Let R a (comm ring)

Def A R-module is an abelian gp M w/ map

$$R \times M \longrightarrow M$$
$$(a,x) \longmapsto ax$$

So, $1 \cdot x = x$

$a(x+y) = ax + ay$

$a \cdot (bx) = (ab) \cdot x$

Def If M & N are R-mods. A R-mod homo is a fnc $\varphi : M \to N$ that is comp w/ $+$ & sc mult.

1) $\varphi(x+y) = \varphi(x) + \varphi(y)$

2) $\varphi(ax) = a\varphi(x)$

R mod isom = bij R-mod homo

E.g. 1) R is an R mod via $a \cdot x = ax$

2) $\mathbb{Z}[i]$ is a $\mathbb{Z}$-mod

More gen, if $\varphi : R \to S$ is a ring homo

$\Rightarrow S$ is a R-mod

via $a \cdot x = \varphi(a)x$

mult is S

Even more gen if N is a S-module w/ φ as abv

N is a R mod by $a \cdot x = \varphi(a) \cdot x$

sc mult in S

"Restriction of sc"

3) If $I \subseteq R$ is an ideal $\Rightarrow$ R-mod via normal mult

as closed under mult.

Def If M is a R-mod then an R-submod of M is a subgrp of M closed under mult

In fact, ideals $\to$ R-submod of R

4) If M is an abelian grp define $2x = x+x$.
So M is a $\mathbb{Z}$-module.
It is the unique $\mathbb{Z}$ mod struct as

$2x = (1+1)\cdot x = 1\cdot x + 1\cdot x = x + x$

So, $\mathbb{Z}$-modules $\longleftrightarrow$ abelian grps!

5) R^n is an R module $\to$ Column vectors & std operations

6) $\mathbb{Z}/2\mathbb{Z}$ is a $\mathbb{Z}$-mod
In general, if M is a R-module & $N \subseteq M$ R-sub mod
$\Rightarrow M/N$ a R mod
$a(x+N) = ax + N$.

Many constructions apply to modules

1) $\varphi : M \to N$ homo of modules,
$\ker \varphi$ = submod of M
$\operatorname{im} \varphi$ = " " N

1st iso thm $M/\ker\varphi \cong N$ $\to$ isom of R mods by φ

2) Direct sums, if M, N 2 R-modules
the direct sum $M \oplus N$ is the R-module w/ elts $M \times N$
& coordwise addition & mult!

E.g.) $R^n \cong \bigoplus_{i=1}^{n} R$

3) R module maps $R^m \to R^n$ described by $n\times m$ matrices in R
If, $A \in M_{n,m}(R)$
$\hookrightarrow$ $n\times m$ matrices w/ entry in R

$Ax \in R^n$ def by usual formula $R^m \to R^n$ R-mod map
$x \mapsto Ax$

Say M is an R-module

Def) A (finite) basis for M is elts $e_1, \dots, e_r \in M$ so every elt in M can be uniquely written as comb of

e.g.) $M = R^n$ & e_i $1 \le i \le n$ usual $\to$ this is a basis!

Def) M is a free R module if it has a basis

e.g.) $\mathbb{Z}/2\mathbb{Z}$ is not a free R-mod

$\hookrightarrow$ only possible basis is $\bar{1} \in \mathbb{Z}/2\mathbb{Z}$

but not basis as $0 \cdot \bar{1} = 0 = 2 \cdot \bar{1}$

but $0 \ne 2$ in $\mathbb{Z}$ so not unique

Rmk) Say M is free w/ (fin) basis $e_1, \dots, e_n$ then the map

$$\varphi : R^n \xrightarrow{\sim} M$$
$$\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \longmapsto \sum_{i=1}^{n} a_i e_i$$

is a R mod isom

e.g.) $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ not a free module as it is not torsion free

$2 \cdot (0, \bar{1}) = (0, 0)$

Def) R = domain M = R-mod

An elt $x \in M$ is torsion if $\exists a \ne 0 \in R$ so $ax = 0$

Rmk) $M_{tors} = \{ x \in M \mid x \text{ is tors} \}$

is a submod of M

Pf) any nontrivial is closed under vec +

let $x, y \in M_{tors}$ $\exists a, b$ nonzero $\Rightarrow ax = by = 0$

$\Rightarrow \underbrace{ab}_{\ne 0 \text{ as domain!}}(x+y) = 0$

$\square$

If R a domain & M is a free R-mod then $M \cong R^n$
and so $M_{tors} = 0$ i.e M is torsion-free

Rmk If M is a f.g $\mathbb{Z}$-mod
that is torsion free
$\Rightarrow$ M is free
(same holds if $\mathbb{Z}$ is a PID)