Motivation,

On $\mathbb{R}^n$ $\exists$ std dot prod
$\rightsquigarrow$ have positivity $v \cdot v \geq 0$ $\forall v$ $\rightsquigarrow$ often important

On $\mathbb{C}^n$, can also consider std dot prod.
But not positive! $\rightsquigarrow$ $v = \sqrt{i}\, e_1$ $v \cdot v = i$.

But, consider.

$$\langle , \rangle : \mathbb{C}^n \times \mathbb{C}^n \longrightarrow \mathbb{C}$$

$$\left\langle \sum_{i=1}^n x_i e_i , \sum y_i e_i \right\rangle \longmapsto \sum_{i=1}^n x_i \overline{y_i} \rightsquigarrow \text{conj}$$

if $y_i = x_i$ $\rightarrow$ $\sum_{i=1}^n |x_i|^2 \geq 0$

Properties.

1) complex linear in 1st var:
$\langle \alpha v + \beta v', w \rangle = \alpha \langle v, w \rangle + \beta \langle v', w \rangle$
for $\alpha, \beta \in \mathbb{C}$ & $v, v', w \in \mathbb{C}^n$

2) Conjugate linear in 2nd var:
$\langle v, \alpha w + \beta w' \rangle = \overline{\alpha} \langle v, w \rangle + \overline{\beta} \langle v, w' \rangle$
$\forall v, w, w' \in \mathbb{C}^n$, $\alpha, \beta \in \mathbb{C}$

3) $\langle v, w \rangle = \overline{\langle w, v \rangle}$ $\forall v, w \in \mathbb{C}^n$

4) $\langle v, v \rangle \in \mathbb{R}$ and ≥ 0 $\forall v \in \mathbb{C}^n$
with eq iff $v = 0$.

Defn if V is a $\mathbb{C}$ v.s then a Hermitian form on V is a form $\langle , \rangle : V \times V \to \mathbb{C}$ if it satisfies 1, 2, 3

it is pos def if 4.

E.g.) let $V = C^0$ on $[0,1] \to \mathbb{C}$.
then $\langle f,g \rangle = \int_0^1 f(x)\overline{g(x)}\,dx$

is a pos def hermitian form!

Most stuff abt gram bilinear forms holds true (with minor modifications)

Let V a hermitian v.s.
Choose a basis $e_1, \dots, e_n$ and let $a_{ij} = \langle e_i, e_j \rangle$

$A \to n \times n$ matrix with entries (a_{ij}) determine the form

$\Rightarrow$ note $a_{ij} = \overline{a_{ji}}$ so A is hermitian matrix.
$\Leftrightarrow A = A^*$

Defn) For an $n \times n$ matrix in $\mathbb{C}$ called B, let $B^* = \overline{B}^t$

Defn) A $n \times n$ complex matrix is unitary if
$B^* = B^{-1}$ $\to$ complex analogue of orth mat

Unitary matrices pres std Hermitian form on $\mathbb{C}^n$
[form a grp $U(n)$].

Def) $v, w \in V$ is orthogonal if $\langle v, w \rangle = 0$

- V always has an orthogonal basis. $v_1, \dots, v_n$
- If V non-degen can normalize v_i so $\langle v_i, v_i \rangle = \pm 1$.
  $\Rightarrow$ as $\langle \alpha v_i, \alpha v_i \rangle = \alpha\bar{\alpha}\langle v_i, v_i \rangle$
  axiom 3 $\Rightarrow \langle v_i, v_i \rangle \in \mathbb{R}$.
  $\Rightarrow$ can't change sign!
- Analogue of sylvesters law holds!

Fix fin dim pos def herm sp V.

Consider a linear operator $T: V \to V$

Since pos def $\to$ fix orthonormal basis $e_1, \dots, e_n$ of V

Look at matrix A of T.

Facts.

1) A is hermetian $(A = A^*)$

$\iff \langle Tv, w \rangle = \langle v, Tw \rangle$ $\forall v, w$ $\to$ T is a hermetian operator (self adj)

2) A is unitary $(AA^* = I)$

$\iff$

$\langle v, w \rangle = \langle Tv, Tw \rangle$ $\forall v, w$ $\to$ T is unitary operator

Spectral Thm.

1) If T is a hermetian operator on V

$\implies$ exist an O.N.B of eigenvectors.

2) If A is a $n \times n$ hermetian matrix

$\implies$ can diagonalize A by a unitary matrix.

i.e. BAB^{-1} is diag for B unitary.

Pf) Let T be given. Let v_1 be an eigenvector for T. $\to$ eval λ

(scale so norm 1) $\leftarrow$ Can do as $\mathbb{C}$ is algebraically closed. (Call ops on fin dim $\mathbb{C}$ v.s have at least one e-val)

Choose $v_2, \dots, v_n$ so $v_1, \dots, v_n$ onb.

Matrix of T wrt $v_1, \dots, v_n$

$$\begin{pmatrix} \lambda & * & * & \cdots & * \\ 0 & & & & \\ \vdots & & * & & \\ 0 & & & & \end{pmatrix}$$

as this matrix is hermetian $\implies$ all zero's! (except λ).

$\text{span}(v_1, \ldots, v_n) = v_1^\perp$

Above observation $\Rightarrow$ T preserves this $n-1$ dim'l sp.

By induction. $\exists$ ONB $w_2, \ldots, w_n$ of eigen vect of T for $v_1^\perp$

So $v_1, w_2, \ldots, w_n$ is ONB ~~for~~ V & evects for T!

2) equiv to 1st statement.

Prop) ~~If~~ T is a hermitian operator on V

$\Rightarrow$ All e.v of T are real.

Pf) Say λ is an eigenvalue for T with eigen vect v.

$$\lambda\langle v,v\rangle = \langle Tv, v\rangle \underset{\text{herm}}{=\!=} \langle v, Tv\rangle = \bar{\lambda}\langle v,v\rangle$$

$$\Rightarrow \boxed{\lambda = \bar{\lambda}} \text{ as } v \neq 0 \Rightarrow \langle v,v\rangle \neq 0$$

$$\Rightarrow \boxed{\lambda \in \mathbb{R}}$$

Back to Reals

let V be a n-dim'l vect sp with pd of symm bilinear form.

let, $T: V \to V$ be a lin op.

- T is symmetric $\Rightarrow$ if $\langle Tv, w\rangle = \langle v, Tw\rangle$

$\Leftrightarrow$ matrix for T wrt ONB, is symmetric.

$\mathbb{R}$ spectral thm

1) If T is a symm op on V $\exists$ orthonormal eigen basis

2) If A $n\times n$ symm with real entries $\exists$ orthogonal B so BAB^{-1} is diag!

RR) . Since A is a sym real matrix it is
also a hermetian matrix

($\Rightarrow$) all eigenvalues of A (or of T)
are <u>real</u> #'s (all roots of char ply
are real)

<u>Now</u>, <u>mimic</u> previous proof!

DT induces hermetian operator on <u>$T \otimes \mathbb{C}$</u> .