Recall | V k dim distribution on a smooth mfd M

i.e. $\dim V(p) = k$ $\forall p$.

Recall V is integrable if $\forall$ X,Y v-fields tangent to V

(i.e. $\forall p$, $X(p) \in V(p)$, $Y(p) \in V(p)$) then $[X,Y]$ is tangent to V

Frobenius | Thm V is integrable the $\exists$ coord charts (around each $p \in M$)

(U, φ) $\varphi : U \to \mathbb{R}^n$

so $D\varphi(V) = \mathbb{R}^k \subseteq \mathbb{R}^n$

$$\mathbb{R}^k = T_q\mathbb{R}^k \qquad \mathbb{R}^n = T_q\mathbb{R}^n$$

and then $\varphi(U)$, $D\varphi(V)$ is tangent to the foliation by

$$\left\{ (x_1, \dots, x_n) \;\middle|\; \begin{aligned} x_{k+1} &= \varphi(p)_{k+1} \\ &\vdots \\ x_n &= \varphi(p)_n \end{aligned} \right\}$$

e.g. | $\mathrm{Heis} = \left\{ \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} \right\}$

$V(1) = \left\langle \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \right\rangle$ → left inv v.f. X, Y

$V(g) = DL_g(V(1)) \Rightarrow V(g) = \langle X(g), Y(g) \rangle$

Then $[X,Y] = Z$ → left inv v.f. def by $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$XY - YX = Z$

Last time | Frobenius is true if V is spanned by k lin ind vf $X_1, \dots, X_k$ so they commute.

→ so flows commute! i.e.

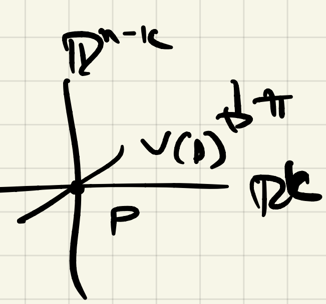
To prove Frobenius in general find k lin ind comm v.f.:

① This is local $\Rightarrow$ can work in $\mathbb{R}^n$.

② Wlog can assume that $V(p)$ tran to $\mathbb{R}^{n-k}$

even $\forall q \in$ nbdh of p $V(q) \pitchfork \mathbb{R}^{n-k}$

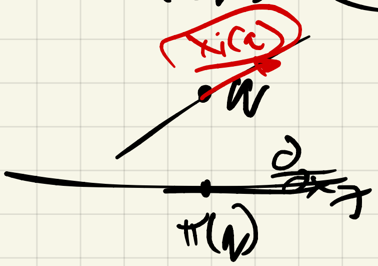
$\pi : \mathbb{R}^n \to \mathbb{R}^{n-k} \Rightarrow D\pi : V(q) \to \mathbb{R}^k$ is bij



$\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k}$ on $\mathbb{R}^k$

$Y_1, \dots, Y_k$ commute! → so these commute.

$X_i(q) = (D\pi)^{-1}\left(\frac{\partial}{\partial x_i}(\pi(q))\right)$ → pull back



⇒ X_i & Y_i are π related! thus! as Y_i commute, X_i commute!

$[X_i, X_j]$ π rel to $[Y_i, Y_j] = 0$ ⇒ $[X_i, X_j] = 0$

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let G a Lie group (C^∞)

look at {left invariant v. field} → Lie algebra of G

i.e. $V(g) = DL_g(V(1))$ → left mult by g

→ Claim this is V space on G.

$(V_1 + V_2)(g) = DL_g(V_1(1) + V_2(1))$ ✓

$\dim\{\dots\} = \dim G$

Note: could just as well use right invariant!

$T_1(G) \xrightarrow[1-1]{}$

Sometimes can define $\mathrm{Lie}\, G = T_1 G$!

Additional Structure

X, Y left inv v.f ⇒ $[X,Y](g) \overset{wts}{=} DL_g([X,Y](1))$

note: $D\varphi([X,Y]) = [D\varphi(X), D\varphi(Y)]$ diffeo

apply to $\varphi = L_g$

Denote by $\mathrm{Lie}\, G$ or $\mathfrak{g}$ → for eg G

$\mathfrak{g}$ endowed with $[\,,\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$

$(X, Y) \mapsto [X, Y]$

Can check $[\,,\,]$ is bilinear ! $\to$ Use derivations

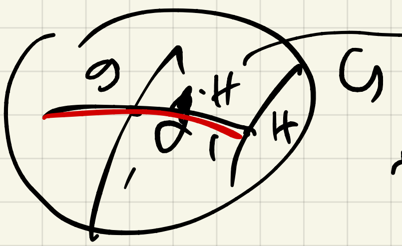
$\hookrightarrow$ it is also skew-symmetric

$\hookrightarrow$ alternating !

Jacobi identity

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

G is a Lie grp, $H \subseteq G$ Lie subgrp i.e. immersed submanifold



$\to g \cdot H$ partition into submanifolds $\hookrightarrow$ foliation !

transversal ?

$V(g) = T_g(g \cdot H)$ distribution

note $DL_g(V(1)) = V(g) \to$ left inv

$\Rightarrow$ left inv tangent to $V(g)$ (equiv tangent to gH)

$\mathfrak{h} \subseteq \mathfrak{g} \to$ integrable too

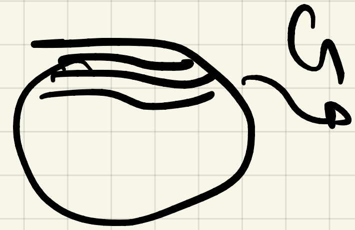
$X, Y \in \mathfrak{h} \Rightarrow [X,Y] \subseteq \mathfrak{h}$

Thm) if H is a lie subgrp $\subseteq G$.
Then $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalgebra i.e $[\mathfrak{h},\mathfrak{h}] \subset \mathfrak{h}$
Conversely if $\mathfrak{h} \subset \mathfrak{g}$ & $[\mathfrak{h},\mathfrak{h}] \subseteq \mathfrak{h}$
Then $\exists H \subseteq G$ a Lie subgrp
s.t $\mathfrak{h} = T_g(g \cdot H)$, i.e $\mathfrak{h}$ = left inv v.f coming from H

Pf) $\Rightarrow$ done

$\Leftarrow$ so $\mathfrak{h}$ is int use Frobenius

$$\boxed{H = \mathcal{F}(1)}$$



$\to$ leaves of foliation $\mathcal{F}$ s.t left invariant under G.