Let G be a finite grp

Goal: # conj classes = # isom classes of irred!

To prove: Let $V_1, \dots, V_s$ irreds
$\chi_1, \dots, \chi_s$ their chars

We'll show χ forms a basis for the space of class func

$\Rightarrow s = \dim \mathcal{C}(G) = \#$ conj classes

Say (V, ρ) is a rep of G
& $\varphi : G \to \mathbb{C}$ is a func define

$$\rho(\varphi) : V \to V$$
$$\rho(\varphi)v \mapsto \sum_{g \in G} \varphi(g)\, g \cdot v$$

Equiv $\rho(\varphi) = \sum_{g \in G} \varphi(g) \rho(g)$

→ linear comb of lin op
$\Rightarrow \rho(\varphi)$ is a lin op.

Lemma) if φ is a class func
$\Rightarrow \rho(\varphi)$ is G-equiv.

Pf)
$$\rho(\varphi) hv = \sum_{g \in G} \varphi(g)\, ghv$$
$$= h \sum_{g \in G} \varphi(g)\, h^{-1}ghv \qquad g' = h^{-1}gh$$
$$= h \sum_{g' \in G} \varphi(hg'h^{-1})\, g'v$$
class
$$= h \sum_{g' \in G} \varphi(g')\, g'v = \boxed{h\, \rho(\varphi) v}$$

Lemma) $tr(\rho(\varphi)) = \langle \overline{\chi_V}, \varphi \rangle$

Pf)
$$tr\, \rho(\varphi) = \sum_{g \in G} \varphi(g)\, tr(\rho(g)) \longrightarrow \text{trace is linear}$$
$$= |G| \cdot \langle \overline{\chi_V}, \varphi \rangle = |G| \cdot \langle \chi_{V^*}, \varphi \rangle$$

Prop) If V is irred $\Rightarrow \rho(\varphi) = c \cdot \mathrm{id}_V$
where $c = \frac{|G|}{\dim V} \langle \overline{\chi_V}, \varphi \rangle$

g lin op on irred is scalar

Pf) By Schur, $\rho(\varphi) = c \cdot \mathrm{id}_V$ for some $c \in \mathbb{C}$
So take tr $\quad \mathrm{tr}(\rho(\varphi)) = c \cdot \dim V$

$$\mathrm{tr}(\rho(\varphi)) = |G| \cdot \langle \overline{\chi_V}, \varphi \rangle$$

$$\Rightarrow c = \frac{|G|}{\dim V} \langle \overline{\chi_V}, \varphi \rangle$$

Thm) $\chi_1, \ldots, \chi_s$ is a basis of $\mathcal{C}(G)$

Pf) Schur orth $\Rightarrow$ lin ind

To show they span, enough to show

if $\langle \chi_i, \varphi \rangle = 0 \ \forall i \Rightarrow \varphi = 0$ (for $\varphi \in \mathcal{C}(G)$)

perp sp dim 0

Say $\varphi \in \mathcal{C}(G)$ satisfies $\langle \chi_i, \varphi \rangle = 0 \ \forall i$ wts $\boxed{\varphi = 0}$

By previous prop $\rho(\varphi) = 0 \ \forall$ irred reps (V, ρ)

$\Rightarrow \rho(\varphi) = 0 \ \forall$ rep (V, ρ) $\rightarrow$ rep breaks down into dir sum, so does operator

key pt: if (V, ρ) & (W, σ) rep, then,

$(V \oplus W, \rho \oplus \sigma)$ $\quad (\rho \oplus \sigma)(\varphi) = \rho(\varphi) \oplus \sigma(\varphi)$

Now apply this to the regular rep (V, ρ)

$V = \mathbb{C}[G]$ An elt of V is a formal sum

$\sum_{g \in G} c(g) \cdot [g]$ $\quad c(g) \in \mathbb{C}$ $\quad [g]$ formal symbol

& $h \sum_{g \in G} c(g) \cdot [g] = \sum c(g) [h \cdot g]$

$\longrightarrow$

Since $\rho(\varphi) = 0 \Rightarrow 0 = \rho(\varphi)[i]$

but $\rho(\varphi)[i] = \sum_{g \in G} \varphi(g)\, g \cdot [i]$ — def of action

(on the other hand) — def

$= \sum_{g \in G} \varphi(g)\, [g]$

$= \varphi(g) = 0 \ \forall g$

$= \boxed{\varphi = 0}$

Tensor Products

Work with vec sp over a field k.

let V, W vec sp

Def A tensor products of V, W is a pair (T, φ) where T is a vec sp & φ is a bilinear map $\varphi : V \times W \to T$ is bilinear & univ

i.e. if (T', φ') is another pair w/ T' a v.s & $\varphi' : V \times W \to T'$ bilin $\exists!$ linear map so that $\psi : T \to T'$

$$\begin{tikzcd} V \times W \arrow[r, "\varphi"] \arrow[dr, "\varphi'"'] & T \arrow[d, "\psi"] \\ & T' \end{tikzcd}$$

Succintly giving a bilinear map is equiv to giving a linear map from T!

Uniqueness

Say (T,φ) and (T',φ') are two tensor prod

Univ prop for T gives lin map $\psi : T \to T'$

— T' gives lin m $\psi' : T' \to T$

So the following commutes!

$$\begin{tikzcd}
V \times W \arrow[r, "\varphi"] \arrow[dr, "\varphi'"] \arrow[ddr, "\varphi"'] & T \arrow[d, "\psi"] \\
& T' \arrow[d, "\psi'"] \\
& T
\end{tikzcd}$$

By the uniqueness of the map in the univ prop $\Rightarrow \psi, \psi'$ $\psi' \circ \psi$ are id!

By uniqueness we can speak of **the** tensor prod. of V and W (assuming) it exists

Notation $T = V \otimes W$ & $v \otimes w$ is the img of $(v,w) \in V \times W$ under φ

$\therefore$ Universality new notation

Given bilinear $\varphi' : V \times W \to T'$

$\exists!$ unique lin map $\psi : V \otimes W \to T'$

s.t. $\varphi'(v,w) = \psi(v \otimes w)$

Special case!

given a bilinear form on V, $\langle , \rangle : V \times V \to k$

$\exists!$ lin map $\psi : V \otimes V \to k$ so $\psi \in (V \otimes V)^*$

$\langle v,w \rangle = \psi(v \otimes w)$

Existence

Consider the vector space $\tilde{T}$ that has a basis indexed by $V \times W$

write $[v,w]$ for the basis elt corr to (v,w)

→ want this to be linear in each elt

Define $R \subset \tilde{T}$ to be subsp spanned by

- $[\alpha v + \beta v', w] - \alpha[v,w] - \beta[v',w]$ $\quad \forall \alpha,\beta \in k$, $v,v' \in V$, $w \in W$
- same in 2nd var

$T = \tilde{T}/R$ is $\otimes$ prod $\left(\varphi : V \times W \to T,\ (v,w) \mapsto \text{img of } [v,w] \right)$

Thm if $v_1, \dots, v_n$ is a basis for V
$w_1, \dots, w_m$ — — — for W

$\Rightarrow \{[v_i, w_j]\}\ \forall i \in \mathbb{N}_n, j \in \mathbb{N}_m$ is a basis for T!

$\Rightarrow \dim V \otimes W = \dim V \dim W$