General comment

$M \curvearrowleft \Gamma \to$ discrete $\qquad M/\Gamma$

e.g. $\mathbb{RP}^n = S^n/\mathbb{Z}_2$
$\mathbb{CP}^n = S^{2n-1}/S^1$

If Γ leaves a "structure" on M invariant then it induces the kind of structure on M/Γ
in this case orientability is important!

e.g. Spse M is C^∞ diff & Γ acts by C^∞ diff maps
$\Rightarrow M/\Gamma$ is diff.

e.g. If M has a Riemannian Metric i.e. $\langle , \rangle_p$ on T_pM and Γ acts by isometries $\gamma \in \Gamma$

i.e. $\langle D\gamma_p(v), D\gamma_p(w)\rangle_{\gamma p} = \langle v, w\rangle_{\gamma p}$

$\quad\uparrow \qquad\qquad \uparrow$

$T_{\gamma(p)}M \qquad T_{\gamma(p)}M$

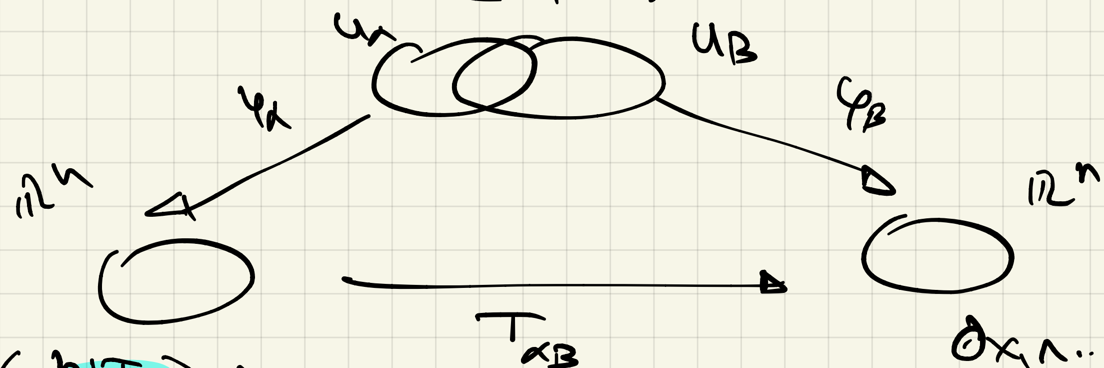
$\Rightarrow M/\Gamma$ inherits a Riemannian Metric

Thm M is smooth then TFAE
a) M is orientable (i.e. $\exists$ non vanishing section of $\Lambda^n M$) $\qquad \Lambda^n(T^*M) \xrightarrow{} \Lambda^n M$, p, $\downarrow M$

b) $\exists$ charts $(U_\alpha, \varphi_\alpha)$ so that the transition maps $T_{\alpha,\beta} : \hat{U} \overset{open}{\subseteq} \mathbb{R}^n \to \hat{V} \overset{open}{\subseteq} \mathbb{R}^n$
has positive det $\to$ i.e. $(DT_{\alpha,\beta})_p$ has pos det

Last time
If we have $\varphi: \mathbb{R}^n \to \mathbb{R}^n$ diffeo $\to$ (or open sets)
$\qquad\quad y_1,\dots,y_n \qquad x_1 \dots x_n$, p
want to know $dx_1 \wedge \dots \wedge dx_n$
wanna know that $D\varphi^*(dx_1 \wedge \dots \wedge dx_n)$
$\to \; = \det D\varphi \; dy_1 \wedge \dots \wedge dy_n$

b => a) we have $(U_\alpha, \varphi_\alpha)$ &



$dx_1 \wedge \dots \wedge dx_n$

$(\det T_{\alpha,B}) dy_1 \wedge \dots dy_n \xleftarrow{T_{\alpha,B}^*}$ (positively valid)

$\vee$
0 by my assn

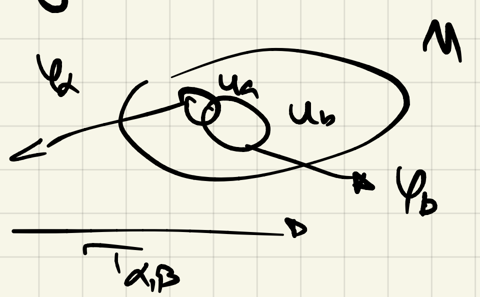
=> $dx_1 \wedge \dots \wedge dx_n$ pull back to positively oriented n-forms on $U_\alpha \cap U_B$

↳ pick partition of unity $\{\eta_i\}$

on U_α get an n-form σ_α

Consider $\sum \eta_\alpha \sigma_\alpha$

a => b) given an orientation



Call φ_α positive if a pull back form on U_α

$\sigma_\alpha = \varphi_\alpha^*(dx_1 \wedge \dots \wedge dx_n)$ is $f \cdot T$ on U_α $f \geq 0$

If all φ_α are positive => get coor charts as in b

$(T_{\alpha,B})^* = \underbrace{\varphi_\alpha^*}\ \underbrace{\varphi_B^*}$ both pos so $T_{\alpha,B}$ is pos!

If not all φ_α positive, flip negative ones.

i.e $\varphi_\alpha' := (-1, \dots, 1) \circ \varphi_\alpha$

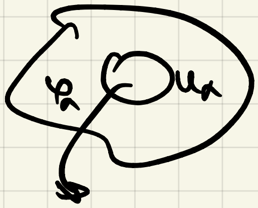
Why bother? if $f: M \to \mathbb{R}$ smooth, compt smooth. Can now def $\int_M f$ ↳ on $\mathbb{R}^n$ use Riemann / Lebesgue integrals ↳ [key: we know volume of cube!]

On a arb mfd → an n-form gives a Parallelpiped.

$$\int_M f \cdot \tau$$

↳ volume form (non-vanishing n form).

How to actually do this?



M

Spse $f : M \to \mathbb{R}$ supp on U_α (0 outside U_α)

want: $\int_M f \cdot \tau = \int_M f \cdot \tau$

Could we take $\int_{V_\alpha} (f \circ \varphi_\alpha)^{-1} \cdot \varphi_\alpha^{*}(\tau)$ ← Similar

$$V_\alpha \supseteq \mathbb{R}^n = \int_{V_\alpha} g_\alpha \cdot dx_1 \wedge \dots \wedge dx_n$$

well defined? Dep on choice of $\{(U_\alpha, \varphi_\alpha)\}$

need change of variables on $\mathbb{R}^n$ $\quad h : \mathbb{R}^n$

$$\int_A h \, dx_1 \dots dx_n = \int_B (h \circ T) \cdot |\mathrm{Jac}| \, dy_1 \dots dy_n$$

↳ $A = T \cdot B$
↳ transform

↳ n forms change by Jac det

↳ doesn't matter how we pull