Reminder | exam Monday.

Thm | if F is a field $\Rightarrow$ F has an alg closure & is unique up to F-isom.

Lemma | let $\{E_i/F\}_{i\in I}$ is a family of alg extn.

$\exists$ alg extn K/F s.t. each E_i admits an F-emb $E_i \to K$.

Pf | Say each E_i/F is finite

let $x_{i,1}, \dots, x_{i,n(i)}$ be gens of E_i (as extn of F).

$$\left.\begin{array}{ccc} F[T_{i,1},\dots,T_{i,n(i)}] & \longrightarrow & E_i \\ T_{i,j} & \longmapsto & x_{i,j} \end{array}\right\} \text{surj ring homo}$$

let $\mathfrak{a}_i$ be the kernel of abv.

let $R = F[T_{ij}]_{\substack{i\in I \\ j\in \mathbb{N}_{n(i)}}}$ $\quad Q =$ ideal gen'd by $\mathfrak{a}_i$

Note $F[T_{i,1},\dots,T_{i,n(i)}] \subset R$ have

$E_i = F[T_{i,1},\dots,T_{i,n(i)}]/\mathfrak{a}_i \longrightarrow R/Q$

$\hookrightarrow$ is inj

Show Q prop ideal

let Q' be max'll ideal containing $Q \supsetneq$ same

get map $E_i \to R/Q'$ now R/Q' is a field.

im of T_{ij} are alg (same as x_{ij}) & gen R/Q'.

Rmk | $R/Q \cong \bigotimes_{i\in I} E_i$ tensor prod $/F$.

e.g. | $F = \mathbb{R}$, $E_1 = E_2 = \mathbb{C}$ $\quad R = \mathbb{R}[T_1, T_2]$

$Q = \langle T_1^2+1, T_2^2+1\rangle$

$\Rightarrow R/Q = \mathbb{C}[\sqrt{-1}] = \mathbb{C}^2 = \mathbb{C}\otimes_{\mathbb{R}}\mathbb{C}$

Pf thm | Constr of a closure

let $\{f_i\}_{i \in I}$ set of all poly w/ coeff in F.

let E_i be a spl. f for f_i over F.

$\Rightarrow \{E_i / F\}_{i \in I}$ let F_1 be alg extn of F s.t. each

$\boxed{\text{lemma}} \leftarrow$ $E_i \hookrightarrow F_1$ by a F embedding.

Define F_2 by the same procedure starting with F_1

$F \subset F_1 \subset \cdots$ $\qquad \bigcup_{n \geq 1} F_n$ is an alg closure of F.

(Actually F_1 already closed & $F_i = F_1$, $i \geq 1$ (ess dont need))

$\to$ alg as F_2 alg $/ F_1$ ... & tower of alg.

Eg | Algebraic closure of $\mathbb{F}_p$ $\longrightarrow$ precisely all alg extn that contain $\mathbb{F}_p$.

$\mathbb{F}_p \subset \mathbb{F}_{p^{2!}} \subset \mathbb{F}_{p^{3!}} \subset \mathbb{F}_{p^{4!}} \subset \cdots$

Union of these is algebraic closure of $\mathbb{F}_p$.

Uniqueness | **Prop** | let Ω be alg closure of F.

let E/F any alg extn

$\Rightarrow \exists$ F embedding of $E \longrightarrow \Omega$.

Pf | First E gen'd by single elt. $E = F(a)$

let f min poly of a $\quad \exists$ root a' of f in Ω

$\exists!$ (stem field) F embedding $E \longrightarrow \Omega$

$a \longmapsto a'$

Second Case | $E = F(a_1, \dots, a_n)$

$$\begin{array}{c} F(a_1, \dots, a_n) \\ | \\ \vdots \\ F(a_1, a_2) \\ | \\ F(a_1) \\ | \\ F \end{array} \longrightarrow \Omega$$

$\boxed{\text{alg}}$

use case 1 repeatedly

General | Zorn's lemma w/ this!

If Ω, Ω' 2 alg closures get F-isom by F-embedding in both directions!

Pf) $\exists$ F embedding $\Omega' \longrightarrow \Omega$

$$\begin{matrix} \Omega \\ | \\ \Omega' \\ | \\ F \end{matrix}$$

$\Rightarrow$ Ω alg extn of Ω'

but Ω' alg closed $\Rightarrow$ $\Omega = \Omega'$!

Recall | If E/F extn say $x \in E$ is transc $/F$ if not alg (ie $\nexists$ $f(t) \in F[t]$ so $f(x) = 0$)

Prop) Say $a \in E$ transc. $/F$

$\Rightarrow$ $\exists$ F isom $F(a) \cong F(x)$

subfield gen'd by a

field of rational functions

Pf) Consider ring homo $\varphi : F[x] \longrightarrow E$

$f \longmapsto f(a)$

φ is injective b/c if $f \in \ker \varphi$ non triv $\Rightarrow$ a alg $\Rightarrow\!\Leftarrow$

φ induces injection on $\mathrm{Frac}(F[x]) \longrightarrow E$

w/ img is $F(a)$.

Now, spse $a, b \in E$ are trans

Is $F(a,b) \cong F(x,y)$?

No! $a = b$.

What if $b \notin F(a)$? Still not true if b alg $/ F(a)$

(e.g. $F = \mathbb{Q}$, $a = \pi^{1/2}$, $b = \pi^{1/3}$) $\longrightarrow$