Last time

M is grp of rigid motions of plane P

we proved every elt $m \in M$, $m = t_a \rho_\theta r^i$ $i \in \{0,1\}$

good b/c it describes all elts of M

bad depends on choice of origin

e.g. rotation about not origin is weird from v

let $b \in P$. rotation by θ around b

$\leadsto$ conjugate rotation by t_b

$$m = t_b \rho_\theta t_{-b}$$
$$= t_b \underbrace{\rho_\theta t_{-b} \rho_\theta^{-1}} \rho_\theta$$
$$= t_b t_{-\rho_\theta(b)} \rho_\theta = t_{b-\rho_\theta(b)} \rho_\theta$$

Obs if $\rho_\theta \neq 1$ (not $2\pi n$)

$\Rightarrow$ $t_a \rho_\theta$ is rotation by θ about some point in the plane.

Pf find b so $t_a \rho_\theta = t_b \rho_\theta t_{-b}$

i.e find b so $b - \rho_\theta b = a$

identify $\mathbb{R}^2 \cong \mathbb{C}$ then ρ^θ is mult by $z = e^{2\pi i \theta}$

$\Rightarrow (1-z) b = a \Rightarrow b = \dfrac{a}{1-z}$ $z \neq 1$

Prop | Classification of elt in M.

Every elt in M is one of the following:

1) orientation preserving
   a) translation
   b) rotation about some pt
2) orientation reversing
   a) reflections (through some line)
   b) glide (translate along line then reflect through it)

Pf idea

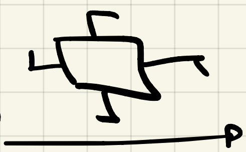
take $t_a \rho_\theta r^i$ realize as one of the four.

Goal: Understand Symmetries of plane figures + subgrps of M

First Step finite subgrps

eg D_n is the dihedral grp of regular n-gon
↳ $2n$ elts → n rotations by $\frac{2\pi k}{n} = \rho_{\frac{2\pi k}{n}}$ $k \in \mathbb{Z}/n\mathbb{Z}$
• n reflections $\rho_{\frac{2\pi k}{n}} \cdot r$ → rotate to x-axis then flip.

Eg $\mathbb{Z}/n\mathbb{Z}$ is a subgrp of M
↳ as it is a subgrp of D_n (just rotations)
↳ e.g. $\mathbb{Z}/4\mathbb{Z}$ realized as



→ break reflectional symmetry

Thm| Every finite subgrp of $M \cong \mathbb{Z}/n\mathbb{Z}$ or D_n
(in fact conjugate to one of the examples written above)

Let G be a finite subgrp of M.

Strategy: Note that stuff above fixes origin
↳ find point in P fixed by $G = \{g_1, \dots, g_n\}$
look at $\{g_1 x, \dots, g_n x\}$ for any $x \in P$.
Let y be the center of mass (identify $P \cong \mathbb{R}^2$ take average)

Lemma| This is the fixed pt for G.

Pf 1| y is uniquely characterized by minimizing dist to $\{g_1 x, \dots, g_n x\}$.
Since g preserves distances and $\{g_1 x, \dots, g_n x\}$
↳ ie $g\{hx \mid h \in G\} = \{hx \mid h \in G\}$
$gy = y$

Pf 2| Fix $g \in G$. Mult by g def of perm of G
$\exists\, \sigma \in S_n$ so $g \cdot g_{(i)} = g_{\sigma(i)}$
$g = t_b \cdot A$ $A = \rho_\theta r^i$ → linear

$$g \cdot g_i(x) = g_{\sigma(i)} x$$

$$\overset{\shortparallel}{A(g_i(x)) + b} \Rightarrow A(g_i(x)) = g_{\sigma(i)} x - b$$

$$\Rightarrow g(y) = Ay + b = \frac{1}{n}\left(Ag_1 x + \dots + Ag_n x\right) + b$$

$$= \frac{1}{n}\left(A g_{\sigma(1)}(x) - b + \dots + A(g_{\sigma(n)}(x) - b)\right) + b$$

$$= A(g_{\sigma(1)}(x)) + \dots + A(g_{\sigma(n)}(x))$$

$$= y$$

$\Rightarrow$ G fixes y $\Rightarrow$ $t_{y^{-1}} G t_y$ fixes 0.

WLOG assume G fixes 0 $\Rightarrow$ $G \leq O(2)$ CM

So reduced to classifying finite subgrps of $O(2)$.

Lemma Every finite subgrp of $SO(2)$ $\cong$ $\mathbb{Z}/n\mathbb{Z}$ for some n

Pf let ρ_θ be the elt of $G \subseteq SO(2)$ with $0 < \theta < 2\pi$ minimal
↳ (say $G \neq 1$)

Claim $G = \langle \rho_\theta \rangle$

Spse $\rho_\varphi \in G$, $0 \leq \varphi < 2\pi$

can find $k \in \mathbb{Z}_{\geq 0}$ st $0 \leq \varphi - k\theta < \theta$

$\Rightarrow$ $\varphi = k\theta$ by the minimality of θ this is zero

$\Rightarrow$ $\rho_\varphi = \rho_\theta^k$

Prop A finite subgrp $G \subseteq O_2$ is conj to $\mathbb{Z}/n$ conj to $\mathbb{Z}/n\mathbb{Z}$ or D_n

Pf $G \cap SO_2$ has index 1 or 2 in G (subgrp)
↳ if index 1 then G is fin subgrp of $SO(2)$ $\Rightarrow$ $G = \mathbb{Z}/n\mathbb{Z}$

↳ if index 2 $\rightarrow$ G has a reflection
↳ D_n