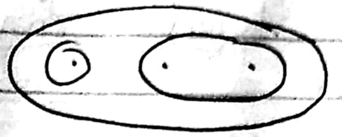
Review Questions (Midterm Feb 25) Feb 17

Let $X = \{a,b,c\}$. For each of the following topols, ~~for~~ write down a path from a to c, if one exists, or ~~write~~ show "no path exists".

(1) $\mathcal{T} = \{\emptyset, \{a\}, \{b,c\}, \{a,b,c\}\}$



$\{a\}$ and $\{b,c\}$ seperation of X
not connected $\to$ not path-conn.

- but b and c are path-connected via:

$f : [0,1] \to X$ (continuous) $(f^{-1}(\{b,c\}) = [0,1])$

$$f(t) = \begin{cases} b & t \le \frac{1}{2} \\ c & t > \frac{1}{2} \end{cases}$$

- thus, a is not in the same path component as c. (otherwise X would be path-conn.)

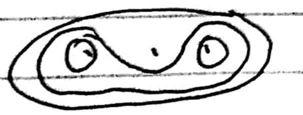
(2) $\mathcal{T} = \{\emptyset, \{a\}, \{a,b\}, \{a,b,c\}\}$

$f : [0,1] \to X$ continuous

$$f(t) = \begin{cases} a & t < 1 \\ c & t = 1 \end{cases}$$

$f^{-1}(\{a\}) = f^{-1}(\{a,b\}) = [0,1)$ open in $[0,1]$
$f^{-1}(\{a,b,c\}) = [0,1]$ open in $[0,1]$

(3) $\mathcal{T} = \{\emptyset, \{a\}, \{c\}, \{a,c\}, \{a,b,c\}\}$



$f : [0,1] \to X$ continuous

$$f(t) = \begin{cases} a & t < \frac{1}{2} \\ b & t = \frac{1}{2} \\ c & t > \frac{1}{2} \end{cases}$$

$f^{-1}(\{a\}) = [0, \frac{1}{2})$ open
$f^{-1}(\{c\}) = (\frac{1}{2}, 1]$ open
$f^{-1}(\{a,c\}) = [0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$ open

Feb 18

- Review Ques for Midterm

§29 Local Compactness

Def. A space X is locally compact at $x \in X$ if there is some compact subspace C of X that contains a nbhd of x.
If X is locally compact at every point in X, then X is locally compact.

ex. $\mathbb{R}$ is locally compact.
$x \in (a,b) \subseteq [a,b]$ compact.

non-ex. $\mathbb{Q}$ not locally compact. (Exer.)
basis for $\mathbb{Q}$: $(a,b) \cap \mathbb{Q}$
Recall compact $\Rightarrow$ limit pt compact.

ex. Any compact space is locally compact.

ex. $\mathbb{R}^n$ $\quad \vec{x} \in (a_1,b_1) \times \cdots \times (a_n,b_n) \subseteq \prod_{i=1}^{n} [a_i,b_i]$ compact.

non-ex. $\mathbb{R}^\omega$ not locally compact.
$B = (a_1,b_1) \times (a_2,b_2) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \cdots$
If $B \subseteq$ Compact C, then $\overline{B}$ compact, but $\overline{B}$ not compact
$\overline{B} = [a_1,b_1] \times [a_2,b_2] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \cdots$
(Recall closed $\overline{B} \subseteq C$ compact $\Rightarrow \overline{B}$ compact).

ex. A simply ordered set X having least upper bound property is locally compact.
Given basis elt for X, $x \subseteq$ closed interval in X, which is compact (by the same thm showing $[a,b] \subseteq \mathbb{R}$ compact).

A subspace of a metrizable space is metrizable
But a subspace of a compact Hausdorff space need not be compact. (ex. $(a,b) \subseteq [a,b]$ in $\mathbb{R}$)

* Any locally compact Hausdorff space can be embedded in a compact Hausdorff space.

Thm. X is locally compact Hausdorff iff there exists a space Y satisfying the following:

existence

(1) X is subspace of Y
(2) $Y \setminus X$ is a single point
(3) Y compact Hausdorff space.

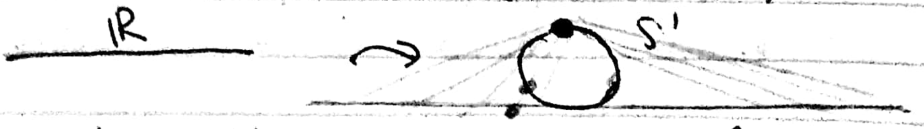
uniqueness

Moreover, if Y and Y' are two spaces satisfying these conditions, then there is a homeo $Y \to Y'$ that equals the identity map on X.

Pf (next time)

Def. We call Y the one-point compactification of X: if Y is compact Hausdorff space and $X \subsetneq Y$, $\overline{X} = Y$, $Y \setminus X$ = single pt.

ex. The one-point compactification of $\mathbb{R}$ is homeo to the circle. (check)



What about the one-pt compactification of $\mathbb{R}^2$?

$\mathbb{R}^2 \to S^2$ unit 2-sphere