

Let G be a finite grp

Goal: # conj classes = # isom classes of irred!

To prove Let v_1, \dots, v_s irreds
 χ_1, \dots, χ_s Their chars

We'll show χ form a basis for the space of class func

$$\Rightarrow S = \dim S(G) = \# \text{ conj classes}$$

say (V, ρ) is a rep of G
& $\psi : G \rightarrow \mathbb{C}$ is a func defn

$$\begin{aligned} \rho(\psi) : V &\rightarrow V \\ \rho(\psi)v &\mapsto \sum_{g \in G} \psi(g) g.v \end{aligned}$$



Equiv $\rho(\psi) = \sum_{g \in G} \psi(g) \rho(g)$

linear comb of lin op
 $\Rightarrow \rho(\psi)$ is a lin op-

Lemma if ψ is a class func

$\Rightarrow \rho(\psi)$ is G -equiv.

Pf $\rho(\psi)hv = \sum_{g \in G} \psi(g) ghv$

$$= h \sum_{g \in G} \psi(g) h^{-1}ghv$$

$$\Rightarrow g' = h^{-1}gh$$

$$= h \sum_{g \in G} \psi(h^{-1}gh) v$$

$$= h \sum_{g \in G} \psi(g) v$$

$$= h \rho(\psi)v$$

Lemma $\text{tr}(\rho(\psi)) = \langle \chi_V, \psi \rangle$

Pf $\text{tr} \rho(\psi) = \sum_{g \in G} \psi(g) \text{tr}(\rho(g)) \rightarrow \text{trace is linear}$

$$= |G| \cdot \langle \chi_V, \psi \rangle = |G| \cdot \langle \chi_{V^*}, \psi \rangle$$

Prop) If v is irred $\Rightarrow \varphi(v) = c \cdot \text{id}_v$

where $C = \frac{\|A\|}{\dim V} \prec \widehat{x_v}, \forall v$

og lin op om irred is skriv

PR By Schur, $\varphi(\psi) = c \cdot \text{id}_V$ for some $c \in \mathbb{C}$

$$C_0 \text{ take } \underline{\pi} \quad \pi(\varphi(\psi)) = C \cdot \dim V$$

1. $\text{G}_1 \cdot \alpha \sqrt{v}, e \rangle$

$$\Rightarrow C = \frac{1}{\dim V} \alpha \widehat{x_V}, \forall x$$

True $\gamma_1, \dots, \gamma_N$ is a basis of $\mathcal{L}(n)$

Schur orth \Rightarrow lin ind

Perp sp

To show they span, enough to show
 if $\langle x_i, \varphi \rangle = 0 \forall i \Rightarrow \varphi = 0$ (for $\varphi \in C_0$)

Space $\mathcal{E} \in \mathcal{B}(C)$ satisfies $\langle x_i, e \rangle = 0 \quad \forall i$ with $e = 0$

By previous prop $P(\emptyset) = 0$ & irred rego (V, P)

$\Rightarrow p(v) = 0$ & rep. (v, p) \rightsquigarrow rep. breaks down into
dir. sum, so does open

key pt: if $(v, e) \neq (w, e)$ or , then,

$$(V \otimes W, \rho \otimes \sigma) \quad (\rho \oplus \sigma)(\varphi) = \rho(\varphi) \oplus \sigma(\varphi)$$

Now apply this to the regular rep (V, P)

$V = C[G]$ An erft by V is a formal sum

$\sum_{g \in S} C(g) \cdot [g] \quad C(g) \in \mathbb{C}[\Sigma g] \text{ for all symbols } g)$

$$\& \quad n \sum_{g \in G} c(g) \cdot [g] = \sum c(g) [n \cdot g]$$

since $\rho(\varphi) = 0 \Rightarrow 0 = \rho(\varphi)[\cdot]$

$$\begin{aligned}
 \text{but } \rho(\varphi)[\cdot] &= \sum_{g \in G} \varphi(g) g \cdot [\cdot] \\
 (\text{on the other hand}) \xrightarrow{\rho(\varphi)} &= \sum_{g \in G} (\varphi(g) \Sigma g) \cdot [\cdot] \quad \xrightarrow{\text{def of action}} \\
 &= \varphi(g) = 0 \quad \forall g \\
 &\Rightarrow \boxed{\varphi = 0}
 \end{aligned}$$

Tensor Products

Work with vec sp over a field k .

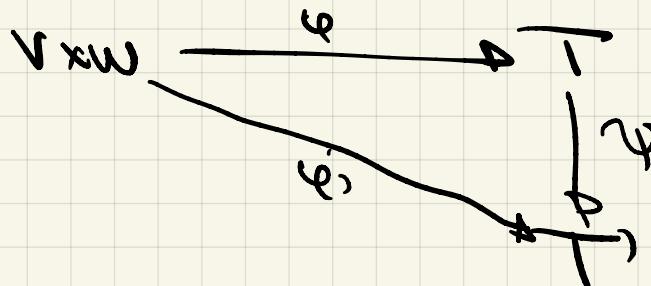
let $v, w \in \underline{\text{vec sp}}$

Def A tensor product of v, w is a pair

(T, ℓ) where T is a vec sp &
 ℓ is a bilinear map $\ell: V \times W \rightarrow T$ is
bilinear & univ

i.e. if (T', ℓ') is another pair
 $w(T)$ a v.s $\cong \ell': V \otimes W \rightarrow T'$ bilin

\exists linear map ψ to T such that $\psi: T \rightarrow T'$



Succinctly giving a bilinear map is equiv to giving
a linear map from T !

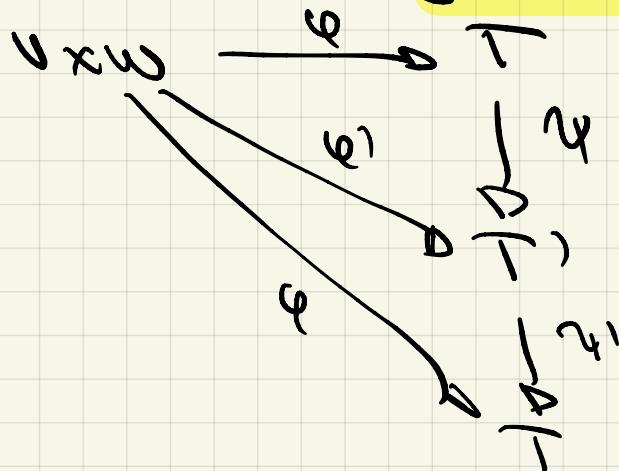
Uniqueness

Say (T, ℓ) and (T', ℓ') are two tensor prod

Univ prop for T gives lin map $\psi: T \rightarrow T'$

T' gives lin m $\psi': T' \rightarrow T$

Do the following commute!



By the Uniques of the map
in the Univ prop
 $\Rightarrow \psi \cdot \psi' = \psi' \cdot \psi$ are id!

By

Uniques we can speak of the tensor prod.
of V and W (assuming) it exists

Notation $T = V \otimes W$ & $v \otimes w$ is the img of
 $(v, w) \in V \times W$ under ℓ

Universality now notables

Given bilinear $\varphi: V \times W \rightarrow T$

\Rightarrow 1 unique lin map $\varphi: V \otimes W \rightarrow T$

$$\text{s.t } \varphi(v, w) = \varphi(v \otimes w)$$

Special case

Given a bilinear form on V , $\alpha, \gamma: V \times V \rightarrow K$

\Rightarrow 1 lin map $\varphi: V \otimes V \rightarrow K$ s.t

$$\varphi(v, w) = \varphi(v \otimes w) \quad ((V \otimes V)^*)$$

Existence

Consider the vector space \mathbb{F}

Want to have a basis
indexed by $\sqrt{v,w}$

write $\{v,w\}$ for the basis elt corr to (v,w)
(so want this to be like in each set)

Define $R \subset \mathbb{F}$ to be subspace spanned by

$$\circ [xv + yv', w] - \alpha[v, w] - \beta[v', w] \quad \forall \alpha, \beta \in \mathbb{K} \\ v, v' \in V, w \in W$$

• same in 2nd var

$T = \tilde{T}/R$ is \otimes pro $(\varphi: V \otimes W \rightarrow$
 $(v,w) \mapsto \text{img } \otimes \{v,w\})$

$\boxed{\text{Tr}}$ if v_1, \dots, v_n is a basis for V
 w_1, \dots, w_m --- for W

$\Rightarrow \{(v_i, w_j)\} \quad \forall i \in \{1, \dots, n\}, j \in \{1, \dots, m\}$
is a basis for T .

$$\Rightarrow \dim V \otimes W = \underline{\dim V \dim W}$$