Recall

Generating function of hitting time of rw

→ Branching process

Z_n = # branches at n & $Z_0 = 1$

we see

$$G_{Z_n}(s) = \underbrace{G_{Z_1} \circ \cdots \circ G_{Z_1}}_{n\text{-times}}$$

$$E(Z_n) = E(Z_1)^n$$

Can also compute var(Z_n) in terms of $E(Z_1)$ & var(Z_1)

- P of extinction at n is $P(Z_n = 0)$

set $\eta_n = P(Z_n = 0)$

- prob of ultimate ext ⇒ $\lim_{n\to\infty} \eta_n$

$\Updownarrow$

→ $\exists N$ st $Z_n = 0$

Prop1 $\eta_n \uparrow \eta$ we have $G_{Z_1}(\eta) = \eta$ → fixed pt

ult ext $= \bigcup_{n=1}^{\infty} \{Z_n = 0\}$ we note $\{Z_n = 0\} \subseteq \{Z_{n+1} = 0\}$

↳ trivial

↳ nested sequence of events → union is ult ext

⇒ η_n is mono increasing sequence (as the events are nested)

$$\eta = \lim_{n\to\infty} \eta_n \qquad \square$$

Alt) $\eta_{n+1} = G_{Z_{n+1}}(0) = G_{Z_1}(G_{Z_n}(0))$

$= G_{Z_1}(\eta_n) \geq \eta_n$

Note $G_{Z_1}'(s) \geq 0 \quad \forall s \in [0,1]$ so $\square$

why is y a fixed pt? $G_{Z_1}(y) = y$

$y = \lim_{n\to\infty} y_n$ & G_{Z_1} is cts

So, $G_{Z_1}(y) = \lim_{n\to\infty} G_{Z_1}(y_n) = \lim_{n\to\infty} y_{n+1} = y$

we've got a fixed pt baby!!

$\therefore$ to find y, just solve G_{Z_1} instead of taking limit ....

is it unique? if >1 what should y be?

we note y solves $G_{Z_1}(s) = s$

we see $G_{Z_1}'(s) = \sum_{n=1}^{\infty} P(Z_1 = n)\, n\, s^{n-1} \geq 0 \quad \forall s \in [0,1]$

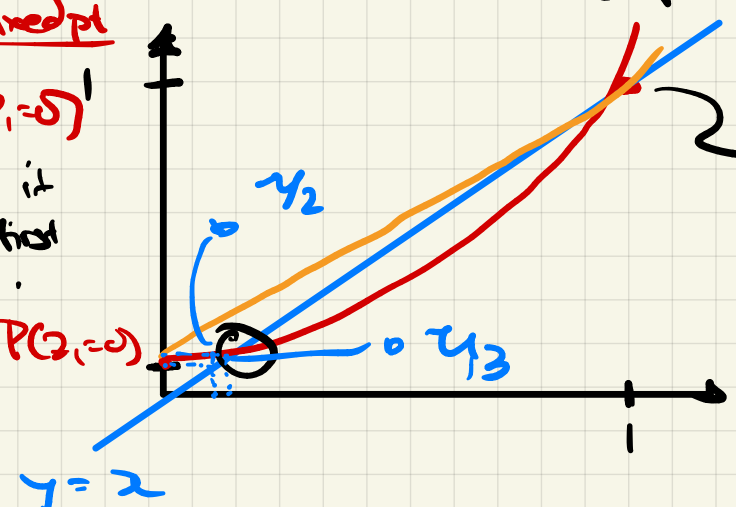
$G_{Z_1}''(s) \geq 0 \quad \forall s \in [0,1]$ so it is like super convex

$\therefore$ graph $G_{Z_1}(s)$

which fixed pt

$y_1 = P(Z_1=0)$

we see it converges to first fixed pt.



$G_{Z_1}(1) = 1$ by earlier

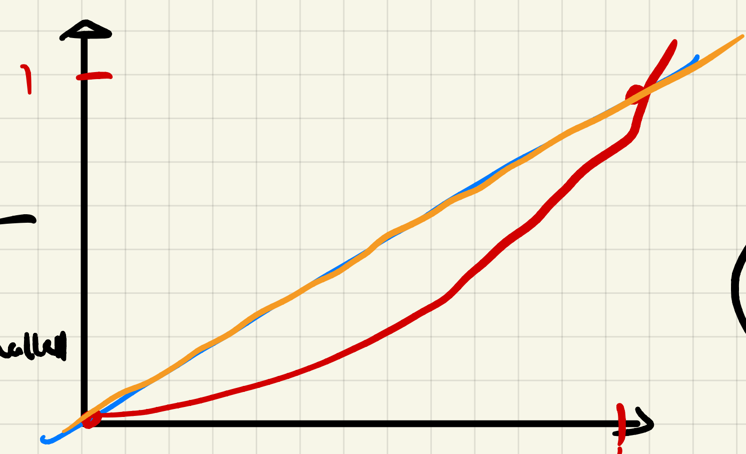
already have a fixed pt

$G_{Z_1}(0) = P(Z_1=0)$

case >0

in this case 1 fixed pt in $[0,1]$ so $y = 1$

in this case 2 fixed pt in $[0,1]$ so y = circle



$y = 0$

in general, it is the smallest f.p

$P(Z_1 = 0) = 0$

logically, never extinct as $\Rightarrow P(Z_1=0)$ coefficient

each branch gives 1 of next gen

E.g.) Geometric Branching process

$Z_1 = X - 1$ where $X \sim \text{Geo}(p)$

Find prob of ult ext.?

$$G_{Z_1}(s) = \frac{p}{1 - s + ps}$$

$$p = s(1 - s + ps) = s - s^2 + ps^2 - p = 0$$

always smaller root

$$s^2(p-1) + s - p = 0$$

$$\cancel{\frac{-1 \pm \sqrt{1 + 4p(p-1)}}{2(p-1)}}$$

1 always a fixed pt so, we can factor

$$(s-1)(\underline{s(p-1) + p})$$

$$s(p-1) + p = 0$$

$$= \frac{p}{1-p}$$

So, $\min\{1, \frac{p}{1-p}\} = \eta$

Characteristic Func

Gen func kinda nice so far, but its kinda restricting

→ usually works "well" for disc r.v. taking val in $\mathbb{Z}_{\geq 0}$

Def let X be a r.v.

The moment gen func of X

$$M_X(t) := E(e^{tX})$$ → where defined

Prop ① $E(X^k) = M^{(k)}(0)$

② $M(t) = \sum_{k=0}^{\infty} \frac{E(X^k)}{k!} t^k$

these hold in radius of conv

③ If X, Y ind $M_{X+Y}(t) = M_X(t) M_Y(t)$

Def let X be a rv. The char func of X.

$$\phi_X(t) : \mathbb{R} \to \mathbb{C}$$
$$t \mapsto E(e^{itX})$$

Rmk $\phi_X(t)$ is always well def

$$|\phi(t)| \leq 1$$

e.g. X is cts

$$\left| \int_{\mathbb{R}} e^{itx} f_X(x) \right| \leq \int_{\mathbb{R}} |e^{itx} f_X(x)| \, dx$$

unit circ so

$$= \int_{\mathbb{R}} |e^{itx}| f_X(x) \, dx$$

$$= \int_{\mathbb{R}} f_X(x) \, dx = 1$$

e.g. $X \sim$ Bernoulli (p)

$$\phi_X(t) = E(e^{itX}) = \sum_{n=0}^{\infty} e^{itn} \underbrace{P(X=n)}_{0 \text{ when nonzero}}$$

$$= e^0 (1-p) + p e^{it}$$

$$= \boxed{1 - p + pe^{it}}$$

2) $X \sim$ Binom (will help when $\phi_{X+Y} = \phi_X \phi_Y$ for ind X, Y)

$$\phi_X(t) =$$

3) $X \sim$ Exp (λ)

4) $X \sim N(0,1)$

$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{itx} e^{-\frac{x^2}{2}} \, dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{1}{2}(x^2 - 2itx - t^2 + t^2)}$$

(actually cont int over $\mathbb{C}$)

$$= \frac{e^{-t^2/2}}{\sqrt{2\pi}} \int e^{-\frac{1}{2}(x - it)^2} \, dx = e^{-t^2/2}$$

kinda $\int \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} dy$

→ skip deets