Recall

- Bayes → $P(A|B)P(B) = P(B|A)P(A)$
- partition → if $B_1 \dots B_n$ partition $P(A) = \sum P(A|B_i)P(B_i)$
- independence → A, B ind $\overset{def}{\iff} P(A\cap B) = P(A)P(B)$
- random walk ⟷ today

Random Walk

E.g. Random Walk (Gambler's Ruin)

A gambler plays fair coin tossing with a boss.

If he wins ("heads") he wins \$1

If he loses ("tails") he loses \$1

He has \$k.

He stops if → he gets \$N or \$0

death barrier

survival barrier

P(Bankrupt)?

Soln

$$P_k = P\left(\begin{matrix}\text{Bankrupt starting} \\ \text{with } \$k\end{matrix}\right)$$

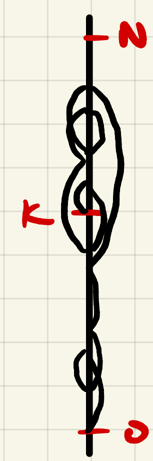
$$P_k = P\left(\begin{matrix}\text{BR start} \\ \text{with } k\end{matrix} \,\middle|\, 1^{st} \text{ is head}\right) P(\text{head})$$

$$+ P\left(\text{BR start w } k \,\middle|\, 1 \text{ is tail}\right) P(\text{tail})$$

$$= 0.5\, P_{k+1} + 0.5\, P_{k-1}$$

Why is it random walk

he either steps up or down randomly



$$\frac{1}{2}P_k + \frac{1}{2}P_k = \frac{1}{2}P_{k+1} + \frac{1}{2}P_{k-1}$$

$$P_{k+1} - P_k = P_k - P_{k-1}$$

repeat pattern

$$= P_{k-1} - P_{k-2} = \dots\dots = P_1 - P_0 := a$$

$ka + P_0$

$$P_k = P_k\left(-P_{k-1} + P_{k-1}\right) \dots\dots \left(-P_0 + P_0\right) \xrightarrow{\text{rearrange slightly}} = (P_k - P_{k-1}) + (P_{k-1} - P_{k-2}) \dots + P_0$$

$\Rightarrow P_0 = 1$

$\therefore P_k = ka - 1$ now $a = P_1 - P_0$, $P_N = 0$

so, $Na + 1 = 0 \Rightarrow a = -\frac{1}{N}$

$P_k = ka - 1 = \frac{N-k}{N}$

$\lim_{n \to \infty} P_k = 1$

Redo problem when $P(W) = p$ & $P(+) = (1-p)$

$P_k = p P_{k+1} + (1-p) P_{k-1}$

$p P_k + (1-p) P_k = p P_{k+1} + (1-p) P_{k-1}$

$p P_{k+1} - p P_k = (1-p) P_k - (1-p) P_{k-1} \Rightarrow \frac{(1-p)[P_k - P_{k-1}]}{p}$

$[P_{k+1} - P_k] = p\ P_{k-1} - (p) P_{k-2}$

$P_{k+1} - P_k = \frac{[1-p]}{p} [P_k - P_{k-1}] = \left(\frac{[1-p]}{p}\right)^2 (P_{k-1} - P_{k-2})$

$\ldots\ldots \left[\frac{1-p}{p}\right]^k [P_1 - P_0] := \alpha^k a$

$P_k = (P_k - P_{k-1}) + (P_{k-1} - P_{k-2}) \ldots\ldots (P_1 - P_0) + P_0$

$= a \cdot \alpha^{k-1} + a\alpha^{k-2} \ldots + a + 1$

$P_k = a \frac{1-\alpha^k}{1-\alpha} + 1 \rightarrow$ geometric sequence $\qquad P_N = 0$

$\Rightarrow a = \frac{\alpha - 1}{1 - \alpha^N}$

$P_k = -\frac{\cancel{\alpha - 1}}{1 - \alpha^N} \frac{1 - \alpha^k}{\cancel{1 - \alpha}} + 1 = 1 - \frac{1 - \alpha^k}{1 - \alpha^N}$

Random Variable

e.g. Tossing a coin 3 times

↳ count # head

$\Omega = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

X counting function — (of heads)

$X(HHH) = 3$, $X(HTH) = 2$ ..... $X(TTT) = 0$

Def) let (Ω, F, P) be a prob sp.

A random variable (RV) X is a function

$X : \Omega \to \mathbb{R}$

s.t, $\forall a \in \mathbb{R}$

$\{\omega \in \Omega : X(\omega) \leq a\} \in \mathcal{F}$

↳ is an event!

back to eg) # of heads

$\Omega = \{ \ldots \}$

$F = \mathcal{P}(\Omega)$

$X : \Omega \to \mathbb{R}$

We can verify. For $a = \frac{3}{2}$, $X^{-1}(\frac{3}{2}) = \{TTT, HTT, THT, TTH\}$

we note $X^{-1}(\frac{3}{2}) \in \mathcal{F}$ trivially as $\mathcal{F} = \mathcal{P}(\Omega)$

Def) let X be a random var. A function $F : \mathbb{R} \to [0,1]$ is a distribution of X if, $\forall a \in \mathbb{R}$

$F(a) = P(X^{-1}(a))$

↳ $X^{-1}(a) := \{\omega \in \Omega \mid X(\omega) \leq a\}$