Thm $A \subseteq \mathbb{R}^n$ is cmpct $\Leftrightarrow$ A is closed and bdd under in $(\mathbb{R}^n, d)$ or $(\mathbb{R}^n, \rho)$
(recall d is euclidean, $\rho(x,y) = \max_{i \in \mathbb{N}_n} \{|x_i - y_i|\}$

Rmk Doesn't hold in arb metric sp!

Pf ($\Rightarrow$) $A \subseteq \mathbb{R}^n$ compact $\Rightarrow$ A is closed as $\mathbb{R}^n$ is Hausdorff!

Recall $\rho(x,y) \leq d(x,y) \leq \sqrt{n}\, \rho(x,y)$ $\forall x,y \in \mathbb{R}^n$

$\therefore$ suffices to show bdd wrt 1 of them (we will work w ρ)

Consider $\mathcal{E} = \{ B_\rho(0,n) \mid n \in \mathbb{N} \}$ as an open cover of A $(\mathbb{R}^n)$

By compactness, $\exists$ $\mathcal{E}'$ finite subset of $\mathcal{E}$ that covers A

$B_\rho(0, m_1), \dots, B_\rho(0, m_j\}$ take $\max_{i \in \mathbb{N}_j} \{m_i\} = M \to A \subseteq B_\rho(0, 2M)$

($\Leftarrow$) Spse A is closed and bdd (in $(\mathbb{R}, \rho)$)

Suffices to show, A is contained in compact subset of $\mathbb{R}^n$ (as closed subsets of compact sets are cmpct)

$\Rightarrow$ bdd implies $\rho(x,y) \leq N$ $\forall$ $N \in \mathbb{N}$ $\forall x,y \in A$ $\Rightarrow$ $\forall x \in A$
special $x_0 \in A$ so $\rho(x_0, 0) = b$ $\rho(x,0) \leq N + b = M$

So, $A \subseteq B_\rho(0, M) \subseteq \overline{B_\rho(0, M)}$ So, A is a closed subset of a compact set ($\overline{B_\rho(0,M)}$ is compact as closed & bdd)

e.g. 1) $S^{n-1} \subseteq \mathbb{R}^n$ compact!
2) $\overline{B^n} \subseteq \mathbb{R}^n$ closed unit ball compact!
3) $A = \{ x \times (\frac{1}{x}) \mid 0 < x \leq 1 \} \subseteq \mathbb{R}^2$ not compact not bdd!

Uniform Continuity Thm

Def $f: X \to Y$ is uniformly cts pt $\forall \varepsilon > 0$ $\exists \delta > 0$ so that $\forall x_0, x_1 \in X$ if $d_X(x_0, x_1) < \delta \Rightarrow d_Y(f(x_0), f(x_1)) < \varepsilon$

Thm (Uniform cont Thm). Let $f: X \to Y$ cts, if X cmpct metric & Y metric
$\Rightarrow$ f is unif cts!

Def (X, d_X) metric & $A \subseteq X$ nonempty

$\forall x \in X$, $d(x, A) = \inf_{a \in A} d(x,a)$ **Claim:** $d: X \to A$, $x \mapsto d(x,A)$ is cts

Pf $\forall x,y \in X$, $a \in A$, $d(x,A) \leq d(x,a) \leq d(x,y) + d(y,a)$
$d(x,A) - d(x,y) \leq d(y,a)$ $\forall a \in A$ so
as symmetric add abs val outside $|d(x,A) - d(y,A)| \leq d(x,y)$ $\Rightarrow$ lipschitz!

Def Same set up earlier, if A bdd then
$$\text{diam } A = \sup\{ d_X(a_1, a_2) \mid a_1, a_2 \in A\}$$

Lemma (Lebesgue Number Lemma) Let $\mathcal{A}$ be an open covering of a metric space if X is compact, there is $\delta > 0$ so for every $\forall A \subseteq X$, $\text{diam } A < \delta$ there exists an element of the covering that contains A. δ is the leb number for covering $\mathcal{A}$

Pf (Unif Ct Thm) Let $f: X \to Y$, X, Y metric, X compact, f cts.

let $\varepsilon > 0$ be given consider covering $\mathcal{E} = \{ B_{d_Y}(y, \frac{\varepsilon}{2}) \mid y \in Y\}$ of Y

let $\mathcal{A} = \{ f^{-1}(C) \mid C \in \mathcal{E}\}$ is open cov of X!

By above, can choose $\delta > 0$ for $\mathcal{A}$ of space X so if $d_X(x_1, x_2) < \delta$ then $x_1, x_2 \in A$ for some $A \in \mathcal{A}$

$\hookrightarrow$ as $\{x_1, x_2\} \subseteq X$ & $\text{diam}\{x_1, x_2\} < \delta$

$\Rightarrow \{f(x_1), f(x_2)\} \subseteq f(A) \subseteq B_{d_Y}(y, \frac{\varepsilon}{2})$ for some $y \in Y$

So, $d_Y(f(x_1), f(x_2)) < d_Y(f(x_1), y) + d_Y(f(x_2), y) < \varepsilon$ $\square$