We know that:

$\mathbb{Q}(2^{1/3})$ is Not a galois extn!

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Say that F is a char 0 field & F contains all $p^{th}$ roots of unity (p prime)

**Prop** $\forall a \in F$ ($a \neq p^{th}$ power)

$F(a^{1/p})$ is galois (since we alr have roots of 1)

$Gal(F(a^{1/p})/F) \cong \mathbb{Z}/(p)$

**Pf)** $a^{1/p}$ is a root of $f(t) = t^p - a = \prod_{i=1}^{p} (t - \zeta^i a^{1/p})$

By assump, $\zeta^i \in F \ \forall i$

So all roots of $f(t)$ belong to $F(a^{1/p})$

$\Rightarrow F(a^{1/p})$ is the splitting field of $f$

Let $\mu_p = \{ \omega \in F \mid \omega^p = 1 \} \cong \mathbb{Z}/p\mathbb{Z}$ ⟶ non canonical

have fnc $\varphi : Gal(F(a^{1/p})/F) \longrightarrow \mu_p$

$\varphi(\sigma) \longmapsto \dfrac{\sigma(a^{1/p})}{a^{1/p}}$

**Note:** $\sigma(a^{1/p})$ this is another $p^{th}$ root of a

$\Rightarrow$ of the form $\zeta^i a^{1/p}$, $\varphi(\sigma) = \zeta^i$

$\varphi$ is grp homo, $\boxed{\text{reason}}$

$\sigma\gamma(a^{1/p}) = \varphi(\sigma\gamma) \cdot a^{1/p}$

$\quad \|$

$\sigma(\varphi(\gamma) a^{1/p}) = \sigma[\varphi(\gamma)\, \sigma(a^{1/p})] = \varphi(\tau) \cdot \varphi(\sigma) \cdot a^{1/p}$

**Note**
$\varphi(\gamma) \in F$ by assump $\Rightarrow$ fixed by $\sigma$

$\varphi$ is inj b/c any $\sigma$ is det by $\sigma(a^{1/p})$

$\varphi$ is surj b/c is is not trivial & $\mu_p$ has no proper non triv subgp.

⟵ ext not triv as $a^{1/p} \notin F$

**Prop** Spose $\mu_p \subset F$ (have roots) & $E/F$ be Galois w/ $\mathrm{gal}(E/F) \cong \mathbb{Z}/(p)$
$\Rightarrow E \cong F(a^{1/p})$ for some $a \in F$

**Pf** let $\sigma$ gen. $\mathrm{Gal}(E/F)$ consider $F$ lin operator $E \to E$
$\sigma^p = 1$ as $\sigma \in \mathbb{Z}/(p)$
1° eigen values of $\sigma$ are $p^{th}$ roots of unity
↳ in particular belong to $F$.
2° can diagonalize $\sigma | F$ (can diag finite order matrices + its evs are in $F$)

we know $\sigma \neq 1$
$\exists$ eval $\omega \in \mu_p$ of $\sigma$ so $\omega \neq 1$
Say $b \in E$ is an eigenvect
$\sigma(b) = \omega b \Rightarrow \sigma(b)^p = (\omega b)^p = b^p$
$\sigma(b)^p = \sigma(b^p)$
as $\sigma$ gens gal grp $\Rightarrow$ gal grp fixes $b^p \Rightarrow b^p \in F$.
Note: $b \notin F$ as $\sigma(b) \neq b$ as $\omega \neq 1$
$E = F(b)$ (b/c degree of inter divides $p$ → $[E:F]$ but $F(b) \neq F$ as $b \notin F \Rightarrow F(b) = E$)
$= F(a^{1/p})$
$\boxed{a = b^p}$

Kummer Thy → spread.

**More Gen** | Say $n > 0$ pos int & $\mu_n \subset F$
$\Rightarrow F(a^{1/n})$ is a gal ext of $F$
+ gal gp $\subseteq \mu_n$
↳ not equal $\mathbb{Z}/n\mathbb{Z}$

Converse true too

Hilbert thm 90

**Prop)** Let $E/F$ be a fin ext (char 0)

$\exists$ a galois extn $M/F$, and $F$-emb $E \hookrightarrow M$

st if $M'/F$ is gal ext w $F$ emb $E \hookrightarrow M'$

$\Rightarrow M \hookrightarrow M'$ $F$ emb

Moreover $M$ is unique up to $F$-isom

**Def)** an $M$ is the galois closure of $E/F$

**Pf)** Let $\Omega/F$ be an alg closure & choose $F$ emb $E \hookrightarrow \Omega$

$E = F(a_1, \dots, a_n)$, $f_i(x)$ min poly of $a_i$

Let $M \subset \Omega$ be gen'd over $F$ by all the roots of $f_i$

$M$ is splitting of $\prod f_i(x) \Rightarrow$ $M/F$ is Galois (also get emb).

Let $M'$ be given. Choose $E$ emb $M' \hookrightarrow \Omega$

Since $M'/F$ gal & $f_i(x)$ has one root in $M'$ (will get $a_i$)

$\Rightarrow f_i(x)$ splits in $M'$

$\Rightarrow M \subset M'$ as gen'd by roots

**E.g.)**

$$\begin{array}{l} F(\sqrt{a+b\sqrt{d}}) = E \\ \quad | \\ F(\sqrt{d}) \\ \quad | \\ F \end{array}$$

$a, b \neq 0$ gal w/ $\mathbb{Z}/2\mathbb{Z}$ (top)

gal w/ $\mathbb{Z}/2\mathbb{Z}$ (bottom)

Might not be gal.

Spse $E/F$ is galois.

Let $\sigma \in Gal(E/F)$ so $\sigma(\sqrt{d}) = -\sqrt{d}$

Put $x = \sqrt{a+b\sqrt{d}} \in E$

$x^2 = a + b\sqrt{d}$

$\sigma(x^2) = a - b\sqrt{d}$

$\sigma(x) = \sqrt{a - b\sqrt{d}}$

$\Rightarrow a - b\sqrt{d}$ square in $E$

↳ in fact converse holds

if $E/F$ not galois $\Rightarrow$ galois closure $F(\sqrt{a+b\sqrt{d}}, \sqrt{a-b\sqrt{d}})$

$$\begin{array}{ll} M & \\ \mid & \Big\} \text{ gal w/ } \mathbb{Z}/2 \oplus \mathbb{Z}/2 \\ F\sqrt{d} & \\ \mid & \Big\} \text{ gal } \mathbb{Z}/2 \\ F & \end{array}$$

ord 8 galois usually not abelian

Rmk 1 if $E/F$ is gal $\Rightarrow$ $\mathrm{Gal}(E/F) = \mathbb{Z}/4$

$F(\sqrt{d})$ sits inside a $\mathbb{Z}/4$ ext iff $d$ = sum of 2 squares in $F$