Review!

Expectation: for cts rv X with density $f_X(x)$

$$E(X) = \int_{\mathbb{R}} x f_X(x)\, dx$$

def when $\int_{\mathbb{R}} |x| f_X(x)\, dx < +\infty$

if g is good, $E(g(x)) = \int_{\mathbb{R}} g(x) f_X(x)\, dx$

More generally! $g: \mathbb{R}^2 \to \mathbb{R}$ nice

$$E(g(X,Y)) = \int_{\mathbb{R}}\int_{\mathbb{R}} g(x,y) f_{X,Y}(x,y)\, dx\, dy$$

(Cor) if X, Y ind, $E(XY) = E(X)E(Y)$

Indep! $F_{X,Y}(x,y) = F_X(x) F_Y(y)$ if cts

$\Rightarrow f_{X,Y}(x,y) = f_X(x) f_Y(x)$

Marginals! $\lim_{y \to \infty} F_{X,Y}(x,y) = F_X(x)$ → general!

cts $\Rightarrow f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy$ disc $P(X=x) = \sum_y P(X=x, Y=y)$

Prop! ① Expectation is linear

② If $X \geq Y \Rightarrow E(X) \geq E(Y)$

③ $E(1) = 1$

Conditional! Condit Dist for cts r.v.

$$P(X \leq x \mid Y = y) = F_{X|Y}(x|y) = \int_{-\infty}^{x} \frac{f_{X,Y}(x,y)}{f_Y(y)}\, dx$$

when $\neq 0$

$$\Rightarrow f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

Condit Exp

$$E(X \mid Y=y) = \int_{\mathbb{R}} x f_{X|Y}(x|y)\, dx = \phi(y)$$

$\Rightarrow E(X|Y) = \phi(Y)$ a rv! $E(E(X|Y)) = X$

Sum) $f_{X+Y}(z) = \int_{-\infty}^{\infty} f_{X,Y}(x, z-x)\, dx$

Cor) if X,Y ind $f_{X+Y}(z) = (f_X * f_Y)(z)$ → conv

Make a table : Disc rv value
: Cts (exp, gamma, chi-sq, normal)

expvar, dist, density, gen func, mom gen fn, char func!

gen funct → $E(s^X)$ → def for $x \geq 0$ works will for X in $\mathbb{Z}_{\geq 0}$ eg binom, rw...

mom gen → $E(e^{tX})$ → works more generally, radius of conv can be 0 tho

char → $E(e^{itX})$ → always conv!

① X, Y same dist ⟺ funct same! (any one of them)

② gen / mom gen / char of sums of ind rv is the prod of the ind functions!

③ $M_{cX}(t) = M_X(ct)$, $\phi_{cX}(t) = \phi_X(ct)$

④ Levy → $X_n \xrightarrow{D} X$ iff $\phi_{X_n} \to \phi_X$ ptwise

Ex ①

② $E(Z_{m+n} Z_m)$ spose Y is the rv for Z_m

we note

$E(E(Z_{m+n} Z_m \mid Z_m))$

$E(Z_{m+n} Z_m \mid Z_m) = \varphi(Z_m)$

$\varphi(y) = E(Z_{m+n} Z_m \mid Z_m = y) = y\, E(Z_{m+n} \mid Z_m = y)$

$= y\, E\left(\sum_{i=1}^{y} Z_n\right)$

$= y^2 E(Z_n)$

$E(\varphi(Y)) = E(Y^2 E(Z_n))$

$= E(Z_n)\, E(Y^2) = \mu^n E(Z_m^n)$

μ^n