Metric Topology

- Every metric space is Hausdorff. Let (X,d) be metric. Let $\varepsilon = \frac{d(x,y)}{57}$ for, $x,y \in X$, $x \neq y \Rightarrow \varepsilon > 0$. And $B(x,\varepsilon) \cap B(y,\varepsilon) = \emptyset$ and they are open & contain x, y respectively.

- Countable product of metrizable spaces is metrizable. (pf similar to $\mathbb{R}^\omega$)

- If A subspace of X metrizable. A is also metrizable using $d|_{A \times A}$ & induces the subsp top on A!

- Next: cts functions in metric spaces.

Thm (X, d_X) and (Y, d_Y) metric spaces.

$f: X \to Y$ is cts $\Longleftrightarrow$

1) $\forall x_0 \in X$, $\varepsilon > 0$ $\exists \delta > 0$ s.t. $\forall x \in B_{d_X}(x_0, \delta)$ then $f(x) \in B_{d_Y}(f(x_0), \varepsilon)$ $[d_X(x, x_0) < \delta \Rightarrow d_Y(f(x), f(x_0)) < \varepsilon]$

Pf ($\Rightarrow$) We have that f is cts.

Let x, $\varepsilon > 0$ be given. By continuity $f^{-1}(B_{d_Y}(f(x), \varepsilon))$ is open. Furthermore $x \in f^{-1}(B_{d_Y}(f(x), \varepsilon))$. As it is open $\exists \delta > 0$ so that $B_{d_X}(x, \delta) \subseteq f^{-1}(B_{d_Y}(f(x), \varepsilon))$

$\Rightarrow f(B_{\delta_X}(x,\delta)) \subseteq B_{\delta_Y}(f(x),\varepsilon))$

↳ this is the result

($\Leftarrow$)

Spose that $f: X \to Y$ satisfies ε-δ argument. Let us show that it is cts. It suffices to show that inverse elements of balls are open.

For some $y \in Y$ take $B_{\delta_Y}(y,\varepsilon)$. If the inverse image of this is empty, we are done!

Else, spose $f^{-1}(B_{\delta_Y}(y,\varepsilon))$ is non empty.

So, grab $x \in f^{-1}(B_{\delta_Y}(y,\varepsilon)) \Rightarrow f(x) \in B_{\delta_Y}(y,\varepsilon)$ is open. So, by openness, $\exists\, \varepsilon_x > 0$ s.t. $B_{\delta_Y}(f(x),\varepsilon_x) \subseteq B_{\delta_Y}(y,\varepsilon)$. By "$\varepsilon,\delta$", $\exists\, \delta_x > 0$ so that

$f(B_{\delta_X}(x,\delta_x)) \subseteq f(B_{\delta_Y}(f(x),\varepsilon_x))$

Claim: $\longrightarrow$ don't need union, have none!

$$f^{-1}(B_{\delta_Y}(y,\varepsilon)) = \bigcup_{x \in f^{-1}(B_{\delta_Y}(y,\varepsilon))} B_{\delta_X}(x,\delta_x)$$

easy to prove:

$\Rightarrow f^{-1}(B_{\delta_Y}(y,\varepsilon))$ is open $\square$

Recall

def $x_n \to x$ if $\forall$ nbhd $U \ni x$ we have that $\exists N \in \mathbb{N}$ so $\forall$ $n > N$ we get $x_n \in U$

Sequence lemma

let $A \subset X$

If there exist a sequence of points in A conv to x $\Rightarrow$ $x \in \overline{A}$

Converse is true if X is metrizable

Pf) ($\Rightarrow$) $a_n \to x$

$\Rightarrow$ $\forall$ nbdn of $x \in U$ we have $\exists N$ so $a_N \in U$ $\Rightarrow$ $A \cap U \supseteq \{a_n\}$

$\Rightarrow x \in \overline{A}$

($\Leftarrow$) Spose X metrizable (X, d)

let $x \in \overline{A}$

let $n \in \mathbb{N}$ be given,

we know $B_d(x, \frac{1}{n}) \cap A \neq \emptyset$

Pick a_n so we get that

$a_n \to x$ by archemedian etc.

Thm) If $f: X \to Y$ cts function

if $\underbrace{x_n \to x}_{\text{in } X}$, $\underbrace{f(x_n) \to f(x)}_{Y}$

converse is true if X is metrizable

Pf) ($\Rightarrow$) Spose f is cts.

Choose $x_n \to x$

let $\mathcal{U}$ be a nbhd of $f(x)$.

we note $f^{-1}(\mathcal{U})$ is a nbhd of

x. $\therefore \exists N \in \mathbb{N}$ so that $\forall n > N$

$x_n \in f^{-1}(\mathcal{U})$

$\Rightarrow f(x_n) \in f(f^{-1}(\mathcal{U})) \subseteq \mathcal{U}$

So, $\forall n > N$

$f(x_n) \in \mathcal{U} \Rightarrow f(x_n) \to f(x)$

($\Leftarrow$) Spose X is a metric space. This gives us uniqueness of convergence!

Now spose $\forall x_n \to x$ [uniquely]

we have $f(x_n) \to f(x)$

For cts, equiv to show

$\forall A \subseteq X \quad f(\overline{A}) \subseteq \overline{f(A)}$

by prev lemma, if $x \in \overline{A}$

$\Rightarrow \exists x_n \to x$ where $\{x_n\} \in A$

By assumption, $f(x_n) \to f(x)$
$\qquad\qquad\qquad\quad \cap$
$\qquad\qquad\qquad\quad f(A)$

$\implies$

$f(x) \in \overline{f(A)}$

$\implies f(\overline{A}) \subseteq \overline{f(A)}$ and we have continuity!

Def Let $f_n : X \to Y$ be a seq of functions where Y is a metric space w/ metric d.
We say $(f_n) \xrightarrow{\text{unif}} f$ if $\forall \varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}_n$ we have that
$d(f_n(x), f(x)) < \varepsilon$ $\underline{\forall x \in X}$

Thm (Unif limit thm) Let $f_n : X \to Y$ be cts. Spose that $f_n \xrightarrow{\text{unif}} f$. Then f is cts

metric space w/ d

Pf Let V be open in Y.
WTS $f^{-1}(V)$ open in X!
If we can show $\forall x \in X$ $\exists$ open $U_x \ni x$ s.t. $U_x \subseteq f^{-1}(V) \iff f(U) \subseteq V$
Spose $f^{-1}(V)$ non-empty & take $x_0 \in f^{-1}(V)$

As V is open $\exists\ \varepsilon>0$ so

$B(f(x_0),\varepsilon) \subseteq V$. By unif conv.

$\exists\ N \in \mathbb{N}$ so $\forall n \in \mathbb{N}_N$ we have that

$\forall x \in X$

$d(f_n(x), f(x)) < \frac{\varepsilon}{3}$

no not exactly "ε-δ" set preimage of the open

By continuity of f_n & "ε-δ" con dit

$\exists$ Nbhd $U \ni x_0$ so that $\overbrace{f_N(U) \subseteq B(f_N(x_0), \tfrac{\varepsilon}{3})}$ $\to$ ①

<u>Claim</u>

$f(U) \subseteq B(f(x_0), \varepsilon)$

If $x \in U$, we have

$$d(f(x), f_N(x)) < \frac{\varepsilon}{3}$$

$$d(f_N(x), f(x_0)) < \frac{\varepsilon}{3}$$

$$d(f_N(x_0), f(x_0)) < \frac{\varepsilon}{3}$$

by Δ

$$d(f(x), f(x_0)) < \varepsilon \qquad \text{D}$$