Quiz Correction

$$F_{T_2}(z) = 1 - e^{-\lambda z} - e^{-\lambda z}\lambda z$$

$$f_{T_2}(z) = \lambda^2 e^{-\lambda z} z \quad \text{when } z > 0, \ 0 \text{ otherwise}$$

Recall

Let X be rv with non negative values

$G_X(s) = E(s^X)$

if X is disc with $\mathbb{N}$ val $\;\; = \sum_{n=0}^{\infty} P(X=n)\, s^n$

$a = a_n$

$G_a = \sum_{n=0}^{\infty} a_n s^n$

whenever con

Thm) $E(X(X-1)\cdots(X-k+1)) = \lim_{s \to 1^-} G_X^{(k)}(s)$

in particular, if radius of conv $R > 1$

$$E(X(X-1)\cdots(X-k+1)) = G_X^{(k)}(1)$$

Series

$\sum_{n=0}^{\infty} a_n s^n$ radius of conv $R = \dfrac{1}{\limsup a_n^{1/n}}$

if X is rv (discrete & $\mathbb{N}$ valued)

$G_X(s) = \sum_{n=0}^{\infty} P(X=n)\, s^n$ if $a_n = P(X=n) \leq 1$

$\therefore R = \dfrac{1}{\limsup(a_n^{1/n})} \geq 1$

Corr) We know k^{th} moment $E(X^k)$ from $G_X(s)$

$= E(X^2) = E(X(X-1)) + E(X)$

$G_X^{2}(1) + E(X) = G_X''(1) + G_X'(1)$

Prop | $G_{X+Y}(s) = G_X(s)\, G_Y(s)$

Pf | X, Y disc $\mathbb{N}$ valued

convolution

$$G_{X+Y}(s) = \sum_{n=0}^{\infty} P(X+Y=n)\, s^n$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} P(X=k, Y=n-k)\, s^n$$

$$= \sum_{n=0}^{\infty} s^n \sum_{k=0}^{n} P(X=k)\, P(Y=n-k)$$

$$G_X(s)\, G_Y(s) = \left(P(X=0) + P(X=1)s + P(X=2)s^2 + \cdots\right)\left(P(Y=0) + P(Y=1)s + P(Y=2)s^2 + \cdots\right)$$

$$= P(X=0)P(Y=0) + s\left(P(X=0)P(Y=1) + P(X=1)P(Y=0)\right)$$

$$+ s^2\left(P(X=0)P(Y=2) + P(Y=1)P(X=1) + P(Y=0)P(X=2)\right)$$

$$\cdots$$

D

In general |

If $X_1 \cdots X_n$ independent

$$G_{X_1+\cdots+X_n}(s) = \prod_{i=1}^{n} G_i(s)$$

Pf | induct but be careful about $X_1 + X_2$ is ind from X_3

E.g. | $X \sim \text{binomial}(n, p)$

$G_X(s) = ?$

$X = X_1 + \cdots X_n$ is bernoulli & ind

by thm $G_X(s) = \prod_{i=1}^{n} G_{X_i}(s)$ $= \left[G_{X_1}(s)\right]^n$

$$= \left[(1-p) + ps\right]^n$$

SSRW) Symmetric simple rw w 2 barries 0 & N N > 0

Spose $\boxed{W_k}$ is the first barrier hitting the starting from k.

Q) G_{W_k}

we see $G_{W_0}(s) = 1 = G_{W_N}(s)$

$0 < k < N$

$$P(W_k = n) = P(W_k = n \mid \text{1st left}) P(\text{1 left}) + P(W_k = n \mid \text{1st step right}) P(\text{1st right})$$

$$= P(W_{k+1} = n-1)\frac{1}{2} + P(W_{k-1} = n-1)\frac{1}{2}$$

$$G_W(s) = \sum_{n=0}^{\infty} P(W=n) s^n$$

$$G_{W_k}(s) = \sum_{n=0}^{\infty} P(W_k = n) s^n$$

$$= \sum_{n=0}^{\infty} s \frac{P(W_{k-1} = n-1)}{2} s^{n-1} + s\frac{P(W_{k+1} = n-1)}{2} s^{n-1}$$

$$= \frac{s}{2} G_{W_{k-1}}(s) + \frac{s}{2} G_{W_{k+1}}(s)$$ if $N=5$

$\to$ solve by poly

$G_{W_0} = G_{W_5} = 1$

$G_{W_1} = \frac{s}{2} + \frac{s}{2} G_{W_2}(s)$

$G_{W_2} = \frac{s^2}{4} + \frac{s^2}{4} G_{W_2} + \frac{s}{2} G_{W_3}$ but by symmetry $W_2 = W_3$

$G_{W_2} = \frac{s^2}{4} + \frac{s^2}{4} G_{W_2} + \frac{s}{2} G_{W_2}$

$4x = s^2 + s^2 x + \frac{s}{2} x$

$x = \frac{s^2}{4 - 2s - s^2} = G_{W_2}$

use G_{W_2} to set

- $P(W_2 = 3)$
- $E(W_2)$

$$E(W_2) = G'_{W_2}(1) = \frac{\partial}{\partial s}\left(\frac{s^2}{4-2s-s^2}\right)\bigg|_{s=1} = \frac{5}{4}$$

$P(W_2 = 3)$

$G_{W_2}(s) \to$ taylor series & compare coeff $\to = \frac{1}{8}$

$$\frac{s^2}{4-2s-s^2} = \frac{s^2}{4} \frac{1}{1-\frac{2s+s^2}{4}} = \frac{s^2}{4}\left(1 + \frac{2s+s^2}{4} + \frac{(2s+s^2)^2}{4^2} \dots\right)$$

$$= \frac{2s\,s^2}{4^2} = \boxed{\frac{1}{8}s^3}$$