Exam: Tuesday 4-6 in class

Format: 5 choose 4, closed book

Sylow Thms.

Defn let G a finite grp of order $p^e \cdot m$ for $p \nmid m$
A p-Sylow subgrp of G is a subgrp of order p^e

Sylow thm

1) Any finite grp has a p-sylow!
2) Any 2 p-sylows are conjugate
   precisely if $H \leq G$ subgrp & K is a p-sylow of G
   $\Rightarrow \exists g \in G$ so $gKg^{-1} \cap H$ is a p-sylow!
3) If s is the number of p-sylows of G
   $\Rightarrow s \mid m$ & $s \equiv 1 \pmod{p}$

Rmk $s = 1$ iff the p-sylow is normal!

Typical application Show a grp isn't simple by producing a normal subgrp via Sylow Thm!

6.9
- p-sylow of S_n for $(n \leq p^2)$ (read from lecture)
- p-sylow of D_n
- p-sylow of $GL_n(\mathbb{F}_q)$ for small n!

II Free Grps & Pres

- Free grp F_n on n generators!
- Mapping prop of F_n:

  If G is any grp and $g_1, \dots, g_n \in G$ are any elts
  $\exists!$ gp homo $\varphi : F_n \to G$ so $\varphi(i^{th}$ gen$) = g_i$

- Presented group $\langle a_1, \dots, a_n \mid r_1, \dots, r_m \rangle$

  formal sym (gen) ; r_i ↪ elts of free gp gen'd by $a_1, \dots, a_n$ (reln)

  $= F_n / N \to$ smallest normal subgrp containing $r_1, \dots, r_m$

↳ also has mapping prop! $\exists!$ gp hom $\varphi : F_n/N \to G$
give $g_1, \dots, g_n$ ($a_1, \dots, a_n$ img) s.t reln holds!

- A presentation of a grp G is an isomorphism
  $G \cong \langle a_1, \dots, a_n \mid r_1, \dots, r_m \rangle$

e.g. $D_n \cong \langle a, b \mid a^2, a^n, abab \rangle$ (a refl b rot)
(2 pts)

Mapping prop gives a homo $\varphi : \quad \to D_n$
$b \mapsto$ rot $\frac{2\pi}{n}$
$a \mapsto$ refl

surj as rot $\frac{2\pi}{n}$ & refl gen $D_n \Rightarrow |\langle | \rangle| \geq |D_n| = 2n$

inj | work with reln to show $|\langle \ | \ \rangle| \leq 2n$

Done | i.e $2n$ words that rep every elt!
(b^i, ab^i)

III Bilinear forms

Defn) Basic def of a bilinear form (symmetric, skew-sym, altern)

(left) null sp of a form $\{v \mid \langle v,w\rangle = 0\}$

- Matrix assoc w/ form $V \to /K$

  If $\langle\,,\rangle$ bilin of V w/ basis $v_1, \dots, v_n$

  the matrix A has (i,j) entry $\langle v_i, v_j\rangle$

  form symmetric $\iff$ A symm

- Discriminant of form $\langle\,,\rangle = \det(\text{matrix})$ only well defd mod $(K^\times)^2$

  if A is the matrix for $\langle\,,\rangle$ in v basis

  A' ——— $\langle\,,\rangle$ in another basis

  $A' = BAB^t$ B change of basis matrix.

  $\det A' = \det B^2 \det A$

- Discr $\neq 0 \iff$ form is not degen (null sp $= 0$),

  (in this case Discr $\in K^\times/(K^\times)^2$

- Quad sp V is vs w/ symm bilinear form $/K$ (char $\neq 2$)

  $\exists$ orthogonal basis $v_1, \dots, v_n$ so $\langle v_i, v_j\rangle = 0$ for $i \neq j$

  Pf idea $v \in V$ (note?) $v^\perp = \{w \in V \mid \langle v,w\rangle = 0\}$ (assum non degen)

  if v isotropic, $\langle v,v\rangle = 0 \Rightarrow v \in v^\perp$

  find first vector that's not isotropic

  $V = v^\perp \oplus \operatorname{span}(v)$ → continue by induction

  → an aniso vector $\exists$ since

  let $q(v) = \langle v,v\rangle$ → if every vector iso $\Rightarrow q \equiv 0$ → form $\equiv 0$

  can recover bilinear form from $q \to q(v+w) - q(v) - q(w) = 2\langle v,w\rangle$

Classification of non dege

- $\mathbb{C}$ → any thing of same dimension are iso
- $\mathbb{R}$ → same dimension & signature
- $\mathbb{F}_p$ → dimension & discr $\in \mathbb{F}_p^\times / (\mathbb{F}_p^\times)^2$

(of order 2)

D/so either square or not

IV Rep Theory! (finite grps / $\mathbb{C}$)

- Maschke's Thm: $\exists$ complementary reps
- $\Longrightarrow$ all things are dir