# Representation.

let $G$ - a grp & $K$ - field.

**Defn** Def of a rep:

1) A gp homo $G \to GL(n,K)$

2) A rep of $G$ is a pair $(V,\rho)$ where $V$ is a $K$-vec sp & $\rho: G \to GL(V)$ grp hom

$GL(V)$ → $\{$bij lin trans $V \to V\}$
→ grp under compos

2) A rep of $G$ is a v.s equipped with a linear action of $G$. I.e. an action $G \times V \to V$ s.t

$\forall g \in G$ the map $V \to V$, $v \mapsto gv$ is linear

To go ② ⇒ ① assume $V$ is fin dim & pick a basis of $V$ get iso from $GL(V) \to GL(n,K)$.

②③ are completely equiv (fin dim not needed)

given $(V,\rho)$ as in ② consider lin act by

$$g \cdot v := (\rho(g))(v)$$

$GL(V)$ $\boxed{\text{plug in}}$

**e.g.** (i) $K = \mathbb{R}$ & think $O(n) \subseteq GL(n,\mathbb{R})$

consider inclusion for ① → called std repr. for any subgrp of $GL(n,\mathbb{R})$

(2) $G = \mathbb{C}^\times$ mult $\underline{K = \mathbb{R}}$ have a 2-d rep of $G$

$G \longrightarrow GL(2,\mathbb{R})$

$a+ib \longmapsto \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$

(3) $G = S_n$ $K$ = any field $G$ has an $n$ dim'l rep ← perm rep

$G \curvearrowright K^n$ by permuting basis vect $g: e_i \to e_{g(i)}$

• $G \longrightarrow GL(n,\mathbb{R})$

$\sigma \longmapsto$ assoc coord perm matrix

(4) $G = S_n$ $\quad k$ any field.

$\text{sgn} : G \to \{\pm 1\} \subset GL_1(k)$ — if $k$ char 2 problem

$\Rightarrow$ Can think of sgn as 1-d rep of $S_n$.

(5) $G$ – any grp $\quad K$ any field. $\quad V$ any vector sp.

Let $g$ act trivially on $V \to g \cdot v := v$

↪ a linear action

$\longrightarrow$ trivial action!

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**Main Goal)**

Given grp $G$ & field $K$ describe what the repr. look like.

↪ their structure!

**Def)** Say $V$ = rep of $G$.

A subrepresentation of $V$ is a subspace $W$ that is stable by $G$.

i.e. $\Rightarrow \forall v \in W \quad g v \in W \quad \forall g \in G$

**Rmk)** If $W \subseteq V$ is a subrep $G$ acts naturally on $V/W$

**Rmk)** Say $V$ fin dim & $W \subseteq V$ subrep.

Let $v_1, \dots, v_m$ be basis of $W$, extend to basis of $V$ $\quad v_1, \dots, v_m, v_{m+1}, \dots, v_n$

If $1 \leq i \leq m$ & $g \in G \Rightarrow g v_i \in W = \text{sp}(v_1, \dots, v_m)$

↪ $\Rightarrow$ matrix for $g$ in this basis has the form

$$g \mapsto \begin{array}{c} \overset{m}{\phantom{x}} \\ \left(\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right) \end{array}$$

$\to$ matrix for quotient

$\to$ matrix for subrep

think abt this!

**Defn** A representation $V$ is irreducable if the only subreps are $0$ & $V$ (and $V \neq 0$)

**Point** These are building blocks of reps

**Prmly** In matrix form $\rho: G \to GL(n,\mathbb{R})$ is reducable if $\exists A \in GL_n(k)$ if $A \rho A^{-1}$ is non tri block upp $\Delta$

to give prev stuff $\leftarrow$ change of basis

**Break prob into 2**

① Describe all irreducable rep

② Understand how gen rep is built from irreducable repr.

e.g. ① $G = GL_n(\mathbb{C})$ let $k = \mathbb{C}$

let $V = \mathbb{C}^n$ with std rep of $G$ ($gv$ = matrix mult).

is this irreducable.

This is irreducable

let $W \subseteq V$ is a nonzero irreducable subrep. Show $W = V$

$\exists w \in W$ non zero

$GL(n,\mathbb{C})$ acts trasitively on nonzero vect

$\exists g_1, \ldots, g_n \in GL(n,\mathbb{C})$ s.t. $g_i w = e_i \in$

$\Rightarrow$ as $W$ subrep $g_i w \in W \Rightarrow W = V$ as sub vec sp

② $G = S_n$ $k = \mathbb{C}$

$V = \mathbb{C}^n$ with perm repr.

This is not irreducable $\to$ (Pt. $n \neq 1$) $n \geq 2$ $\to$ 1 dim always irredu

let $x = e_1 + \cdots + e_n = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ is inv under $S_n$

$\Rightarrow gx = x \ \forall g \in S_n \to$ by lin $\alpha x$ inv $\forall \alpha \in \mathbb{C}$.

$\Rightarrow sp(x)$ is a subrep (non trivial).

Consider the linear map $\varepsilon : V \to \mathbb{C}$, $e \mapsto 1$ (kinda like dot prod)

for $g \in S_n$ obs $\varepsilon(gv) = \varepsilon(v)$

$\Rightarrow \ker(\varepsilon)$ is a subrep.

Explicitly, $\ker(\varepsilon)$ is spanned by $e_i - e_j$ $1 \leq i, j \leq n$

Prop These $\ker \varepsilon$ representations are irred ($n \geq 2$)

Let $W \subseteq \ker(\varepsilon)$ nonzero subrep.

Let $w \in W$ non-zero vector.

write $w$ out in basis $= \sum a_i e_i$ $a_i \in \mathbb{C}$

$0 = \varepsilon w = \sum a_i \Rightarrow$ not all $a_i$s equal as $w \neq 0$

pick 2 indices so $a_i \neq a_j$

consider $y = \underset{\underset{S_n}{\in}}{(i\,j)} \cdot x \in W$ by subrep

$\Rightarrow x - y \in W$ by vec sp

$$x - y = (a_i - a_j) e_i + (a_j - a_i) e_j \in W$$

$\Rightarrow$ div by $a_i - a_j$ $\Rightarrow$ $e_i - e_j \in W$

act by $S_n$ to get any $k \neq l$ to get

$g(e_i - e_j) = e_k - e_l \in W \Rightarrow W = \ker(\varepsilon)$!