Last time: Representation Theory.

Pb: Problems of understanding reps of G breaks down to:

1) Understanding irred reps

2) How a gen rep is built from irred.

Today) We will solve ② when G is finite + $k = \mathbb{C}$

↳ this is nice!

Defn) Let V - rep of G & $W \subset V$ is a subrep.

A complementary subrep is a subrep $U \subseteq V$ that is comp to W in vec sp sense

↳ i.e. $U \cap W = \{0\}$ $V \cong W \oplus U$ ~ so int dir sum like

In matrix Say $\rho: G \to GL_n(k)$ is the homo corr to rep V w basis $v_1, \dots, v_n$

if W subrep. choose basis so $v_1, \dots, v_m$ basis of W

$\Rightarrow \rho$ has upper triangular form. $\rho = \begin{array}{c|cc} & m & n-m \\ \hline m & * & * \\ n-m & 0 & * \end{array}$

if U is comp to W & $v_{m+1}, \dots, v_n$ is a basis for $U \Rightarrow \rho = \left(\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array}\right)$ → block diag

e.g. 1) $G = S_n$ $k = \mathbb{C}$ $V = \mathbb{C}^n$ perm rep of S_n.

↳ from last time These are subrep $W = \text{span}(e_1 + \dots + e_n)$ is a subrep

$U = \{v \in \mathbb{C}^n \mid v = \sum a_i e_i, \sum a_i = 0\}$

dim $n-1$ $= \text{span}(e_i - e_j \mid 1 \le i < j \le n)$

↳ These are complementary subrep!

let $g = (1\ 2\ 3)$ what is $\rho(g)$ look like

First std basis $e_1, \dots, e_5$ $g e_1 = e_2$ $g e_2 = e_3$ $g e_3 = e_1$ fixes rest!

So, $\rho(g) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ ↳ but this isn't a <u>great basis</u>!

Better basis

$v_1 = e_1 + \cdots + e_5$

$v_2 = e_1 - e_2$

$v_3 = e_2 - e_3$

$v_4 = e_3 - e_4$

$v_5 = e_4 - e_5$

<u>know</u>

$$\begin{array}{cc} 1 & 4 \end{array}$$

$$\rho(g) = \left(\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array}\right)$$

$g v_1 = v_1$

$g v_2 = e_2 - e_3 = v_3$

$g v_3 = e_3 - e_1 = -(v_2 + v_3)$

$g v_4 = e_1 - e_4 = v_2 + v_3 + v_4$

$g v_5 = v_5$

$$= \rho(g) = \left(\begin{array}{c|cccc} 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

<u>inf grp</u>

e.g. 2) <u>No comp.</u> $G = \mathbb{Z}$ $\quad k = \mathbb{C}$ $\quad V = \mathbb{C}^2$

$\rho(n) = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$

↳ $\rho(n)\rho(m) = \begin{pmatrix} 1 & n+m \\ 0 & 1 \end{pmatrix}$ ✓

$W = \mathrm{span}(e_1)$ is a <u>subrep</u>.

↳ but W has no compl <u>subrep</u>! <u>Eigenstuff</u>!

↳ if one <u>existed</u> $\Rightarrow$ $\rho(n)$ would be diagble but it <u>isn't!</u> (if <u>$n \neq 0$</u>)

e.g. 3) $G = \mathbb{Z}/p\mathbb{Z}$ $\quad k = \mathbb{F}_p$ $\quad V = \mathbb{F}_p^2$ $\quad \rho(a) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$

span (e_1) has no <u>compl</u>!

Maschke

Thm) if G is a finite grp & V is a fin dim rep then any subrep has a compl. ($k=\mathbb{C}$)

Rmk) Showed if $k=\mathbb{C}$ and G is inf — false in gen

$k=\mathbb{F}_p$ & G is fin — false in gen

Thm holds if $\dim V = \infty$

Cor) if V is a fin dim'l rep of a finite grp G (over $\mathbb{C}$)

$\Rightarrow \exists$ irreducible subreps $W_1, \dots, W_r$ so $V \cong \bigoplus_{i=1}^{r} W_i$

→ complete reducibility. → fin dim

First lin alg. Let V – v.s / field k.

A projection operator is a linear op $e: V \to V$ so $e^2 = e$

if e is proj $Id - e$ is a proj.

$(Id-e)^2 = Id - 2e + e^2 = Id - e$ ✓.

Say $v \in V$ then $v = (e + (1-e))v = \underbrace{e(v)}_{\operatorname{Im} e} + \underbrace{(v - e(v))}_{\ker e}$

$\Rightarrow V = \operatorname{im}(e) + \ker(e)$ → so $\ker e$ is compl. to $W = \operatorname{im} e$

→ this is a direct sum actually.

as $e|_{\operatorname{im}(e)} = id$ → if $v \in \operatorname{im}(e) \Rightarrow v = e(w)$

$e(v) = e^2(w) = e(w) = v$

if $v \in \operatorname{Im} e \cap \ker e \Rightarrow 0 = e(v) = v$

let $v_1, \dots, v_n$ basis of V so

$v_1, \dots, v_m$ basis of $\operatorname{Im} e$ & $v_{m+1}, \dots, v_n$ is basis for $\ker e$.

$\Rightarrow$ matrix for e is

$$\begin{array}{c|c} & \begin{matrix} m & n-m \end{matrix} \\ \begin{matrix} m \\ n-m \end{matrix} & \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & 0 \end{array}\right) \end{array}$$

→ get compl.

do this by choosing good basis and dropping later basis

Note: if $W \subseteq V$ subsp $\exists$ proj on $e: V \to W$ so $\operatorname{Im} e = W$

Pf of Maschke Let V - fin dim $\mathbb{C}$ rep of G
W-subrep of V. → also G-equivariant.

Def A G-linear proj op $e: V \to V$ is a lin map s.t.
$e^2 = e \to$ proj
$e(gv) = g \cdot e(v) \quad \forall g \in G, v \in V$

Obs If e is such a thing then $\mathrm{im}(e)$ & $\ker(e)$ comp sub reps.

Pf $v \in \ker(e) \Rightarrow e(v) = 0$
let $g \in G \Rightarrow e(gv) = g(e(v)) = 0$
$\Rightarrow gv \in \ker e$.
→ Similiar arg for $v \in \mathrm{im}(e)$

Upshot If can find a G-lin proj op $e: V \to V$
so $\mathrm{im}(e) = W \Rightarrow \ker(e)$ is a compl. subrep!

Let $e_0: V \to V$ be any proj op s.t. $\mathrm{im}(e_0) = W$
Let $e: V \to V$
$$v \longmapsto \frac{1}{|G|} \sum_{g \in G} g e_0(g^{-1} v)$$
→ using char of field.
→ using finite grp

Claim e is a G-linear proj op whose im is W.
→ ok not well def
→ **Obs:** holds if G is finite & $\# G \neq 0$ in k $\mathbb{C}$

1) let us show it is a proj with $\mathrm{im}(e)$
2) let us show it is G-equivariant.

① $\mathrm{im}(e) \subset W$ as $\mathrm{im}(e_0) = W \Rightarrow g\, e_0(w) \in W$
$\Rightarrow e(v) \in W$

Next wts $e|_W = \mathrm{id}_W$ let $w \in W$ be given
$g^{-1} w \in W \Rightarrow e_0(g^{-1} w) = g^{-1} w \Rightarrow g\, e_0(g^{-1} w) = w$
$\Rightarrow e(w) = w \qquad \Rightarrow e$ is a proj! as $e(e(v)) = e(v)$

2) Let us show it is G equivariant. Let $h \in G$, $v \in V$ be given we see

$$e(hv) = \frac{1}{|G|} \sum_{g \in G} g\, e_0(g^{-1} h v)$$

Change ~~of~~ var
write $g = hg'$
$\Rightarrow g^{-1} h = g'^{-1}$

$$= \frac{1}{|G|} \sum_{g' \in G} h g' (g'^{-1} v)$$

grp action linear

$$= h \left(\frac{1}{|G|} \sum_{g' \in G} g' (g'^{-1} v) \right)$$

$$= h\, \underline{e(v)}$$