Midterm: Wed Oct 13 6-8pm, Angell Hall Auditorium C

Recall)

Variance → $Var(X) = E((X-E(X))^2) = E(X^2) - E(X)^2$

Covariance → $cov(X,Y) = E((X-EX)(Y-EY))$

$= E(XY) - E(X)E(Y)$

Lemma) if X, Y are ind,

$E(XY) = E(X)E(Y)$ => Pf in textbook

Cor) if X, Y are ind $cov(X,Y) = 0$

Correlation → $\rho(X,Y) = \dfrac{cov(X,Y)}{\sqrt{Var(X)Var(Y)}}$

Thm) Cauchy-Schwartz

$E(XY)^2 \leq E(X^2)E(Y^2)$

Cor) $|\rho(X,Y)| \leq 1$

Pf) CS.

Assume $E(X^2) > 0$ (in other case use how as $P(X^2=0)=1$)
=> both side 0

$0 \leq E((tX+Y)^2)$ ⟶ non neg rv exp.

$= t^2 E(X^2) + 2t E(XY) + E(Y^2)$

non negative quadratic in t ∴ $b^2 - 4ac \leq 0$

$4E(XY)^2 - 4E(X^2)E(Y^2) \leq 0$

$4E(XY)^2 \leq 4E(X^2)E(Y^2) \leq E(X^2)E(Y^2)$

(cool) $E(X,Y)$ inner prod of rs or rv

e.g. X Y are drv s.t joint mass func is

↳ $f_{X,Y}(x,y)$

$y \backslash x$	-1	1
-1	$1/3$	$1/6$
1	$1/6$	$1/3$

get E, var, cov, ρ. Determine if they're ind.

Ans| Using the shit on the left|

$E(X) = (-1)P(X=-1) + 1\, P(X=1)$

$$P(X=-1) = f_X(-1)$$
$$= f_{X,Y}(-1,1) + f_{X,Y}(-1,-1)$$
$$= \frac{1}{6} + \frac{1}{3} = \frac{1}{2}$$

$P(X=1) = \frac{1}{2}$

$\therefore E(X) = 0$

Similarly $E(Y) = 0$

$Var(X) = E(X^2)$ (as $E(X) = 0$)

X^2 only take val 1 so

$E(X^2) = 1$

$Var(Y) = 1$

$$Cov(X,Y) = E(XY)$$
$$= 1 \cdot \underbrace{P(XY=1)}_{x=-1,\,y=-1;\ x=1,\,y=1} + (-1)P(XY=-1)$$
$$= \frac{2}{3} - \frac{1}{3}$$
$$= \frac{1}{3}$$

Recall

$F_{X,Y}(x,y) = P(X \le x, Y \le y)$

if X Y disc

$f_{X,Y}(x,y) = P(X=x, Y=y)$

Prop

① $\lim_{x \to \infty} F_{X,Y}(x,y) = F_Y(y)$ → $X \le x$ eventually reach Ω

② $\lim_{y \to \infty} F_{X,Y}(x,y) = F_X(x)$

↳ marginal dist of X, Y

When X, Y disc,

$$\sum_{y \in \mathbb{R}} f_{X,Y}(x,y) = f_X(x)$$

→ This is countable

$$\sum_{x \in \mathbb{R}} f_{X,Y}(x,y) = P(X=x, \bigsqcup_{y \in \mathbb{R}} \{Y=y\})$$
$$P(X, x) = f_X(x)$$

$$\sum_{x \in \mathbb{R}} f_{X,Y}(x,y) = f_Y(y)$$

→ marginal mass func

$\implies X, Y$ not ind as $\text{cov}(X,Y) \neq 0$

$\rho(X,Y) = \dfrac{1/3}{\sqrt{1}} = \dfrac{1}{3}$

Conditional Expectation

Defn) let X be a disc rv. let $B \in \mathcal{F}$ s.t $P(B) > 0$

Conditional Exp of X given B

$$E(X \mid B) = \sum_{x \in \Omega} x \, P(X = x \mid B)$$

Prop) $B_1 \ldots B_n \in \mathcal{F}$ s.t $\bigsqcup_{i=1}^{n} B_i = \Omega$, $P(B_i) > 0$ then

$$E(X) = \sum_{i=1}^{n} E(X \mid B_i) P(B_i) \quad // \text{ez pf.}$$

e.g.) Roll a fair die

X = # rolls before seeing the 1^{st} 6

Let B_i event where the first roll is an i

$\bigsqcup_{i=1}^{6} B_i = \Omega$

$$E(X) = \sum_{i=1}^{6} E(X \mid B_i) P(B_i) = \frac{1}{6} \sum_{i=1}^{6} E(X \mid B_i)$$

we note $E(X \mid B_6) = 1$

$i \neq 6 \implies E(X \mid B_i) = E(X) + 1$ $\Rightarrow$ we restart the process but say roll

So, $E(X) = \frac{1}{6}\left[5(E(X)+1) + 1\right]$

$E(X) = \frac{1}{6}\left[5E(X) + 6\right] \implies \frac{E(X)}{6} = 1 \implies E(X) = 6$

e.g. 2 | Tossing a coin

$X \to$ number of tosses before seeing 2 consec heads.

Partition based on first 2 tosses

$B_1 \to TT$, $B_2 \to TH$ ... $B_4 \to HH$

$$E(X) = E(X \mid HH)P(HH) + \underbrace{E(X \mid HT)P(TH)}_{\boxed{D}} \cdots E(X \mid TT)P(TT)$$

$$= 2\left(\frac{1}{4}\right) + \boxed{\text{junk}}_{\text{somoding}} + (1+E(X))\left(\frac{1}{2}\right)$$

$$E(X) = E(X \mid HH)P(HH) + E(X \mid HT)P(HT) + E(X \mid T)P(T)$$

$$E(X) = 2\left(\frac{1}{4}\right) + (E(X)+2)\frac{1}{4} + \frac{1}{2}(E(X)+1)$$

$$\frac{E(X)}{4} = \frac{3}{2} \Rightarrow E(X) = 6$$

Redo eg 1 but $\boxed{HT}$

$$E(X) = E(X \mid T)P(T) + E(X \mid H)P(H)$$

$$+ \sum_{i=1}^{\infty} \left(\frac{1}{2^i}\right) E(X \mid \underbrace{H \cdots H}_{i}) + \frac{1}{2^i}\, i$$

$$= (E(X)+1)\frac{1}{2} + 1 + 2$$

$$\frac{E(X)}{2} =$$