Aside: If has a C^1 struct then it has a compatible C^∞ struct

[Assume mfd diffble

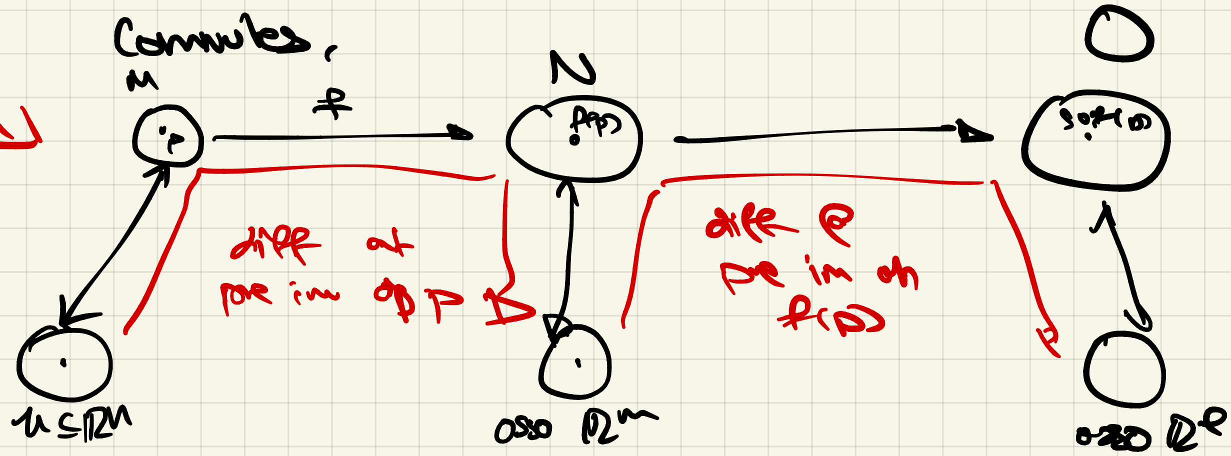
Recall Suppose $f: M \to N$, $g: N \to O$, M, N, O mfds (→ diffble). Have $p \in M$. Can look at $g \circ f(p) \in O$

Chain Then $g \circ f$ is diffble and

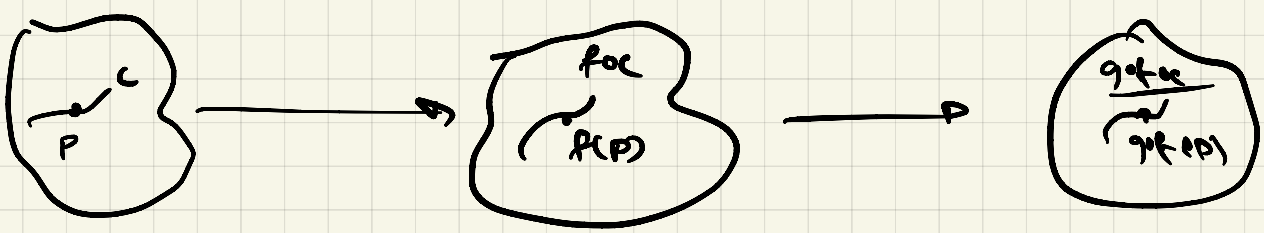
$$
\begin{tikzcd}
D_p(g\circ f) : T_pM \arrow[rr] \arrow[dr, "Df_p"'] & & T_{(g\circ f)(p)}O \\
& T_{f(p)}N \arrow[ur, "Dg_{f(p)}"'] &
\end{tikzcd}
$$

Commutes,

Idea



Using curves



$[c] \longrightarrow [f\circ c] \longrightarrow [g\circ f\circ c]$

What is a diffeo?

$f: M \to N$ which is diffble & invertible where f^{-1} is diffble

So $f: M \to N$ and $g = f^{-1}: N \to M$

$Id = D_p(id_M) = D_p(f^{-1}\circ f) = D_{f(p)} f^{-1} \circ D_p f$

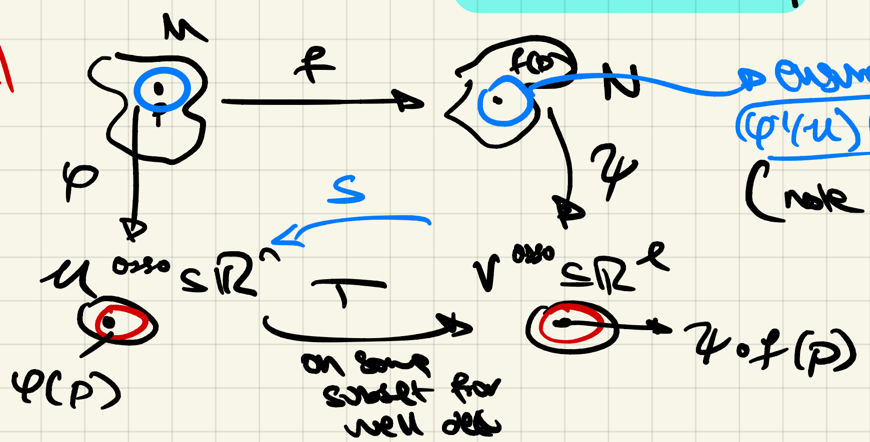
Cor $D_p f : T_pM \to T_pN$ has an inverse if f is a diffeo

Defn | $f: M \to N$ is a local diffeo at $p \in M$ if $\exists$ open nbhd U of p and V of $f(p)$ s.t. $f|_U : M \to V$ is a diffeo

Thm | Suppose $f: M \to N$ is C^1 and for $p \in M$ and $D_p f : T_pM \to T_pN$ is invertible as a linear map.

$\Rightarrow f$ is a local diffeo at p → inverse function thm

Pf |



→ ensure this is open by intersecting $(\varphi^{-1}(U)) \cap (f^{-1}(\psi^{-1}(V)))$

(note $n = \ell$ as Df_p invertible)

$U \overset{open}{\subseteq} \mathbb{R}^n$, $V \overset{open}{\subseteq} \mathbb{R}^\ell$, $\psi \circ f(p)$, $\varphi(p)$

T on some subset for well def.

note by chain $D_{\varphi(p)} T$ invertible

So by inv funct thm T is locally inv. Call this S.

The local inversion is $\varphi^{-1} \circ S \circ \psi$. Show well def

Defn Spse M is a C^1 mfd. $S \subseteq M$ is an embedded (regular) C^1 submfd of M if:

$\forall p \in S$ $\exists$ nbhd U & coord chart $\varphi: U \to \mathbb{R}^m$ (coord chart for M)

$\varphi|_S$ maps into $\mathbb{R}^k \subset \mathbb{R}^n$ (embed) and $S = \{q \in U \mid \varphi(q) = (\underbrace{x_1, \dots x_k}_{k}, 0, \dots 0)\}$

↳ adapted charts

Cor | S is a C^1 k-mfd in its own right with charts $\{(S \cap U_\alpha, \varphi|_{S \cap U_\alpha})\}_{\alpha \in I}$

e.g. | $\mathbb{R}^\ell \subseteq \mathbb{R}^n$ $\ell \le n$

$S^\ell \subseteq S^n$ $\ell \le n$ sphere

$\mathbb{RP}^{\ell} \subset \mathbb{RP}^n$

$\mathbb{R}^2/\mathbb{Z}^2 = S^1 \times S^1 = T$

Non-eg |

$\subseteq \mathbb{R}^2$

$\subseteq \mathbb{R}^2$

too close

$\alpha \notin \mathbb{Q}$ → $p(\alpha x)$ dense in T ↳ not submanifold

map $\mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$

x axis is first S_1

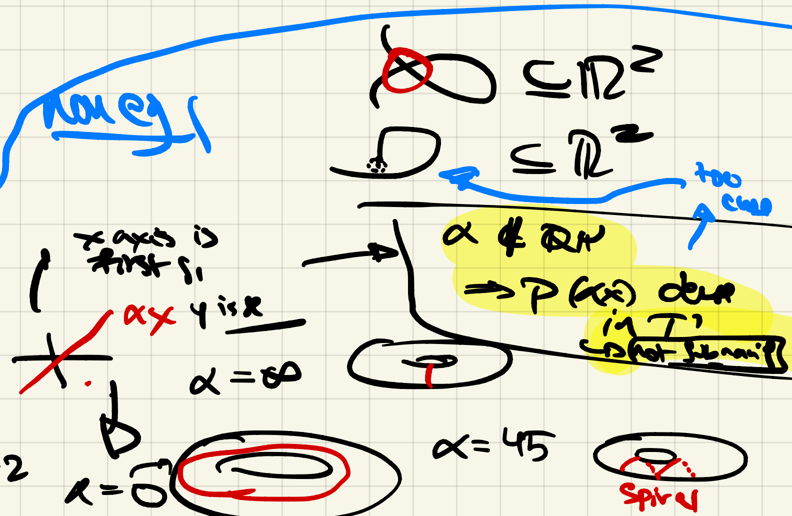
αx, y is x

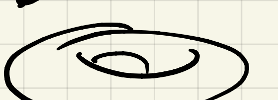
$\alpha = \infty$

$\alpha = 0$

$\alpha = 45$

spiral





Defn We call things like the torus. $f: M \to N$ C^1 map
then f is an immersion if
$\forall p \in M$ Df_p is injective

Cor $f(M)$ is "locally" a submfd