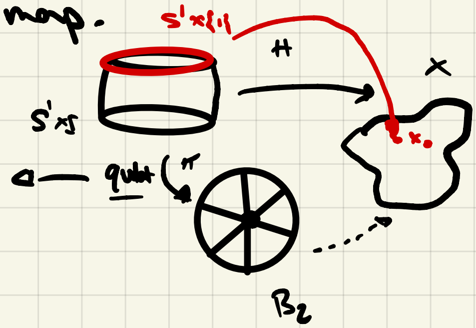
Lemma $h : S^1 \to X$ cts. TFAE
1) h is null homotopic
2) h extends to cts map of B^2, $k : B^2 \to X$
3) h_* is a trivial homomorphism on the fundamental grp

Pf (1) ⇒ (2) Suppose h is null homotopic. $\exists H : S^1 \times I \to X$ that is a homotopy between h & const map.



$H|_{S^1 \times \{1\}} \equiv e_{x_0}$ ∴ restriction is constant!

we want to collapse $S^1 \times \{1\}$ to a single pt $0 \in B^2$ → cts surj, quotient map!

$\pi : S^1 \times I \to B^2$
$\pi(x,t) \longmapsto (1-t)x$

$\exists k : B^2 \to X$ (see section on quotient topology) which extends h

(2) ⇒ (3) for $h : S^1 \to X$ cts for which $\exists$ extension $k : B^2 \to X$

If $j : S^1 \to B^2$ inclusion map. Then $h = k \circ j$ ⇒ $h_* = k_* \circ j_*$
$a \longmapsto a$ → trivial as simply conn

But, $j_* : \pi_1(S_1, p_0) \to \pi_1(B_2, k)$

⇒ j_* trivial, ⇒ h_* trivial

(3) ⇒ (1) Textbook!

Cor The inclusion $j : S^1 \to \mathbb{R}^2 \setminus \{0\}$ is not null homotopic.
The identity map $id_{S_1} : S_1 \to S_1$ is not null homotopic.

Pf There is a retraction $r : \mathbb{R}^2 \setminus \{0\} \to S^1$ $\left(r(x) = \frac{x}{\|x\|} \right)$

⇒ j_* is injective (by lemma from last class)

⇒ $j_* : \pi_1(S_1, b_0) \to \pi_1(\mathbb{R}^2 \setminus \{0\})$ non trivial (as $\pi_1(S_1, b_0)$ non trivial)

Similarly, $id_* : \pi_1(S_1, b_0) \to \pi_1(S_1, b_0)$ is the identity homomorphism. so non trivial.

Fundamental Theorem of Algebra

Thm) A polynomial eqn with $\mathbb{C}$ coeff. $x^n + a_{n-1}x^{n-1} + \dots + a_0 = 0$, $n > 0$ has at least 1 root.
↳ identify to

Pf) **Step 1** $f: S^1 \to S^1$ $f(z) = z^n$ induces a homomorphism "mult by n" (Hw)

$\Rightarrow$ f_* is injective!

Step 2 $g: S^1 \to \mathbb{R}^2 \setminus \{0\}$, $g(z) = z^n$. Then g is not null homotopic.

$g = j \circ f$, $f: S^1 \to S^1$ as given & $j: S^1 \to \mathbb{R}^2 \setminus \{0\}$ inclusion

So $g_* = j_* \circ f_*$, j_* injective by corollary $\Rightarrow$ g_* injective

$\Rightarrow$ g_* is non-trivial as $\pi_1(S^1, b_0) \cong \mathbb{Z}$ $\Rightarrow$ g not null homotopic!

Step 3 Special case. $\sum_{i=0}^{n-1} |a_i| < 1$ show poly eqn has a root in B^2

If not, define a map from $k: B^2 \to \mathbb{R}^2 \setminus \{0\}$ $\to$ p is the poly
$h: S^1 \to \mathbb{R}^2 \setminus \{0\}$ $\quad x \longmapsto p(x)$

if $h = k|_{S^1}$, h is null-homotopic as it has ext to B^2

can define homotopy between h, g which is contra as h null homotopic & g not

$F: S^1 \times I \to \mathbb{R}^2 \setminus \{0\}$
$(z, t) \to z^n + t\left(\sum_{i=0}^{n-1} a_i x^i\right)$ never hit 0 on RHS as

$\|F(z,t)\| \geq |z^n| - |t|\,|a_{n-1}z^{n-1} + \dots + a_0|$ $\quad z \in S^1$

rev triangle $\quad \geq 1 - t(|a_{n-1}| + \dots + |a_0|) > 0$

General case let $x = cy$ for $c \in \mathbb{R}_+$ $\quad (cy)^n + a_{n-1}(cy)^{n-1} + \dots + a_0 = 0$

$\Rightarrow y^n + \left(\frac{a_{n-1}}{c}\right)y^{n-1} + \dots + \frac{a_0}{c^n} = 0$

If has root y_0 then original poly has root $c\,y_0$.

Choose c large enough so $\sum \frac{|a_i|}{c^{n-i}} < 1$ $\Rightarrow$ special case