Last time

- $G \subset Aut(K)$ finite subgrp*. K/K^G is galois w/ grp G
- If E/F is any finite ext $\#\{F \text{ emb } E \to J\} = [E:F]$

from a while ago

$\Rightarrow \#Gal(E/F) \leq [E:F]$

$\Rightarrow$ if E/F splitting field $\Rightarrow E/F$ galois.

E/F is galois $\Rightarrow E^{Gal(E/F)} = F$

Prop Say E/F is finite, TFAE,

a) E/F galois

b) E/F is a splitting field

c) if $f(x) \in F[x]$ irred + f has a root in E
$\Rightarrow f$ has all roots in E

Pf $b \Rightarrow a$ abv

$a \Rightarrow c$ let $f(x) \in F[x]$ be irred + Spse $\exists\, a \in E$
So $f(a) = 0$.
Let $\sigma \in Gal(E/F)$ then $\sigma(a)$ is a root of f
as $f(\sigma(a)) = \sigma(f(a)) = \sigma(0) = 0$.
let $a_1, \ldots, a_n$ be the orbit of a under $Gal(E/F)$
Let $g(x) = \prod_{i=1}^{n}(x - a_i)$ we know $g(x) \mid f(x)$

important idea

key pt coeff of $g(x)$ are symmetric poly in $a_1 \ldots a_n$
$\Rightarrow$ coeff are fixed by $Gal(E/F) \Rightarrow$ coeffs in F
$\Rightarrow g(x) \in F[x] \Rightarrow g(x) = f(x)$ as irred. $\square$.
The a_i's are all the roots of f & in E

c $\Rightarrow$ b | Since E/F finite $E = F(a_1 \_\_\_ a_n)$
let $f_i(x)$ be min poly of a_i in $F(x)$ is irred.
By c, $f_i(x)$ splits $/E$ into linear factors.
$\Rightarrow$ same is true for $\prod_{i=1}^{n} f_i(x)$
$\therefore$ E splitting of $\prod_{i=1}^{n} f_i(x)$

Cor | (a$\Rightarrow$b) if E/F is galois & K intermediate
$\Rightarrow E/K$ is galois.

$$\begin{tikzcd} E \arrow[d, dash] \\ K \arrow[d, dash] \\ F \end{tikzcd}$$

gal. gal $\boxed{X}$ $\exists$ nils $\in$

Pf) E is spl. of $f(x) \in F[x]$
$\Rightarrow E$ is spl. of $f(x) \in K[x]$ same one
Split. $\Leftrightarrow$ galois $\square$.

Eg |

$$\begin{tikzcd} \mathbb{Q}(\sqrt[3]{2}, \zeta) \arrow[d, dash] \\ \mathbb{Q}(\sqrt[3]{2}) \arrow[d, dash] \\ \mathbb{Q} \end{tikzcd}$$

not gal

Main thm) if E/F gal w grp G
$\exists$ bij

$$\{\text{subgrps of } G\} \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \{\text{intermediate fields}\}$$

$\Phi(H) = E^H$
$\Psi(K) = \mathrm{Gal}(E/K)$ } w/s Φ, Ψ inverse

Spse | $H < G$ w/s $\Psi(\Phi(H)) = H$

$$\Psi(\Phi(H)) = \mathrm{Gal}(E/E^H) \overset{\text{last class}^*}{=} H \quad \square$$

next, $\Phi(\Psi(K)) := \Phi(\mathrm{Gal}(E/K)) := E^{\mathrm{Gal}(E/K)} = K$ $\square$

as E/K is galois

The corr is order reversing

if $H_1 \subset H_2 \subset G \Rightarrow \Phi(H_1) \supset \Phi(H_2)$ | Clear from defn

if $F \subset K_1 \subset K_2 \subset E \Rightarrow \Psi(K_1) \supset \Psi(K_2)$
inter

$\Psi(K_1) = Gal(E/K_1)$, $\Psi(K_2) = Gal(E/K_2)$

Let $H \subset G$ & $K = \Phi(H) = E^H$

wts H is normal subgrp $\Leftrightarrow$ K/F is galois.

Pf) Spse, H is normal.

Obs | every elt G maps $K = E^H$ into itself,

Pf) let $x \in E^H$ wts $\sigma(x) \in E^H$

let $\gamma \in H$ $\gamma(\sigma(x)) = \sigma \underbrace{\sigma^{-1} \gamma \sigma}_{\text{in } H \text{ as } H \text{ normal}} (x)$

$= \sigma(x)$

$\Rightarrow \sigma^{-1} \gamma \sigma(x) = x$ as $x \in E^H$

$\Rightarrow \sigma x \in E^H$

Now show K/F galois. Spse $f(x) \in F$ galois with $a \in K$ a root)

we know if $a_1, \ldots, a_n$ is the G-orbit of a

$\Rightarrow$ These are all the roots of f (in E)

but since $a \in E^H \xRightarrow{\text{obs}} a_1, \ldots, a_n \in K$

$\Rightarrow f$ splits $/K$ so K is galois $/F$.

Spse, K/F is galois want $H = Gal(E/K)$ is normal

it is once again true, K mapped to itself by elts of G.

K is splitting field of some $f(x)$. It is the unique one contained in E. $\Rightarrow$ if $\sigma \in G \Rightarrow \sigma K$ still spl. field of $f \Rightarrow \sigma K = K$

$\exists$ homo, $\mathrm{Gal}(E/F) \xrightarrow{\varphi} \mathrm{Gal}(K/F)$

$\sigma \longmapsto \sigma|_K$

Kernel of this map is $\mathrm{Gal}(E/K)$

$\Rightarrow$ $\mathrm{gal}(E/K)$ is normal.

This is surjective by counting & transitivity of degree.

This is exact seq of gr

$$0 \longrightarrow \mathrm{Gal}(E/K) \longrightarrow \mathrm{Gal}(E/F) \xrightarrow{\varphi} \mathrm{Gal}(K/F) \longrightarrow 0$$

holds in gen w/ following tower

$$\text{gal}\left\{\begin{array}{c} E \\ | \\ \left.\begin{array}{c} K \\ | \\ F \end{array}\right\}\text{gal} \end{array}\right.$$