Suppose $\varphi : R \to S$ is a ring homo

Restr of Scal: if M is an S-module, can give R-mod str

Call $\varphi^*(M)$ $\leftarrow$ $r \cdot x = \varphi(r) \cdot x$ for $r \in R$, $x \in M$

Exten. of Scalar: If M is an R module & makes an S-mod of M for $R \to S$

Say M is finitely pres R-module

$\exists$ map $f : R^m \to R^n$ so $M \cong \operatorname{coker} f = \dfrac{R^n}{\operatorname{im}(f)}$

$\hookrightarrow$ fin gen

Concretely f is spec by $n \times m$ matrix, entries in R.

let $f' : S^m \to S^n$ be map given by taking $\hookrightarrow$ matrix & applying φ to each entry.

Define $\varphi_*(M)$ to be the S module $\operatorname{coker}(f')$

Exer 1 this is canonically ind of pres.

e.g. 1) $\varphi_*(R) = S \longrightarrow R = R/0 = \operatorname{coker}(0 \to R)$

$\Rightarrow \varphi_*(R) = \operatorname{coker}(0 \to S)$

$= S$

$\underbrace{\varphi^*(\varphi_*(S))}$ $= S$ thought of as an R-module

$\hookrightarrow$ these aren't inverses to one another

if $N = S$-mod $\varphi_* \varphi^*(N) = N \otimes_S \varphi_*(\varphi^*(S))$

2) $\varphi_*(M \oplus N) \cong \varphi_*(M) \oplus \varphi_*(N)$

$\varphi_*(R^n) = S^n$

3) $\varphi : \mathbb{Z} \hookrightarrow \mathbb{Q}$ $M = \mathbb{Z}/3\mathbb{Z}$

$\varphi_*(M) = \mathbb{Q}/3\mathbb{Q} = 0$

$f : \mathbb{Z} \xrightarrow{\times 3} \mathbb{Z}$ $M = \operatorname{coker} f$

$f' : \mathbb{Q} \xrightarrow{\times 3} \mathbb{Q}$ surj

$\operatorname{coker} f' = 0$

4) $\varphi : \mathbb{Q} \to \mathbb{C}$

$\varphi_*(\mathbb{Q}^n) = \mathbb{C}^n$ — upgrade

V is n dim $\mathbb{Q}$ vs then $\varphi_*(V)$ is n dim $\mathbb{C}$ vs

some notn do funct $\Bigg\{$ If $T : V \to V$ is a $\mathbb{Q}$ lin operator

$\Rightarrow \exists$ induced op $\varphi_*(T) : \varphi_*(V) \to \varphi_*(V)$

$\varphi_*(T)$ is the "same" matrix as T

5) If $\varphi : R \to R/I \Rightarrow \varphi_*(M) = M/IM$

$\otimes$ prod of $\mathbb{Z}$-modules.

If M, N are $\mathbb{Z}$ modules we say a tensor prod is a $\mathbb{Z}$-mod T equipped with a $\mathbb{Z}$-bilinear map $b : M \times N \to T$. That is universal

i.e. if $M \times N \xrightarrow{b'} T'$ is another bilin map.

$\exists !$ $\mathbb{Z}$-mod homo $T \xrightarrow{a} T'$ so

$$\begin{tikzcd} M \times N \arrow[r, "b"] \arrow[dr, "b'"'] & T \arrow[d, "a"] \\ & T' \end{tikzcd}$$

easy to see that unique if it exists

$\otimes$ prod always exists

Constr $\left(\dfrac{\text{free } \mathbb{Z} \text{ mod w/ basis symbol } [m,n] \ \substack{m \in M \\ n \in N}}{\text{submod gend by } [\alpha_1 m_1 + \alpha_2 m_2, n] = \alpha_1 [m_1, n] + \alpha_2 [m_2, n] \ \cdots \ [m, \alpha_1 n_1 + \alpha_2 n_2] = \alpha_1 [m, n_1] + \alpha_2 [m, n_2]} \right)$

Notation $M \otimes N$ or $M \otimes_{\mathbb{Z}} N$

for $m \in M, n \in N$ $m \otimes n = [m,n]$ in $M \otimes N$ → pure tens

[illegible]

$f : M \to M'$

$g : N \to N'$

$f \otimes g : m \otimes n \mapsto f(m) \otimes f(n)$

Prop 1
1) $\mathbb{Z} \otimes M \cong M$
2) $M \otimes N \cong N \otimes M$
3) $(M_1 \oplus M_2) \otimes N \cong (M_1 \otimes N) \oplus (M_2 \otimes N)$

$\mathbb{Z}^n \otimes \mathbb{Z}^m = \mathbb{Z}^{nm}$

4) $M \otimes \_$ is a right exact

<u>i.e</u> $0 \to N_1 \xrightarrow{f_1} N_2 \xrightarrow{f_2} N_3 \to 0$

is SES of $\mathbb{Z}$-mods then

$$M \otimes N_1 \xrightarrow{id_M \otimes f_1} M \otimes N_2 \xrightarrow{id_M \otimes f_2} M \otimes N_3 \to 0$$

exact

e.g 1 $\mathbb{Z}/3\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/5\mathbb{Z}$

$$0 \to \mathbb{Z} \xrightarrow{3} \mathbb{Z} \to \mathbb{Z}/3 \to 0$$

$$\rightsquigarrow \quad \mathbb{Z}/5 \otimes \mathbb{Z} \xrightarrow{id \otimes} \mathbb{Z}/5 \otimes \mathbb{Z} \to \mathbb{Z}/5 \otimes \mathbb{Z}/3 \to 0$$

$\mathbb{Z}/5 \otimes \mathbb{Z} = \mathbb{Z}/5$, $\mathbb{Z}/5 \otimes \mathbb{Z} = \mathbb{Z}/5$

<u>surj</u> $\Rightarrow$ coker$(\ ) \cong \mathbb{Z}/5 \otimes \mathbb{Z}/3$ exactness

coker$(\ ) = 0$

or $m \otimes n \in \mathbb{Z}/3 \otimes \mathbb{Z}/5$

$3(m \otimes n) = m \otimes n + m \otimes n + m \otimes n$
bilin $= (3m) \otimes n = 0$

$5(m \otimes n) = m \otimes (5n) = 0$

$\Rightarrow M \otimes N$ killed by $(3,5) = 1$

R general ring M, N 2-R mods $M \otimes_R N$: recieves univ R-bilin fm $M \times N$

If $\varphi : R \to S$ is a ring homo.

Then S is naturally an R module (rests of sc)

For an R-mod M $\quad \varphi_*(M) = S \otimes_R M$

Reason) Pick a presentation for M

similar to eene

$$\begin{tikzcd}
 & R^m \arrow[r, "f"] & R^n \arrow[r] & M \arrow[r] & 0 \\
\Rightarrow & S \otimes_R R^m \arrow[r, "f"] \arrow[d, equal] & S \otimes_R S^n \arrow[r] \arrow[d, equal] & S \otimes_R M \arrow[r] \arrow[d, equal] & 0 \\
 & S^m \arrow[r, "f"] & S^n \arrow[r] & \varphi_*(M) \arrow[r] & 0
\end{tikzcd}$$

Struct on pure tens $\quad s \cdot (s' \otimes m) = (ss') \otimes m$

If $S \subset R$ mult set $\quad \varphi : R \to S^{-1}R$, $\; r \mapsto \frac{r}{1}$

$\varphi_*(M) = S^{-1}M = S^{-1}R \otimes_R M$

Def) An R-module M is flat if

$M \otimes_R -$ is an exact functor