# Multivariable Stuff

$V, W$ fdips / $\mathbb{R}$
↳ gives geom
↳ gives topology

**Lemma)** let $T \in \hom(V, W)$
$T$ is cts at $\bar{v} = \bar{0}_V$

**Pf)** fix $\varepsilon > 0$

By prev work $\exists\, c \in \mathbb{R}$ st $\forall\, v \in V$
$\|T(\bar{v})\|_W \leq c\,\|v\|_V$

Choose $\delta = \dfrac{\varepsilon}{c+1}$ $\square$

**Corr)** Let $V, W$ be fdips
$T \in \hom(V, W)$, then $T : V \to W$ is cts!

**Pf)** fix $v \in V$
take a seq $n \mapsto v_n$ that conv to $\bar{v}$
Task $n \mapsto T(v_n)$ conv to $T(\bar{v})$

Note $n \mapsto (v_n - v)$ conv to $0_V$

we also know $n \mapsto T(v_n - v)$ conv to $0_W$ by prev

But $T(v_n - v) = T(v_n) - T(v)$
↳ conv to $0_W \iff n \mapsto T(v_n)$ conv to $T(\bar{v})$ $\square$

$\therefore T$ is cts!

**Lemma)** Vector Addition is cts. Let $V$ be a fgips

$$V \times V \longrightarrow V$$
$$(v, w) \longmapsto v + w$$

is cts. Immediate $\square$

**Lemma)** Scalar Mult is cts

$$\mathbb{R} \times V \longrightarrow V$$
$$(\alpha, v) \longmapsto \alpha v$$

Pf: ex

**Fact)** See hubbard

Let $V$ be fgips & let $W_1, W_2 \dots W_m$ be fgips.

Let $X \subseteq V$. Spose we have some func

$f_j : X \to W_j$ $\to$ cts

then

$$F : X \longrightarrow W_1 \times W_2 \dots W_m$$
$$x \longmapsto (f_1(x), \dots f_m(x))$$

is cts

**Pf)** Actually iff $\to$ virtue of prod topology

e.g. $(0, \infty) \longrightarrow \mathbb{R}$

$$t \longmapsto (\ln(t), e^t, \cos(t))$$

is cts!

Caution!

$f: \mathbb{R}^2 \to \mathbb{R}$

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{cases} \dfrac{xy}{x^2+y^2}, & \begin{pmatrix} x \\ y \end{pmatrix} \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Claim! $f$ isn't cts at $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

The problem is that the domain is $\mathbb{R}^2$ and cod $\mathbb{R}$

$f$ is cts at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ pt

$\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. if

$$d\left(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) < \delta$$

$$d\left(f\begin{pmatrix} x \\ y \end{pmatrix}, f\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) < \varepsilon$$

note if $x=0$ or $y=0$ then $f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

but, we have more than 2 direc of approach

Consider $y = 2x$

$$\begin{pmatrix} x \\ 2x \end{pmatrix} = \frac{2x^2}{5x^2} = \frac{2}{5} \quad \text{oops!}$$

$\hookrightarrow$ So, no uniform behavior at origin

$$\lim_{\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}} f\begin{pmatrix} x \\ y \end{pmatrix} \text{ d.n.e}$$

# Differentiation

Goal: replace complex nonlinear things with simple linear things

Recall: The derivative is the main tool to linearize a func in 1 var

$f: \mathbb{R} \to \mathbb{R}$ $\quad a \in \mathbb{R}$

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

how do we generalize to a higher dim?

Same def doesn't work (can't divide by vector)

Neither does dividing by $\|h\|$ (different limit!)

↳ eg $f(x) = |x|$ would be diff'ble with $f'(x) = 1$ $\forall x \in \mathbb{R}$ (even 0)

Recast to the following

Let $X$ be an open subset of $\mathbb{R}$.

Let $f: X \to \mathbb{R}$ be a func

Then $f$ is diff'ble at $a \in \mathbb{R}$ with derivative $m \in \mathbb{R}$ iff

$$\lim_{h \to 0} \frac{1}{h} \left( f(a+h) - f(a) - (m \cdot h) \right) = 0$$

↳ this is linear as a func of $h$!

**Defn** Spse $V, W$ are fdips / $\mathbb{R}$.

let $U \subseteq V$ be open

Spse $f: U \to W$ is a func & $\bar{a} \in U$

If $\exists\, T_{\bar{a}} \in \text{hom}(V, W)$ s.t.

$$\lim_{h \to 0} \frac{\| f(a+h) - f(\bar{a}) - T_{\bar{a}}(\bar{h}) \|_W}{\| h \|_V} = 0$$

then $f$ is diff'ble at $a$ with derivative $T_{\bar{a}} \in \text{hom}(V, W)$

if $f$ is diffble at each $a \in U$

$Df : U \to \hom(V, W)$

$\bar{a} \to T_a : V \to W$

is the "total derivative" & $f$ is diffble.

Notation, we use $D_a f$ or $T_{\bar{a}}$ for $Df(\bar{a})$

Wait! Does this actually make sense?

$f : \mathbb{R} \to \mathbb{R}$

$x \mapsto \cos(x)$

fix $a \in \mathbb{R}$ $\quad f'(a) = -\sin(a)$

$T_{\bar{a}} \in \hom(\mathbb{R}, \mathbb{R})$

$T_a : \mathbb{R} \to \mathbb{R}$

$h \mapsto (-\sin(a))\, h$

Remarks!

1. We need to show that, when it exists, $T_a$ is unique
2. for $F : \mathbb{R} \to \mathbb{R}$ & $a \in \mathbb{R}$ if $v$ is a basis of $\mathbb{R}$ then
   ↳ non zero!

   ${}_v[T_a]_v = [f'(a)]$

   → 295 def
3. By choosing $U \subseteq V$ to be open, no endpts & boundary pts
4. Our def is ind of $\|\cdot\|_V$ & $\|\cdot\|_W$

   wait... we certainly used them in the def

   ↳ But nw says all norms are equiv
5. 

→

5 $$\lim_{\bar{h} \to 0_V} \frac{\| f(a+h) - f(\bar{a}) - T_{\bar{a}}(\bar{h}) \|_W}{\|\bar{h}\|_V} = 0$$

$$\iff$$

$$\lim_{h \to 0_V} \frac{f(a+h) - f(a) - T_{\bar{a}}(\bar{h})}{\|\bar{h}\|_V} = 0_W$$

$W \to \mathbb{R}$

$w \mapsto \|\bar{w}\|_W$ is cts

$\hookrightarrow$ justifies 5