Lec 5

Setup H subgroup of G. $g \sim g' \iff g = g'h$ for some $h \in H$, $g' \in G$

1) Reflexive $\to$ $g \sim g$ as $1 \in H$
2) Symmetric $\to$ $g \sim g' \Rightarrow g = g'h \Rightarrow g' = gh^{-1}$
   so $g' \sim g$
3) Transitive $g \sim g'$ $g' \sim g''$ $\Rightarrow g = g'h$ $g' = g''h'$
   $\Rightarrow g = g''h'h$
   $\Rightarrow g \sim g''$

We write $gH = \{g' \mid g' \sim g\} = \{g' \mid g' = gh \text{ for } h \in H\}$
$\hookrightarrow$ left coset of H

e.g. $G = \mathbb{Z}$ $H = 2\mathbb{Z}$ $\to$ cosets are evens and odds
$0 + 2\mathbb{Z}$ $\quad$ $1 + 2\mathbb{Z}$

$G = \mathbb{Z}$ $H = n\mathbb{Z}$
cosets are $i + n\mathbb{Z}$ $\quad 0 \leq i \leq n-1$ $\to$ division algorithm

Thm | Lagrange Thm

If $H \leq G$ subgrp both finite $\Rightarrow |H| \mid |G|$

Claim for $g, g' \in G$ $|gH| = |g'H| = |H|$. set $\varphi : H \to gH$, $h \mapsto gh$ bij
surj ✓ $gh_1 = gh_2 \Rightarrow h_1 = h_2$

since cosets are disjoint (equiv classes), $\to$ partition
we get $S = \{gH \mid g \in G\}$ S is finite and, by above
$|G| = |S| \cdot |H| \Rightarrow |H| \mid |G|$

Def | Right cosets $Hg = \{hg \mid h \in H\}$

Rmk | $N \leq G$ normal $\iff gN = Ng$ $\forall g \in G$

Pf ($\Rightarrow$) if $N \leq G$ normal we see $\forall\, g \in G$, $n \in N$ $(gng^{-1}) \in N$
$\Rightarrow gn \in Ng$ certainly $gn \in Ng$ $\quad \subseteq$ $\quad \supseteq$ same

$\Leftarrow$ spse $gN = Ng$ $\forall g \in G$.

let $n \in N$, $g \in G$ wts $gng^{-1} \in N$.

$gn \in Ng \Rightarrow \exists\, n' \in N$ so $gn = n'g$

righ mult by g^{-1} $\Rightarrow$ $gng^{-1} = n' \in N$