

last time

- $G \subset \text{Aut}(K)$  finite subgroup.  $K/\mathbb{Q}_G$  is Galois w/ grp  $G$
  - If  $E/F$  is any finite ext  $\Rightarrow \{\text{F emb } \mathbb{Z} \rightarrow \mathcal{O}_E\} = [E:F]$
- $\rightarrow \#\text{Gal}(E/F) \leq [E:F]$
- From a while ago
- $\rightarrow$  if  $E/F$  splitting field  $\Rightarrow E/F$  Galois.
- $E/F$  is Galois  $\Rightarrow E^{\text{Gal}(E/F)} = F$

Prop Say  $E/F$  is finite, TFAE

- $E/F$  Galois
- $E/F$  is a splitting field
- if  $f(x) \in F[x]$  irred +  $f$  has a root in  $F$   
 $\Rightarrow f$  has all roots in  $E$

Pf b  $\Rightarrow$  a abv

a  $\Rightarrow$  c let  $f(x) \in F[x]$  be irred + Spec  $\exists a \in E$   
 $\text{so } f(a) = 0$ .

let  $\sigma \in \text{Gal}(E/F)$  then  $\sigma(a)$  is a root of  $f$   
as  $f(\sigma(a)) = \sigma(f(a)) = \sigma(0) = 0$ .

let  $a_1, \dots, a_n$  be the orbit of  $a$  under  $\text{Gal}(E/F)$

let  $g(x) = \prod_{i=1}^n (x - a_i)$  we know  $g(x) | f(x)$

important idea

key pt coeff of  $g(x)$  are symmetric poly in  $a_1, \dots, a_n$   
 $\Rightarrow$  coeff are fixed by  $\text{Gal}(E/F)$   $\Rightarrow$  coeff in  $F$   
 $\Rightarrow g(x) \in F[x] \Rightarrow g(x) = f(x)$  as irred.  $\square$ .

The  $a_i$ 's are all the roots of  $f$  & in  $E$

C  $\Rightarrow$  b) Since  $E/F$  finite  $\Leftrightarrow F(a_1, \dots, a_n) = E$

Let  $f_i(x)$  be min poly of  $a_i$  in  $F(x)$  & irred.

By C,  $f_i(x)$  splits  $E$  into linear factors.

$\Rightarrow$  same is the for  $\prod_{i=1}^n f_i(x)$

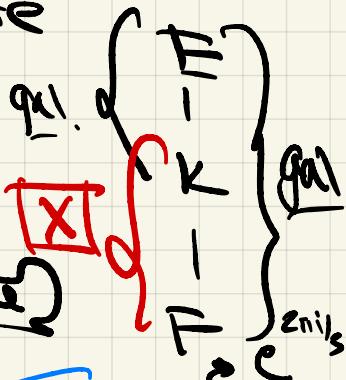
$\therefore E$  splitting by  $\prod_{i=1}^n f_i(x)$

Cor (a2  $\Rightarrow$  b) If  $E/F$  is galois &  $K$  intermediate  
 $\Rightarrow E/K$  is galois.

RA)  $E$  is spl. of  $f(x) \in F[x]$

$\Rightarrow E$  is spl. of  $f(x) \in K[x] \Rightarrow$  same as

Spl.  $\Leftrightarrow$  galois D.



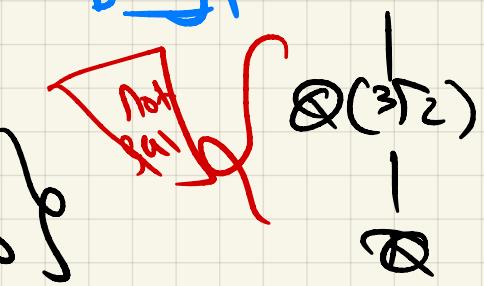
Main thm) If  $E/F$  gal  $\Leftrightarrow$  gp  $G$

$\{\text{subgps of } G\} \xrightarrow{\Phi} \{\text{intermediate fields}\}$

$$\Phi(H) = E^H$$

$$\Psi(K) = \text{Gal}(E/K)$$

w/  $\Phi, \Psi$  inverse



Spec!  $H \subset G \Leftrightarrow \Psi(\Phi(H)) = H$

$$\text{Gal}(E/E^H) \stackrel{\text{last class}}{=} H^*$$

next,  $\Phi(\text{Gal}(K)) := \Phi(\text{Gal}(E/K)) := E^{\text{Gal}(E/K)}$

$\text{as } E/K \text{ is galois} \Rightarrow K = E^{\text{Gal}(E/K)} \Rightarrow \Phi(\text{Gal}(E/K)) = K$

The core is order reversing

$$\text{if } H_1 \subset H_2 \subset G \Rightarrow \Phi(H_1) \supset \Phi(H_2) \quad \boxed{\text{clear from defn}}$$

$$\text{if } F \subset K_1 \subset K_2 \subset E \xrightarrow{\text{inter}} \Psi(K_1) \supset \Psi(K_2)$$

$$\text{Gal}(E|K_1) \quad \text{Gal}(E|K_2)$$

Let  $H \subset G$  &  $K = \overline{\Phi}(H) = E^H$

wts  $H$  is normal subgroup  $\Leftrightarrow K/F$  is galois.

Pf Since,  $H$  is normal.

Obs every elt  $G$  maps  $K = E^H$  into itself,

Pf let  $x \in E^H$  wts  $\sigma(x) \in E^H$

$$\text{let } \tau \in H \quad \Psi(\tau(x)) = \underbrace{\tau \sigma^{-1} \tau \sigma}_{\text{in } H \text{ as } H \text{ normal}}(x)$$

$$= \sigma(x)$$

$$\Rightarrow \tau^{-1} \tau \sigma(x) = x \text{ as } x \in E^H$$

$$\Rightarrow \tau x \in E^H$$

Now show  $K/F$  galois. Since  $f(x) \in F$  galois with  $a \in K$  a root

we know if  $a_1, \dots, a_n$  is the  $G$ -orbit of  $a$

$\Rightarrow$  these are all the roots of  $f$  (in  $E$ )

but since  $a \in E^H \xrightarrow{\text{obs}} a_1, \dots, a_n \in K$

$\Rightarrow f$  splits  $/K \Leftrightarrow K$  is egalois  $/F$ .

Spose,  $K/F$  is galois want  $H = \text{Gal}(E|K)$  is normal

it is once again true,  $K$  mapped to itself by elts of  $G$ .

$K$  is splitting field of some  $f(x)$ . It is the unique one contained in  $E$ :  $\Rightarrow$  if  $\tau \in G \Rightarrow \tau K$  still spl. field by  $f \Rightarrow \tau K = K$

$$\exists \text{ ram}, \text{ Gal}(E/F) \xrightarrow{\varphi} \text{Gal}(K/F)$$

$$F \longrightarrow \mathcal{G}|_K$$

Kernel of this map is  $\text{Gal}(E/K)$

$\Rightarrow \text{gal}(E/K)$  is normal.

This is surjective by counting &

transitivity of degree.

$\hookrightarrow$  This is exact seq if

$$0 \longrightarrow \text{Gal}(E/K) \longrightarrow \text{Gal}(E/F) \xrightarrow{\varphi} \text{Gal}(K/F) \rightarrow 0$$

holds in gen w/ following notes

