Deformation Retracts

Lemma) Let $h, k : (X, x_0) \to (Y, y_0)$ are cts maps so $h(x_0) = k(x_0) = y_0$. If h, k homotopic & image of $x_0 \in X$ stays fixed at $y_0 \in Y$ during homotopy then $h_* = k_*$ (same induced iso)

Pf) have $H : X \times I \to Y$ homotopy $h \mapsto k$, $H(x_0, t) = y_0$ $\forall t \in I$

If $[f] \in \pi_1(X, x_0)$. Claim: $h_*([f]) \simeq_p k_*([f])$

WTS $h \circ f \simeq_p k \circ f$. Consider,

$$I \times I \xrightarrow{f \times id} X \times I \xrightarrow{H} Y$$
$$(t, s) \longmapsto (f(t), s) \longmapsto H(f(t), s)$$

Check: path homotopy betw $[h \circ f]$, $[k \circ f]$

Since $h_*([f]) = k_*([f])$ same homomorphism)

Thm) The inclusion map $j : S^n \to \mathbb{R}^{n+1} \setminus \{0\}$ induces isomorphism of fund groups!

Pf) have homomorphism need isomorphism (bijectivity needed)

$\hookrightarrow$ goal collapse $\mathbb{R}^2 \setminus \{0\}$ to S^1 $x \mapsto \frac{x}{\|x\|}$ this is a retraction.

Let $X = \mathbb{R}^{n+1} \setminus \{0\}$ pick $x_0 = e_1 \in X$ also $x_0 \in S^n$. Consider retraction $r : x \mapsto \frac{x}{\|x\|}$ (identity now

note $r \circ j : S^n \to S^n$ also $r \circ j = Id_X$ so $r_* \circ j_* = id_*$

$\Rightarrow j_*$ injective! Must show surjectivity. Consider,

$j \circ r : X \to X$. Claim $j_* \circ r_*$ also identity. Claim: $j \circ r$ homo to id.

$$H : X \times I \to X$$
$$(x, t) \longmapsto (1-t)x + t\,x \quad \rightsquigarrow \text{works!}$$

$\Rightarrow$ as it $j \circ r(x_0) = id(x_0)$ & $H(x_0, t) = x_0$ $\forall t$

$\therefore$ by lemma $(j \circ r)_* = id_*$ (same homo. but iso so)

$\Rightarrow j_*$ is surj.

$\Rightarrow j_*$ iso!

More generally!

Def) $A \subset X$. Then A is a def retract of X if $id_X : X \to X$ is homotopic to a map that retracts X onto A so it fixed points on A & pts of A fixed during homo.

That is $\exists$ cts $H : X \times I \to X$ so, $H(x,0) = 0$,

retract $H(x,1) = r(x)$, $X(a_0, t) = a_0$ $\forall a_0 \in A$, $t \in I$.

H is a deformation retraction of X onto A

from H, can get retraction $\tilde{r} : X \to A$, $x \mapsto H(x,1)$ so $\tilde{r} \equiv r$

H is a homotopy bt $j \circ r$ where $j : A \to X$ incl.

Thm) let A be a deformation ret of X. let $x_0 \in A$ the inclusion map $j : (A, x_0) \to (X, x_0)$ induces isomorphism between $\pi_1(A, x_0)$ & $\pi(X, x_0)$

Note to self: Might be able to show $\forall$ $x_0 \in X$ $\exists$ $a_0 \in A$ so $\exists$ path $x_0 \to a_0$, $p(t) = H(x_0, t)$

$\Rightarrow \pi_1(X, x_0) \cong \pi_1(X, a_0)$ — interesting

no info about $\pi_1(A, a_0)$

e.g. $X = \mathbb{R}^3 - \{z \text{ axis}\}$ $A = (\mathbb{R}^2 \setminus \{0\}) \times \{0\}$

in fact get def retract.

$H : X \times I \to X$

$(x,t) \to (x_1, x_2, (1-t)x_3)$

$\Rightarrow \pi_1(\mathbb{R}^3 \setminus \{z \text{ axis}\}, \tilde{x}) \cong \pi_1(\mathbb{R}^2 \setminus \{0\}) \cong \pi_1(S^1) \cong \mathbb{Z}$

E.g. $\mathbb{R}^2 \setminus \{p, q\}$ collapse to figure 8 $\cong \mathbb{Z}^2$