Lec 26 |

Suppose V is a $\underbrace{\text{quadratic sp}}_{\text{fin dim}}$ / field K

$\exists$ orthogonal basis $v_1, \dots, v_n$ say $\langle v_i, v_i \rangle = a_i$

$$V = [a_1, \dots, a_n]$$

Defn | given $b_1, \dots, b_n \in K$ let $[b_1, \_\_, b_n]$ be quad sp with orth basis $v_1, \dots, v_n$ & $\langle v_i, v_i \rangle = b_i$

matrix with $v_1, \dots, v_n$ basis

$$\begin{pmatrix} b_1 & & 0 \\ & \ddots & \\ 0 & & b_n \end{pmatrix} \rightsquigarrow \text{disc} \Rightarrow = b_1 \cdots b_n \pmod{(K^*)^2}$$

non degen $\iff b_i \neq 0 \ \forall i$

Problem | Classify quad sp / K up to isom know have form $[b_1, \dots, b_n] \rightsquigarrow$ understand isom between them

First $K = \mathbb{C}$

Claim $[b_1, \dots b_n] \cong [1, \dots, 1]$

all $b_i \neq 0$

let $V = [b_1, \dots, b_n]$ with basis $v_1, \dots, v_n$

let $w_i = \dfrac{v_i}{\sqrt{b_i}}$ have $\langle w_i, w_j \rangle = 0$ if $i \neq j$

$$\langle w_i, w_i \rangle = \frac{\langle v_i, v_i \rangle}{b_i} = 1$$

$\Rightarrow w_1, \dots, w_n \rightarrow$ orth norm basis $\Rightarrow V \cong [1, \dots, 1]$

Rephrase if V, W 2 non degen quad sp over $\mathbb{C}$

$V \cong W \iff$ same dimension

General Obs | over K

$$[b_1, \dots, b_n] \cong [c_1^2 \cdot b_1, c_2^2 \cdot b_n, \dots c_n^2 \cdot b_n]$$

for $c_1, \dots, c_n \in K \setminus \{0\}$

(Idea scale basis by $c_1, \dots, c_n$)

$$[b_1, \dots b_n] \cong [b_{\sigma(1)}, \dots b_{\sigma(n)}] \quad \sigma \in S_n$$

Second) $k=\mathbb{R}$

Above observation shows that (assuming b_i's $\neq 0$)

$\{b_1, \dots, b_n\} \cong [\underbrace{+1, \dots, +1}_{p}, \underbrace{-1, \dots, -1}_{q}]$

Sylvester's Law of inertia

- p, q above are well def
- if V is a non degen quadratic sp $/\mathbb{R}$ $\exists!\ p, q \geq 0$ s.t. $p+q = n = \dim V$

$V \cong [\underbrace{+1, \dots, +1}_{p}, \underbrace{-1, \dots, -1}_{q}]$

Defn) The signature of V is (p, q)

(sometimes sign includes no of zeros if degen)

Rmk) If V is non degen s.t. (p, q)

$p+q = \dim V$ & $(-1)^q = \operatorname{disc} V$

Obs) if $V = [+1, \dots, +1]$

$\langle x, y\rangle = x \cdot y$ dot prod.

key prop, if $x \neq 0$ $\langle x, x\rangle > 0$ $\to$ positive definite

Similarly $V = [-1, \dots, -1]$ $\Rightarrow$ neg def

if mix $\to$ indefinite

$\Leftrightarrow$ $\exists$ non zero $x \in V$ s.t. $\langle x, x\rangle = 0$

E.g. $V = [+1, -1]$ with basis v_1, v_2

$x = v_1 + v_2$ $\quad \langle x, x\rangle = \langle v_1 + v_2, v_1 + v_2\rangle$

$= \langle v_1, v_1\rangle + 2\langle v_1, v_2\rangle + \langle v_2, v_2\rangle$

$= +1 + 0 - 1 = \boxed{0}$

These are isotropic.

Pf) Say $V = [\underbrace{+1, \dots, +1}_{p}, \underbrace{-1, \dots, -1}_{q}]$

$W = [\underbrace{+1, \dots, +1}_{r}, \underbrace{-1, \dots, -1}_{s}]$

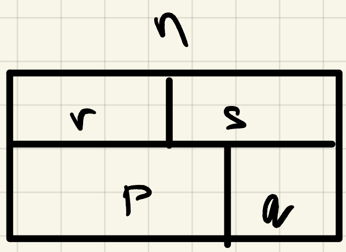
Spse $V \cong W$ but $(p, q) \neq (r, s)$ $\to$ know $p+q = r+s$

$\to$

Let $\varphi : V \to W$ be isom $v_1, \dots, v_n$ basis of V
$w_1, \dots, w_n$ basis of W

$$W = \underbrace{\text{span}(w_1, \dots, w_r)}_{\text{pos def}} \oplus \underbrace{\text{sp}(w_{r+1}, \dots, w_n)}_{\text{neg def}}$$

$$= \underbrace{\text{sp}(\varphi(v_1) \dots \varphi(v_p))}_{\text{pos def}} \oplus \underbrace{\text{sp}(\varphi(v_{p+1}), \dots, \varphi(v_n))}_{\text{neg def}}$$



Claim! $r+q > n$ or $p+s > n$]
$\Rightarrow (p+s) + (r+q) = 2n$
if Claim false $p+s = n$ $r+q = n$
$\Rightarrow (p,q) = (r,s)$

WLOG] $r+q > n$
by lin alg $\text{span}(w_1, \dots, w_r) \cap \text{sp}(\varphi(v_{p+1}), \dots, \varphi(v_n))$
is nonzero
Gs $\Rightarrow\Leftarrow$ as first is pos def second is neg def
& subsp of pos def is pos def &
subsp of neg def is neg def.
$\Rightarrow$ int is both!

Def V = quad sp $/K$
- $x \in V$ is isotropic if $\langle x, x \rangle = 0$
- V is anisotropic if 0 is the only isotropic vec
(e.g $K = \mathbb{R}$, pos def or neg def are anisotropic)
if $K = \mathbb{C}$ then only 1 dim is anisotropic

$\mathbb{H}$ (hyperbolic): 2-d quad sp with $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Prop' If V is non-degen $\Rightarrow$ $V \cong \mathbb{H}^{\oplus n} \oplus V'$ ↳ anisotropic

Pf If V' anisotropic then done $(n=0)$ → char $k \neq 0$

else $0 \neq x \in V$ is isotropic

Claim $\exists y \in V$ so $\langle x, y\rangle = 1$

By non degen $\exists y' \in V$ $\langle x, y'\rangle \neq 0$ $y = \frac{y'}{\langle x, y'\rangle}$

as $\langle x, x\rangle = 0$ & $\langle x, y\rangle = 1 \Rightarrow y \notin sp(x)$ ↳ else ip $= 0$

$W = sp(x, y) = V$ → W is 2-d

Matrix for W is $\begin{pmatrix} 0 & 1 \\ 1 & a \end{pmatrix}$ $a = \langle y, y\rangle$

def $z = y + \beta x$ $\langle x, z\rangle = \langle x, y + \beta x\rangle$

want → z isotropic $= \langle x, y\rangle + \langle x, \beta x\rangle = 1$

$0 = \langle y + \beta x, y + \beta x\rangle = a + 2\beta$ → (char $k \neq 0$)

$\Rightarrow \boxed{\beta = -\frac{a}{2}}$

with basis $W = sp(x, y) = sp(x, z)$

↳ matrix is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\Rightarrow W \cong \mathbb{H}$ → true since W non degen $\Rightarrow W^\perp$ non-degen

So, $V \cong W \oplus W^\perp$

$\cong \mathbb{H} \oplus \boxed{W^\perp}$ → repeat on this by induction

in $\mathbb{R}$ Proof of prop shows

if V non-degen & 2 dimension & has non zero isotropic vector $\Rightarrow V \cong \mathbb{H}$

$\Rightarrow$ over any K $[1, -1] \cong K$ $\cong \mathbb{H}^{\oplus q} \oplus \underbrace{[+1, \dots, +1]}$

$\Rightarrow$ over $\mathbb{R}$ (p, q) sign $p \geq q$ $[+1, \dots, +1, -1, \dots, -1]$