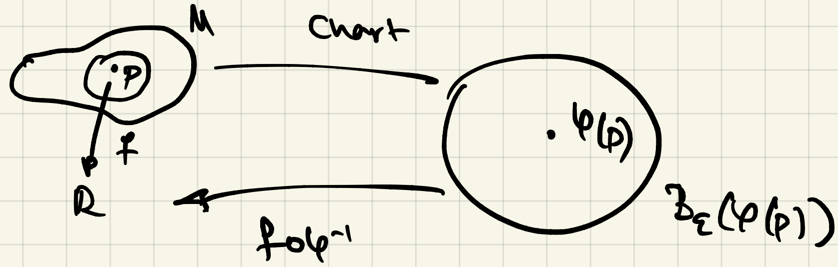
Last time



$f, g \in C^\infty(M)$

Really want

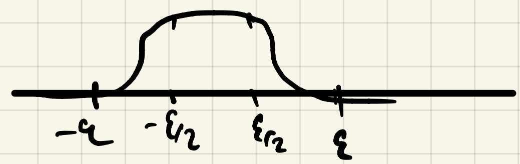
$f \longrightarrow \overline{f}$

which is zero outside $\varphi^{-1}(B_\varepsilon(p)) \to$ defined otherwise on $\varphi^{-1}(B_\varepsilon(p))$

For this take bump. $\rho: \mathbb{R}^n \to \mathbb{R}$ $\to$ for $n=1$

$\rho = 0$ outside $B_\varepsilon(\varphi(p))$

$\rho = 1$ on $B_{\varepsilon/2}(\varphi(p))$



Back to immersion, submersions, reg value

- M, N – C^∞ Manifolds $f: M \to N$

$\to$ recall $q \in N$ is a reg value of f if

$Df(p): T_pM \to T_qN$ surjects $\forall\, p \in f^{-1}(q)$

$\to$ conv. if $f^{-1}(q) = \emptyset$ then reg value

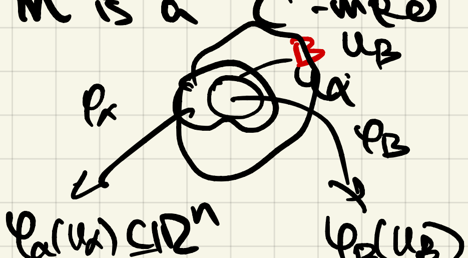
Measure

$A \subseteq \mathbb{R}^n \to \mathrm{vol}(A) = \int_A 1\, dx \in [0, \infty]$

Need A to be measurable (if we use Lebesgue measure)

$\to$ say A has measure 0 if $\mathrm{vol}\, A = 0$

M is a C^1-mfd



consider $B \subseteq U_\alpha$. B has measure 0 if $\varphi_\alpha(B)$ has measure 0.

$\to$ need to show well def.

$\to$ consistent under coord charts

Suppose φ_β, U_β is another coord chart, & $B \subseteq U_\beta$.

WTS $\varphi_\alpha(B)$ measurable $\Rightarrow \varphi_\beta(B)$ measurable.

also $\mathrm{vol}(\varphi_\alpha(B)) = 0 \Rightarrow \mathrm{vol}(\varphi_\beta(B)) = 0$

$B \subseteq U_\alpha, U_\beta$

$\varphi_\beta(B) = \underbrace{\varphi_\beta \circ \varphi_\alpha^{-1}}_{\text{transition map } C^1}(\underbrace{\varphi_\alpha(B)}_{\to \text{measurable}})$ } image of measurable under C^1 is measurable!

$$\mathrm{vol}(\varphi_\beta(B)) = \int_{\varphi_\beta(B)} 1 \, dx = \int_{x \in \varphi_\alpha(B)} \det(D(\varphi_\beta \circ \varphi_\alpha^{-1})_x) \, dx$$

if $\varphi_\alpha(B)$ has zero measure this is zero!

↳ volume in general not well defined. But zero volume is!

Def) $B \subseteq M$ has zero measure / volume if $\forall \varphi_\alpha, U_\alpha$
$\mathrm{vol}(\varphi_\alpha(B \cap U_\alpha)) = 0$

Def) $A \subseteq M$ has full measure if $M \setminus A$ is measure zero

Sard's Thm)
$f : M \to N$ (both C^∞ mfd, and C^∞ fn)
$\Rightarrow$ the set of regular values has full measure

Warning!: needs C^∞ → not work for C^1

Application) makes calculation unnecessary!

e.g.) $SL(n, \mathbb{R})$ mfd $= \det^{-1}(1)$
↳ good enough to show 1 is a regular value.
↳ note: $\det(\lambda A) = \lambda^n \det A$
Claim: if 1 is not regular: neither is λ^n

$$\mathrm{Mat}(n \times n) \xrightarrow{\lambda} \mathrm{Mat}(n,n) \xrightarrow{\det} \mathbb{R}$$

if $\lambda \neq 0$ this is invertible (divide by λ^n)

$$D(\det \circ \mathrm{mult}_\lambda) = D\det \circ D\,\mathrm{mult}_\lambda$$ inv

$\Rightarrow$ λ^n is reg $\Leftrightarrow$ 1 regular

if 1 not regular $\Leftrightarrow$ λ^n not regular

oops, $\{\lambda^n \mid \lambda \in \mathbb{R}\}$ doesn't have measure 0!

Immersions - normal form

① simplest immersion $\mathbb{R}^k \subseteq \mathbb{R}^n$ $= k+(n-k)$ $k \le n$

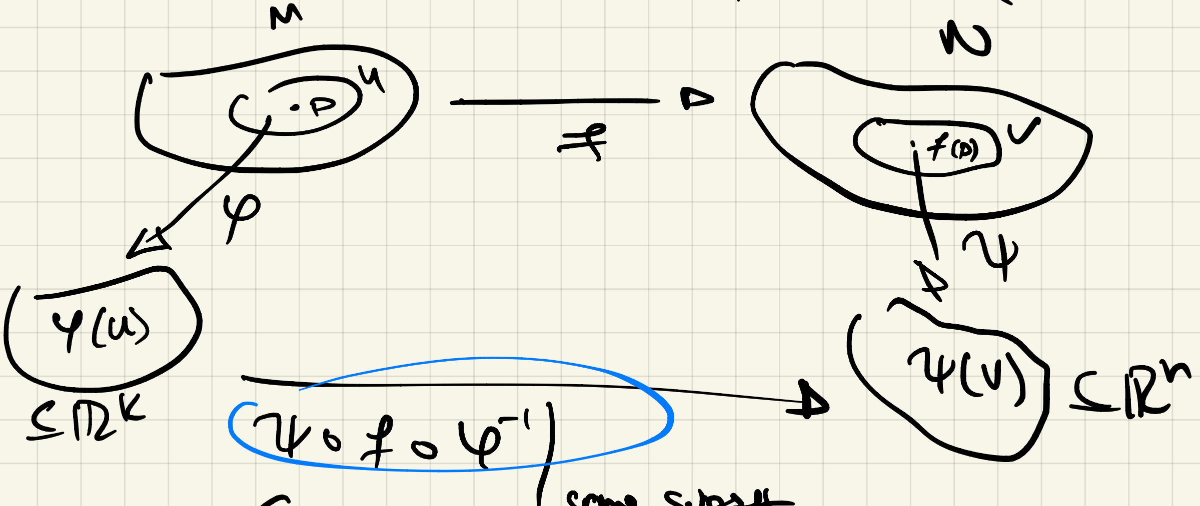
$(x_1, \dots x_k) \mapsto (x_1, \dots, x_k, \overbrace{0, \dots, 0}^{n-k})$

Normal form :

$f : M^k \to N^n$ an immersion at p $k \le n$

$Df_p : T_pM \to T_{f(p)}N$ injective.

Thm $\exists$ charts (U, φ) at p and (V, ψ) at $f(p)$

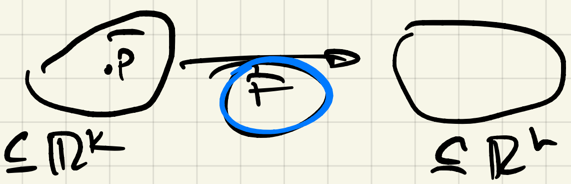


$\hookrightarrow$ looks like some normal form.

get map, locally invertible, and use inverse function thm

immersion trick

$M \times \mathbb{R}^{n-k} \to N$ locally diffeo $\to$ good enough to work on charts $(U, \varphi), (V, \psi)$



$DF_{\bar{p}}$ is injective

$DF_p = A \to$ $n \times k$ matrix $\hookrightarrow$ injectivity

$= \begin{pmatrix} \hat{A} \\ \dots \end{pmatrix} \to$ wlog first k invertible

compose with

$(\hat{A})^{-1} \circ F : \mathbb{R}^k \to \mathbb{R}^n$

extend to $\mathbb{R}^n \to \mathbb{R}^n$

$\hat{\hat{A}}(x_1 \dots, x_n) = \hat{A} \circ (F(x_1, \dots x_k), x_{k+1}, x_n))$ $D = \begin{pmatrix} id & ? \\ 0 & id \end{pmatrix}$

$\hookrightarrow$ local diffeo $: \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^n$