Generalization of 1/a

Let R = comm ring

Def A multiplicative set in R is a $S \subseteq R$ s.t. $1 \in S$ & $x, y \in S \Rightarrow xy \in S$

Def Given a mult set $S \subseteq R$, the localization $S^{-1}R$ is $R[\frac{1}{s}]_{s \in S}$ iterated version of $R[\frac{1}{a}]$ from earlier

E.g. 1) Say R is a domain & $0 \notin S$
$\Rightarrow S^{-1}R$ is the subring of $\mathrm{Frac}(R)$ consisting of elts $\frac{x}{s}$ w/ $x \in R, s \in S$

2) $R = \mathbb{Z}$, fix prime p S = all integers coprime to p

$\mathbb{Z}_{(p)} = S^{-1}R$ is subring $\mathbb{Q}$ of fractions w/ denom coprime to p

Q What are the ideals of $\mathbb{Z}_{(p)}$?
They are (p^n) for $n \geq 0$

(cool) $\mathbb{Z}_{(p)} \subset \mathbb{Z}_p$ ⇝ p-adics in fact $\mathbb{Z}_p \cap \mathbb{Q} = \mathbb{Z}_{(p)}$

Last time:
$\frac{\mathbb{F}_7[x]}{(x^2+1)}$ this is a field with 49 elts!

Q When is R/I a field? (what does this say abt I?)

Def A maximal ideal of a ring R is a proper ideal that is not strictly contained in any other proper ideal!

Lemma A ring is a field $\iff$ it has 2 ideals $(0), (1)$

$\Rightarrow$ if F is a field & I is a nonzero ideal.
$\Rightarrow x \in I$ $x \neq 0$ $\Rightarrow x x^{-1} = 1 \in I \Rightarrow I = (1)$

$\Leftarrow$ Say F is a ring w/ exactly 2 ideals.
Must be (0) and (1). Say x is a nonzero elt.

we note $(x) \neq (0) \Rightarrow (x) = (1) \Rightarrow \exists y$ so $xy = 1$ $\rightarrow y = x^{-1}$ $\square$

Prop) Spse $I \subset R$ ideal
I is maximal $\iff R/I$ is a field

Pf) Ideal corr thm:

$$\left\{ \begin{array}{c} \text{Ideals of } R \\ \text{containing } I \end{array} \right\} \longleftrightarrow \{ \text{ideals of } R/I \}$$

This has 2 elts $\iff I$ is maximal

has 2 elts $\iff R/I$ a field

Exercise) What are the ideals of $\mathbb{Z}$?
Every ideal of $\mathbb{Z}$ has the form (n) for some n
$\mathbb{Z}/n\mathbb{Z}$ is a field $\iff n$ is prime
$\Rightarrow$ Max ideals of $\mathbb{Z} = \{(p) \mid p \text{ prime}\}$

Exercise) $R = \mathbb{Q}[x]$

$$\frac{\mathbb{Q}[x]}{x^2+1} = \mathbb{Q}[i] = \mathbb{Q}(i)$$

$\mathbb{Q}[i] = \{a+bi \mid a,b \in \mathbb{Q}\}$

$\mathbb{Q}(i)$: smallest subfield in $\mathbb{C}$ containing $\mathbb{Q}$ & i

$\Rightarrow (x^2+1)$ is maximal as $\mathbb{Q}(i)$ a field!

Aside)

$$\frac{\mathbb{Q}[x]}{((x^2+1)(x-1))} \cong \frac{\mathbb{Q}[x]}{(x^2+1)} \times \frac{\mathbb{Q}[x]}{(x-1)}$$

non-irred $\rightarrow$ generalized chinese rem thm

$\Rightarrow$ not a field so ideal not maximal

Prop) if R a ring & $I \subset R$ a proper ideal
$\Rightarrow \exists J$ maximal ideal in R so $I \subset J$

Pf) Zorn's Lemma stuff. let $\Sigma = \{\text{proper ideals } J \text{ s.t. } J \supseteq I\}$
Partial order by containment $\rightarrow$ wts Σ has max elt

Suppose $J_1 \subset J_2 \subset \cdots$ is a chain Σ

Put $J = \bigcup_{i=1}^{\infty} J_i$ ~> bdd by this

$\Rightarrow$ this contains I

$\hookrightarrow$ but none J_i have 1 as they are proper

$\Rightarrow 1 \notin J_i \ \forall i \Rightarrow 1 \notin J \Rightarrow J$ proper

$\Rightarrow J$ bounds chain $\Rightarrow$ Can apply Zorn! ☺

Consider $R = \mathbb{C}[x_1, \ldots, x_n]$

- $(x_1, \ldots, x_n)$ is maximal as $\dfrac{R}{(x_1, \ldots, x_n)} = \mathbb{C} \to$ field!

- Given some $a \in \mathbb{C}^n$ $J_a = (x_1 - a_1, \ldots, x_n - a_n)$
  as $R/J_a = \mathbb{C}$

Thm Every maximal ideal R is one of the J_a's

Pf Let J be a maximal ideal.
Let $K = R/J$ a field!

$\mathbb{C} \xhookrightarrow[\text{incl.}]{} R \xrightarrow[\text{surj}]{\pi} K$ (composite is injective as $\mathbb{C}$ is a field)

$\mathbb{C} \subset K$ by above embedding!

What fields contain $\mathbb{C}$ as a subfield?

E.g. $\mathbb{C}$, $\mathbb{C}(x)$, $\mathbb{C}(x_1, \ldots, x_n)$
ratl' fns
$\hookrightarrow$ this is the smallest