Defn | E/F field extn. $a_1, \dots, a_n \in E$. We say that they are alg ind if $\Phi(a_1, \dots, a_n) = 0 \Rightarrow \Phi = 0$ for $\Phi \in F[T_1, \dots, T_n]$

More gen any set $S \subset E$ is alg ind if any finite sso A is alg ind

e.g. | $E = \mathbb{C}(x,y)$ $F = \mathbb{C}$ then x, y alg ind

if $a_1, \dots, a_n \in E$ alg ind $\Rightarrow F(a_1, \dots, a_n) \cong F(T_1, \dots, T_n)$

e.g. | $\exists$ inf sso of $\mathbb{C}$ that is alg ind $/\mathbb{Q}$

Constr by ind.

Spse $a_1, \dots, a_n \in \mathbb{C}$ alg independent. Consider all elts of $\mathbb{C}$ alg $/\mathbb{Q}(a_1, \dots, a_n)$. This is a ctbl subfield of $\mathbb{C}$ so $\exists$ such an elt. $\Rightarrow \exists a_{n+1} \in \mathbb{C}$ trans $/\mathbb{Q}(a_1, \dots, a_n)$

$\Rightarrow a_1, \dots, a_n, a_{n+1}$ alg ind $/\mathbb{Q}$

Def | A transcendental base for E/F is a max'll alg ind. set.

Fact | If S is trans base for $E/F \Rightarrow E/F(S)$ algebraic (makes sense if we find transc. elt can ext S)

Fact | Any ext has a trans base (pf uses Zorn)

Def | The transc. degree of E/F, denoted $\mathrm{trdeg}(E/F)$ is the card of any transc. base.

e.g. |

1) $\mathrm{trdeg}(E/F) = 0 \iff E$ is alg $/F$
2) $\mathrm{trdeg}(\mathbb{C}(T_1, \dots, T_n)) = n$
3) given K/E, E/F $\mathrm{trdeg}(K/F) = \mathrm{trdeg}(E/F) + \mathrm{trdeg}(K/E)$
4) $F = \mathbb{C}$ $E = \mathrm{Frac}\left(\dfrac{\mathbb{C}(x,y)}{(y^2 - x^3 - x)}\right)$ $\mathrm{trdeg}(E/F) = 1$

trans base is any elt of E not in $F(x, y, xy, \dots)$

$y=\sqrt{x^3-x}$

$$\begin{tikzcd} & E \arrow[dl, dash, "2"'] \arrow[dr, dash, "3"] & \\ \mathbb{C}(x) \arrow[dr, dash] & & \mathbb{C}(y) \arrow[dl, dash] \\ & F=\mathbb{C} & \end{tikzcd}$$

$E/\mathbb{C}(y)$ gen'd by x min'l poly is $x^3-x=y^2$

$X=\{(x,y)\in\mathbb{C}^2 \mid y^2-x^3+x\}$

E is the fnc field for X

$$\begin{tikzcd} & X \arrow[dl, "x"'] \arrow[dr, "y"] & \\ \mathbb{C} & & \mathbb{C} \end{tikzcd}$$

$(x,y)\mapsto x$, $(x,y)\mapsto y$

usually $2\to 1$ $\qquad$ usually 3 to 1

Application / Use in Alg geom

$R=\dfrac{\mathbb{C}[x_1,\dots,x_n]}{(f_1,\dots,f_n)}$ (domain) $\quad \boxed{\text{alg varieties}}$

geom $R \longleftrightarrow$ simultaneous zero locus of $f_1,\dots,f_n$ in $\mathbb{C}$

Defn) $\dim(X) = \operatorname{trdeg}(\operatorname{Frac}(R)/\mathbb{C})$

Write $\mathbb{C}(X)=\operatorname{Frac}(R)$ $\quad$ (fnc field of X)

Say $f:X\to Y$ is a map of varieties.

gives a field extn $\mathbb{C}(X)\supseteq\mathbb{C}(Y)$

Rmk) If f is gen d to 1 (d inv image)

$\Rightarrow [\mathbb{C}(X):\mathbb{C}(Y)]=d$

App of trdeg

Thm) If E/F fin gen ext. Then every int field is fin gen $/F$.

Cor) let $\overline{F}=\{a\in E \mid a \text{ alg } /F\}$ $\Rightarrow \overline{F}$ fin extn of F

Defn) E/F is purely trans if $E=F(a_1,\dots,a_n)$ w/ $a_1,\dots,a_n$ alg ind

Q) is every subextn of a purely trans extn still? (say F alg closed)

A) $n=1$ Yes (Lüroth thm)

$n=2$ maybe in char $=0$

but $n\geq 3$ no!