Ch 4

Thm Urysohn Metrization Thm. Every regular sp X w a countable basis is metrizable!

Thm Tietze Ext Thm. Let X be normal, $A^{\text{closed}} \subseteq X$.

a) cts $f: A \to [a,b] \subseteq \mathbb{R}$ can be extended to $\tilde{f}: X \to [a,b]$

b) Any cts $g: A \to \mathbb{R}$ can be ext $\tilde{g}: X \to \mathbb{R}$

Thm If X is a compact m-mfd, then X can be embedded in $\mathbb{R}^N$, $N \in \mathbb{Z}$

$\exists f: X \hookrightarrow \mathbb{R}^N$ inj, cts & homeo onto $f(X)$

Def m-dim'l mfd X: Hausdorff, second ctble (countable basis)

$\forall x \in X$ $\exists x \in U_x^{\text{open}} \subseteq X$ so U_x homeo to $V_x \subseteq \mathbb{R}^m$

E.g. S^2 is a 2-mfd in $\mathbb{R}^3$

§32 Normal Spaces ~ Separate & disj closed sets!

Thm Every metrizable space is normal.

Pf Sk. Spse (X, d) metric space. Let A, Z closed disj sets.

$\forall a \in A$ choose ε_a so $B(a, \varepsilon_a) \cap Z = \emptyset$ (as $a \in Z^c$ which is open)

$\forall z \in Z$ choose ε_z so $B(z, \varepsilon_z) \cap A = \emptyset$

Let $U = \bigcup_{a \in A} B(a, \frac{\varepsilon_a}{2})$, $V = \bigcup_{z \in Z} B(z, \frac{\varepsilon_z}{2})$.

$U^{\text{open } X} \supseteq A$ and $V^{\text{open } X} \supseteq Z$

Note $U \cap Z = \emptyset$ & $V \cap A = \emptyset$.

By triangle, can get $U \cap V = \emptyset$ ✓ $\square$

Thm Every regular space with ctbl basis (second countable) is normal!

↪ HW: Lindelöf is sufficient!

Pf X regular sp with ctbl basis $\mathcal{B} = \{B_n\}$

Let A, Z closed disjoint.

$\forall a \in A$, $\exists\, U_a \overset{open}{\subseteq} X$ so $U_a \cap Z = \emptyset$ (as $a \in Z^c$)

By regularity and thm from last class,

$\exists$ nbhd $W_a \ni a$ so $\overline{W_a} \subseteq U_a$

Choose $B_a \in \mathcal{B}$ so $a \in B_a \subseteq W_a$ and look at $\{B_a \mid a \in A\}$ is countable (counting repeats) open cover of A so every set disj from Z. Relabel as $\{B_n\}_{n \in \mathbb{N}}$

Also, $\overline{B_n} \subseteq \overline{W_n}$ disjoint from Z.

We get a similar open cover for Z, V_n so $\overline{V_n} \cap A = \emptyset$.

Consider $V = \bigcup_{n \in \mathbb{N}} V_n$, $U = \bigcup_{n \in \mathbb{N}} B_n$

Both open, contain Z, A respectively. Must show V, U disj

Can modify $\{V_n\}, \{B_n\}$ to ensure disj.

Let $B_n' = B_n \setminus \bigcup_{i \leq n} \overline{V_i}$ and $V_n' = V_n \setminus \bigcup_{i \leq n} \overline{B_i}$

B_n', V_n' are open as $\bigcup_{i \leq n} \overline{V_i}$ closed! Still cover A, Z resp.

Now, $U' = \bigcup_{n \in \mathbb{N}} B_n'$, $V' = \bigcup_{n \in \mathbb{N}} V_n'$ works!

Open and contain, A, Z. Let $x \in U' \cap V'$. Spse, $x \in B_j' \cap V_k'$

Wlog $j \leq k$, note $B_j' \subseteq \overline{B_j}$ but $\overline{B_j} \cap V_k' = \emptyset$. Oops! $\blacksquare$

For HW: Show closed subsp of Lindelöf is Lindelöf

Thm) Compact Hausdorff spaces are normal!

Thm) Every well ordered set with order top is normal

$\hookrightarrow$ every subset has least elt (actually well-ord not needed)