Recall:

① Discrete RV: only takes countably many values!

↳ mass func $f(k) = P(X=k)$

② Cts RV: distribution func is an integration

$$F(a) = \int_{-\infty}^{a} f(t)\,dt \longrightarrow \text{density func}$$

↳ F is cts

e.g. Toss a coin n-times & count # of heads,

$P(\text{head}) = p$, $P(\text{tail}) = (1-p)$

$X :=$ # heads in n tosses $X(\omega) = 0, 1, \ldots, n$

Mass funct:

$$f(k) = P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$$

↳ Binomial Distribution

e.g. Poisson Distribution Fix $\lambda \in \mathbb{R}$

X is a rv that takes values in $\{0, 1, \ldots\ldots\}$

mass $f(k) = \frac{\lambda^k}{k!} e^{-\lambda} = P(X=k)$

e.x) $\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} = 1$ ⇒ taylor exp of e^x where $x = -\lambda$

Focus on discrete RV.....

Independence of RV

Recall: 2 events A & B are ind $\iff P(A \cap B) = P(A)P(B)$

Defn 2 discrete rv, X & Y, are independent if
$\{X=x\}$ & $\{Y=y\}$ are ind $\forall\, x,y \in \mathbb{R}$
↳ $P(X=x, Y=y) = P(X=x)\, P(Y=y)$

Defn let $X_1 \dots X_n$ be rv.
The joint distribution func is def as
$F_{X_1 \dots X_n} : \mathbb{R}^n \to [0,1]$
$(x_1, \dots, x_n) \mapsto P(X_1 \le x_1, X_2 \le x_2, \dots, X_n \le x_n)$

If $X_1, X_2, \dots, X_n$ are discrete. We can def joint mass func.
$f_{X_1 \dots X_n} : \mathbb{R}^n \to [0,1]$
$(x_1, \dots x_n) \longmapsto P(X_1 = x_1, X_2 = x_n, \dots X_n = x_n)$

Reformulate Defn

X, Y (disc) are ind iff
$f_{X,Y}(x,y) = f_X(x)\, f_Y(y) \quad \forall\, x,y \in \mathbb{R}$

non.eg. X = no of head in n tosses → $P(\text{head}) = p$
Y = no of tail in n tosses $P(\text{tail}) = 1-p$

Q Are X, Y ind
$P(X=k, Y=n-k) = P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$
but $P(Y=n-k) \ne 1$
So, X, Y not ind.

Thm Let X, Y be disc rv, let $f, g: \mathbb{R} \to \mathbb{R}$ be func.
if X & Y are ind $\Rightarrow$ $f(X), g(Y)$ are ind
↳ composing

Pf) $P(f(X)=a,\ g(Y)=b) = P(f(X)=a)\,P(g(Y)=b)$ $\forall a,b\in\mathbb{R}$

$$\{f(X)=a\} = \{X = f^{-1}(\{a\})\} = \bigcup_{\{x_i:\ f(x_i)=a\}} \{X = x_i\}$$

or $x_i \in f^{-1}\{a\}$

$$\{g(Y)=b\} = \bigcup_{y_j \in f^{-1}(\{b\})} \{Y = y_j\}$$

Rmk

$$\left(\bigcup A_i\right) \cap \left(\bigcup B_j\right) = \bigcup_{i,j} (A_i \cap B_j)$$

$$P(f(X)=a,\ g(X)=b)$$

$$= P\Big(\bigcup_{x_i\in f^{-1}(a)} \{X=x_i\} \cap \bigcup_{y_j\in f^{-1}(b)} \{Y=y_j\}\Big)$$

$$= P\Big(\bigsqcup_{\substack{x_i\in f^{-1}(a)\\ y_j\in f^{-1}(b)}} \{X=x_i,\ Y=y_j\}\Big)$$

ends up being disjoint union

$$= \sum_{\substack{x_i\in f^{-1}(a)\\ y_j\in f^{-1}(b)}} P(X=x_i,\ Y=y_j) \overset{\text{ind}}{=} \sum_{\substack{x_i\in f^{-1}(a)\\ y_j\in f^{-1}(b)}} P(X=x_i)\,P(Y=y_j)$$

distributive things

$$= \sum_{x_i\in f^{-1}(a)} P(X=x_i) \cdot \sum_{y\in f^{-1}(b)} P(Y=y_j)$$

$$= P(f(X)=a)\cdot P(g(Y)=b) \quad \therefore \text{ind} \quad \square$$

Expectation

e.g. Flip a fair coin 1000 times

expect → #H = #T = 500

Defn) let X be a disc RV → equiv: exp value, mean val, ave val

The expectation of X $\Longleftrightarrow$

$$E(X) = \sum_{x_i\in X(\Omega)} x_i\, P(X=x_i)$$

whenever the series conv absolutely

↳ image of X is countable

↳ if not in image, $P(X=x)=0$!

e.g. X = # of head in 1000 toss

$$E(x) = \sum_{i=0}^{1000} i\, P(x=i) = \sum_{i=0}^{1000} i \binom{1000}{i} \left(\tfrac{1}{2}\right)^{i} \left(\tfrac{1}{2}\right)^{1000-i}$$

$$= \frac{1}{2^{1000}} \sum_{i=0}^{1000} i \binom{1000}{i}$$

$i=0$ doesn't do any thing.

$$= \frac{1}{2^{1000}} \sum_{i=1}^{1000} \frac{1000 \cdot 999!}{(i-1)!\,(999-(i-1))!}$$

$$= \frac{1000}{2} \underbrace{\sum_{i-1=0}^{999} \binom{999}{i-1} \frac{1}{2^{i-1}} \frac{1}{2^{999-(i-1)}}}$$

$$= 500 \cdot \left(\tfrac{1}{2} + \tfrac{1}{2}\right)^{999}$$

binom expan

$$= 500$$

Defn X is disc RV

1. k^{th} moment of $X \to E(X^k)$
2. $\mathrm{var}(X) = E((X - EX)^2)$
3. $\mathrm{dev}(X) = \sqrt{\mathrm{Var}(X)}$