Rep of S_n

Recall Conj cl. of S_n $\xleftrightarrow{\text{bij}}$ partition of n

elt $\longmapsto$ cycle decomp gives part!

irred rep = # conj cl = # parts of n

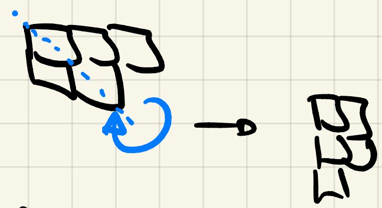
Young Diagram

↳ given a partition of $n = \lambda_1 + \dots + \lambda_r$) convention
$\lambda_1 \geq \lambda_2 \geq \lambda_r \dots$

assoc. the diagram

λ_1 boxes in 1st row
λ_2 boxes in 2nd row
⋮
λ_r ——— rth row

e.g. $5 = 3+2$



↳ reflecting along diagonal axis gives another Young Diag!
↳ The transpose diagram!

Dominance Order!

λ & μ are parts of n

we say μ dominates λ, write $\lambda \trianglelefteq \mu$

if $\mu_1 \geq \lambda_1$, $\mu_1 + \mu_2 \geq \lambda_1 + \lambda_2$, ...

In terms of Young Diagrams $\lambda \trianglelefteq \mu$ means μ is obtained from λ by moving boxes ↑

e.g. $n = 5$

$5 = 5$
$= 4+1$
$= 3+2$
$= 3+1+1$
$= 2+2+1$
$= 2+1+1+1$
$= 1+1+1+1+1$

↑ dominance order!

this is a total ord on S_5

in general its a partial ord on S_n

↳ eg $S_6 \to$ $4+1+1$, $3+3$

(can't comp)

Recall
- $S_n \curvearrowright \mathbb{C}^n$ by basis perm rep $\cong$ triv $\oplus$ std $= \mathbb{C}[S]$
  ↳ $S = \{1, \ldots, n\}$
  ↳ part $(n-1)$ 1
- $n=5$ $\quad X = \{2$ elt ss of $\{1, \ldots, 5\}\}$

  $S_5 \curvearrowright \mathbb{C}[X]$ 10-d rep

  Same as partition of $\{1, \ldots, 5\}$ into set of 2 & 5!

  $\mathbb{C}[X] = $ 5-d irred $\oplus$ triv $\oplus$ std

  ↳ $M_{(3,2)}$

General constr

let λ part of $n = \lambda_1 + \ldots + \lambda_r$

let $X_\lambda = \{$ part of $\{1, \ldots, n\}$ into r sso size $\lambda_1, \ldots, \lambda_r \}$ ↺ S_n acts on this!

let $M_\lambda = \mathbb{C}[X_\lambda]$ this is a perm

↳ e.g. - std perm rep $S_n \curvearrowright \mathbb{C}^n$ is $M_{(n-1),1}$
- earlier we had $M_{(3,2)}$

Rmk
- (n) is max. partn in dom order

  $M_{(n)} = $ triv rep $\quad (X_{(n)} = 1$ pt set$)$
- $X_{(n-1,1)} = \{1, \ldots, n\}$ $\quad M_{(n-1,1)} = \mathbb{C}^n$
- $(1^n) = (1 \ldots 1)$ is min partn in dom ord

  $X_{(1^n)} = \{(a_1 \ldots a_n) \mid \underset{a_i \text{ dist}}{1 \le a_i \le n}\} \cong S_n$

  $\Rightarrow M_{(1^n)} \cong \mathbb{C}[S_n]$ → reg rep

Thm $M_\lambda = (\text{irred}) \oplus (\text{sum of irreds})$

↳ appear in M_μ w/ $\mu \ngeq \lambda$

$\boxed{\text{new}}^\lambda$

↳ call it S_λ

These give the irred of S_n!

Rmk $S_\lambda \otimes \text{sgn} = S_{\lambda^t}$

$\underbrace{}_{\text{mult by sgn}}$ $\quad \hookrightarrow$ transpose by λ

e.g. $\begin{array}{|c|c|c|c|}\hline \ & \ & \ & \ \\ \hline \ \\ \cline{1-1} \ \\ \cline{1-1} \end{array}$ is self transposing

S_λ was the 6-d rep & was self signing

$\hookrightarrow S_\lambda \cong S_\lambda \otimes \text{sgn}$

Young Tableau

A Young Tableau is a Young diagram labeled $1, \dots, n$ (each num app once)

e.g. $\begin{array}{|c|c|c|}\hline 3 & 4 & 1 \\ \hline 5 & 2 \\ \cline{1-2} \end{array}$ $\rightarrow$ is a Young Tableau with shape $(3,2)$

Let T be a Young Tableau of shape λ.

Let S_1 = nums in 1st row of T

S_2 = nums in 2nd row of T

$\vdots$

S_r = ______ T

Then $S_1, \dots, S_r$ partition of n into r subsets of size $\lambda_1, \dots, \lambda_r$

$(S_1, \dots, S_r) \in X_\lambda \Rightarrow$ corr basis vect in M_λ

$\hookrightarrow$ call this e_T

Rmk $e_T = e_{T'}$ if T' is obt by perm within rows of T

e.g. $T = \begin{array}{|c|c|c|}\hline 3 & 4 & 1 \\ \hline 5 & 2 \\ \cline{1-2} \end{array}$ $\qquad T' = \begin{array}{|c|c|c|}\hline 4 & 3 & 1 \\ \hline 2 & 5 \\ \cline{1-2} \end{array}$

$e_T = e_{T'}$ $\qquad C_T = \{1, (35), (24), (35)(24)\}$

Defn $x_T \in M_\lambda$

$$x_T = \sum_{\sigma \in C_T} \text{sgn}\, \sigma \; e_{\sigma T}$$

C_T = "col stab" of T

(leave col each as set)

Thm) $\operatorname{span}(x_T) = S_\lambda \subseteq \underline{M_\lambda}$

as T varies over all tab of sp λ

e.g.) $\begin{array}{|c|c|c|c|c|}\hline & & & & \\ \hline \\ \cline{1-1}\end{array}$ $\lambda = (n-1, 1)$

$M_\lambda = \mathbb{C}^n$

$T = \begin{array}{|c|c|c|}\hline i & \cdots & \\ \hline j \\ \cline{1-1}\end{array}$ $x_T = e_i - e_j$ yoooo

A tableau is called <u>std</u> if the rows & cols in increasing order!

Thm) x_T w/ T std gives a <u>basis</u> of S_λ

e.g. <u>S_5</u>

$\lambda = \begin{array}{|c|c|c|}\hline & & \\ \hline & \\ \cline{1-2}\end{array}$ saw last time S_λ 5 D

$\Rightarrow$ exp 5 std tableau

1. $\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2}\end{array}$

2. $\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2}\end{array}$

3. $\begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2}\end{array}$

4. $\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2}\end{array}$

5. $\begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2}\end{array}$