Recall X space $x_0 \in X$.

$\pi_1(X, x_0) = \{$ path homotopy classes of loops based at $x_0\}$

↳ grp with concat!

Thm If there is a path in X from x_0 to x_1, $\pi_1(X,x_0) \cong \pi_1(X,x_1)$

Pf let $\alpha(x_0) = x_1$ be this path. def

$$\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1) \quad \rightsquigarrow \text{can check well def}$$
$$[f] \to [\bar{\alpha}] * [f] * [\alpha]$$

$\hat{\alpha}$ is a grp homomorphism

$$\hat{\alpha}([f]) * \hat{\alpha}([g]) = \left([\bar{\alpha}] * [f] * \underbrace{[\alpha]\right) * \left([\bar{\alpha}]}_{\mathrm{Id}} * [g] * [\alpha]\right)$$
$$= [\bar{\alpha}] * [f] * [g] * [\alpha]$$
$$= \hat{\alpha}([f] * [g])$$

By symmetry: can def homomorphism $\hat{\alpha}' : \pi_1(X, x_1) \to \pi_1(X, x_0)$
$$[f] \mapsto [\alpha] * [f] * [\bar{\alpha}]$$

Since we have backward homo ↝ & $\hat{\alpha} \circ \hat{\alpha}' = \mathrm{Id}(\pi_1(X, x_1))$

$\Rightarrow \pi_1(X, x_0) \cong \pi_1(X, x_1)$ & $\hat{\alpha}' \circ \hat{\alpha} = \mathrm{Id}(\pi_1(X, x_0))$

Cor If X is path con $\pi_1(X, x_0) \cong \pi_1(X, x_1)$ $\forall x_0, x_1 \in X$

Def A space X is simply connected if it is path conn & has trivial fundamental grp! ↳ $\pi_1(X, x_0) = 0$ for some (thus any) $x_0 \in X$

The fundamental grp is a top inv!

Notation $h : (X, x_0) \to (Y, y_0)$ a cts map $X \to Y$ so $h(x_0) = y_0$

Def let $h : (X, x_0) \to (Y, y_0)$ cts. We define

$$h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$$
$$[f] \to [h \circ f]$$

h^* is the homomorphism induced by h relative to x_0!

→

Check h_* is well def. That is, if $f \simeq_p f'$ that $h \circ f \simeq_p h \circ f'$

Let F be path homotopy between f, f'

$\Rightarrow$ $h \circ F$ path homotopy between $h \circ f$, $h \circ f'$ $\Rightarrow [h \circ f] = [h \circ f']$

- h_* grp homo as $(h \circ f) * (h \circ g) = h \circ (f * g)$ ⟵ Check

Functorial prop of homeo on π_1

Thm) If $h: (X, x_0) \to (Y, y_0)$, $k: (Y, y_0) \to (Z, z_0)$

$k \circ h: (X, x_0) \to (Z, z_0)$ but

$(k \circ h)_* = k_* \circ h_*$

If $i_X: (X, x_0) \to (X, x_0) \Rightarrow i_*$ is identity homo on $\pi_1(X, x_0)$

Pf) By def $(k \circ h)_*([f]) = [k \circ h \circ f]$

$= k_*([h \circ f]) = k_* \circ h_* (f)$

Top inv of π_1

Cor) if $h: (X, x_0) \to (Y, y_0)$ is a homeo $\Rightarrow$ h_* is an iso $\pi_1(X, x_0) \cong \pi_1(Y, y_0)$

Pf) h_* is a homeo $\pi_1(X, x_0) \to \pi_1(Y, y_0)$

$(h^{-1})_*$ is a homeo $\pi_1(Y, y_0) \to \pi_1(X, x_0)$

If we show h_*, $(h^{-1})_*$ are inv we are done!

$h_* \circ (h^{-1})_* ([f]) \overset{\text{Func}}{=} (h \circ h^{-1})_* ([f]) = i_Y([f])$

Similar for $h^{-1}_* \circ h_*$. $\therefore$ h_* is an iso!

e.g. $\pi_1(S^1, x_0) \cong \mathbb{Z}$ e.g. S^n for $n \geq 2$ is simply connected!

e.g.