Last time

We looked at properties & applications of generating functions.

e.g. Symmetric simple rw.

Start from 0.

Z_k is the 1st hitting time for k

Q) $G_{Z_k}(s)$?

$$G_{Z_1}(s) = \sum_{n=0}^{\infty} P(Z_1 = n)\, s^n$$

note $P(Z_1 = 0) = 0$

$P(Z_1 = 1) = 1/2$

for $n \geq 2$,

condit on first step. If first step to the right

$$P(Z_1 = n) = \frac{1}{2} P(Z_2 = n-1)$$

since you move left on your first step. At -1 want to get n 1

$$G_{Z_1}(s) = \frac{1}{2}s + \sum_{n=2}^{\infty} P(Z_1 = n)\, s^n$$

$$= \frac{s}{2} + \frac{1}{2}s \sum_{n=2}^{\infty} P(Z_2 = n-1)\, s^{n-1}$$

$$= \frac{s}{2} + \frac{s}{2} \sum_{n=1}^{\infty} P(Z_2 = n)\, s^n$$

$$G_{Z_1}(s) = \frac{s}{2} + \frac{s}{2} G_{Z_2}(s) \Rightarrow G_{Z_2}(s) = \frac{2}{s}\left[G_{Z_1}(s) - \frac{s}{2}\right]$$

observe) $P(Z_2 = n) = \sum_{k=0}^{n} P(Z_1 = k) P(Z_1 = n-k)$

disc conv

$\underbrace{}_{\text{conv}}$ $a_n * a_n$, $a_n = P(Z_1 = n)$

2 is repeating 1 twice

if $a = (a_n)$

$b = (b_n)$, $c = (c_n)$

s.t) $c = a * b$

$c_n = \sum_{k=0}^{n} a_k b_{n-k}$

$G_c(s) = G_a(s)\, G_b(s)$

Thus,

$$G_{Z_2}(s) = \left(G_{Z_1}(s)\right)^2$$

$$G_{Z_1}(s) = \frac{s}{2} + \frac{s}{2} G_{Z_1}(s)^2 \Rightarrow 0 = \frac{s}{2}x^2 - x + \frac{s}{2}$$

$$\Rightarrow G_{Z_1} = \frac{2 \pm \sqrt{4 - 4s^2}}{2s} = \frac{1 \pm \sqrt{1-s^2}}{s}$$

note: $G_{Z_1}(0) = P(Z_1 = 0) = 0$

Consider $\lim_{s \to 0} \frac{1 + \sqrt{1-s^2}}{s} = \infty$

$\lim_{s \to 0} \frac{1 - \sqrt{1-s^2}}{s} = 0$

$\therefore G_{Z_1}(s) = \frac{1 - \sqrt{1-s^2}}{s}$

E.x.) 1) Use taylor series of $\sqrt{1+x}$ to figure out $P(Z_1 = n)$

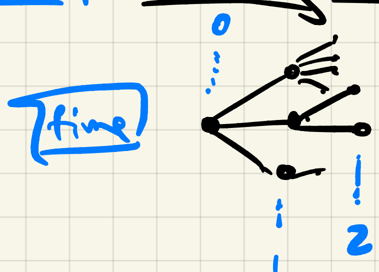
& verify hitting time thm for Z_1

2) find $G_{Z_2}(s) = ?$

$\vdots$

$G_{Z_k}(s) = ?$ **Note:** $G_{Z_k} = G_{Z_{k-1}} \, G_{Z_1}$

E.g.) Branching Process A tree with a root (original branch)



Each branch gives rise to X of next gen branches, X discrete rv in $\mathbb{N} \cup \{0\}$

Z_k = # branches at time k

Q1) What is $E(Z_k)$?

Q2) $P(Z_n = 0) = ?$ (extinction) or Ultimate extinction $\lim_{n \to \infty} P(Z_n = 0) = ?$

Lemma) $X_1, X_2, \dots$ i.i.d rv with the same dist as rv X.

Defn) i.i.d: independent, identically dist

$N \geq 0$ is a RV taking integer values

$S = X_1 + \dots X_N \implies G_S(t) = G_N(G_X(t))$

Pf of lem 1

$$G_S(t) = E(t^S) = E(E(t^S \mid N))$$ S is finite sum

$$= \sum_{n=0}^{\infty} P(N=n)\, E(t^S \mid N=n)$$

if $X_1, X_2, \dots X_m$ ind.

$$G_{X_1+X_2+\dots+X_n}(t) = G_{X_1}(t)\, G_{X_2}(t) \dots$$

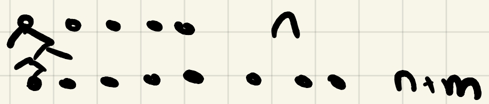
$$= \sum_{n=0}^{\infty} P(N=n)\, G_{X_1}(t) \cdots G_{X_n}(t)$$

$$= \sum_{n=0}^{\infty} P(N=n) \left(G_X(t)\right)^n$$

$$= G_N(G_X(t)) \quad \blacksquare$$

Cor 1 $G_{Z_{n+m}}(t) = G_{Z_n}(G_{Z_m}(t))$ (Z from Q)

Pf)



for everything at n+m, there is a unique ancestor

We start branching from Z_n roots

We realize Z_{n+m} is a sum of Z_n r.v where the rv are i.i.d as Z_m

$\Rightarrow$ by lemma

$$G_{Z_{n+m}}(t) = G_{Z_n}(G_{Z_m}(t)) \quad \blacksquare$$
$$= G_{Z_m}(G_{Z_n}(t))$$

Cor 2) $G_{Z_2}(t) = G_{Z_1} \circ G_{Z_1}(t)$ let's figure this out

$\vdots$

$$G_{Z_n}(t) = \underbrace{G_{Z_1} \circ G_{Z_1} \circ \dots \circ G_{Z_1}}_{n \text{ times}}(t)$$

Thm if $E(Z_1) = \mu$ $Var(Z_1) = \sigma^2$

$E(Z_n) = \mu^n$ $Var(Z_n) = \ldots\ldots$

$E(Z_n) = G'_{Z_n}(1)$

note $G_{Z_1}(1) = \sum_{n=0}^{\infty} P(Z_1 = n) 1^n$
$= 1$

$\therefore$ any comp thereof is 1

$G_{Z_n}'(1) =$

$$G_{Z_{n-1}}'(1) \cdot \overset{E(Z_1)}{G_{Z_1}'(1)} = \mu^n$$

induct

$$Var(Z_n) = E(Z_n^2) - E(Z_n)^2$$

$$E(Z_n(Z_n - 1)) + E(Z_n) - E(Z_n)^2$$

$$G''_{Z_n}(1) + \mu^n - \mu^{2n}$$

$$G_{Z_n}(t) = G_{Z_1}(G_{Z_{n-1}}(t))$$

$$G_{Z_n}'(t) = G_{Z_1}'(G_{Z_{n-1}}(t)) \cdot G'_{Z_{n-1}}(t)$$

d prod

$$G_{Z_n}''(t) = G''_{Z_1}(G_{Z_{n-1}}(t)) \cdot [G'_{Z_{n-1}}(t)]^2 + G'_{Z_1}(G_{Z_{n-1}}(t)) \cdot G''_{Z_{n-1}}(t)$$

$$G_{Z_n}''(1) = G''_{Z_1}(1) \cdot G'_{Z_{n-1}}(1)^2 + G'_{Z_1}(1) \cdot G''_{Z_{n-1}}(1)$$

recursion solve D

Now, let us look at the probability of extinction

$$P(Z_n = 0) = G_{Z_n}(0) = \underbrace{G_{Z_1} \circ \cdots \circ G_{Z_1}}_{n \text{ of these}}(0)$$

Ultimate ext $\lim_{n\to\infty} \underbrace{G_{Z_1} \circ \cdots \circ G_{Z_1}}(0)$

by analysis $u_n = P(Z_n = 0)$

Prop) u_n is a non-decreasing seq converging to u

↳ bounded above by 1

Pf) $G_{Z_1}'(t) = \sum_{n=1}^{\infty} P(Z_1 = n)\, n\, t^{n-1}$

$\forall t \in [0,1)\ G_{Z_1}'(t) \geq 0$ all are ≥ 0

$\therefore G_{Z_1} \uparrow$ also, $G_{Z_1}(0) = P(Z_1 = 0) \geq 0$

$\underbrace{G_{Z_1}(G_{Z_1}(0))}_{u_2} \geq \underbrace{G_{Z_1}(0)}_{u_1} \geq 0$