Character Table of S_4

Conjugacy Classes of S_4 $\longrightarrow$ partitions of 4

		rep	# C_i
$1+1+1+1$	C_1	(1)	1
$2+1+1$	C_2	(12)	6
$2+2$	C_3	$(12)(34)$	3
$3+1$	C_4	(123)	8
4	C_5	(1234)	6

	C_1 (#1)	C_2 (#6)	C_3 (#3)	C_4 (#8)	C_5 (#6)	
χ_1	1	1	1	1	1	$\to$ trivial
χ_{sgn} (sign)	1	-1	1	1	-1	
χ_{std}	3	1	-1	0	-1	$\leftarrow$ $\chi_{sgn}\cdot\chi_{std}$
χ_4	3	-1	-1	0	1	$\to$ by pointwise char multiply to get new irred
χ_5	2	0	2	-1	0	$\to$ check $\langle \chi_5, \chi_5 \rangle = 1$

in general: $S_n \curvearrowright \mathbb{C}^n = P$ by permuting basis vect

We know $P = \text{triv} \oplus \text{std}$

triv $\to$ span$(e_1, \dots, e_n)$

std $\to$ span$(e_i - e_j)$ $\to$ always irred

$$\chi_P = \chi_{triv} + \chi_{std} \implies \chi_{std} = \chi_P - 1$$

	C_1	C_2	C_3	C_4	C_5
χ_P	4	2	0	1	0

look at matrix for action: $\begin{pmatrix}1&&&\\&1&&\\&&1&\\&&&1\end{pmatrix}$ $\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$ $\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}$ check

look later

$\chi_P(g) = $ # fixed pts in permutation!

Aside: $\mathcal{C}(G)$ = class functions

def $\varphi \in \mathcal{C}(G)$ is an (irred) char if $\varphi = \chi_V$ for V (irred)]

Prop ① if φ & ψ are char, so is $\varphi+\psi$ as
$\varphi=\chi_V$, $\psi=\chi_W$ $\Rightarrow$ $\varphi+\psi=\chi_{V\oplus W}$

② If φ is a char so is $\overline{\varphi}$ (dual rep)
$=\chi_{V^*}$

③ If $\varphi=\chi_V, \psi=\chi_W \Rightarrow \overline{\varphi}.\psi=\chi_{\mathrm{Hom}(V,W)}$ is a char

② ③ $\Rightarrow$ $\varphi.\psi$ is a char

Rmk if φ is an irred char & ψ is a 1-D char
then $\varphi.\psi$ is irred → in gen have not irred.

Pf $\psi\overline{\psi}=1 \Rightarrow \langle\varphi\psi,\varphi\psi\rangle=\langle\varphi,\varphi\rangle=1$
Schur & converse

Row below not irred

	C_1 1	C_2 6	C_3 3	C_4 8	C_5 6
χ_3^2	9	1	1	0	1

$\chi_3^2=m_1\chi_1+m_2\chi_2+m_3\chi_3+m_4\chi_4+m_5\chi_5$

$m_i=\langle\chi_i,\chi_3^2\rangle$

$m_1=\langle\chi_{triv},\chi_3^2\rangle=\frac{1}{24}(9+6\cdot1+3\cdot1+0+6)=1$

$m_2=\langle\chi_{sgn},\chi_3^2\rangle=\frac{1}{24}(9-6+3+0-6)=0$

$m_3=\langle\chi_{std},\chi_3^2\rangle=\frac{1}{24}(27+6-3+0-6)=1$

$m_4=\langle\chi_4,\chi_3^2\rangle=\frac{1}{24}(27-6-3+0+6)=1$

note $\langle\chi_3^2,\chi_3^2\rangle=\frac{1}{24}(81+6+3+0+6)=4$

on the other hand, Schur gives $m_1^2+m_2^2+m_3^2+m_4^2+m_5^2=$

$\Rightarrow$ $\boxed{m_5=1}$ can solve ($\chi_5=\chi_3^2-\chi_1-\chi_3-\chi_4$)

Permutation reps

let G finite gp $G \curvearrowright X$ finite set

$\mathbb{C}[X] = \mathbb{C}$ vs w/ basis $[x]$ with $x \in X$

An elt of $\mathbb{C}[X]$ is a formal sum $\sum_{x \in X} c_x \cdot [x]$

Can define a rep of G on $\mathbb{C}[X]$

↳ enough to show for basis $g \cdot [x] \longmapsto [gx]$.

e.g. if $G = S_n$, $X = \{1, 2, \dots, n\}$

then $\mathbb{C}[X] = \mathbb{C}^n$ with S_n permuting basis vectors!

Prop $\chi_{\mathbb{C}[X]}(g) = \#$ fixed points of g in X ⇝ general grp

↳ pf from matrix of $g \curvearrowright$ on $\mathbb{C}[X]$

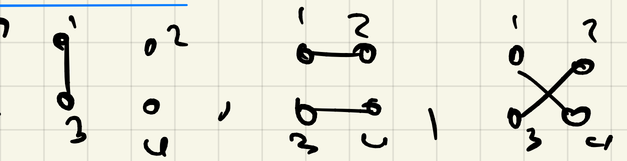
→ this is a permutation matrix (moves bases around)

↳ diag are 0 or 1 w/ 1's fixed basis vect!

Note:
- $\sum_{x \in X} [x]$ is a G-invariant elt of $\mathbb{C}[X]$!
- $\epsilon : \mathbb{C}[X] \longrightarrow \mathbb{C}$, $[x] \longmapsto 1$ is a map of reps when $\mathbb{C}$ is trivial 1-D rep!

augmentation

$$\mathbb{C}[X] = \mathrm{span}\Big(\sum_{x \in X} [x]\Big) \oplus \ker \epsilon$$

e.g. $X = \{$  $\}$ → matchings

$S_4 \curvearrowright X$ by permuting the labels!

Fact in this case $\ker(\epsilon)$ is irred with char χ_5

e.g.] **Regular Rep**

G acts on itself by left multiplication!

$\xrightarrow[\text{perm rep}]{} \mathbb{C}[G]$ = regular rep

$\dim \mathbb{C}[G] = \# G$

$\chi_{reg} = \chi_{\mathbb{C}[G]}$

Prop: $\chi_{reg}(g) = \begin{cases} \# G & \text{if } g = 1 \\ 0 & \text{if } g \neq 1 \end{cases}$

PF fixed pointcount $\to g \curvearrowright G$ by left mult!

$L_1, \dots, L_s$ are irred reps of G

$\chi_1, \dots, \chi_s$ are chars!

$$\mathbb{C}[G] = L_1^{\oplus m_1} \oplus \cdots \oplus L_s^{\oplus m_s}$$

$$m_i = \langle \chi_i, \chi_{\mathbb{C}[G]} \rangle \underset{\text{usually zero}}{=} \chi_i(1) = \dim L_i$$

$$\mathbb{C}[G] = L_1^{\oplus \dim L_1} \oplus \cdots \oplus L_s^{\oplus m_s}$$

$\boxed{\text{Schur}}$

$$\# G = (\dim L_1)^2 + (\dim L_2)^2 + \cdots (\dim L_s)^2$$