Wanna discuss $G_{\mathbb{Q}}$

It has some special elts.

$\overline{\mathbb{Q}}$ is an abstract alg closure of $\mathbb{Q}$

$\exists$ $\mathbb{Q}$-embedding $i : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ $\quad$ $i(\overline{\mathbb{Q}})$ is stable by complex conj c

$\hookrightarrow$ stay in set &

$\therefore \; i^{-1} \circ c \circ i \in G_{\mathbb{Q}}$

Spse j is a second embedding.

$i(\overline{\mathbb{Q}}) = j(\overline{\mathbb{Q}})$ = set of alg nums in $\mathbb{C}$

$\Rightarrow \sigma = j^{-1} \circ i \in G_{\mathbb{Q}} \Rightarrow i = j \circ \sigma$

$\Rightarrow j^{-1} c j = \sigma^{-1} \underbrace{(i^{-1} c i)}_{\text{conj to } i^{-1}ci \text{ in } G_{\mathbb{Q}}} \sigma$ $\Longrightarrow$ Upshot } $\exists$ conj class in $G_{\mathbb{Q}}$ corresp. to $\mathbb{C}$ conjugation.

$\exists$ embedding $i : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ as $\mathbb{Q} \hookrightarrow \mathbb{Q}_p$

$i(\overline{\mathbb{Q}}) \circlearrowleft$ by $\mathrm{Gal}(\overline{\mathbb{Q}}_p / \mathbb{Q}_p) = G_{\mathbb{Q}_p}$ (as earlier we emb $\mathbb{Q}$-ok & use its gal grp)

$\rightsquigarrow$ get homo $G_{\mathbb{Q}_p} \longrightarrow G$, $\sigma \longmapsto i^{-1} \sigma i$ } well def up to conj by $G_{\mathbb{Q}}$

$$\begin{array}{ccccc} & \overline{\mathbb{Q}}_p & & & \\ \text{gal gp is } I_p \le G_{\mathbb{Q}_p} \text{ inertia subgrp} & | & & & \\ & \mathbb{Q}_p^{un} & \xleftrightarrow{\text{corr}} & \overline{\mathbb{F}}_p & \\ \text{gal grp } \hat{\mathbb{Z}} & | & & | & \text{Galois grp } \hat{\mathbb{Z}} \\ & \mathbb{Q}_p & & \mathbb{F}_p & \end{array}$$

$\mathbb{Q}_p^{un}$ $\rightarrow$ Maximal unramified extn.

$\hookrightarrow$ generator of this is Frob_p $\quad$ Frobenius map

Suppose $K/\mathbb{Q}$ is a finite galois extn.

So, $G_{\mathbb{Q}} \xrightarrow{\pi} Gal(K/\mathbb{Q})$ (by surj by restr.)

$G_{\mathbb{R}} \subset G_{\mathbb{Q}} \supset G_{\mathbb{Q}_p}$

$$\begin{array}{c} \overline{\mathbb{Q}} \\ | \\ K \\ | \\ \mathbb{Q} \end{array}$$

Fact for all but finitely many p, $\pi(I_p) = 1$ — inertial subgp feels like acting sq root.

K is unramified at p if this holds.

e.g. $\mathbb{Q}(i)$

- $2 = (1+i)^2$ up to units } → ramified at 2
- If $p \equiv 1\ (4) \Rightarrow p = \pi \cdot \bar{\pi}$ in $\mathbb{Z}[i]$
- if $p \equiv 3\ (4) \Rightarrow p$ prime in $\mathbb{Q}(i)$.

} unramified else at p

$\forall$ odd p get a well def elt $Frob_p \in Gal(\mathbb{Q}(i)/\mathbb{Q})$
$\cong \mathbb{Z}/(2)$

$Frob_p$ is id if $p \equiv 1\ (4)$

$\neq id$ if $p \equiv 3\ (4)$

Fix finite set Σ of primes.

$\exists$ max alg extn $\overline{\mathbb{Q}}^{\Sigma}$ of $\mathbb{Q}$

s.t. $\forall p \notin \Sigma$, p is unramified in $\overline{\mathbb{Q}}^{\Sigma}$

Let $G_{\mathbb{Q},\Sigma} = Gal(\overline{\mathbb{Q}}^{\Sigma}/\mathbb{Q})$

$$G_{\mathbb{Q},\Sigma} = \left(G_{\mathbb{Q}} / \text{normal subgp gen'd by } I_p,\ p \notin \Sigma \right)$$

for $p \notin \Sigma$ $\exists$ well def conj cl. of elts $Frob_p \in G_{\mathbb{Q},\Sigma}$

Also have cx conj $c \in G_{\mathbb{Q},\Sigma}$

Thm (Chebotarev density thm).

1) The Frob_p $(p \notin \Sigma)$ dense in $G_{\mathbb{Q},\Sigma}$ (topologically as profinite)

2) If $K/\mathbb{Q}$ is a finite gal extn $\Rightarrow$ every elt $G(K/\mathbb{Q})$ is of the form Frob_p for some prime p.

Let $K/\mathbb{Q}$ be a finite Galois extn unram outside Σ.

Consider repr of $\rho : \mathrm{Gal}(K/\mathbb{Q}) \longrightarrow GL_n(\mathbb{C})$

finite grp $\hookleftarrow$ ct

for $p \notin \Sigma$ have a conj cl. Frob_p in $\mathrm{Gal}(K/\mathbb{Q})$

$\Rightarrow$ get a cx # $\chi_\rho(\mathrm{Frob}_p) \in \mathbb{C}$

(ρ character

Now consider ct rep $\rho : G_{\mathbb{Q},\Sigma} \longrightarrow GL_n(\mathbb{C})$

for $p \in \Sigma$ can again consider $\chi_\rho(\mathrm{Frob}_p) \in \overline{\mathbb{Q}}_\ell$

E.g. Recall have cyclotomic character

$$\chi_\ell : G_{\mathbb{Q},\{\ell\}} \longrightarrow \mathbb{Z}_\ell^\times \subset GL_1(\mathbb{Q}_\ell) = \mathbb{Q}_\ell^\times$$

$$\sigma(\zeta) \longmapsto \zeta^{\chi_\ell(\sigma)}$$ for an ℓ-power root of 1 ζ

Can think of χ_ℓ as a 1-diml ℓ-adic repr.

$\Rightarrow$ assigning some ℓ-adic number to p

$\chi_\ell(\mathrm{Frob}_p) = p$

Given some $\rho : G_{\mathbb{Q},\Sigma} \longrightarrow GL_n(\overline{\mathbb{Q}}_\ell)$

$\exists$ modular form s.t. the number $\chi_\rho(\mathrm{Frob}_p)$ are the coeff of f (of f)

e.g. $\Delta(q) = q \prod_{n \geq 1} (1 - q^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n$ → Ramanujan Tau

$q = e^{2\pi i z}$, $z \in \mathfrak{h}$.

$\exists$ repr $\rho : G_{\mathbb{Q}} \to GL_2(\mathbb{Z}_\ell)$ so $\mathrm{tr}\, \rho(\mathrm{Frob}_p) = \tau(p)$

$\det \rho(\mathrm{Frob}_p) = p^{11}$