Lec 2

Recall

We discussed Ω as a sample space

F a σ-field as $\subseteq P(\Omega)$ a set of all possible evt

Q Chance of each event occurring.

Def A probability on (Ω, F) is a func

$P : F \to [0,1]$ satisfying the following

1) $P(\emptyset) = 0$

2) $P(\Omega) = 1$

3) If $A_1, A_2, \ldots$ are pairwise disjoint $(A_i \cap A_j = \emptyset$ if $i \neq j)$

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

(Ω, F, P) is called a probability space

Remark

If $A_1, A_2 \ldots A_n$ are pairwise disjoint event

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$$

e.g. flip a coin

$\Omega = \{H, T\}$, $F = \{\emptyset, \{H\}, \{T\}, \Omega\}$

$P(\emptyset) = 0$, $P(\Omega) = 1$, $P(\{H\}) = p$, $P(\{T\}) = 1-p$

Corr if $A \in P$, $P(A) + P(A^c) = 1$

In general, if $\Omega = \{x_1, x_2 \ldots x_n\}$ (finite) $\Rightarrow$ F is finite

& $p_1, p_2 \ldots p_n \in [0,1]$ & $p_1 + p_2 \ldots + p_n = 1$

$\exists$ probsp $(\Omega, 2^{\Omega}, P)$ satis $\forall i \in \mathbb{N}_n$ $P(x_i) = p_i$

if $P_1 = P_2 \cdots = P_n$. P is called uniform (also $P(x_i) = \frac{P_i}{n}$)

e.g.) N pets, including Luna & Max

run to N food bowls in a row

Each pet runs to each bowl equally likely

What is the probability that Luna is on the left of Max? (not nec immediately)

$\hookrightarrow$ $\Omega = \{$ pos of N pets $\}$

$|\Omega| = N!$

Prob on Ω is unif

A → event that Luna is on the left

A^c → Luna is on the right of Max

$P(A) + P(A^c) = 1$

consider $f: \Omega \to \Omega$ swap Luna & max

f is a bijection & $f(A) = A^c$ & $f(A^c) = f(A)$

$\Rightarrow P(A) = P(A^c)$ as $|A| = |A^c| \Rightarrow P(A) = \frac{1}{2}$

follow up

Luna is next left to max.

A = event that Luna is next left to max

we want $|A|$. We note there are $n-1$ choices for L&M.

$(n-2)!$ for the rest.

So, $|A| = (n-1)(n-2)!$

Since prob is unif $P(A) = \frac{|A|}{|\Omega|} = \frac{1}{n}$

Properties)

(a) $P(A^c) = 1 - P(A)$

(b) if $B \supset A \implies P(B) = P(A) + P(B \setminus A)$
$\geq P(A)$

(c) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

(d) $P(A \cup B \cup C) = P(A) + P(B) + P(C)$
$- P(A \cap B) - P(B \cap C) - P(C \cap A)$
$+ P(A \cap B \cap C)$

Pf) a) $P(A^c) + P(A) = P(A^c \cup A) = P(\Omega) = 1$

b) As $B \setminus A$ is disj A.

$P(B) = P(A \cup (B \setminus A)) = P(A) + P(B \setminus A)$

c) $P(A \cup B) = P(A \setminus B) + P(A \cap B) + P(B \setminus A)$

$P(A) + P(B) - P(A \cap B)$

d) follow from c

Lemma)

a) if $A_1 \subset A_2 \subset \ldots$ if $A = \bigcup_{i=1}^{\infty} A_i$

Then $P(A) = \lim_{n \to \infty} P(A_i)$

b) If $B_1 \supset B_2 \supset \ldots$ if $B = \bigcap_{i=1}^{\infty} B_i$

$P(B) = \lim_{n \to \infty} P(B_i)$

Pf) ex

monotone cv

Def) Conditional prob $(\Omega, \mathcal{F}, P)$.

Let $B \in \mathcal{F}$ be an event s.t. $P(B) > 0$

The conditional prob.

a given B

$\overset{\triangle}{P}(A|B)$ is the prob. A occurs given B occurs.

and is defined as $P(A|B) = \dfrac{P(A\cap B)}{P(B)}$

e.g. 1

1) A = event that Luna is next left to Max.

B = event that Max is not left most

$$P(A|B) = \frac{1}{N-1}$$

$$A\cap B = A \Rightarrow P(A) = P(A\cap B) = P(A|B)\,P(B) = \frac{1}{N-1}\,\frac{N-1}{N} = \frac{1}{N}$$

2) HW 5b