# Lec 17 — Derivations

Derivation at $p \in M$ is a map $\partial : C^\infty(M) \to \mathbb{R}$ st
$\partial(f \cdot g) = \partial(f) \cdot g(p) + f(p) \cdot \partial g$

**Lemma)** $v \in T_pM$, $\partial_v$ = direct deriv then $\partial_v$ is a deriv $\to D\rho_p(v)$
moreover if $\partial_v = \partial_w$ then $v = w$ as elts of $T_pM$

take coord chart $(U_p, \rho)$ around $p$ so $\rho(p) = 0$

want $D\rho_p(v) = \frac{\partial}{\partial x_1}$, $D\rho_p(w) = \frac{\partial}{\partial x_2}$ can do unless $w = av$, $a \in \mathbb{R}$

indeed $D\rho_p(v), D\rho_p(w)$ lin ind in $\mathbb{R}^n$ $\to$ get linear map $A$ so these map to $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}$

$\Downarrow$

$\to$ use comp to get $\rho = A \circ \rho$

look at func $x_1 : \rho(U_\alpha) \to \mathbb{R}$

$\alpha : x = (x_1, \dots, x_n) \mapsto x_1$

$\frac{\partial}{\partial x_1}\alpha = 1$, $\frac{\partial}{\partial x_2}\alpha = 0$ $\Rightarrow$ $\frac{\partial}{\partial x_1} \neq \frac{\partial}{\partial x_2}$ are diff

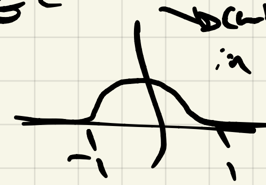
are we done? No. This is local $x_1 \circ \rho$, $x_2 \circ \rho$ only on nbhd of $p$.

consider $C^\infty(M) \xrightarrow{\text{restriction}} C^\infty(U_\alpha)$

bump it to be well def. — 2D comes up a lot

$\in C^\infty(\mathbb{R})$ — part of in CW

on $\mathbb{R}$ we want a function that looks like

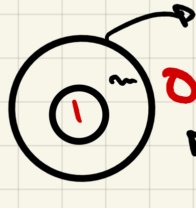


$$\psi(x) = \begin{cases} e^{-\frac{1}{1-x^2}} & x \in (-1,1) \\ 0 & \text{otherwise} \end{cases}$$

on $\mathbb{R}^n$ want 0 outside ball of rad 1

$\bar{\psi}(\vec{x}) = \psi(\|\vec{x}\|^2)$

Really want $\psi : \mathbb{R}^n \to \mathbb{R}$



rad 1

rad 1/2

try $\phi(x) = \bar{\psi}(x)\, \psi\left(\frac{1}{2\|x\|}\right)$

Now we can go from $C^\infty(U_p) \to C^\infty(M)$

So, $\partial_v = \partial_w \iff v = w$ in $T_pM$

(a) $\exists$ $n$ linearly independent derivations!

**Lemma)** derivations of $p$ are an $n$ dimensional vector sp.

**Pf)** a) $\geq n$ dim v sp $\{\partial_v \mid v \in T_pM\}$

$\Rightarrow \dim \geq n$

b) $f \in C^\infty(M) \to$ look locally at $C^\infty(U_p)$

so $f(p) = 0$

$I_p = \{f \in C^\infty \text{ near } p \mid f(p) = 0\} \subseteq C^\infty(\ldots)$

$I_p^2 = \{\sum f_i g_i \mid f_i, g_i \in I_p\}$

if $f, g \in I_p$ $\quad \delta(f \cdot g) = \delta(f)\underbrace{g(p)}_{0} + \underbrace{f(p)}_{0} \cdot \delta(g)$

$= 0$

$\Rightarrow \delta$ vanishes on $I_p^2$ !

deriv $\underset{\text{\ldots}}{\hookrightarrow}$ $(I_p / I_p^2)^*$

Claim = lemma) $\dim (I_p / I_p^{2*}) = n$

**Cor)** derivations at $p = \{\partial_v \mid v \in T_pM\}$

$\hookrightarrow$ dim finite & equal

**lemma)** Suppose $f$ is $C^\infty$ real valued function $f : U \overset{\text{open}}{\ni} 0 \to \mathbb{R}$

$\Rightarrow \exists$ $C^\infty$ func $f_i$ on $U$ (or $\mathbb{R}^n$)

so $f(x) = f(0) + \sum x_i f_i$ $\quad \to f_i(0) = \frac{\partial f}{\partial x_i}(0)$

by ftc $\quad f(x) = f(0) + \int_0^1 \frac{\partial}{\partial t} f(tx)\, dt$

$$= f(0) + \int_0^1 \sum_{i=1}^n x_i \frac{\partial f}{\partial x_i}(tx)\, dt$$

$$= f(0) + \sum_{i=1}^n x_i \underbrace{\int_0^1 \frac{\partial f}{\partial x_i}(tx)\, dt}_{\to f_i}$$

$\longrightarrow$

**Lemma)** $f$ is $C^\infty$ ($U$ near $0$ in $\mathbb{R}^n$) then $\exists$ $C^\infty$ fncs $f_{ij}$ on $\mathbb{R}^n$ (near $0$) so

$$f(x) = f(0) + \sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i}(0) + \sum_{i,j=1}^{n} x_i x_j \underbrace{f_{ij}(x)}_{\hookrightarrow \text{ zero when } i \neq j}$$

↪ apply earlier to $f_i(x)$     ↪ only thing that matters is the term

apparently done

## Dealing with germs of functions

let $p \in M^n$ mfd

spse $f$ is $C^\infty$ function on an open nbhd of $U_f$ of $p$

spse $g$ is $C^\infty$ .... on $U_g$ of $p$.

Say $f, g$ define the same germ if $f|_{U_f \cap U_g} = g|_{U_f \cap U_g}$

**Note|** $f_1, f_2 \rightsquigarrow$ same germ

$g_1, g_2 \rightsquigarrow$ same germ

$\Rightarrow$ $f_1 + g_1$, $f_2 + g_2 \rightsquigarrow$ same germ

$f_1, f_2$ defines an equiv reln of fnc around $p$!

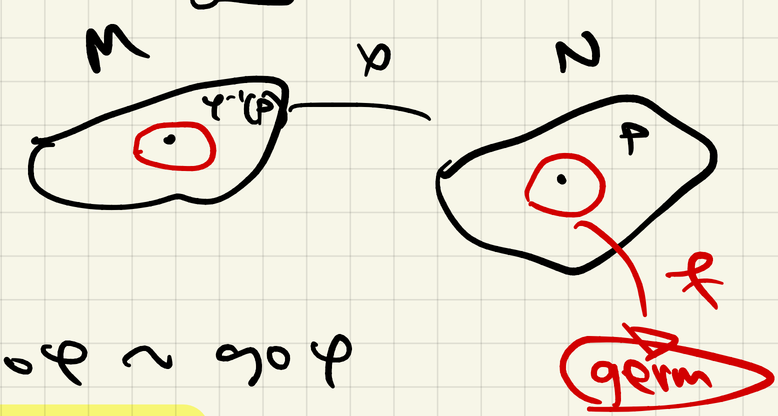
**Note** we have globally $\subseteq$ locally defined functions open nbhd of $p$

$\subseteq$ then we have germs of $p$ (super local)

$\subseteq$ then we have collection of formulas at $p$ (infinitesimal knowledge)

**Note|** $I_p, I_p^2$ make sense for germs.

$\dim I_p^{\text{germ}} / I_{p^2}^{\text{germ}} = n$




**Given** $\varphi: M \to N$ $C^\infty$

if $f$ a germ on $N$

$\Rightarrow$ $f \circ \varphi$ is a germ on $N$

moreover if $f \sim g \Leftrightarrow f \circ \varphi \sim g \circ \varphi$

$\varphi^*: I_{\varphi(p)} / I^2_{\varphi(p)} \longrightarrow I_p / I_p^2$

## Duality

$$T_pM \xrightarrow{D\varphi = \varphi_*} T_{\varphi(p)}M$$

$$I_p/I_p^2 \xleftarrow{\varphi^*} I_{\varphi(p)}/I_{\varphi(p)}^2$$

$$T_pM \xleftrightarrow{\text{dual}} I_p/I_p^2$$
$$\parallel$$
$$T_pM^*$$

$\Rightarrow$ so can identify $T_pM^* = I_p/I_p^2$

dual $\varphi^* : T^*N \longrightarrow T^*M$

$D\varphi : TM \longrightarrow T$