Compactness

Def) A collection $\mathcal{A}$ is a covering of X if $\bigcup_{A \in \mathcal{A}} A = X$. It is an open covering if all elts are open

Def) A space X is compact if any open covering admits a finite sub-cover

e.g. 1 $\mathbb{R}$ not compact $\to$ $\mathcal{B} = \{(n, n+1) \mid n \in \mathbb{Z}\}$

2 Any finite space is compact $\Rightarrow$ $\mathcal{T}$ of finite X is finite

3 $(0,1] \subseteq \mathbb{R}$ not $\to$ $\{(\frac{1}{n}, 1] \mid n \in \mathbb{N}\}$

4 $\{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\} \subseteq \mathbb{R}$ is compact (closed & bounded)

Let $\mathcal{A}$ be an open cover of X (above)

$\exists A \in \mathcal{A}$ so that $0 \in A$. $A = X \cap \mathcal{U}$ for open $\mathcal{U}^{\text{open}} \in \mathbb{R}$ as $\mathcal{U}$ is open $\exists N \in \mathbb{N}$ so $\forall n > N$ $\frac{1}{n} \in \mathcal{U}$ so $\frac{1}{n} \in A$.

$\therefore$ we have a finite subcovering

Def) If $Y \subseteq X$ is a subspace. A collection $\mathcal{A}$ of sso X covers Y if $\bigcup_{A \in \mathcal{A}} A \supseteq Y$. Open cover if each $A \in \mathcal{A}$ is open.

Lemma) $Y^{\text{subsp}} \subseteq X$ is compact (wrt subsp top)

$\iff$ $\forall$ open covering of Y in X admits a finite subcov

Pf) ($\Rightarrow$) Spose Y comp. Let $\mathcal{A}$ be open cov of Y in X.

Let spose $\mathcal{A} = \{A_\alpha\}_{\alpha \in I}$ let $\mathcal{A}' = \{A_\alpha \cap Y\}_{\alpha \in I}$

This is open cover of Y in Y. So $\exists$ fin subset $I' \subseteq I$ so $\{A_\alpha \cap Y\}_{\alpha \in I'}$ covers Y $\Rightarrow$ $\{A_\alpha\}_{\alpha \in I'}$ covers Y

($\Leftarrow$) Similar let $\mathcal{B}$ open cover of Y in Y. Let $\mathcal{B} = \{C_i\}_{i \in I}$

$\forall i \in I$ $\exists C_i'$ open in X so that $C_i = X \cap C_i'$

$\therefore$ $\{C_i'\}_{i \in I}$ open cover of Y in X. By hypothesis $\exists I' \overset{\text{fin}}{\subseteq} I$

So, $\{C_i'\}_{i \in I'}$ covers Y $\to$ $\{C_i\}_{i \in I'}$ covers Y

Thm | Every closed subspace of compact space is compact!

Pf) 295 final. Let $Y^{\text{closed}} \subseteq X$. Show Y is compact.

Let $\mathcal{A} \subseteq \mathcal{T}_X$ be an open cover of Y

Note $\mathcal{A} \cup \{X \setminus Y\}$ is an open cover of X.

$\exists$ finite subcover of this that covers $X \Rightarrow$ remove $\{X \setminus Y\}$ & done!

Theorem | Image of a compact space under a cts map is compact.

Pf) Let $f : X^{\text{cmpct}} \to Y$ be cts. & wlog surjective (only care about im)

Let $\mathcal{E}$ be an open cover of Y. $\mathcal{E} = \{C_i\}_{i \in I}$

note $\mathcal{E}' = \{f^{-1}(C_i)\}_{i \in I}$ is an open cover of X as

$\bigcup_{i \in I} C_i = Y \Rightarrow \bigcup_{i \in I} f^{-1}(C_i) = f^{-1}(Y) = X$!

So $\exists$ finite $I' \subseteq I$ so $\{f^{-1}(C_i)\}_{i \in I'}$ covers X.

Claim: $\{C_i\}_{i \in I'}$ covers Y. Yes as surjective!

Thm | Every compact subspace of a Hausdorff space is closed!

Pf) Spse $Y^{\text{compact}} \subseteq X^{\text{hausdorff}}$ wts $X \setminus Y$ is open.

Note: If $X \setminus Y = \emptyset, X$ done. So, spse not. Let $x_0 \in X \setminus Y$.

Will show $\exists\, \mathcal{U} \ni x_0$ so $\mathcal{U} \cap Y = \emptyset$.

$\forall y \in Y \ \exists\, \mathcal{U}_y, V_y$ so $y \in \mathcal{U}_y$, $x_0 \in V_y$ so $\mathcal{U}_y \cap V_y = \emptyset$

$\{U_y\}_{y \in Y}$ is an open cover of Y $\exists$ finite $I \subseteq Y$ so

$\{U_i\}_{i \in I}$ covers Y. Claim, $\bigcap_{i \in I} V_i$ is the nbhd we seek.

Since each V_i is open & contains x_0 as I is finite it is a nbhd of x_0.

Need $\bigcap_{i \in I} V_i \cap Y = \emptyset$. Note $\bigcap_{i \in I} V_i \cap U_i = \emptyset \ \forall i \in I$

As $Y \subseteq \bigcup_{i \in I} U_i$ is immediate!

Cor (eg) | $\mathbb{R}$ Hausdorff $\Rightarrow$ (a,b), $(a,b]$ not compact (contrapos of above)

Thm) Let $f: X \to Y$ be a continuous bijection. If X is compact & Y hausdorff f is a homeo.

Pf) Need to show f is closed.

Suppose $V \subseteq X$ closed $\Rightarrow$ V is cmpct as X is.

So $f(V)$ cmpt $\Rightarrow$ $f(V)$ closed as Y compact $\square$