Gen rmk on reps

G - any grp and k - any field.

Def If V, W are 2 reps of G then a map of reps is a k-linear map $f: V \to W$ that is also G-linear. If bijective its an iso of reps

↳ a.k.a G-equivariant ......

i.e. $f(gv) = g f(v) \quad \forall v \in V$

Notation 1) $\mathrm{Hom}_G(V,W)$ is the set of all maps of reps.

↳ this is a vector space / K

f_1 & f_2 maps of reps $\Rightarrow$ $f_1 + f_2$ is too.

So is scalar mults.

2) $\mathrm{End}_G(V) = \mathrm{Hom}_G(V,V)$

↳ endomorphism.

Fact if $f: V \to W$ is a map of reps.

$\Rightarrow$ $\ker f$ is a subrep of V

↳ if $f(v) = 0 \Rightarrow f(gv) = g f(v) = g0 = 0$

$\Rightarrow gv \in \ker f$

$\Rightarrow$ $\mathrm{Im}\, f$ is a subrep of W similarly

If V, W are two reps of G $\Rightarrow$ $V \oplus W$ carries a natural representation. let $(v,w) \in V \oplus W$ let $g(v,w) = (gv, gw)$

if we pick basis $v_1, \dots, v_n$ of V & $\rho: G \to GL_n(K)$ now

basis $w_1, \dots, w_m$ of W & $\sigma: G \to GL_m(K)$ now

Then $v_1, \dots, v_n, w_1, \dots, w_m$ basis of $V \oplus W$ ↳ w/ $\rho \oplus \sigma$

$\Rightarrow$ Mat for g is $\begin{pmatrix} \rho(g) & 0 \\ 0 & \sigma(g) \end{pmatrix}$

$\longrightarrow$

Reformulation of Complete Irreducibility

if V is a finite dim comp. rep of finite grp G
$\exists$ irred reps of G $W_1, \dots, W_r$ & an iso of reps
$$V \cong W_1 \oplus \cdots \oplus W_r$$

Thm (Schur's Lemma)

Let V fin dim irred complex rep of gp G.
$\Rightarrow$ any map of reps from $V \to V$ is scalar mult
i.e $End_G(V)$ is 1 dim'l $\cong \mathbb{C}$

Pf) Let f be a map of rep $V \to V$
because we are over $\mathbb{C}$ $\exists$ eigen value. $f(v) = \lambda v$
$\lambda \in \mathbb{C}$ and $v \neq 0$ nonzero

wts $f = \lambda \cdot Id$

Let $f' = f - \lambda Id$ (show this is 0)
This is a map of reps $V \to V$ & $v \in \ker f'$
So sine V irred & $\ker f'$ is a non empt subrep
it must be $\ker f' = V \Rightarrow f' \equiv 0$
$\Rightarrow f \equiv \lambda \cdot Id$ $\square$

Complementary

Let V_1, V_2 are fin dim irred $\mathbb{C}$ rep of G

Then $Hom_G(V_1, V_2)$ has dim 1 if $V_1 \cong V_2$ iso as reps
dim 0 if Not!

Pf) If $V_1 \cong V_2$ $\Rightarrow Hom_G(V_1, V_2) \cong End_G(V_1)$ done from above.

if $V_1 \ncong V_2$ & proceed by contra & spose $f: V_1 \to V_2$ nonzero map of reps.

$\overset{1}{\circ}$ $im(f)$ is non-zero subrep of $V_2 \Rightarrow im(f) = V_2$ as V_2 irred

$\overset{2}{\circ}$ $ker(f)$ is a subrep of V_1, it is not everything as f onto $\Rightarrow ker(f) = 0$ as V_1 irred

①② $\Rightarrow$ f is iso ... oops!

Abelian!

<u>Setup</u>: • G is a finite abelian gp
ok $= \mathbb{C}$
• work with fin dim reps of <u>G</u>

Thm) Every irred of G is 1 <u>dim'll</u>.

Pf) let V be fin dim'll irred rep of G

<u>Claim</u>: every gp elt acts by a <u>scalar</u>.

<u>reason</u> if $g \in G$ then $L_g : V \to V$, $v \mapsto gv$

is a map of reps as <u>G is abelian</u>

$L_g(hv) = ghv = hgv$

$\Rightarrow$ by Schur $\Rightarrow L_g = \lambda_g \cdot Id$ for some λ_g

$\Rightarrow$ every subspace is a subrep of V !

$\Rightarrow$ $\dim V = 1$ by <u>irred</u>!

<u>Now say</u> $G = \langle g \rangle$ is cyclic of ord <u>n</u>.

1-D reps always irred & by abel irred $\Rightarrow$ 1D

A <u>1-D rep</u> of G is a homo $G \xrightarrow{\rho} GL_1(\mathbb{C}) = \mathbb{C}^\times$

ρ is det by $\xi = \rho(g)$ & ξ can be any elt of $\mathbb{C}^\times$ so that $\xi^n = 1$ $\to$ n^{th} roots of unity $e^{\frac{2\pi i m}{n}}$

$\therefore \rho_m : G \longrightarrow GL(1, \mathbb{C})$
$g^t \longmapsto e^{\frac{2\pi i m}{n}}$ $\Rightarrow L_m = (\mathbb{C}, \rho_m)$ is a 1, d rep

$\therefore$ the irred reps of G are $L_0, L_1, \ldots, L_{n-1}$

$\Rightarrow$ every irred is iso to one of & no two are isomorphic. (note $L_0 = L_n$)

So, every fin dim $\mathbb{C}$ rep of G is isom to $\oplus$ of L_m's
So if $A \in GL_d(\mathbb{C})$ so $A^n = 1$ → d dim rep of G.

$\Rightarrow$ this matrix can be diagonalized!