last time

classified finite subgrps of M $(D_n, \mathbb{Z}/n)$

can we classify all subgrps?

we note $\vec{T} < M$, $T \cong \mathbb{R}^2$ so classifying subgrps of M at least as hard as $\mathbb{R}^2$ at least as hard as $\mathbb{R}$

Subgrps of $\mathbb{R}$?

- $\mathbb{Q}$
- $\mathbb{Q}[\sqrt{2}] \cong \mathbb{Q}$ $\longrightarrow$ repeating this to inf gives $\mathbb{R}$ is inf dim vs / $\mathbb{Q}$
- $\mathbb{Z}$, $\mathbb{Z}[\frac{1}{2}]$

$\hookrightarrow$ very hard to do... too many.

Thus its super hard to classify all subgrps of $\mathbb{R}^2 \longrightarrow M$

Reals are hard due to density. We want disc subgrps.

Def| A subgrp L of $\mathbb{R}$ is discrete if $\exists \varepsilon > 0$ so $\forall x \in L \setminus \{0\}$, $|x| > \varepsilon$

Prop| If L is disc subgrp of $\mathbb{R}$. $L = \{0\}$ or $L = \mathbb{Z} \cdot a$, $a \in \mathbb{R}$

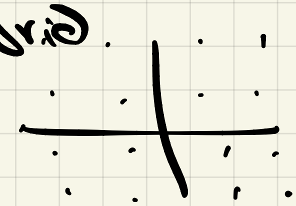
Pf| Pick smallest pos elt of L (if $L \neq 0$) and subtract down

Def| A subgrp L of $\mathbb{R}^2$ is discrete if $\exists \varepsilon > 0$ so $\|v\| > \varepsilon$ $\forall x \in L$

Prop| Let L be disc subgrp of $\mathbb{R}^2$. Then one of the following,

1) $L = 0$
2) $L = \mathbb{Z} \cdot v$ for nonzero $v \in \mathbb{R}^2$
3) $L = \mathbb{Z}v + \mathbb{Z}w$ for $v, w \in \mathbb{R}^2$ lin ind $\longrightarrow$ L is a lattice

$\hookrightarrow$ grid.



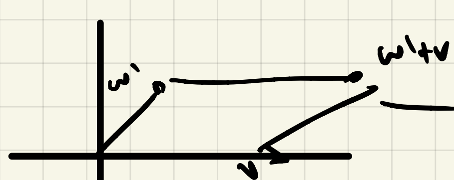
Pf| If $L \subseteq$ line in $\mathbb{R}^2$, then it's essentially a subgrp of $\mathbb{R}$ $\Rightarrow$ (a), (b)

Now suppose $\exists\, v', w' \in L$ that are lin ind.

Consider $L \cap \mathbb{R} \cdot v'$ (disc subgrp of $\mathbb{R} \cdot v'$) $\Rightarrow$ $L \cap \mathbb{R} \cdot v' = \mathbb{Z} \cdot v$

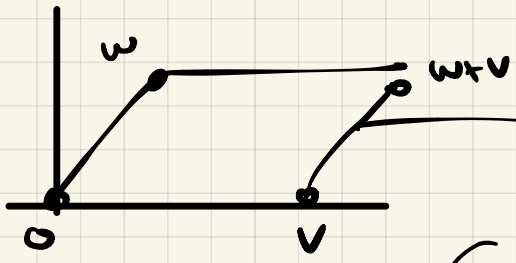
$\hookrightarrow$ for some $v \in \mathbb{R} \cdot v'$

to pick w consider



$\longrightarrow$ only fin many of L in here due to compact-

Let w be the vector in the parallelogram not on $\mathbb{R}\cdot v$ but minimal dist to the line $\mathbb{R}\cdot v$



$\to$ no elt of L in the interior
↳ points in the interior
↳ closer to $\mathbb{R}\cdot v$ $\to$ violates minimality of w
↳ only pts on bdry are $0, w, v, w+v$ similarly.
↳ a bit more careful was needed.

Claim: $L = \mathbb{Z}v + \mathbb{Z}w$. $\supseteq$ immediate.
$x \in L$. Use v, w basis of $\mathbb{R}^2$. $x = \alpha v + \beta w$.
add or subtract disc v or w to make $\alpha, \beta \in [0,1]$
$\Rightarrow x'$ in parallelogram
$\Rightarrow x = 0, v, w, v+w$

Def A subgrp G of M is discrete if
$\exists \varepsilon > 0$ so
a) If $a \in \mathbb{R}^2$ is non-zero and $t_a \in G \Rightarrow \|a\| > \varepsilon$
b) If $\rho \in G$ that is rotation by $\theta \in (-\pi, \pi]$ about some pt. Then $|\theta| > \varepsilon$

Fix a discrete subgrp of G. We assoc to G two assoc grps.
1) $L_G = \{a \in \mathbb{R}^2 \mid t_a \in G\} \to$ disc subgrp of $\mathbb{R}^2$
$\simeq G \cap T$
↳ $L_G \cong 0$ or $\mathbb{Z}$ or $\mathbb{Z}^2$
↳ in this case called wallpaper grp

2) Recall have a homo from $M \to O_2 \cong M/T$
define the point grp of G, $\overline{G}$, to be img of G in O_2

e.g. Let G be the symmetry grp of $\mathbb{Z}e_1 + \mathbb{Z}e_2$
$L_G = \mathbb{Z}e_1 + \mathbb{Z}e_2 \leq \mathbb{R}^2$, $\overline{G} = D_4$,
in general L_G normal subgrp of G. $G/T\cap G \cong \overline{G}$ $\quad T\cap G \cong L_G$
↳ first isom thm

Prop $\bar{G}$ is a disc subgrp of O_2.

Pf Suppose $\bar{G}$ has a rotation by small θ

any inver img of this in G has the form

$t_a \rho_\theta$ for some a. This is rotation by θ around

some pt (can't be arb small as G not disc)

Cor $\bar{G}$ finite subgrp of O_2

Pf Discrete subset of O_2 is finite

Cor $\bar{G} \cong \mathbb{Z}/n$ or D_n $\longrightarrow$ disc classification of finite subgrp of O_2 earlier

Prop If $a \in L_G$. $\bar{g} \in \bar{G} < O_2$

$\bar{g} \cdot a \in L_G$

Pf Say $\bar{g} = \rho_\theta$. Let $g \in G$ be inv img of quotient map.

$\Rightarrow g = t_b \rho_\theta$ for some $\mathbb{R}^2$

$a \in L_G \Rightarrow t_a \in G$

$G \ni g t_a g^{-1} = t_b \rho_\theta t_a \rho_\theta^{-1} t_b^{-1} = t_b t_{\rho_\theta(a)} t_b^{-1} = t_{\rho_\theta(a)}$

$\Rightarrow$ as $t_{\bar{g}(a)} \in G \Rightarrow \bar{g}(a) \in L_G$ $\qquad = t_{\bar{g}(a)}$

e.g. Spse $L_G = \mathbb{Z}e_1 + \mathbb{Z}e_2 \subseteq \mathbb{R}^2$

$\to$ says $\bar{G}$ $\longrightarrow$ fin subgrp of O^2 that preserves L_G.

$\Rightarrow \bar{G} \leq D_4$