Let

vec sp of class func!

G be a finite grp & $\mathcal{C}(G) = \{ f : G \to \mathbb{C} \mid f(ghg^{-1}) = f(h) \}$

Let $s = \#$ of conj classes $C_1, \dots, C_s$ pick $g_i \in C_i$ rep

$C_1 = \{1\}$

$\Rightarrow$ class func are uniquely determined by vals on g_i

no constr. on the val

Reason | given $h \in G \Rightarrow h \in C_i \Rightarrow h$ conj to g_i
$\Rightarrow \varphi(h) = \varphi(g_i)$ where φ class fn

Since there are no constr of values $\varphi(g_i)$, so

$$\mathcal{C}(G) \longrightarrow \mathbb{C}^s$$
$$\varphi \longmapsto \begin{pmatrix} \varphi(g_1) \\ \vdots \\ \varphi(g_s) \end{pmatrix}$$

is an isomorphism $\Rightarrow \dim \mathcal{C}(G) = s$

Recall | if $\varphi, \psi \in \mathcal{C}(G)$ we def

$$\langle \varphi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\varphi(g)}\, \psi(g)$$

This is a pos def herm form on $\mathcal{C}(G)$

as class fn

Note ! $\langle \varphi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \overline{\varphi(g)}\, \psi(g) = \frac{1}{|G|} \sum_{i=1}^{s} \sum_{g \in C_i} \overline{\varphi(g)}\, \psi(g)$

val is same on each conj class

$$= \frac{1}{|G|} \sum_{i=1}^{s} \# C_i\, \overline{\varphi(g_i)}\, \psi(g_i)$$

Kinda like a weighted dot product! on $\mathbb{C}^s$

Now let V be a rep of G (fin dim $\mathbb{C}$)

Recall The char χ_V of G is def by $\chi_V(g) = \mathrm{tr}(g \mid V)$

g acting on V as lin map

As trace is inv on conj $\chi_V \in \mathcal{C}(G)$

Last time! $\dim(V^G) = \frac{1}{|G|}\sum_{g\in G}\chi_V(g)$

$= \langle 1, \chi_V\rangle$ — const fnc 1 on G, $1 = \chi_{triv} \to$ triv 1-d rep

Say W a second rep

$\dim \operatorname{Hom}(V,W)^G = \langle \chi_V, \chi_W\rangle$

|| (defn)

$\dim \operatorname{Hom}_G(V,W)$ =

Schur orth! if V, W irred $\Rightarrow \langle \chi_V, \chi_W\rangle = \begin{cases} 1 & \text{if } V\cong W \\ 0 & \text{else.}\end{cases}$

Spse! $L_1, \dots, L_r$ are non isom irred

let $\chi_i = \chi_{L_i}$

Schur orth $\Rightarrow \{\chi_i\}$ are orth $\Rightarrow$ lin ind

$\Rightarrow$ at most $\dim \mathcal{C}(G) = s$ of these!

$\Rightarrow \boxed{r \le \dim \mathcal{C}(G)} = \#$ conj classes

So $\exists$ finitely many isomorphism classes of irreds!

Let $L_1, \dots, L_r$ be reps of isom classes of irred reps

We know $r \le s$ [in fact $r = s$]

Let V be an arb rep

$V \cong V_1 \oplus \dots \oplus V_n$ where each V_i is irred!

each $V_i \cong L_j$ as irred!

Define $m_j = \#\{ i \mid V_i \cong L_j\} \to$ multiplicity of L_j in V

$\Rightarrow V \cong L_1^{\oplus m_1} \oplus L_2^{\oplus m_2} \dots \oplus L_r^{\oplus m_r}$

L_1 dir sum with itself m_1 times

char of dir. sum = sum of char

$\Rightarrow \chi_V = m_1\chi_1 + m_2\chi_2 + \dots + m_r\chi_r$

Since each χ_i lin ind $\Rightarrow m_i$ unique $\Rightarrow$ multiplicity of L_i in V well def

In fact **Schur orth** $\Rightarrow m_i = \langle \chi_i, \chi_V \rangle$
(gives that decomp into irred is unique up to iso of each irred)

The Character table!

	C_1 ($\# C_1$)	C_2 ($\# C_2$)	$\cdots$	C_s ($\# C_s$)
χ_1				
χ_2		$\chi_i(c_j)$		
$\vdots$				
χ_r				

e.g. 1) $\mathbb{Z}/2\mathbb{Z} = \{1, \sigma\}$

	$C_1 = \{1\}$	$C_2 = \{\sigma\}$
χ_1	1	1
χ_2	1	-1

Char of 1-d rep
$\mathbb{Z}/2 \to GL_1(\mathbb{C})$
$1 \to 1$
$\sigma \to -1$

size prove
$\#$ rep $= \#$ conj class $\to$ done!

2) $\mathbb{Z}/3\mathbb{Z} = \{1, \sigma, \sigma^2\}$

$\zeta = e^{2\pi i/3}$

	$C_1 = \{1\}$	$C_2 = \{\sigma\}$	$C_3 = \{\sigma^2\}$
χ_1	1	1	1
χ_2	1	ζ	ζ^2
χ_3	1	ζ^2	ζ

1 d rep
$1 \to 1$
$\sigma \to \zeta$
$\sigma^2 \to \zeta^2$

3) S_3. Conjugacy classes are 3 $\to$ symmetric grp is partition of n

$3 = 3$
$= 2+1$
$1+1+1$

$C_1 = \{1\}$
$C_2 = \{(12), (13), (23)\}$
$C_3 = \{(123), (132)\}$

$\to$

	#1	#3	#2
	c_1	c_2	c_3
χ_1	1	1	1
$sgn = \chi_2$	1	-1	1
χ_3	2	0	-1

Schur orth weighted dot of rows as vec

sgn rep

1-d rep

$S_n \to \{\pm 1\} \subseteq GL_1(\mathbb{C})$

$S_3 \to \mathbb{C}^3$ by perm bas vect

If $V \subseteq \mathbb{C}$ $V = sp(e_i - e_j)$

shown V is irred!

In fact $P = V \oplus \text{triv}$

$\text{triv} = sp(e_1+e_2+e_3)$

So, we have to compute $\chi_p(1)$, $\chi_p((12))$, $\chi_p((123))$

Mat

$(1) \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\text{trace}} \chi_p(1) = 3$

$(12) \longmapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \to \chi_p((12)) = 1$

$(123) \longmapsto \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \to \chi_p((123)) = 0$

$\Rightarrow \chi_p = \chi_V + \chi_{\text{triv}}$

$\Rightarrow \chi_V = \chi_p - 1$

$\langle \chi_3, \chi_3 \rangle = \frac{1}{6}\left(1 \cdot 2^2 + 3 \cdot 0^2 + 2(-1)^2\right) = 1 \Rightarrow$ irred

size of grp

Prop) Let V be a rep of G.
Let m_i be the multiplicity of C_i in V
$$\langle \chi_V, \chi_V \rangle = \sum_{i=1}^{r} m_i^2$$

In part V is irred $\iff \langle \chi_V, \chi_V \rangle = 1$

Pf) follows from schur orth.

$\langle \chi_V, \chi_V \rangle = \sum m_i^2$ as $\chi_V = m_1 \chi_1 + \dots + m_r \chi_r$

χ_j orth & res follows