Thm Let Ω/F is an alg closure. (char 0)

K/F finite extn.

The number of F embeddings : $K \to \Omega$ $= [K:F]$

Pf First spse (K,α) is a stem field for the irred $f(x) \in F[x]$

Note $[K:F] = \deg f$

By stem field theory, given F-emb of $K \to \Omega$

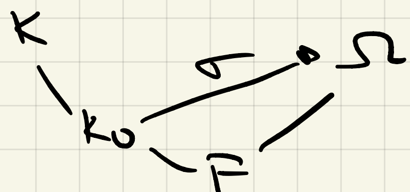
$\iff$ choosing a root of $f(x)$ in Ω..

The number of distinct roots is $\deg(f(x))$

$\Leftarrow$ separable in char 0

b/c Ω alg closed $f(x)$ no repeated roots.

General let K_0 be an inter field to K/F that is stem $/F$.



Pick F embedding $\sigma : K_0 \to \Omega$

Think Ω as alg closure of K_0

by ind on degree,

number of K_0 emb of $K \to \Omega$ $= [K:K_0]$ } Ω/K_0 ext via σ

What is a K_0 embedding of $K \to \Omega$?

This is a field homo $\tau : K \to \Omega$ so on K_0 agree w/ σ.

$\tau|_{K_0} = \sigma$

$\Rightarrow$ as σ Femb $\Rightarrow$ τ is F-emb

$\therefore$ each F-emb $\sigma : K_0 \to \Omega$ admits $[K:K_0]$ F-emb $K \to \Omega$.

By abv, have $[K_0:F]$ such σ.

$\#$ F emb $K \to \Omega$ $= \#$ F-emb $K_0 \to \Omega$

$\cdot$ $\#$ ways of ext. those to K

$= [K_0:F][K:K_0] = [K:F]$

Corollary $\#\mathrm{Gal}(K/F) \leq [K:F]$

Pf Fix an F embedding $\sigma: K \to \Omega$

if $\gamma \in \mathrm{Gal}(K/F)$ → can recover γ from emb

Then $\sigma \circ \gamma : K \to K \to \Omega$ is another F embedding. → as σ inj

$\Rightarrow$ have an injection $\mathrm{Gal}(K/F) \to \{F \text{ emb } K \to \Omega\}$

Cor Say K/F splitting field of $f(x) \in F[x]$.

Then K/F is a Galois extn.

Pf WLOG, assume $K \subseteq \Omega$

Spse, $\sigma: K \to \Omega$ is an F emb.

Cl. $\sigma(K) = K$.

if $a_1, \ldots, a_n$ roots of $f(x)$ in K

$\Rightarrow \sigma(a_1), \ldots, \sigma(a_n)$ also roots of $f(x)$ in K

$K = F(a_1, \ldots, a_n)$, $\sigma(K) = F(\sigma(a_1), \ldots, \sigma(a_n)) = K$

$\Rightarrow$ each emb is an automorphism of K

$\#\mathrm{Gal}(K/F) = \#F \text{ emb } K \to \Omega = [K:F]$

Rmk Pf shows if K/F spl. field & σ, γ F emb: $K \to \Omega$

$\Rightarrow$ they have the same img! $\sigma(K) = \gamma(K)$

→ in fact! an ext K/F w/ this prop is splitting field.

Thm let K be a field & let G be a finite grp of aut of K. Put $F = K^G$ ⤳ fixed field

$\Rightarrow K/F$ is galois w/ $\mathrm{Gal}(K/F) \cong G$.

PF) We know $G \subseteq \mathrm{Gal}(K/F)$ (by defn G fixes all in F)

$\Rightarrow [K:F] \geq \#G$

It suffices to show $\#G \geq [K:F]$ (sandwiches all)

Put $m = \#G$ wts $[K:F] \leq m$

Will show if $n > m$ & have $\alpha_1, \dots, \alpha_n \in K$

$G = \{\sigma_1, \dots, \sigma_m\}$ $\Rightarrow \alpha_1, \dots, \alpha_n$ F-lin dep!

Consider the following system of eqns

$$\left.\begin{aligned} \sigma_1(\alpha_1)x_1 + \cdots + \sigma_1(\alpha_n)x_n &= 0 \\ &\vdots \\ \sigma_m(\alpha_1)x_1 + \cdots + \sigma_m(\alpha_n)x_n &= 0 \end{aligned}\right\} \text{ more var than eqn}$$

$\therefore \exists$ nontrivial soln as $n > m$ in K → need to show in F

to get F lin dep ←

Consider a soln of $(c_1, \dots, c_n) \in K^n$

(choose this w/ as many zeros as possible (no trivial))

Wlog $c_1 = 1$ (permute & scale).

Cl: $(c_1, \dots, c_n) \in F^n$

Spse not $\Rightarrow \exists i$ so $c_i \notin F \Rightarrow \exists \sigma \in G$ so $\sigma c_i \neq c_i$ — not fixed by G

Obs $(\underbrace{\sigma c_1}_{1}, \dots, \sigma c_n)$ is still soln to system (this'll be perms!)

Consider $(c_1, \_\_, c_n) - (\sigma c_1, \_\_, \sigma c_n)$ is a soln

$= (0, ?, \underbrace{c_i - \sigma c_i}_{\neq 0}, ?)$ ← added a new zero & nontrivial vec soln

↳ $\sigma(\text{zero}) = \text{zero}$

$\Rightarrow \sigma(\text{zero}) - \sigma(\text{zero}) = 0$

⨳

$\therefore$ first eqn gives F lin dep ⨳

Eg) $K = \mathbb{C}(x_1, \dots, x_n)$ let $S_n \curvearrowright K$ by permuting vars.

$F = K^{S_n} \implies K/F$ is galois w/ grp S_n & degree $\underline{n!}$.

Write, $\prod_{i=1}^{n} (t - x_i) = \sum_{i=1}^{n} \underbrace{c_i(x_1, \dots, x_n)}_{S_n \text{ inv}} t^i = f(t)$

Fact: $F \supseteq \mathbb{C}(c_0, \dots, c_{n-1})$

K/F has degree $n!$ .... get $\subseteq$ so equal.

Rmk) if G any finite grp then $G \subseteq S_n$ for some S_n some n.

$K = \mathbb{C}(x_1, \dots, x_n)$ $F = K^G \implies K/F$ is gal extn w/ gp G.