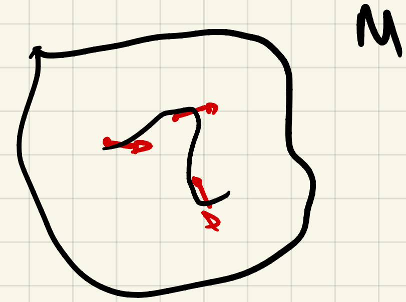
Lec 20



X vector field $\longleftrightarrow$ φ_t flow

↳ curves tangent to v.f.

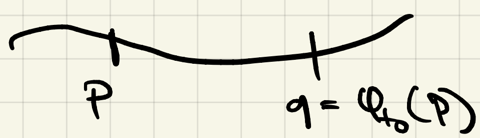
$$\frac{\partial}{\partial t}\Big|_{t=t_0} \varphi_t(q) = X(t_0)$$

Thm if X C^1, Lipschitz, $\exists!$ soln curves on small intervals

↳ if X is complete, flow defined for all time.

Thm M compact $\Rightarrow$ any vector field is complete.

X v.f



$\longrightarrow$ by uniqueness of ODE $\varphi_s(q) = \varphi_{t_0+s}(p)$

$\Rightarrow \varphi_{s+t_0}(p) = \varphi_s(\varphi_{t_0}(p))$

Provided φ_{s+t_0} is defined

$\rightarrow$ sufficient if X complete

$\Rightarrow \varphi_{s+t_0} = \varphi_s \circ \varphi_{t_0}$

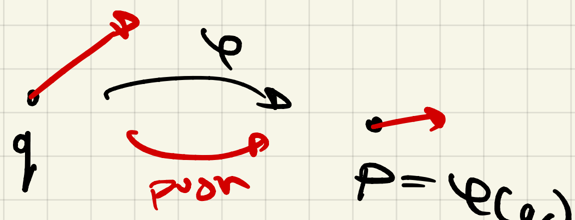
↳ defines an action of $\mathbb{R}$

($\mathbb{R}$ actions called flows.)

Spse X v.f

$F: M \to M$ a diffeo (could have free map)

$F_*(X) =$ $\longrightarrow$



$DF_q(X(q)) =: F_*(X)(p)$

Def X, Y are F related (suff for full rank instead of diffeo)

↳ i.e. $\exists F$ so $X = F_*(Y)$

Also X, Y are C^k conjugate if F is C^k.

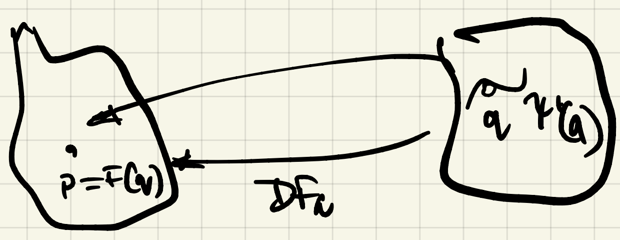
Spse, as above, φ_t is flow of X, ψ_t flow of Y.

If $\boxed{F_*(Y) = X}$ chain deriv of ψ_t $M \to X$ v.f $N \to Y$ v.f

$$\frac{\partial}{\partial t}\Big|_{t=0} F(\psi_t(q)) = DF_{\psi_t(q)}(Y(\psi_t(q)))$$

soln curve $\rightarrow$ $= X(F(\psi_t(q)))$

$\varphi_t = F \circ \psi_t \circ F^{-1}$ $\leftarrow$ conjugate



More general

$$M \xrightarrow{\pi} N, \quad \varphi_t \circlearrowright M, \quad \psi_t \circlearrowright N$$

→ saying nothing abt π.

$(\psi_t \circ \pi)(x) = (\pi \circ \varphi_t)(x)$ is interesting

Suppose X, Y are vector fields on M.

$[X,Y]$

↳ $[X,Y]$ a vec field.

Recall X, Y vf ⟷ derivations (one to one corr) ⟺ $X : C^\infty(M) \to C^\infty(M)$

$Y : C^\infty(M) \to C^\infty(M)$

$[X,Y] = X \circ Y - Y \circ X$ → is a derivation

↳ ∴ is a vec field

Check

lin clear

$$(X \circ Y)(fg) = X(Y(f \cdot g)) = X(Y(f) \cdot g + f \cdot Y(g))$$
$$= (X \circ Y)(f) \cdot g + Y(f) \cdot X(g)$$
$$+ X(f) \cdot Y(g) + f \cdot (X \circ Y)(g)$$

not quite right

$$Y \circ X(f \cdot g) = (Y \circ X)(f) \cdot g + X(f) \cdot Y(g)$$
$$+ Y(f) \cdot X(g) + f(Y \circ X)(g)$$

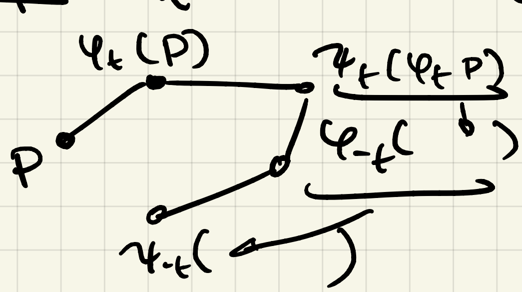
$$\Rightarrow ((X \circ Y) - (Y \circ X))(fg) = (X \circ Y)(f) \cdot g - (Y \circ X)(f) \cdot g$$
$$+ f(X \circ Y)(g) - f(Y \circ X)(g)$$
$$= (X \circ Y - Y \circ X)(f)\, g + f(X \circ Y - Y \circ X)(g)$$

↳ derivation!

$[X,Y]$ → measures how much X, Y don't commute

In terms of vector fields (rather than derivations)?

Suppose φ_t → flow of X, ψ_s → flow of Y.



Note: geom as $t \to 0$ by continuity we end at p. → careful

↳ if $X = \frac{\partial}{\partial x}$, $Y = \frac{\partial}{\partial y}$

↳ $\forall t$ end where we started