Setup: $T: V \to W$ linear and let $\mathcal{V} = (v_1, \dots, v_n)$ $\mathcal{W} = (w_1, \dots, w_m)$

Choice of basis gives $i_{\mathcal{V}} : F^n \xrightarrow{\sim} V$, $i_{\mathcal{W}} : F^m \xrightarrow{\sim} W$

$$\begin{tikzcd}
V \arrow[r, "T"] & W \arrow[d, "i_{\mathcal{W}}^{-1}"] \\
F^n \arrow[u, "i_{\mathcal{V}}"] \arrow[r] & F^m
\end{tikzcd}$$

By last time,
$T' = T_A$ for $A \in \mathrm{Mat}(m \times n)$

$$T' = i_{\mathcal{W}}^{-1} \circ T \circ i_{\mathcal{V}}$$

Defn) Above A is the matrix for T wrt $\mathcal{V}$ and $\mathcal{W}$

Direct Sums

Given V, W /$\mathbb{F}$ vec sp. Their external direct sum

$$V \oplus W = \{(v,w) \mid v \in V, w \in W\} \text{ with obvious operations}$$

$$(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2), \quad c(v,w) = (cv, cw)$$

Given V /$\mathbb{F}$ vector sp. let $\mathcal{U}, W \le V$ subsp. we say V is the internal direct sum of $\mathcal{U}, W$ if

① $V = \mathcal{U} + W = \{u + w \mid u \in \mathcal{U}, w \in W\}$

② $\mathcal{U} \cap W = \{0\}$

These are equiv notions.

Start with $\mathcal{U}, W$ → $V = \mathcal{U} \oplus W$ is the internal direct of
$\overline{\mathcal{U}} = \mathcal{U} \times \{0\}$ $\overline{W} = \{0\} \times W$

Thm) let $\mathcal{U}, W$ fin dim vs $\mathbb{F}$. let $\underline{u} = (u_1, \dots, u_n)$
$\underline{w} = (w_1, \dots, w_m)$ bases

Claim $\mathcal{V} = \{(u_i, 0) \mid i \in \mathbb{N}_n\} \cup \{(0, w_i) \mid i \in \mathbb{N}_m\}$ is a basis

Cor)
$$\begin{array}{l}
\dim V = \dim \mathcal{U} + \dim W = n + m \\
\shortparallel \\
\dim(\mathcal{U} \oplus W)
\end{array}$$

Let $T : \mathcal{U} \to W$ be a linear transformation.

Def $\ker T = \{u \in \mathcal{U} \mid Tu = 0\}$ $\quad$ $\operatorname{im} T = \{Tu \mid u \in \mathcal{U}\}$

Claim $\ker(T) \to$ subspace of $\mathcal{U}$ $\quad$ $\operatorname{im} T \to$ subsp of V

Fact T is injective $\Leftrightarrow$ $\ker T = \{0\}$

Def) The nullity of T $= \dim \ker T$

The rank of T $= \dim \operatorname{im} T$

Thm) Rank Nullity

Suppose $\mathcal{U}$ fin dim. Then $\dim \mathcal{U} = \operatorname{rank}(T) + \operatorname{nullity}(T)$

Def) Let $\mathcal{U}$ vector sp. Let $\mathcal{U}_1$ is a subsp.

A complementary space to $\mathcal{U}_1$ is a subsp $\mathcal{U}_2$ so

$\mathcal{U} = \mathcal{U}_1 \oplus \mathcal{U}_2$

Lemma) Given any $\mathcal{U}_1 \subseteq \mathcal{U}$ complementary space exists.

Pf) Let $u_1, \dots u_n$ basis for $\mathcal{U}_1$. Can extend to basis of $\mathcal{U}$ called $u_1, \dots, u_n, u_{n+1}, \dots, u_{n+m}$

Claim: $\mathcal{U}_2 = \operatorname{span}(u_{n+1}, \dots, u_{n+m})$ is the comp sp.

Clearly, $\mathcal{U} = \mathcal{U}_1 + \mathcal{U}_2$ and $\mathcal{U}_1 \cap \mathcal{U}_2 = \{0\}$ by lin ind

$\Rightarrow \mathcal{U} = \mathcal{U}_1 \oplus \mathcal{U}_2$

Warning: these spaces, generally, aren't unique

Eg) $V = \mathbb{R}^2$ $\quad \mathcal{U}_1 = \operatorname{span}(e_1)$, $\mathcal{U}_2 = \operatorname{sp}\left(\begin{pmatrix} 0 \\ 3 \end{pmatrix}\right)$

or $\operatorname{sp}\left(\begin{pmatrix} 0 \\ 57 \end{pmatrix}\right)$

Pf of rn) $T : \mathcal{U} \to W$. $\mathcal{U}_1 = \ker(T)$, ~~pick~~ $\mathcal{U}_2$ comp sp

$\Rightarrow \mathcal{U} = \mathcal{U}_1 \oplus \mathcal{U}_2 \Rightarrow \dim \mathcal{U} = \dim \mathcal{U}_1 + \dim \mathcal{U}_2$

Show $\mathcal{U}_2 \cong \operatorname{im} T$ i.e $T|_{\mathcal{U}_2} : \mathcal{U}_2 \to \operatorname{im} T$ is an iso

$\Rightarrow \dim \mathcal{U}_2 = \dim \operatorname{im} T = \operatorname{rank}(T)$ and we're done!

<u>Show claim</u> $T|_{U_2} : U_2 \to \operatorname{im} T$

$u \in \ker T|_{U_2} \implies u \in U_2$ and $T|_{U_2} u = Tu = 0$

$\implies u \in U_2 \cap U_1 = \{0\} \implies u = 0$

so $T|_{U_2} : U_2 \to \operatorname{im} T$ is injective (trivial ker)

<u>Surj</u> let $w \in \operatorname{im} T$, $w = T(u)$ for $u \in U$

$u = u_1 + u_2$ for $u_1 \in U_1$, $u_2 \in U_2$.

$w = T(u) = T(u_1 + u_2) = T(u_2)$

$\implies T|_{U_2} : U_2 \to \operatorname{im} T$ surjective.

$\implies U_2 \cong \operatorname{im} T$

Linear Operation

Defn let V be a vs. A linear operator on V is a lin tran $T : V \to V$

E.g. $V = \mathbb{R}[x]$ $\quad T = x^3 \frac{\partial^2}{\partial x^2} + x \frac{\partial}{\partial x}$

$\ker T$ is the collection of poly soln this ODE $T = 0$

Defn A eigenvector for T with eigenvalue $\lambda \in F$.

Is a nonzero $v \in V$ so $Tv = \lambda v$

For some $\lambda \in F$ fixed. The λ-eigensp $= \{v \in V \mid Tv = \lambda v\}$

$= V_\lambda$

Check subsp

$= \ker(T - \lambda \cdot \mathrm{id}_V)$

A λ-eigen vect is $v \in V_\lambda \setminus \{0\}$

λ is an eigenval if $V_\lambda \neq \{0\}$

λ is e-val $\iff \ker(T - \lambda \cdot \mathrm{id}_V) \neq 0 \iff T - \lambda \cdot \mathrm{id}_V$ not invertible

inj $\iff$ surj since $T : V \to V$

$\implies$ eg rank-nullity