Recall

- Prob of ultimate ext in branching process
- characteristic function of a rv
  - $\Phi_x : \mathbb{R} \to \mathbb{C}$
    $t \mapsto E(e^{ixt})$
  - → defined everywhere, $|\varphi_x(t)| \leq 1$
- If $x_1, \dots, x_n$ ind $\Rightarrow \Phi_{x_1 + x_2 \cdots x_n}(t) = \prod_{n=1}^{N} \phi_{x_n}(t)$
- $\Phi_{aX+b}(t) = e^{itb} \Phi_x(at)$
- examples: $X \sim N(0,1) \Rightarrow \Phi_x(t) = e^{-t^2/2}$

Coming up

- Char functions determine dist fnc of a RV ← first
- Levy's continuity Thm

→ Law of large numbers & central limit Thm!

Rmk]

If X, Y have the same dist then $\Phi_x(t) = \Phi_Y(t)$ $\forall t \in \mathbb{R}$

Conversely — lemma]

let X, Y be rv st $\Phi_x = \Phi_Y$ $\forall t \in \mathbb{R}$ $\int_{-\infty}^{\infty} |\Phi_x(t)| \, dt < +\infty$

$\Rightarrow$ X, Y have the same dist funct!

↳ corollary of the following Thm

Thm] X is rv with dist F char Φ. Assume $\int_{-\infty} |\Phi(t)| \, dt < +\infty$

$\Rightarrow \forall a, b \in \mathbb{R}$ $\overline{F}(b) - \overline{F}(a) = \lim_{N \to \infty} \int_{-N}^{N} \frac{e^{-ita} - e^{-itb}}{2\pi i t} \Phi_x(t) \, dt$

$\overline{F}(x) = \frac{1}{2}\left(F(x) + \lim_{y \to x^-} F(y)\right)$ → note: right cts, if F cts at a then two are equal

Sketch of thm $\Rightarrow$ prop

$\forall a, b \in \mathbb{R}$ F_X and F_Y are cts at a & b $F_X(a) = F_Y(a)$

$$F_X(b) - F_X(a) = \lim_{N\to\infty} \left[\int \varphi_X(t) \right] = \lim_{N\to\infty} \left[\int \varphi_Y(t) \right] = F_Y(b) - F_Y(a)$$

let $a \to -\infty \Rightarrow F_Y(b) = F_X(b)$ $\forall\, b \in \mathbb{R}$ s.t F_X, F_Y cts at b

The set of discts pts of F_X or F_Y is countable (measure 0)

by right continuity of dist func, we get that $F_X = F_Y$

Cor | if X is cts r.v. s.t.

$$\int_{-\infty}^{\infty} |\varphi_X(t)| < +\infty$$

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \varphi_X(t)\, dt$$

→ inverse of fourier transform

φ_X is fourier trans of $f_X(x)$

Sketch | $f_X(x) = \frac{\partial}{\partial x}(F(x) - F(0))$

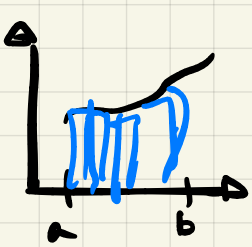
$$\text{Thm} = \frac{\partial}{\partial x} \left(\lim_{N\to\infty} \int_{-N}^{N} \frac{1 - e^{-itx}}{2\pi i t} \varphi_X(t)\, dt \right)$$

$$= \lim_{N\to\infty} \int_{-N}^{N} \frac{it\, e^{-itx}}{2\pi i t} \varphi_X(t)\, dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \varphi_X(t)\, dt$$

Riemann Integral

$f : [a,b] \to \mathbb{R}$ cts



$$\int_a^b f = \text{limit of riemann sums}$$

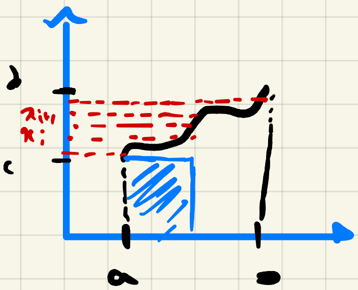
↪ take a partition of $[a,b]$

→ also works for piecewise cts func! BUT NO MORE e.g. $\int_a^b \mathbb{1}_{\mathbb{Q}\cap[a,b]}$ not def as Riemann Integral

Lebesgue Integral

Special Case $f:\mathbb{R}\to\mathbb{R}$ Lebesgue-Stieltjes Integral

take a partition of the image instead of the domain



for each subinterval consider the inverse image of it (what gets mapped to it)

approx by min of the subinterval.

take the limit of partitions (finer)

$$\lim_{n\to\infty} \sum_{i=1}^{n} x_i \, |f^{-1}(x_i, x_{i+1})| =: \int_a^b f$$

↳ how define?

we note if R int exists they agree

This is more general

$$\int_a^b \mathbb{1}_{\mathbb{Q}\cap[a,b]} \, dx = 1 \, |f^{-1}(x_{n-1}, 1)| = 0$$

→ if exists, f is measurable

measure how large a set is $f^{-1}(a,b)$ $\forall a,b\in\mathbb{R}$

In prob context this is just in def $P(a \le x \le b)$

Thm $(\Omega, \mathcal{F}, P)$ prob sp.

$X:\Omega\to\mathbb{R}$ is a rv

Then

$\int_\Omega X \, dP$ can be defined if $\int_\Omega |X| \, dP < +\infty$

Special case X is discrete taking values $x_1, x_2 \ldots$

$A_1, \ldots A_n \ldots$ is a partition of Ω

$A_1 = X^{-1}(x_1)$, $A_n = X^{-1}(x_n)$

$$X = \sum_{i=1}^{\infty} x_i \mathbb{1}_{A_i}$$

$$\int_\Omega X\,dP = \sum_{i=1}^{\infty} x_i\,P(A_i) = \sum x_i\,P(X=x_i) = E(X)$$

only the intervals that have x_i contribute

special case X is CB

$$E(X) = \int_{-\infty}^{\infty} x\,f_X(x)\,dx$$

on $(a, a+\Delta a)$ $\quad \int_a^{a+\Delta a} x\,f_X(x)\,dx \simeq a\cdot P(a\le X\le a+\Delta a)$

$\qquad a\cdot P(X^{-1}(a, a+\Delta a))$

handwave $\quad \int_\Omega X\,dP = E(X)$

in general let $(\Omega, \mathcal{F}, P)$ be PS

def) let X be a a rv

$$E(X) = \int_\Omega X\,dP \quad \text{whenever} \quad \int_\Omega |X|\,dP < +\infty$$

Prop 1 get linearity of Exp by lin of integral

Some inequalities let X be a rv s.t. $E(|X|) < \infty$

Markov Inequ

$$P(X\ge a) \le \frac{E(X)}{a} \quad \forall a>0$$

Pf) $X \ge X\cdot \mathbb{1}_{\{X\ge a\}} \ge a\cdot \mathbb{1}_{\{X\ge a\}}$ as a fn

$$E(X) \ge E(a\cdot \mathbb{1}_{\{X\ge a\}}) = a\cdot P(X\ge a)$$

~~Chebyshev's Ineq~~

$\cancel{P(|X|\ge a) \le \frac{E(X^2)}{a^2} \quad \forall a>0}$ $\quad$ ~~Ex)~~ $\cancel{X\sim N(0,\sigma^2)}$ $\cancel{P(|X|>\sigma)\le 0.7\ldots}$

Generality if h is fn $\quad P(h(X)\ge a) \le \frac{E(h(X))}{a} \quad \forall a\ge 0$

Pf) like above $\quad h(\omega) \ge a\cdot \mathbb{1}_{\{h(X)\ge a\}}$