Lec 8

Cool: for prime p, get $GL_n(\mathbb{F}_p)$ is a finite group under multiplication

Recall If V vs $/F$. Then span of $\{v_i\}_{i\in\mathbb{N}_n}$ is

$$\{\sum_{i\in\mathbb{N}_n} \alpha_i v_i \mid \alpha_i \in F\} \subseteq V$$

Def V is finite dim if $\exists \{v_i\}_{i\in\mathbb{N}} \subseteq V$ so $\text{span}(\{v_i\}) = V$

Def We say $v_1 - v_n$ is linear independent if any lin comb is trivial

i.e if $\sum_{i\in\mathbb{N}_n} \alpha_i v_i = 0 \Rightarrow \alpha_i = 0 \ \forall i \in \mathbb{N}_n$

Def $v_1 \dots v_n$ is a basis of V if it is spanning & lin ind

e.g. $V = F^n$ (col vect). let $e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i$

Claim: $\{e_i\}_{i\in\mathbb{N}_n}$ is a basis.

let $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in F^n$ let $\alpha_i = x_i \ \forall i \in \mathbb{N}_n$ $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \sum_{i=1}^{N} \alpha_i v_i$

spse $\sum_{i=1}^{n} \alpha_i v_i = 0 \Rightarrow \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = 0 \Rightarrow \alpha_i = 0 \ \forall i \in \mathbb{N}_n$

Def $\{e_i\}$ is the std basis for F^n

Prop Any fin dim'l vs has a finite basis

Pf let $\{v_i\}_{i\in I}$ be a finite spanning set.

$\quad\quad\quad\quad$ ↳ finite sp set

pick $J \subseteq I$ minimal so $\{v_i\}_{i\in J}$ still span.

$\quad\quad\quad\quad$ ↳ containment

↳ This is a basis (just need to show lin ind) spse not

$\exists \alpha_i$ not all zero so

$\sum_{i\in J} \alpha_i v_i = 0$ let $\alpha_{i_0} \neq 0$. Then $v_{i_0} = \sum_{i\in J\setminus\{i_0\}} -\frac{\alpha_i}{\alpha_{i_0}} v_i$

$\Rightarrow \{v_i\}_{i\in J\setminus\{i_0\}}$ is spanning $\Rightarrow J$ not minimal $\quad \square$

Prop If V is fin dim'l. Then any lin ind set can be extended to a basis.

Prop| Let $V = \text{span}(v_1, \dots v_n)$ any lin ind st has at most n elements.

Pf| Contrapos if $m > n$, $w_1, \dots, w_m \in V$ then they are linearly dependent.

$$w_j = \sum_{i=1}^{n} \alpha_{i,j} v_i \qquad \begin{bmatrix} a_{11} & \cdots & \alpha_{1m} \\ \alpha_{n1} & \cdots & a_{nm} \end{bmatrix} \; n \times m$$

$= 0$

↳ Think as a homog system of m variab n equations.

as $m > n$ have nontrivial soln

$$(x_1 \cdots x_m) = (\beta_1, \dots, \beta_m)$$

$$\Rightarrow \beta_1 w_1 + \dots + \beta_m w_n = 0$$

↳ coeff of v_i in this expression vanish $\Rightarrow$

Prop| If V fin dim vs. Then any 2 bases have same # of ell.

Pf| Let $v_1, \dots v_n$ and $w_1 \dots w_m$ be basis

By above $m \le n$, $n \le m \Rightarrow m = n$

Def| If V vs fin dim. $\dim(V) = |B|$ B is a basis for V

e.g.| $\dim F^n = n$

Prop| Let V fin dim vs, $W \subseteq V$ sub sp. Then W fin dim with $\dim W \le \dim V$.

Pf|