Product topology on $\prod_{\alpha \in J} X_\alpha$

e.g. if $J = \mathbb{N}_n$ $= X_1 \times \cdots \times X_n$ finite cartesian prod

e.g. if $J = \mathbb{N}$ $= X_1 \times X_2 \cdots$ infinite cartesian prod

e.g. if $J = \mathbb{Z}^+$ $X_i = \mathbb{R}$ $\mathbb{R}^\omega = \prod_{n \in \mathbb{Z}^+} X_n$

Def) Box topology on $\prod_{\alpha \in J} X_\alpha$ has basis:

$$\mathbb{B} = \{ \prod_{\alpha \in J} U_\alpha \mid U_\alpha \text{ open in } X_\alpha \text{ for } \alpha \in J \}$$

Def) The product topology on $\prod_{\alpha \in J} X_\alpha$ has basis;

$$\mathbb{B} = \{ \prod_{\alpha \in J} U_\alpha \mid U_\alpha \text{ is open in } X_\alpha,\ U_\alpha = X_\alpha \text{ for abfm } \alpha \in J \}$$

Cor) It is immediate that the box & product topology coincide in the finite case

Rmk) Box topology is always finer than the product top

We will assume $\prod_{\alpha \in J} X_\alpha$ has prod topology on it, unless stated otherwise!

Thm) (Hold for both box & product)

① If X_α has a basis $\mathcal{B}_\alpha$, then

- $\{ \prod_{\alpha \in J} B_\alpha \}$ is a basis for $\prod_{\alpha \in J} X_\alpha$ with box
- $\{ \prod_{\alpha \in J} B_\alpha \}$ ($B_\alpha = X_\alpha$ for abfm α) is a basis for prod topology!

② If A_α subspace for X_α, then $\prod_{\alpha \in J} A_\alpha$ is a subspace of $\prod_{\alpha \in J} X_\alpha$ (box \ top coincides with subsp) (uniform)

③ If X_α is hausdorff box/prod is also hausdorff

Thm let $f : A \to \prod_{\alpha \in J} X_\alpha$ be given by $f(a) = (f_\alpha(a))_{\alpha \in J}$

$f_\alpha : A \to X_\alpha$ let $\prod_{\alpha \in J} X_\alpha$ be given prod top — write in terms of coor func!

Then f is cts $\iff$ each f_α is cts! — doesn't hold for box

e.g. Consider $\mathbb{R}^\omega = \prod_{i \in \mathbb{N}} \mathbb{R}$

let $f : \mathbb{R} \to \mathbb{R}^\omega$, $f(t) = (t, t, \ldots\ldots)$

each $f_i : \mathbb{R} \to \mathbb{R}$, $f_i \cong Id_{\mathbb{R}}$, so $f : \mathbb{R} \to \mathbb{R}^\omega$

with prod top is cts (by Thm)

Now consider $\mathbb{R}^\omega$ with box. Claim, f not cts!

Take $B = \prod_{j \in \mathbb{N}} \left(-\frac{1}{j}, \frac{1}{j}\right)$ is open in $\mathbb{R}^\omega$ (basis elt)

but $f^{-1}(B) = \{0\} \to$ not open in $\mathbb{R}$!

$\forall \varepsilon > 0 \quad (-\varepsilon, \varepsilon) \not\subseteq \{0\}$

So, f not cts!

Metric Spaces

Def A metric on X is a func $d : X^2 \to \mathbb{R}$ s.t

- $d(x,y) \geq 0 \ \forall x, y \in X$ with equality iff $x = y$
- $d(x,y) = d(y,x)$
- $d(x,y) \leq d(x,z) + d(z,y) \ \forall x, y, z \in X$

e.g. $\mathbb{R}^n$, euclidean metric

$$d(x,y) = \sqrt{\sum_{i \in \mathbb{N}_n} (x_i - y_i)^2} \qquad x = (x_1, \ldots, x_n)$$

e.g. Square metric on $\mathbb{R}^n$ $\to$ balls are squares!

$$\rho(x,y) = \max\{|x_i - y_i| \mid i \in \mathbb{N}_n\} \to \text{sup norm}$$

Def given X & d metric on X, $\varepsilon \in \mathbb{R}$

$B_d(x, \varepsilon) = \{y \in X \mid d(x,y) < \varepsilon\} \to$ epsilon ball around x

Def If d is a metric on X. The collection of all ε balls is a basis for top on X. This gives the metric top

$$\mathcal{B} = \{ B_d(x, \varepsilon) \mid x \in X, \varepsilon > 0 \}$$

Pf of Def Show it is a basis

① for $x \in X$, $x \in B_d(x, 1)$

② Spse $B_d(x_1, \varepsilon_1) \cap B_d(x_2, \varepsilon_2) \neq \emptyset$ let y in here

let $\delta = \min\{\varepsilon_1 - d(x_1, y), \varepsilon_2 - d(x_2, y)\}$

both are > 0.

By triangle, $B_d(y, \delta) \subseteq B_d(x_1, \varepsilon_1) \cap B_d(x_2, \varepsilon_2)$

U open in metric $\iff \forall y \in U \; \exists \delta > 0$ so $B_d(y, \delta) \subseteq U$

E.g Set X

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & x \neq y \end{cases}$$

gives discrete topology! $B(x, 0.5) = \{x\}$

E.g $\mathbb{R}$, $d(x,y) = |x-y|$, metric $\cong$ order $\cong$ std!

easy to show!