Dual Rep

Say V is a fin dim v.space w/ basis $v_1 \dots v_n$

V^* — dual space w/ basis $v_1^*, \dots, v_n^*$

Recall, if $T: V \to V$ a linear op

the (i,j) matrix entry of T in this basis <u>matrix</u> $\langle v_i^*, T(v_j)\rangle$

let ρ be a rep of G on V $\rho: G \to GL(V) \overset{\text{via } v_1,\dots,v_n}{\cong} GL(n,\mathbb{C})$ (<u>w basis</u>)

let $\rho^*: G \to GL(V^*) \overset{\text{via dual basis}}{\cong} GL(n,\mathbb{C})$ The dual rep!

Prop $\rho^*(g)$ is the transpose inverse of the matrix $\rho(g)$

Pf The (i,j) matrix entry of $\rho^*(g)$ is

$$\langle v_i^{**}, g\cdot v_j^*\rangle = \langle g v_j^*, v_i\rangle = \langle v_j^*, g^{-1} v_i\rangle$$

canonical iso

defn of dual rep.

The (j,i) entry of $\rho(g^{-1}) = \rho(g)^{-1}$

Hom Representation!

Let V, W be 2 reps of G.

$\mathrm{Hom}(V,W) = \{T: V \to W \mid T \text{ linear}\} \rightsquigarrow \boxed{\text{vec space}}$

$\dim \mathrm{Hom}(V,W) = \dim V \cdot \dim W$!

This holds for any vec sp but the rep struct of V, W gives a rep on $\mathrm{Hom}(V,W)$!

Define a rep of G on $\mathrm{Hom}(V,W)$ by, for $T \in \mathrm{Hom}(V,W)$

$$(gT)(v) = gT(g^{-1}v) \rightsquigarrow \text{check works!}$$

Special cases • $\mathrm{Hom}(V, \mathbb{C}) \rightsquigarrow$ (1 dim rep with trivial action)

$= \text{Dual Rep } V^*$

• $\mathrm{Hom}(\mathbb{C}, V) \cong V$ (as rep)

Property! ◦ $\mathrm{Hom}(V, W_1 \oplus W_2) \cong \mathrm{Hom}(V, W_1) \oplus \mathrm{Hom}(V, W_2)$
& in first component too!

Prop $\chi_{\mathrm{Hom}(V,W)}(g) = \overline{\chi_V(g)} \cdot \chi_W(g)$ (exercise)

Pf Pick basis $v_1 \dots, v_n \in V$ & $w_1, \dots, w_m \in W$
$T_{ij} : V \to W$ so $T_{ij}(v_k) = \begin{cases} w_j & \text{if } k=i \\ 0 & \text{else} \end{cases}$ } basis for $\mathrm{Hom}(V,W)$!

$$\chi_{\mathrm{Hom}(V,W)}(g) = \sum_{ij} \underbrace{\langle T_{ij}^*, g T_{ij} \rangle}$$

↳ show this is $\langle v_i^*, g^{-1} v_i \rangle$. $\langle w_j^*, g w_j \rangle$

$$= \chi_V(g^{-1}) \chi_W(g)$$
$$= \overline{\chi_V(g)} \chi_W(g)$$

Invariants!

If V is a rep of G the invariant sp $V^G = \{v \mid gv = v \ \forall g \in G\}$

subrep of V where G acts trivially on V^G { ↳ vector subsp of V. ↳ closed under action of G

The averaging operator is a G-lin proj onto V^G

Defn $\mathrm{avg} : V \to V$
$$v \mapsto \frac{1}{|G|} \sum_{g \in G} gv$$

Prop - avg is linear (sum of linear)
◦ avg is g linear! $\mathrm{avg}(hv) = \frac{1}{|G|} \sum_{g \in G} ghv$
$$= \frac{1}{|G|} \sum_{g' \in G} g'v = \mathrm{avg}(v)$$
$$= h \cdot \mathrm{avg}(v)$$

• $\mathrm{im}(\mathrm{avg}) \leq V^G$
◦ $\mathrm{avg}(v) = v$ if $v \in V^G$
⇒ Proj op onto inv sp!

Lemma Let $e : V \to V$ be proj (i.e. $e^2 = e$)

Then $\dim \operatorname{im}(e) = \operatorname{tr}(e)$

Pf $V = \operatorname{im}(e) \oplus \ker(e)$

$v_1 \dots v_m \qquad v_{m+1} \dots v_n$ basis

Matrix for e $= \begin{array}{cc} \left(\begin{array}{c|c} I_m & 0 \\ \hline 0 & 0_{n-m} \end{array}\right) & \begin{array}{c} m \\ n-m \end{array} \\ \begin{array}{cc} m & n-m \end{array} & \end{array}$

$\operatorname{tr}(\text{mat for } e) = m$

Thm Let V be a rep of G.

$$\dim V^G = \frac{1}{|G|} \sum_{g \in G} \chi_V(g)$$

Pf $\dim(V^G) = \operatorname{tr}(\text{avg}) \to$ as avg is a proj on V^G

$\text{avg}(v) = \frac{1}{|G|} \sum_g gv$

So as an op $\text{avg} = \frac{1}{|G|} \sum g$

So, $\operatorname{tr}(\text{avg}) = \frac{1}{|G|} \sum_g \underbrace{\operatorname{tr}(g|V)}_{\chi_V(g)}$

tr is linear

Say V, W are two reps! We have $\operatorname{Hom}(V, W) \to$ all lin maps

Can also look at $\operatorname{Hom}_G(V, W) = G$-eq maps $V \xrightarrow{T} W$

$T(gv) = gT(v)$

Obs $T \in \operatorname{Hom}(V, W)$ is G-eq iff it is G-inv for action of G on hom rep!

Reason If $g \cdot T = T$ then $gT(g^{-1}v) = T(v)$ $\forall v \in V$

if $v' = g^{-1}v \implies gT(v') = T(gv')$ $\forall v' \in V$

$\implies T$ is G-equiv

So) $\mathrm{Hom}_G(V,W) = (\mathrm{Hom}(V,W))^G$

Thm) $\dim \mathrm{Hom}_G(V,W) = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_V(g)}\, \chi_W(g)$

Pf) Apply prev thm to rep $\mathrm{Hom}(V,W)$
& note $\mathrm{Hom}(V,W)^G = \mathrm{Hom}_G(V,W)$.

Recall $\mathcal{C}(G)$ is the space of class func = $G \to \mathbb{C}$ inv under conj

for $\varphi, \psi \in \mathcal{C}(G)$ we define

$$\langle \varphi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\varphi(g)}\, \psi(g)$$

This is a pos def hermitian form on $\mathcal{C}(G)$

P $\dim \mathrm{Hom}_G(V,W) = \langle \chi_V, \chi_W \rangle$

Thm) (Schur Orthogonality)
Characters of irreducible reps are orthonormal!
wrt $\langle , \rangle$

Expand) if V, W are irred reps of G then

$$\langle \chi_V, \chi_W \rangle = \begin{cases} 1 & \text{if } V \cong W \\ 0 & V \not\cong W \end{cases}$$

Pf) $\langle \chi_V, \chi_W \rangle = \dim \mathrm{Hom}_G(V,W) = \begin{cases} 1 & \text{if } V \cong W \\ & V \not\cong W \end{cases}$

by Schur.

Cor) # isomorphism classes of irred reps is $\leq \dim \mathcal{C}(G)$
= conj classes in G