Last time | If M is a f.g. $\mathbb{Z}$ mod $\Rightarrow M = \bigoplus$ cyclic $\mathbb{Z}$ mods

Uniqueness | Note that this decomp is not unique.
As mods,

$$\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \quad \text{CRT.}$$

$\hookrightarrow$ cyclic $\qquad \hookrightarrow$ cyclic decomp.

A uniqueness Result

Let M be a f.g. $\mathbb{Z}$-mod. Then $\exists$ iso $M \cong \bigoplus_{i=1}^{r} C_i$
where $C_i = \mathbb{Z}$ or a finite cyclic grp of prime order
(unique up to perm).

If $M = \bigoplus_{j=1}^{s} C_j'$ is another such decomp
$\Rightarrow s = r$ & $\exists\, \sigma \in S_r$ so $C_{\sigma(j)}' = C_i$

"Pf" | Let $S = \mathbb{Z}/\{0\}$, then $\#\{i \mid C_i \cong \mathbb{Z}\} = \dim_{\mathbb{Q}}(S^{-1}M) = d$
$\Rightarrow \#\{i \mid C_i' \cong \mathbb{Z}\} = d$.

Assume $C_1 = \cdots = C_d = \mathbb{Z}$ (reorder as above is inv)
$C_1' = \cdots = C_d' = \mathbb{Z}$

Q) $\bigoplus_{i=d+1}^{r} C_i \cong \bigoplus_{j=d+1}^{s} C_j'$ yes! as both $\cong M_{\text{tors}}$)

By looking at p-power torsion, we reduce to the case where C_i, C_j' are cyclic & of p power order.

Why are following not iso.

$$\mathbb{Z}/p^5\mathbb{Z} \oplus \mathbb{Z}/p^5\mathbb{Z} \oplus \mathbb{Z}/p^6\mathbb{Z} \not\cong \mathbb{Z}/p^4\mathbb{Z} \oplus \mathbb{Z}/p^6 \oplus \mathbb{Z}/p^6$$

$\hookrightarrow$ 1 elt of ord 6 $\qquad$ 2 elt ord 6.

Another Result M fg $\mathbb{Z}$-mod $(d_i \geq 2)$

$\Rightarrow \exists$ isom $M \cong \mathbb{Z}^r \oplus \mathbb{Z}/d_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_n\mathbb{Z}$

w/ $d_2 | d_1$, $d_3 | d_2$ ....

This is unique & $r = \text{rank } M$

Pf idea Start by writing $M = \bigoplus (\mathbb{Z}$ or cyclic of pr power order$)$

Say $p_1, \ldots, p_m$ primes appearing.

Say we have $\mathbb{Z}/p_1^{e_1}, \mathbb{Z}/p_2^{e_2}, \ldots$ among somands with $e_1, \ldots, e_m$ max'l

$\mathbb{Z}/p_1^{e_1} \oplus \cdots \oplus \mathbb{Z}/p_m^{e_m} \overset{CRT}{\cong} \mathbb{Z}/d_1$

where $d_1 = \prod_{i=1}^{m} p_i^{e_i}$

Next say $\mathbb{Z}/p_1^{f_1}, \ldots \mathbb{Z}/p_m^{f_m}$ are the biggest remaining cyclic mods

$d_2 = \prod_{j=1}^{m} p_j^{f_j}$ $\longrightarrow$ rinse & repeat.

Struct Thm for PIDs

Thm Let R a PID & $M =$ fg R-mod

$\Rightarrow M$ is a $\oplus$ of R & $R/(a)$

Rmk 1) if $a, b \in R$ coprime $R/(ab) \underset{CRT}{\cong} R/(a) \oplus R/(b)$

$\Rightarrow$ if $a = \pi_1^{e_1} \cdots \pi_r^{e_r}$ distinct primes π_j

$\Rightarrow R/(a) \cong R/(\pi_1^{e_1}) \oplus \cdots \oplus R/(\pi_r^{e_r})$

2) Same uniqueness results hold!

e.g. $R = \mathbb{C}[t]$ is a PID (in general F a field $F[t]$ is a pid & primes are irred poly)

If $R = \mathbb{C}[t]$ the irred/prime elts are $t - \alpha$, $\alpha \in \mathbb{C}$

Pf) Say $h(t)$ a monic irred poly $\in \mathbb{C}[t]$.
As $\mathbb{C}$ alg closed $\exists \alpha$ so $h(\alpha) = 0 \Rightarrow t-\alpha \mid h(t)$
so $h(t)$ irred $\Rightarrow h(t) = t-\alpha$

So, Struct thm for f.g mods sew,s

$\hookrightarrow$ f.g R mods $\bigoplus \mathbb{C}[t]$'s & $\mathbb{C}[t] / \langle (t-\alpha)^n \rangle$

Appl/ Jordan Normal form

Recall a Jordan Block is a matrix of the form

$$\begin{bmatrix} \alpha & 1 & & 0 \\ & \alpha & \ddots & \\ & & \ddots & 1 \\ 0 & & & \alpha \end{bmatrix} \quad \text{E.g.} \begin{bmatrix} \alpha & 1 \\ 0 & \alpha \end{bmatrix}, [\alpha], \ldots$$

Jordan Normal form Thm)

Let M be a $\mathbb{C}$-matrix. $\exists$ inv A s.t
AMA^{-1} is a block diag matrix where the blocks are Jordan Block

$$AMA^{-1} = \begin{bmatrix} J_1 & 0 & 0 & 0 \\ 0 & J_2 & 0 & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & J_v \end{bmatrix}$$

Rmk) if all J_i are 1×1 matrices $\Rightarrow A$ diagble!

Def) V be a fin dim'l $\mathbb{C}$-vs & $T: V \to V$ a lin op
we can give V a structure of a $\mathbb{C}[t]$ module
by $t \cdot v = T(v)$

[more generally, if $f(t) \in \mathbb{C}[t]$
$f(t) \cdot v = \underbrace{f(T)}_{\text{lin op } T \to T}(v)$ eg $f = x^2+1$, $f^2 = (T^2+1)$]

As $\mathbb{C}[t]$ mod $V \cong \mathbb{C}[t]^r \oplus \mathbb{C}[t]/\langle (t-\alpha_1)^{e_1} \rangle \oplus \cdots \oplus \mathbb{C}[t]/\langle (t-\alpha_s)^{e_s} \rangle$

first step | r (num of $\mathbb{C}[t]$ in V) is 0
as $\dim_{\mathbb{C}} V < \infty$ but $\dim_{\mathbb{C}} \mathbb{C}[t] = \infty$

Now Claim each $\mathbb{C}[t] / \langle (t-\alpha_i)^{e_i} \rangle$ gives jordan block.

E.g. | $V = \mathbb{C}[t]/(t^e)$
let $\varphi : V \xrightarrow{\sim} \mathbb{C}[t]/(t^e)$ isom
key pt φ is isom of $\mathbb{C}$ vs st $\varphi(Tv) = t \cdot \varphi(v)$
$1, t, \dots, t^{e-1}$ is a $\mathbb{C}$-basis for $\mathbb{C}[t]/(t^e)$
$x_1 \quad x_2 \quad \cdots \quad x_e$
& $t x_i =$