Product Topology

Def if X & Y are topological spaces, the product topology on $X \times Y$ has a basis,

$$\mathcal{B} = \{ U \times V \mid U \text{ open in } X, V \text{ open in } Y \}$$

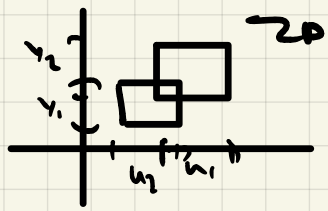
① $x \times y \in X \times Y \in \mathcal{B}$

② spose $x \times y \in U_1 \times V_1, U_2 \times V_2$, $x \times y \in (U_1 \cap U_2) \times (V_1 \cap V_2)$ (open in X, open in Y)

$\Rightarrow (U_1 \cap U_2) \times (V_1 \cap V_2) \subseteq U_1 \times V_1, U_2 \times V_2$

↳ equal to intersection!

Note: $\mathcal{B}$ itself is not a topology



→ dotted

→ union should be open

Thm $\mathcal{B}_x$ is basis for X, $\mathcal{B}_Y$ is a basis for Y, then

$\mathcal{B} = \{ B \times C \mid B \in \mathcal{B}_x, C \in \mathcal{B}_Y \}$ is a basis for prod top of $X \times Y$

ex The std top on $\mathbb{R}^2$ is the same as the product top on $\mathbb{R}^2$

$\mathcal{B} = \{ U \times V \mid U, V \in \mathcal{T}_{\mathbb{R}} \}$ or $\mathcal{B} = \{ (a_1, b_1) \times (a_2, b_2) \mid a_1, a_2, b_1, b_2 \in \mathbb{R} \}$

ex if X, Y are top sp with the discrete top, what is prod top on $X \times Y$? → discrete singletons in basis $\{x \times y\}$ open in $X \times Y$

ex (HW2) Order top on $\mathbb{R}^2$ is the same as the product top on $\mathbb{R}_{\text{discrete}} \times \mathbb{R}_{\text{std}}$

Subspace Topology

Def let $(X, \mathcal{T})$ top sp with $Y \subseteq X$, the collection

$\mathcal{T}_Y = \{ U \cap Y \mid U \in \mathcal{T} \}$ is the subspace topology on Y (sub sp of X)

ex $[0,1] \subseteq \mathbb{R}$, $[0, \frac{1}{2}) \in \mathcal{T}_{[0,1]}$ but not in $\mathcal{T}_{\mathbb{R}}$

Lemma if $Y^{\text{open}} \subseteq X$, then U open in $Y \Rightarrow U$ open in X.

Pf U open in $Y \Rightarrow \exists V^{\text{open}} \subseteq X$ s.t. $\underbrace{Y \cap V}_{\in \mathcal{T}_X} = U \Rightarrow U \in \mathcal{T}_X$

Lemma | If $\mathbb{B}$ is a basis for X, $Y \subseteq X$ then
$\mathbb{B}_Y = \{B \cap Y \mid B \in \mathbb{B}\}$ is a basis for Y giving the subsp top

e.g. | $[0,1] \subseteq \mathbb{R}$

Gets a basis look like (a,b), $[0,b)$, $(a,1]$ $\quad a,b \in [0,1]$
$[0,1]$, $\emptyset$

Same as the order topology on $[0,1]$!

e.x. (HW2) subsp top $\neq$ order topology

ex) $\underbrace{[0,1] \times [0,1]}_{=:\, I^2} \subseteq \mathbb{R}^2$, subsp top & lex-order top are different

$\{\frac{1}{2}\} \times (\frac{1}{2}, 1]$ open in order but not subspace!

① $\{\frac{1}{2}\} \times (\frac{1}{2}, 1] = I^2 \cap (\{\frac{1}{2}\} \times (\frac{1}{2}, 2))$

Think about this, I think we are using $\mathbb{R}^2$ with order as ambient space)