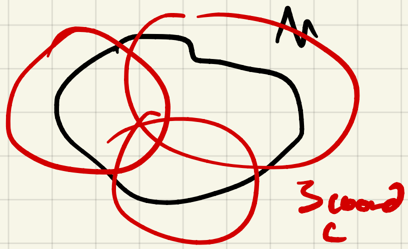
Lec 28

Embedding of manifolds into $\mathbb{R}^N$

Thm) M compact $\exists\, f: M \to \mathbb{R}^N$, f is an embedding



$\{U_\alpha, \varphi_\alpha\}$ coord charts $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$

$M \longrightarrow \prod_\alpha \mathbb{R}^n = \mathbb{R}^N \Rightarrow N = n \cdot \#(\alpha)$

$\cup$

funky function bad (not smooth / C^∞) $\longleftarrow$ $x \in U_{\alpha_1}, \dots U_{\alpha_j} \longmapsto (\varphi_{\alpha_1}(x), \dots, \varphi_{\alpha_j}(x), 0, \dots, 0)$

say

Make it smooth by tampering with a partition of unity of U_α $\to$ call $\{\tau_\alpha\}_\alpha$ $\to$ $\sum \tau_\alpha = 1$ $\tau_\alpha : M \to [0,1)$.

So,

$x \longmapsto (\tau_{\alpha_1} \cdot \varphi_{\alpha_1}(x), \dots, \tau_{\alpha_j} \varphi_{\alpha_j}(x), 0, \dots, 0)$.

also $\tau_{\alpha_i} \equiv 1$ on $V_{\alpha_i} \subseteq U_{\alpha_i}$

$\hookrightarrow$ note $\tau_{\alpha_i} \varphi_{\alpha_i}$ on V_{α_i} $\equiv \varphi_{\alpha_i}$

repeat process with covering of V_α instead of U_α. $\longleftarrow$ it is injective here $\longleftarrow$

Q) M not compact idea

$M \longrightarrow \prod_{\alpha_i} \mathbb{R}^n \xrightarrow{\text{proj}}$ large $\mathbb{R}^N$

(infinite)

$\hookrightarrow$ show $\exists\, N$ large enough to allow this.

Look at $\mathbb{R}^n = V$

$\Lambda^k V = \{$ multilinear alternating functionals $\}$

$\lambda : \overbrace{V \times \cdots \times V}^{k} \longrightarrow \mathbb{R}$

s.t. $\lambda(\ldots, v_i, \ldots)$ linear in all v_i &

$\lambda(\ldots, v_i, \ldots, v_j \ldots) = -\lambda(\ldots, v_j, \ldots, v_i, \ldots)$

Thm $\dim \Lambda^n \mathbb{R}^n = 1$

$\dim \geq 1$ as det in $\Lambda^n \mathbb{R}^n$ show $\dim \leq 1$ by scaling

geom $|\det|$ is volume.

① $\Lambda^1 V = V^*$

↳ dual in the sense of lin alg.

② $\Lambda^2 \mathbb{R}^n$? let $e_1, \ldots, e_n$ basis for $\mathbb{R}^n$ $(n \geq 2)$

$\lambda \in \Lambda^2 \mathbb{R}^n$ $\lambda(v, w) = \lambda(\sum \alpha_i e_i, \sum \beta_j e_j)$

$$= \sum_i \alpha_i \lambda(e_i, \sum_j \beta_j e_j)$$

$$= \sum_i \alpha_i \left(\sum_j \beta_j \lambda(e_i, e_j) \right) \qquad \lambda(e_i, e_j) = -\lambda(e_j, e_i)$$

$$= \sum_{i<j} (\alpha_i \beta_j - \alpha_j \beta_i) \lambda(e_i, e_j)$$

$\Rightarrow \dim \Lambda^2 \mathbb{R}^n = \binom{n}{2}$ basis elts

in gen define $e_i \wedge e_j$ as elt which assigns

$$e_i \wedge e_j \left(\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \right) = x_i y_j - x_j y_i$$

↳ check alternating.

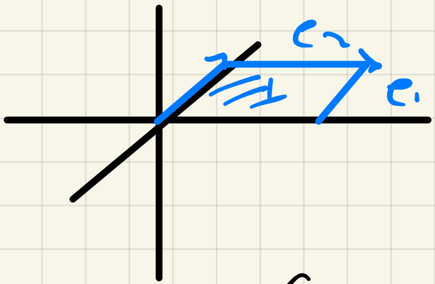
③ $\Lambda^k \mathbb{R}^n$ has basis $\{e_{i_1} \wedge \cdots \wedge e_{i_k}\}$ $i_1 < \cdots < i_k$

↳ as def above.

$\Rightarrow \dim \Lambda^k \mathbb{R}^n = \binom{n}{k}$

Meaning:

$$\mathbb{R}^3 \to \quad e_1 \wedge e_2 \left(\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix} \right) = 1\cdot 1 - 0\cdot 0 = \underline{1}$$



↳ area of sq.

$$e_1 \wedge e_2 \left(\begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right) = 0$$

intersected area with affine sp given by sp(e_1, e_2)

meaning ↳ the k-volume wrt k-subsp sp given by the vectors in wedge

signed

↳ of p-pped