Field Theory — Study of field extensions

Let F a field

Defn A field extension of F is a pair where E a field & $i: F \to E$ is a field homo

Rmk i is inj, as $\ker i$ is an ideal of F. & i field homo
$\to \ker i$ is a proper ideal of $F \Rightarrow \ker i = (0)$

So $F \subseteq E$ subfield

We write E/F for E an extension of F, (i is left impt)

Let E/F be given. An elt $\alpha \in E$ is algebraic on F if $\exists$ nonzero $f \in F[x]$ so $f(\alpha) = 0$

Say α is transcendental if α not alg.

e.g. 1) $\mathbb{C}/\mathbb{Q}$ → algebraic elts in $\mathbb{C}$ are exactly the alg numbers.

2) $\mathbb{C}/\mathbb{R}$ → every elt of $\mathbb{C}$ is alg $/\mathbb{R}$.
If $z \in \mathbb{C}$ we see $z + \bar{z}, z\bar{z} \in \mathbb{R}$
$\Rightarrow z$ root $(x - z)(x - \bar{z}) \in \mathbb{R}[x]$

Defn E/F is alg if every $\alpha \in E$ is alg.
Is transc. if not alg.

e.g.
- $\mathbb{C}/\mathbb{R}$ is alg
- $\mathbb{Q}(i)/\mathbb{Q}$ is alg
- $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5})/\mathbb{Q}$ is alg → $\sqrt{2}, \sqrt{3}, \sqrt{5}$ are alg
every elt is in alg subfield of $\mathbb{C}$

- $\mathbb{C}/\mathbb{Q}$ trans ext
  Cb card alg $\mathbb{Q}[x]$ ctbl, $\mathbb{C}$ not.

If E/F Think E as a v.s $/F$ (add mult via $F \subseteq E$)

Def The degree E/F denoted $[E:F]$ is $\dim_F E$

Prop if E/F is finite (finite degree) then it is alg.

Pf given $\alpha \in E$ the elts $(1, \alpha, \alpha^2, \ldots)$ must be f lindep.
as in a fin dim F vs

$\Rightarrow \exists c_i \in F$ st

$c_0 + \alpha c_1 + \cdots \alpha^\ell c_\ell = 0$:) $\rightarrow$ got poly in $F[x]$

eg let $\overline{\mathbb{Q}} \subseteq \mathbb{C}$ be the alg numbers,
we have show $\overline{\mathbb{Q}}$ subfield of $\mathbb{C}$

$\Rightarrow \overline{\mathbb{Q}}/\mathbb{Q}$ but $[\overline{\mathbb{Q}}:\mathbb{Q}] = \infty$

So $\overline{\mathbb{Q}}/\mathbb{Q}$ is an alg extn of ∞ of inf degree.

Prop (Transitivity of deg) given E/F, K/E denoted

$\Rightarrow [K:F] = [K:E][E:F]$

$$\begin{array}{c} K \\ | \\ E \\ | \\ F \end{array}$$

Pf see notes but obvious :)

Cor Spose K/F ext E is intermediate field (K/E, E/F)

$\Rightarrow [E:F], [K:E] \mid [K:F]$

Cor if $[K:F]$ prime $\Rightarrow$ only inter field K, F.

eg) $K = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ $\rightsquigarrow$ $\mathbb{Q}$ basis $1, \sqrt{2}, \sqrt{3}, \sqrt{6}$

$\Rightarrow \deg_{\mathbb{Q}} = 4$

let $F = \mathbb{Q}(\sqrt{2})$

$\Rightarrow 4 = \underbrace{[K:F]}_{2} \cdot \underbrace{[F:\mathbb{Q}]}_{2}$ $\rightsquigarrow$ need to show $\sqrt{2}, \sqrt{3}$ lin ind.

$K = F(\sqrt{3})$, $1, \sqrt{3}$