Last time

Actions + G-sets

e.g. $G \curvearrowright G$ in 3 ways $g \cdot h \mapsto gh,\ hg^{-1},\ ghg^{-1}$

More generally

$H \leq G$ subgrp $G/H = \{gH \mid g \in H\}$ (note: grp iff H is normal else set)

$G \curvearrowright G/H$

$g' \cdot gH = g'gH$

note Stabilizer of $H \in G/H$ is H
(if $gH = H$ then $g \in H$ and conversely)

Stabilizer of $gH = gHg^{-1}$

Generally $X = G$-set, $x \in X$, $g \in G$

$G_{gx} = g G_x g^{-1}$

let $h \in G_x$ wts $ghg^{-1} \in G_{g(x)}$

$ghg^{-1}(gx) = ghx = (gx)$ ✓

G/H is transitive! So it has just one orbit.

$g \cdot H = gH$! $O_H = G/H$

Defn: A map of G sets $f: X \to Y$ so
$f(gx) = g f(x)$ $\forall g \in G, x \in X$
(called G-map, G-equivariant map)

Defn An isomorphism of G sets is just a bijective map of G-sets.

Classification of G-sets

Let X be a G-set. Consider O_x = orbit of $x \in X$ [closure]

Obs O_x is a G-subset of X. $y \in O_x \Rightarrow y = hx$
$\Rightarrow gy = ghx \in O_x$

O_x is a transitive G-set.

Spose $x_1, \dots, x_n$ are rep of distinct G-orbits on X.

$\Rightarrow X = \bigsqcup_{i \in \mathbb{N}_n} O_{x_i} \to$ disjoint

$\hookrightarrow$ all G-sets are disjoint unions of transitive G-sets!

$\Rightarrow$ suffices to classify transitive G-sets!

Prop) let X be a trans G-set.

Then X is isomorphic to G/H (as a G-set)

for some $H \leq G$

In fact $x \in X$ get iso $G/G_x \to X$

$gG_x \mapsto gx$

Pf) Pick $x \in X$, say $H = G_x$

define $f: G/H \longrightarrow X$

$gH \longmapsto gx$

Claim $\to$ isomorphism and well def.

1) f well def $gH = g'H \Rightarrow g = g'h$ for $h \in H = G_x$

$\Rightarrow gx = g'\underset{G_x}{\underset{\cap}{h}}x = g'x$ ✓

2) show it is a g-map.

$g(f(g'H)) = gg'x =: f((gg')H) = f(g \cdot g'H)$

3) show surjectivity

let show $y \in X \Rightarrow$ trans $\exists g'$ so $g'x = y$

$\Rightarrow f(g'H) = g'x = y$ ✓

4) show injectivity.

Spose $f(gH) = f(g'H) \Rightarrow gx = g'x$

$\Rightarrow x = g^{-1}g'x \Rightarrow g^{-1}g' \in H \Rightarrow gh = g'$

$\Rightarrow gH = g'H$

Def X is transitive G-set $G/G_x \xrightarrow{\sim} X$ for $x \in X$

$G/G_y \xrightarrow{\sim} X$ for $y \in X$

if $y = gx \Rightarrow G_y = g G_x g^{-1}$

$\Rightarrow G/G_x \cong G/gG_xg^{-1}$

Prop Suppose $H_1, H_2 \leq G$ subgroups

G/H_1 is isom to G/H_2 as G-sets

$\Longleftrightarrow H_1$ conj to H_2

Cor $\{$trans G-sets$\}/$isom $\xleftrightarrow{\text{bij}}$ $\{$conjugacy classes of subgroups$\}$

Pf of Prop $\Rightarrow$ if $H_2 = gH_1g^{-1}$ $X = G/H_1$ $x = gH_1$

$\Rightarrow G_x = gH_1g^{-1} = H_2$

prev prop $\Rightarrow \exists$ iso

$$\begin{array}{ccc} G/G_x & \longrightarrow & G/G_1 \\ \| & & \| \\ G/H_2 & & G/H_1 \end{array}$$

$\Leftarrow$ suppose $f: G/H_1 \to G/H_2$ is an isom of G-sets

$f(H_1) = gH_2 \in G/H_2$

Stab of $f(H_1)$ = stab of H_1 (b/c is is isom)

$= H_1$

Stab of $gH_2 = gH_2g^{-1}$

$\Rightarrow H_1 = gH_2g^{-1}$

Prop G - fin grp. $X = G$-set $x \in X$ (counting)

then $\# O_x \cdot \# G_x = \# G$

Pf O_x is transitive G-set $\Rightarrow$ $G/G_x \xrightarrow{\sim} O_x$ isom of G-set

$\Rightarrow |O_x| =$ number of cosets $= \dfrac{|G|}{\# G_x}$

$\Rightarrow \# O_x \cdot \# G_x = \# G$

Defn let p prime.
A p-grp is a finite grp whose order is p^n

Prop If G is a nontrivial p-grp the $Z(G)$ is non trivial
centre

Def $Z(G) = \{g \in G \mid gh = hg \ \forall h \in G\}$

$g \in G$ $C_g =$ conj class of $g = \{hgh^{-1} \mid h \in G\}$

centraliser $Z_g = \{h \in G \mid hg = gh\} \to hgh^{-1} = g$

$Z_g = G \iff g \in Z(G)$

Note $G \curvearrowright G$ by conjugation
$g \cdot h = ghg^{-1}$
O_g for $g \in G$ is C_g
stabiliser of $g \in G$ is Z_g

$\hookrightarrow$ counting gives $\#C_g = \dfrac{\#G}{\#Z_g}$

Pf) Prop $\#G = p^n$ $n > 0$
let $g_1, \dots, g_r$ be reps of conj classes
$\Uparrow$

$G = C_{g_1} \sqcup \dots \sqcup C_{g_r}$ → class eqn

$\Rightarrow p^n = \#G = \#C_{g_1} + \dots + \#C_{g_r}$

$\#C_{g_1} = \dfrac{\#G}{\#Z_g}$ This is $\boxed{1 \iff g_i \in Z(G)}$
if $g_i \notin Z(G)$ then
reduce class eqn mod p $\#C_{g_i} = p^k$, $n > k > 0$
by divisibility

$0 \equiv \#C_{g_1} + \dots + C_{g_r} \pmod p$

$\hookrightarrow$ This is just zeros and 1s, as $\#C_{g_1} = 1 \Rightarrow$ some other must be nonzero $\Rightarrow$ central elt