# Seifert van Kampen

**Thm)** Let $X = U \cup V$ for $U, V$ open in $X$. Assume $U, V$, $U \cap V$ all path connected. Let $x_0 \in U \cap V$

$$j : \pi_1(U, x_0) * \pi_1(V, x_0) \to \pi_1(X, x_0)$$

be the homomorphism extending $j_1, j_2$ induced by incl.

$$\boxed{\begin{tikzcd}
 & \pi_1(U, x_0) \arrow[dr, "j_1"] & \\
\pi_1(U, x_0) * \pi_1(V, x_0) \arrow[rr, "j"'] & & \pi_1(X, x_0) \\
 & \pi_1(V, x_0) \arrow[ur, "j_2"'] &
\end{tikzcd}}$$

Then $j$ is surjective & its kernel is generated by all elts of the free prod $i_1(g)^{-1} i_2(g)$ for all $g \in \pi_1(U \cap V, x_0)$, & conj

$$\boxed{\begin{tikzcd}
 & \pi_1(U, x_0) \\
\pi_1(U \cap V, x_0) \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & \\
 & \pi_1(V, x_0)
\end{tikzcd}}$$

**Free Product)** of groups $G_1, G_2$.

$G_1 * G_2 = \{$ finite words of letters chosen from $G_1, G_2 \} / \sim$

group operation $*$ is concat

$a * b \sim c * d \iff a, b, c, d \in G_1$ & $ab = cd$

or          or

$a * b \sim ab$     " , $G_2$ & " "

Thus $e_{G_1} \sim$ empty word $\sim e_{G_2}$ (identity)

**Pf of surjectivity)** from §59 ($\pi_1(S^n), n \geq 2$) the images of $j_1, j_2$ generate $\pi_1(X, x_0) \Rightarrow j$ surjective!

**Pf of ker $j$)** Requires decomposing loops & combinatorial manipulation!

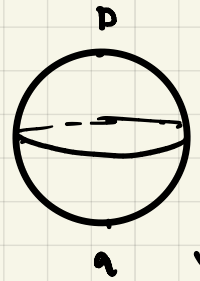
**Special Case]** If $\pi_1(U \cap V$ is simply connected$)$. Then,

$j : \pi_1(U, x_0) * \pi_1(V, x_0) \longrightarrow \pi_1(X, x_0)$ is injective as $\ker j$ is of the form $i_1(g)^{-1} i_2(g)$ for inclusion $i_1, i_2$ $g \in \pi_1(U \cap V, x_0) = \{e_1\}$.

$\implies$ $j$ is an isomorphism (in general, modding out $\sim$ kernel gives an iso)

**Eg]**

1) $S_2$



let $U = S^2 \setminus \{p\}$ (deformation ret to point) $V = S^2 \setminus \{q\}$

$U \cap V = S^2 \setminus \{p, q\}$

$\pi_1(U, x_0) = \{e\}$, $\pi_1(V, x_0) = \{e\}$

Van Kampen says $\exists$ surj

$j : \underbrace{\pi_1(U, x_0) * \pi_1(V, x_0)}_{\text{trival}} \longrightarrow \pi_1(S^2, x_0)$

trival $\implies$ $\pi_1(S^2, x_0)$ is trivial too

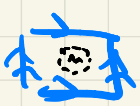
2) Fig 8 $\infty$ $X$ or $S^1 \vee S^1$ (wedge of 2 circles)

$U = \infty$ $V = \infty$ $U \cap V = \times$ $\rightarrow$ Star convex & contractible $\implies$ trivial

$\implies$ $U \cap V$ simply connected $\implies$ $j$ is a isomorphism

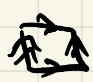
$\implies$ $\pi_1(X, x_0) \cong \pi_1(U, x_0) * \pi_1(V, x_0) \cong S^1 * S^1 \cong \mathbb{Z} * \mathbb{Z}$

deformation retract to $S^1$ | seq of loop around $a$ then $b$

3) $S^1 \times S^1$  let $U =$ (dashed disk) $V =$ Complement with some overlap

$U \cap V$ is an open annulus

$U$ deformation retracts to a point $\implies$ $\pi_1(U, x_0) \cong \{e_1\}$

$V$ deformation retract to  $\cong \infty$ figure 8 $\implies \pi_1(V, x_0) \cong \mathbb{Z} * \mathbb{Z}$