last time

$R = \mathbb{C}[x_1, \dots, x_n]$ $\quad a \in \mathbb{C}^n$ $\quad J_a = (x_1 - a_1, \dots, x_n - a_n)$ max ideal

Consider the ring homomorphism $\quad ev_a : R \longrightarrow \mathbb{C}$

$$f \longmapsto f(a)$$

Claim $\quad J_a = \ker(ev_a)$

$\subseteq$ clear

$\supseteq$ make a linear COV to reduce to $a = 0$ $\quad \mathbb{C}[x_1 - a_1, \dots, x_n - a_n]$

↳ also J_a is a maximal ideal and so $\ker(ev_a)$ bigger not R

Prop) The J_a's are all maximal ideals of R

Pf) let $J \subseteq R$ be a max ideal

↳ last time $R/J \cong \mathbb{C}$

Consider the map $\quad \varphi : R \longrightarrow R/J \cong \mathbb{C}$

let $a_i = \varphi(x_i) \in \mathbb{C}$

$J_a \subset \ker(\varphi)$ $\implies$ thus equal as J_a maximal!

$(f) \subseteq J_a$

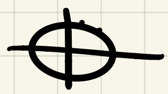
Let $f \in R$ $\quad f \in J_a \iff f(a) = 0$ by above!

Consider) $\{z \in \mathbb{C}^n \mid f(z) = 0\} \subseteq \mathbb{C}^n$

↳ this set is naturally in bij w/ set of maximal ideals $R/(f)$

↳ maximal ideals are J_a w/ $(f) \subseteq J_a$ i.e. $f(a) = 0$

Eg) $n = 2$ $\quad f = x_1^2 + x_2^2 - 1$

$\{z \mid f(z) = 0\}$ 

$\mathbb{C}$ pts of unit circle $\longleftrightarrow$ max ideals of $\dfrac{\mathbb{C}[x,y]}{(x^2 + y^2 - 1)}$

Let R = ring

Closed / indexing set

Def) $\operatorname{MaxSpec}(R) = \{\max \text{ ideals of } R\}$

For ea $x \in \operatorname{MaxSpec}(R)$ let $m_x \subset R$ for corr maximal

$k_x = R/m_x$

For $f \in R$ for each $x \in \operatorname{MaxSpec}(R)$

Define $f(x) \in k_x$ to be the img of f in R/m_x

A carries a natural topology

Given an ideal I,

$V(I) = \{x \in \operatorname{MaxSpec} \mid I \subseteq m_x\}$

e.g. If $R = \mathbb{C}[x_1, \ldots, x_n]$ $I = (f)$

$V(I) = \{J_a \mid f(a) = 0\}$

The $V(I)$'s are exactly the closed sets of $\operatorname{MaxSpec}(R)$

Prop) $R = \mathbb{C}[x_1, \ldots, x_n]$

let $f_1, \ldots f_r \in R$

TFAE (a) $(f_1, \ldots, f_r) = (1)$

(b) $\{z \in \mathbb{C}^n \mid f_1(z) = \ldots = f_r(z) = 0\}$

Pf) (a) $\Rightarrow$ (b)

$1 = \sum_{i=1}^{r} g_i f_i \quad g_i \in R$

Given $z \in \mathbb{C}_n$ $1 = \sum_{i=1}^{r} g_i(z) f_i(z)$

so not all f_i can vanish else RHS $= 0$.

(b) $\Rightarrow$ (a) Contra so $\neg(a) \Rightarrow \neg(b)$

Assume $(f_1, \ldots, f_r) \neq (1)$

$\Rightarrow (f_1, \ldots, f_r) \subseteq$ maximal ideal $= J_a$ for $a \in \mathbb{C}_n$

$\Rightarrow$ each $f_i \in J_a \Rightarrow f_i(a) = 0 \rightarrow$ common zero

Factorization

Fundamental Thm of Arithmetic

If n is a nonzero int, $\exists$ factorization
$n = \pm p_1 \cdots p_r$ where p_i is prime
↳ Unique up to perm order of mult

Q) is there analogue of this in gen ring?
A) Sometimes .... it's subtle!

e.g.) 1) $F[x]$ F is a field
There is theory of unique factorization
2) Similar holds in $F[x_1, \dots, x_n]$
3) $\mathbb{Z}[i]$ has unique factorization
4) $F(\sqrt{-5})$ doesn't $\to$ $6 = 2 \cdot 3 = (1+\sqrt{-5})(1-\sqrt{-5})$ ← Not $\sim$
↳ but in $\mathbb{Z}$ $6 = 2 \cdot 3 = (-2)\cdot(-3)$
↳ these shouldn't be different as they differ by units!

Def) From here on R is a domain.
For $x, y \in R$, we say $x \mid y$ x divides y
if $y = wx$ for $w \in R \Rightarrow y \in (x) \Leftrightarrow (y) \subset (x)$

Lemma) if $x \mid y$ & $y \mid x \Rightarrow x = u \cdot y$ for a unit u

Pf) $x = u \cdot y$, $y = v \cdot x$ for $u, v \in R$

$\Rightarrow x = u \cdot v \cdot x \Rightarrow x(1 - u \cdot v)$

trivial one $\xleftarrow{x \neq 0}$ $\Rightarrow 1 - uv = 0 \Rightarrow uv = 1$ → $\{u, v \text{ units}\}$

Def) We say x and y are associates $(x \sim y)$ if
$x = u \cdot y$ for a unit u.

Def | An elt π of R is irreducible if ($\pi \neq 0$, $\pi \neq$ unit)
$\pi = xy \Rightarrow x$ unit or y unit

e.g. | 1) $R = \mathbb{Z}$; ireds are $\pm$ prime

2) $R = F[x]$; ireds are ired polyn !

Def | A unique factorization