Lec 6 – Quotient Grps

Let G a grp and $N \trianglelefteq G$ normal subgrp.

We'll define grp G/N, the quotient group.

The elements of this groups are $\{gN \mid g \in G\} = \{Ng \mid g \in G\}$

where operation is $g_1N \cdot g_2N = (g_1 \cdot g_2)N$

First to check (this is well def). Suppose g_1, g_2 def same coset & h_1, h_2 def same coset. WTS

$$g_1N \cdot h_1N = (g_1 \cdot h_1)N = (g_2 \cdot h_2)N = g_2N \cdot h_2N$$

$\exists n, n' \in N$ so $g_1 = g_2 n$, $h_1 = h_2 n'$

$$g_1 h_1 = g_2 n h_2 n' = g_2 h_2 \underbrace{h_2^{-1} n h_2}\, n'$$

conjugation N is closed

$$= g_2 h_2 n''$$

$$\Rightarrow g_2 h_2 N = g_1 h_1 N$$

Group axioms for G/N straightforward.

Def) let G grp, N normal subgrp. Consider

$$\pi: G \longrightarrow G/N$$
$$g \longmapsto gN$$

Claim π is grp homo. $\ker(\pi) = N$

Pf 1) $\pi(g)\pi(h) = gN\,hN = ghN = \pi(gh)$

Pf 2) $\pi(g) = gN = \underset{\text{identity elt}}{N} \iff g \in N$.

e.g. $G = (\mathbb{Z}, +)$, $N = n\mathbb{Z}$

$G/N = \mathbb{Z}/n\mathbb{Z} = \{\bar{0}, \bar{1}, \ldots \overline{n-1}\}$

$\bar{a} + \bar{b} := \overline{a+b} = \bar{c}$ where $0 \le c < n$ and $c \equiv a+b \mod n$

$\mathbb{Z}/n\mathbb{Z} = \langle \bar{1} \rangle$ so it is cyclic.

if $n = 12$ then $\langle \bar{5} \rangle$, $\langle \bar{7} \rangle$

e.g $G = GL_2(\mathbb{C}) \supseteq N = \left\{ \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} \mid \alpha \in \mathbb{C} \setminus \{0\} \right\}$

Claim N is normal $\rightarrow g \in G$ and $n \in N$ $gn = ng$

n is central et

$gng^{-1} = n \in N$

$GL_2(\mathbb{C})/N = PGL(\mathbb{C})$ projective gen lin grp

$SL_2(\mathbb{C}) < GL_2(\mathbb{C}) \xrightarrow{\pi} PGL_2(\mathbb{C})$

Ex| get $\ker(\pi|_{SL_2(\mathbb{C})}) \rightarrow$ set of coordinate permutations

e.g. $G = \mathbb{C}^\times$, $N = \{ z \in \mathbb{C} \mid |z| = 1 \}$

$G/N \cong \mathbb{R}_+ = \{ x > 0 \mid x \in \mathbb{R} \}$ (under mult)

$\varphi : G/N \longrightarrow \mathbb{R}_+$ is it

$zN \longmapsto |z|$

e.g $G = S_3$ $N = \langle (1,2,3) \rangle = A_3 = \ker(\text{sgn})$ by below $G/N \cong \mathbb{Z}/2\mathbb{Z}$

Def| $[G:N]$ is the index of N in G (number of cosets)

Aside| $H \leq G$ subgrp $G/H = \{ gH \mid g \in G \}$, $H \backslash G = \{ Hg \mid g \in G \}$ quotient sets

There is a natural bijection bet $G/H \rightarrow H \backslash G$

$gH \mapsto Hg$ not well def! $\rightarrow$ if $g = g'h$ for $h \in H$

$\Rightarrow gH = g'H$ but not nec $Hg = Hg'$

we want $gH \mapsto Hg^{-1}$

$\hookrightarrow$ ex!

Def| The index of H in G, denoted $[G:H]$ is the number of left or right cosets.

Lagrange's Thm Pt 2

G finite grp $H \leq G$ Then $[G:H]|H| = |G|$

Pf| $G/H = \{ g_1 H, \ldots, g_n H \}$ no dup

$n = [G:H]$ $\#(g_i H) = \#H$ $\forall i \in \mathbb{N}_n$

Thm) First Isomorphism Thm

Let $\varphi : G \to H$ is surj homeo.

Let $N = \ker(\varphi)$

Then φ induces an isom $\psi : G/N \to H$

$gN \to \varphi(g)$

PP) First ψ well def?

1) Spose $g_1N = g_2N \Rightarrow g_1 = g_2 n$ for some $n \in N$

$\Rightarrow \varphi(g_1) = \varphi(g_2 n) = \varphi(g_2)\varphi(n)$ $\quad n \in \ker(\varphi)$

$= \varphi(g_2)$

$\Rightarrow \psi(g_1N) = \psi(g_2N)$ ✓

2) Homo: let $\psi(gN) \cdot \psi(nN) = \varphi(g)\varphi(n) = \varphi(gn)$

$= \psi(gnN) = \psi((gN)\cdot(gH))$

3) Surj. let $h \in H$. $\exists g \in G$ so $\varphi(g) = h$

$\Rightarrow \psi(gN) = \varphi(g) = h$

4) Inj. let $gN \in \ker \psi \Rightarrow \varphi(g) = 1 \Rightarrow g \in \ker \varphi$

$\Rightarrow g \in N$ so $gN = N$ ✓

e.g.) $\varepsilon : S_n \to \{\pm 1\}$ sign $n \geq 2$

Recall $A_n = \ker(\varepsilon)$ is alt grp $A_n = \ker \overset{\text{surj}}{\varepsilon}$

By Thm $S_n / A_n \cong \{\pm 1\}$