

- Def**) A ring is a set R equipped with 2 binary op + (addition) & \cdot (Multi) s.t
- 1) $(R, +)$ is an abelian grp (id elt 0) (additive inv x is $-x$)
 - 2) (R, \cdot) is a monoid (id elt 1) (if \exists mult inv x is x^{-1})
 - 3) Distributivity $x \cdot (y+z) = xy + xz$ (and right too)

Def) A ring is called commutative if Mult commutes!

- E.g.**
- \mathbb{Z} is a ring
 - $\mathbb{Z}[x]$ is a ring \rightarrow in fact any (comm)ring coeff works!
↳ can add more vars
 - Any field a ring
 - Let R be a ring $M_{nn}(R)$ is a non comm ring (in gen)
 - The zero ring $R = \{0\}$ ($1 = 0$ here)
↳ if $1 = 0$ in R then R is zero ring

Exer/Exm

- $x \cdot 0 = 0 \quad \forall x \in R$
- $(-1) \cdot x = -x$
- $(-x) \cdot y = -x \cdot y$

Def) Let R, S rings then a ring hom is a fun that
is compat with $+$ and \cdot & takes $1_R \mapsto 1_S$
↳ to not include $f(x)=0$

Rmk) A elt x of ring R is a unit if it has a
mult inv i.e.
 $\exists y$ so $xy = yx = 1$

The set R^\times of units is a grp under mult

If $f: R \rightarrow S$ is a ring hom it induces a gp hom
on R^\times \Rightarrow this is why we want $f(1)=1$

Eg Ring homo

- $\text{id}_R : R \rightarrow R$ is a ring homo
 - $\forall R \exists!$ ring homo $R \rightarrow 0 \xrightarrow{\text{zero ring}}$
 - $\exists 0$ is a final object in category of rings
 - If ring $R \exists!$ ring homo from $\mathbb{Z} \rightarrow R$ (initial obj)
- $\forall n \in \mathbb{N} \mapsto \underbrace{1 + 1 + \dots + 1}_{n}$

unique outgo

Rank

- in \mathbb{Z} trivial grp is initial + final
- in fields has no init or final
- in fields of char 0, \mathbb{Q} is initial!
- $\mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z}$ quot map is a ring homo
- Complex conj is a ring homo! (field homo)

$\circ R$ commutative ring \Rightarrow Ring homo

$$R[x] \longrightarrow R$$

$$a_0 + \dots + a_n x^n \mapsto a_0 + \dots + a_n$$

$$\varphi \longmapsto \varphi(1)$$

in general if $c \in R$ get ring homo $R[x] \rightarrow R$

$$\varphi \longmapsto \varphi(c)$$

Thm Mapping Prop for poly rings

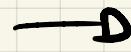
R, S com rings

Giving a ring homomorphism $\varphi : R[x] \rightarrow S$

Same as giving ring homo $\varphi : R \rightarrow S$ & $c \in S$

$\varphi_0 = \varphi|_R$ (thinking $R \subset R[x]$ as const poly)

$$c = \varphi(x)$$



Def $\varphi_0: R \rightarrow S$ & $C \in S$ define
 $f: R[\Sigma x_i] \rightarrow S$
 $(\sum a_i x^i) \mapsto \sum_{i=1}^n \varphi_0(a_i) C^i$

Can show this is a well def homo & inv
 to previous construction

Another expt given $\varphi_0: R \rightarrow S$ as $C \in S$
 $R[\Sigma x_i] \rightarrow S[\Sigma x_i] \rightarrow S$
 $\sum a_i x^i \rightarrow \sum \varphi_0(a_i) x^i$
translate $f \rightarrow f(C)$
evaluate

Rank $R[\Sigma x, y] \cong R[\Sigma x][y]$ in gen
Co ring iso

$\cong R, S$ com

Let $f: R \rightarrow S$ be a ring homo.

Def $\ker f = \{r \in R \mid f(r) = 0\}$

this is an additive subgroup of R

f is inj $\Rightarrow \ker(f) = 0$

Lemma if $x \in \ker f$ & $y \in R$ then $xy \in \ker(f)$

PF $f(xy) = f(x)f(y) = 0 \cdot f(y) = 0$

Def A ideal I of R is an additive subgroup that
 is closed under all scalar mult

$(\forall a \in R \ \forall c \in I \Rightarrow ac \in I)$

Prop above \ker of ring homo is ideal!

Right

in non comm case have
left ideal (left mult by arb elt)
right ideal (right mult by arb elt)
2-sided ideal

Spec $I \subset R$ is an ideal can think of R/I
as an abelian grp (addition commutes)
by

$$(a+I)(b+I) = (ab+I)$$

Must show $| ab+I = (a+x)b+I \text{ for } x \in I$

$$\Rightarrow \text{need } bx \in I \rightarrow \text{okay as } I \text{ ideal}$$

This makes

R/I as a comm ring

$\pi : R \rightarrow R/I$ is a ring homo

$$\text{with } \ker \pi = I$$

\Rightarrow each ideal ker of homo