Def | $\langle a_1, \dots, a_n \mid r_1, \dots, r_m \rangle = F/N$

$\downarrow$ symbols for free grp

$\hookrightarrow$ words in a's and a^{-1}'s

where F is free grp on $a_1, \dots a_n$

N is intersection of all Normal subgrps of F containing $r_1, \dots, r_m$.

($=$ subgrp of F gen'd by conj of $r_1, \dots, r_n$)

A finite presentation of a grp G is an isom

$$G \cong \langle a_1, \dots, a_n \mid r_1, \dots, r_m \rangle$$

$\downarrow$ generators

$\hookrightarrow$ relations

Mapping Prop G – any grp.

To give a grp homo $\langle a_1, \dots a_n \mid r_1, \dots, r_m \rangle \xrightarrow{\varphi} G$ you give $g_1, \dots g_n \in G$ corr to $a_1, \dots a_n$ so that the reln hold i.e. changing a's to g's in r_i gives $1 \in G$.

Pf | $\langle a_1, \dots, a_n \mid r_1, \dots, r_m \rangle = F/N$

By mapping prop for F $\exists!$ grp homo $\tilde{\varphi} : F \to G$

$a_i \longmapsto g_i$

We want

$$\begin{tikzcd} F \arrow[r, "\tilde{\varphi}"] \arrow[d, "\text{quot}"'] & G \\ F/N \arrow[ur, dashed] & \end{tikzcd}$$

$\tilde{\varphi}(r_i) = r_i$ w/ a's changed to gs

$= 1$ by assump

$\Rightarrow r_i = \ker(\tilde{\varphi}) \Rightarrow N \subseteq \ker(\varphi)$

$\Rightarrow \exists! \, \varphi$ s.t. diagram commutes!

let $\varphi(nN) = \tilde{\varphi}(n)$ is well def as

How to work with $\langle a_1, \dots a_n \mid r_1, \dots, r_m \rangle$

- Upper bd: work hard on with the relation to show every elt is rep by certain kind of word.
- Lower bd: using the mapping property → find nontrivial homo to grp shows $N \neq F$

E.g. D_n. A = rot by $\frac{2\pi}{n}$, B = reflc

$BAB^{-1} = A^{-1} \Rightarrow BABA = 1$

$B^{-1} = B$ $\Rightarrow (BA)^2 = 1$

Claim $D_n \cong \langle a, b \mid a^n, b^2, (ba)^2 \rangle = F/N = G$ all 1

mapping prop gives homo $\varphi : G \longrightarrow D_n$

$a \longmapsto A$

$b \longmapsto B$

(okay since $A^n = B^2 = (BA)^2 = 1$

also φ is surj as A, B gen D_n

$\Rightarrow \#G \geq 2n$

This idea shows F/N not trivial

(map to some non 1 elt (A,B)

In D_n every elt A^i or BA^i $0 \leq i < n$

Show: analogous statement in G.

In G, we have elts a, b that gen G, satisfy $a^n = b^2 = (ab)^2 = 1$

$\Rightarrow abab = 1 \Rightarrow ab = ba^{-1}$ implies every elt in G looks like

$b^i a^j$ $i, j \in \mathbb{Z}$ → by restrictions, $i \in \{0, 1\}$, $0 \leq j < n$

$\Rightarrow \#G = 2n$ → have the iso

E.g $\langle a, b \mid aba^{-1}b^{-1} \rangle = G$

In G a, b gen and $aba^{-1}b^{-1} = 1 \Rightarrow ab = ba$

UB $\Rightarrow$ every elt in G as some $a^i b^j$ by commutation, $i \in \mathbb{Z}$, $j \in \mathbb{Z}$

no redundancy

by mapping prop $\exists$ grp homo

$\varphi : G \longrightarrow \mathbb{Z}^2$

$a \longmapsto (1,0) = x$

$b \longmapsto (0,1) = y$

Check $xy = yx$

$\Rightarrow \varphi(a^i b^j) = (i, j)$ $\ker \varphi = 0$

e.g.| Coxeter Presentation for S_n

$s_1 = (1\,2)$ $s_2 = (2\,3)$ ... $s_i = (i\ i{+}1)$ $s_{n-1} = (n{-}1\ n)$

Go these gen S_n

Reln| $s_i^2 = 1$

$(s_i s_{i+1})^3 = 1$

$(s_i s_j)^2 = 1$

$|i-j|>1$

$s_i s_{i+1} = (i\ i{+}1)(i{+}1\ i{+}2) = (i\ i{+}1\ i{+}2)$

if i, j more than 2 apart, they commute

braid reln

e.g.| $B_n = \langle s_1, \dots, s_{n-1}$

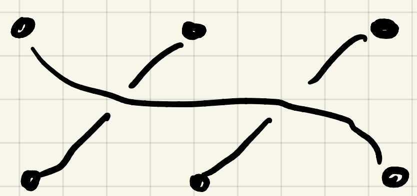
$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$

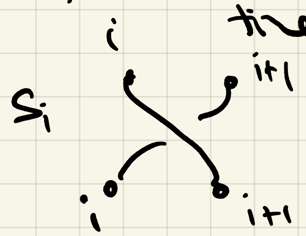
$s_i s_j = s_j s_i \longrightarrow |i-j|>1$

take away 1st reln above

these are same as ②, ③.

Braid grp $\to$ tracing perm through time

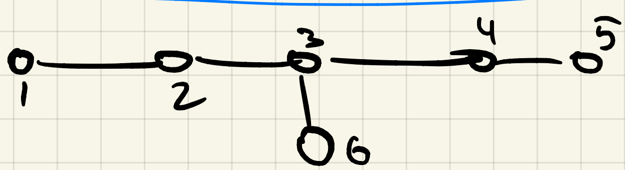




By mapping $\exists$ surj $B_n \to S_n$

$s_i \mapsto (i\ i{+}1)$

e.g.| More General Coexter Grp



Grp gen by s_i where i are vertices

- $s_i^2 = 1$
- $s_i s_j = s_j s_i$ if not conn
- $(s_i s_j)^3 = 1$ if i, j are conn