Def Let M a R module. Given any $S \subset M$ the R-submod gen by S is the smallest submodule containing S

$$= \bigcap_{S \subset N} N$$

The elts of this submod are $a_1x_1 + \dots + a_nx_n \quad a_i \in R, \; x_n \in S$

Def We say M is fin gen if gen'd by some finite subset.

E.g 1) if R a field & $M = R$ mod (i.e vec sp) then M f.g as R mod $\iff$ it fin gen'd!

2) R^n is f.g w/ basis $e_1, \dots, e_n$

3) If M, N fin gen so is $M \oplus N$

4) $R[x]$ is not a fg R mod $(R \neq 0)$

5) $\mathbb{Q}$ is not f.g as a $\mathbb{Z}$-mod.

Mapping Prop for free modules

Given any R-module M & $x_1, \dots, x_n \in M$ $\exists!$ map of R-mods

$$\varphi : R^n \to M$$
$$e_i \to x_i$$

Moreover $\operatorname{im}\varphi$ = sub mod gen by $\{x_i\}_{i \in \mathbb{N}_n}$

Pf $\varphi(a_1e_1 + \dots + a_ne_n) = a_1x_1 + \dots + a_nx_n$

Cor M is fin gen $\iff$ $\exists$ surj map of R-mod $\varphi : R^n \to M$

Say M is fg module.

Choose surj $\varphi : \mathbb{R}^n \to M$

1^{st} isom $\Rightarrow M \cong \mathbb{R}^n / \ker\varphi \to$ if $\ker\varphi$ is gen'd $y_1, \dots y_m$ — Might not be fg

$$\Rightarrow M = \frac{\mathbb{R}^n}{\langle y_1, \dots, y_m \rangle}$$

Def We say M is finitely presented if $M \cong \frac{R^n}{\langle y_1, \dots, y_m \rangle}$ for some n & some $y_1, \dots, y_m \in R^n$ ↳ presentation

E.g. 1) $R = \mathbb{Z}$

$M = \mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/7\mathbb{Z} \quad (\cong \mathbb{Z}/35\mathbb{Z})$

gen by $(1,0)$ & $(0,1)$ $\Rightarrow$ M is f.g.

we have

$$\begin{aligned} \mathbb{Z}^2 &\xrightarrow{\varphi} M \\ e_1 &\longmapsto (1,0) \\ e_2 &\longmapsto (0,1) \end{aligned}$$

$\ker \varphi = \{k5e_1 + j7e_2\} = \langle 5e_1, 7e_2 \rangle$

$M = \frac{\mathbb{Z}^2}{\langle 5e_1, 7e_2 \rangle} \left(\cong \frac{\mathbb{Z}}{\langle 35 \rangle} \right)$

2) $R = \mathbb{C}[x,y]$

$M = (x,y) \subset R$

M is fin gen as it is gen'd by x & y.

$$\begin{aligned} \varphi : R &\longrightarrow R \\ e_1 &\longmapsto x \\ e_2 &\longmapsto y \end{aligned} \qquad M = \operatorname{im} \varphi$$

is $\ker \varphi$ f.g?

multiple

$-ye_1 + xe_2 \in \ker \varphi$

as $\varphi(-ye_1 + xe_2) = -y\,\varphi(e_1) + x\,\varphi(e_2)$

$= -yx + xy = 0$

In general $x^n y^m e_1 - x^{n+1} y^{m-1} e_2 \in \ker(\varphi)$ by factorizing

if $fe_1 + ge_2 \in \ker \varphi \Rightarrow \varphi(fe_1 + ge_2) = 0$

$\Rightarrow fx + gy = 0 \Rightarrow x \mid gy \Rightarrow x \mid g$ and work to show

So, $\ker \varphi$ is multiples of $-ye_1 + xe_2$

$\Rightarrow \ker \varphi = \langle -ye_1 + xe_2 \rangle$ so M is fin pres

Say $M \cong \dfrac{R^n}{\langle y_1, \dots, y_m \rangle}$

Let $\psi : R^m \longrightarrow R^n$
$e_j \longrightarrow y_j$

$\operatorname{im}(\psi) = \langle y_1, \dots, y_m \rangle \subset R^n$

so $M \cong \operatorname{coker}(\psi)$ [Recall: $\varphi : M \to N$
$\operatorname{coker} \varphi = N/\operatorname{im} \varphi$

↳ given by $m \times n$ matrix
w/ coeff in R (presentation matrix)

e.g. 1) $R = \mathbb{Z}$
$M = \mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/7\mathbb{Z}$

$$M \cong \frac{\mathbb{Z}^2}{\langle 5e_1, 7e_2 \rangle} \cong \frac{\mathbb{Z}}{\langle 35 \rangle}$$

pres $\begin{bmatrix} 5 & 0 \\ 0 & 7 \end{bmatrix}$ pres $[35]$

2) $R = \mathbb{C}[x,y]$ $M = (x,y)$

$$M = \frac{R^2}{\langle -ye_1 + xe_2 \rangle}$$

$\psi : R \longrightarrow R^2$
$e_1 \longrightarrow -ye_1 + xe_2$

$\begin{bmatrix} -y \\ x \end{bmatrix}$ is pres matrix

3) Let $R = \mathbb{Z}$

M is module w/ pres matrix

$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

$M = \operatorname{coker} \psi$ $\qquad \psi : \mathbb{Z}^2 \to \mathbb{Z}^2$

$e_1 \mapsto e_1 + 2e_2$

$e_2 \mapsto 2e_1 + e_2$

So $M \cong \dfrac{\mathbb{Z}^2}{\langle e_1 + 2e_2, 2e_1 + e_2 \rangle}$

Claim: Not zero as $\det A = -3$ — any int mult lin sum of coeff mult 3, so it's not inv over $\mathbb{Z}$

Claim: $M \cong \mathbb{Z}/3\mathbb{Z}$

Say M is a fin gen R-mod

$\exists\ \varphi : R^n \twoheadrightarrow M$ surj

For M to be fin presented want $\ker \varphi$ fin gen.

(this is not true in general)

Problem! It's possible that submod of fg mod not fg

Ex) $R = \mathbb{C}[x_1, x_2, \ldots]$

$M = (x_1, x_2, \ldots)$

R fin gen as R module but M not

$R/M \cong \mathbb{C}$ (only const stay)

$\mathbb{C}$ is fin gen but not fin presented!