Lec 4

Recall We discussed isomorphism

Defn A group homomorphism from groups $G \to H$ is a function that is compatible with the grp law

Lemma Let $\varphi : G \to H$ grp homo. Then
1) $\varphi(1) = 1$
2) $\varphi(x^{-1}) = \varphi(x)^{-1}$

Pf 1) $\underline{\varphi(1)} = \varphi(1 \cdot 1) = \underline{\varphi(1) \cdot \varphi(1)} \Rightarrow 1 = \varphi(1)$

2) $1 = \varphi(1) = \varphi(x \cdot x^{-1}) = \varphi(x) \cdot \varphi(x^{-1}) \Rightarrow \varphi(x^{-1}) = \varphi(x)^{-1}$

e.g. 1) $G = GL_n(\mathbb{R})$ $\quad$ det : $G \to \mathbb{R}^{\times}$ → multiplicative

true since $\det(AB) = \det(A)\det(B)$

2) sgn : $S_n \to \{\pm 1\}$

Characterized → transpositions go to -1

3) G a grp. $g \in G$ consider $\varphi : \mathbb{Z} \to \langle x \rangle$
$n \longmapsto x^n$

φ is an iso iff x has inf order

Get two auxiliary grps. The image & kernel. $\varphi : G \to H$

Obs im φ is a subgrp of H.

1) If $x, y \in \operatorname{im}\varphi \Rightarrow \exists x_1, y_1$ so $\varphi(x_1) = x$, $\varphi(y_1) = y$
$\Rightarrow \varphi(x_1 y_1) = xy \in \operatorname{im}\varphi$

2) $1 = \varphi(1) \in \operatorname{im}\varphi$

3) $x \in \operatorname{im}\varphi \Rightarrow x = \varphi(x_1) \Rightarrow x^{-1} = \varphi(x_1^{-1}) \in \operatorname{im}\varphi$

Lemma ker φ a subgrp

1) let $x, y \in \ker(\varphi) \Rightarrow \varphi(x), \varphi(y) = 1$
$\Rightarrow \varphi(xy) = \varphi(x)\varphi(y) = 1$
$\Rightarrow xy \in \ker(\varphi)$ . . . .

Prop | φ is inj $\iff \ker(\varphi) = \{1\}$

Pf | If φ is inj trivial as $\varphi(1) = 1$ always

$\Leftarrow$ suppose $\ker(\varphi) = \{1\}$ say $\varphi(x) = \varphi(y)$

$\Rightarrow \varphi(x)\varphi(y)^{-1} = 1 \Rightarrow \varphi(x)\varphi(y^{-1}) = 1$

$\Rightarrow \varphi(xy^{-1}) = 1 \Rightarrow xy^{-1} \in \ker\varphi = \{1\}$

$\Rightarrow xy^{-1} = 1 \Rightarrow x = y$

Def | G is a grp & $N \leq G$ subgrp. We say N is normal if $g \in G$ $n \in N$ Then $gng^{-1} \in N$ (N is closed under comp by elt of g)

Prop | For any grp homo. $\ker\varphi$ is a normal subgrp of G

Pf | let $n \in \ker\varphi$ and $g \in G$

$\varphi(gng^{-1}) = \varphi(g)\,\cancel{\varphi(n)}\,\varphi(g^{-1}) = \varphi(g)\varphi(g^{-1}) = 1$

$\Rightarrow gng^{-1} \in \ker\varphi$

E.g. |

- $SL_n(\mathbb{R}) = \ker(\det : GL_n(\mathbb{R}) \to \mathbb{R}^*)$
- $A_n = \ker(\operatorname{sgn} : S_n \to \{\pm 1\})$ (even perms)
- If G is a commutative grp all subgrps are normal

Non eg | $G = GL_2(\mathbb{R})$ $H = \left\{\begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix}\right\}_{r \in \mathbb{R}} < G$

H not normal $h = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $g = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$ghg^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \notin H$

- $G = S_4$ $h = (1\,2\,3)$ $H = \langle h \rangle = \{1, h, h^2\}$ $h^2 = (1\,3\,2)$

$\sigma \in S_4$ $\sigma h \sigma^{-1} = (\sigma(1)\ \sigma(2)\ \sigma(3))$

$\sigma = (1\,4) \Rightarrow \sigma h \sigma^{-1} = (4\ 2\ 3) \notin H$

Def Equiv reln.

Let S a set. A equiv reln on S is a binary reln $\sim$ that is reflexive, symmetric, transitive.

e.g $\cong$ (isom) is an equiv reln

- $S = \mathbb{Z}$ def $n \sim m$ if they have same parity $\iff 2 \mid n-m$

General Say $\varphi : S \to T$ a function. Define equiv on S if x, y $x \sim y \iff \varphi(x) = \varphi(y)$

Def Let S set with $\sim$ equiv.

The equiv class $C_x = \bar{x} = [x] = \{y \in S \mid y \sim x\}$

Note: $x = [x]$

Prop for $x, y \in S$. Then either $[x] = [y]$ or $[x] \cap [y] = \emptyset$

Cor $[x] = [y] \iff x \sim y$

Defn The quotient of S by $\sim$ is the set of all equiv classes

$S/\sim = \{[x] \mid x \in S\}$

E.g if $S = \mathbb{Z}$ $\sim$ is parity $S/\sim = \{[0], [1]\} = \{[57], [60]\}$

Name Quotient map $\pi : S \to S/\sim$

$x \mapsto [x]$