## Lec 15

Last time: $G \leq M$ discrete subgrps.

assoc grps → $L_G = \{a \in \mathbb{R}^2 \mid t_a \in G\}$

$\overline{G} = \mathrm{im}(G \to O_2)$

- $L_G$ is a discrete subgrp of $\mathbb{R}^2 \cong \{0\}, \mathbb{Z}, \mathbb{Z}^2$
- $\overline{G}$ is finite and $\cong \mathbb{Z}/n\mathbb{Z}$ or $D_n$
- $\overline{G}$ preserves $L_G$. $\bar{g} \in \overline{G}$ and $a \in L_G \Rightarrow \bar{g} \cdot a \in L_G$

## Crystallographic Restriction

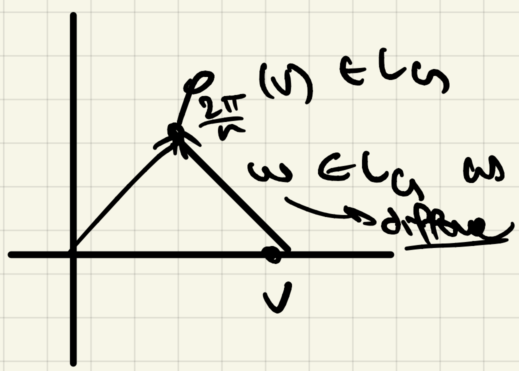
if $L_G$ is a lattice ($\cong \mathbb{Z}^2$)
then $\overline{G}$ is $\leq D_n$ for $n \in \{1,2,3,4,6\}$

Pf| it suffices to show $\overline{G}$ doesn't contain rotation by $\frac{2\pi}{n}$ for $n \geq 7$ or $n=5$

$\hookrightarrow \rho_{\frac{2\pi}{n}}$

can find due to discreteness & except ball around $\vec{0}$

let $v \in L_G$ be a nonzero vector of min length
suppose $\rho_{\frac{2\pi}{n}} \in \overline{G}$



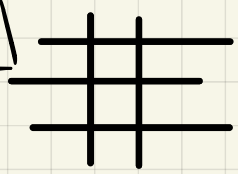

Must have $\frac{2\pi}{n} \geq \frac{2\pi}{6}$ (ie $n \leq 6$)
else $w$ would be shorter than $v$.

Exclude $n=5$: use $\rho_{\frac{4\pi}{n}}$ and add to get $w'$
if $n=5$ $w'$ shorter than $v$

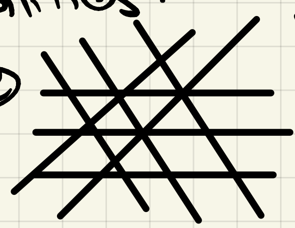
## Classification of Crystallographic grps

up to isom, 17 possibilities.

Ex.:




(2)

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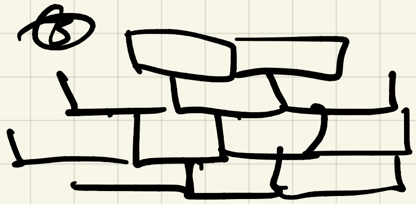


**Def** $G$ - gp. $X$ a set.

An action of $G$ on $X$ is a func
$G \times X \to X$ so that $\forall x \in X$ $1 \cdot x = x$
$(g, x) \mapsto gx$ $g \cdot (h x) = (g \cdot h) \cdot x$

**Def** A $G$-set is a set equipped with an action on $G$

**Main idea** actions repr symmetry.

**e.g.** • $M$ = rigid motions acts on the plane $P$.

• $S_n$ acting on $\mathbb{N}_n$ by permutation (up to order)

• $G$ acts on itself in 3 way

1) $g \cdot h = gh$

2) $g \cdot h = ghg^{-1}$

3) $g \cdot h = hg^{-1}$

• For any field $F$, $GL_n(F) \curvearrowright F^n$

• $O_2 \curvearrowright S^1$

Say that $G$ acts on $X$ and let $x \in X$

• The $G$-orbit of $x$ is
$O_x = \{gx \mid g \in G\} \subseteq X$

• Stabilizer of $x$ is
$G_x = \mathrm{Stab}(x) = \{g \in G \mid gx = x\} \subseteq G$

**Obs** $G_x$ is a subgrp.

**Obs** the orbits of $G$ partition $x$.

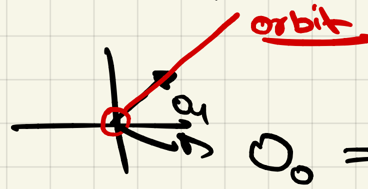
ETS if $O_x \cap O_y \neq \emptyset \Rightarrow O_x = O_y$

Spse $z \in O_x \cap O_y \Rightarrow z = gx = hy$
$\Rightarrow y = h^{-1}gx \Rightarrow y \in O_x \Rightarrow O_y \subseteq O_x$
similarly $x = g^{-1}hx \Rightarrow O_x \subseteq O_y$

e.g. $G = \{x \in \mathbb{R} \mid x > 0\}$ under mult

$X = \mathbb{R}^2$

$G \curvearrowright X$ scalar mult




$O_x$

$G_x$ = trivial $= \{1\}$

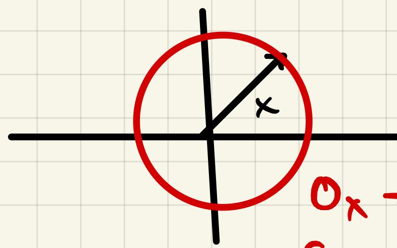
$O_0 = \{0\}$

$G_0 = G$

Ex 2 $G = \{z \in \mathbb{C} \mid |z| = 1\}$

$X = \mathbb{C}$

$G$ acts by scalar mult

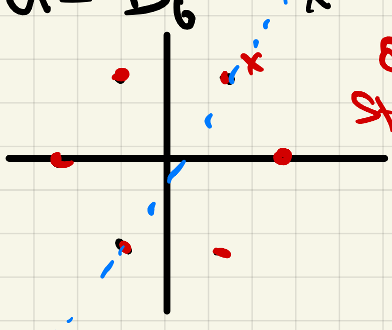



$O_x = \{z \mid |z| = |x|\}$

$G_x = \{1\}$

if $x = 0$ then $G_x = G$

Ex 3 $G = D_6$ $X$ = vertices of hexagon, $\#X = 6$




$O_x = X$

$Stab_x$ = reflection along and 1

Def An action of $G \curvearrowright X$ is transitive if there is only 1 orbit.

Eg. $S_n \curvearrowright \{1, \dots, n\}$ $G_n$ looks like $S_{n-1}$