Lec7 Linear Alg?

$\mathbb{R}^n = \left\{ \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \mid a_i \in \mathbb{R} \right\}$

Two operations that we care about. Addition & scalar mult!

Def/ A $\mathbb{R}$-vector space. Is a set V with two operations

- $+ : V \times V \to V$
- $\circ : \mathbb{R} \times V \to V$

so 1) $(V, +)$ is an abelian group!

2) $a \cdot (b \cdot \vec{v}) = (a \cdot b) \cdot \vec{v} \quad \forall a, b \in \mathbb{R}, v \in V$

3) $1 \cdot \vec{v} = \vec{v} \quad \forall \vec{v} \in V$

4) $a \cdot (\vec{v} + \vec{w}) = a\vec{v} + a\vec{w} \quad \forall a, b \in \mathbb{R}, \vec{v}, \vec{w} \in V$

$(a+b)\vec{v} = a\vec{v} + b\vec{v}$

Prop/ Let V be a real vector space

1) $0 \cdot \vec{v} = \vec{0} \quad \forall v \in V$

2) $-1 \cdot \vec{v} = -\vec{v} \quad \forall v \in V$

3) $a \cdot \vec{0} = \vec{0} \quad \forall a \in \mathbb{R}$

Pf/ $0 \cdot \vec{v} + 0\vec{v} = (0+0)\vec{v} = 0\vec{v}$

since abelian (vec addition) $0\vec{v} = \vec{0}$

e.g/ 1) $V = \mathbb{R}^n$

2) $V = \mathbb{C}$

3) $V = \mathbb{R}[x]_{\leq 5}$

4) $V =$ all cts real valued fnc on $[0,1] \to \mathbb{R}$

Def/ A commutative Ring is a set R with two binary operations $+, \circ$ s.t. (refer to mult)

1) $(R, +)$ is an abelian grp (write 0 for $+$ identity)

2) $(R, \circ)$ is a commutative monoid (identity is 1)

3) $x(y+z) = xy + xz \quad \forall x, y, z \in R$ (grp → might not have inverse)

e.g. 1) $\mathbb{R}, \mathbb{Q}, \mathbb{C}$ — a field too

2) $\mathbb{Z}$ → not field

3) Polynomials in $\mathbb{C}$ → $\mathbb{C}[t]$

4) $\mathbb{C}(t)$ → rational function $\left\{ \frac{p(t)}{q(t)} \mid p, q \in \mathbb{C}[t] \right\}$ — a field

5) $\mathbb{Z}/n\mathbb{Z}$ for all int n — × field iff n prime — Check well def

↳ mlt here is $(a+n\mathbb{Z})(b+n\mathbb{Z}) = (ab+n\mathbb{Z})$

Def | A commutative ring is a field if
- $(R \setminus \{0\}, \cdot)$ is an abelian grp
  $\forall x \neq 0 \in R$, x has inverse called x^{-1}
- $1 \neq 0$

Prop | Say $n \geq 2$ then $\mathbb{Z}/n\mathbb{Z}$ is a field $\iff$ n prime

Pf | n not prime.

Factor $n = ab$ where $1 < a, b < n$

$\Rightarrow \bar{a}\,\bar{b} = 0$ in $\mathbb{Z}/n\mathbb{Z}$ and $\bar{a}, \bar{b} \neq 0$

gives contradiction. If field mult both sides by $\bar{a}^{-1}$

so

$\bar{a}^{-1}\bar{a}\,\bar{b} = \bar{a}^{-1} 0 \Rightarrow \bar{b} = 0$ oops

$\Leftarrow$ Spse n is prime. let $\bar{a} \neq 0$ in $\mathbb{Z}/n\mathbb{Z}$

$\Rightarrow$ a not div by n. $\Rightarrow \gcd(a, n) = 1$ (rel prime)

bizout

$\Rightarrow ba + cn = 1$ for $b, c \in \mathbb{Z}$

$\Rightarrow \bar{1} = \overline{ba} + \overline{cn} = \overline{ba} = \bar{b}\,\bar{a} \Rightarrow \boxed{\bar{b} = \bar{a}^{-1}}$

Def | For prime p. Write $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$

Def | let F be a field. Define characteristic of F.

as minimal int $n \geq 1$ s.t.

$\underbrace{1 + \cdots + 1}_{n} = 0$. Or 0 if no such exist!

e.g. $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ char 0, $\mathbb{F}_p$ → char p, $\mathbb{F}_p(t)$ char p — rational func

Ex 1) if F is a field. Then char $F = 0$ or prime #

2) give an example of a field with 4 elts

Def Let R be a commutative ring an **R-module** is a set M with two operations

a) addition $+ : M \times M \to M$

b) scalar mult $\cdot : R \times M \to M$

satisfying axioms analogous to real vs

Def If F is a field an **F-vector space** is an **F-module**

Ex Convince that $\mathbb{Z}$-modules are the same as abelian grps.

Let F a field

eg 1) $F^n = \left\{ \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \middle| f_n \in F \right\}$

2) $F[x]$

3) $\mathrm{Mat}_{n \times m}(F)$

Def Let V be an F-vector space. A **subspace of V** is a nonempty subset of V closed under $+$ and scalar mult

$\hookrightarrow$ gives additive inverses

Cons Let V be F-vs. Let $v_1, \dots, v_n \in V$.

A linear comb of $v_1, \dots, v_n$ is $\sum_{i \in \mathbb{N}_n} \alpha_i v_i$ where $\alpha_i \in F$.

Observe, the set of all lin comb $\{v_i\}_{i \in \mathbb{N}_n}$ is a subsp of V.

assoc, comm +, dist

As $\sum_{i \in \mathbb{N}_n} \alpha_i v_i + \sum_{i \in \mathbb{N}_n} \beta_i v_i = \sum_{i \in \mathbb{N}_n} (\alpha_i + \beta_i) v_i \in V$

dist

$\gamma \sum_{i \in \mathbb{N}_n} \alpha_i v_i = \sum_{i \in \mathbb{N}_n} \gamma \alpha_i v_i \in V$

Def $\mathrm{span}(v_1, \dots, v_n) = \left\{ \sum_{i \in \mathbb{N}_n} \alpha_i v_i \middle| \alpha_i \in F \right\}$ subsp of V.