Quiz

$$X_i = \begin{cases} +1 & i\text{th step moves to the right} \\ -1 & i\text{th step moves left} \end{cases}$$

$$X = \sum_{i \in \mathbb{N}_n} X_i$$

$$E(X_i) = 0$$

$$\therefore E(X) = 0$$

$$Var = E(X^2)$$

$$X^2 = (X_1 + \dots + X_n)^2$$

$$= X_1^2 + \dots + X_n^2 + 2\sum_{i<j} X_i X_j$$ independent for $i \neq j$

$$\therefore E(X^2) = E(X_1^2) \dots + (X_n^2) + 2 \sum_{i<j} \cancel{E(X_i)E(X_j)} \rightarrow 0$$

$$= 1 + 1 \dots + 1$$

$$= n$$

Recall - (Ω, F, P) prob space.

X: dis RV $\quad B \in F$ s.t $P(B) > 0$

$$E(X|B) = \sum_{x \in \mathbb{R}} x \, P(X = x | B)$$

e.g. # tosses until HT

X = # tosses until HT

$$E(X) = E(X \mid \text{1st head}) P(\text{1st head}) + E(X \mid \text{1st tail}) P(\text{1st tail})$$

$$= \frac{1}{2} E(X|H) + \frac{1}{2} E(X|T)$$

$$= \frac{1}{2} \underbrace{E(X|H)}_{E(Y)+1} + \frac{1}{2}(E(X) + 1)$$

Y = no of tosses until 1 tail

$$E(X) = \frac{E(Y) + E(X)}{2} + 1$$

So,

$E(X) = E(Y) + 2$

$= \underline{E(X) = 4}$

$E(Y) = E(Y \mid \text{tail})\, P(t)$
$\quad + E(Y \mid \text{head})\, P(h)$

$E(Y) = \frac{1}{2} + \frac{(E(Y)+1)}{2}$

$E(Y) = 2$

Def) Conditional Distribution

X, Y rv:

Conditional Distribution

$$F_{X|Y}(x,y) = P(X \le x \mid Y = y)$$

whenever $P(Y=y) > 0$

Def) Conditional Mass Func

X, Y are discrete

Conditional Mass func

$$f_{X|Y}(x,y) = P(X = x \mid Y = y)$$

Defn) let X, Y be discrete RVs

$$\psi(x) = E(Y \mid X = x)$$

Compare

$\psi(X) := E(Y \mid X) \longrightarrow$ conditional expectation of $Y \mid X$ is a random variable

$E(Y \mid X) := \psi(X) \longrightarrow \psi \circ X$

$$\begin{array}{ccccc} \omega & \overset{X}{\longmapsto} & x & \overset{\psi}{\longmapsto} & \psi(x) \\ \Omega & & \mathbb{R} & & \mathbb{R} \end{array}$$

So, $E(Y \mid X)(\omega) = \psi(X(\omega)) = E(Y \mid X = X(\omega))$

$\longrightarrow$

Properties X, Y, Z dis R.V. $g : \mathbb{R} \to \mathbb{R}$

1) $E(aY + Z \mid X)$
2) $E(Y \mid X) \geq 0$ if $Y \geq 0$
3) $E(1 \mid X) = 1$
4) If X, Y are ind $\Rightarrow$ $E(Y \mid X) \equiv E(Y)$ → const fund
5) $E(Y \cdot g(X)) \mid X) = g(X)\, E(Y \mid X)$

lemma) $B \in \mathcal{F}$, $P(B) > 0$

1) $E(X + Y \mid B) = E(X \mid B) + E(Y \mid B)$
2) $E(cX \mid B) = cE(X \mid B)$

Pf ex

Combine linearity of exp with def!

1) let $\omega \in \Omega$ be given

$E(aY + Z \mid X)(\omega) = E(aY + Z \mid X = X(\omega))$

$= aE(Y \mid X = X(\omega)) + E(Z \mid X = X(\omega))$

$= (aE(Y \mid X) + E(Z \mid X))(\omega)$

$\therefore E(aY + Z \mid X) = aE(Y \mid X) + E(Z \mid X)$

2) Spose $Y \geq 0$ $\forall \omega \in \Omega$

let $\omega \in \Omega$

$E(Y \mid X)(\omega) = E(Y \mid X = X(\omega)) = \sum_{y \in \mathbb{R}} \underbrace{y}_{\geq 0} \underbrace{P(Y = y \mid X = X(\omega))}_{\geq 0}$

≥ 0

4) $\forall \omega \in \Omega$

$E(Y \mid X)(\omega) = E(Y \mid X = X(\omega)) = \sum_{y \in \mathbb{R}} y\, P(Y = y \mid X = X(\omega))$

$= \sum_{y \in \mathbb{R}} y\, P(Y = y)\, P(X = X(\omega))$

$= E(Y)$

Thm) $E(\underbrace{E(Y|X)}_{\psi(X)}) = E(Y)$

Partition

$E(\psi(X)) = \sum_{x\in\mathbb{R}} \psi(x)P(X=x) \Rightarrow \sum_{x\in\Omega} E(Y|X=x)P(X=x)$

lemma

$= E(Y)$

E.g) Throw a fair die ...

X = # throws until first 2 consec 6

Y = first time we get 6

idea $E(X) = E(E(X|Y))$

$\Rightarrow E(X|Y) = \psi(Y)$

$\psi(y) = E(X \mid Y=y)$

y^{th} toss, 6 shows up

$y+1$ toss, 1-5 :(, 6 :)

$E(X|Y=y) = \frac{1}{6}(y+1) + \frac{5}{6}[E(X)+y+1]$

$\psi(y)=E(X|Y=y) = 1+y+\frac{5}{6}E(X)$

$\therefore E(X|Y) = \psi(Y) = 1+Y+\frac{5}{6}E(X)$

$\therefore$ By linearity

$E(E(X|Y)) = E(1+Y+\frac{5}{6}E(X)) = E(1)+E(Y)+E(\frac{E(X)}{6})$

$= 1 + \frac{E(X)}{6} + E(Y)$

$E(X) = 6 + 6E(Y)$

$E(Y) = \sum_{y=1}^{\infty} y\, P(Y=y) = \sum_{y=1}^{\infty} y \left(\frac{5}{6}\right)^{y-1}\left(\frac{1}{6}\right)$

$= \,?\ \Rightarrow$ done!

#5 hw X = lifetime

$E((X-t) \mid X>t) = ?$

$E(X \mid X>t) - t$

Important dis RVs

1) Bernoulli

$X \sim Ber(p)$ if $X = \begin{cases} 1 & \text{prob } p \\ 0 & \text{— } 1-p \end{cases}$

$X \sim Binom(n, p)$ p.t

$X = X_1 + \cdots + X_n$)

$X_i \sim Ber(p)$ with each X_i independent!