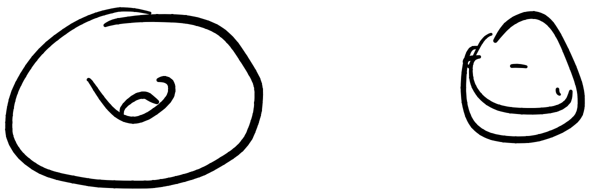
Feb 2 Ch3 . §23 Math 590

Connected Spaces

an intuitive concept

①



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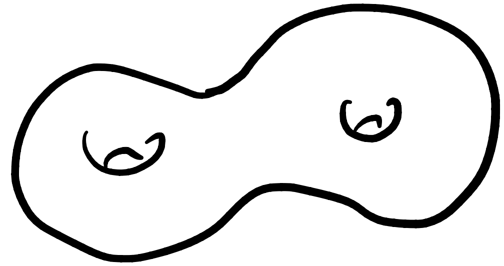
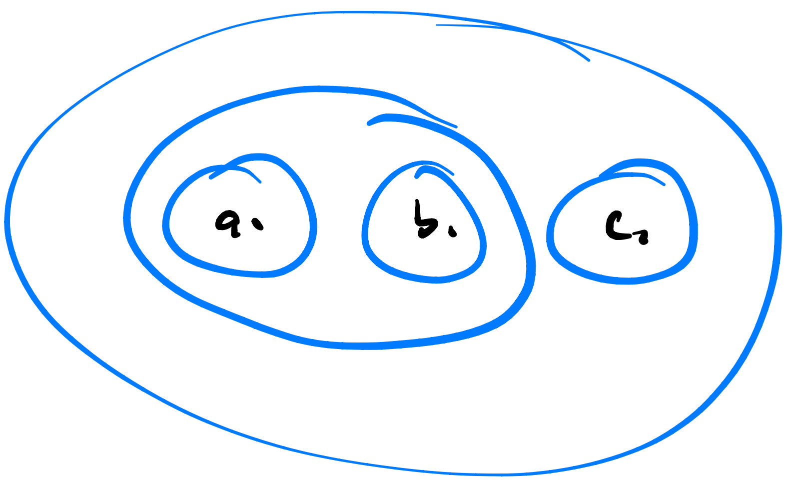


Is this topological space connected?

③



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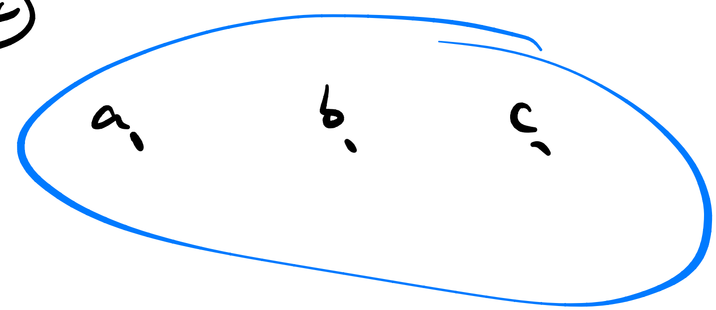


Connectedness

Def. A separation of X (space) is a pair U, V of disjoint nonempty open subsets of X whose union is X.

A space X is connected if there does not exist a separation of X.

★ Equivalently: A space X is connected iff the only subsets of X which are both open and closed are $\emptyset$ and X.

Pf. $\Rightarrow$ $\emptyset \subsetneq A \subsetneq X$

A open and closed

A closed $\Rightarrow X \setminus A$ open

$\{A, X \setminus A\}$ form a separation of X.

$\Leftarrow$ Supp. X not connected.

U, V separation of X

$U = X \setminus V$ closed + open

$\square$

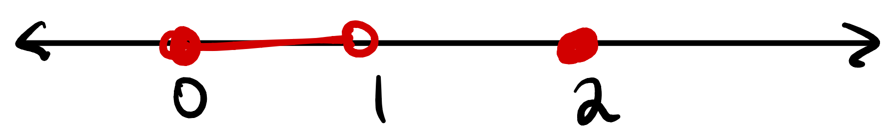
Connectedness of a subspace via limit points

Lemma. $Y \subseteq X$ (subspace).

A separation of Y is a pair of disjoint, nonempty sets A and B whose union is Y, neither of which contains a limit point of the other.

ex. $X = \mathbb{R}$

$Y = \underbrace{[0,1)}_{A} \cup \underbrace{\{2\}}_{B}$



ex. $X = \mathbb{R}$

$Y = (0,1)$

$A = (0, \tfrac{1}{2}]$ $\quad$ $B = (\tfrac{1}{2}, 1)$

Connectedness of a subspace via limit points

Lemma. $Y \subseteq X$ (subspace).
A separation of Y is a pair of disjoint, nonempty sets A and B whose union is Y, neither of which contains a limit point of the other.

Pf. (old defn $\to$ new defn) Supp. A, B form separation of Y. Then A is both open + closed in Y. The closure of A in Y is $\bar{A} \cap Y$. ($\bar{A}$ in X)
Since A is closed in Y, $A = \bar{A} \cap Y$. $\Rightarrow B \cap \bar{A} = \emptyset$.

$\bar{A} = A \cup \{\text{limit pts in } X \text{ of } A\}$, B contains no limit pts of A.

Similarly, A contains no limit pts of B.

Connectedness of a subspace via limit points

Lemma. $Y \subseteq X$ (subspace).

A *separation* of Y is a pair of disjoint, nonempty sets A and B whose union is Y, neither of which contains a limit point of the other.

Pf. (old defn $\Leftarrow$ new defn) Supp. A, B pair of disjnt. nonempty sets A and B whose union is Y, neither of which contains a limit point of the other.

$$A \cap \overline{B} = \emptyset \quad \text{and} \quad B \cap \overline{A} = \emptyset.$$

$$\Rightarrow \quad Y \cap \overline{B} = B \quad \text{and} \quad Y \cap \overline{A} = A \quad \rightarrow \quad A, B \text{ closed in } Y$$

$$A = Y \setminus B \text{ open in } Y \quad \Rightarrow \quad A, B \text{ separation of } Y.$$

$$B = Y \setminus A \text{ open in } Y$$

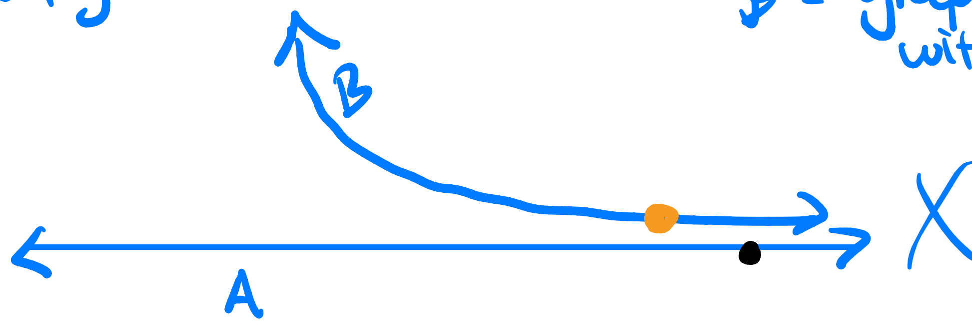
$\square$

ex. In $\mathbb{R}^2$, consider the subset $X \subseteq \mathbb{R}^2$

$$X = \{ x \times y \mid y = 0 \} \cup \{ x \times y \mid x > 0 \text{ and } y = \tfrac{1}{x} \}$$

A = graph of $y = 0$ $\qquad$ B = graph of $y = \frac{1}{x}$ with $x > 0$

Q: Is X connected?



Not connected

Use lemma A B disjoint nonempty, $A \cup B = X$

Each A, B does not contain limit pt of the other set

<u>ex.</u> $(\mathbb{R}^\omega$, box topology$)$ is <u>not connected</u>.

$\mathbb{R}^\omega = A \cup B$ separation of $\mathbb{R}^\omega$.

where $A = \{$unbounded sequences$\}$ are disjoint and <u>open</u>; ✓
$B = \{$bounded sequences$\}$.
$b = (b_i)$, $b_i \le M$ integer $\forall i$

If $a = (a_i) \in A$, then

$$a \in U = (a_1 - 1, a_1 + 1) \times (a_2 - 1, a_2 + 1) \times \cdots \subseteq A$$

If $a = (a_i) \in B$, then

$$a \in U = (a_1 - 1, a_1 + 1) \times (a_2 - 1, a_2 + 1) \times \cdots \subseteq B$$

<u>ex.</u> $(\mathbb{R}^\omega$, product topology$)$ connected (it turns out...)

<u>ex.</u> $\mathbb{R}$ connected (next time...)

Thm. A union of connected subspaces of X that have a point in common is connected.

Thm. Let A be a connected subspace of X. If $A \subseteq B \subseteq \bar{A}$ then B is connected.

Thm. A finite Cartesian product of connected spaces is connected.

ex. $\mathbb{R}^n$ connected

Thm. The image of a connected space under a continuous map is connected.

Thm. The image of a connected space under a continuous map is connected.

Pf. Let $f: X \to Y$ continuous, X connected.

Want to show $f(X)$ is connected.

The (surjective) map $g: X \to f(X)$, $g(a) = f(a)$ $\forall a \in X$, is also continuous.

By way of contradiction, suppose $f(X)$ has a separation A, B.

Then $g^{-1}(A) \cup g^{-1}(B) = X$, and $g^{-1}(A)$ and $g^{-1}(B)$ are disjoint sets. By continuity of g, $g^{-1}(A)$ and $g^{-1}(B)$ open in X.

Since g surj, $g^{-1}(A)$, $g^{-1}(B)$ are nonempty.

Thus, X has separation $g^{-1}(A)$, $g^{-1}(B)$. $\square$