## Recall Convergence

Pointwise => almost surely => in prob => in dist.

None of the converses hold

- if $X_n \xrightarrow{D} c$ for $c \in \mathbb{R}$ => $X_n \xrightarrow{P} c$ } special case conv!

**Def** Given $\{X_n\}_{n \in \mathbb{N}}$ that are independent & have the same dist

$E(X_i) = \mu$, $\text{var}(X_i) = \sigma^2$

1) The seq $\{X_n\}$ is said to satisfy the weak law of large no (weak LLN) if $\frac{X_1 + \cdots + X_n}{n} \xrightarrow{P} \mu$

2) $(X_n)$ is said to satisfy Strong LLN if

$$\frac{X_1 + \cdots + X_n}{n} \xrightarrow{a.s} \mu$$

3) $(X_n)$ is said to satisfy Central lim thm [CLT] if

$$\frac{X_1 + \cdots + X_n - n\mu}{\sqrt{n}\,\sigma^2} \xrightarrow{D} N(0,1)$$

**Thm** If $E(|X_i|) < \infty$ then (same assumption)

1) (Khindchin) $(X_n)$ satisfies weak LLN

2) (Kolmogorov) $(X_n)$ satisfies Strong LLN

**Ex** Assume $E(X_i^2) < \infty$ => $E(|X_i|) < \infty$ ← Cauchy Schwartz / Hölder

Prove weak LLN!

following steps

① Use def of conv in prob to write out what we need

② Use Chebyshev

$P(\{\omega \in \Omega \mid |X_n(\omega) - \mu| \geq \varepsilon\}) \xrightarrow{n \to \infty} 0$

$P(|X - E(X)| \geq a) \leq \frac{\text{var}(X)}{a^2}$ → $P(X_n - E(X_n) \geq \varepsilon)$

$$Y_n = \frac{X_1 + X_2 \dots + X_n}{n} \qquad E(Y_n) = \mu$$

$$\therefore P(|Y_n - \mu| \geq \varepsilon) \leq \frac{\operatorname{var}(Y_n)}{\varepsilon^2} = \frac{\sigma^2}{\varepsilon^2} = \frac{\sigma^2}{n\varepsilon^2}$$

$n \to \infty \quad \downarrow 0$

Now let us get rid of stronger assumption ($E(X_i^2) < \infty$)

But first Levy continuing Thm

$(X_n)$ seq of RV, $X$ is an rv

$\Phi_{X_n}(t) \to \Phi_X(t) \quad \forall t \in \mathbb{R}$ ch functions

Then $X_n \xrightarrow{D} X$ in dist

Pf skip. On canvas

Stronger version

if $\Phi_{X_n}(t) \to \Phi(t)$ & $\Phi$ is cts at $0$

$\exists$ RV $X$ s.t. $X_n \xrightarrow{D} X$ & $\Phi_X = \Phi$

easier version, if $\int_{\mathbb{R}} |\Phi_X|\, dx < +\infty$

use levy inversion thm $\Rightarrow \bar{F}_{X_n} \to \bar{F}_X$

If $F_X$ cts at $t$ $\quad \bar{F}_X(t) = F_X(t)$

Remaining issue: $F_{X_n}$ may not be cts at $t$

ex: Resolve.

Pf) WLLN with only $E(|X_i|) < \infty$

First show $\frac{X_1 + \dots + X_n}{n} \xrightarrow{D} \mu$ & use earlier thm to get result

by levy, show $Y_n = \frac{X_1 + \dots + X_n}{n}$

$\Phi_{Y_n}(t) \to \Phi_\mu(t) = e^{it\mu}$

Recall that $\Phi_{cX}(t) = \Phi(ct)$

$\Phi_{X_1 + \dots + X_n}(t) = \prod \Phi_{X_i}(t)$ if $X_i$ ind

$$\Rightarrow \phi_{Y_n}(t) = \phi_{\Sigma X_n}\left(\frac{t}{n}\right) \overset{\text{indep}}{=} \prod_{i=1}^{n} \phi_{X_i}\left(\frac{t}{n}\right)$$

$$= \left[\phi_{X_1}\left(\frac{t}{n}\right)\right]^n \quad \text{same dist}$$

**Recall** taylor exp of $\phi_X$

if $E(|X|^k) < \infty$

$\Rightarrow \phi_X$ has taylor exp to order $k$

$\Rightarrow \phi_X(t) = E(e^{itX})$

$$= \sum_{j=0}^{k} \frac{E(X^j)}{j!}(it)^j + o(t^k)$$

taylor order 1

$$= \left[1 + E(X_1)\, i\frac{t}{n} + o\left(\frac{t}{n}\right)\right]^n$$

$$= \left[1 + \frac{it\mu}{n} + o\left(\frac{t}{n}\right)\right]^n$$

if we ignore last term we done!

$$\lim_{n\to\infty}\left(1+\frac{t}{n}\right)^n = e^t$$

ex: Show that $\left(1 + \frac{x}{n} + o\left(\frac{1}{n}\right)\right) \to e^x$

↳ ratio

**Thm** (Lindeberg - Levy)

If $E(X_i^2) < \infty$ & $\operatorname{var}(X_i) = \sigma^2 > 0$

$\Rightarrow (X_n)$ sat CLT

**Pf** Show

$$\frac{X_1 + \dots + X_n - n\mu}{\sqrt{n\sigma^2}} \to N(0,1)$$

if $X \sim N(0,1) \Rightarrow \phi_X(t) = e^{-t^2/2}$

By Levy we only need $\phi(t) \to e^{-t^2/2}$

let $Y_i = \frac{X_i - \mu}{\sigma}$ ($Y_i$ are ind & same dist)

$E(Y_i) = 0$ & $\operatorname{var}(Y_i) = E(Y_i^2) = 1$

$$\frac{X_1 + \dots + X_n - n\mu}{\sqrt{n\sigma^2}} = \frac{Y_1 + \dots + Y_n}{\sqrt{n}}$$

$$\phi_{\frac{Y_1+\dots+Y_n}{\sqrt{n}}}(t) = \left[\phi_{Y_1}\left(\frac{t}{\sqrt{n}}\right)\right]^n = \left[1 + 0 + \frac{1}{2!}\left(\frac{it}{\sqrt{n}}\right)^2 + o\left(\frac{t^2}{\sqrt{n}^2}\right)\right]$$

same dist

taylor 2nd moment

$$= \left[1 - \frac{t^2}{2n} + o\left(\frac{t^2}{n}\right)\right]$$

argue as earlier $= e^{-t^2/2}$

e.g. Rolling a fair die $n$ times

$S_n$ total score of $n$ rolls

Estimate

① $P(|S_n - \frac{7}{2}n| < \varepsilon n)$

② $P(|S_n - \frac{7}{2}n| < 5\sqrt{n})$

using WLLN & CLT & Chebyshev & Bernstein