Recall Inequalities

1. **Markov**: $X \geq 0$ $\quad E(X) < +\infty$

$$P(X \geq a) \leq \frac{E(X)}{a} \quad \forall a > 0$$

2. **Chebyshev**: $E(X^2) < +\infty$

$$P(|X - EX| \geq a) \leq \frac{var(X)}{a^2}$$

3. **Bernstein**: $X_1 \dots X_n$ indep $E(X_i) = 0$
$|X_i| \leq 1$

$$P(X \geq a) \leq e^{-\frac{a^2}{2n}} \quad \forall a > 0$$

$$P(X \leq -a) \leq e^{-\frac{a^2}{2n}}$$

$\Big\}$ Concentration

4. **Holder**:

$$E(|XY|) \leq (E(|X|^p))^{1/p} \, (E(|Y|^q))^{1/q}$$

for $p, q \geq 0$ s.t. $\frac{1}{p} + \frac{1}{q} = 1$

$\Big\}$ $p = q = 2$ Cauchy Schwartz

5. **Minkowski**:

$$E(|X+Y|^p)^{1/p} \leq E(|X|^p)^{1/p} + E(|Y|^p)^{1/p}$$

e.g.) Holder: If $E(|X|^3) < +\infty \Rightarrow E(X^2)$ finite

$$E(X^2) = E(X^2 \cdot 1) \leq \left(E((X^2)^{3/2})\right)^{2/3} E(1^3)^{1/3}$$

$$\Downarrow$$

$$E(X^3)^{2/3} \cdot 1 < +\infty$$

Similar, if $E(|X|^m) < +\infty \Rightarrow E(|X|^n) < +\infty$

$\forall n \in \mathbb{N}_m$

Models of Convergence

Def Let (Ω, F, P) be a prob space. Let (X_n) be a seq of r.v. Let X be a r.v.

① $X_n \to X$ Pointwise $\quad X_n(\omega) \to X(\omega) \quad \forall \omega \in \Omega$

② (X_n) conv to X almost surely, " $X_n \xrightarrow{a.s} X$ " if

$$P(\{\omega \in \Omega \mid \lim_{n\to\infty} X_n(\omega) = X(\omega)\}) = 1$$

$\Big]$ → prob of conv pts is 1

③ (X_n) conv to X in Prob "$X_n \xrightarrow{P} X$" if

$$P\{\{\omega \in \Omega \mid |X_n(\omega) - X(\omega)| \geq \varepsilon\}\} \xrightarrow{n\to\infty} 0$$

" set where conv fails gets smaller "

④ (X_n) conv to X in distribution "$X_n \xrightarrow{D} X$" if

$F_{X_n}(z) \to F_X(z) \quad \forall z \in \mathbb{R}$ s.t F_X is cts at z

Rmk

1) Conv in dist does not make X_n & X have same prob sp.

2) Why do we need to necessitate conv at only cts pts?

e.g. $X_n = \frac{1}{n} \cdot \mathbb{1}_\Omega \to$ constant
$X = 0$
} we really want $(X_n) \xrightarrow{D} X$

$$F_{X_n}(z) = \begin{cases} 0 & z < \frac{1}{n} \\ 1 & z \geq \frac{1}{n} \end{cases}$$

$$F_X(z) = \begin{cases} 0 & z < 0 \\ 1 & z \geq 0 \end{cases}$$

$$\lim_{n\to\infty} F_{X_n}(z) = \begin{cases} 0 & z \leq 0 \\ 1 & z > 0 \end{cases}$$

oops.

Differ at exactly at pt of discontinuity

e.g. HW11 #5 X_n unif in $\{1, \dots, n\}$

$Y \sim \text{Unif}([0,1])$, $Y_n = \frac{X_n}{n}$

$Y_n \xrightarrow{D} Y$

note: Prob sp for Y_n is $(\underbrace{\{\frac{1}{n}, \frac{2}{n}, \dots \frac{n}{n}\}}_{\Omega_n}, 2^{\Omega_n}, P)$

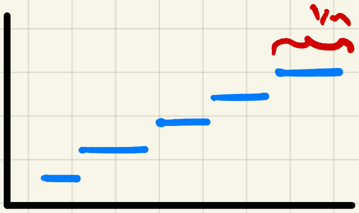
$Y_n : \Omega_n \to \mathbb{R}$

$Y_n(\omega) = \omega$

Prob sp for Y is $([0,1], \mathcal{B}^{\text{Borel}}, \text{Leb})$

$Y : [0,1] \to \mathbb{R}$
$Y(\omega) = \omega$

$$F_{Y_n}(z) = \begin{cases} 0 & \text{if } z<0 \\ \frac{\lfloor nz \rfloor}{n} & \text{if } z \in [0,1) \\ 1 & \text{if } z \geq 1 \end{cases}$$



$$F_Y(z) = \begin{cases} 0 & \text{if } z<0 \\ z & \text{if } 0 \leq z < 1 \\ 1 & \text{if } z \geq 1 \end{cases}$$

we only care about middle lim

$\frac{nz-1}{n} \leq \frac{\lfloor nz \rfloor}{n} \leq \frac{nz}{n}$] lim gives result by squeeze

Thm for the 4 types of Conv

① $\Rightarrow$ ② $\Rightarrow$ ③ $\Rightarrow$ ④ (but not converse)

trivial (under ①⇒②), trivial (under ②⇒③)

↳ Find counter eg

Pf ① ⇒ ② trivial

② ⇒ ③

$$A_n = \bigcup_{m \geq n} \{\omega \in \Omega \mid |X_m(\omega) - X(\omega)| \geq \varepsilon\}$$

$$A_n \supset A_{n-1} \cdots \quad A_\infty = \bigcap_{n=1}^{\infty} A_n$$

$$\lim_{n \to \infty} P(A_n) = P(A_\infty)$$

$$\text{if } \omega \in A_\infty \Rightarrow X_n(\omega) \not\to X(\omega)$$

$$\Rightarrow P(A_\infty) = 0$$

subset goes to

$$\Rightarrow P(\{\omega \mid |X_n(\omega) - X(\omega)| \geq \varepsilon\}) \leq P(A_n) \leq 0$$

lim

$$\therefore X_n \xrightarrow{P} X$$

③ $\Rightarrow$ ④

$$F_{X_n}(x) \to F_X(x)$$

$$P(X_n \le x) \to P(X \le x)$$

disjointification

can add $X_n \le n$

$$\{X_n \le x\} \subseteq \{X_n \le x, |X_n - X| < \varepsilon\} \cup \{|X_n - X| \ge \varepsilon\}$$

$$\cap\!\!| \quad \{X \le x+\varepsilon\}$$

$$P(\{X_n \le x\}) \le P(X \le x+\varepsilon) + P(|X_n - X| \ge \varepsilon)$$

we can swap X_n & X by symmetry & x by $x-\varepsilon$

$$P(\{X \le x-\varepsilon\}) \le P(X_n \le x) + P(|X_n - X| \ge \varepsilon)$$

Combine

$$P(X \le x-\varepsilon) + \underbrace{P(|X_n - X| \ge \varepsilon)}_{\to 0} \le P(X_n \le x) \le P(X \le x+\varepsilon) + \underbrace{P(|X_n - X| \ge \varepsilon)}_{\to 0 \text{ as conv in prob}}$$

$\downarrow$ lim $n \to \infty$

$$P(X \le x-\varepsilon) \le \lim_{n\to\infty} P(X_n \le x) \le P(X \le x+\varepsilon)$$

let $\varepsilon \to 0$, squeeze

$$\lim_{n\to\infty} P(X_n \le x) = P(X \le x)$$

as we only look at points of cts of F_X

$\square$

Thm 2

If (X_n) let $c \in \mathbb{R}$ if $X_n \xrightarrow{D} c \Rightarrow X_n \xrightarrow{P} c$

So, if const, then conv holds

Pf $\forall \varepsilon > 0$

$$P(|X_n - c| \ge \varepsilon) = P(X_n \ge c+\varepsilon) + P(X_n \ge c-\varepsilon)$$

let $n \to \infty$

$$= 1 - \lim_{y \to c+\varepsilon} F_{X_n}(y) + F_{X_n}(c-\varepsilon)$$

right cts

if we can swap lim

$$= 1 - \lim_{y\to c+\varepsilon} \lim_{n\to\infty} F_{X_n}(y) + F_X(c-\varepsilon) \to 0$$

as X const

cts

$$= 1 - 1 - 0$$

But lets do it

$$\lim_{n\to\infty} \lim_{y\to (c+\varepsilon)} F_{X_n}(y)$$

$\forall \delta > 0$, $\exists N$ s.t $\forall n \geq N$

$$F_{X_n}(c+\varepsilon/2) \geq 1-\delta$$

as F_{X_n} non decreasing

$$F_{X_n}(y) \leq 1-\delta \quad \forall\ c+\frac{\varepsilon}{2} < y < c$$

$$\lim_{y\to (c+\varepsilon)^-} F_{X_n}(y) \geq 1-\delta \quad \forall n \geq N$$

let $N \to \infty$ let $\delta \to 0$

$$\lim_{N\to\infty} \lim_{y\to (c+\varepsilon)^-} F_{X_n}(y) \geq 1$$ $\longrightarrow$ **and we get result**