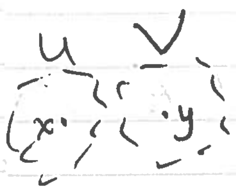
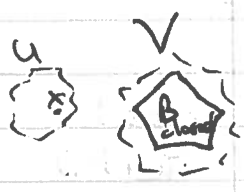
March 9

§31 Separation Axioms (e.g. Hausdorff)

~~Def.~~ ~~Supp.~~ (Not "separation" from connectedness def; union of two disjoint open sets need not be entire space)

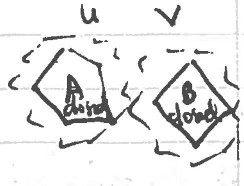


Recall why Hausdorff useful:
- limits are unique
- locally compact Hausdorff $\Leftrightarrow$ has a one-point compactification



Def. Supp single-pt sets are closed in X. (T_1)

- X is **regular** if
  [for each pair consisting of a point $\in X$ and a closed set B disjoint from x,] there are disjoint open sets $U \ni x$ and $V \supseteq B$.



- X is **normal** if
  [for each pair of disjoint closed sets A, B] there exist disjoint opens $U \supset A$ and $V \supset B$.

Rem. Clear that
Regular $\Rightarrow$ Hausdorff
Normal $\Rightarrow$ Regular.

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<u>Lemma</u>. Let X be a ~~T~~ space satisfy T_1 (one-pt sets are closed).

(a) X is regular iff given $x \in X$ and a nbhd U of x, there is a nbhd V of x s.t. $\overline{V} \subset U$.

(b) X is normal iff given a closed set A and an open set U containing A, there is an open set V containing A s.t. $\overline{V} \subset U$.

Pf. (a)($\Rightarrow$) Supp X regular. Let $x \in X$, U nbd of x. $X \setminus U$ closed. By regularity of X, $\exists$ open disjoint sets $V \ni x$ and $W \supset (X \setminus U)$.
Then $\overline{V} \subseteq U$ since $V \subset U$ and if $y \in X \setminus U$, then W is a nbd of y disjoint from V, (y is not a limit pt of V).

($\Leftarrow$) Let $x \in X$ and B closed set not containing x. Then $X \setminus B$ is a nbhd of x. By hypothesis, there is a nbd V of x s.t. $\overline{V} \subseteq X \setminus B$. The open sets V and $X \setminus \overline{V}$ are disjoint open sets containing x and B, resp.

(b) Similar; replace the point x by the set A. $\square$

<u>Thm</u>. · A subspace of a Hausdorff space is Hausdorff.
· A product of Hausdorff spaces is Hausdorff.

(skip)

Pf. · $\underset{\text{open}}{x \in U}$, $\underset{\text{open}}{y \in V}$ $\Rightarrow$ $\underset{\text{open in } Y}{x \in U \cap Y}$, $\underset{\text{open in } Y}{y \in V \cap Y}$
U, V disjoint + disjoint

· $x = (x_\alpha) \neq y = (y_\alpha)$ $\exists \beta$ s.t. $x_\beta \neq y_\beta$.
$U_\beta \ni x_\beta$ $V_\beta \ni y_\beta$. disjoint open nbds
$\pi_\beta : \prod X_\alpha \to X_\beta$ $\pi_\beta^{-1}(U) \cap \pi_\beta^{-1}(V) = \emptyset$.

↙ no analogous thm for normal spaces ~~proof~~ (Y closed $\subseteq X$ normal $\Rightarrow Y$ normal)

<u>Thm.</u> (1) A subspace of a regular space is regular.
(2) A product of regular spaces is regular.

<u>Pf.</u> (1) Let $Y \subseteq X$ regular (reg.)
Let $x \in Y$ and B closed in Y disjoint from x.
$B = \overline{B} \cap Y$, where $\overline{B}$ is closure in X.
$\Rightarrow x \notin \overline{B}$. By reg of X, $\exists$ disjoint open U and V of X containing x and $\overline{B}$.
Then $U \cap Y \ni x$ and $V \cap Y \supset B$ open disjoint in Y.

(2) X_α regular. Show $X = \prod X_\alpha$ regular.
Let $x = (x_\alpha)$ point of X, U nbd of x.
We use the preceding lemma.
Choose a basis elt $x \in \prod U_\alpha \subseteq U$.
Choose, for each α, a nbd V_α of x_α s.t. $\overline{V_\alpha} \subseteq U_\alpha$. (if $U_\alpha = X_\alpha$, let $V_\alpha = X_\alpha$.)
Then $V = \prod V_\alpha$ is a nbd of x.
Since $\overline{V} = \prod \overline{V_\alpha}$, $\overline{V} \subseteq \prod U_\alpha \subseteq U$.
$\therefore$ X is regular. $\square$

<u>ex.</u> $\mathbb{R}_K$ basis $\left\{ \begin{array}{l} (a,b) \\ (a,b) - K \end{array} \right.$, $K = \{\frac{1}{n} \mid n \in \mathbb{Z}_+\}$

- $\mathbb{R}_K$ Hausdorff ($\mathbb{R}_K$ is finer than $\mathbb{R}_{std}$)

- $\mathbb{R}_K$ not regular

An interesting pair of a point and a closed set ⟶ K is closed in $\mathbb{R}_K$ and <u>doesn't</u> contain 0.

Supp. disjoint open sets $U \ni 0$ and $V \supset K$ exist.
A basis elt containing 0 and lying in U must be of the form $(a,b) - K$
$\exists$ some n s.t. $\frac{1}{n} \in (a,b)$.
Since $\frac{1}{n} \in V$, $\exists$ a basis elt (c,d) s.t. $\frac{1}{n} \in (c,d) \subset V$.
Now find $z \in U \cap V$: choose z: $\max(c, \frac{1}{n+1}) < z < \frac{1}{n}$.

ex. $\mathbb{R}_\ell$ is normal, but the product $\mathbb{R}_\ell \times \mathbb{R}_\ell$ is not normal.

- one-pt sets closed in $\mathbb{R}_\ell$ (topology on $\mathbb{R}_\ell$ finer than $\mathbb{R}_{std}$).
- Let A, B disjoint closed sets in $\mathbb{R}_\ell$.

$B = \bar{B}$, $A = \bar{A}$ $\Rightarrow$ $\forall a \in A$, choose a basis elt $[a, x_a)$ disjoint from B.

$\forall b \in B$, choose a basis elt $[b, x_b)$ disjoint from A.

Let $U = \bigcup_{a \in A} [a, x_a)$, $V = \bigcup_{b \in B} [b, x_b)$

are open, contain A, resp. B, and disjoint. (Why?) (Clear from construction) $\square$

a x_a b x_b

cannot intersect.