Recall • bilinear form $\langle , \rangle : V \times V \to F$ ⟶ V sp / F

• Choice of basis of V ⟶ assoc matrix A ⟶ changes to $x^T A x$!

• $\operatorname{disc}(\langle , \rangle) = \det(A)(F^\times)^2 \in F^\times/(F^\times)^2$

⟶ discriminant ⟶ can distinguish bilinear form by this inv.

Defn A quadratic space is a pair $(V, \langle , \rangle)$ where V is a vec sp. & $\langle , \rangle : V \times V \to F$ is a skew-symmetric bilinear form! ⟶ A is symmetric

Def A isometry betw quadr spaces $(V, \langle , \rangle)$, $(V', \langle , \rangle')$ is a isomorphism $\varphi : V \to V'$ so that $\forall v, w \in V$

$\langle v, w \rangle = \langle \varphi(v), \varphi(w) \rangle'$

Def We say 2 elts $v, w \in V$ are orthogonal if $\langle v, w \rangle = 0$

for any $W \leq V$ vector subspace def orth comp as

$W^\perp = \{ v \in V \mid \langle v, w \rangle = 0 \ \forall w \in W \}$

⟶ check $W^\perp$ is vec subsp!

Def Let $(V, \langle , \rangle)$ & $(V', \langle , \rangle')$ quad sp.

Define orthogonal direct sum $V \oplus V'$ with $(\langle , \rangle \oplus \langle , \rangle')$ ⟶ matrix is:

$$\begin{bmatrix} A & 0 \\ 0 & A' \end{bmatrix}$$

def as

$V \oplus V' \times V \oplus V' \longrightarrow F$

$((v, v'), (w, w')) \longmapsto \langle v, w \rangle + \langle v', w' \rangle'$

Note that: $(v, w') \longmapsto \langle v, 0 \rangle + \langle 0, w' \rangle' = 0$

$\Longrightarrow V \times \{0\} \subseteq (\{0\} \times W)^\perp$

Def The null sp of $(V, \langle , \rangle) = V^\perp$

We say V non-degen if $V^\perp = \{0\}$

Char $F \neq 2$
$\dim V < \infty$

Lemma) $(V, \langle , \rangle)$ is the orthogonal direct sum
- its null space
- A non degenerate sp.

Pf) Let U be any vector subsp of V s.t. $V = V^{\perp} \oplus U$ as a v.sp
Check orthogonal direct sum (as $U \subseteq V$) $\Rightarrow V^{\perp}$ is orth to U — basis can extend basis
→ form on U is non degen
Compute $U^{\perp}$ (inside U) $= \{u \in U \mid \langle u, u' \rangle = 0 \ \forall u' \in U\}$

$\oplus$ $\ = \{u \in U \mid \langle u, u' + v' \rangle = 0 \ \forall u' \in U, v' \in V\}$ — $u'+v'$ in V — dir sum

$= \{u \in U \mid \langle u, v \rangle = 0 \ \forall v \in V\}$

$= U \cap V^{\perp} = \{0\}$ as $U \oplus V^{\perp} = V$ as v sp

e.g. degen sp $\left\langle \begin{pmatrix} x_1 \\ \vdots \\ x_4 \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_4 \end{pmatrix} \right\rangle = x_1 y_1 + x_2 y_2$

Prop) if $\langle , \rangle$ is non degen on $W \subseteq V$ subsp Then $V = W \oplus W^{\perp}$

Pf) W non degen $\Rightarrow W \cap W^{\perp} = \{0\}$ as
WTS $W \oplus W^{\perp} = V$ → we have sum is orthogonal by trivial int
→ show $W + W^{\perp}$ → spanning!

Lemma) $W \to \hom(W, F)$
$v \mapsto (w \mapsto \langle v, w \rangle)$ is an isomorphism

Pf) non degen $\Rightarrow$ map above is injective between same dim
$\Rightarrow$ isomorphism!

Let $v \in V$ consider $W \to F$ obtained by $w \mapsto \langle w, v \rangle$
is linear func on W!
By lemma $\exists! w' \in W$ so $\langle w, w' \rangle = \langle w, v \rangle \ \forall w \in W$!
$\Rightarrow \langle w, w' - v \rangle = 0 \ \forall w \in W \Rightarrow w' - v \in W^{\perp}$
$\Rightarrow v = w' + w''$ / $w'' \in W^{\perp}$ $\Rightarrow \boxed{W + W^{\perp} = V}$!

Prop) if $\langle , \rangle$ is not identically 0 $\Rightarrow \exists v \in V$ s.t. $\langle v, v \rangle \neq 0$

Call this "v is anisotropic"

Pf) $\exists x, y \in V$ so $\langle x, y \rangle = \langle y, x \rangle \neq 0$

$\boxed{\text{Note}}$ $\langle x+y, x+y \rangle = \langle x, x \rangle + \underbrace{2\langle x, y \rangle}_{\text{not } 0 \text{ as char } F \neq 2} + \langle y, y \rangle$

$\hookrightarrow$ thus, at least one of $\langle x+y, x+y \rangle, \langle x, x \rangle, \langle y, y \rangle$ non zero!

Def) An orthogonal basis of a quadratic space is a basis $(e_1, \dots, e_n)$ so $\langle e_i, e_j \rangle = 0$ if $i \neq j$

$\hookrightarrow$ matrix is diagonal!

Thm) Every quadratic sp (not over F of char 2) has orth basis

Pf) write $V = V^\perp \oplus W$ for W non degen!

W non deg $\Rightarrow$ W not $\equiv$ 0

$\Rightarrow \exists$ anisotropic $v_1 \in W$

$\Rightarrow W = Fv_1 \oplus \{Fv_1\}^\perp$ — non degen as anisotropic

$\Rightarrow \exists v_2 \in (Fv_1)^\perp$ anisotropic

$\hookrightarrow$ get orthog basis of W!

$\hookrightarrow$ concat with any basis of $V^\perp$ to get it!

Cor) If A is a symmetric matrix — assoc with quadratic form!

$\exists B$ s.t. $B^T A B$ is diagonal

moreover, diag entries are exactly $\langle e_i, e_i \rangle$ of orth basis elt!

e.g.) back to $F = \mathbb{F}_p$ — matrix $\to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \xrightarrow{\text{disc}} [-1] \in \mathbb{F}_p^* / (\mathbb{F}_p^*)^2$

• $\langle , \rangle : \mathbb{F}_p^2 \times \mathbb{F}_p^2 \to \mathbb{F}_p$

$\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right) = x_1 y_1 + x_2 y_2$