Last time

Given $p: E \to B$ covering $p(e_0) = b_0$. We defined lifting corresponding

loop @ b_0 $\varphi: \pi_1(B, b_0) \longrightarrow p^{-1}(b_0)$ end pt path lift (will start at e_0)

$[f] \longmapsto \tilde{f}(1)$

Also, last time, if E is simply connected $\Rightarrow$ φ is bijective!

Thm The fundamental grp of $S^1 \cong (\mathbb{Z}, +)$

Pf Let $p: \mathbb{R} \to S^1$ be the std covering map $t \mapsto (\cos(2\pi t), \sin(2\pi t))$

coord not interval

Let $e_0 = 0 \in \mathbb{R}$ then $p(e_0) = b_0 = (1,0) \in S^1$

We see $p^{-1}(b_0) = \mathbb{Z}$! Since $\mathbb{R}$ is simply connected, $\varphi: \pi_1(S^1, b_0) \to \mathbb{Z}$ bij

Claim: this map φ is also a isomorphism (must check homomorphism)

Let $[f], [g] \in \pi_1(S^1, b_0)$ show $\varphi([f * g]) = \varphi([f]) + \varphi([g])$

lift starting at 0!

Let $\tau_n: \mathbb{R} \to \mathbb{R}$, $x \mapsto x+n$. Then $\tau_n \circ \tilde{f}$ is a lift of f starting at n

$\rightsquigarrow \tau_n \circ \tilde{f}(0) = n$

The lift of $f * g$ to a path in $\mathbb{R}$ starting at 0 $\tilde{f}(1) = \varphi([f])$

$\widetilde{f * g} = \tilde{f} * (\tau_n \circ \tilde{g})$ when $n = \tilde{f}(1)$ to allow $*$

(can check this commutes)

with end pt $\tilde{f} * (\tau_n \circ \tilde{g})(1) = \tilde{f}(1) + \tilde{g}(1)$

$\Rightarrow \varphi([f * g]) = \varphi([f]) + \varphi([g])$ $\square$

Retractions & Fixed Points

Def If $A \subseteq X$, a retraction of X onto A is $f: X \to A$ is cts & $f|_A = \mathrm{Id}_A$. A is called a retract of X!

e.x. $x_0 \in X$, $f: X \to \{x_0\}$ is a retraction!

① $r: S_1 \times S_1 \to S_1 \times \{b_0\}$, $(x,y) \mapsto x \times b_0$, ② $r: \mathbb{R} \to \mathbb{R} \times \{0\}$, $x \times y \mapsto x \times \{0\}$

Lemma If A is a retract of X then the inclusion map $j: A \to X$, $a \mapsto a$ induces an injective homomorphism

$$j_*: \pi_1(A, a) \longmapsto \pi_1(X, a)$$
$$[f] \longmapsto [j \circ f]$$

Pf) Already have homomorphism. must show injective. Have $r: X \to A \Rightarrow r \circ j: A \to A$ is exactly $id_A : A \to A$

$\Rightarrow r_* \circ j_* : \pi_1(A, a) \to \pi_1(A, a) = id_{A_*} \Rightarrow j_*$ injective!

Thm) No retraction of B^2 onto S^1

Pf) If S^1 were a retract of B^2. Then the inclusion map $j: S_1 \to B_2$ would induce injection $j_* : \pi_1(S_1, b_0) \to \pi_1(B^2, b_0)$

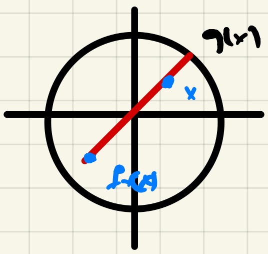
but $\pi_1(B^2, b_0)$ is trivial as B^2 simply connected.

Can't have injective homo from non-trivial grp to trivial $\square$

Brower Fixed pt thm for B^2

TR $f: B^2 \to B^2$ cts $\exists$ fixed point! $\exists\, x \in B^2$ so $f(x) = x$

Pf) Suppose $f: B^2 \to B^2$ cts so f has no fixed point! $f(x) \neq x \;\forall\, x \in B^2$



$\forall\, x \in B^2$ def $g(x)$ = endpt to ray from $f(x)$ to x to S^1.

$\Rightarrow g$ is a retraction of X onto A $\rightarrow\leftarrow$