Unless we say otherwise

fin dim $\mathbb{C}$ reps of finite Grps!

Defn) let (V, ρ) be a rep of G. The character of V is

$(\chi_\rho =)$ $\chi_V : G \longrightarrow \mathbb{C}$

$\chi_V(g) = \operatorname{tr} \rho(g)$

Obs! 1) $\rho(e) = I_{\mathbb{C}^n} \Rightarrow \chi_V(e) = \underline{\dim(V)}$!

2) χ_V is conjugation invariant.

$\Rightarrow \chi_V(ghg^{-1}) = \chi_V(h)$

$\Leftarrow$ $\chi_V(ghg^{-1}) = \operatorname{tr} \rho(ghg^{-1}) = \operatorname{tr} \rho(g)\rho(h)\rho(g^{-1})$

$= \operatorname{tr} \rho(h)$

Defn) if a func $\varphi : G \to \mathbb{C}$ is conj inv

$\Leftrightarrow$ it is a class function!.

$\Leftrightarrow \chi_V$ is a cl func!

$\mathcal{C}(G) = \{$ all class func $G \to \mathbb{C} \}$ — vecal feeling

3) If V & W are isom reps $\chi_V \equiv \chi_W$

Reason if $f : V \to W$ isom

$\Rightarrow f \circ \sigma(g) = \rho(g) \circ f$

$\Rightarrow \sigma(g) = f \circ \rho(g) \circ f^{-1}$

$\Rightarrow \operatorname{tr} \rho(g) = \operatorname{tr} \rho(g)$

Amazing Thm) $\chi_V \equiv \chi_W \Rightarrow \underline{V \text{ \& } W \text{ are isom}}$!

(we prove this soon)

ii) We can recover char poly of $\rho(g)$ from χ_V as $\chi_V(g^i) = \mathrm{tr}(\rho(g)^i)$ if we know for all i $\Rightarrow$ we can recover char poly of $\rho(g)$!

Prop) Let $\overset{\rho}{V}$ and $\overset{\sigma}{W}$ be reps of G. $V \oplus W$ is then a rep of G

$$\chi_{V \oplus W} = \chi_V + \chi_W$$

PS) Pick basis for V, W then the matrix for grp elt g on $V \oplus W$ is

$$\left(\begin{array}{c|c} \rho(g) & 0 \\ \hline 0 & \sigma(g) \end{array}\right) \Rightarrow \mathrm{tr}(\quad) = \mathrm{tr}\,\rho(g) + \mathrm{tr}\,\sigma(g)$$

Duals)

Let V a vector sp / $\mathbb{C}$

$$V^* = \{\underbrace{f : V \to \mathbb{C} \text{ linear}}_{\text{functions!}}\}$$

$\to$ this is a vec sp!

Notation) $\langle\,,\rangle : V^* \times V \to \mathbb{C}$

$$(\lambda, v) \mapsto \lambda(v)$$

Let $v_1, \dots, v_n$ basis then defn $v_i^*(v_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else} \end{cases} = \delta_{i,j}$

Lemma) $v_1^*, \dots, v_n^*$ basis of $\mathbb{C}$

Warning)
- Whole constr. dep on basis
- No natural iso from $V \to V^*$

Fix $v_1, \dots, v_n$ basis $\rightsquigarrow$ get $v_1^*, \dots, v_n^*$ of V^*

Let $v \in V \Rightarrow v = \alpha_1 v_1 + \dots + \alpha_n v_n$

$$\alpha_1 = v_1^*(v) \Rightarrow v = \sum_{i=1}^{n} \langle v_i^*, v\rangle v_i$$

let $T: V \to V$ be a lin op.

let $A \to$ matrix for T in v_i basis

A_{ij} = coeff of v_i in $T(v_j)$

$= \langle v_i^*, T(v_j) \rangle$

$$tr(T) = \sum_{i=1}^{n} \langle v_i^*, T(v_i) \rangle$$

Rmk) $\exists$ natural isom $V \to (V^*)^*$

The map $V \to V^{**}$

$v \mapsto (\underbrace{\lambda \mapsto \langle \lambda, v \rangle}_{\text{func } V^* \to \mathbb{C}})$

Dual Repr

V is a rep of G

We'll constr a natural rep of G on V^*

no choice (i.e. basis)

Given $g \in G$ & $\lambda \in V^*$ we wanna def $g\lambda \in V^*$

$(g\lambda)(v) = \lambda(g^{-1}v)$

$\to$ why?

$(g \cdot (h\lambda))(v) = (h\lambda)(g^{-1}v) = \lambda(h^{-1}g^{-1}v)$

$\neq (gh)\lambda(v)$

What is the char of V^*?

let $v_1, \dots, v_n$ basis of V have $v_1^*, \dots, v_n^*$ for V^*

Rmk) $v_1 \dots v_n$ is dual basis for $(V^*)^*$ wrt $v_1^*, \dots, v_n^*$ of V^*

$\chi_{V^*}(g) = \mathrm{tr}(g \mid V^*)$ — function of $\lambda \to g\lambda$

$$= \sum_{i=1}^{n} \langle g v_i^*, v_i \rangle$$

$$= \sum_{i=1}^{n} \langle v_i^*, g^{-1} v_i \rangle$$ defn

$$= \mathrm{tr}_V(g^{-1} \mid V) = \chi_V(g^{-1})$$

Obs) $\chi_V(g) = \mathrm{tr}\, \rho(g)$

= sum of eigenval of $\rho(g)$

B/c g has fin order so does $\rho(g)$

$\Rightarrow \rho(g)$ has eigen values roots of unity

$\Rightarrow$ if ζ is ev then $\zeta^{-1} = \overline{\zeta}$

$\Rightarrow \mathrm{tr}(\rho(g^{-1})) = \overline{\mathrm{tr}\, \rho(g)}$

$\Rightarrow \chi_{V^*}(g) = \overline{\chi_V(g)}$