Last time

- Finite gen
- Finite pres ✓ better

Recall A finitely gen module may not be fin pres!

Say M is fin gen $\Rightarrow$ $\exists$ surj $\varphi: R^n \to M$

Prob $\ker \varphi$ may not be fin gen

Defn A R-mod M is noetherian if the follow equiv condt holds

(1) Every submod of M is fin gen

(2) ACC holds for submods

i.e $N_1 \subset N_2 \subset \dots M$ is ascending chain of submod

$\exists\, n \in \mathbb{N}$ so $\forall\, m \geq n$ $N_m = N_n$

Rmk R is noeth (as a ring) iff R is noeth (as an R-mod)

Spose R^n is a noeth R mod $\forall\, n \in \mathbb{N}$

$\Rightarrow$ every fin gen R-mod is fin presented!

(in fact converse holds for contra pos take the quotient module!)] ?

Prop Spose R is noeth, then every fin gen R-module is noeth mod

[In particular R^n is noeth $\Rightarrow$ every fin gen mod is fin pres as $\ker \varphi$ fin gen]

Defn Consider maps of R-mods

$$M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2} M_3$$

we say seq abv is exact (at M_2)

if $\operatorname{im} \varphi_1 = \ker M_3$

- $0 \xrightarrow{\varphi} M_2 \xrightarrow{\psi} M_3$ exact at $M_2 \iff \ker \psi = 0$ $\Rightarrow \psi$ inj

- $M_1 \xrightarrow{\varphi} M_2 \xrightarrow{\psi} 0$ exact at $M_2 \iff \varphi$ surj

Defn! A short exact sequence is a seq is a seq

$$0 \to M_1 \xrightarrow{\varphi} M_2 \xrightarrow{\psi} M_3 \to 0$$

exact at M_1, M_2, M_3

$\Rightarrow$
- φ is inj
- ψ is surj
- $M_3 \cong \frac{M_2}{\varphi(M_1)} = \operatorname{coker} \varphi$ by 1^{st} isom

We say M_2 is an extension of M_3 by M_1

$\hookrightarrow$ dir sum always

e.g. ① $R = \mathbb{Z}$, $M_1 = M_3 = \mathbb{Z}/2\mathbb{Z}$ possibility of extension

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \xrightarrow{\text{reduce}} \mathbb{Z}/2\mathbb{Z} \to 0$$

$$1 \longmapsto 2$$

since everything is finite $|M_2| = |M_1| \cdot |M_2|$

$$0 \longmapsto \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

M_3 is a quotient of M_2 & M_1 is a subgrp of M_2

② $$0 \to \mathbb{Z}/3\mathbb{Z} \to \mathbb{Z}/15\mathbb{Z} \xrightarrow{\text{red mod } 5} \mathbb{Z}/5\mathbb{Z} \to 0$$

$$1 \longmapsto 5$$

since $\mathbb{Z}/15\mathbb{Z}$ unique extension as unique grp of ord 15

$\hookrightarrow$ note can flip $\to$ not always symm

③ $R = \mathbb{Z}$ $M_1 = \mathbb{Z}/2\mathbb{Z}$ $M_3 = \mathbb{Z}$

Only ext is $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$

Pick $x \in M_2$ w/ $\varphi(x) = 1$ $\Rightarrow M_2 = \varphi(\mathbb{Z}/2) \oplus \langle x \rangle$

but if flip!

$M_2 = \mathbb{Z}$ φ = mult by 2

ψ = red mod 2

or $M_2 = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$

$\to$ not symm

e.g.| if $R = F$ is a field
Suppose $0 \to V_1 \to V_2 \to V_3 \to 0$
is a SES of fin dim VS
$\Rightarrow$ ① $\dim V_1 - \dim V_2 + \dim V_3 = 0$ $\to$ cause of modding out
↳ kind like rank nullity
② $0 \to V_3^* \to V_2^* \to V_1^* \to 0$
is SES of duals

Lemma| Suppose we have a SES
$$0 \to M_1 \xrightarrow{\varphi} M_2 \xrightarrow{\psi} M_3 \to 0$$
s.t. M_1 & M_3 are fin gen $\Rightarrow$ M_2 SES.
equiv
if $M_2 \supseteq M_1$ where M_1 fin gen $\Rightarrow$ M_2/M_1
fin gen $\Rightarrow$ M_2 fin gen'.

Pf| Pretend φ is inclusion by replacing M_1 w/ $\varphi(M_1)$
↳ so assume $M_1 \subset M_2$ & φ incl
let $x_1 \ldots\ldots x_n$ gens for $M_1 \subset M_2$
$\overline{y_1} \ldots\ldots y_m$ gens for M_3
Choose $y_i \in M_2$ so $\psi(y_i) = \overline{y_i}$ (by surj)
Cl. $x_1, \ldots, x_n$ & $y_1, \ldots, y_m$ gen M_2
let $z \in M_2$ be given
we see $\psi(z) = b_1\overline{y_1} + \cdots + b_m\overline{y_m}$ where $b_j \in R$
Note $\psi(z - b_1 y_1 + \cdots + b_m y_m) = 0$
$\Rightarrow z - b_1 y_1 + \cdots + b_m y_m \in \ker \psi = M_1$
$\Rightarrow z - b_1 y_1 + \cdots + b_m y_m = a_1 x_1 + \cdots + a_n x_n$

Lemma) Spse we have SES

$$0 \to M_1 \xrightarrow{\varphi} M_2 \xrightarrow{\psi} M_3 \to 0$$

if M_1, M_3 noeth $\Rightarrow$ M_2 noeth

Cor) if M_1, M_3 noeth, so is $M_1 \oplus M_3$ as it is an ext

Pf) Assume $M_1 \subset M_2$, φ incl

let $N_2 \subset M_2$ submod

Put $N_3 = \psi(N_2) \subset M_3$

$\ker(\psi : N_2 \to N_3) = \underbrace{N \cap M_1}_{N_1}$

have SES

$$0 \to N_1 \to N_2 \to N_3 \to 0$$

by Noeth of $M_1, M_3 \Rightarrow N_1, N_3$ fin gen

$\Rightarrow$ by prev N_2 fin gen $\Rightarrow$ M_2 noeth

Lemma) Spse have a SES (try submod & quot of noeth are noeth)

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

s.t M_2 noeth $\Rightarrow$ M_1 noeth, M_3 noeth

Pf) Any submodule of M_1 is submod of M_2

$\Rightarrow$ it is fin gen as M_2 is noeth

$\Rightarrow$ M_1 is noeth

Say $N_3 \subset M_3$ is submod

$\Rightarrow \psi^{-1}(N_3) \twoheadrightarrow N_3$ by surj of ψ

fin gen b/c M_2 is noeth

fin gen as quotient of f.g

$\Rightarrow$ M_3 noeth

Pf of Prop 1 Let M = fin gen R mod

$\exists$ surj $\varphi : R^n \twoheadrightarrow M$

$R^n = \underbrace{R \oplus \cdots \oplus R}_{n}$ is noeth (b/c $\oplus$ of noeth is noeth)

$\Rightarrow$ M is noeth as quotient of R^n

Eg) Any $\mathbb{Z}$ submod of $\mathbb{Z}$ is fin gen!