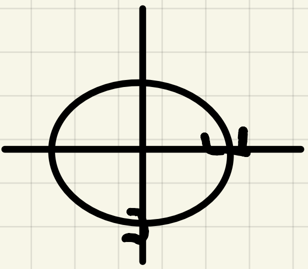
Quiz 1



X, Y indep or unif

$$F_{X,Y}(x,y) = P(X \le x)\, P(Y \le y)$$
$$= \left(\frac{2}{3\pi}\right)^2 x \cdot y$$

$$f_{X,Y}(x,y) = \begin{cases} \frac{4}{9\pi^2} & (x,y) \in (0, \frac{3\pi}{2}]^2 \\ 0 & \text{otherwise} \end{cases}$$

$$\{(x,y) \in A_z\} = \{Z \le z\}$$

if $z < 0$ $\quad P(Z \le z) = 0$

$z > \pi$ $\quad P(Z \le z) = 1$

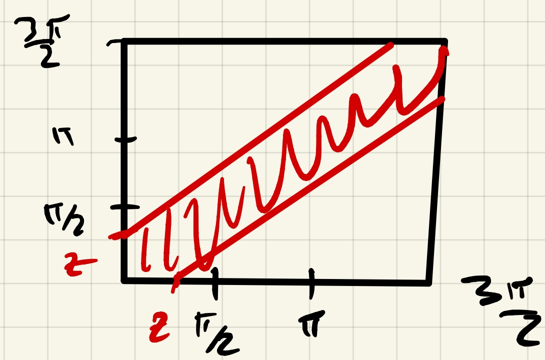
$z \in [0, \pi]$

$$P(Z \le z) = \int_A f_{X,Y}(x,y) = \frac{4}{9\pi^2}(\text{Area of } A_z)$$

first split

$z \in [0, \frac{\pi}{2})$

Claim =



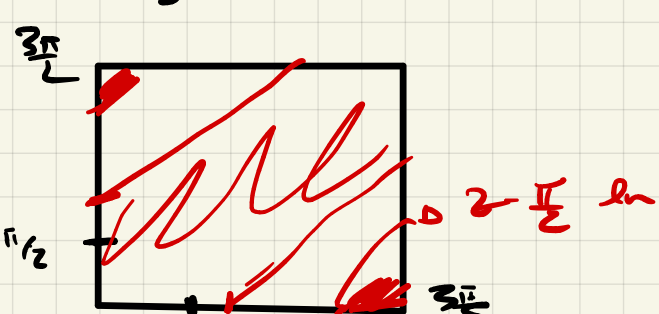
$$\text{Area} = \frac{9\pi^2}{4} - \left(\frac{3\pi}{2} - z\right)^2$$
$$= \frac{6\pi z}{2} - z^2$$

$$P(Z \le z) = 1 - \left(1 - \frac{2z}{3\pi}\right)^2$$

$0 \le x \le z$
$0 \le y \le z$ $\Big\}$ $|y - x| \le z$

$z \in [\pi/2, \pi)$

$$\text{Area} = \frac{9\pi^2}{4} - \left(\frac{3\pi}{2} - z\right)^2 - \left(z - \frac{\pi}{2}\right)^2$$



$$f_Z(z) = \begin{cases} \frac{4}{3\pi}\left(1-\frac{2z}{3\pi}\right) & [0, \pi/2] \\ \frac{8}{9\pi} & [\pi/2, \pi) \\ 0 & \text{otherwise} \end{cases}$$

$$E(Z) = \int_0^\pi z f_Z(z)\, dz =$$

Conditional density & Exp

X, Y cts RX

$$P(a_2 \le X \le a_1 \mid b_2 \le Y \le b_1) = \frac{P(a_2 \le X \le a_1,\ b_2 \le Y \le b_1)}{P(b_2 \le Y \le b_1)}$$

A $\qquad P(B) > 0 \to$ doesn't make sense $Y = b$

$\Delta x << 1$

$$P(Y \le y \mid x \le X \le x + \Delta x) = \frac{P(Y \le y,\ x \le X \le x+\Delta x)}{P(X \in (x, x+\Delta x))}$$

Suppose

$f_{X,Y}$ is cts in x

&

f_X is cts & $f_X(x) \neq 0$

$$= \frac{\int_{-\infty}^{y} f_{X,Y}(x,u)\, \cancel{\Delta x}\, du}{f_X(x)\, \cancel{\Delta x}} = \frac{1}{f_X(x)} \int_{-\infty}^{y} f_{X,Y}(x,u)\, du$$

Defn) Condition distribution of Y given $X = x$

$$F_{Y|X}(y|x) = \frac{1}{f_X(x)} \int_{-\infty}^{y} f_{X,Y}(x,u)\, du = P(Y \le y \mid X = x)$$

Defn) Conditional Density

$$f_{Y|X}(y|x) := \frac{1}{f_X(x)} f_{X,Y}(x,y) \qquad \forall x \text{ s.t. } f_X(x) > 0$$

Defn) Conditional Exp

$$E(Y|X=x) = \int_{-\infty}^{\infty} y \, f_{Y|X}(y|x) \, dy$$

$\hookrightarrow$ this is a function of $x \to \psi(x)$

Defn) $E(Y|X) = \psi(X)$

Thm) $E(E(Y|X)) = E(Y)$

Pf) $E(\psi(X))$

$$= \int_{-\infty}^{\infty} \psi(x) f_X(x) \, dx$$

$$= \int_{\underset{=A}{\{x \mid f_X(x) > 0\}}} \psi(x) f_X(x) \, dx = \int_A \int_{-\infty}^{\infty} y \, f_{Y|X}(y|x) \, dy \, f_X(x) \, dx$$

$$= \int_{-\infty}^{\infty} y \int_A f_X(x) f_{Y|X}(y|x) \, dx \, dy$$

$$= \int_{-\infty}^{\infty} y \int_A f_{X,Y}(x,y) \, dx \, dy$$

marginal

$$= \int_{-\infty}^{\infty} y \, f_Y(y) \, dy = E(Y)$$