Struct thm for f.g $\mathbb{Z}$-mods.

Any f.g $\mathbb{Z}$ mod is $\cong$ fin $\oplus$ $\mathbb{Z}$'s or $\mathbb{Z}/n\mathbb{Z}$'s

Pf Strat 1

Let M = f.g $\mathbb{Z}$ mod. (so fin pres)

Choose a pres $\mathbb{Z}^m \xrightarrow{\varphi} \mathbb{Z}^n \longrightarrow M \to 0$

$M \cong \text{coker}\,\varphi = \dfrac{\mathbb{Z}^n}{\ker\varphi}$

φ corr to $n \times m$ matrix

Prove that you can 'diag' it using certain row + col ops (pf Artin).

Strat 2 (More Module Theoretic)

Step 1 A finitely gen'd torsion free $\mathbb{Z}$-mod F is free (showed by Thm)

Pf Q1 · find submod of F $\mathbb{Z}x \subset F$, $\mathbb{Z}x \cong \mathbb{Z}$

· Claim $F/\mathbb{Z}x$ is "smaller" than F & free by induction.

· $F \cong \mathbb{Z}x \oplus F/\mathbb{Z}x$

discuss what this is ....

($\to$ consider localizing to a $\mathbb{Q}$ vs.)

Q1 What does smaller mean?

A1 We'll take $\mathbb{Q}$-dim of localization.

Q2 How to choose x so $F/\mathbb{Z}x$ is tors-free. Eg $x=2$, $F=\mathbb{Z}$ bad

Eg 2 $F = \mathbb{Z}^2$

$x = \binom{6}{4} \in F$

$x = 2y$ where $y = \binom{3}{2}$ so $2\bar{y}$ is zero in $F/x\mathbb{Z}$ $\longrightarrow$ not tors free.

$\bar{y} \neq 0$ in $F/x\mathbb{Z}$

Let F = f.g tors free $\mathbb{Z}$ mod.

Pick $x \in F$ non-zero.

Define L in F all elts that are "rational multiples" of x.

$$L = \{ y \in F \mid \exists n, m \in \mathbb{Z} \text{ s.t. } ny = mx \quad n \neq 0 \}$$

So, $S = \mathbb{Z} \setminus \{0\}$ $\quad S^{-1}\mathbb{Z} = \mathbb{Q}$

If M is a $\mathbb{Z}$ mod then $S^{-1}M$ is a $\mathbb{Q}$ vs.

have

$$M \longrightarrow S^{-1}M$$
$$x \longmapsto \frac{x}{1}$$

which is inj if M not tors.

note M f.g $\Rightarrow$ $S^{-1}M$ f.g as

$$\mathbb{Z}^n \twoheadrightarrow M \Rightarrow S^{-1}\mathbb{Z}^n \twoheadrightarrow S^{-1}M$$
$$S^{-1}\mathbb{Z}^n \cong \mathbb{Q}^n$$

So

$$\mathbb{Q}x \subset S^{-1}F \longrightarrow \text{fin dim } \mathbb{Q} \text{ vs}$$
$$F \subset S^{-1}F$$

$$L = \mathbb{Q}x \cap F$$

Cl 1) L is f.g as $\mathbb{Z}$-mod

Pf) $\mathbb{Z}$ is noeth + F is f.g $\mathbb{Z}$ mod so all submod f.g (including L)

Cl 2) $\exists y \in L$ s.t $L = \mathbb{Z}y$

Lemma) If N is a f.g $\mathbb{Z}$-mod submod of $\mathbb{Q}$, $N = \mathbb{Z}y$ for some $y \in N$

Pf) let $z_1, \dots, z_r \in N$ be gens. Pick a non-zero $a \in \mathbb{Z}$ so $az_i \in \mathbb{Z}$ $\forall i$

$aN \subset \mathbb{Z}$. Since $\mathbb{Z}$ a PID $\xRightarrow{aN \text{ is } \mathbb{Z} \text{ submod}}$ $aN = b\mathbb{Z} \Rightarrow N = \frac{b}{a}\mathbb{Z}$

Pf) L is a f.g $\mathbb{Z}$ submod of $\mathbb{Q}x \cong \mathbb{Q}$. Cl follows

Claim 3 | F/L is torsion-free

Pf) Spose $\bar{z}$ is a torsion elt of F/L

i.e. $n\bar{z} = 0$ $n \in \mathbb{Z}$ $n \neq 0$

let z be a lift of $\bar{z}$ to F

as $n\bar{z} = 0 \Rightarrow nz \in L$

In $S^{-1}F \Rightarrow nz$ is rat'l mult of x

$\Rightarrow z$ is ________ x

$z \in L \Rightarrow \bar{z} = 0$

For a fin gen $\mathbb{Z}$ module, M let

$\partial(M) = \dim_{\mathbb{Q}}(S^{-1}M)$

we showed if M fin gen $\Rightarrow \partial(M) < \infty$

Cl 4) $\partial(F/L) < \partial(M)$

Pf) $S^{-1}(F/L) \cong S^{-1}F/_{S^{-1}F}$ → vec sp

→ by functoriality.

so $\dim S^{-1}(F/L) = \partial(S^{-1}F) - \underbrace{\dim(S^{-1}L)}_{1}$

$L \cong \mathbb{Z}$

$\Rightarrow S^{-1}L \cong \mathbb{Q}$

Pf of Step 1 | {F f.g tors free $\Rightarrow F$ free}

By induction on $\partial(F)$

Base Case | $\partial(F) = 0$

$\partial(F) = 0 \Rightarrow S^{-1}F = 0 \Rightarrow F = 0$ b/c F tors-free

$F \hookrightarrow S^{-1}F$ inj

Inductive Step | $0 \neq x \in F$

let $L = (\mathbb{Q}x) \cap F \subset S^{-1}F$

by cl 2 $L \cong \mathbb{Z}$

cl 3 F/L tors free

cl 4 $\partial(F/L) < \partial(F)$

} by induction F/L is free.

Situation 1

$$0 \longrightarrow L \longrightarrow F \longrightarrow F/L \longrightarrow 0$$

(free) (free)

WTS: F free.

Lemma) R is any ring $M \subset N$ R mods so

$E = N/M$ free

$\Rightarrow N \cong E$