Expectation

if 1. X is discrete rv $\quad E(X) = \sum_{x \in \mathbb{R}} x\, P(X=x) \in \mathbb{R}$

2. X is discrete rv $\quad B \in \mathcal{F}$ s.t $P(B) > 0$

$$E(X \mid B) = \sum_{x \in \mathbb{R}} x\, P(X=x \mid B) \in \mathbb{R}$$

3. X, Y discrete r.v $\quad E(X \mid Y) = \phi(Y) \longrightarrow$ rv

$$\phi(y) = E(X \mid Y=y) \in \mathbb{R}$$

Thm $E(E(X \mid Y)) = E(X)$

Summaries of important d.r.v.

1) Bernoulli (p) $\longrightarrow p \in [0,1]$

$$X = \begin{cases} 1 & \text{w/ prob } p \\ 0 & \text{w/ prob } 1-p \end{cases}$$

$\therefore E(X) = p$

$\therefore Var(X) = p - p^2$

Key e.g. Tossing a coin & see if we get head or not!

2) Indicator RV $A \in \mathcal{F}$ $\quad \omega \quad P(A) = p$

$$\mathbb{1}_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}$$

2) Binomial RV (n, p) $\Rightarrow p \in [0,1]$

$$X = X_1 + \dots + X_n$$

where $X_i \sim$ Bernoulli (p) rv & $X_1 \dots X_n$ independent

$$E(X) = E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i) = np$$

$$Var(X) = E(X^2) - E(X)^2 = n(p - p^2)$$

Key e.g. No of head in n tosses.

$$f_X(k) = P(X=k) = \binom{n}{k} (p)^k (1-p)^{n-k}$$

3. Geometric (p), $p \in [0,1]$

X w/ mass func given by

$f_X(k) = (1-p)^{k-1} p$ (for $k \geq 1$, $k \in \mathbb{N}$)

e.g. $E(X) = \frac{p}{(1-p)^2}$ Var(X)

Key eg: No of tosses until first head

4. Poisson (λ), $\lambda > 0$

X is r.v with mass

$f_X(k) = e^{-\lambda} \frac{\lambda^k}{k!}$ $(k \in \mathbb{Z}, k \geq 0)$

Show $E(X) = \text{Var}(X)$

Key eg! # of occurrences of certain phenomenon in a time interval.

Key reln with binom

When we increase no of trials of a binomial rv $(n \to \infty)$ but want to keep $E(X) = np$ (fixed) $\hookrightarrow \lambda$

Claim: $\forall k$

$$f_{\text{Binomial}(n, \frac{\lambda}{n})}(k) \xrightarrow{n \to \infty} f_{\text{Poisson}(\lambda)}(k)$$

$$f_{\text{binom}(n, \frac{\lambda}{n})} = \binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k}$$

Recall

$\left(1 + \frac{x}{n}\right)^n \xrightarrow[n \to \infty]{} e^x$

$$\frac{(n)(n-1)\cdots(n-k+1)}{k!\, n^k} \lambda^k = \underbrace{\binom{n}{k}\left(\frac{\lambda}{n}\right)^k}\left(1 - \frac{\lambda}{n}\right)^n \left(1 - \frac{\lambda}{n}\right)^{-k}$$

$\hookrightarrow e^{-\lambda}$ $\hookrightarrow$ $n \to \infty$ $\frac{\lambda}{n} \to 0$

$1 \cdot \frac{\lambda^k}{k!}$

$$= 1 \cdot \frac{\lambda^k}{k!} \quad e^{-\lambda} \cdot 1 = e^{-\lambda} \frac{\lambda^k}{k!}$$

Ex. $X_1 \sim \text{Poisson}(\lambda_1)$, $X_2 \sim \text{Poisson}(\lambda_2)$, X_1, X_2 independent!

a) $X_1 + X_2 \sim \text{Pois}(\lambda_1 + \lambda_2)$

((hint: $f_{X_1+X_2}(k) = P(X_1 + X_2 = k) = \sum_{j=0}^{k} P(X_1 = j, X_2 = k-j)$

Infact | Raikov Thm (1937)

if $X_1 + X_2 \sim Pois(\lambda)$ with X_1, X_2 ind

$\Rightarrow X_1, X_2 \sim Pois$

b) Prove

$$E(X_1 \mid X_1 + X_2 = k) = k \cdot \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

hint Rhs | $E\left(\text{binomial}\left(k, \frac{\lambda_1}{\lambda_1+\lambda_2}\right)\right)$

Thm | let (Ω, F, P) be a prob sp

If X, Y are rv $\Rightarrow X+Y$ rv

Pf | Spose X, Y rv

$\Rightarrow \forall r \in \mathbb{R} \quad \{X \le r\} \in F, \{Y \le r\} \in F$

consider $\{X+Y \le r\}$ → by nw suffices $\{X+Y < r\}$

trivial

$$= \left[\bigcup_{a \in \mathbb{R}} \underbrace{\left(\{X < a\} \cap \{y \le r-a\}\right)}_{\in F}\right]$$

take sequence of rational $q_n \uparrow a$ $y \le r - q_n$

$$\overset{?}{=} \bigcup_{b \in \mathbb{Q}} \{X \le b, Y \le c-b\}$$

$\supseteq$ trivial

lets say $\omega \in \Omega$ st $(X+Y)(\omega) \le r$

$= X(\omega) + Y(\omega) \le r \Rightarrow Y(\omega) \le r - a$

where $X(\omega) = a$

finish later ...

Symmetric RW

$$X_i = \begin{cases} 1 & i\text{th step to right} \quad p = 0.5 \\ -1 & i\text{th step to left} \quad p = 1 \end{cases}$$

Position after n step starting at a

$$= a + \sum_{i=1}^{n} X_i$$

Assume, $a = 0$ .....

Let $Y_n = \sum_{i=1}^{n} X_i$ → pos after n steps!

let $b \in \mathbb{Z}$

The hitting time, Z_b = the first time $Y_n = b$

Thm The mass func of Z_b is

$$P(Z_b = n) = \frac{|b|}{n} \underbrace{P(Y_n = b)}_{\text{binomial}}$$

prove by induction on n | WLOG $b \geq 0$ (by symmetry)

if $\underline{n = 1}$:

if $|b| > 1$ then $P(Z_b = 1) = 0$ & $P(Y_1 = b) = 0$

if $b = 0, 1$ true as both sides are 0 or $P(Z_1 = 1) = P(Y_1 = 1) = 0.5$!

Spse holds for $n = k$