§24 Connected Subspaces of Real line Feb 4, 2022

Last time: Connected spaces
Today: $\mathbb{R}$, intervals, rays are connected

Def. An ordered set A has the least upper bound property if every nonempty subset $A_0 \subseteq A$ that is bounded above has a least upper bound (called the supremum) in A.

$\exists b \in A$ s.t. $x \le b \; \forall x \in A_0$

ex. $\mathbb{R}$ has LUB property. So does $(0,1)$.
Q: What is a set that does not have LUB property?
- $A = (0,1) \cup (1,2)$, $A_0 = (0,1) \subseteq A$
- $\mathbb{Q}$ (take a set of rationals approximating irrational #) $A = \{x \in \mathbb{Q} \mid x < \sqrt{2}\}$

Def. A simply ordered set L having more than one elt is called a linear continuum if:
(1) L has the least upper bd property
(2) If $x < y$, $\exists$ z s.t. $x < z < y$.

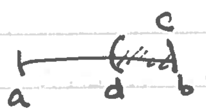
non-ex. not (2): $\mathbb{Z}$

Thm. If L is a linear continuum, w. the order topology, then L is connected, and so are the intervals and rays in L.
Cor. $\mathbb{R}$ is connected and so are intervals and rays in $\mathbb{R}$.
Pf of Thm.
We will prove if Y convex subspace of L, then Y is connected. ($\forall a,b \in Y$, $[a,b]_L \subseteq Y$.)
Supp. $Y = A \cup B$, A, B disjoint, nonempty open sets in Y.
Choose $a \in A$, $b \in B$, WLOG $a < b$.
Y convex $\Rightarrow$ $[a,b] \subseteq Y = A \cup B$
$\Rightarrow [a,b] = A_0 \cup B_0$ union of disjnt sets
$A_0 = [a,b] \cap A$, $B_0 = [a,b] \cap B$, $a \in A_0$, $b \in B_0$
A_0, B_0 open in $[a,b]$ (in subspace (of Y) = order topol).
$\rightarrow A_0, B_0$ separation of $[a,b]$.
Let $c = \sup A_0$ (exists since L is linear contin. and A_0 has an upper bd $b \in [a,b]$.)

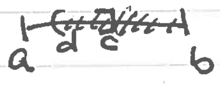
We show $c \notin A_0$ and $c \notin B_0$, contradicts $c \in [a,b]$.

<u>Case 1.</u> Supp. $c \in B_0$. Then $c \neq a$, so either $c = b$ or $a < c < b$.

$c \in B_0$ open in $[a,b]$ $\Rightarrow$ $(d, c] \subseteq B_0$



- If $c = b$, then d is smaller upper bd on A_0 than c, contradiction. ($c = \sup A_0$).



- If $a < c < b$, then $(c, b] \cap A_0 = \emptyset$ since c is upper bd on A_0.

  $\Rightarrow (d, b] = (d, c] \cup (c, b] \subseteq B_0$.

  Again, d is smaller upper bd on A_0 than c.

<u>Case 2.</u> Supp $c \in A_0$. Then $c = a$ or $a < c < b$.

$c \in A_0$ open in $[a,b]$ $\Rightarrow$ $[c, e) \subseteq A_0$.

By (2) of linear continuum, $\exists$ pt. $z \in L$ s.t. $c < z < e$. Then $z \in A_0$, contradiction ($c = \sup A_0$) $\square$

<u>Application:</u>

<u>Thm</u> (Intermediate Value Thm): Let $f: X \to Y$ be a continuous map, X connected, Y ordered w. order topol. If $a, b \in X$ and $f(a) < r < f(b)$ for some $r \in Y$, then $\exists c \in X$ s.t. $f(c) = r$.

<u>ex.</u> $f: [a,b] \to \mathbb{R}$ continuous satisfies IVT.

<u>Pf of Thm.</u> If no $c \in X$ s.t. $f(c) = r$:

$f(X) = A \cup B$, $A = f(X) \cap \underbrace{(-\infty, r)}_{\text{open in } Y}$, $B = (r, \infty) \cap f(X)$.

A, B disjoint, nonempty ($f(a) \in A$).

Each is open in $f(X)$, since $(-\infty, r)$ open in Y.

$\Rightarrow f(X)$ has separation A, B.

But image of a connected space under contin map is connected $\square$.

ex. linear continuum $I \times I$ dictionary order
$I = [0,1]$. (can check or see Munkres)

Another useful criterion that implies connectedness:
Path connectedness. ($\Rightarrow$ connected).

Def. Given $x, y \in X$, a path from x to y is a continuous map $f : \underset{\subseteq \mathbb{R}}{[a,b]} \to X$ s.t. $f(a) = x$, $f(b) = y$.

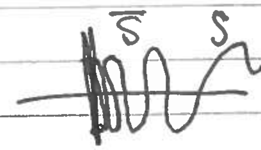
A space X is path connected if every pair of pts of X can be joined by a path in X.

also $\to$ ex. unit ball $B^n \subset \mathbb{R}^n$ (straight line paths)

Path-connected $\Rightarrow$ connected: Let $f : [a,b] \to X$ a path.
$[a,b]$ conn'td $\Rightarrow$ $f([a,b]) \subseteq X$ connected.
If X not conn, find $X = A \cup B$ separation of X
$\Rightarrow f([a,b])$ lies entirely either in A or B
(otherwise $A \cap \operatorname{im} f$ and $B \cap \operatorname{im} f$ separate $\operatorname{im} f$).
$\exists$ no path in X joining any $a \in A$ to $b \in B$.
contradiction

connected $\not\Rightarrow$ path-conn.

ex. Topologist's sine curve $\bar{S} \subseteq \mathbb{R}^2$



$S = \{ x \times \sin(\tfrac{1}{x}) \mid 0 < x \leq 1 \}$
S is image of $(0,1]$ under continuous map $\Rightarrow$ S conn.
Closure $\bar{S}$ is also conn. $\bar{S} = S \cup \{0\} \times [-1,1]$.

$\to$ ex. $\mathbb{R}^n \setminus \{\vec{0}\}$, $n > 1$ is path-connected.
(If $x, y \in \mathbb{R}^n$, take straight line path if it does not go through the origin. Else, let z be a pt not on that line from x to y, take $x \to z \to y$.)