Metric top

Def | A topological space X is metrizable pt $\exists$ a metric on X that induces $\mathcal{T}_X$

e.x. $\mathbb{R}^n$, $d(x,y) = \sqrt{\Sigma (x_i - y_i)^2}$

$\mathbb{R}^n$, $\rho(x,y) = \max_{i \in \mathbb{N}_n} \{|x_i - y_i|\}$

Thm | topologies on $\mathbb{R}^n$ induced by d & ρ are the same!
And same as $\mathbb{R}^n$ with prod topology

Lemma | let d, d' metrics on X that induce top $\mathcal{T}, \mathcal{T}'$

$\mathcal{T}'$ is finer than $\mathcal{T}$ iff $\forall x \in X$, $\varepsilon > 0$ $\exists \delta > 0$ so,

$B_{d'}(x, \delta) \subseteq B_d(x, \varepsilon)$

Pf lemma | Suppose $\mathcal{T}'$ finer than $\mathcal{T}$, ($\Rightarrow$)

let $x \in X$, $\varepsilon > 0$ we have $B_d(x, \varepsilon) \in \mathcal{T} \subseteq \mathcal{T}'$

Need to show equiv of this def

$\therefore$ as $x \in B_d(x, \varepsilon)$ can find $\exists \delta > 0$ so $B_{d'}(x, \delta) \subseteq B_d(x, \varepsilon)$

($\Leftarrow$) suffices to show basis elements of $\mathcal{T}$ open in $\mathcal{T}'$

Consider $B_d(x_0, \varepsilon)$, let $x \in B_d(x_0, \varepsilon)$ be given.

note $\exists \varepsilon' > 0$ so $B_d(x, \varepsilon') \subseteq B_d(x_0, \varepsilon)$

By ε-δ hyp $\exists \delta > 0$ so $B_{d'}(x, \delta) \subseteq B_d(x, \varepsilon') \subseteq B_d(x_0, \varepsilon)$

So, $B_d(x_0, \varepsilon) \in \mathcal{T}'$ So $\mathcal{T}'$ finer than $\mathcal{T}$ $\square$

Pf of thm | let $\varepsilon > 0$. <u>Claim</u>

$B_d(x, \varepsilon) \subseteq B_\rho(x, \varepsilon) \subseteq B_d(x, \sqrt{n}\,\varepsilon)$

If we show this, Then $\mathcal{T}_d = \mathcal{T}_\rho$

Note, if $\rho(x,y) = \varepsilon \Rightarrow \max_{i \in \mathbb{N}_n} \{|x_i - y_i|\} \Rightarrow d(x,y) \geq \varepsilon$

Similarly show $d(x,y) \leq \sqrt{n}\, \rho(x,y)$

As $\rho(x,y) \leq d(x,y) \leq \sqrt{n}\, \rho(x,y)$

Now, show $\mathcal{T}_p$ is equiv to prod top on $\mathbb{R}^n$ $(\mathcal{T})$

let B be basis of $\mathcal{T}$ so, $B = \prod_{i \in \mathbb{N}_n} (a_i, b_i)$ $\Leftarrow$

Pick $x \in B$ $\forall i \in \mathbb{N}_n$ $a_i < x_i < b_i$

So, $\exists \varepsilon_i$ so, $(x_i - \varepsilon_i, x_i + \varepsilon_i) \subseteq (a_i, b_i)$

let $\varepsilon := \min_{i \in \mathbb{N}} \varepsilon_i \Rightarrow x_i \in (x_i - \varepsilon, x_i + \varepsilon)$ $\forall i \in \mathbb{N}_n$

So, $x \in \underbrace{\prod_{i \in \mathbb{N}_n} (x_i - \varepsilon, x_i + \varepsilon)}_{B_p(x, \varepsilon)} \subseteq B$

So, $B \in \mathcal{T}_p$

Since $B_p(x, \varepsilon)$ basis elt of $\mathcal{T}_p$ this is so it is basis of prod top.

So, $\mathcal{T}_p = \mathcal{T}$

Q) Is $\mathbb{R}^\omega$ metrisable?

Attempt $d(x,y) = \left(\sum_{i=1} (x_i - y_i)^2\right)^{1/2}$ but convergence .....

$\rho(x,y) = \sup_{i \in \mathbb{N}} \{|x_i - y_i|\}$ $\rightarrow$ might not be bdd above ...

Def) $\bar{\rho}(x,y) = \sup\{\min\{|x_i - y_i|, 1\}\}$ $\rightarrow$ this is well def

$\rightarrow$ can check it is a metric!

$\bar{\rho} \rightarrow$ uniform metric on $\mathbb{R}^\omega$ inducing unif top

Thm) On $\mathbb{R}^\omega$ box top finer than unif top which is finer than prod top!

but

Thm) let $\bar{d}(a,b) = \min\{|a-b|, 1\}$ metric on $\mathbb{R}$.

If $x, y \in \mathbb{R}^\omega$ define $D(x,y) = \sup_{i \in \mathbb{N}} \left\{\frac{\bar{d}(x_i, y_i)}{i}\right\}$

Pf Show D metric. Everything trivial but triangle. Show

$$\forall i \quad \frac{\bar{\partial}(x_i, z_i)}{i} \leq \underbrace{\frac{\bar{\partial}(x_i, y_i)}{i} + \frac{\bar{\delta}(y_i, z_i)}{i}}_{\text{upper bd}} \leq D(x,y) + D(y,z)$$

$$\Rightarrow D(x,z) \leq \frac{\bar{\partial}(x_i, y_i) + \bar{\partial}(y_i, z_i)}{i}$$

compare $\mathcal{T}_D = \mathcal{T}_\pi$

($\subseteq$) let $\mathcal{U} \in \mathcal{T}_D$ and $x \in \mathcal{U}$, need to find $V \in \mathcal{T}_p$ so

$x \in V \subseteq \mathcal{U}$

can find ε so $B_D(x, \varepsilon) \subseteq \mathcal{U}$. Choose N so $\frac{1}{N} < \varepsilon$

let $\underbrace{V}_{\text{basis}} = (x_1 - \varepsilon, x_1 + \varepsilon) \times \cdots \times (x_N - \varepsilon, x_N + \varepsilon) \times \mathbb{R} \times \mathbb{R} \cdots$

Note $\forall i \geq N \quad \bar{\partial}(x_i$