Last time

Noeth modules

R noeth ring $\iff$ any f.g mod is a noeth mod

{Cor: if R noeth then any f.g mod is fin pres}

Hilbert Basis Thm

if R is noeth ring so is $R[x]$

Cor| R noeth $\Rightarrow$ $R[x_1, \dots, x_n]$ noeth

Cor| if R is noeth, so is any quotient ring of R (saw last time is we consider quot module)

Cor| if R is noeth $\Rightarrow$ any $\underbrace{\text{fin gen } R\text{-alg}}_{\frac{R[x_1 \dots x_n]}{(I)}}$ is noeth

Pf| Let, $I \subset R[x]$ be given

if $I \neq (0)$ choose $f \in I$ of min degree. Maybe it goes? No.

Prob: We'd like to use div alg, if $g \in I$

Then $\exists\, h$ s.t $g - hf$ has smaller degree than g.

$\hookrightarrow$ problem in $\mathbb{Z}$ is if leading coeff are coprime

e.g| $g = bx^n + \dots$ $\quad f = ax^m + \dots$

need $a | b$ or $b \in (a)$ in $\mathbb{Z}$ not always

(so | to cancel)

Motivation

Say $f_1 = a_1 x^{n_1} + \dots \quad f_2 = a_2 x^{n_2} + \dots \in I$

and $g = bx^n + \dots$

if $b \in (a_1, a_2)$ and $n \geq n_1, n_2$

$\Rightarrow \exists\, h_1, h_2$ s.t $g - h_1 f_1 - h_2 f_2 \Rightarrow$ leading cancels

Def The initial coef of some $f \neq 0 \in R[x]$ is just the leading coeff $in(f)$ $(a, a \neq 0)$

Def The initial ideal of I is
$$in(I) = \{in(f) \mid f \in I \setminus \{0\}\} \cup \{0\}$$

Lemma is an ideal of R

Say $a, b \in in(I)$ not zero

$a = in(f)$ $\quad f = ax^n + \cdots$

$b = in(g)$ $\quad g = bx^m + \cdots$

let $c \in R$, if $ca = 0 \Rightarrow ca \in in(I)$

else $ca = in(c \cdot f) \in in(I)$

We can assume $n = m$ (add x^{n-m})

$a + b$ either zero or $a + b = in(f + g) \in in(I)$

In Hilbert,

as R is Noeth, $in(f)$ is f.g

Choose $f_1, \ldots, f_r \in I$ so $in(f_1), \ldots, in(f_r)$ gen $in(I)$

Lemma Given $g \in I$ s.t $\deg(g) \geq \deg(f_i)$ $\forall i$

$\exists h_1, \ldots, h_r \in R[x]$ so

$\deg(g - h_1 f_1 - \cdots - h_r f_r) < \deg(g)$

Pf let $g = bx^m + \cdots$ $\quad f_i = a_i x^{n_i} + \cdots$

$a_1, \ldots, a_r$ gen $in(I)$ & $b \in in(I)$

$\Rightarrow b = c_1 a_1 + \cdots + c_r a_r$ for $c_r \in R$

let $h_i = c_i x^{m - n_i}$ $\longrightarrow$ reduce degree

until N $\longrightarrow$

Cor if $N = \max_i \{\deg(f_i)\}$

$I_{\le N} = \{f \in I \mid \deg(f) \le N\}$

$\hookrightarrow$ not an ideal but closed under + & scalar mult

$\hookrightarrow$ $I_{\le N}$ is an R-submodule of I

$I = (f_1, \dots, f_r) + I_{\le N}$ by prev lemma

$$I_{\le N} = \underbrace{R\cdot 1 \oplus Rx \oplus \cdots \oplus Rx^N}_{\text{f.g } R \text{ mod } (\cong R^{N+1})}$$

since R noeth $\Rightarrow$ $I_{\le N}$ fin gen

if $(f_1', \dots, f_s')$ gen $I_{\le N}$ as R-mod

$\Rightarrow I = (f_1, \dots f_r, f_1', \dots f_s') \Rightarrow I$ fin gen $\square$

Aside

Invariant theory. $G \curvearrowright \mathbb{C}[x_1, \dots x_n]$ act by ring homo

Problem describe $\mathbb{C}[x_1 \dots x_n]^G$ (G-invariant poly)

E.g. consider homo deg 2 poly in 2 vars

$ax^2 + bxy + y^2 = F$

given $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathbb{C})$

let $gF = f(\alpha x + \beta y, \gamma x + \delta y) = a(\alpha x + \beta y)^2 + b(\alpha x + \beta y)(\gamma x + \delta y) + c(\gamma x + \delta y)^2$

x^2 term is $(a\alpha^2 + b\alpha\gamma + c\gamma^2)$

$G = SL_2(\mathbb{C})$ $\mathbb{C}[a, b, c]$

$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ $g \cdot a = a\alpha^2 + b\alpha\gamma + \gamma^2$

$\mathbb{C}[a, b, c]^G$ is poly fnc of the coeff of quad forms invariant under linear cov.

$\cong \mathbb{C}[t]$, $t = b^2 - ac$

e.g. Consider higher degree

$F = a_0 x^n + a_1 x^{n-1} y + \cdots + a_n y^n$

$G = SL_2(\mathbb{C}) \curvearrowright \mathbb{C}[a_1, \ldots, a_n]$ as above

We said,

$n=2$ inv ring is poly ring in 1 gen

$n=3$ ———————— 2 gen

Paul Gordan proved for this family the inv ring is a fin gen $\mathbb{C}$-alg $\forall n$.

Hilbert for a large class of G the inv ring is always a fin gen $\mathbb{C}$ algebra

→ reductive grp

Topics Midterm

Ring

- defn of ring, subrings : ideals & quot rings
- presentation of Rings eg $\mathbb{Z}[i] = \dfrac{\mathbb{Z}[x]}{(x^2+1)}$
- adjoining elts satisfying eqn $\dfrac{R[x]}{(f(x))}$ → [deg d]
- 2 cases
  - f monic ⟹ → free R-module with basis $1, x, \ldots x^{d-1}$
  - f not monic → $ax-1$ $R[1/a]$

$\dfrac{\mathbb{C}[x]}{(x^2+1)} \cong \mathbb{C} \times \mathbb{C}$ → not domain

$\dfrac{\mathbb{F}_2[x]}{(x^2+1)} \cong \dfrac{\mathbb{F}_2[u]}{(u^2)}$ → nilpotent so quotient not reduced

- Frac fields of a domain
- Max'l ideals (quotient is field)

- Classified Max'l ideals of $\mathbb{C}[x_1, \ldots, x_n]$
  All of form $(x_1 - a_1, \ldots, x_n - a_n)$ $a_i \in \mathbb{C}$

Factorization (R is domain)

- Units, divisibility, associate elts → differ by unit in dom
- irreducibles & primes! defn of UFD
- If R noetn & irred are prime ⇒ R UFD!
- PID are UFD by
- Euc rings are PIDs
- Gaussians are euclidean → understood primes!