Lecture 3

Recall

↳ we defined probability measure $P : \mathcal{F} \to [0,1]$ with some prop

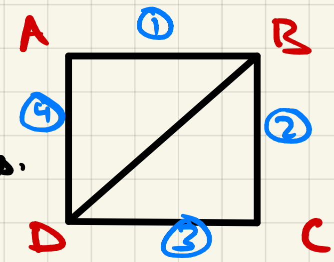
↳ we defined conditional prob $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

↳ if $P(B) > 0$

E.g

→ Same set up as hw1 5b

→ what is the prob C has power given BD works.



↳ $P(C \mid BD^c) = 2p^2 - p^4$

Lemma 1

① $P(A \mid \Omega) = P(A)$ → Pf) immediate

② Bayes if $P(A), P(B) > 0 \Rightarrow P(A|B)P(B) = P(B|A)P(A)$

Lemma 2)

① B is an evt s.t $0 < P(B) < 1$ for some event

$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c)$

② If $B_1 \dots B_n$ forms a partition of Ω $P(B_i) > 0 \ \forall i \in \mathbb{N}_n$

$\Rightarrow P(A) = \sum_{i=1}^{n} P(A|B_i)P(B_i)$

↳ pt $\Omega = \bigsqcup_{i=1}^{n} B_i$

Pf) 1) $P(A|B)P(B) + P(A|B^c)P(B^c)$

↳ $P(A \cap B) + P(A \cap B^c)$

$= P((A \cap B) \cup (A \cap B^c)) = P(A)$

→ distributivity

2) $\sum_{i=1}^{n} P(A|B_i)P(B_i) = \sum_{i=1}^{n} P(A \cap B_i)$ bcuz disjoint

$= P(A \cap (\bigsqcup_{i=1}^{n} B_i)) = P(A \cap \Omega) = P(A)$

$\square$

E.g.) Drug testing

Test whether someone is using meth.

Test is 90% sensitive => users are identified 90% of the time

↳ $P(\text{+ive test} \mid \text{user}) = 90\%$

&

it is 80% specific => non users are identified 80% of the time

↳ $P(\text{-ive test} \mid \text{non-user}) = 80\%$

There are 100 people with 7 unidentified users (93 non-users)

what is the prob tha a person is a user given that they are positive?

→ $$P(\text{user} \mid \text{+ test}) = \frac{P(\text{+ test} \mid \text{user})\, P(\text{user})}{P(\text{+ test})}$$

$$= \frac{0.9 \cdot 0.07}{P(\text{+test})}$$

$$= \frac{0.9 \cdot 0.07}{0.249} = \boxed{0.253}$$

P(+ test) = P(+ test | user) P(user) + P(+ test | non user)Pn

$$= 0.9 \cdot 0.07 + 0.2 \cdot 0.93$$

$$= 0.249$$

Def) Independent

① Two events A, B are ind if $P(A \cap B) = P(A)P(B)$

② A family of events $\{A_i\}_{i \in I}$ → ind set
is said to be ind if

$$P\left(\bigcap_{i \in J} A_i\right) = \prod_{i \in J} P(A_i) \quad \forall \text{ finite } J \subseteq I$$

Rmk)

1) NOT $A \cap B = \emptyset$ → disjoint events

2) NOT " A & B are ind " $\iff$ $P(A|B) = P(B)$
↳ issue since $P(B) > 0$ nec

Lemma) if A & B are ind & $P(B) > 0$
then $P(A|B) = P(A)$

Rmk → the conv holds (if note $P(B) > 0$)

Pf) $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{P(A)\cancel{P(B)}}{\cancel{P(B)}} = P(A)$

e.g. Roll 2 fair dice

A = first die shows 3

B = second shows 4

C = sum of 2 die is 7

$P(A) = \frac{1}{6}$, $P(B) = \frac{1}{6}$, $P(C) = \frac{1}{6}$

$P(A \cap B) = \frac{1}{36} = P(A)P(B) \Rightarrow A, B$ are ind

Note : $A \cap B = A \cap C = B \cap C \Rightarrow A, C$ are ind , B, C are ind

But A, B, C are not ind $A \cap B \cap C = B \cap C$
$\Rightarrow P(A \cap B \cap C) = \frac{1}{36}$

E.g) Simple Random walk (Gambler's Ruin)

A gambler is playing coin tossing (fair) to a loser

if head → gambler wins \$1

tail → gambler loses \$1

The gambler has \$K

↳ he stops if he reaches \$N > K or \$0.

What is the prob that he is bankrupt?