Path Connected

Thm | Path conn $\Rightarrow$ conn

Pf | X, let $f:[a,b] \to X$ path. $[a,b]$ conn $\Rightarrow f([a,b])$ conn

Proceed by contr. Suppose $X = A \cup B$ separation of X.

$\Rightarrow f([a,b])$ in exactly one of A, B else $\operatorname{im} f \cap A$, $\operatorname{im} f \cap B$ gives a sep of $f([a,b])$

But for $x \in A$, $y \in B$ (both non-empty) $\exists f:[a,b] \to X$ cts so $f(a) = x$, $f(b) = y$ $\Rightarrow$ $\operatorname{im} f \cap A$, $\operatorname{im} f \cap B \neq \emptyset$ $\perp$

Note | conn $\not\Rightarrow$ path conn

e.g. $I \times I$ dict order!

e.g. Top sine curve $\bar{S} \subseteq \mathbb{R}^2$ $\quad S = \{(x, \sin(\frac{1}{x})) \mid x \in (0,1]\}$

$S = \operatorname{im} f$ for $f:(0,1] \to \mathbb{R}^2$, $x \mapsto \binom{x}{\sin(\frac{1}{x})}$ $\to$ CD

$(0,1]$ conn $\Rightarrow f((0,1])$ conn $\Rightarrow S$ conn $\Rightarrow \bar{S}$ conn

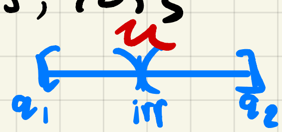
§ Connected Components & Local Conn

Def | let X top sp, define an equiv reln by: $x \sim y$ if $\exists$ conn subsp of X containing x, y $\to$ check

The equiv classes $X/\sim$ are the conn comp of X!

ex | $\mathbb{R} \supseteq Y = (0,1) \cup \{2\}$ $\quad Y/\sim = \{(0,1), \{2\}\}$

$\mathbb{Q} \subset \mathbb{R}$ $\quad \mathbb{Q}/\sim = \{\{q\} \mid q \in \mathbb{Q}\}$



Thm | The components of X are ① conn, ② disj, ③ whose union is X s.t. any connected subset of X lies in exactly 1 comp! ④

Pf | ② ③ As partition by equiv reln

④ let $A^{\text{nonemp conn}} \subseteq X$ so, $A \cap C_1, A \cap C_2 \neq \emptyset$

let $a \in A \cap C_1$, $b \in A \cap C_2 \Rightarrow a \sim b \Rightarrow C_1 = C_2$!

① Pick $x_0 \in C_1 \in X/\sim$ $\forall x \in C_1 \Rightarrow x_0 \sim x \Rightarrow \exists C_x^{\text{conn}}$

So, $x, x_0 \in C_x$. By above $C_x \subseteq C_1$. Claim:

$C_1 = \bigcup_{x \in C_1} C_x$ $\to$ conn with overlap x_0 $\Rightarrow C_1$ conn!

Path Conn Components (Path components)

transitive uses pasting lemma

Def) If X top sp defn equiv reln: $x \sim y$ if $\exists$ path from $x \to y$

$X/\sim$ are the path components of X

Thm) The path comp of X are p.conn (1), disj (2), whose union is X (3) s.t. any connected subset of X lies in exactly 1 comp! (4)

Pf) Same-ish as earlier.

Rmk) (1) Each component of X is closed in X (by thm)
(why? Closure of conn set to closed implies $\forall C \in X/\sim$, $C = \overline{C}$)

(2) If X has finitely many components, then each comp is also open.
c.o. complement of each comp is finite union of closed $\Rightarrow$ closed
$\to$ not always true
e.g. $\mathbb{Q} \subseteq \mathbb{R}$

(3) Path comp need not be open or closed
Eg. Top Sine Curve

Local Conn

Def) (1) X is locally conn at x if $\forall$ nbhd $\mathcal{U}$ of x, there is a conn nbhd $V \ni x$ so, $V \subseteq \mathcal{U}$

(2) locally path conn at x similarly!

(3) If X locally con at x $\forall x \in X \to X$ is locally conn

(4) X locally path conn similarly!

Local path conn $\Rightarrow$ locally conn

e.g. (1) Intervals and Rays in $\mathbb{R}$ $\to$ path conn, locally path conn (all 4)
(2) $[-1,0) \cup (0,1] \subseteq \mathbb{R}$ $\to$ locally path conn
(3) $\mathbb{Q} \subset \mathbb{R}$ $\to$ none?
(4) Top sine curve $\to$ conn, but not locally!