Lec 11

Say $T: V \to V$ lin op. V has basis $(v_1, \dots, v_n)$

$$\begin{tikzcd} V \arrow[r, "T"] & V \arrow[d, "i_v^{-1}"] \\ F^n \arrow[u, "i_v"] \arrow[r, "T_A"] & F^n \end{tikzcd}$$

$T_A = i_v^{-1} \circ T \circ i_v$

A is the matrix of T wrt v

Let w be a basis. $i_v = i_w \circ T_C$ where $T_C : F^n \to F^n$ inv

Say B is the matrix of T wrt w

conj by C^{-1}

$T_A = i_v^{-1} \cdot T \cdot i_v = T_C^{-1} \circ T_B \circ T_C = T_{C^{-1}BC} \Rightarrow A = C^{-1}BC$

So the matrix for T is well defined ~~by~~ up to conjugation by $GL_n(F)$

↳ this notion lets us get lin trans invariants.

Recall | $A, B \in M_{n\times n}(F) \Rightarrow \det(A) \in F$ and $\det(AB) = \det(A)\det(B)$

$\Rightarrow \det(C^{-1}BC) = \det B$ if $C \in GL_n(F)$

So we define $\det(T) = \det(A)$ $\big(= \det(B)\big)$

lin transf — mat wrt v — ↳ mat wrt w

also $tr(T) = tr(A)$ $(= tr\ A)$

Def The characteristic polynomial denoted $\chi_T(z)$ is $\det(T - z\, id_V)$. If A = matrix for T then $\chi_T(t) = \chi_A(t)$

Last time $\lambda \in F$ is an eigenval for $T \iff \chi_T(\lambda) = 0$

$\{\text{eigenval of } T\} = \{\text{roots of } \chi_T \text{ in } F\}$

e.g. $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $F = \mathbb{R}$

$\chi_A(t) = \det(A - t \cdot id) = \det\begin{pmatrix} -t & 1 \\ -1 & -t \end{pmatrix} = t^2 + 1$

↳ no roots in $\mathbb{R}$

$\Rightarrow$ no $\mathbb{R}$ eval

Alg for finding eval, evec

1. pick basis $\mathbb{V}$ and compute matrix A
2. compute $\chi_T(z) = \chi_A(z)$
3. find roots of $\chi_A(t)$ in F. eval
4. compute eigsp $\ker(A - \lambda \cdot id) = V_\lambda$ eigenspace $\forall \lambda$

Diagonalization

Def | A linear operator T is diagonalizable such that $\exists$ basis so that the matrix of T wrt $\mathbb{V}$ is diagonal

Def | A $n \times n$ matrix A is diagonalizable iff $\exists C \in GL_n(\mathbb{R})$ so that CAC^{-1} is diagonal.

Rmk | two def are compatible. That is, T_A is diagonalizable $\iff A$ is diagonalizable

Prop | A linear operator $T: V \to V$ diagble iff $\exists$ basis consisting of eigen vectors for V.

Pf | $\Leftarrow$ Spse $\mathbb{V} = (v_1, \dots, v_n)$ is a bsb of evecs.

So $Tv_i = \lambda_i v_i \quad \forall i \in \mathbb{N}_n$

note ith col of A is the coord vect for $T(v_i)$

$$\Rightarrow A = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 \ddots & \\ 0 & & \lambda_n \end{pmatrix} \to \text{diagonal}$$

$\Rightarrow$ Spse diagonalizable $\therefore \exists$ basis $\mathbb{V}$ so matrix of T wrt A is diagonal. $\begin{pmatrix} \lambda_1 & \dots & \lambda_n \end{pmatrix}$

but immediate v_i has eval λ_i.

Nonex | $V = \mathbb{C}^2 \quad F = \mathbb{C}$

$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. $A \mid T_A$ not diag'ble.

$\chi_A(t) = (1-t)^2 \Rightarrow 1$ is the only eval

if v_1, v_2 were lin ind evect

$\Rightarrow T_A(v_1) = v_1$

$T_A(v_2) = v_2$

$\Rightarrow T_A = id_A \Rightarrow A = id \Rightarrow \Leftarrow$

Prop Let $T: V \to V$ lin op, $\dim V = n$. If $\chi_T(z)$ has n distinct roots in F. Then T is diagble.

Lemma Let $v_1, \dots, v_m$ evect for T with distinct evals $\lambda_1, \dots, \lambda_m$.
$\Rightarrow v_1, \dots, v_m$ are lin ind

Pf Suppose $\sum \alpha_i v_i = 0$ $\to$ new lin reln

$\Rightarrow \sum \alpha_i T(v_i) = 0 \Rightarrow \sum \alpha_i \lambda_i v_i = 0$

$\Rightarrow \sum \alpha_i \lambda_i^j v_i = 0 \quad \forall j \in \mathbb{N} \cup \{0\}$

Cs these eqn $\Rightarrow \alpha_i = 0 \Rightarrow$ lin ind

e.g. if $m=2$

① $\alpha_1 v_1 + \alpha_2 v_2 = 0$ & $\alpha_1 \lambda_1 v_1 + \alpha_2 \lambda_2 v_2 = 0$ ②

λ① $-$ ② $\Rightarrow \alpha_2 \underbrace{(\lambda_1 - \lambda_2)}_{\neq 0} v_2 = 0 \Rightarrow \alpha_2 = 0$

in gen
$$\det \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_m \\ \vdots & & & \\ \lambda_1^{m-1} & \lambda_2^{m-1} & \cdots & \lambda_m^{m-1} \end{bmatrix} = \pm \prod_{1 \le i < j \le m} (\lambda_i - \lambda_j) \neq 0 \text{ as } \lambda_i \neq \lambda_j$$
Vandermonde mat

Pf prop Say $\lambda_1, \dots, \lambda_n$ are distinct roots of χ_T.
Let $v_1, \dots, v_n$ be corr evects.
By lemma $v_1, \dots, v_n$ are lin ind $\Rightarrow$ they are a basis
So we have a basis of evect $\Rightarrow$ diagble

Trailer Let $\mathbb{R}^2$ be the plane.
A rigid motion of $\mathbb{R}^2$ is a bijection that preserves distance
e.g. rotations, translations, reflections,
The collection of all rigid motions in $\mathbb{R}^2$ is a grp! M.
Given $X \subseteq \mathbb{R}^2$ a subset (a "plane figure")
The symmetry group of X is $\{g \in M \mid g(X) = X\}$ grp

E.g. Symmetry grp of reg hexagon $\to$ 6 rot 6 refl $\to$ dihedral D_6
what are possible sym grps?