Recall → Top sp Defn

Basis for Topology

Instead of describing the entire topology $\mathcal{T}$ explicitly, use a smaller collection of subsets to gen the topology.

Defn If X is a set, a basis $\mathcal{B}$ for a top on X is a collection s.t.

1) $\forall x \in X$, $\exists B \in \mathcal{B}$ s.t $x \in B$
2) If $\exists B_1, B_2 \in \mathcal{B}$ s.t $x \in B_1 \cap B_2 \Rightarrow \exists B_3 \in \mathcal{B}$ s.t $B_3 \subseteq B_1 \cap B_2$

Given a basis $\mathcal{B}$ on X, $U \subseteq X$ is open $\Leftarrow\Rightarrow$ $\forall x \in U$ $\exists B \in \mathcal{B}$ s.t $x \in B \subseteq U$

E.g. If X is any set, consider $\mathcal{T}_{disc}$ has basis $\mathcal{B}_{disc}$

$\mathcal{B}_{disc} = \{\{a\} \mid a \in X\}$ = collection of singletons

Claim $\mathcal{T}_{disc} = \mathcal{T}_{\mathcal{B}_{disc}}$

Pf ($\supseteq$) trivial $U \in \mathcal{T}_{\mathcal{B}_{disc}} \Rightarrow U \subseteq X \Rightarrow U \in \mathcal{T}_{disc}$

($\subseteq$) let $U \in \mathcal{T}_{disc}$ let $x \in U$ we note $\{x\} \in \mathcal{B}_{disc}$ & $\{x\} \subseteq U$ $\square$

Lemma let X be a set. Let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X.

$\mathcal{T}$ = collection of all unions of basis elements of $\mathcal{B}$

Pf ($\supseteq$) Note basis elements are open.
$\Rightarrow$ any type of union is open

($\subseteq$) let $U \in \mathcal{T}$ we note $\forall x \in U$ we have $\exists B_x \in \mathcal{B}$ s.t $B_x \in \mathcal{B}$. Claim so $B_x \subseteq U$

$$U = \bigcup_{x \in U} B_x$$ $\supseteq$ trivial $\subseteq$ trivial

$\therefore U$ in collection of all unions $\square$

e.x. The standard topology on $\mathbb{R}$ is gen by

$\mathcal{B} = \{(a,b) \mid a,b \in \mathbb{R}\}$

e.x. $\mathcal{B}_\ell = \{[a,b) \mid a,b \in \mathbb{R}\}$ lower limit topology $\mathbb{R}_\ell$

e.x. $\mathcal{B}_K = \{(a,b)\} \cup \{(a,b) \setminus K\}$ $K = \{\frac{1}{n} \mid n \in \mathbb{Z}_{>0}\}$

K topology $\mathbb{R}_K$

Def If $\mathcal{T}$ & $\mathcal{T}'$ are topologies on X

① If $\mathcal{T} \subseteq \mathcal{T}'$ ($\subsetneq$) $\Rightarrow$ $\mathcal{T}'$ is finer than $\mathcal{T}$ (strictly finer)

② If $\mathcal{T}' \subseteq \mathcal{T}$ ($\subsetneq$) $\Rightarrow$ $\mathcal{T}'$ is coarser than $\mathcal{T}$ (strictly coarser)

③ If neither are true $\Rightarrow$ $\mathcal{T}, \mathcal{T}'$ are not comparable

ex. Trivial top $\mathcal{T}_{indisc}$ is the coarsest topology on X

$\mathcal{T}_{disc}$ is the finest topology on X

Lemma $\mathcal{B}, \mathcal{B}'$ are bases for $\mathcal{T}, \mathcal{T}'$ on X.

TFAE:

1) $\mathcal{T}'$ is finer than $\mathcal{T}$

2) $\forall x \in X$ each basis elt $B \in \mathcal{B}$ $\exists B' \in \mathcal{B}'$ s.t $x \in B' \subseteq B$

Pf $1 \Rightarrow 2$ trivial as B is open in $\mathcal{T} \Rightarrow$ open in $\mathcal{T}'$

①

$2 \Rightarrow 1$ let $U \in \mathcal{T}$ WTS $U \in \mathcal{T}'$

② let $x \in U$, $\Rightarrow \exists B \in \mathcal{B}$ s.t $x \in B \subseteq U$ but also,

$\hookrightarrow$ $\exists B' \in \mathcal{B}'$ s.t $x \in B' \subseteq B \subseteq U \Rightarrow U \in \mathcal{T}'$