Lec 11

Def let $M \xrightarrow[f]{} N$ and f is C^1 with M, N at least C^1.

f is a submersion $\Leftrightarrow$ every value $y \in N$ is a regular value.
(note. $y \notin f(M)$ automatically means y is regular)

$\underline{\forall x \in f^{-1}(y) \text{ ie fiber}}$, we want that

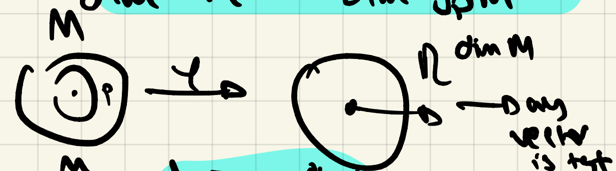
$$Df_x : T_x M \to T_y N \quad \text{is surjective}$$

$\Rightarrow \dim T_x M \geq \dim T_y N \Rightarrow \dim M \geq \dim N$

Reg Value Thm

Suppose $f : M \to N$ is C^1 and $q = f(p)$ is a regular value
Then $f^{-1}(q)$ is a embedded submfld.

dim M = dim T_pM



$\Rightarrow \dim T_{f(p)}\mathbb{R}^{\dim M} = \dim \mathbb{R}^{\dim M}$
$= \dim M$

$T_pM \cong T_{f(p)}\mathbb{R}^{\dim M}$

Def $S \subseteq M$ (C^{1} mfd) is a embedded C^1 submfd if we can find adapted coord charts
$\forall p \in S$ $\exists$ coord chart for M
$p \in U_\alpha$, $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$ and $S \cap U_\alpha = \varphi_\alpha^{-1}(\mathbb{R}^k \subseteq \mathbb{R}^n)$
$\hookrightarrow$ last $n-k$ coord 0

Def $f : M \to N$ is an immersion if $\forall p \in M$
$Df_p : T_pM \to T_{f(p)}N$ is injective.

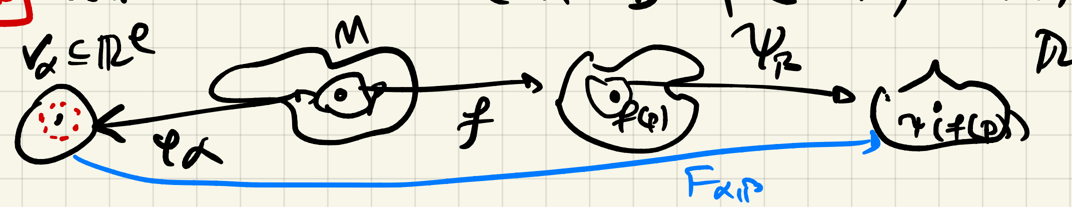
Def $S \subseteq M$ considering inclusion $i : S \to M$, $s \mapsto m$

S is an immersed submfd if i is an immersion

Application of Thm

1) $SL(n, \mathbb{R})$ is a submfd of $GL(n, \mathbb{R})$ as $\forall p \in \det^{-1}(1)$
$D\det_p : T_pM \to T_1\mathbb{R}$ is surj
$\Rightarrow 1$ is regular so $\det^{-1}(1)$ is regular

Pf idea Work with coord chart of $p \in M$, U_α, φ_α



Note it suffices to check claim for the transfer map $F_{\alpha,\beta}$

need to show $F_{\alpha,\beta} : \mathcal{U}^{\text{open}} \mathbb{R}^\ell \to \mathcal{V}^{\text{open}} \mathbb{R}^n$

<u>**Wts**</u> $F_{\alpha,\beta}^{-1}(\psi(f(p))$

$$\begin{tikzcd} (DF_{\alpha,\beta})_{\varphi_\alpha(p)} : T_{\varphi_\alpha(p)}\mathbb{R}^\ell \arrow[r] \arrow[d, "\text{isom}"] & T_{\psi(f(p))}\mathbb{R}^n \arrow[d, "\text{isom}"] \\ \mathbb{R}^\ell \arrow[r] & \mathbb{R}^n \end{tikzcd}$$

is surj $\Rightarrow \ell \geq n$

Let

$\overline{G} = (DF_{\alpha,\beta})_{\varphi(p)} \circ B$

$\mathbb{R}^\ell \to \mathbb{R}^n$

$\ker \overline{G} = \mathbb{R}^{\ell-n}$

$(DF_{\alpha,\beta})_{\varphi(p)} : \mathbb{R}^\ell \to \mathbb{R}^n$

$\hookrightarrow \ker (DF_{\alpha,\beta})_{\varphi(p)} \subset \mathbb{R}^\ell$

$\hookrightarrow$ rank nullity

$\Rightarrow \dim \_\_ = n-\ell$

put $\ker(DF_{\alpha,\beta})_{\varphi(p)}$ into $\mathbb{R}^\ell$ as last $\ell-n$ std vectors

$\exists$ lin map $B^{-1}(\ker(DF_{\alpha,\beta})_{\varphi(p)})$

$= \mathbb{R}^{\ell-n} \subseteq \mathbb{R}^\ell$

$B : \mathbb{R}^\ell \to \mathbb{R}^\ell$

Def extended map,

$\overline{\overline{G}} : \mathbb{R}^\ell \to \mathbb{R}^n \times \mathbb{R}^{\ell-n}$

$(x_1, \ldots, x_\ell) \mapsto (\overline{G}_{\alpha,\beta}(x_1, \ldots, x_\ell), x_{\ell-n+1}, \ldots, x_\ell)$

$$D\overline{\overline{G}}_{\varphi_\alpha(p)} = (D\overline{G}_{\varphi(p)}, id_{\mathbb{R}^{\ell-n}})$$

$$= \left(\begin{array}{c|c} * & * \\ \hline & id_{\mathbb{R}^{\ell-n}} \end{array}\right)$$

Use inverse function thm on $\overline{\overline{G}} \Rightarrow \overline{\overline{G}}$ local diffeo

$(\overline{\overline{G}}^{-1})(\psi(f(p))) = \ldots\ldots$ <u>more details</u> D

Appl 2

$SO(2) = \{A \in Mat_{n\times n} \mid AA^T = I, \det A = 1\}$ submfd

Set $A \to AA^T$