Def) A ring is a set R equipped with 2 binary op $+$ (addition) & $\cdot$ (multi) s.t

1) $(R,+)$ is an abelian grp (id elt 0) (additive inv x is $-x$)
2) $(R,\cdot)$ is a monoid (id elt 1) (if $\exists$ mult inv x is x^{-1})
3) Distributivity $x\cdot(y+z) = xy + xz$ (and right too)

Def) A ring is called commutative if mult commutes!

e.g.)
- $\mathbb{Z}$ is a ring
- $\mathbb{Z}[x]$ is a ring $\longrightarrow$ in fact any (comm) ring coeff works!
  - $\hookrightarrow$ can add more vars
- Any field a ring
- let R be a ring $M_{n\times n}(R)$ is a non comm ring (in gen)
- The zero ring $R = \{0\}$ ($1=0$ here)
  - $\hookrightarrow$ if $1=0$ in R then R is zero ring

Exer) (on hw)
- $x\cdot 0 = 0 \quad \forall x\in R$
- $(-1)\cdot x = -x$
- $(-x)\cdot y = -x\cdot y$

Def) let R,S rings then a ring homo is a fun that is compat with $+$ and $\cdot$ & takes $1_R \longrightarrow 1_S$

$\hookrightarrow$ to not include $f(x)=0 \;\; \forall x$

Rmk: A elt x of ring R is a unit if it has a mult inv i.e
$\exists y$ so $xy = yx = 1$

The set $R^\times$ of units is a grp under mult

If $f: R\to S$ is a ring homo it induces a gp homo on $R^\times$ $\longrightarrow$ this is why we want $f(1)=1$

Eg Ring homo

- $id_R : R \to R$ is a ring homo
- $\forall R\ \exists !$ ring homo $R \to 0$ → zero ring
  {0 is a final object in category of rings} ← unique arrow
- $\forall$ ring $R\ \exists !$ ring homo from $\mathbb{Z} \to R$ (initial obj)
  $n > 0 \quad n \mapsto \underbrace{1 + 1 \cdots + 1}_{n}$

Rmk
- in grps trivial grp is initial + final
- in fields have no init or final
- in fields of char 0, $\mathbb{Q}$ is initial!

- $\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ quot map is a ring homo
- complex conj is a ring homo! (field homo)
- R commutative ring $\exists$ Ring homo
  $R[x] \to R$
  $a_0 + \cdots + a_n x^n \mapsto a_0 + \cdots + a_n$
  $\varphi \longmapsto \varphi(1)$
  in general if $c \in R$ get ring homo $R[x] \to R$
  $\varphi \mapsto \varphi(c)$

Thm Mapping Prop for poly rings
R, S comm rings
Giving a ring homomorphism $\varphi : R[x] \to S$
same as giving ring homo $\varphi : R \to S$ & $c \in S$
$\varphi_0 = \varphi|_R$ (thinking $R \subset R[x]$ as const poly)
$c = \varphi(x)$

→

Pf idea) Given $\varphi_0 : R \to S$ & $c \in S$ define
$\varphi : R[x] \to S$
$\left(\sum_i a_i x^i\right) \longmapsto \sum_{i=1}^{n} \varphi_0(a_i) c^i$

Can show this is a well def homo & inv to previous construction

Another expl) given $\varphi_0 : R \to S$ and $c \in S$

$R[x] \longrightarrow S[x] \longrightarrow S$
$\sum a_i x^i \xrightarrow{\text{translate}} \sum \varphi(a_i) x^i$
$f \xrightarrow{\text{evaluate}} f(c)$

Rmk) $R[x,y] \cong R[x][y]$ in gen
(→ ring iso

→ R, S com

Let $f : R \to S$ be a ring homo.

Def) $\ker f = \{ r \in R \mid f(r) = 0 \}$
This is an additive subgrp of R
f is inj $\iff \ker(f) = 0$

Lemma) if $x \in \ker f$ & $y \in R$ then $xy \in \ker(f)$
Pf) $f(xy) = f(x) f(y) = 0 \cdot f(y) = 0$

Def) A ideal I of R is an additive subgrp that is closed under all scalar mult
(→ $a \in R$, $b \in I \Rightarrow ab \in I$

By above: ker of ring homo is ideal!

Rmk) in non comm case have
left ideal (left mult by arb elt)
Right ideal (right mult by arb elt)
2-sided ideal

Spse $I \subset R$ is an ideal can think of R/I
as an abelian grp (addition commute)
by
$(a+I)(b+I) = (ab+I)$

Must show | $ab + I = (a+x)b + I$ for $x \in I$
$\Rightarrow$ need $bx \in I$ $\to$ okay as I ideal)

This makes
R/I as a comm ring

$\pi : R \to R/I$ is a ring homo
with $\ker \pi = I$

$\Rightarrow$ each ideal ker of homo