Limit (aka cluster/accumulation) Point

Def) Let $A \subseteq X^{\text{top sp}}$ & $x \in X$. We say x is a limit pt of A if every nbhd of x intersects A in some point other than x

equivalently $x \in \overline{A \setminus \{x\}}$

(x may or may not be in A)

e.g in $\mathbb{R}$, let $B = \{\frac{n+1}{n} \mid n \in \mathbb{Z}^+\}$ 1 is the only limit pt of B

in $\mathbb{R}$, let $A = (0,1]$, set of limit pts is $[0,1]$

in $\mathbb{R}$, let $\mathbb{Q} \subseteq \mathbb{R}$. limit points is entire set of $\mathbb{R}$

Another way to define the closure of a set.

Thm) $A \subseteq X$ top sp. let $A' :=$ set of lim pts of A. Then

$$\overline{A} = A \cup A'$$

Pf) $(\supseteq)$ $\overline{A} \supseteq A$ trivially. let $x \in A'$ $\forall$ nbhd of x intersects A, thus it is in the closure!

$(\subseteq)$ let $x \in \overline{A}$. If $x \notin A$ every nbhd of x intersects A (and it does so in a point that isn't x), so it is in A'!

↳ Thm aka closure from last class

Cor) A subset of a top sp is closed $\Leftrightarrow$ it contains all of its limit points!

Pf) A closed $\Leftrightarrow A = \overline{A} \Rightarrow \overline{A} = A \cup A'$ so $A' \subseteq \overline{A} = A$

← $\{b\}$ not closed

Hausdorff Spaces

Usually singletons are closed. But not always! → eg indiscrete top, many more!

↳ if this doesn't hold, convergence may not be unique

Const b conv to a, b, c

Def) $\{x_n\} \subseteq X$ conv to $x \in X$ iff

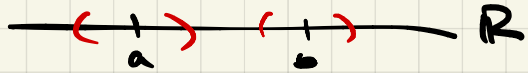
$\forall \mathcal{U}^{\text{open}} \subseteq \mathcal{T}_X$ s.t $x \in \mathcal{U}$ $\exists N \in \mathbb{N}$ so $\forall n > N$ $x_n \in \mathcal{U}$

↳ nbhd

Def) A top sp X is hausdorff if, for each $x_1, x_2 \in X$, $x_1 \neq x_2$

$\exists\, U_1^{\text{open}} \ni x_1$ & $U_2^{\text{open}} \ni x_2$ so $U_1 \cap U_2 = \emptyset$

E.g.)



Thm) If X is hausdorff every finite set is closed.

Pf) It suffices to show singletons are closed.

let $x \in X$. $\forall\, y \in X,\ y \neq x$ $\exists\, U_y \ni y$ st $x \notin U_y$

So, Claim $X \setminus \{x\} = \bigcup_{y \in X \setminus \{x\}} U_y$

$\supseteq$ trivial $\subseteq$ trivial. $\therefore$ $X \setminus \{x\}$ open $\Rightarrow \{x\}$ closed $\square$

Def) A space in which singletons are closed satisfies the T_1 Axiom

$\hookrightarrow$ equiv finite sets

we showed hausdorff $\Rightarrow$ satisfies T_1 axiom (hausdorff is stronger)

(converse doesn't hold) No, cofinite topology!

Thm) Convergence is unique in Hausdorff spaces.

let $\{x\}_n$ conv to x. let us show it can't conv to $y \neq x$.

since $y \neq x$ $\exists\, U_x^{\text{open}} \ni x$ $U_y^{\text{open}} \ni y$ so $U_x \cap U_y = \emptyset$

$\exists\, N \in \mathbb{N}$ st $\forall\, n > N$ $x_n \in U_x \Rightarrow x_n \notin U_y$ $\forall n > N$

So $\{x_n\}$ can't conv to y!

Thm) Subspace of Hausdorff space is Hausdorff (using subsp top)

Product is also hausdorff.

Every simply ordered set w order top is Hausdorff

P

Thm Let X be a top sp satisfying the T_1 axiom.

Let $A \subseteq X$.

$x \in X$ is a lim pt of A $\iff$ every nbhd of x contains infinitely many points of A.

Pf $\Leftarrow$ immediate since it will have 1 'non'-x point!

$\Rightarrow$ let $x \in X$ be a limit pt of A.

Let U be a nbhd of x. Suppose $U \cap A$ is finite.

Then $U \cap (A \setminus \{x\})$ is finite.

By T_1 axiom $U \cap (A \setminus \{x\})$ is closed.

Let $V = X \setminus U \cap (A \setminus \{x\})$ is an open nbhd of x

But $U \cap V$ is an open neighborhood that doesn't intersect $A \setminus \{x\}$ oops! x not limit point!