Review 1

1. Probability space (Ω, F, P)

Ω = sample space $\to$ set of all possible outcomes

F = σ field $\to$ set of all possible events

P = $P : F \to [0,1]$ (prob measure)

$F \subseteq 2^{\Omega}$ is a σ field if

$\hookrightarrow$ $\emptyset \in F$, $A \in F \Rightarrow A^c \in F$, F closed under countable unions!

Prop 1 σ field closed under set difference, countable int / union

$P : F \to \mathbb{R}$ is a prob measure if

a) $P(\emptyset) = 0$

b) $P(\Omega) = 1$

c) if $A_1, A_2, \dots \in F$ are disjoint $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$ $\to$ σ additivity

Prop 2

1) $P(A^c) = 1 - P(A)$ $\forall A \in F$

2) σ-additivity holds for fin unions.

3) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$

4) Nesting. If $A_1 \subset A_2 \subset A_3 \dots$

$P(\bigcup_{i=1}^{\infty} A_i) = \lim_{n \to \infty} P(A_i)$

demorgan

if $B_1 \supset B_2 \supset B_3 \dots$

$P(\bigcap_{i=1}^{\infty} B_i) = \lim_{i \to \infty} P(B_i)$

2. Conditional Probability

A, B events with $P(B) > 0$

$$P(A|B) := \frac{P(A \cap B)}{P(B)}$$

Prop 1. If $B_1, B_2 \ldots$ partition Ω

$$P(A) = \sum_{i=1}^{\infty} P(A|B_i) P(B_i)$$

2. Bayes $P(A|B)P(B) = P(B|A)P(A) \; (= P(A \cap B))$

3. Independence

$A, B \in \mathcal{F}$ are ind p.t $P(A \cap B) = P(A)P(B)$

$\Leftrightarrow P(A|B) = P(A)$ (when $P(B) > 0$)

A family $\{A_i\}_{i \in I}$ is ind p.t

$\forall J \subseteq I$ finite $\quad P(\bigcap_{i \in J} A_i) = \prod_{i \in J} P(A_i)$

Remark: family is ind is distinct from pairwise ind.

Random Var

A rv is a funct $X : \Omega \to \mathbb{R}$ p.t $\forall a \in \mathbb{R} \; \{X \leq a\} \in \mathcal{F}$

Prop 1 If X is a rv

$\forall a, b \in \mathbb{R}$, the following are in $\mathcal{F}$

$\{X < a\}, \{X > a\}, \{X \geq a\}, \{X = a\}, \{a < X < b\} \ldots\ldots$

- we can replace the condition $\forall a \in \mathbb{R} \; \{X \leq a\} \in \mathcal{F}$ the following.....

① $\forall a \in \mathbb{R} \; \{X < a\} \in \mathcal{F}$

② $\forall a \in \mathbb{R} \; \{X > a\} \in \mathcal{F}$ or geq

③ $\forall a, b \in \mathbb{R} \; \{a < X < b\} \in \mathcal{F}$ (can make leq)

- Sum of RV is RV
- Prod of RV is RV

Distribution

X RV. The dist func of X

$$F : \mathbb{R} \to [0,1)$$

$$F(x) = P(X \leq x)$$

Prop 1

1) $F(x) \leq F(y)$ if $x \leq y$
2) $\lim_{x \to -\infty} F(x) = 0$, $\lim_{x \to \infty} F(x) = 1$
3) $\forall y \in \mathbb{R}$, $\lim_{x \to y^+} F(x) = F(y)$

characterize dist func!

4) $P(a < X \leq b) = F(b) - F(a)$
5) $P(X = a) = F(a) - \lim_{x \to a^-} F(x)$ → if F is cts this is 0.
6) $P(X > a) = 1 - F(a)$

Joint Distribution

if X, Y rv

joint distribution $F_{X,Y} : \mathbb{R}^2 \to [0,1]$

$$(x, y) \longmapsto P(X \leq x, Y \leq y)$$

Marginal

$$\lim_{x \to +\infty} F(x, y) = F_Y(y)$$ (as $X \leq x \to \Omega$)

Discrete RV

A rv X is discrete p.t X only takes countably many values.

Mass func $\rightarrow$ $f_X(x) = P(X = x)$

Cts RV

A rv X is cts p.t its dist func can be expressed as

$$F(x) = \int_{-\infty}^{x} f(t)\, dt$$

for some f integrable. f is the **probability density func!**

Rmk | If X is a cts rv then F_X is cts
Converse not true!

In fact, if F_X is cts & **diff'ble everywhere except possible at countably many points**, then X is cts

Density $f_X(x) = F_X'(x)$

Given 2 RV, X, Y, are ind p.t $\forall a, b \in \mathbb{R}$,

$\{X \leq a\}$, $\{Y \leq b\}$ are ind!

6. Expectation

X is discrete rv.

$$E(X) = \sum_{x \in \mathbb{R}} x\, P(X = x)$$ whenever this abs conv!

Prop |

- $E(aX + Y) = aE(X) + E(Y)$ $\rightarrow$ linear! $\forall c \in \mathbb{R}$
- $E(1) = 1$
- $E(X) \geq 0$ if $X \geq 0$

Variance & cov

$$\mathrm{Var}(X) = E((X - E(X))^2) = E(X^2) - E(X)^2$$

$$\mathrm{Cov}(X, Y) = E((X - E(X))(Y - E(Y))) = E(XY) - E(X)E(Y)$$

Prop

Cov is symmetric & bilinear!

If X, Y ind $\Rightarrow$ $\text{Cov}(X,Y) = 0$

$\hookrightarrow$ converse not true!

If $X_1 \ldots X_n$ pairwise ind

$\text{Var}(X_1 + \ldots + X_n) = \text{Var}(X_1) + \ldots + \text{Var}(X_n)$

Conditional Exp

$B \in \mathcal{F}$ & $P(B) > 0$, X disc R_X

$$E(X|B) = \sum_{x \in R} x \, P(X = x \mid B)$$

Prop 1) linear, $E(1 \mid B) = 1$, $E(X \mid B) \geq 0$ if $X \geq 0$

2) $B_1, B_2 \ldots$ form a partition of Ω so

$$E(X) = \sum_{i \in \mathbb{N}} E(X|B_i) P(B_i)$$

Condititon Exp on 2 RX

X, Y rv

$E(X|Y) = \phi(Y)$ is a rv $\phi(y) = E(X \mid Y = y)$

So $E(X|Y) : \Omega \to \mathbb{R}$

$$\omega \mapsto E(X \mid Y = Y(\omega))$$

Properties ① (as before see hw)

② $E(E(X|Y)) = E(X)$

7.1 Popular Disc RV

- Bernoulli
- Binomial
- Poisson
- Geometric!