Thm) The finite product of compact spaces is compact

Pf) It suffices to show X, Y compact $\Rightarrow X \times Y$ compact (& induct)

Let $\mathcal{A}$ be open cover, fix $x_0 \in X$

Since $\{x_0\} \times Y$ (homeo to Y) is compact, a finite subcollection of $\mathcal{A}$ covers $\{x_0\} \times Y$

Those set, being open, cover a "tubular" nbhd of $\{x_0\} \times Y$ (slice)

Call this $U_{x_0} \times Y \Rightarrow \forall x$ U_x covers X $\exists$ finite sub $U_{x_1} \dots U_{x_n}$ that covers X.

Then, a finite subcover of sets in $\mathcal{A}$ covers: $\bigcup_{i=1}^{n} U_{x_i} \times Y = X \times Y$

Compact Subspaces of the Real Line

We will show $[a,b] \subseteq \mathbb{R}$ is cmpct. As a cor, we get EVT

Thm) (EVT) let $f: X \to Y$ cts, order top on Y.

If X is compact, $\exists\, c, d \in X$ so $\forall x \in X$ $f(c) \leq f(x) \leq f(d)$

Pf) $A = f(X) \to$ it is compact.

Since A has no largest, this covers

Suppose A has no largest element. We have a covering $\{(-\infty, a) \mid a \in A\}$

A is compact, so $\exists$ finite subcover from rays $I_1, I_2 \dots I_n$

let $a = \max\{a_i\}_{i \in \mathbb{N}_n}$ at A but not in fin subcov. Oops

So, we have max. Min argument identical

Rec) Recall, $\mathbb{R}$ is a linear cont that satisfies LUB prop.

Recall, A closed $\subseteq X$ compact $\Rightarrow A$ cmpct & $B^{\text{cmplt}} \subseteq X^{\text{hausdorff}} \Rightarrow B$ closed

Thm) X simply ordered set with LUB. In order top on X, each closed interval in X is cmpct!

by convexity of $[a,b]$

Outline) ⓪ let $\mathcal{A}$ be open cover of $[a,b]$ (note subsp top coinc w ord top)

① $x \in [a,b) \Rightarrow \exists\, y > x$ so $[x,y]$ is cov by ≤ 2 elts of $\mathcal{A}$

② $C = \{y \in [a,b] \mid [a,y]$ has fin subcov from $\mathcal{A}\}$ is non empt

③ $c = \sup C$ (exist by hyp), show $c \in C$, $c = b$!

Pf) ① Let $x \in [a,b]$

cov ≤ 2 elts of A

- If x has an immediate successor $y \in X \Rightarrow [x,y] = \{x,y\}$
- else, choose $\mathcal{U} \in \mathcal{A}$ st $x \in \mathcal{U} \xrightarrow{\mathcal{U} \text{ open}} \mathcal{U}$ contains some $[x,c)$ where $c \in [a,b]$
- choose $y \in (x,c) \Rightarrow [x,y] \subseteq [x,c) \subseteq \mathcal{U}$ → 1 elt of A

② Consider C, by 1 we have C non empty by taking $x=a$ and getting y so $[x,y] = [a,y]$ cov by 1/2 elts of $\mathcal{A}$

③ Let $\gamma = \sup C$ (as C is nonempty & bdd by b)

Goal Show $\gamma \in C \Rightarrow [a,\gamma]$ has fin subcover from $\mathcal{A}$

Spse, $\gamma \notin C$ $\exists A_\gamma \in \mathcal{A}$ s.t. $\gamma \in A_\gamma$ (as $\mathcal{A}$ is a cov & $\gamma \in [a,b]$)

A_γ open $\Rightarrow \exists$ basis elt B that contains $(d,\gamma] \subseteq B \subseteq A_\gamma$

But, as $\gamma \in C$ $\exists$ $z \in C$ so $z \in (d,c)$ (since $\gamma \geq a$)

So, $\exists$ fin subcover $A_1, \dots A_n$ covers $[a,z]$ but $A_1, \dots, A_n, A_\gamma$ covers $[a,z] \cup (d,\gamma] = [a,\gamma] \Rightarrow \gamma \in C$!

④ Show $\gamma = b$, suppose $c < b$

Let $x = \gamma$ in 1 gives us $y > c$ so $[c,y]$ covered by 2 elts of a.

$\Rightarrow y > \gamma$ & $y \in C$ but $\gamma = \sup C$ oops ▷