Last time: P-adic

($p \neq 2$)

Fact: if $a \in \mathbb{Z}$ rel prime to p and $a = \square$ in $\mathbb{F}_p$
(Hensel's lemma) $\Rightarrow a = \square$ in $\mathbb{Z}_p$

Fact $\mathbb{Z}_p^* = \{x \in \mathbb{Z}_p \mid \exists y \in \mathbb{Z}_p \text{ so } xy = 1\}$
units of $\mathbb{Z}_p$. Note $x \in \mathbb{Z}_p \Rightarrow |x|_p \leq 1$

$= \{x \in \mathbb{Q}_p \mid |x| = 1\}$

Fact Every non-zero number of $\mathbb{Q}_p$ can be written uniquely as
$x = p^n \cdot u$ $n \in \mathbb{Z}$, $|u|_p = 1$ → pull out all p's

$\Rightarrow \mathbb{Q}_p^\times \cong \mathbb{Z} \times \mathbb{Z}_p^*$

$p^n u \longleftarrow (n, u)$

$\Rightarrow \mathbb{Q}_p^\times / (\mathbb{Q}_p^\times)^2 \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}_p^\times / (\mathbb{Z}_p^\times)^2$

Hensel reduction mod p — Hensel

$\Rightarrow \mathbb{Z}_p^\times / (\mathbb{Z}_p^\times)^2 \xrightarrow{\sim} \mathbb{F}_p^\times / (\mathbb{F}_p^\times)^2 \cong \mathbb{Z}/2\mathbb{Z}$

So, $\mathbb{Q}_p^\times / (\mathbb{Q}_p^\times)^2 \cong \underbrace{\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}}$ → 4 sq classes

Explicitly Choose $a \in \mathbb{Q}_p$ so $|a|_p = 1$ & $a \neq \square$ in $\mathbb{F}_p^*$
$\Rightarrow$ 4 square classes are

$\boxed{1, a, p, pa}$

Point if V is a non degen quadratic sp / $\mathbb{Q}_p$
$\Rightarrow \exists$ 4 choices for disc(V).

Warning if V, W q sp / $\mathbb{Q}_p$ st $\dim V = \dim W$
& $\operatorname{disc} V = \operatorname{disc} W \not\Rightarrow V \cong W$

Def For $a,b \in \mathbb{Q}_p^\times$ define $\curvearrowright$ Norm on a sp $\{a,b\}$.

$$(a,b) = \begin{cases} +1 & \text{if } 1 = ax^2 + by^2 \text{ for some } x,y \in \mathbb{Q}_p \\ -1 & \text{otherwise.} \end{cases}$$

Properties. (a,b) only dep on $a,b \in \mathbb{Q}_p^\times / (\mathbb{Q}_p^\times)^2$

↳ it is symmetric.

↳ it is bilinear.

$$(a_1 a_2, b) = (a_1, b)(a_2, b).$$

Rmk $V = \mathbb{Q}_p^\times / (\mathbb{Q}_p^\times)^2$

Can think V as vf over $\mathbb{F}_2$ ⤳ it is 2 d.

↳ Hilbert symbol is a symmetric bilinear form on V.

Defn of Hasse Inv

V = non-degen quad sp $/ \mathbb{Q}_p$.

$\cong [a_1, \dots, a_n]$ $a_i \in \mathbb{Q}_p^\times$

The Hasse inv of V is

$$\mathrm{Hasse}(V) = \prod_{1 \le i < j \le n} (a_i, a_j) \in \{\pm 1\}$$

Remarkable that Hasse(V) is well def.

i.e. $[a_1, \dots, a_n] \cong [b_1, \dots, b_n]$

$$\Rightarrow \prod_{1 \le i < j \le n} (a_i, a_j) = \prod_{1 \le i < j \le n} (b_i, b_j)$$

Thm If V, W are quadratic sp $/ \mathbb{Q}_p$ then

$$V \cong W \iff \left.\begin{array}{l} \dim(V) = \dim(W) \\ \mathrm{disc}(V) = \mathrm{disc}(W) \\ \mathrm{Hasse}(V) = \mathrm{Hasse}(W) \end{array}\right]$$

But easy to compute (a_i, a_j) as only 4 choices for each! $\Leftarrow (\mathbb{Q}_p^\times)/(\mathbb{Q}_p^\times)^2$ grp of 4.

E.g. $p=3$, $a=-1$ is a non sq in $\mathbb{Z}_p^\times$.

$\Rightarrow \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2$ is rep by $1, -1, p, -p$

Hilb sym!

$(1,b)=1$ always

$(-1,-1)$?

$\Updownarrow$

$-1 = x^2+y^2$

let $x=1$

$y=\sqrt{-2}$

as $-2 = 1$ in $\mathbb{Z}_3$

	1	-1	p	$-p$
1	1	1	1	1
-1	1	1	-1	
p	1	-1		
$-p$	1			

$(-1,p)$?

$1 = -x^2+py^2$

not solvable in $\mathbb{Z}_3$

as reducing mod p

$\Rightarrow 1 = -x^2$

$\Leftrightarrow$ $\boxed{-1 \text{ is a sq.}}$

Say $x = \frac{a}{p^2}$

$y = \frac{b}{p^2}$

$a, b \in \mathbb{Z}_p$

$-x^2+py^2 = \frac{-a^2+pb^2}{p^4}$

$\Leftarrow$ can't get result but can't cancel

Cor Any non-degen q sp. $V/\mathbb{Q}_p$ of dimension ≥ 5 is isotropic.

(i.e. has isotropic vector $\langle v,v\rangle = 0$). $v \neq 0$.

Rmk $[1, a, a, pa]$ is anisotropic

$a \in \mathbb{Z}_p^\times$ is not sqr..

Now, work over $\mathbb{Q}$:

Say, V (dim n) is a quad sp $/\mathbb{Q}$

Pick orthogonal basis of $V \cong [a_1, \dots, a_n]$ $a_i \in \mathbb{Q}^\times$.

$\mathbb{Q} \subset \mathbb{Q}_p$, $\mathbb{Q} \subset \mathbb{R}$. $\longrightarrow$ this is $V \otimes \mathbb{Q}_p$ (tensor prod)

Can consider, $[a_1, \dots, a_n]_p \rightsquigarrow$ quad sp / $\mathbb{Q}_p$

$[a_1, \dots, a_n]_{\mathbb{R}} \longrightarrow$ quad sp / $\mathbb{R}$. $\rightsquigarrow V \otimes \mathbb{R}$.

Thm (Hasse - Minkowski)

V, W are non degen q sp / $\mathbb{Q}$

TFAE

1) $V \cong W$

2) $V \otimes \mathbb{Q}_p \cong W \otimes \mathbb{Q}_p$ $\forall p$ and $V \otimes \mathbb{R} \cong W \otimes \mathbb{R}$

3) $\dim V = \dim W$

$\operatorname{disc} V = \operatorname{disc} W$

$\operatorname{Hasse}(V \otimes \mathbb{Q}_p) = \operatorname{Hasse}(W \otimes \mathbb{Q}_p)$ $\forall p$

$\operatorname{sig}(V \otimes \mathbb{R}) = \operatorname{sig}(W \otimes \mathbb{R})$

Also, can list all possibilities for $\dim(V)$, $\operatorname{disc}(V)$, $\operatorname{Hass}(V \otimes \mathbb{Q}_p)$'s and $\operatorname{sig}(V \otimes \mathbb{R})$.

Defn A q sp V / F represents $a \in F$ if $\exists\, 0 \neq v \in F$ s.t. $\langle v, v \rangle = a$.

Thm if V non degen quad sp / $\mathbb{Q}$ then TFAE.

1) V reps a

2) $V \otimes \mathbb{Q}_p$ reps a $\forall p$, $V \otimes \mathbb{R}$ reps a.