Recall

we defined conditional distribution, density & exp

Thm) $E(X) = E(E(X|Y))$

let $X_1, X_2 \dots X_n$ be rv

Def) Sample mean : $\overline{X} = \frac{1}{n}(X_1 + \dots + X_n)$

Sample variance : $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$

Thm) if $X_1 \dots X_n \sim N(\mu, \sigma^2)$ independent

a) $\overline{X} \sim N(\mu, \frac{\sigma^2}{n})$

b) $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$

c) $\overline{X}$ & S^2 independent

Missing ingredient in Pf (b&c)

$\to$ generalization of the following

"If g is strictly ↑, diff'ble, X cts rv

$f_{g(X)}(z) = (g^{-1})'(z)\, f(g^{-1}(z))$

if $z \in \mathrm{Im}(g)$, 0 otherwise"

$\to$ to a joint density form

let $X_1 \dots X_n$ be rv.

let $g: \mathbb{R}^n \to \mathbb{R}^n$ is diff'ble &

let $(Y_1 \dots Y_n) = g(X_1 \dots X_n)$ | sol

$\Rightarrow f_{Y_1 \dots Y_n}(y_1 \dots y_n) = f_{X_1 \dots X_n}(g^{-1}(y_1 \dots y_n))$

det of Df $\leftarrow$ $\mathrm{Jac}(g^{-1})(y_1 \dots y_n)$

Generating function

Defn) let $(a_n) \in \mathbb{R}$ be a seq.

The generating function of (a_n) is

$$G_a(s) = \sum_{n=0}^{\infty} a_n s^n$$

whenever this converges

E.g. 1 let $p \in (0,1)$ $a_0 = 0$ $a_n = (1-p)^{n-1} p$

$$G_a(s) = \sum_{n=1}^{\infty} (1-p)^{n-1}(p)\, s^n$$

$$= \frac{ps}{1 - s(1-p)}$$

geometric

when $|s| < \frac{1}{1-p}$

Radius of conv) $R = \sup_r \{ r : G_a(s) \text{ converges } \forall\, |s| < r \}$

Prop) let $(a_n) = a$ & $b = (b_n)$ be seq

Suppose R = radius of conv of $G_a(s)$

1) $G_a(s)$ conv $\forall\, s < |R|$

2) G_a is diff'ble $\forall\, s < |R|$ & $G_a'(s) = \sum_{n=1}^{\infty} n a_n s^{n-1} \Rightarrow C^{\infty}$

3) If $\exists R > r > 0$ s.t. $G_a(s) = G_b(s)$ $\forall |s| < r$

$\Rightarrow G_a = G_b$ $\longrightarrow$ They are $\underline{C^\omega}$ $\Rightarrow a_n = b_n$ $\forall n \in \mathbb{N}$

Observe 1 in previous e.g. $a_n = (1-p)^{n-1} p$ if $n > 0$, 0 otherwise

let X be a RV following geometric (p)

$a_n = (1-p)^{n-1} p = P(X = n)$

Def let X be disc rv taking values in $\mathbb{N}$

The prob generating func of X.

$$G_X(s) = \sum_{n=0}^{\infty} P(X = n) s^n$$

Obs 1 X is disc RV taking values of in $\mathbb{N}$

$$\underbrace{E(s^X)}_{\text{num}} = \sum_{x=0}^{\infty} s^x P(X = x) = G_X(s)$$

$\longrightarrow$ we get generalization

Def let X be a rv, the prob generating func of X is

$$G_X(s) := E(s^X) \quad \text{whenever defined}$$

E.g. 1 let $X \sim \text{Pois}(\lambda)$ find gen func of X

Determine the radius of conv.

$$G_X(s) = \sum_{n=0}^{\infty} s^n e^{-\lambda} \frac{\lambda^n}{n!} = e^{-\lambda} \sum_{n=0}^{\infty} \frac{(s\lambda)^n}{n!}$$

$$= e^{-\lambda} e^{s\lambda} = e^{\lambda(s-1)}$$

it conv for all $\underline{s \in \mathbb{R}}$ $R = \infty$

Property (a) $E(X) = G_X'(1)$ if $R > 1$ if $R = 1$ take $\lim_{s \to 1^-} G_X(s)$

(b) $E(\underbrace{X(X-1)\dots(X-k+1)}_{k \text{ terms}}) = G_X^{(k)}(1)$ if $R > 1$, if $\underline{R = 1 \text{ take limit}}$

Remark: use linearity to extract $\underline{k^{th} \text{ moment}}$

Pf) a) if $\underline{R \geq 1}$, $\underline{X \text{ disc}}$ → warmup

$$G_X(s) = \sum s^n P(X=n)$$

$$G'_X(s) = \sum_{n=1} n s^{n-1} P(X=n)$$

$$\Rightarrow G'_X(1) = \sum_{n=1}^{\infty} n P(X=n) = E(X)$$

→ if $1 < R$

if X not disc

$G_X(s) = E(s^X)$ let us diff wrt s

$G_X'(s) = E(X s^{X-1})$ if $1 < R$ $= \underline{E(X)}$

should hold in general

Demo'd in class for cts

if $\underline{R=1}$

unif conv

$$\lim_{s \to 1} G'_X(s) = \lim_{s \to 1^-} E(X s^{X-1}) \overset{?}{=} E(\lim_{s \to 1^-} X s^{X-1}) = E(X)$$

b) Discrete → ex

lemma | X cts → unif conv

$$G_X^{(k)}(s) \overset{?}{=} \int_{-\infty}^{\infty} \frac{\partial^{(k)}}{\partial s^{(k)}} (s^x f_X(x))\, dx$$

$$= \int_{-\infty}^{\infty} (x)(x-1)\cdots(x-k+1) s^{x-k} f_X(x)\, dx$$

$$= E(X(X-1)\cdots(X-k+1) s^{X-k})$$

if $\underline{s=1}$ done!