To align with book $(\Lambda^n M)_p := \Lambda^n(T_p^* M)$ $=$ alt k-maps -mult maps $T_p M \times \dots \times T_p M \to \mathbb{R}$

e.g.) $(dx_1 \wedge dx_2)(\partial x_i, \partial x_j)$

$= \partial x_1(\partial x_i) \cdot \partial x_2 \partial(x_j) - \partial x_1(\partial x_j)\, \partial x_2(\partial x_i)$

if $(i,j) = (1,2)$ then 1 if $(j,i) = (1,2)$ then -1

else 0

Recall) α a k-form is smooth, $X_1, \dots, X_k$ are smooth v.f.

then $p \mapsto \alpha_p(X_1(p), \dots, X_k(p))$ is smooth.

or $\alpha = \sum \alpha_i\, \partial x_{i_1} \wedge \dots \wedge \partial x_{i_k}$ $\longrightarrow$ $\alpha_i : M \to \mathbb{R}$.

$\hookrightarrow$ smooth

look at $\quad \partial x_1 \wedge \partial x_2 \left(\sum \alpha_i \partial x_i , \sum \beta_j \partial x_j \right)$

most vanish $\quad = \partial x_1 \wedge \partial x_2 (\alpha_1 \partial x_1, \beta_2 \partial x_2) + \partial x_1 \wedge \partial x_2(\alpha_2 \partial x_2, \beta_2 \partial x_2)$

$= \alpha_1 \beta_2 - \alpha_2 \beta_1$ $\rightarrow$ some sort determinant

$$\begin{vmatrix} \alpha & \beta_1 \\ \beta_2 & \alpha_2 \end{vmatrix}$$

Recall) $\alpha \in \Lambda^k V$, $\beta \in \Lambda^\ell V$

$$(\alpha \wedge \beta)(v_1, \dots, v_k, v_{k+1}, \dots, v_{k+\ell}) = \frac{1}{k!\,\ell!} \sum_{\sigma \in S(k+\ell)} (-1)^\sigma \alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)})\, \beta(v_{\sigma(k+1)}, \dots, v_{\sigma(k+\ell)})$$

$\hookrightarrow$ increasing perms!

e.g.) $k = \ell = 1$

$(\alpha \wedge \beta)(v_1, v_2) = \alpha(v_1)\beta(v_2) - \alpha(v_2)\beta(v_1)$

Prop) if $\alpha \in \Lambda^k V$, $\beta \in \Lambda^\ell V$ then,

$\alpha \wedge \beta \in \Lambda^{k+\ell} V$ (check this is legit).

Important) wedge product is a multipliable operation.

e.g.) $\dim V = n$ $\quad k + \ell = n$ $\quad \alpha \in \Lambda^k V$, $\beta \in \Lambda^\ell V$

$\alpha \wedge \beta \in \Lambda^{k+\ell} V = \Lambda^n V \cong \mathbb{R}$.

E.g. 2) let M → dim n smooth mfd. oriented

$L: \Lambda^n M \cong \{f: M \to \mathbb{R} \mid \underline{\text{smooth}}\}$

$\hookrightarrow \alpha$

$\Lambda^n M \underset{\pi}{\overset{\sigma}{\rightleftarrows}} M$ → section σ never vanishing ↪ call a volume form!

Why? $\mathcal{J} \in \Lambda^n M$ $\quad \mathcal{J}(p) = \underbrace{f(p)}_{\in \mathbb{R}} \cdot \sigma(p)$

↪ f uniquely det

⇒ $\Lambda^n M \rightsquigarrow f: M \to \mathbb{R}$

↪ if $f: M \to \mathbb{R} \rightsquigarrow f(p) \cdot \sigma(p)$ for corr!

Note: $C^\infty M = \Lambda^0 M$ so arrow $\Lambda^0 M \to \Lambda^n M$.

Properties! of wedge.

1) $\alpha \wedge \beta = (-1)^{k \cdot l} \beta \wedge \alpha$

2) $(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$

Orientation) aside

M space have coord charts $\{U_\alpha, \varphi_\alpha\}$ so that transition map $T_{\alpha_1, \alpha_2}: \mathbb{R}^n \to \mathbb{R}^n$ on $U_{\alpha ij}$ take

→ Note D^n or since $dx_1 \wedge \dots \wedge dx_n$ never vanish

$dx_1^{\alpha_i} \wedge \dots \wedge dx_n^{\alpha_i}$

$\downarrow J_{\alpha_i \alpha_j}$

$dy_1^{\alpha_i} \wedge \dots \wedge dy_n^{\alpha_i}$ ↪ want! $dy_1^{\alpha_j} \wedge \dots \wedge dy_n^{\alpha_j}$

Zo smooth functions pull back n-forms

$J_{\alpha_i \alpha_j}$ pullback ↪ $g(x)\, dx_1^{\alpha_i} \wedge \dots \wedge dx_n^{\alpha_j}$

≥ 0?

(1) Pull back $\varphi: M \to N$

let $\beta \in \Lambda^k N$ $\quad \varphi^* \beta \in \Lambda^k M$ $\quad \varphi^* \beta(v_1, \dots, v_n)$

$\beta(D\varphi(v_1) \dots D\varphi(v_n))$