Last time

if $X, Y \sim N(0,1)$ indep

$\Rightarrow X+Y \sim N(0,2)$

in general $X \sim N(\mu_1, \sigma_1^2)$, $Y \sim N(\mu_2, \sigma_2^2)$

$\Rightarrow X+Y \sim N(\mu_1+\mu_2, \sigma_1^2 + \sigma_2^2)$

Poisson Process

Def) X cts rv with density f.

$$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

whenever this is defined!

Def) Variance of X cts rv

$$\mathrm{Var}(X) = E((X - E(X))^2)$$

$$k^{th} \text{ moment} = E(X^k)$$

e.g.) $X \sim N(\mu, \sigma^2)$

$y = x - \mu$

$$E(X) = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx$$

$$= \int_{-\infty}^{\infty} (y+\mu) \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}}\, dy$$

$$= \int_{-\infty}^{\infty} \frac{y e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}\, dy + \mu \int_{-\infty}^{\infty} \frac{e^{-\frac{y^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}\, dy$$

odd func

$$= 0 + \mu \cdot 1$$

this is just the density

$$= \mu$$

exercise) ① Exp & var of rv of following

- normal
- unif
- exp
- gamma

Prop) a) $E(X+Y) = E(X) + E(Y)$

b) $E(cX) = cE(X) \quad \forall c \in \mathbb{R}$

c) $E(X+c) = E(X) + c$

Pf) a) $E(X+Y)$

$$= \int_{-\infty}^{\infty} z \, f_{X+Y}(z) \, dz$$

$$= \int_{-\infty}^{\infty} z \int_{-\infty}^{\infty} f_{X,Y}(s, z-s) \, ds \, dz$$

$t = z - s$

$\Rightarrow s = z - t$

$z = s + t$

$dz = dt$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (t+s) \, f_{X,Y}(s,t) \, ds \, dt$$

marginal!

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} t \, f_{X,Y}(s,t) \, ds \, dt + \int \int_{-\infty}^{\infty} s \, f_{X,Y}(s,t) \, dt \, ds$$

$$= \int_{-\infty}^{\infty} t \, f_Y(t) \, dt + \int_{-\infty}^{\infty} s \, f_X(s) \, ds$$

$$= E(Y) + E(X)$$

b) Exercise

(c) Let $c \in \mathbb{R}$ $\longrightarrow$ density of X

$$F_{X+c}(y) = P(X + c \le y) = P(X \le y - c)$$

$$= \int_{-\infty}^{y-c} f_X(t) \, dt$$

take derivative

$$f_{X+c}(y) = f_X(y-c)$$

So,

$$E(X+c) = \int_{-\infty}^{\infty} y \, f_{X+c}(y) \, dy \qquad dy = du$$

$$= \int_{-\infty}^{\infty} y \, f_X(y-c) \, dy \qquad u = y - c$$

$$= \int_{-\infty}^{\infty} (u+c) f_X(u) \, du = \int_{-\infty}^{\infty} u f_X(u) \, du + c \int_{-\infty}^{\infty} f_X(u) \, du$$

$$= E(X) + c$$

Prop) X cts RV w/ density f

g (cts) appropriate function

($g(x)$ is a cts rv)

$$E(g(X)) = \int_{-\infty}^{\infty} g(x) f(x)\, dx$$

Cor) $$E(X^k) = \int_{-\infty}^{\infty} x^k f(x)\, dx$$

Eg.) $X \sim N(\mu, \sigma^2)$

$Var(X) = E((X - E(X))^2)$ — exp and stuff

$= E(X)^2 - E(X^2)$

for $X \sim N(\mu, \sigma) \Rightarrow E(X)^2 = \mu^2$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx$$

$y = \frac{(x-\mu)}{\sigma}$

$\sigma\, dy = dx$

$$= \int_{-\infty}^{\infty} (\sigma y + \mu)^2 \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi\cancel{\sigma^2}}} \cancel{\sigma}\, dy$$

$$= \int_{-\infty}^{\infty} (\sigma^2 y^2 + 2\sigma y \mu + \mu^2) \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}\, dy$$

$$= \int_{-\infty}^{\infty} \frac{\sigma^2 y^2 e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} + \int_{-\infty}^{\infty} 2\sigma\mu y \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}\, dy + \mu^2 \int_{-\infty}^{\infty} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}$$

odd

$$= \int_{-\infty}^{\infty} \sigma^2 y^2 \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} \quad 0 \quad + \quad \mu_2$$

$$= \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{y}_{u} \cdot \underbrace{y \cdot e^{-y^2/2}\, dy}_{dv}$$ — IBP $\Rightarrow v = -e^{-y^2/2}$

$$= \frac{\sigma^2}{\sqrt{2\pi}} \left[y e^{-y^2/2} \Big|_{-\infty}^{+\infty} - \int_{-\infty} (-e^{-y^2/2})\, dx \right]$$ — looks like full int but not exp

$$= \frac{\sigma^2}{\sqrt{2\pi}} \left[0 \text{ apply lim} + \sqrt{2\pi} \right] = \sigma^2$$

$\therefore \ var(X) = (\sigma^2 + \mu^2) - \mu^2 = \sigma^2$

Lemma 1 If X is cts rv taking only non-neg values.

Then $E(X) = \int_0^{+\infty} P(X > x)\, dx$

$= \int_0^{+\infty} (1 - F_X(x))\, dx$

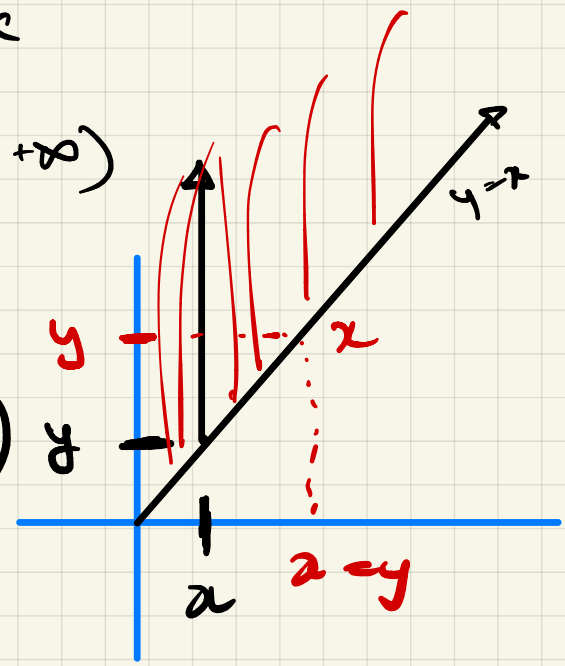
Pf) RHS

$\int_0^{\infty} P(X > x)\, dx \longrightarrow X \in [x, +\infty)$

$= \int_0^{\infty} \int_x^{+\infty} f_X(y)\, dy\, dx$

switch order

$= \int_0^{\infty} \int_0^{y} f_X(y)\, dx\, dy$



$= \int_0^{\infty} f_X(y) \int_0^{y} dx\, dy$

non negative rv

$= \int_0^{\infty} y f_X(y)\, dy \longrightarrow = E(X)$

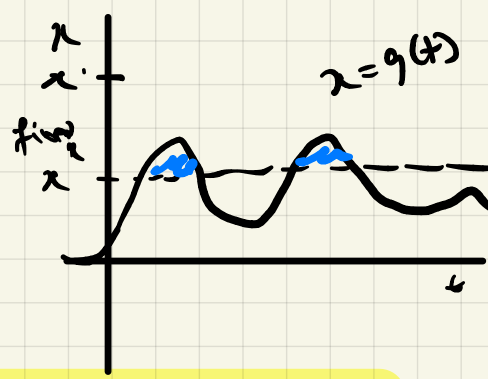
Pf of Prop) $E(g(X)) = \int_{-\infty}^{\infty} g(x) f(x)\, dx$

Assume $g \geq 0 \Rightarrow Y = g(X) \geq 0$

$\therefore E(Y) = \int_0^{\infty} P(Y > x)\, dx$

$= \int_0^{\infty} P(g(X) > x)\, dx$

$= \int_0^{\infty} \int_{\{g(t) > x\}} f_X(t)\, dt\, dx$



$= \int_{-\infty}^{\infty} \int_0^{g(t)} f_X(t)\, dx\, dt = \int_{-\infty}^{\infty} f(t) g(t)\, dt$

Conditional Exp

Goals: Define

- $E(X \mid Y = y)$
- $E(X \mid Y)$
- $E(X) = E(E(X \mid Y))$

Next $\rightarrow$ conditional dist & density