Lec 9

Prop let V be an F-vector sp. let $v_1, \dots, v_n$ vectors
TFAE
1) They are a basis
2) $\forall\, v \in V \ \exists!\ \alpha_i$ of F so $v = \sum_i \alpha_i v_i$

Pf ① $\Rightarrow$ ②
Since $\{v_i\}_{i \in \mathbb{N}}$ basis. $w \in \operatorname{span}(v_i)$.
Suppose $w = \sum_i \alpha_i v_i = \sum_i \beta_i v_i$
$\Rightarrow 0 = \sum_i (\alpha_i - \beta_i) v_i \Rightarrow \alpha_i = \beta_i$
② $\Rightarrow$ ① similar

This lets us get well def coord on V wrt fixed basis
i.e $w = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \Rightarrow w = \sum_i \alpha_i v_i$

e.g $V = \mathbb{R}^2$ $(e_1, e_2) \to$ std coord system
choose new basis $v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$
$v_1 + v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}_{v_i}$

The system of coordinates gives a bijection
$i : F^n \to V$
$\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \mapsto \alpha_1 v_1 + \dots + \alpha_n v_n$ $\rightarrow$ is this an "isomorphism"

Def V, W vs over F. Then $f : V \to W$ is a linear transformation p.t
$\forall\, v, v' \in V$, $c \in F$ $\quad f(v + v') = f(v) + f(v')$
$f(cv) = c f(v)$

Def A linear isomorphism is a bijective linear transformation

Obs $i : F^n \to V$ given above is an iso
Pf trivial

Upshot | Any n dimensional F-vec space $V \cong F^n$

Note: i satisfies $i(e_j) = v_j$ (basis to basis)

Prop | V is a vec sp with basis $(v_1, \dots v_n)$
W is a vec sp with elt $(w_1, \dots, w_n)$ $\Big]$ over F

$\exists!$ linear map $f: V \to W$ s.t $f(v_i) = w_i$ $\forall i \in \mathbb{N}_n$

Moreover, if w_i a basis f is an iso!

PR | To begin define $T: V \to W$ by

$$T\left(\sum_i \alpha_i v_i\right) = \sum_i \alpha_i T(v_i) = \sum_i \alpha_i w_i$$

$\hookrightarrow$ well def by legwork. Trivially lin transf

Check uniqueness. lets say $T': V \to W$ so $v_i \mapsto w_i$

$$T(v) = T\left(\sum \alpha_i v_i\right) \overset{\text{def}}{=} \sum \alpha_i T(v_i) = \sum \alpha_i T'(v_i)$$

$$\underset{\text{linearity}}{=} T'\left(\sum \alpha_i v_i\right) = T'(v)$$

If w_i basis,

WTS v is an iso. Construct inverse.

By above $\exists!$ $U: W \to V$ so $U(w_i) = v_i$

$T \circ U = Id_W$, $U \circ T = Id_V \to$ bij ✓

$w_i \mapsto w_i$ $\quad$ $v_i \mapsto v_i$ $\to$ by uniqueness it is the identity!

Say A is an $n \times m$ matrix with entries in F.

We define

$$T_A : F^n \to F^m$$
$$\vec{v} \longmapsto A v$$

Claim | Every lin transf $T: F^n \to F^m$ is of the form $T_A : F^n \to F^m$ for a unique A.

Pf) Let $v_i = T(e_i)$. Claim $A = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$

By comp $T_A(e_i) = v_i$. By uniqueness from prev prp $T_A = T$

Uniqueness is easy to show as $T_A(e_i) = \text{col } i$

Set up) $A = n \times m$ mat, $B = l \times m$ mat

$$\begin{tikzcd} F^m \arrow[r, "T_A"] \arrow[rr, bend right, "T_{BA}"'] & F^n \arrow[r, "T_B"] & F^l \end{tikzcd}$$

Claim: $T_B \circ T_A = T_{BA}$ → matrix pro

Pf) Compute image of $\{e_i\}_{i \in \mathbb{N}_m}$

Cor) if $A = n \times n$ matrix then T_A is isom iff A is inv

i.e. isom from $F^n \to F^n$ = $Aut(F^n) \cong GL_n(F)$ → group isom

$T_A \leftarrow\!\shortmid A$

Say $V = (v_i)_{i \in \mathbb{N}}$, $W = (w_i)_{i \in \mathbb{N}}$ basis for V vs.

$i_V : F^n \to V$ $\qquad$ $i_W : F^n \to V$

$\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \mapsto \sum \alpha_i v_i$

Get an automorpha

$$\begin{tikzcd} F^n \arrow[r, "i_V"] \arrow[rr, bend right, "T_A"'] & V \arrow[r, "i_W^{-1}"] & F^n \end{tikzcd}$$

$i_W^{-1} \circ i_V = T_A$

$\Rightarrow i_V = i_W \circ T_A$ → change of basis

The automorphism T_A lets us know how different V, W are.

e.g. $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ $\quad i_V = i_W \cdot T_A$ eval on e_1, e_2

$v_1 =$