Cts functions

Def let X, Y be top spaces. $f: X \to Y$ is cts p.t $\forall A^{open} \subseteq Y$ $f^{-1}(A)$ is open!

to show continuity, suffices to show preimage of basis els are open!

if $V = \bigcup_\alpha B_\alpha$, $f^{-1}(V) = f^{-1}(\bigcup_\alpha B_\alpha) = \bigcup_\alpha f^{-1}(B_\alpha)$

e.g. $f: \mathbb{R} \to \mathbb{R}$ In analysis, "ε-δ" agrees with ↑

E.g. $\mathbb{R}_\ell$ has a basis given by $\mathcal{B} = \{[a,b) \mid a, b \in \mathbb{R}\}$ not open

$f: \mathbb{R} \to \mathbb{R}_\ell$ given by $f(x) = x$ → not cts! $f^{-1}([a,b)) = [a,b)$

but $g: \mathbb{R}_\ell \to \mathbb{R}$ $g: y \mapsto y$ is cts!

Thm If $f: X \to Y$. TFAE

1) f is cts

2) for every subset $A \subseteq X$, $f(\bar{A}) \subseteq \overline{f(A)}$

3) for all closed $E \subset Y$, $f^{-1}(E)$ is closed in X!

4) for all $x \in X$, each nbhd V of $f(x)$ there is a nbhd of $\mathcal{U}$ of x so $f(\mathcal{U}) \subseteq V$

Pf (1) ⟹ (4)

$\mathcal{U} = f^{-1}(V)$ is open & $f(\mathcal{U}) \subseteq V$ ☐
↳ nbhd of x as $f^{-1}(V) \ni f(x)$

(4) ⟹ (1)

let V be open in Y. let $x \in f^{-1}(V) \Rightarrow f(x) \in V$
so, $\exists\, \mathcal{U}_x^{open} \ni x$ so $f(\mathcal{U}_x) \subseteq V \Rightarrow \mathcal{U}_x \subset f^{-1}(V)$

Claim: $f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} \mathcal{U}_x$ ← open ☐

Immediate $\subseteq$ $\supseteq$

Rest: Munkres!

Def Let $f: X \to Y$ be bijective.

If f & f^{-1} are cts then f is a homeomorphism

equiv a bijection $f: X \to Y$ is a homeo iff

f, f^{-1} are open, or

$f(U)$ open $\Leftarrow$ U open

A homeo $f: X \to Y$ induces a bij btw the top on X & Y

e.g. $f: \mathbb{R} \to \mathbb{R}$, $f(x) = 3x+1$ is homeo

e.g. $f: (a,b) \to (0,1)$ $\quad f(x) = \dfrac{x-a}{b-a}$

Thm Rules for constructing cts function.

$f: A \to X$, $A \subseteq X$

$a \mapsto a$

$Id_X|_A$

1) Constant functions are always cts

2) Inclusion maps $f: A \to X$, $A \subseteq X$ are cts

3) Compositions of cts func are cts

4) $f: X \to Y$ cts $\Rightarrow$ $f|_A$ $A \subseteq X$ is cts

5) local formulation of continuity.

If $X = \bigcup_\alpha U_\alpha$, U_α open in X. $f: X \to Y$ is cts iff $f|_{U_\alpha}$ is cts $\forall \alpha$

Thm (pasting lemma)

let $X = A \cup B$, A, B closed in X. let $f: A \to Y$ cts, $g: B \to Y$ cts

That agree on $A \cap B$

Then f, g glue together to get cts func $h: X \to Y$

$$x \mapsto \begin{cases} f(x) & x \in A \\ g(x) & \end{cases}$$

Pf (pasting)

Check h is cts. Let C be closed in Y.

$C \cap A$, $C \cap B$ are closed

$h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$ by def of h & $X = A \cup B$

as f is cts $f^{-1}(C)$ is closed in $A \Rightarrow f^{-1}(C)$ closed in X as A closed ! $\Rightarrow$ absolutely closed

Similarly $g^{-1}(C)$ closed in X.

$\therefore h^{-1}(C)$ closed in X

$\square$

e.g. 1