Lec 24 - Bilinear Forms

Motivation: $\mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$

$\langle x, y \rangle \longmapsto \sum x_i y_i$

→ Linear in both entries $\langle x+y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ & similar

→ Symmetric $\langle x, y \rangle = \langle y, z \rangle$

→ pos def $\langle x, x \rangle \geq 0$ with equality iff $x = 0$

Def Let V be a F-v space. Then, a bilinear form is

$\langle \ , \ \rangle : V \times V \to F$

So, 1) $\langle \ , \ \rangle$ is F linear in each variable

Adjective

1) Symmetric: $\langle v, w \rangle = \langle w, v \rangle$

2) Skew-Symmetry: $\langle v, w \rangle = -\langle w, v \rangle$ (Alternating)

2') Alternating — $\langle v, v \rangle = 0$

$\Rightarrow$ skew sym $\to 0 = \langle v+w, v+w \rangle$

$= \langle v, v \rangle + \langle v, w \rangle + \langle w, v \rangle + \langle w, w \rangle$

$= \langle v, w \rangle + \langle w, v \rangle$ ✓

Skew Sym $\Rightarrow$ Alternating if F not char 2

3) Nondegen $\to \nexists\, v \neq 0$ s.t. $\forall w$ $\langle v, w \rangle = 0 \Rightarrow$

or $\langle v, v \rangle = 0$

e.g. 1) $V = \mathbb{F}^2$ $\quad \langle \ , \ \rangle : V \times V \to F$

$\left\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\rangle = \det \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}$

$= x_1 y_2 - y_1 x_2$

skew symmetric, Alternating, vanishing.

2) $V = \mathbb{F}^n$ dot prod

3) $V = \{ f : [0,1] \to \mathbb{R}$ is cts$\}$ on $\mathbb{R}$

$\langle f, g \rangle = \int_0^1 f(x) g(x)\, dx$.

4) $V = M_n(\mathbb{R})$ $\quad \langle A, B \rangle = tr(AB)$ → Symmetric

as $tr(A) = tr(BAB^{-1})$

$tr(AB) = tr(BABB^{-1})$

pretty easy to use defs to show this

e) $\mathbb{Q}^4 \times \mathbb{Q}^4 \to \mathbb{Q}$ dot prod.

for $a \in \mathbb{Z}$

$\exists!$ x s.t $\langle x, x \rangle = a$? Yes!

$x = \begin{pmatrix} x_1 \\ \vdots \\ x_4 \end{pmatrix} = x_1^2 + \cdots + x_4^2$ (Lagrange)

Matrix assoc to bilinear form

Let $\langle , \rangle : V \times V \to F$ let $(e_1, \ldots, e_n)$ of V

Def $A : (\langle e_i, e_j \rangle) \to \boxed{n \times n}$

Then $\langle \sum_i a_i e_i , \sum_j b_j e_j \rangle = \sum_i a_i \sum_j \langle e_i, e_j \rangle b_i$

Q: how does A dep on choice of basis

$= (a_1 \ldots a_n) A \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$

$= \boxed{a^t A b}$

$\boxed{\begin{array}{l} \text{if bilinear is symmetric} \Rightarrow A \text{ sym} \\ \text{skew symm} \Rightarrow A \text{ skew sym} \end{array}}$

if $\{e_1, \ldots, e_n\}$ $\{f_1, \ldots, f_n\}$ basis with X cov. E to F

$\quad\; = E \qquad\qquad = F$

$a = [v_1]_E \Rightarrow a' = [v_1]_F = Xa$

$b = [v_2]_E \qquad b' = [v_2]_F = Xb$

For A' want $(Xa)^T A (Xb) = \langle v_1, v_2 \rangle = a^T A b$

$a X^T A X b = a^T A b$

$\Rightarrow \boxed{X^T A X = A}$

$\langle , \rangle \xrightarrow{\text{basis}} A \to \det(A)$ well def? No

change basis then $\det(A)$ changes by square $\hookrightarrow \det(X)^2$.

disc $\langle , \rangle \xrightarrow{\text{basis}} A \to \det(A) \in F^* / (F^*)^2 \cup \{0\}$ if $\det A = 0$

if $F = \mathbb{R}$ $\rightarrow$ it detects sign! $\mathbb{Z}/2\mathbb{Z}$ valued

$F = \mathbb{C}$ $\rightarrow$ bets are off

does discr determine bilinear form? No

dot prod dot prod mult 2

But for $\mathbb{F}_p$ yes!