Let

$\Omega^k(M) \longrightarrow$ smooth k-forms on M

exterior der: $\Omega^k M \xrightarrow{\partial} \Omega^{k+1} M$ Prop we want ⓪ ∂_k linear!

① $\Omega^0 M := C^\infty(M) \longrightarrow \partial f := df$ direc deriv

$df(v) = df(v)$ $v \in T_pM$

② $\partial^2 \equiv 0$

③ $\partial(\alpha \wedge \beta) = \partial\alpha \wedge \beta + (-1)^k \alpha \wedge \partial\beta$ if $\alpha \in \Omega^k M$!

Prop! $\exists!$ ∂ satisfying the above!

① $M = \mathbb{R}$ $\Omega^0 M = C^\infty(\mathbb{R})$

let $f \in \Omega^0 M$ $\partial f = df$

let $\alpha \in \Omega^1 M \Rightarrow \alpha = f \wedge dx$ $\to C^\infty$

if $A \in \Omega^0(M)$, $B \in \Omega^2 M$ $\Rightarrow A \wedge B = A \cdot B$

③ $\Rightarrow \partial\alpha = \partial f \wedge dx + \cancel{f \wedge \partial(dx)} \to 0$

$= \partial f \wedge dx \in \Omega^2 \mathbb{R} \to \boxed{0 \text{ vec sp}}$

② $M = \mathbb{R}^2$ ∂ on Ω^0, Ω^2 simple!

let $\alpha = f(x,y)\, dx$ ie $f(x,y) \wedge dx$

$\Rightarrow \partial\alpha \overset{③}{=} \partial f \wedge dx = \left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy\right) \wedge dx$

$= \frac{\partial f}{\partial y} dy \wedge dx$

Gen case on mfd

Claim: ∂ is "local" i.e. sort out what happens in coord charts!

i.e. if U, φ chart $\partial(\alpha|_U) = \partial\alpha|_U$

Pf: Use φ bump func on U & $V \supseteq U$ open chart

$\varphi \equiv 0$ outside V & $\varphi \equiv 1$ in U $\varphi \geq 0$ everywhere!

Consider $\varphi \cdot \alpha = \begin{cases} \alpha|_U & \text{on } U \\ 0 & \text{on } M \setminus V \\ \text{smooth} & \text{on } V \end{cases}$

$$\partial(\varphi \cdot \alpha) = \partial\varphi \wedge \alpha + \varphi \wedge \partial\alpha$$

on $U \to \partial\varphi = 0 \xRightarrow{\boxed{\varphi \text{ is } 1}} \partial(\varphi \cdot \alpha) = \partial\alpha$

So $\partial(\alpha|_U) = \partial\alpha|_U$

Upshot: Calculation on ∂ on wfd is calc on coord!

Let $F: M \to N$ smooth let $\alpha \in \Omega^k N \Rightarrow F^*\alpha \in \Omega^k M$

What is $\partial(F^*\alpha) \overset{?}{=} F^*(\partial\alpha)$? Go show indep of coord chart!

Lemma 1 Above holds!

Pf) By locality suffices to show $\mathbb{R}^n \to \mathbb{R}^m$

Cool! Let $\dim M = n$

$$0 \to \Omega^0 M \xrightarrow{\partial} \Omega^1 M \xrightarrow{\partial} \cdots \xrightarrow{\partial} \Omega^n M \xrightarrow{\partial} 0$$

also $\partial \circ \partial = 0 \Rightarrow \operatorname{im} \partial_k \subseteq \ker \partial_{k+1}$

$\Rightarrow$ can look at $\ker \partial_{k+1} / \operatorname{im} \partial_k =: H^k_{dR} M$

What is $H^k_{dR} \mathbb{R}^n$?

① $M = \mathbb{R}$ $\quad 0 \to \Omega^0(\mathbb{R}) \to \Omega^1(\mathbb{R}) \to 0$

$$C^\infty(\mathbb{R}) \xrightarrow{\partial} \{f \cdot dx \mid f \in C^\infty(\mathbb{R})\}$$

$$g \longmapsto \partial g = g' \, dx$$

$\Rightarrow \operatorname{im} \partial = \{f \cdot dx \mid f = g' \text{ for some}\} = f$ ones C^∞ func! (integral!)

$\Rightarrow \partial : \Omega^0(\mathbb{R}) \to \Omega^1(\mathbb{R})$ onto

$\Rightarrow H^1_{dR}\mathbb{R} = 0$! $\quad H^0_{\mathbb{R}} = \ker \partial = \{g \mid g' = 0\}$

$=$ const func! $\cong \mathbb{R}$

Prop) M is a smooth mfd!

$H^0 M = \mathbb{R}$?

Pf) if M conn H^0_M is $\{f \in \Omega^0 \mid df = 0\}$

= locall const

= const func! (real val) $\cong \mathbb{R}$

So, since can vary on each conn comp

$H^0 M = \mathbb{R}^{\#\text{ conn comp } M}$

(3) $M = \mathbb{R}^n$ $\quad H^0 \mathbb{R}^n \cong \mathbb{R}$ $\quad H^k \mathbb{R}^n \cong 0$ $\quad 1 \le k \le n$

Poincare lemma

Defn) if $\alpha \in \Omega^k M$ then α is closed if $d\alpha = 0$

α is exact if $\alpha = d\beta$ for some $\beta \in \Omega^{k-1} M$

Note: exact $\Rightarrow$ closed

on $\mathbb{R}^n$ converse holds!

also $H^k = \{\text{closed } k \text{ forms}\} / \{\text{exact } k \text{ forms}\}$

Idea of Poincare lemma

Actually: works for any star-shaped open region in $\mathbb{R}^n$? $\to$ convexity holds at a point!

(will construct a map $I_k : \Omega^k A \to \Omega^{k-1} A$

$$\boxed{\begin{tikzcd} \bullet \arrow[r, shift left, "\partial_{k-1}"] & \bullet \arrow[l, shift left, "I_k"] \arrow[r, shift left, "\partial_k"] & \circ \arrow[l, shift left, "I_{k+1}"] \\ \Omega^{k-1} & \Omega^k & \Omega^{k+1} \end{tikzcd}}$$

will show $\partial \circ I + I \circ \partial = id$

with subscr

$\partial_{k-1} \circ I_k + I_{k+1} \circ \partial_k = id$

I is called chain homotopy!

Disgusting thing $\to$ get I but lets say we have it!

now spose $\alpha \in \Omega^k$ closed $\Rightarrow d\alpha = 0$

$\alpha = d \circ I(\alpha) + I \circ \cancel{d\alpha}^{=0} \Rightarrow d(I(\alpha)) \Rightarrow$ exact!