e.g. $\mathbb{Z}/p^r\mathbb{Z}$ is inj as a $\mathbb{Z}/p^r\mathbb{Z}$ (not as $\mathbb{Z}$ mod)

$\mathbb{Q}$, $\mathbb{Q}/\mathbb{Z}$ is inj as $\mathbb{Z}$ mod

If F is a field $\Rightarrow$ any vs is inj (and proj) as F-mod.

$\mathbb{C}(x)$ and $\mathbb{C}(x)/\mathbb{C}[x]$ are inj $\mathbb{C}[x]$-mod.

$\mathrm{Frac}(R)$, $\mathrm{Frac}(R)/R$ for R PID.

Baer's Crit | let R - any comm ring.

An R-mod M is inj $\iff$ if $\varphi : I \to M$ is a map of R modules with I ideal of R $\exists$ extension $\psi : R \to M$ so $\psi|_I = \varphi$

↳ R mod hom

e.g. $\mathbb{Q}$ inj as a $\mathbb{Z}$ module.

let $\varphi : I \to \mathbb{Q}$ be given.

we know $I = n\mathbb{Z}$ for some $n > 0$. (if $I = 0$ take $\psi = 0$)

let $x = \varphi(n) \in \mathbb{Q}$

define $\psi : \mathbb{Z} \to \mathbb{Q}$ by $\psi(1) = \frac{x}{n}$

$\Rightarrow \psi(n) = n \cdot \psi(1) = x = \varphi(n)$ & ✓ $\psi|_n$ (any div mod inj)

e.g. $\mathbb{Z}$ not inj as $\mathbb{Z}$ module.

Take $I = 2\mathbb{Z}$ ($I \cong \mathbb{Z}$ as $\mathbb{Z}$ mod)

Define $\varphi : I \to \mathbb{Z}$ $\quad \varphi(2n) = n$

$2 \mapsto 1$

φ doesn't extend to $\psi : \mathbb{Z} \to \mathbb{Z}$

$2 \cdot \psi(1) = \psi(2) = \varphi(2) = 1$

but $\frac{1}{2} \notin \mathbb{Z}$ $\Rightarrow \Leftarrow$

e.g. $\mathbb{Z}/p\mathbb{Z}$ as $\mathbb{Z}/p\mathbb{Z}$ mod. well this is vs ✓

E.g. $\mathbb{Z}/p^2\mathbb{Z}$ as $\mathbb{Z}/p^2\mathbb{Z}$-mod.

let $\varphi : I \longrightarrow R$ be given.

by ideal corr ideals of $\mathbb{Z}/p^2\mathbb{Z}$ are $(0), (1), (p)$

Case 1 $I = (0)$ then $\varphi = 0$ so take $\psi = 0$

Case 2 $I = (1) \Rightarrow I = R$ so $\psi = \varphi$

Case 3 $I = (p)$ φ is det by $\varphi(p)$

$$p\,\varphi(p) = \varphi(p^2) = \varphi(0) = 0$$

$\Rightarrow \varphi(p) = p \cdot a$ for $a \in \mathbb{Z}/p^2\mathbb{Z}$

(as p killed it in $\mathbb{Z}/p^2\mathbb{Z}$)

So $\psi : R \longrightarrow R$

$\quad 1 \longmapsto a$

$\therefore \psi(p) = p \cdot a = \varphi(p)$ $\square$

E.g. $\mathbb{Z}/p^r\mathbb{Z} = R$ & R inj (so finished structure thm of $\mathbb{Z}$-mods)

let $\varphi : I \to R$ given

By ideal corr $I = (p^s)$ $0 \leq s \leq r$

$$p^{r-s}\varphi(p^s) = \varphi(p^r) = 0$$

$\therefore \varphi(p^s) = p^s a$ for $a \in \mathbb{Z}/p^r\mathbb{Z}$

let $\psi : R \to R$

$\quad 1 \mapsto a$

Pf of Baer $\rightarrow$

Pf) ($\Leftarrow$) let M be given & spse the criterion holds.

wts M is injective module

Let $i : M \to N$ be inj of R-mods.

wts $i(M)$ has compl mod in N

ETS, $\exists\, s : N \to M$ st $s \circ i = id_M$ (compl mod is ker(s))

To begin def s on $i(M)$ to be inverse of i (of inj)

i.e $s(i(x)) = x \;\forall x \in M$.

We'll keep enlarging domain of $s : i(M) \to M$

Spse we have a submod $i(M) \subset N_1 \subset N$

s.t we have $s : N_1 \to M$ def'd

let $x \in N$, $x \notin N_1$, put $N_2 = N_1 + Rx$

We want to extend s to $N_2 = N_1 + Rx$

let $I = \{a \in R \mid ax \in N_1\}$ ideal

This is an ideal of R

we can define $\varphi : I \to M$

$a \mapsto s(ax)$

By crit $\exists\, \psi : R \to M$ so $\psi|_I = \varphi$

let $s : N_2 \to M$

$n_1 + ax \longmapsto s(n_1) + \psi(a)$ (so we patch on overlap of $Rx \cap N_1$)

Now Zorn's Lemma:

$$\Sigma = \left\{ (s, N') \;\middle|\; \begin{array}{l} i(M) \subset N' \subset N \\ s : N' \to M \text{ inv to } i \end{array} \right\}$$

$$(s_1, N_1') \leq (s_2, N_2') \quad \text{if } N_1' \subset N_2' \;\&\; s_2|_{N_1} = s_1$$

Zorn's Lemma $\Rightarrow \exists$ max elt. By prev para max has $N' = N$.

D.

Pf] ($\Leftarrow$) Now suppose M is inj wrt ext modules.

Let $\varphi : I \to M$ be given.

$$\begin{tikzcd} M \arrow[rr, dotted, "s"] & & N \\ I \arrow[u, "\varphi"] \arrow[r, hook] & R \arrow[ul, "\psi"'] \arrow[ur, "r"'] & \end{tikzcd}$$

work in categ since bottom inj

$\to$ can check injective.

$\hookrightarrow$ M inj $\Rightarrow \exists$ one sided inv $s : N \to M$

$\psi = s \circ r$

Pushout of above. $N = \dfrac{M \oplus R}{\{(\varphi(x), 0) - (0, x) \mid x \in I\}}$

The point is $(\varphi(x), 0) = (0, x)$ in N.

Pf of Struct thm]

M - fg $\mathbb{Z}$ module.

$\hookrightarrow$ M / M_{tors} is fg & tors free $\Rightarrow$ free (fg work)

- Since it is free, the SES

$$0 \to M_{tors} \to M \to M / M_{tors} \to 0$$

splits $\Rightarrow M = M_{tors} \oplus \underbrace{M / M_{tors}}_{\text{free}}$.

- M_{tors} fin gen ($\mathbb{Z}$ noeth) & tors.

$\Rightarrow$ finite, it's a $\mathbb{Z}/N\mathbb{Z}$ module for $N \geq 1$

- CRT $M_{tors} = \bigoplus_p M_p$ $\to$ a $\mathbb{Z}$ module killed by power of p

$M_p = 0$ almost all p.

- Fix p. Let r be min s.t. $p^r M_p = 0$. Choose $x \in M_p$ of ord $p^r \Rightarrow \langle x \rangle \cong \mathbb{Z}/p^r\mathbb{Z}$

Since $\mathbb{Z}/p^r\mathbb{Z}$ inj as a mod of itself

So $M_p \cong \langle x \rangle \oplus$ (smaller)

$\hookrightarrow$ sum of cyclic