# Midterm

① let $f = xy(x+y) - 1$

Write down max'l ideals of $\frac{\mathbb{C}[x,y]}{(f)}$

1) Ideal corr thm  max'l ideals of $\frac{\mathbb{C}[x,y]}{(f)} \xleftrightarrow{\text{bij}}$ ideals of $\mathbb{C}[x,y]$ cont $(f)$.

2) Max ideals of $\mathbb{C}[x,y]$ are $(x-a, y-b) = \ker(ev_a : \mathbb{C}[x,y] \to \mathbb{C}$, $g \mapsto g(a,b))$

3) find $(a,b) \in \mathbb{C}^2$ so $\underline{f(a,b)} = 0$.

② $\varphi : R \to S$ surj ring map s.t $\ker \varphi$ is made up of nilp elts

$\underline{\text{wts}}$ for $s \in S$ unit $\exists x \in R$ $\varphi(x) = s$ $x$ $\underline{\text{unit}}$!

$\exists y \in S$ so $yy' = 1$ can pick $x, x' \in R$ so

$\varphi(x) = y$, $\varphi(x') = y'$

So $\varphi(xx') = yy' = 1 = \varphi(1)$

$\Rightarrow xx' = 1 + z$ for $z \in \ker \varphi$

$\hookrightarrow$ nilpotent

$= 1 + z$ $\boxed{\text{unit}}$

$\Rightarrow x \underbrace{x'(1+z)^{-1}}_{x^{-1}} = 1$ $\Rightarrow$ $x$ unit

③ $\mathbb{Z}[\frac{1}{6}] \subseteq \mathbb{Q}$

consider $(60) \subseteq \mathbb{Z}[\frac{1}{6}]$

Then describe $J = (60) \cap \mathbb{Z}$ i.e find gen!

elt of ideal looks like $\frac{60a}{6^n}$ $\to$ can get $\underline{5}$.

so $\underline{5 \in J}$ $\Rightarrow (5) \subseteq J$,

but $1 \notin J$ $\underline{\text{so}}$ as $(5)$ is $\underline{\text{max'll}}$ $\underline{J = (5)}$.

gen fact if

$R[\frac{1}{fg}] = R[\frac{1}{f}, \frac{1}{g}]$ $\quad (\frac{1}{f} = g \cdot \frac{1}{fg})$

④ factor $4+7i$ in $\underline{\text{gaussians}}$.

$N(4+7i) = 4^2 + 7^2 = 65 = \underline{13 \cdot 5}$.

$5 = (2+i)(2-i)$

$13 = (3+2i)(3-2i)$

⑤ PID w/ 3 max'll $\underline{\text{ideals}}$.

$\mathbb{Z}[\frac{1}{p}]_{p \neq \underline{2,3,5}}$.

Let $I < \mathbb{Z}[\frac{1}{p}]_{p \neq 2,3,5}$

$\underline{\text{Consider}}$ $\mathbb{Z} \cap I$ $\to$ ideal will work

$= n\mathbb{Z}$ $\Rightarrow n \in I$ $\Rightarrow (n) \subset I$

Can show $nR = I$ by clearing $\underline{\text{denom}}$!

given $x = \frac{a}{b} \in I$ $\Rightarrow bx \in I \cap \mathbb{Z}$ so $= nc$

$\Rightarrow x = \frac{n \cdot c}{b}$ $\Rightarrow \frac{c}{b} \in R$ $\Rightarrow x \in n \cdot R$

$\Rightarrow \underline{I = nR}$.

Recall | R-domain & S mult set ($0 \notin S$ so no degen).

$S \subseteq R$ is mult set $S^{-1}R = \left\{ \frac{x}{s} \mid \begin{array}{l} x \in R \\ s \in S \end{array} \right\} \subset \mathrm{Frac}(R)$

For R-mod M $S^{-1}M$ has elts $\frac{x}{s}$ (${\in M}$) & $\frac{x_1}{s_1} = \frac{x_2}{s_2}$

if $\exists s \in S$ $\underline{s(s_2 x_1 - s_1 x_2) = 0}$

If $f: M \longrightarrow N$ is a map of R-mod

There is an induced map $S^{-1}f : S^{-1}M \longrightarrow S^{-1}N$

$$\frac{x}{s} \longmapsto \frac{f(x)}{s}$$

Lemma | if $f$ is injective so is $S^{-1}f$.

Pf) Say $\frac{x}{s} \in S^{-1}M$ in $\ker(S^{-1}f)$

$\Rightarrow 0 = (S^{-1}f)(\frac{x}{s}) = \frac{f(x)}{s} \sim \frac{0}{1}$

$\Rightarrow \exists s' \in S$ by equiv reln so

$s'(f(x) \cdot 1 - 0 \cdot s) = 0$ in $N$

$= s'(f(x)) \overset{\text{f mod map can drag.}}{=} f(s'x) = 0$

as $f$ is injective $s'x = 0$ in $M$

$\Rightarrow \underline{\frac{x}{s}} = 0$ in $S^{-1}M$

so, $\ker(S^{-1}f) = 0$

lemma) $f$ surj $\Rightarrow$ $S^{-1}f$ surj

Pf) given $\frac{y}{s} \in S^{-1}N$

$\exists x \in M$ s.t $f(x) = y$ $\Rightarrow (S^{-1}f)\left(\frac{x}{s}\right) = \frac{y}{s}$

lemma) Say $M$ submod $N$ & $f: M \hookrightarrow N$ (so inj)

can consider, as $S^{-1}f$ inj, $S^{-1}M \subseteq S^{-1}N$

$$\frac{S^{-1}N}{S^{-1}M} \cong S^{-1}(N/M)$$

Pf) let $\pi : N \to N/M$ quot map

then $S^{-1}\pi : S^{-1}N \to S^{-1}(N/M)$ this is surj by above.

Cl) $\ker(S^{-1}\pi) = S^{-1}M$ (then done by 1st iso)

Pf) $S^{-1}M \subset \ker(S^{-1}M)$

if $x \in M$, $s \in S$ $(S^{-1}\pi)\left(\frac{x}{s}\right) = \frac{\pi(x)}{s} = 0$ ($x \in \ker \pi$)

- Say $\frac{x}{s} \in \ker(S^{-1}\pi)$

so, $0 = (S^{-1}\pi)\left(\frac{x}{s}\right) = \frac{\pi(x)}{s}$ by eqvir

$\Rightarrow \exists s' \in S$ so $s'(\pi(x)) = 0$ in $N/M$

ring map $\Rightarrow$ $\pi(s'x) = 0$ so $\underset{\substack{\shortparallel \\ y}}{s'x} \in \ker \pi = M$

$\Rightarrow \frac{x}{s} = \frac{s'x}{ss'}$ in $S^{-1}M$

$\in S^{-1}M$ ✓

**Rephrase** let

$$0 \longrightarrow \overset{M}{M_1} \longrightarrow \overset{N}{M_2} \longrightarrow \overset{M/N}{M_3} \longrightarrow 0$$

is a SES of R modules.

$$\Rightarrow 0 \longrightarrow S^{-1}M_1 \longrightarrow S^{-1}M_2 \longrightarrow S^{-1}M_3 \longrightarrow 0$$

is SES of $S^{-1}R$ mods

↳ so localization is an exact functor.