Let R - domain

Recall an elt $x \in M$ ~ R-module is torsion if $\exists\, 0 \neq a \in R$ so $ax = 0$

An R-module is torsion if every elt is torsion.

If M is a free R-module $\Rightarrow$ M is torsion free, i.e. $M_{tor} = 0$

e.g. $\mathbb{Z} \oplus \mathbb{Z}/_{2\mathbb{Z}}$ is not free as $\mathbb{Z}$ module as it has non-zero

Free $\Rightarrow$ torsion free but reverse implication not always true.

E.g. The ideal (x,y) in $\mathbb{C}[x,y]$ is a torsion free $\mathbb{C}[x,y]$ module but not free.

Thm if M is a fin gen'd torsion-free $\mathbb{Z}$ module
$\Rightarrow$ M is free
(can repl. $\mathbb{Z}$ PID.)

Rmk Finite gen'd is necessary.

eg $\mathbb{Q}$ is a $\mathbb{Z}$-mod
Its torsion free
Not free (Reason: any 2 elts satisfy lin reln
eg $3 \cdot \frac{1}{3} + (-2) \cdot \frac{1}{2} = 0$
$\Rightarrow$ basis could only have 1 elt
$\Rightarrow$ $\mathbb{Q} \cong \mathbb{Z}$ oops $\square$

Defn R-ring $M_1, \ldots$ R-mods

inf dir prod $\leftarrow$ $\prod_{n=1}^{\infty} M_n$ = all tuples $(x_1, x_2, \ldots)$ $x_i \in M_i$ & coordwise ops.

inf dir sum $\leftarrow$ $\bigoplus_{n=1}^{\infty} M_n \subseteq \prod_{n=1}^{\infty} M_n$ is the ab fin $\neq$ subset.

eg $\prod_{i=1}^{\infty} \mathbb{Z}$ is not free

$\bigoplus_{n=1}^{\infty} R$ is free
basis $(0 \ldots, 1, 0 \ldots)$

$\hookrightarrow$ std basis doesn't work as req fin sums for spanning

R-domain – Localize

Recall $Frac(R)$ is a field, and all elts have form $\frac{a}{b}$, $a,b \in R$ $b \neq 0$.

A multiplicative set $S \subset R$ is a subset containing 1 and closed under mult

localize

$S^{-1}R = \{ \frac{a}{b} \mid a \in R, s \in S \} \subset Frac(R)$ (say $0 \notin S$) else boring

ideal given an R mod M
Create an $S^{-1}R$-mod $S^{-1}M$

Def elts of $S^{-1}M$ are rep'd by expression

$\frac{m}{s}$ $m \in M$, $s \in S$ say $\frac{m_1}{s_1} = \frac{m_2}{s_2}$

if $\exists$ $s \in S$ if $s(s_2 m_1 - s_1 m_2) = 0$

since unit·sm=0 ⇒ sm=0 as s unit

intuition (since $\frac{m}{1} = \frac{sm}{s}$ $\forall m$

so $\frac{m}{1} = 0$ if $\exists s \in S$ if $sm = 0$

m torsion by sm in S

This is a $S^{-1}R$ module via

$$\frac{a}{s_1} \cdot \frac{m}{s_2} = \frac{am}{s_1 s_2}$$

$$\frac{m_1}{s_1} + \frac{m_2}{s_2} = \frac{s_2 m_1 + s_1 m_2}{s_1 s_2}$$

E.g ① $R = \mathbb{Z}$ $S = \{1, 2, 4, \dots\}$

$M = \mathbb{Z}/2\mathbb{Z}$

$S^{-1}M = 0$ $\forall x \in M$ $2x = 0 \Rightarrow \frac{x}{1} = 0$ in $S^{-1}M$ as $2 \in S$.

so $\frac{x}{s} = \frac{1}{s} \cdot \frac{x}{1} = \frac{1}{s} \cdot 0 = \boxed{0}$

② $R = \mathbb{Z}$ $S = \{1, 2, 4, \dots\}$

$M = \mathbb{Z}/3\mathbb{Z}$

$S^{-1}M = \mathbb{Z}/3\mathbb{Z}$ as S already units in $\mathbb{Z}/3\mathbb{Z}$

$\frac{x_1}{s_1} = \frac{1}{2} \stackrel{?}{=} \frac{2}{1} = \frac{x_2}{s_2}$ $s_2 x_1 - s_1 x_2 = 1 \cdot 1 - 2 \cdot 2 = -3 = 0$

So $\frac{1}{4} = \frac{4}{1} = \frac{1}{1}$

So all elt of $S^{-1}M$ is $0, \frac{1}{1}, \frac{2}{1}$! (think)

③ $R = \mathbb{Z}$ $S = \{1, 2, 4, \dots\}$

$M = \mathbb{Z}/6\mathbb{Z}$

$S^{-1}M \cong \mathbb{Z}/3\mathbb{Z}$

④ $R = \mathbb{Z}$ $S = \mathbb{Z} \setminus \{0\}$

$M = \mathbb{Z}^n$

$S^{-1}M = \mathbb{Q}^n$ $S^{-1}R = \mathbb{Q}$

Rmk | R is a domain & $S = R \setminus \{0\}$ $K = S^{-1}R = \mathrm{Frac}(R)$

if M is any R-module

$\Rightarrow$ $S^{-1}M$ is a $S^{-1}R$ module i.e. K vec space!

Spse | $f : M \longrightarrow N$ is a map of R-mods homo.

This induces homomorphism of $S^{-1}R$ modules.

$$S^{-1}M \xrightarrow{S^{-1}f} S^{-1}N$$
$$\frac{x}{s} \longmapsto \frac{f(x)}{s}$$

fun

If $g : N \to L$ is another R-mod map

$\Rightarrow$ $S^{-1}(g \circ f) = (S^{-1}g) \circ (S^{-1}f)$

$$\begin{tikzcd} S^{-1}M \arrow[r, "S^{-1}f"] \arrow[rr, bend right, "S^{-1}(g\circ f)"'] & S^{-1}N \arrow[r, "S^{-1}g"] & S^{-1}L \end{tikzcd}$$