Inf Galois thy

Problem: find intermediate field b/w $\mathbb{F}_p$ and $\overline{\mathbb{F}_p}$

Consider: $\mathbb{F}_p \subset \mathbb{F}_{p^2} \subset \mathbb{F}_{p^4} \subset \mathbb{F}_{p^8} \subset \cdots$

The union is intermediate b/w $\mathbb{F}_p$ & $\overline{\mathbb{F}_p}$

It's not all $\overline{\mathbb{F}_p}$ b/c it doesn't have $\mathbb{F}_{p^3}$

or $\bigcup_{n\geq 1} \mathbb{F}_{p^{3\cdot 2^n}}$

or, $q_1, q_2, \ldots$ primes $\equiv 1 \pmod 4$

$\bigcup_{n\geq 1} \mathbb{F}_{p^{q_1\cdots q_n}}$

Consider, a formal product $n = \prod_{\text{all primes } q} q^{e(q)}$ w/ $e(q) \in \mathbb{N}\cup\{\infty\}$

Given $m \in \mathbb{N}$ we say the power of q in m is $\leq e(q)$ $\forall$ pr q

$$\mathbb{F}_{p^n} := \bigcup_{m|n} \mathbb{F}_{p^m}$$

What is $\mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p)$?

have $\overline{\mathbb{F}_p}$ — $\mathbb{F}_{p^n}$ — $\mathbb{F}_p$, restriction map

$$\mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p) \to \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p) \cong \mathbb{Z}/n\mathbb{Z}$$

$$\sigma \longmapsto \sigma|_{\mathbb{F}_{p^n}}$$

this is well def as $\mathbb{F}_{p^n}$ is adding any root of poly which is pres by aut

This is a surj grp homom.

Fix a prime q, consider

$$\begin{tikzcd}
 & & \mathrm{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p) \arrow[dl, "\text{rest}"'] \arrow[d] \arrow[dr] & \\
\cdots \arrow[r] & \mathrm{Gal}(\mathbb{F}_{p^{q^3}}/\mathbb{F}_p) \arrow[r, "\text{rest}"] \arrow[d, "\cong"] & \mathrm{Gal}(\mathbb{F}_{p^{q^2}}, \mathbb{F}_p) \arrow[r] & \mathrm{Gal}(\mathbb{F}_{p^q}, \mathbb{F}_p) \\
 \arrow[r] & \mathbb{Z}/q^3 \arrow[r] & \mathbb{Z}/q^2 \arrow[r] & \mathbb{Z}/q
\end{tikzcd}$$

$\vdots$ — $\mathbb{F}_{p^{q^2}}$ — $\mathbb{F}_p$

$\implies$ grp homo $\mathrm{Gal}(\overline{\mathbb{F}}_p / \mathbb{F}_p) \longrightarrow \varprojlim \mathrm{Gal}(\mathbb{F}_{p^n} / \mathbb{F}_p)$

↳ inv lim

So, $\mathrm{Gal}(\overline{\mathbb{F}}_p / \mathbb{F}_p)$

maps $\longleftrightarrow \varprojlim \mathrm{Gal}(\mathbb{F}_{p^n} / \mathbb{F}_p)$

↳ not as easy as you only have map if divisibility

$\cong \varprojlim \mathbb{Z}/n \cong \widehat{\mathbb{Z}}$

↳ profinite compl. of $\mathbb{Z}$

So, $\widehat{\mathbb{Z}} := \varprojlim \mathbb{Z}/n$

(CRT)

$= \varprojlim \prod_p \mathbb{Z}/p^{\mathrm{val}_p(n)}$ → how many times p prime div n

$\cong \prod_p \mathbb{Z}_p$ → inverse limit & products switch

↳ both are categorical limits!

$\mathbb{Z}_p$ → p adic.

This is the absolute Galois Grp of $\overline{\mathbb{F}}_p$.

$\mathbb{Z}_q$ has a topology. → higher powers shrink.

The closed subgrps are 0 & $q^n \mathbb{Z}_q$ $(n \in \mathbb{N})$

"$q^\infty \mathbb{Z}_q$"

We've now observed bijection,

$\left\{ \begin{array}{c} \text{closed subgrps of} \\ \mathrm{Gal}(\overline{\mathbb{F}}_p / \mathbb{F}_p) \end{array} \right\} \longleftrightarrow \left\{ \dfrac{\text{intermediate fields}}{\overline{\mathbb{F}}_p / \mathbb{F}_p} \right\}$

Say, K = char 0

Def | The abs Gal grp of k is $Gal(\bar{k}/k)$. Denoted G_k

k ctbl

Def | let $\{g_n\}_{n \in \mathbb{N}}$ be a seq in G_k & let $h \in G_k$

we say g_n conv to h if

$\bar{k} = \bigcup_{n \geq 1} L_n$, L_n/k is galois & finite. $L_1 \subset L_2$

s.t $g_n|_{L_n} = h|_{L_n}$

↳ live in gal grp

Def | not ctbl.

For each finite gal ext L/k $(L \subset \bar{k})$

have surj, $G_k \xrightarrow{\pi_2} Gal(L/k)$

Ker π_2 are open subgrps of G_k ⤳ gen 'n

G_k is profinite, $G_k \cong$ inv lim of $Gal(L/k)$.

inv lim of finite

(Krull topology).

Thm | $\{$closed subgrps of $G_k\}$ $\longleftrightarrow$ $\{$interm fields $\bar{k}/k\}$

$\{$open subgrps of $G_k\}$ $\longleftrightarrow$ $\{$finite extensions $\bar{k}/k\}$

→ open subgrps are closed in top grp,

Work $\overline{\mathbb{Q}}/\mathbb{Q}$ have $G_{\mathbb{Q}}$

Fix prime p $\zeta_{p^n} = e^{2\pi i/p^n}$

<u>have</u>

$$\begin{array}{c} \vdots \\ | \\ \mathbb{Q}(\zeta_{p^2}) \\ | \\ \mathbb{Q}(\zeta_p) \\ | \\ \mathbb{Q} \end{array} \Bigg\} \; Gal = (\mathbb{Z}/p^3\mathbb{Z})^\times$$

$$G_{\mathbb{Q}} \longrightarrow Gal(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q}) \cong (\mathbb{Z}/p^n\mathbb{Z})^\times$$

Compatible as n varies $\Rightarrow$ get have

$\chi_p : G_{\mathbb{Q}} \longrightarrow \mathbb{Z}_p^\times$ cyclo char

If $\zeta \in \overline{\mathbb{Q}}$ is a $\overset{\text{adic}}{p}$-power root of 1 & $\sigma \in G_{\mathbb{Q}} \Rightarrow \sigma\zeta = \zeta^{\chi_p(\sigma)}$

this is like a repr

$$\mathbb{Z}_p^\times = GL_1(\mathbb{Z}_p) \subseteq GL_1(\mathbb{Q}_p)$$