Lec 6 |

Rmk | if R is a comm ring & I, J are ideals then IJ is the ideal gen'd by ab w/ $a \in I$, $b \in J$

↳ (note $IJ \subseteq I \cap J$)

Last time | defined UFD

Say R is a domain. To be a UFD need

1) Every elt (not 0 or unit) factors into irred

2) This factorization is appropriately unique.

↳ (1) Can fail

Attempted eg $\mathbb{C}[x_1, x_2, \dots]$

↳ this does factor as polynomials uses finitely many variables.

(This is actually a UFD)

E.g. | R = cts fnc $[-1,1] \to \mathbb{C}$

↳ addition is ptwise

↳ multiplication → ptwise → This is not a domain

↳ consider bump fnc.

but, S = analytic $\mathbb{R} \to \mathbb{C}$

Mult ptwise

↳ this is an integral domain.

↳ can show irreducibles are $x-a$ for $a \in \mathbb{R}$ as the units are non zero functions

but if you have infinite zero multiplicity you can keep factoring

$$\sin(x) = x \cdot \frac{\sin(x)}{x} = x \cdot (x - 2\pi) \frac{\sin(x)}{x(x-2\pi)} \cdots$$

↳ (1) fails

(1) Also fails $S = \mathbb{C}[x^{1/2^k}]_{1 \le k} = \dfrac{\mathbb{C}[y_1, \dots]}{(y_i^2 - y_{i-1})}$

$$x = x^{1/2} \cdot x^{1/2} = x^{1/4} \cdot x^{1/4} \cdot x^{1/4} \cdot x^{1/4} = \dots$$

Prop | R any ring TFAE

(a) Ideals of R satisfy the ascending chain condition (ACC)

$I_1 \subset I_2 \subset \cdots \; \exists n$ s.t $I_n = I_{n+1} = \cdots$.

(b) Every ideal is finitely gen'd

Pf | (b) $\Rightarrow$ (a)

Consider ascending chain of ideals!

Put $J = \bigcup_{j \in \mathbb{N}} I_j$ this is an ideal (earlier)

it is fin gen'd $\rightarrow$ $\sum_{i \in \mathbb{N}} a_i x_i \; | x_i \in R \}$

$J = (a_1, \ldots, a_r)$ for $a_1, \ldots, a_r \in R$

$\exists n$ s.t $a_1, \ldots, a_r \in I_n \Rightarrow J \subseteq I_n \Rightarrow J = I_n$

$\Rightarrow I_{n+1} = I_{n+2} = \cdots = J$

!(b) $\Rightarrow$!(a)

$\exists$ ideal J not finitely gen'd $\rightarrow$ construct strict chain.

Pick, $a_1 \in J$ $(a_1) \subsetneq J$ $\therefore \exists a_2 \in J/(a_1)$

$\vdots \; \exists a_3 \in J/(a_1, a_2)$

let $I_j = (a_1, \ldots, a_j)$ $\longrightarrow$ strictly ascending chain by

Def | A comm ring is called Noetherian if above holds!

very important!

E.g | A principle ideal domain (PID every ideal gen'd by single) is Noetherian!

Prop | let R = noetherian domain

let $x_1, x_2, \ldots \in R$ be elts

s.t $x_2 | x_1, \; x_3 | x_2, \ldots$

Then $\exists n$ s.t $x_n \sim x_{n+1}, \; x_{n+1} \sim x_{n+2}$ $\sim$ is associate (diff by unit)

Pf) if $x \mid y \Rightarrow (y) \subseteq (x)$

so

$(x_1) \subseteq (x_2) \subseteq \cdots$

By ACC $\exists\, n$ s.t $(x_n) = (x_{n+1}) = \cdots$

Think $\Rightarrow$ $x_n \sim x_{n+1}$, $x_{n+1} \sim x_{n+2}, \ldots$

Prop) if R is a noeth domain. Then everything factor into irred

Pf) let $x \in R$ be given, say $x \neq 0$, unit

Cl) x divisible by some irred elt

follows from prior

Now, choose irred $\pi_1 \mid x \rightarrow x = \pi_1 x_2$

$\pi_2 \mid x_2 \quad x = \pi_1 \pi_2 x_3$

Go process stops by prev !

Obs) say $p \geq 2$ is an int TFAE. $x, y \in \mathbb{Z}$

(a) if $p = xy$ then $x = \pm 1$ or $y = \pm 1$

(b) $p \mid xy \Rightarrow p \mid x$ or $p \mid y$

Rmk In gen ring R (a) leads to idea of irred

(b) leads to idea of prime elt.

Def) let R ring. Let $\pi \in R$ not 0 or unit.

we say π is prime if

$\pi \mid xy \Rightarrow \pi \mid x$ or $\pi \mid y$.

Obs) prime $\Rightarrow$ irred.

let π prime $\quad \pi = xy \overset{\text{prime}}{\Rightarrow} \pi \mid x$ or $\pi \mid y$

but $x \mid \pi$ and $y \mid \pi$

$\Rightarrow x \sim \pi$ or $y \sim \pi$

Prop Let R be a domain where all elts have irred factors. TFAE

1) R is a UFD

2) All irreds are prime.

Pf (b) $\Rightarrow$ (a) Say $x = \pi_1 \cdots \pi_r = \pi_1' \cdots \pi_s'$ are 2 factorizations of x

as $\pi_r \mid x \Rightarrow \pi_r \mid \pi_1' \cdots \pi_s'$ ← irred is prime.

$\Rightarrow \pi_r \mid \pi_1'$ or $\pi_r \mid \pi_2'$ or $\ldots$

so $\exists i$ so $\pi_r \mid \pi_i' \Rightarrow \pi_r \sim \pi_i'$ ← as irred

Now cancel π_r, π_i' continue by induction

(a) $\Rightarrow$ (b)

Let π be irred

WTS $\pi \nmid x, \pi \nmid y \Rightarrow \pi \nmid xy$

$$\left.\begin{aligned} x &= \sigma_1 \cdots \sigma_r \\ y &= \sigma_1' \cdots \sigma_s' \end{aligned}\right\} \text{irred facts}$$

π cannot be assoc to any σ_i', σ_j

$xy = \sigma_1 \cdots \sigma_r \sigma_1' \cdots \sigma_s'$

Since π not ↑ & this is the only irred fact of xy.

(b/c R is UFD) $\Rightarrow \pi \nmid xy$