W) - \dim(V) + \dim(\ker(T))$

(1) 2 way containment.

$\subseteq$ let $\lambda \in \ker(T^*)$

$\Rightarrow T^*(\lambda) = \lambda \circ T$ is the zero functional.

$\forall v \in V \; T^*(\lambda)(v) = 0$

$\Rightarrow \lambda(T(\bar{v})) = 0 \quad \forall v \in V.$

$\therefore \forall w \in \mathrm{Im}(T), \; \lambda(w) = 0 \Rightarrow \lambda \in (\mathrm{im}(T))^0$

$\supseteq$ let $\lambda \in (\mathrm{im}(T))^0$

let show it is the zero functional in V^* $(T^*(\lambda))$

$\forall v \in V \Rightarrow \lambda(T(\bar{v})) = 0$

$\Rightarrow \lambda \circ T$ is the zero functional

$= T^*(\lambda) \Rightarrow \lambda \in \ker(T^*)$

lemma let V, W fg $\mathbb{F}$.

$T \in \hom(V, W)$

① $\dim(\operatorname{im}(T)) = \dim(\operatorname{im}(T^*))$

② $\operatorname{Im}(T^*) = (\ker(T))^0$

Pf ① we know, by rank-nullity

$\dim(\operatorname{im}(T^*)) = \dim(W^*) - \dim(\ker(T^*))$

by prev $\longrightarrow$ $= \dim(W) - (\dim(W) - \dim(V) + \dim(\ker(T)))$

$= \dim(V) - \dim(\ker(T))$

$= \dim(\operatorname{im}(T))$

② $\operatorname{im}(T^*) = (\ker(T))^0$

Plan: 1 way containment & dimension anal

$\dim(\operatorname{im}(T^*)) = \dim(\operatorname{im}(T)) = \operatorname{codim}(\ker(T))$

$= \dim(\ker(T)^0)$ ⊳

let $\lambda \in \operatorname{im}(T^*)$

$\exists\, \mu \in W^*$ st $T^*(\mu) = \mu \circ T = \lambda$

fix $v \in \ker(T)$ show $\lambda(\bar{v}) = 0$.

$\lambda(\bar{v}) = \mu \circ T(\bar{v}) = \mu(0) = 0$

$\therefore\ \lambda \in \ker(T)^0 \implies \operatorname{im}(T^*) = \ker(T)^0$ D