Rigid Motions

let P – plane. we can identify P with $\mathbb{R}^2$ once we declare an origin

$M \to$ group of rigid motions on P (distance preserving bijection)

fix $O \in P$ and identify $P \cong \mathbb{R}^2$

Some elts of M

1) Translations. $a \in \mathbb{R}^2$, let $t_a : P \to P$
$x \mapsto x + a$

2) Rotations. given θ let $\rho_\theta : P \to P$
rotation about O of θ

3) Reflection. let $r : P \to P$ be reflection abt x-axis (wrt comp)

$\to$ note ρ_θ and r are linear

$$\left.\rho_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad r = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\} \in O(n)$$

$\{\rho_\theta\}$ forms a group SO_2 (orth mat of det 1)

ρ_θ's and r generate O_2

Prop | Every elt of M has the form $t_a \rho_\theta r^i$ $i \in \{0,1\}$
$\hookrightarrow$ see where O goes

Pf define $a = m(0) \to m_1 = t_{-a} m$ $\quad m_1(0) = 0$

$m_1(e_1)$ is on the unit circle (m_1 fixes O and pres distances $\to$ fixes unit circle)

$\Rightarrow \exists \theta$ so $m_1(e_1) = \rho_\theta(e_1)$

$m_2 = \rho_{-\theta}(m_1) \Rightarrow m_2(0) = 0, \quad m_2(e_1) = e_1$

note $\|m_2(e_2) - m_2(e_1)\| = \sqrt{2}$ & $\|m_2(0) - m_2(e_1)\| = 1$

$\Rightarrow m_2(e_2) \in \{\pm e_2\}$

$$m_3 = \begin{cases} m_2 & \text{if } m_2(e_2) = e_2 \\ r m_2 & \end{cases} \Rightarrow \left.\begin{aligned} m_3(0) &= 0 \\ m_3(e_1) &= e_1 \\ m_3(e_2) &= e_2 \end{aligned}\right\} \Rightarrow m_3 = \mathrm{id}$$

How does multiplication work in this grp!

→ $t_a + t_b = t_{(a+b)}$

→ $\rho_\theta t_a(x) = \rho_\theta(x+a) \overset{\text{liner}}{=} \rho_\theta(x) + \rho_\theta(a) = t_{\rho(\theta)a} \circ \rho_\theta(x)$

⟺ $\rho_\theta t_a \rho_\theta^{-1} = t_{\rho_\theta(a)}$ → conjugation

→ $r t_a(x) = r(a+x) = r(x) + r(a) = t_{r(a)} \circ r(x)$

$(r t_a r^{-1} = t_{r(a)})$

→ $\rho_\theta \rho_\varphi = \rho_{\theta+\varphi}$

→ $r^i r^j = r^{i+j}$

→ $r\rho_\theta = \rho_{-\theta} r \implies r \rho_\theta r^{-1} = \rho_{-\theta}$

Subgrps of M

- $T = \{t_a\}$ normal
  $T \cong \mathbb{R}^2$ grp isom
- $R = \{\rho_\theta\}$ grp of rot abt 0
  → not norm

$R = S^1 = SO(2) = \mathbb{R}/2\pi\mathbb{Z}$

So we observe!

1) $\{t_a\}$ is a normal subgroup $T \subset M$

2) ρ_θ are a subgrp but not normal

$t_a \rho_\theta t_{-a}$ → rotation centered at a → a nonzero bed

3) $M/T \cong O_2$ $\quad O_2 \subseteq M$ subgrp fixing origin so

we have natural inclusion map

$$\begin{tikzcd} O_2 \arrow[r, "incl"] \arrow[rr, bend right, "i"'] & M \arrow[r, "quotient"] & M/T \end{tikzcd}$$

→ This is a homomorphism as comp

from $m = t_a \rho_\theta r^i$ every coset in M/T is rep by elt in O_2

$\implies i$ is surj

$\ker(i) = T \cap O_2 = \{1\}$ → trivial $\implies$ inj

RMK $\quad m = t_b \underbrace{\rho_\theta r^i}_{A}$ → $m(x) = Ax + b$

4) $\{t_a \rho_\theta\}$ is a normal subgrp $N \subseteq M$ by correspondence thm
$SO_2 \subseteq O_2$ push back

and $M/N \cong \{\pm 1\} \cong \mathbb{Z}/2\mathbb{Z}$
$r \mapsto -1$

∃ homo from M to $\{\pm 1\}$ → σ
s.t. $\sigma(t_a) = \sigma(\rho_\theta) = 1 \quad \sigma(r) = -1$

⟹ m is orientation pres iff $\sigma(m) = 1$