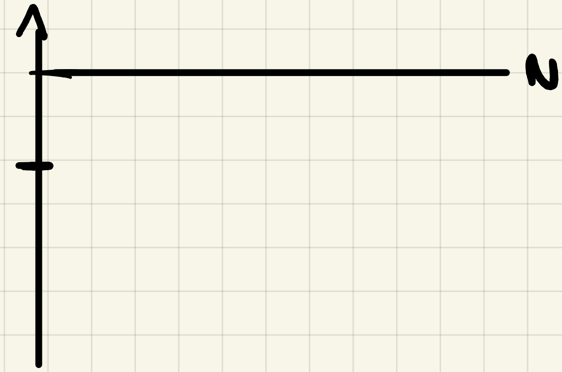
Lec 5

Observe) fair gambler's ruin.



flip picture

$$P(\text{win} \mid \text{start with } k)$$
$$= P(\text{lose} \mid \text{start with } N-k)$$
$$= \frac{N-(N-k)}{N} = \frac{k}{N}$$

Note: doesn't work for biased random walk (not symmetric)

Recall) $(\Omega, \mathcal{F}, P)$ is a prob sp.

1) A func $X : \Omega \to \mathbb{R}$ is a rv pt $\forall a \in \mathbb{R}$

$$\{\omega \in \Omega \mid X(\omega) \le a\} \in \mathcal{F}$$

usually write $\{X \le a\}$ for, $P(X \le a)$ the conceal

2) $F : \mathbb{R} \to [0,1]$ is the cumulative dist func of X pt

$$F(x) = P(\{X \le x\}) = P(\{\omega \in \Omega \mid X(\omega) \le x\})$$

usually write $P(X \le x)$

e.g. Repeatedly tossing a fair coin N times.

each time $\to$ head $\to$ gain 1
$\quad\quad\quad\;\;\to$ tail $\to$ lose 1

Start with k

let X_n be the position after n tosses

$\hookrightarrow \omega(X_n)$ is the position of the random walk at time $= n$

X_n is rv

$n = 3$

$$\Omega = \left\{ \begin{matrix} HHH, \\ \vdots \\ TTT, \end{matrix} \right\}$$

$$\mathcal{F} = \mathcal{P}(\Omega)$$

let $a \in \mathbb{R}$

note: $X_3(\{HHH\}) = \underline{\underline{k+3}}$

$$\{X_3 \le a\} = \begin{cases} \varnothing & a < k-3 \\ \{TTT\} & k-3 \le a < k-1 \\ \text{2 tails} \cup \text{3 tails} & k-1 \le a < k+1 \\ \text{2 tail, 3 tail, 1 tail} & k+1 \le a < k+3 \\ \Omega & k+3 \le a \end{cases}$$

Rmk if $\mathcal{F} = \mathcal{P}(\Omega)$ any func $X : \Omega \to \mathbb{R}$ is a rv

in particular if Ω is countable,

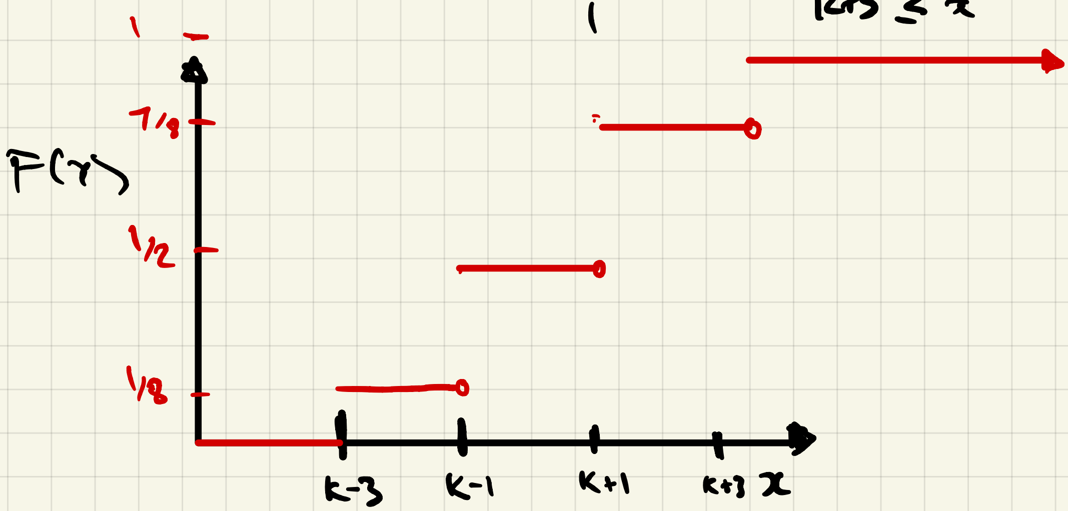
E.g. Dist func

Find distribution func of X_3 $(F : \mathbb{R} \to [0,1])$

$$F(x) = P(X_3 \le x)$$

E.g. $F(k+3) = 1$

$$F(x) = \begin{cases} 0 & x < k-3 \\ 1/8 & k-3 \le x < k-1 \\ 1/2 & k-1 \le x < k+1 \\ 7/8 & k+1 \le x < k-3 \\ 1 & k+3 \le x \end{cases}$$



Lemma let F be a dist func of a rv X.

(a) $F(x) \le F(y)$ if $x \le y$

(b) $\lim\limits_{x \to -\infty} F(x) = 0$

$\lim\limits_{x \to \infty} F(x) = 1$

(c) F is right cts $\forall x \in \mathbb{R}$

$$F(x) = \lim_{h \downarrow 0} F(x+h)$$

h goes to 0 from right $\lim\limits_{h \to 0^+}$

equiv $= \lim\limits_{y \downarrow x} F(y)$

d) $P(X > x) = 1 - F(x)$

e) $P(x < X \leq y) = F(y) - F(x)$

f) $P(X = x) = F(x) - \lim_{y \to x^-} F(y)$

Rmk) d) $\{X > x\}$ is an event since

$\{X > x\} = \{\underset{\in F}{X \leq x}\}^c \in F$

e) $\{x < X \leq y\} = \underset{\in F}{\{X \leq y\}} \setminus \underset{\in F}{\{X \leq y\}} \in F$ → by lemma

f) $\{X = x\}$ → countable int

$= \bigcap_{n=1}^{\infty} \underbrace{\{x - \frac{1}{n} < X \leq x + \frac{1}{n}\}}_{\in F \text{ by e)}} \in F$

alternatively,

$\{X = x\} = \underset{\in F}{\{X \leq x\}} \cap \{X \geq x\}$ ∈F

$\{X \geq x\} = \{\underline{X < x}\}^c$ ∈F

$\{X < x\} = \bigcup_{n=1}^{\infty} \{X \leq x - \frac{1}{n}\}$ ∈F

countable union

Pf) a) if $x \leq y$ — $A \subseteq B$

e) $F(y) = P(X \leq y)$ → $P(B) = P(A) + P(B \setminus A)$

$= P(X \leq x) + P(x < X \leq y)$

$= F(x) + P(x < X \leq y)$

$\geq F(x)$

b) $A_n = \{X \leq -n\} \Rightarrow A_n \supset A_{n+1} \supset A_{n+2}$

$\bigcap_{n=1}^{\infty} A_n = \emptyset$ since contra) if $w \in \bigcap_{n=1}^{\infty} A_n$

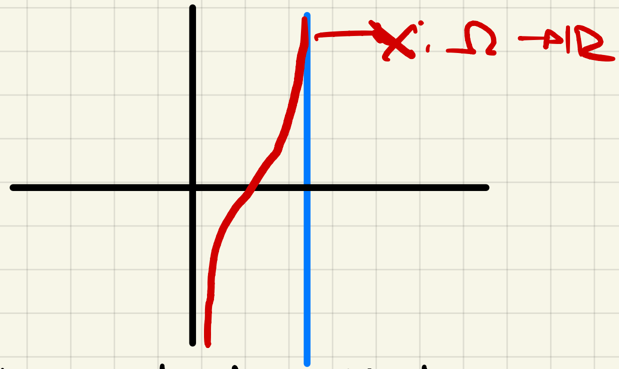
$\Rightarrow X(w) \leq -n \quad \forall n \in \mathbb{N}$

$\therefore X(w) \in \mathbb{R}$ & X not well def.

e.g. aside

if $\Omega = (0,1)$

$X : \Omega \to \mathbb{R}$



we see that $\forall n \in \mathbb{N}$

$\{X \le -n\} \ne \emptyset$

So, we can't claim A_n is eventually const

So, $0 = P(\emptyset) = \lim_{n \to \infty} P(A_n)$

$= \lim_{n \to +\infty} F(-n) = \lim_{n \to -\infty} F(n)$ $\varepsilon - \delta$ and monotonic

b) ii) similar $\Omega = \bigcup_{n=1}^{\infty} B_n$ where $B_n = \{X \le n\}$

c) $F(x) \to P(X \le x)$

$\{X \le x\} = \bigcap_{n=1}^{\infty} \{X \le x + \frac{1}{n}\}$ nested

$F(x) = P(X \le x) = \lim_{n \to \infty} P(X \le x + \frac{1}{n})$

$= \lim_{n \to \infty} F(x + 1/n)$ use A

$= \lim_{y \to x^+} F(y)$

d) $P(X > x) = P(\{X \le x\}^c)$

$= 1$