Quotient Topology

Def | $\sim$ is an equiv reln pt.
- $\sim$ is reflexive $\to x \sim x$
- $\sim$ is symm $\to x \sim y \Rightarrow y \sim x$
- $\sim$ is trans $\to x \sim y,\ y \sim z \Rightarrow x \sim z$

lemma | Two equiv classes are either disjoint or equal

$\to$ If $\sim$ is an equiv reln on A, and $\mathcal{E} = A/\sim$ is a partition of A

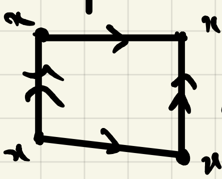
$$A = \bigsqcup_{E \in \mathcal{E}} E$$

First Example: 1) $[0,1]/_{0\sim 1}$ $\to$ 0 is equiv to 1 & nothing else

$\cong \bigcirc^{0,1} \to S^1$

closed unit disk

2)  / pts on bdry $\sim q$ $\cong$  S^2 q

3)  / pts on opp bdy same $\cong$ torus $\to S^1 \times S^1$

4) $[0,1]/_{x \sim y\ \forall x,y} \cong \{1\}$

Def | let X, Y be top spaces $p: X \to Y$ be surjective. The map p is a quotient map if

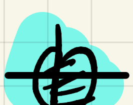
$$p^{-1}(U) \text{ is open in } X \iff U \text{ is open in } Y \Big\}\ \text{Strong cont}$$

Def | Given surj $p: X^{\text{top sp}} \to A$ $\exists!$ top on A so p is a quotient map. This, $\mathcal{J}$ is the quotient top on A.

$$\mathcal{J} = \{ U \subseteq A \mid p^{-1}(U) \text{ is open} \}$$

Def | X be a top space, let X^* be a partition of X, let $p: X \to X^*$ be surj, $x \in X$ maps to $E \in X^*$ so that $x \in E$ (well def)

With quotient top, X^* is a quotient sp of X!

e.g. , $X^* = \{ \{x\} \mid x \in \text{int} X \} \cup \{ \text{Bd} X \} \to$ can show $X^* \cong S^2$

e.g. take comp of $q \circ p^{\text{quotient map}}$ is a quotient map! $(q \circ p)^{-1}(U) = p^{-1}(q^{-1}(U))$

Thm let $p: X \to Y$ quotient map. let Z be a sp & $g: X \to Z$ be constant on each preimage of $p^{-1}(\{y\})$ $\forall y \in Y$

$$\begin{tikzcd} X \arrow[d, "p"'] \arrow[dr, "g"] & \\ Y \arrow[r, dotted, "f"'] & Z \end{tikzcd}$$

g induces a map $f: Y \to Z$ allows this to be well def

$f \circ p = g$

f is cts $\Longleftrightarrow$ g is cts !

f is a quotient map $\Longleftrightarrow$ g is a quotient map

Pf $f: Y \to Z$ for for $y \in Y$ take $x \in p^{-1}(\{y\})$ and take $g(x)$

Concretely, $g(p^{-1}(\{y\})) = z_y \to f(y) = z_y$

0) Check $(f \circ p)(x) = g(x) \longrightarrow$ trivial

1) Spose f is cts $\Rightarrow$ as $g = f \circ p \Rightarrow$ g comp of 2 cts func $\square$

spose g is cts $\Rightarrow$ let $V \subseteq Z$ open, wts $f^{-1}(V)$ open

$\Rightarrow g^{-1}(V)$ open in X $= (f \circ p)^{-1}(V) = p^{-1}(f^{-1}(V))$

$\Rightarrow p$ quotient map so $p^{-1}(f^{-1}(V))$ open

$f^{-1}(V)$ open $\square$

→ only need to show $f^{-1}(U) \Rightarrow U$ open

2) If f is quotient $\Rightarrow$ g quotient as comp of 2 quotient maps $\square$

g is quotient $\Rightarrow$ g surj $\to$ f is surj as $g = f \circ p$

have ↪ let $V \subseteq Z$ V open $\overset{g\text{ quo}}{\Longleftrightarrow} g^{-1}(V)$ open in X

$\Longleftrightarrow p^{-1}(f^{-1}(V))$ open in X

$\overset{p\text{ quo}}{\Longleftrightarrow} f^{-1}(V)$ open in Y

Cor let $p: X \to X^*$ quotient so $X^* = \{g^{-1}(\{z\}) \mid z \in Z\}$

$g: X \to Z$ surj cts !

$$\begin{tikzcd} X \arrow[d, "p"'] \arrow[dr, "g"] & \\ X^* & Z \end{tikzcd}$$

G induces bijective cts $f: X^* \to Z$ st

1) f is homeo iff g is quotient

2) If Z hausdorff so is X^*

Pf We satisfy stuff from thm so g cts $\to \exists f: X^* \to Z$ cts

f is clearly bijective $\to$ g surj $\Rightarrow$ f inj as $g = f \circ p$

(i) If f homeo $\Rightarrow$ f is quotient $\to$ g is quotient