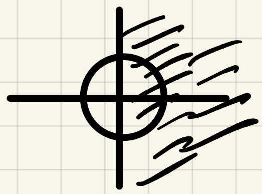
More on homotopy equiv

ex $S^1 \cup (\mathbb{R}_+ \cup \mathbb{R})$



is homotopic equiv to S^1. Retract right half to S^1

Fundamental grp of $S^n \subseteq \mathbb{R}^{n+1}$

Thm | let $X = U \cup V$ where $U, V^{\text{open}} \subseteq X$, $U \cap V$ path connected. let $x_0 \in U \cap V$
let $i_U : U \to X$, $i_V : V \to X$ be incl maps. Then the images of the homomorphism
$i_{U_*}, i_{V_*} : \pi_1(-, x_0) \to \pi_1(X, x_0)$ generate $\pi_1(X, x_0)$

Pf | WTS if f loop at x_0. $f \simeq_p g_1 * g_2 * \dots g_n$ where each
g_n is a loop in either $\pi_1(U, x_0)$ or $\pi_1(V, x_0)$ (in image of homo)

Step 1 | by Leb number lemma $\exists\, a_0 < a_1 < \dots < a_n$ sub div of $[0,1]$
s.t. $f(a_i) \in U \cap V$ & $f^{-1}([a_{i-1}, a_i]) \subseteq U$ or V

req arg
by Leb number lemma $\exists\, b_0 < \dots < b_n$ so $f^{-1}([b_i, b_{i+1}]) \subseteq U$ or V
$\hookrightarrow$ combine sufficiently many intervals to get the above

define $f_i : I \xrightarrow{\text{bij}} X$ by $I \xrightarrow{\text{bij}} [a_{i-1}, a_i] \xrightarrow{f} X$

by choice in step 1, f_i path in U or V. $[f] = [f_1] * \dots * [f_n]$

Need to replace f_i with g_i loop at x_0. Since endpt in $U \cap V$ which
is path conn. let $g_i = \alpha_{i-1} * f_i * \bar{\alpha}_i$ α_i path in intersection x_0 to $f(a_i)$

reverse

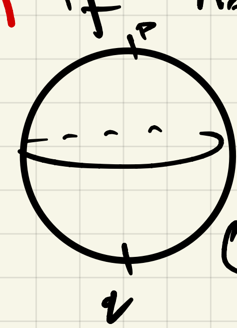
a_0, a_n (must) x_0

g_i in either U or V. $[f] = [g_1] * \dots * [g_n]$ $\square$

Cor | If U, V as above & also U, V simply conn $\Rightarrow X$ simply conn

Thm | if $n \geq 2 \Rightarrow S^n$ simply connected!

Pf |



let $p = (0, \dots, 0, 1)$, $q = (0, \dots, 0, -1)$
let $U = S^n \setminus \{p\}$ $V = S^n \setminus \{q\}$

① If $n \geq 1$, $S^n \setminus \{p\}$ homeo to $\mathbb{R}^n$ $f : S^n \setminus \{p\} \to \mathbb{R}^n$ (stereo proj)
$\mathbb{R}^{n+1} \ni \vec{x} \longmapsto \dfrac{(x_1, \dots, x_n)}{1 - x_{n+1}}$

f is a homeo! (has inv)

$g(x_1, \dots, x_n) = (t(x) x_1, \dots, t(x) x_n, 1 - t(x))$, $t(x) = \dfrac{2}{1 + \|x\|^2}$

$\therefore$ $S^n \setminus \{p\}$ homeo to $\mathbb{R}^n$. Similarly $S^n \setminus \{p\}$ homeo to $S^n \setminus \{q\}$ using
$x \longmapsto (x_1, \dots, -x_{n+1})$. $\Rightarrow$ $S^n \setminus \{q\}$ homeo to $\mathbb{R}^n$ $\Rightarrow$ both simply conn!
Trivially, they are open $S^n \setminus \{p\} = S^n \cap (-\infty, p)$
Show $S^n \setminus \{p\} \cap S^n \setminus \{q\} = S^n \setminus \{p, q\}$ is path conn. Well each
$S^n \setminus \{p\}$, $S^n \setminus \{q\}$ path conn (homeo to $\mathbb{R}^n$) $\Rightarrow$ $S^n \setminus \{p, q\}$ homeo
to $\mathbb{R}^n \setminus \{0\}$. Path con for $n \geq 2$
$\Rightarrow$ by lemma, $\pi_1(S^n) = \{e\}$