Last time

f, g paths in X from x_0 to x_1. $f \simeq_p g$ if $\exists$ path homotopy

$F: I \times I \to X$ so, $F(s,0) = f(s)$, $F(s,1) = g(s)$, $F(0,t) = x_0$, $F(1,t) = x_1$

Def) $*$ on paths with compatible endpts (concat).

$*$ on path homotopy classes $[f]$ $(\{\text{paths}\}/\simeq_p)$

Check $[f]*[g] = [f*g]$ well def. let $f \simeq_p f'$, $g \simeq_p g'$ & show

$[f]*[g] = [f']*[g']$

To show $f*g \simeq f'*g'$ check the following is path homotopy

$$H(s,t) = \begin{cases} F(2s,t) & t \in [0, 1/2] \\ F(2s-1,t) & t \in [1/2, 1] \end{cases}$$

Prop of $f*g$

Note: only def when $f(1) = g(0)$

1) **Assoc** If $[f]*([g]*[h])$ well def so is the following & they are the same $([f]*[g])*[h]$

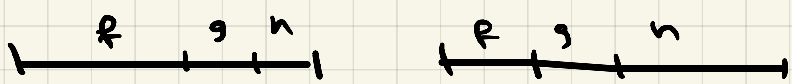
2) **Iden** (ish) $\forall x \in X$ let const path $e_x : I \to X$, $e_x(t) = x$ $\forall t$

Then, $[f]*[e_{x_1}] = [f]$, $[e_{x_0}]*[f] = [f]$ $\forall$ path f from x_0 to x_1

3) **Inv** Given a path from x_0 to x_1, let $\bar{f}(s) = f(1-s)$. $\bar{f}$ is the rev path of f so.

$[f]*[\bar{f}] = [e_{x_0}]$, $[\bar{f}]*[f] = [e_{x_1}]$ $\rightsquigarrow$ some sort inverse

Pf) ①



So not same path. Must show they are the same as repara.

Lemma) $f: I \to X$ path & $p: I \to I$ path from 0 to 1. Claim $f \circ p : I \to X$ (repara of f). Claim $f \simeq_p f \circ p$

(would immediately give us associativity as they are repara)

Pf) Define path homotopy $H(s,t) = f((1-t)s + t\varphi(s))$

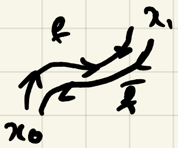
Check that this works

Pf prop) ① Assoc ✓ As $[f]*([g]*[h])$ reparam of $([f]*[g])*[h]$

② Id (isn) $[f]*[e_{x_1}] = \underset{f}{|\!-\!-\!-}\underset{1/2}{|}\overset{e_{x_1}}{-\!-\!-}|$ is a reparam of f (2 times speed)

So by lemma $f * e_{x_1} \simeq_p f$

③ Inv (isn) (idea: partially traverse f then turn around)



Think $H(s,t) = f_t(s) * \bar{f}_t(s)$ if $t=1$ traverse full, else partial only

$\to \; = f(ts) * \bar{f}((1-t)s)$

Can check

□

As opposed to arbitrary paths. Consider loops! ⟶ get grp

Def) X space, $x_0 \in X$. A path that starts & ends at x_0 is a loop based at x_0.

The fundamental grp of X based at x_0 is

$$\pi_1(X, x_0) = \{\text{loops at } x_0\} / \simeq_p \quad \text{with operation } *.$$

Pf) Prev thm $\Rightarrow$ $\pi_1(X, x_0)$ is a grp! with identity $[e_{x_0}]$

ex) $\pi_1(\mathbb{R}^n, x_0)$ for $x_0 \in \mathbb{R}^n$. Any loop at x_0. Use straight line homotopy to shrink to e_{x_0}

$\Rightarrow$ $\pi_1(\mathbb{R}^n, x_0)$ iso to trivial grp.

ex) $\pi_1(B^n, x_0)$ for $x_0 \in B^n$. By same process, $\pi_1(\mathbb{R}^n, x_0) \cong \{e\}$ as B^n convex

How does $\pi_1(X, x_0)$ compare to $\pi_1(X, x_1)$ for $x_0, x_1 \in X$?

Thm) If X top sp with $x_0, x_1 \in X$ st $\exists\, \alpha : I \to X$ path from x_0 to x_1. Then, $\pi_1(X, x_0) \cong \pi_1(X, x_1)$

Pf) Const an iso! Define $\hat{\alpha} : \pi_1(X, x_0) \longrightarrow \pi_1(X, x_1)$

$f \longmapsto [\bar{\alpha}] * [f] * [\alpha]$ (rev of α)

Bijectivity is kinda imm ⟶

Check homomorphism:

$$\hat{\alpha}([f]) * \hat{\alpha}([g]) = \hat{\alpha}([f]*[g])$$

$$[\bar{\alpha}]*[f]*\overset{e_{x_0}}{\cancel{[\alpha]*[\bar{\alpha}]}}*[g]*[\alpha] = [\bar{\alpha}]*[f]*[g]*[\alpha] =$$