Lec 21

Remember $D_n \to$ dihedral grp

$a \in D_n$ rotation by $\frac{2\pi}{n}$

$b \in D_n$ some reflection

any elt of D_n has the form $b^i \cdot a^j$ $i \in \{0,1\}$, $0 \le j < n$ → unique

rem: $a^n = 1$, $b^2 = 1$. Note $\underline{bab^{-1} = a^{-1}}$ → by earlier work $r\,f_\theta\, r^{-1} = f_{-\theta}$

$\Rightarrow a, b$ are generators of D_n

→ these are relations!

Free Group

Let S a set of symbols $S = \{a_1, \dots, a_n\}$ → works if S inf.

A **word** in S is just a finite string of symbols $a_{i_1} a_{i_2} \cdots a_{i_r}$

S^* is the set of all symbols (including empty word).

define multiplication on S^* as concatenation.

this is assoc & has identity of empty word

→ This isn't a grp (have no inv) it is a monoid

Let $\bar{S} = \{a_i, a_i^{-1} \mid 1 \le i \le n\}$ } $2n$ elts
→ formal symbol

A word in $\bar{S}$ is reduced if never see $a_i a_i^{-1}$ next as subword ($a_i^{-1} a_i$)

Starting with any $\bar{S}$ word, can cancel to get to a reduced word.

→ subtlts $S = \{a, b, c\}$ → $b\,a\,b\,b^{-1}\,a^{-1}\,c\,c^{-1}\,a$

$b\,a\,a^{-1}\,c\,c^{-1}\,a$ $\qquad$ $b\,a\,b\,b^{-1}\,a^{-1}\,a$

$b\,c\,c^{-1}\,a$ $\qquad$ $b\,a\,b\,b^{-1}$ $\qquad$ $b\,a$

ba

→ Different a's show up.

There's a unique well def reduced word. But can't backtrack

Key lem | Given any word $w \in \bar{S}^*$. No matter how we cancel, end up with same reduced word.

Pf sk | If word reduced to begin with done!

If not $\exists\ a_i a_i^{-1} \in w$.

Enough to show we can cancel $a_i a_i^{-1}$ first (done by ind)

Note: A some point in any cancellation. At least one of a_i, a_i^{-1} must be cancelled

1) At some point, cancel this $a_i a_i^{-1}$ pair

↳ in this case, just cancel $a_i a_i^{-1}$ first and go on with rest of process.

2) At some point, get to $a_i^{-1} a_i a_i^{-1}$ and cancel red

og.

but note! order of cancellation doesn't matter here

↳ might as well cancel og pair → reduces to case 1

↳ This second case shows why backtracking not well def.

Define an equiv rel'n $\sim$ on $\bar{S}^*$ st $w \sim w'$ iff w and w' have same reduced form!

Lemma | $\sim$ comp with mult. i.e. $w \sim w'$, $v \sim v'$

$\Rightarrow wv \sim w'v'$

Pf | let w_0 = red word of w, w'

v_0 = " " " v, v'

Claim $wv \sim w_0 v$ by doing cancellations in w

$\sim w_0 v_0$ cancellation in v

$\Rightarrow wv \sim w_0 v_0 \sim w'v'$ → similarly

$\Rightarrow wv \sim w'v'$

Def) $F = \bar{S}^*/\sim$ (set of equivalence classes) → unique reduced word in each class

↳ convient to make repr

↳ in bij with set of red words

lem ⇒ mult on $\bar{S}^*$ induces mult on F.

⇒ F is a monoid!

Prop) F is actually a grp. Just check inverses exist

e.g. $S = \{a, b, c\}$ $\langle g \rangle \leq F$ $g = ab^{-1}c$

$g^{-1} = c^{-1} b a$

$gg^{-1} \sim$ empty word $\Rightarrow [gg^{-1}] = 1$

Defn) The free grp on S is $F = \bar{S}^*/\sim$

F_n = free grp on $\{a_1, \dots, a_n\}$.

Mapping Prop) G - any grp. S - some set. F - free grp on S.

have a bijection

$\{$grp homo $F \to G\} \xrightarrow{\sim} \{$arbitrary func $S \to G\}$

$\varphi \longmapsto \varphi|_S$

Pf) $S \subseteq F$ generates F.

If φ, ψ are homomorphism that agree on S (generators)

Then they are equal!

↳ gives injectivity.

↳ analogous to linear map uniquely det by basis is sent!

Let $\varphi_0 : S \to G$ be given $S = \{a_1, \dots, a_n\}$.

let $G \ni g_i = \varphi_0(a_i)$

Def: $\tilde{\varphi} : \bar{S}^* \longrightarrow G$ $\tilde{\varphi}(a_{i_1}^{(\pm)} \cdots a_{i_k}^{(\pm)}) = g_{i_1}^{(\pm)} \cdots g_{i_k}^{(\pm)}$

Claim $\tilde{\varphi}$ compatible with equiv → $\tilde{\varphi}$ induces $\varphi : F \to G$ → well def

↳ this is clear! → cancel on both sides!

By constr, it is a homo! ($\tilde{\varphi}$ is a monoid homo)

↳ also clear $\varphi|_S = \varphi_0$.

e.g.\ let F → free grp on $\{a,b\}$

mapping prop ⟹ G any grp then $A,B \in G$

$\exists!$ grp homo $\varphi : F \to G$ so $\varphi(a) = A$
$\varphi(b) = B$.

$\mathrm{im}(\varphi) = \langle A, B \rangle$ → subgrp of G gen by A, B

⟹ φ surj ⟺ G gen by A, B

e.g.\ Specific

F → free grp on $\{a,b\}$

$G = D_n$ → A → rotation by $\frac{2\pi}{n}$

B → reflection

$\exists!$ grp homo $\varphi : F \to D_n$ so $\varphi(a) = A$
$\varphi(b) = B$

we note $A^n = 1$, $B^2 = 1$, $ABAB = 1$

→ $BAB^{-1} = A^{-1}$

→ $a^n, b^2, abab \in \ker(\varphi)$

Fact $D_n \cong F/N$ N is the smallest normal subgrp of N containing $a^n, b^2, abab$.