Cts RV

Recall: X is a cts rv, dist F, density f

$$F(x) = \int_{-\infty}^{x} f(t)\, dt$$

If g strictly increasing & diff'ble then

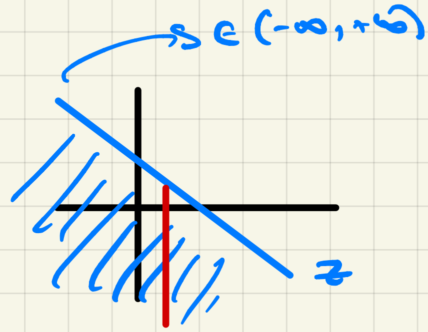
$$\cancel{f_{g(x)}(z) = f(g(z)) \cdot g'(z)} \qquad f_{g(x)}(z) = (g^{-1})'(z)\, f(g^{-1}(z))$$

X, Y rv, joint dist $F_{X,Y}$, joint density $f_{X,Y}$

$A \subseteq \mathbb{R}^2$ be "reasonable" domain (approx by boxes)

$P((X,Y)\in A)$

$$\int_A f_{x,y}(s,t)\, ds\, dt$$



Prop $f_{X+Y}(z) = \int_{-\infty}^{\infty} f(x, z-x)\, dx$

Pf $P(X+Y \le z) = P((X,Y) \in A_z)$

$$= \int_{A_z} f_{x,y}(s,t)\, ds\, dt$$

$u = t+s$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{z-s} f_{x,y}(s,t)\, ds\, dt$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{z} f_{x,y}(s, u-s)\, du\, ds$$

$$= \int_{-\infty}^{z}\int_{-\infty}^{\infty} f_{x,y}(s, u-s)\, ds\, dt$$

$P(X+Y \le z) = \int_{-\infty}^{z} f_{x+y}(u)\, du$

Recall!

$$\Rightarrow f_{X+Y}(u) = \int_{-\infty}^{\infty} f_{x,y}(s, u-s)\, ds$$

$\longrightarrow$

Cor) if X, Y ind $\longrightarrow f_X * f_Y(u)$

$$f_{X+Y}(u) = \int_{-\infty}^{\infty} f_X(s)\, f_Y(u-s)\, ds$$

Def) if $g, h : \mathbb{R} \to \mathbb{R}$ Then the conv of g & h denoted $g * h$ is

↳ convolution

$$g * h(z) = \int_{-\infty}^{\infty} g(x)\, h(z-x)\, dx$$

whenever This exists

Ex) $X, Y \sim N(0,1)$ ind

density func $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

Find density of $X+Y$

we have that

$$f_{X+Y}(u) = \int_{-\infty}^{\infty} f_X(s)\, f_Y(u-s)\, ds$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(u-s)^2}{2}}\, ds$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\frac{-s^2-(u-s)^2}{2}}\, ds$$

$(u-s)^2 + s^2$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\frac{-s^2-u^2+2us-s^2}{2}}\, ds$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\frac{-2s^2-u^2+2us}{2}}\, ds$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{(u^2+2s^2-2us)}{2}}\, ds \rightarrow$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{((u-s)^2+s^2))}{2}}\, ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\frac{-((s-\frac{u}{2})^2+\frac{u^2}{4})}{}}\, ds$$

$$= \frac{1}{2\pi} e^{-\frac{z^2}{4}} \int_{-\infty}^{\infty} e^{-(x-\frac{z}{2})^2}\, ds$$

let $\frac{t}{2} = x - \frac{z}{\sqrt{2}}$

$\frac{ds}{\sqrt{2}} = dt$

$$= \frac{1}{2\pi} e^{-\frac{z^2}{4}} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2}}\, \frac{dt}{\sqrt{2}}$$

• This is just normal without

$$= \frac{1}{2\pi} e^{-\frac{z^2}{4}} \frac{1}{\sqrt{2}} \cdot 1 \cdot \sqrt{2\pi}$$

$$= \frac{1}{2\sqrt{\pi}} e^{-z^2/4}$$

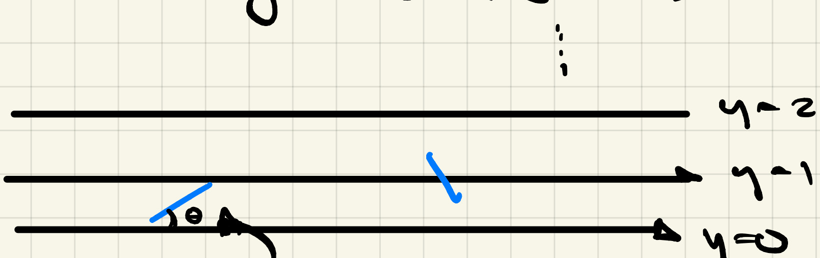
$\therefore\ X+Y \sim N(0,2)$

E.g.) Buffon's needle

on plane w/ line $y=m$ $m \in \mathbb{Z}$

Throw a needle of length 1 at random

P(Needle intersects line)



(x_1, y_1)

(x_2, y_2)

Spose Θ is the rv for the angle

Spose Y is the rv for the y coordinate of the center

WLOG center lies between $\{0 \le y \le 1\}$ uniformly

↳ by translation

$Y \sim \text{Unif}([0,1])$ $\qquad$ $\Theta \sim \text{Unif}([0, \pi])$

π > is a typo

& Y, Θ are ind.

Needle crosses line iff

$$1 \le \frac{1}{2}\sin\Theta + y \quad \text{or} \quad y - \frac{1}{2}\sin\Theta \le 0$$

$$A = \left\{ (y,\theta) \in [0,1] \times [0,\pi] : y \le \frac{1}{2}\sin\theta \text{ or } y \ge 1 - \frac{1}{2}\sin\theta \right\}$$

needle crosses line! $(Y, \Theta) \in A$

$\forall (y,\theta) \in [0,1] \times [0,\pi]$

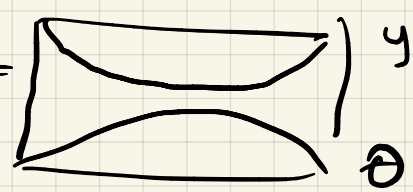
$$P(\text{needle crosses line}) = P((Y,\Theta) \in A)$$

$$= \int_A f_{Y,\Theta}(y,\theta)\, dy\, d\theta \qquad = \text{ind so} = f_Y(y)\, f_\Theta(\theta) = 1 \cdot \frac{1}{\pi}$$

$$= \int_A f_Y(y)\, f_\Theta(y)\, dy\, d\theta$$

density fing

$$= \frac{1}{\pi}\int_A dy\, d\theta$$

A = 

$$= \frac{1}{\pi}\int_0^{\pi} \left(\int_0^{1/2 \sin\theta} dy + \int_{1 - 1/2\sin\theta}^{1} dy \right) d\theta$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin\theta \, d\theta$$

$$= \boxed{2/\pi}$$

E.g.) Poisson Process $\lambda > 0$

Consider # of phone calls that a store gets.

If t_1, t_2 are times.

$X_{[t_1, t_2]}$ = no of calls within $(t_1, t_2]$ time int

want to have a model st. within $[t, t+\Delta t]$ with $\Delta t << 1$

Then the P of getting a call is $\lambda \Delta t$

Assume no more than 1 call at a time

Thus when $\Delta t << 1$ this is

$X_{[t, \Delta t]} \sim Ber(\lambda \Delta t)$

Thus!

$X_{[t_1, t_2]}$ is a sum of $\frac{t_2 - t_1}{\Delta t}$ intervals & Bernoulli rv $\rightarrow \boxed{\lambda \Delta t}$

$\rightarrow$ so it is binom

as we reduce Δt, we approach a Poisson

$X_{(t_1, t_2)} \sim Pois(\lambda(t_2 - t_1))$

Def) This is the Poisson Process

$\rightarrow$ A Poisson process with intensity λ is a family of rv $X_{[s,t]}$ $s, t \in [0, \infty)$ s.t

1. $X_{(s,t]} \sim Pois(\lambda(t-s))$ $\forall$ $s < t \in (0, \infty)$
2. $0 \leq r < s < t < \infty$

$X_{[r,t)} = X_{(r,s]} + X_{[s,t)}$

3. disjoint int are ind r.v.

if $\{ (s_i, t_i] \; \forall i \in I \}$ pairwise disjoint $\Rightarrow \{ X_{(s_i, t_i]} : i \in I \}$ $\boxed{\text{ind}}$

Y = waiting time for the first call

$$P(Y \leq z) = 1 - P(Y > z)$$ first call after z

$$= 1 - P(X_{[0,z]} = 0)$$ → no call in z time

$$= 1 - e^{-\lambda z} \frac{(\lambda z)^0}{0!} \rightarrow 1$$

$$= 1 - e^{-\lambda z}$$

Density!

$$f_Y(z) = \lambda e^{-\lambda z} \longrightarrow \text{exponential dist}$$

$$Y \sim \text{Exp}(\lambda)$$