Lec 21 – Countability & Separation Axioms — Chapter 4

Motivation

- When does a space embed into a metric sp X?
- Embed into $\mathbb{R}^n$?

Urysohn Metrization Thm

If X is second countable & regular, then X can be embedded in some metric space

§30 – Countability Axioms

Def) A space X has a countable basis at $x \in X$ if there is a countable collection $\mathcal{B}$ of nbhds of x so every nbhd of of x contains some $B \in \mathcal{B}$.

If X has a countable basis at all $x \in X$ then, it satisfies the first countability axiom

E.g. 1 If (X,d) metric space, it is first countable. Choose $x \in X$. Take,

$$\mathcal{B} = \{ B(x, \tfrac{1}{n}) \mid n \in \mathbb{N} \}$$

e.g. 2 $\mathbb{R}_\ell$ is first countable. for $x \in X$ take $\{ [x, x+\frac{1}{n}) \mid n \in \mathbb{N} \}$

→ converse

Note: limit pt of seq of first ctbl $\iff$ seq conv to that pt.

Def) If X has a countable basis for its topology, it is second countable

→ let $\mathcal{E}$ basis

Rem 1 Second countable $\Rightarrow$ first countable. Let X be second ctbl, let $x \in X$, $\mathcal{B} = \{ B \mid B \in \mathcal{E}, x \in B \}$

E.g. $\mathbb{R}$ is second countable, $\{ (a,b) \mid a, b \in \mathbb{Q} \}$

E.g. $\mathbb{R}^n$ is second countable with product basis

E.g. $\mathbb{R}^\omega = \mathbb{R} \times \cdots \times \mathbb{R}$ with product top is second ctble. $\left\{ \prod_{i \in \mathbb{N}} U_i \;\middle|\; \begin{array}{l} U_i = (a_i, b_i),\ a_i, b_i \in \mathbb{Q} \\ \text{for finitely many } i, \\ \mathbb{R} \text{ otherwise} \end{array} \right\}$

Non-eg. 1 Uniform top on $\mathbb{R}^\omega$. Comes from metric,

$$\rho(x,y) = \sup_{i \in \mathbb{N}} (\bar{d}(x_i, y_i)) \qquad \bar{d}(x_i, y_i) = \max(1, |x-y|)$$

This is first ctbl as it comes from a metric. But not second ctbl. | dist is 1 for non eq pts

→ set of seq of 0,1's

Consider $\{0,1\}^\omega \subseteq \mathbb{R}^\omega$. Note: $\{0,1\}^\omega$ is uncountable & inherits disc top.
So $\{0,1\}^\omega$ isn't second ctbl $\Rightarrow$ $\mathbb{R}^\omega$ not sec ctbl

Thm on next

Thm 6) If $A \subseteq X$, X is 1^{st} ctbl $\Rightarrow A$ is first ctble
" " 2^{nd} " $\Rightarrow$ " " second "

② Ctbl product topology also behaves well

Pf) Pretty immediate, in textbook

Dense Subset

Def) A subset $A \subseteq X$ is dense in X if $\overline{A} = X$.

Thm) Suppose X second ctbl. Then:

(X is Lindelöf) a) Every countable cover has a ctbl subcover

(X is separable) b) $\exists$ a ctbl subset $A \subseteq X$ that is dense

In general, these two conditions are weaker than second ctbl!

Pf) Let $\{B_n\}$ be a ctbl basis for X.

a) let $\mathcal{A}$ open cover of X.

Let $n \in \mathbb{N}$, choose A_n s.t $B_n \subseteq A_n$ (if possible)

$\mathcal{A}' = \{A_n\} \subseteq \mathcal{A}$. $\mathcal{A}'$ is a ctble subcoll.