Popular cts RV

Note: for cts RV

$P(a < X \le b) = F(b) - F(a)$

$P(-\infty \le X \le \infty) = \int_{-\infty}^{\infty} f(t)\,dt = 1$

Final Dec 13 1:30-3:30

① **Uniform dist** $X \sim Unif([a,b])$

density $f = \frac{1_{[a,b]}}{b-a}$

$$1_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

② **Exponential** $X \sim Exp(\lambda)$ $\lambda > 0$

$\forall x \ge 0$ $f(x) = \lambda e^{-\lambda x}$ note: $P(X > x) = \int_x^{+\infty} \lambda e^{-\lambda t}\,dt = e^{-\lambda x}$

0 otherwise

If $Y \sim geo(1 - e^{-\lambda}) \to p$

$P(Y > k) = \sum_{j=k+1}^{\infty} (1-p)^{j-1} p = \frac{p(1-p)^k}{p} = (1-p)^k$
$= e^{-\lambda k}$

Exponential Dist is the cts version of geometric.

③ **Normal (Gaussian) Distribution**

$X \sim (\mu, \sigma^2)$ $\mu \in \mathbb{R}$

$\forall x \in \mathbb{R}$ $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

why $\int_{-\infty}^{\infty} f(x)\,dx = 1$

④ **Gamma Dist** $X \sim Gamma(\lambda, t)$ $\lambda, t > 0$

density $x \ge 0$ $f(x) = \frac{1}{\Gamma(t)} \lambda^t x^{t-1} e^{-\lambda x}$

where $\Gamma(t) = \int_{-\infty}^{\infty} \lambda^t x^{t-1} e^{-\lambda x}\,dx$ → guarentees integral is 1

change of var $y = \lambda x$

$= \int_{-\infty}^{\infty} y^{t-1} e^{-y}\,dy$

integration by parts

$\Gamma(1) = \Gamma(2) = 1$, ..., $\Gamma(n) = (n-1)\Gamma(n-1)$

Note $\text{Gamma}(\lambda, 1) := \text{EXP}(\lambda)$

def $\text{Gamma}(\frac{1}{2}, \frac{d}{2}) =: \chi^2(d) \to$ Chi-squared dist d degrees of freedom

Fact | if X is a rv, $g: \mathbb{R} \to \mathbb{R}$ "Borel" function

$\Rightarrow$ $g \circ X$ is a rv

Fact | X is cts RV & , $g: \mathbb{R} \to \mathbb{R}$ is cts,

$\Rightarrow$ $g \circ X$ is cts RV!

e.g. | spose X is cts rv with density f_X

Find density of X^2 ?

We have that,

$$P(X \le x) = \int_{-\infty}^{x} f(t)\, dt$$

$$P(X^2 \le x) \Rightarrow P(-\sqrt{x} \le X \le \sqrt{x}) = P(X \le \sqrt{x}) - P(X \le -\sqrt{x})$$

if $x \le 0$ then it is 0 !

$$= \int_{-\sqrt{x}}^{\sqrt{x}} f(t)\, dt$$

$$= \int_{0}^{\sqrt{x}} f(t)\, dt - \int_{0}^{-\sqrt{x}} f(t)\, dt$$

$\boxed{\text{ftc remain}}$

Now, we get that $f_{X^2}(x) = \frac{1}{2\sqrt{x}} f(\sqrt{x}) + \frac{1}{2\sqrt{x}} f(-\sqrt{x})$

Prop | let X be cts rv $g: \mathbb{R} \to \mathbb{R}$ $\uparrow$ & diff'ble .

$g \circ X$ is a cts rv with density.

$$f_{g \circ X}(t) = \underline{\qquad\qquad}$$

if $x \le a$ $F = 0$

if $x \ge b$ $F = 1$

what if g not surj.

not true $\text{im}(g) = (a, b)$

Pf | Distribution

$$F_{(g(X))}(x) = P(g(X) \le x) = P(X \le g^{-1}(x))$$

$$= \int_{-\infty}^{g^{-1}(x)} f(t)\, dt$$

$$\therefore f_{g \circ X}(x) = \frac{\partial}{\partial t}\left(\int_{0}^{g^{-1}(x)} f(t)\, dt + \text{const} \right) = (g^{-1})'(x)\, f(g^{-1}(x))$$

Sums of cts RV

Claim if X, Y cts rv $\Rightarrow$ $X+Y$ too!

Recall joint dist funct

$$F_{X,Y} : \mathbb{R}^2 \to [0,1]$$
$$(x,y) \mapsto P(X \le x, Y \le y)$$

$$\lim_{y \to +\infty} F_{X,Y}(x,y) = P(X \le x) = F_X(x)$$ marginal dsn

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X, Y are cts RV

joint density $f_{X,Y}(s,t)$ is the function satisfying

$$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(s,t) \, dt \, ds$$

Special case, if X, Y ind

$$F_{X,Y}(x,y) = F_X(x) F_Y(y)$$
$$= \int_{-\infty}^{x} f_X(s) \, ds \cdot \int_{-\infty}^{y} f_Y(t) \, dt$$
$$= \int_{-\infty}^{x} \int_{-\infty}^{y} f_X(s) f_Y(t) \, dt \, ds$$

So, in this case $f_{X,Y}(s,t) = f_X(s) f_Y(t)$

X, Y cts

$$F_X(x) = \lim_{y \to \infty} F_{X,Y}(x,y) = \lim_{y \to \infty} \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(s,t) \, dt \, ds$$

monoton conv thm

$$= \int_{-\infty}^{x} \int_{-\infty}^{+\infty} f_{X,Y}(s,t) \, dt \, ds = \int_{-\infty}^{x} f_X(s) \, ds$$

(marginals)

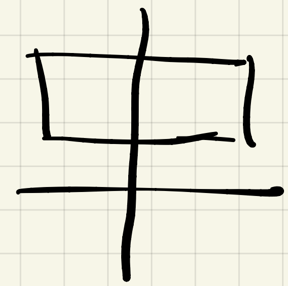
extracting x's density!

So, $f_X(s) = \int_{-\infty}^{+\infty} f_{X,Y}(s,t)\,dt$

$$f_Y(t) = \int_{-\infty}^{\infty} f_{X,Y}(s,t)\,ds$$

Fact 1 1. $P(a_1 \le X \le a_2,\ b_1 \le Y \le b_2)$

$$= \int_{a_1}^{a_2}\int_{b_1}^{b_2} f_{X,Y}(s,t)\,dt\,ds$$

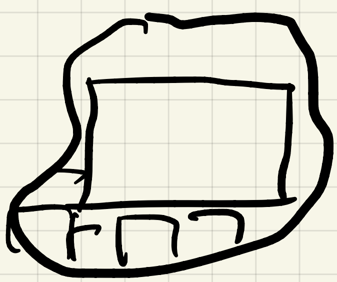


$$P((X,Y)\in A) = \int_A f_{X,Y}(s,t)\,dt\,ds$$

"victory domain"

holds for more general regions by approx by rectangle

A

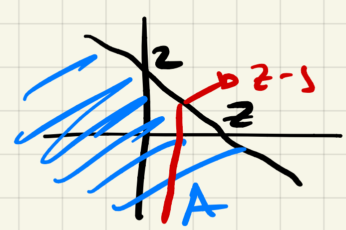


Finally showing $X+Y$ is cts rv

Claim: density of $X+Y$ $\quad f_{X+Y}(z) = \int_{-\infty}^{\infty} f_{X,Y}(x, z-x)\,dx$

Pf) $F_{X+Y}(z) = P(X+Y \le z)$

$$= P((X+Y)\in A)$$



$$= \int_A f_{X,Y}(s,t)\,dt\,ds$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{z-s} f_{X,Y}(s,t)\,dt\,ds = \int_{-\infty}^{\infty}\int_{-\infty}^{u} f_{X,Y}(s,u-s)\,du\,ds$$

$u = t+s$

Fubini's / marginal $= \int_{-\infty}^{u}$