Thm | A space X is locally path connected $\iff$ $\forall$ open $\mathcal{U}$ of X, each path component of $\mathcal{U}$ is open in X.

Rmk | Analogous holds for connected / connected comp

Pf | ($\Rightarrow$) Suppose X is loc path conn. Spose $\mathcal{U}$ is open, let C be a path component of $\mathcal{U}$.

$\forall x \in C \subseteq \mathcal{U}$ $\exists V_x^{\text{open}} \subseteq \mathcal{U}$ so that V path conn. But, since C is a path comp containing x, $V_x \subseteq C$ (by equiv reln).

$\therefore C$ is open as $\forall x \in C$ $\exists V_x^{\text{open}} \subseteq C$

($\Leftarrow$) Suppose $\forall \mathcal{U}$ of X, each path comp of $\mathcal{U}$ is open in X.

Let $x \in X$, and $\mathcal{U}^{\text{open}} \ni x$. Let C be path comp of x contained in $\mathcal{U}$. $\Rightarrow$ C is open & $x \in C \subseteq \mathcal{U}$ $\square$

§ Limit Point Compactness

Defn | A space X is limit point compact if every infinite subset of X has a limit point.

Recall | for $A \subseteq X$, then $x \in X$ is a limit pt of A if every nbhd of x intersects A in some point other than itself. (ie $x \in \overline{A - \{x\}}$)

Thm | Compactness $\Rightarrow$ Limit point compactness (but not conv)

Pf | ($\not\Leftarrow$) $Y = \{a, b\}$ with indisc topology $X = \mathbb{Z}_+ \times Y$ $\to$ subset of $\mathbb{R}$

Claim, X is limit point compact but not compact.

① Claim every non-empty subset of X has a limit pt. $\to$ stronger

let $(n,a) \in A \subseteq X$

let $\mathcal{U}$ nbhd of (n,a) $\ni (n,a) \in B^{\text{basis}} \subseteq A$, $B = \{n\} \times Y$

$\Rightarrow$ (n,b) is a limit point of A! $\Rightarrow$ every nbhd of (n,b) intersects A at (n,a)

② Covering $\to$ $\{\{n\} \times Y \mid n \in \mathbb{Z}_+\}$ $\to$ not cmpct $\Rightarrow$ contra pos

($\Rightarrow$) Suppose X is compact. If $A \subseteq X$ has no lim pt wts A is finite

Since A has no lim pt $\Rightarrow$ it contains its lim pts $\Rightarrow$ A is closed

For each $a \in A$, nbhd $\mathcal{U}_a$ so $\mathcal{U}_a \cap A = \{a\}$ (if no such $\mathcal{U}_a$ $\Rightarrow$ a lim pt)

Consider $\{\mathcal{U}_a \mid a \in A\} \cup \{X \setminus A\}$ open cover of X $\Rightarrow$ A fin

By cmpct $\Rightarrow$ $\exists$ fin subcov $\mathcal{U}_{a_1}, \dots \mathcal{U}_{a_n}, X \setminus A$ $\Rightarrow$ $A \subseteq A \cap (\mathcal{U}_{a_1} \cup \dots \cup \mathcal{U}_{a_n})$

Def) X is sequentially compact if every seq has a conv subsequence.

Recall) $\{y_n\} \in X$ conv to y if $\forall\ \mathcal{U}^{open} \ni y$ $\exists N \in \mathbb{N}$ so $\forall n > N$ $y_n \in \mathcal{U}$

Thm) Let X be metrizable. Then TFAE:

1) Compactness
2) Limit pt cmpct
3) sequentially cmpct

Pf) We have $(1) \Rightarrow (2)$

$(2) \Rightarrow (3)$

Suppose X is limit point compact. Let us show sequentially cmpct.
Let $(x_n) \in X$. WTS $\exists$ conv subseq.
Let $A = \{x_n \mid n \in \mathbb{N}\}$. If A is finite, (x_n) is eventually const.
Else, A has a limit pt $x \in X$. (hypothesis)
Claim: $\exists$ subseq of (x_n) conv to x. Choose

$x_{n_1} \in B(x, 1) \cap A$, $x_{n_2} \in B(x, \frac{1}{2}) \cap A$ ...

Can guarentee $n_1 < n_2$ ... as we are eliminating fin many pts.
i.e. can show $B(x, \frac{1}{i}) \cap A$ is infinite!

Easy to see $\{x_n\} \to x$

$(3) \Rightarrow (1)$ This is in