Recall

Levy's Thm: X is RV w/ char ϕ s.t. $\int_{-\infty}^{\infty} |\phi| \, dt < +\infty$ then

$\forall a, b \in \mathbb{R}$

$$\overline{F}(b) - \overline{F}(a) = \lim_{N \to \infty} \int_{-N}^{N} \frac{e^{-ita} - e^{-itb}}{2\pi i} \phi(t) \, dt$$

Cor) X, Y RV.

If $\phi_x = \phi_y \Rightarrow X, Y$ same dist

Def) $E(X) = \int_{\Omega} X \, dP$

whenever $\int_{\Omega} |X| \, dP < +\infty$

Markov's inequality $E(|X|) < +\infty$ & $X \geq 0 \Rightarrow$

$$P(X \geq a) \leq \frac{E(X)}{a} \quad \forall a > 0$$

Pf) $X \geq X \cdot 1_{\{X \geq a\}}$ for positive X

$\geq a \cdot 1_{\{x \geq a\}}$

$\therefore E(X) \geq E(a \cdot 1_{\{x \geq a\}}) = a \, P(X \geq a)$

Cor) Chebyshev's Inequality

Assume $E(|X|) < +\infty$, $E(X^2)$ [variance is def]

$$P(|X - E(X)| \geq a) \leq \frac{\text{var}(X)}{a^2}$$

Pf) Apply markov to

$[X - E(X)]^2$

markov

$$P(|X - E(X)| \geq a) = P(|X - E(X)|^2 \geq a^2) \leq \frac{E((X - E(X))^2)}{a^2}$$

$$= \frac{\text{Var}(X)}{a^2}$$

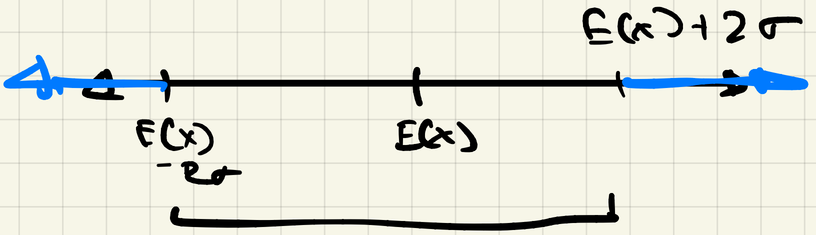
Cor1 Suppose $\operatorname{var}(X) = \sigma^2$

Chebyshev to X & $a = \theta\sigma$

$$P(|X - E(X)| \geq \theta\sigma) \leq \frac{1}{\theta^2}$$

In particular, $\theta = 2$

$$P(|X - E(X)| \geq 2\sigma) \leq \frac{1}{4}$$



→ take value in here $\geq 75\%$.

E.g.1 Rolling a fair die 10000 time. X = total score.

$$E(X) = 3.5 \times 10000 = 35000$$

$$P(|X - 35000| > 1000) \leq \frac{\operatorname{Var}(X)}{1000^2} \leq \frac{35}{1200}$$

var linear as ind

$\to \frac{1}{6}\cdot 1 + \frac{1}{6}\cdot 4 + \frac{1}{6}\cdot 9 + \frac{1}{6}\cdot 16 + \frac{1}{6}\cdot 25 + 6$

$= \frac{91}{6}$ $\qquad \frac{91}{6} - \frac{49}{4} = \frac{35}{12}$

$\Rightarrow \operatorname{var}(X) = \frac{350000}{12}$

Bernstein's Inequality

$X_1 \ldots X_n$ indep $\quad E(X_i) = 0, \ |X_i| < 1$

$X = \sum X_i, \ \forall a \geq 0$

① $P(X \geq a) \leq e^{-\frac{a^2}{2n}}$

② $P(X \leq -a) \leq e^{-\frac{a^2}{2n}}$

Can get ② from ① by applying ① to $(-X)$

Pf \1

$$P(X \ge a) = P(e^{tX} \ge e^{ta}) \overset{\text{markov}}{\le} \frac{E(e^{tX})}{e^{ta}}$$

$\to$ moment gen fnc $\downarrow$ X is sum of ind rv so, moment is prod

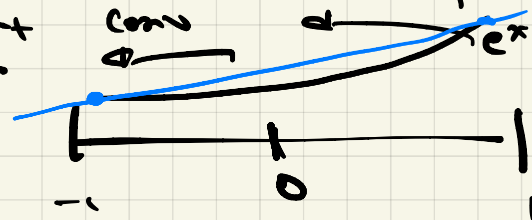
$$= e^{-ta} E(e^{tX_1}) \cdots E(e^{tX_n})$$

<u>Estimate</u>

$$E(e^{tX_j})$$

(slope $\frac{e^t - e^{-t}}{2t}$

int $\frac{e^t + e^{-t}}{2}$



$\forall x \in [-t, t]$

$$e^x \le \frac{e^t - e^{-t}}{2t} x + \frac{e^t + e^{-t}}{2}$$

as $X_i \in [-1, 1]$ $\quad tX_i \in [-t, t]$

$$e^{tX_i} \le \frac{e^t - e^{-t}}{2i} X_i + \frac{e^t + e^{-t}}{2}$$

$$\Rightarrow E(\;) \le E(\;) = 0 + \frac{e^t + e^{-t}}{2}$$

$$e^t = 1 + t + \frac{t^2}{2} + \frac{t^3}{3!} \cdots$$

$$e^{-t} = 1 - t + \frac{t^2}{2} - \frac{t^3}{3!} \cdots$$

$$\Rightarrow \frac{e^t + e^{-t}}{2} = 1 + \frac{t^2}{2} + \frac{t^4}{4!} \cdots\cdots$$

$$e^{\frac{t^2}{2}} = 1 + \frac{t^2}{2} + \frac{t^4}{2^2 \cdot 2!} + \frac{t^6}{2^3 \cdot 3^2}$$

coeff $(2n)!$ vs $(2)^n n!$ we note $\ge$

$$\therefore \Rightarrow e^{t^2/2} \ge \frac{e^t + e^{-t}}{2}$$

holds $\forall t$ where moment gen fnc def

$$\Rightarrow P(X \ge a) \le e^{-ta} \{e^{t^2/2}\}^n = e^{-ta} e^{n\frac{t^2}{2}} = e^{\frac{nt^2}{2} - ta}$$

so minimize $\frac{n}{2}t^2 - ta = -\frac{b}{2a} = \frac{a}{n}$ $\le e^{\frac{-a^2}{2n}}$

E.g.| Back to die problem

Use bernsteins inequality to give an upper bd to

$P(|X - 35000| \geq 1000)$

we note, $\frac{X_i - 3.5}{2.5} =: Y_i$ has our property. (normalize)

→ indep . . . - -

we have $X - 35000 = (Y_i \cdot 1000) 2.5 \to \left(\frac{X_i - 3.5}{2.5}\right) 1000$

$Y \Rightarrow \frac{X}{2.5}$

$P(|X - 35000| \geq 1000) = P\left(|Y| \geq \frac{1000}{2.5}\right)$

$= P(|Y| \geq 400)$

Bernstien

bern

$= P(Y \geq 400) + P(Y \leq 400) = 2P(Y \geq 400)$

$= 2e^{-\frac{400^2}{20000}} = 2e^{-\frac{160000}{20000}}$

$= 2e^{-8}$

Our other bound $\frac{35}{1200}$ (Chebyshev)

This is better

E.g.| SSRV start from 0

Y_n = position at time n

$P(|Y_n| \geq k)$? $k = c\sqrt{n}$? ↩ bernstien

let X_i be the move at each step,

we have $Y_n = \sum_{i=1}^{n} X_i$

$P(|Y_n| \geq k) = 2P(Y_n \geq k) = 2e^{-\frac{k^2}{2n}}$

if $k = c\sqrt{n}$

$P(|Y_n| \geq k) = 2e^{-\frac{c^2 n}{2n}} = 2e^{-\frac{c^2}{2}}$