Review Local Compactness.

X is **locally compact** if for all $x \in X$ $\exists\, C^{\text{compact}} \subseteq X$ that contains a nbhd of x

Thm X locally cmpt, Hausdorff $\exists!$ space Y s.t.

1) $X \subseteq Y$ ↪ 1 pt compactification
2) $Y \setminus X$ is a singleton
3) Y cmpct Hausdorff.

Uniqueness of Y is up to homeomorphism

Pf (Uniqueness) Spose Y, Y' satisfy ①–③ with $\{p\} = Y \setminus X$, $\{q\} = Y' \setminus X$

Define $h : Y \longrightarrow Y'$ with $h|_X = \mathrm{Id}_X$ & $h(p) = q$

immediate that h bij.

It suffices (by symmetry) to show that $U^{\text{open}} \subseteq Y \Rightarrow h(U)$ open

Case 1 $p \notin U \Rightarrow h(U) = U$ & $U \subseteq X \subseteq Y \underset{\text{open in } Y}{\Longrightarrow} U$ open in X

$\Rightarrow U \subseteq X \subseteq Y'$ is open in X, so open in Y' (in Hausdorff singletons closed)

Case 2 $p \in U$ let $C = Y \setminus U$ is closed (subset of Y which is cmpct) so it is compact & $p \notin C$ so $C \subseteq X$

C cmpt in $Y \Rightarrow C$ cmpct in X, $X \subseteq Y' \Rightarrow C$ is compact in Y'

$\Rightarrow C$ closed in Y' (as Hausdorff) $\Rightarrow Y' \setminus C = h(U)$ is open $\square$

Pf (existence) ($\Rightarrow$) Spose X is locally cmpct Hausdorff

① form $Y = X = \{\infty\}$

② $\mathcal{T}_Y = \mathcal{T}_X \cup \{Y \setminus C \mid C \text{ compact in } X\}$

finite union of compct is cmpct ✓

Check ① $\emptyset, Y \in \mathcal{T}_Y$ immediate ② Intersection of 2 sets:

(1) $U_1, U_2 \in \mathcal{T}_X \Rightarrow U_1 \cap U_2 \in \mathcal{T}_X$

(2) $U_1, U_2 \in$ $\Rightarrow U_1 \cap U_2 \in$ ✓

(3) $U_1 \in \mathcal{T}_X$, $U_2 \in$

$U_1 \cap (Y \setminus C_1) = U_1 \cap (\underbrace{X \setminus C_1}_{\substack{\text{open} \\ \text{as } C_1 \text{ closed}}})$ as $p \notin U_1$

$\in \mathcal{T}_Y$ ✓

③ Union

· Can decompose as $\bigcup U_\alpha$ (open in X) $\cup \bigcup (Y \setminus C_\beta)$ (cmpct in X)

$= \underbrace{\bigcup U_\alpha}_{\substack{\text{immediately open} \\ \text{in } Y. \\ \text{call it } U}} \cup (Y \setminus \bigcap C_\beta)$

↪ C_β compct in $X \Rightarrow \bigcap C_\beta$ closed in $X \underset{X \text{ open}}{\Longrightarrow} \bigcap C_\beta$ closed in Y

$\underset{Y \text{ cmpct}}{\Longrightarrow} \bigcap C_\beta$ cmpct in $Y \underset{\bigcap C_\beta \subseteq X}{\Longrightarrow} \bigcap C_\beta$ cmpct in X $= C$

$= U \cup (Y \setminus C) = Y \setminus (C \setminus U)$

↪ $C \setminus U$ closed $\Rightarrow C \setminus U$ cmpct in Y

$\Rightarrow C \setminus U$ cmpct in X $\square$

(3) $X \subseteq Y$ subspace check $\mathcal{T}_X = \{ V \cap X \mid V \in \mathcal{T}_Y \}$

($\subseteq$) immediate from construction that $\mathcal{T}_X \subseteq \mathcal{T}_Y$

($\supseteq$) let $V \in \mathcal{T}_Y$

Case (1) If $V \in \mathcal{T}_X \Rightarrow V \subset X \Rightarrow V \cap X = V \in \mathcal{T}_X$

Y & X differ by 1 pt

Case (2) $V = Y \setminus C$ for C^{compt} in X we note $V \cap X = (Y \setminus C) \cap X$
$= (X \setminus C) \cap X$
$= X \setminus C$

and C cmpct $\Rightarrow C$ closed $\Rightarrow X \setminus C$ open

(4) Y is compact. Let $\mathcal{A}$ be an open cover of Y.

$\exists\, \mathcal{U} \in \mathcal{A}$ so, $\infty \in \mathcal{U}$, furthermore $\mathcal{U} = Y \setminus C^{\text{compt sso } X}$ $(\infty \notin X)$

C cmpt in $X \Rightarrow \mathcal{A}$ has finite subcover $\mathcal{A}'$ that covers C.
$\hookrightarrow$ covers $Y \supseteq X \supseteq C$

$\therefore \{\mathcal{U}\} \cup \mathcal{A}'$ is a finite subcover of Y !

(5) Y is hausdorff. Sorta immediate. As X is hausdorff and $\mathcal{T}_X \subseteq \mathcal{T}_Y$ we just need to show $x \in X$, ∞ are separable.

by local cmpct

$\exists$ compact $C \subseteq X$ so for some $\mathcal{U}^{\text{nbhd of } x} \subset C$. Note $Y \setminus C$ open in Y and $\infty \in Y \setminus C$. $\therefore \mathcal{U} \cap (Y \setminus C) = \emptyset$, both are open &
$x \in \mathcal{U}$, $\infty \in Y \setminus C$

$\Leftarrow$ (if $\exists$ 1 point compactification, X is locally compact & Hausdorff)

- X hausdorff as subsp of hausdorff
- $x \in X$. Choose, $\mathcal{U}^{\text{open}} \ni x$, $V^{\text{open}} \ni y$ with $\mathcal{U} \cap V = \emptyset$ by Y hausdorff
  $C = Y \setminus V$ is closed sso $Y \Rightarrow C$ cmpct in $Y \underset{C \subseteq X}{\Rightarrow} C$ cmpct in X.

  & $x \in \mathcal{U} \subseteq C$ $\square$