Recall: $p: E \to B$ is a covering map. If (surj, cts)
$\forall b \in B$ $\exists$ nbhd U_b s.t. p evenly covers U_b

$\Rightarrow$ $p^{-1}(U_b) = \bigcup V_\alpha$ open disj s.t. $p|_{V_\alpha}$ homeo to U

Thm If $p: E \to B$, $p': E' \to B'$ covering maps. Then

$(p \times p'): E \times E' \to B \times B'$ covering map

Pf $\forall$ $b \in B$, $b' \in B'$, $\exists$ nbhds U, U' s.t.

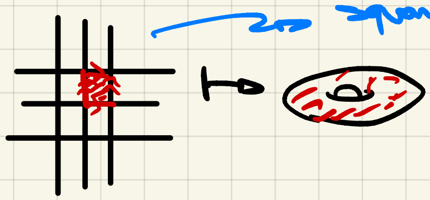
$p^{-1}(U) = \bigcup V_\alpha$, $(p')^{-1}(U') = \bigcup V_\beta'$ (disj open)

Claim: $U \times U'$ is the nbhd of $b \times b'$ that is evenly covered!

$(p \times p')^{-1}(U \times U') = \bigcup (V_\alpha \times V_\beta')$ disj and open!

Also, can show $(p \times p')|_{V_\alpha \times V_\beta'}$ is homeo onto $U \times U'$

ex. last time covering map $p: \mathbb{R} \to S^1$ $(\cos(2\pi x), \sin(2\pi x))$
$\Rightarrow$ by above $p \times p: \mathbb{R}^2 \to S^1 \times S^1$ is a covering map (for torus)



square 1×1 maps around torus once as $[n, n+1]$ map around S^1 once

§ Fundamental Group of the Circle

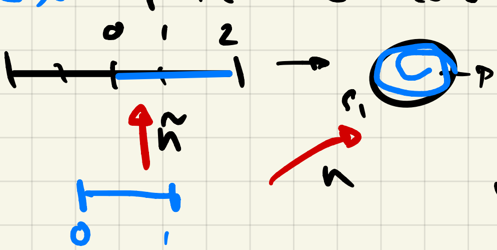
Def Let $p: E \to B$ be a map. If $f: X \to B$ cts then a lift of f is a map $\tilde{f}: X \to E$ so $p \circ \tilde{f} = f$

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & B \end{tikzcd}$$

Key idea ★ When p is a covering map $p: E \to B$, a path in B can be lifted to a path in E

★ path homotopies in B can be lifted to path homotopies in E

e.x. $p: \mathbb{R} \to S^1$ covering map $x \mapsto (\cos(2\pi x), \sin(2\pi x))$



loop in S_1 can be path starting in p going around twice
it is $h: I \to S_1$ $s \mapsto (\cos(4\pi s), \sin(4\pi s))$

h can be lifted to $\tilde{h}: [0,1] \to \mathbb{R}$ $s \mapsto 2s$

Lemma Let $p: E \to B$ covering map. With $p(e_0) = b_0$.
Then any path in B beginning at b_0 lifts to path in E at e_0!

PS Cover even B by open sets $\mathcal{U}$, each evenly covered by p.

Subdivide $[0,1]$ into subintervals by $s_1, \ldots, s_n$
$[s_0, s_1], \ldots [s_{n-1}, s_n]$
s.t $f([s_i, s_{i+1}])$ contained in some $\mathcal{U}$
} by Leb number Lemma

Define $\tilde{f}(0) = e_0$. Suppose $\tilde{f}$ is def for all $0 \le s \le s_i$
We will def it on $[s_i, s_{i+1}]$ (inductive argument)
We have $f([s_i, s_{i+1}]) \subseteq \mathcal{U}$ — evenly covered by p $\Rightarrow p^{-1}(\mathcal{U}) = \bigcup_\alpha V_\alpha$
& $p|_{V_\alpha} : V_\alpha \to \mathcal{U}$ homeomorphism!
$\tilde{f}(s_i)$ lies in exactly one V_α → call it V_0 now, pull it back properly.

$$\tilde{f}(s) = (p|_{V_0})^{-1}(f(s))!$$

By pasting theorem $\tilde{f}$ is cts from $[0,1] \to E$. It is path starting at e_0!

Let us show uniqueness of this lift! $\tilde{g}: [0,1] \to E$ is another lift.
wts $\tilde{g} \equiv \tilde{f}$.

Note $\tilde{g}(0) = e_0$ $\tilde{g}$ lifts $f \Rightarrow \tilde{g}^{-1}([s_i, s_{i+1}]) \subseteq p^{-1}(\mathcal{U}) = \bigcup V_\alpha$
Suppose $\tilde{g} = \tilde{f}$ $\forall s \le s_i$. $\therefore \tilde{g}(s_i) = \tilde{f}(s_i) = V_0$. Then by def
$\tilde{g}(s)$