lec 20

fixed prime

G is a finite grp of order mp^e, $p \nmid m$.

$\leadsto$ a p-sylow subgrp of G is a subgrp of order p^e.

1st Sylow A p-sylow subgrps exists.

Second Sylow If $H \leq G$ is a p-sylow and $K \subset G$ is any subgrp. Then $\exists g \in G$ s.t $K \cap gHg^{-1}$ is a p-sylow of K

Pf • G acts on G/H. The stabilizer in G of gH is gHg^{-1}.

• K also acts on G/H. The stab in K of gH is $K \cap gHg^{-1}$.

• $\#G/H = \frac{mp^e}{p^e} = m$ in particular $p \nmid \#G/H$.

follows (since set is disj union of orbits) $\exists gH \in G/H$ s.t the cardinality of the K-orbit of gH is prime to p.

K-orbit of $gH \cong K/(K \cap gHg^{-1})$ orbit stab w counting

$\Rightarrow p \nmid \# K/(K \cap gHg^{-1})$, $\# K \cap gHg^{-1} \mid gHg^{-1} = p^e$

$\Longrightarrow K \cap gHg^{-1}$ is a p-Sylow of K.

Third Sylow Thm Let s be the number of p-sylow subgps of G.

(a) $s \mid m$ (b) $s \equiv 1 \pmod p$

Pf let $\{H_1, \ldots, H_s\}$ be the p sylow subgrps.

let $H = H_1$

G acts on $\mathcal{X}$ by conj (does so transitively as p-sylows are conjugate by above)

What is stabilizer of H?

It is $\{g \in G \mid gHg^{-1} = H\} =: N_G(H)$ normalizer in G of H (called grp)

rmk $H \subseteq N_G(H)$ also H is normal subgrp of $N_G(H)$

by the counting formula (by trans) $s = \frac{\# G}{\# N_G(H)}$

Since $H \leq N_G(H) \Rightarrow p^e \mid \# N_G(H)$

$\Rightarrow s \mid m$

b) Think of $H \curvearrowright S$ by conjugation (not ness trans)

any H-orbit has size p^k for $0 \leq k \leq e$

$\Rightarrow s \equiv \#\, H$-fixed pts on $\{H_1, \ldots, H_s\}$ (mod p) ← since looking at mod p then don't consider

H is a fixed pt ! let's show no more

Claim H does not fix any other H

Say H fixes H_i for some $2 \leq i \leq s$

$\Rightarrow H \subset \text{Stab of } H_i = N_G(H_i)$

Since H_i is a normal p-Sylow in $N_G(H_i)$ it is a unique sylow subgrp (conjugacy)

$\Rightarrow H = H_i$, oops. $\square$

Thm The only grp of order 15 (up to iso) is $\mathbb{Z}/15\mathbb{Z}$.

Pf Let G be a grp of order 15.

Let $s_3 \to$ number of 3 sylows

$s_5 \to$ number of 5 sylows.

By 3rd sylow $s_3 \mid 5$ and $s_3 \equiv 1 \bmod 3 \Rightarrow s_3 = 1$

$s_5 \mid 3$ & $s_5 \equiv 1 \bmod 5 \Rightarrow s_5 = 1$

$\therefore$ let H be unique 3-sylow, K be unique 5-sylow } both normal because unique sylows.

WTS G = internal direct prod of H, K.

- $\# H \cap K = 1$ as $H \cap K$ divides $\# H$, $\# K$ $\Rightarrow \# H \cap K = 1$
- H and K commute. $k \in K$, $h \in H$, $khk^{-1} \in H \to$ normal

$\Rightarrow khk^{-1}h^{-1} \in H$

but also $hk^{-1}h^{-1} \in K \Rightarrow \in K$

$\Rightarrow khk^{-1}h^{-1} \in H \cap K = \{1\}$

$\Rightarrow kh = hk$

- $G = HK$ as they commute & $\# HK$ multiple of 3×5

So, $S \cong H \times K \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z} \underset{\text{Chinese Remainder}}{\cong} \mathbb{Z}/15\mathbb{Z}$