Quiz 5)

1) Given the first person gets their own.

Reduce to the problem of n-1 people & n-1 hat

$\therefore$ for n-1 people it is $\underline{1}$.

Thus total expection is 2.

2) $E(\#\text{ matches}) = E(\#\text{ match} \mid \text{first match}) P(\text{first match})$
$+ E(\#\text{ match} \mid \text{first not match}) P(\text{not match})$

$$1 = 2 \cdot \frac{1}{n} + \frac{n-1}{n}(a) \Rightarrow a = \frac{n-2}{n-1}$$

Recall)

Last time: Symmetric Simple RW

X_i = move on the ith step $\begin{cases} 1 & \text{right} \\ -1 & \text{left} \end{cases}$

Starting from 0, position of walk of n steps $Y_n = \sum_{i=1}^{n} X_i$

The hitting time Z_b = 1st time that the walk hits b $(b \neq 0)$

Fact (hw))

P(the walk hits b) = 1

Lets try to understand more fine of Z_b

Thm)

$$P(Z_b = n) = \frac{|b|}{n} P(Y_n = b)$$

$\Rightarrow$ Rmk different pf in 3.10 of text

Pf) By inducting on n

wlog $b > 0$ (by symmetry)

base case) $\boxed{n=1}$ if $b > 1 \Rightarrow \overbrace{P(Z_b = 1) = 0 \; ⌣ \; \text{as } P(Y_1 = b) = 0}^{\text{both sides } 0}$

$b = 1 \Rightarrow P(Z_b = 1) = \frac{1}{2} \; ⌣ \; \text{as } P(Y_1 = 1) = 1/2$

Inductive step $\to$ Assume this holds for $x \le n-1$

$$P(Z_b = n) = P(Z_b = n \mid X_1 = 1)\, P(X_1 = 1) + P(Z_b = n \mid X_1 = -1)\, P(X_1 = -1)$$

$$= P(Z_{b-1} = n-1)\,\frac{1}{2} + P(Z_{b+1} = n-1)\,\frac{1}{2}$$

inductive $\downarrow$

$$= \frac{1}{2}\,\frac{b-1}{n-1}\,P(Y_{n-1} = b-1) + \frac{1}{2}\,\frac{b+1}{n-1}\,P(Y_{n-1} = b+1)$$

$\boxed{\text{Consider}}$ $\frac{1}{2} P(Y_{n-1} = b-1)$ shift

$$= P(X_1 = 1)\, P(Y_n = b \mid X_1 = 1)$$

bayes

$$= P(Y_n = b)\, P(X_1 = 1 \mid Y_n = b)$$

$\boxed{\text{Similarly}}$

$$= P(Y_n = b)\, P(X_1 = -1 \mid Y_n = b)$$

$$\text{So, } = \frac{1}{n-1}\left[P(Y_n = b)\left\{\underbrace{(b-1)\,P(X_1 = 1 \mid Y_n = b) + (b+1)\,P(X_1 = -1 \mid Y_n = b)}\right\}\right]$$

$$= \frac{1}{n-1}\left[P(Y_n = b)\left(b - 1 - P(X_1 = -1 \mid Y_n = b) + P(X_1 = 1 \mid Y_n = b)\right)\right]$$

$$= \frac{1}{n-1}\left[P(Y_n = b)\left[b - \underbrace{\{1 \cdot P(X_1 = 1 \mid Y_n = b) + (-1)\,P(X_1 = 1 \mid Y_n = b)}_{E(X_1 \mid Y_n = b)}\right.\right.$$

Observe, $E(X_1 \mid Y_n = b) = E(X_2 \mid Y_n = b)$

$\hookrightarrow$ swapping the steps makes no difference!

$$\Rightarrow E(X_1 \mid Y_n = b) = \underline{E(X_n \mid Y_n = b)}$$

$$\text{So} \Rightarrow n\,\underbrace{E(X_1 \mid Y_n = b)}_{b/n} = E(X_1 + X_2 \cdots X_n \mid Y_n = b)$$

$$= E(Y_n \mid Y_n = b) = b$$

$$= \frac{1}{n-1}\,P(Y_n = b)\left(b - \frac{b}{n}\right) = \frac{b}{n}\,P(Y_n = b) \quad \smile$$

So,

$$E(Z_b) = \sum_{n=1}^{\infty} n P(Z_b = n)$$

sub

$$= b \sum_{n=1}^{\infty} \underbrace{P(Y_n = b)}_{} =$$

k steps are $+1$

$n-k$ steps are -1 $\quad k-(n-k) = b \Rightarrow b = \frac{n+b}{2}$

if $\frac{n+b}{2}$ not int then $P = 0$

else, it is

$\binom{n}{\frac{n+b}{2}}\left(\frac{1}{2^n}\right) \rightarrow$ binomial

$$= b \sum_{n=1}^{\infty} \binom{n}{\frac{n+b}{2}}\left(\frac{1}{2^n}\right) \sim b \sum_{n=1}^{\infty} C(b) \frac{1}{\sqrt{n}}$$

Sterling Formula $\quad n! = \frac{1}{2\sqrt{\pi n}}\left(\frac{n}{e}\right)^n$

$$\sim b\, C(b) \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \rightarrow \text{diverges to } +\infty$$

$E(Z_b) =$ not defined $\;-\; +\infty$