Ch 9 - Fundamental Grp

Q: Given X, Y top sp, are they homeo?

ex 1 Compactness distinguishes $(0,1)$, $[0,1]$.
Connectedness distinguishes $\mathbb{R}$, $\mathbb{R}^2$
What abt $\mathbb{R}^2$ & $\mathbb{R}^3$?
S^2 & torus? } need new techniques!
↳ simply connectedness

Def X is simply connected if every closed curve on X can be shrunk to a const loop
↳ helps distinguish S^2 & Torus

Application of fundamental grp generalizes simply connectedness!

↳ show when 2 spaces are not homeomorphic (invariant)
↳ maps of spheres & their fixed pts!
↳ fundamental theorem of algebra! (every poly in $\mathbb{C}[x]$ has root)

Homotopy of Paths

Def If f, f' are cts from $f, f' : X \to Y$ (top sp) we say they are homotopic if there is a homotopy between them. That is:
$\exists\ F : X \times I \to Y$ $(I = [0,1])$ so,

$$F(x,0) = f(x) \qquad \forall x \in X$$
$$F(x,1) = f'(x)$$

Notation for f homotopic to f': $f \simeq f'$

If f homotopic to const map, f is null homotopic

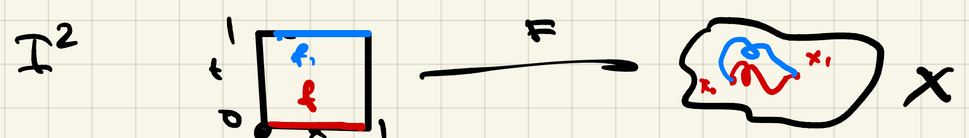
Special case $f : [0,1] \to X$ is a path (so $f(0) = x_0$, $f(1) = x_1$)

Def f, f' are paths $f, f' : [0,1] \to X$ so $f(0) = f'(0) = x_0$, $f(1) = f'(1) = x_1$. They are path homotopic, $f \simeq_p f'$, if (have same init pt & endpt) $\exists$ a cts $F : I^2 \to X$ so that
→ path homotopy

$$\left.\begin{array}{l} F(x,0) = f(x) \\ F(x,1) = f'(x) \end{array}\right\} F \text{ is homotopy between } f, f'$$

stronger ver'n $\left\{\begin{array}{l} F(0,t) = x_0 \\ F(1,t) = x_1 \end{array}\right. \quad \forall t \in I$ } each, for fixed t, $F(x,t)$ is a path! ↳ from x_0 to x_1

∴ clearly $f \simeq_p f' \Rightarrow f \simeq f'$



ex. If $f, g : X \to \mathbb{R}^2$ are homotopic, then can get straight line homotopy

$$F(x,t) = (1-t)f(x) + g(x) \quad \Big] \text{ convex combination}$$

If $f, g : I \to \mathbb{R}^2$ are paths $f(0) = g(0) = x_0$, $f(1) = g(1) = x_1$. F as def above is a straight line path homotopy

Lemma) The relations of homotopic and path homotopic are equiv relns.

Pf) ① $f \simeq f$ take $F(x,t) = f(x)$

② $f \simeq g \Rightarrow g \simeq f$. $f \simeq g \Rightarrow \exists F$ homotopic let $G(x,t) = F(x, 1-t)$

③ $f \simeq f', f' \simeq f'' \Rightarrow f \simeq f''$ take homotopy F bet f, f'
F' bet f', f''

let $G : X \times I \to Y$ so $G(x,t) = \begin{cases} F(x, 2t) & t \in [0, 1/2] \\ F'(x, 2t-1) & t \in [1/2, 1] \end{cases}$

Can check G is well def. Also, G cts (by pasting lemma)

In all cases F, F' path homotopies, so is G. $\therefore$ path homo is equiv reln.

Notation If f is a path in X, denote equiv class under $\underset{p}{\simeq}$ as $[f]$

Def) An operation $*$ on paths:

If f is a path in X from x_0, x_1 & g from x_1 to x_2 then

$h = f * g = \begin{cases} f(2s) & s \in [0, 1/2] \\ g(2s-1) & s \in [1/2, 1] \end{cases}$ path from x_0 to x_2.
(by pasting lemma)

natural glue