Ordered sets & Ordered Topology   Ch 1 §3, Ch 2 §14

Def A relation $<$ on a set X is an **order relation** (or simple / linear order) if it is
1) if $x \neq y \Rightarrow$ either $x<y$ or $y<x$
2) non-reflexivity $\to$ $x<x$ not true $\forall x \in X$
3) transitivity $x<y,\ y<z \Rightarrow x<z$

e.g. $\mathbb{R}$ with usual ordering $<$

ex $\mathbb{R}$, $<_{sq}$ $\xrightarrow{\text{defn}}$ $x <_{sq} y \iff x^2<y^2$ or if $x^2=y^2$ & $x<y$

Def if X set, $<$ order rel'n then

$(a,b) := \{x \in X \mid a<x<b\}$

$(a,b] := \{x \in X \mid a<x\leq b\}$ & $[a,b)$, $[a,b]$ similarly

$\hookrightarrow$ with ~~no~~ intervals too

called "open" / "closed" rays

Def The dictionary or lexi order $\leq$ on $A\times B$ (will use $a\times b$ for elt not (a,b) to avoid confusion)

If $(A,<_A)$ & $(B,<_B)$ ordered, define $<$ on $A\times B$ by

$a_1\times b_1 < a_2 \times b_2 \iff$
① $a_1 <_A a_2$
② $a_1 = a_2$, $b_1 <_B b_2$

Rmk $\to$ extend to finite / countable products of simply ordered sets

ex A = english alphabet $<$ (usual order $a,b,c,\dots$)
in $A^3 \to$ 3 letter words car < cat < cub < dog

If $(X,<)$ is simply ordered, there is a standard topology on X, order top

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Def let $(X,<)$ be simply ordered, $|X|>1$ let

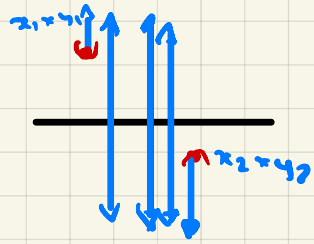
$\mathcal{B} = \{(a,b) \mid a,b \in X\} \cup \{[a_0,b) \mid b\in X\} \cup \{(a,b_0] \mid a \in X\}$

This is the basis of the order topology on X $\mathcal{T}_{\mathcal{B}}$

note: a_0, b_0 are smallest & largest numbers (if exist)

ex.1 Order topology on $\mathbb{R}^2$ (dictionary order)

Basis → $(x_1 \times y_1, x_2 \times y_2)$ looks like this



ex.1 order topology on $\mathbb{Z}_+$

Basis: $\{2\} = (1,3)$, $\{1\} = [1,2)$

↳ get the discrete topology

ex.1 The set $\{1,2\} \times \mathbb{Z}_+$ dictionary ordered

smallest → a_0 → 1×1, no largest

$B = \{[1\times1, a\times b) \mid a\times b \in \{1,2\}\times\mathbb{Z}_+\} \cup \{(a_1\times b_1, a_2\times b_2) \mid \dots\}$

is this the same as the discrete topology on X?

No $\{2\times1\}$ not in order top! Easy to show