DeRham

E.g. $M = S^1$ for $n \geq 2$ $H^n_{dR}(S^1) = 0$

$H^0_{dR}(S^1) = \mathbb{R}^{\# \text{ conn comp}} = \mathbb{R}$

$H^1_{dR}(S^1)$? **Recall** we had for $M = \mathbb{R}$ α closed $\Updownarrow$ $\alpha = d\beta$ for some $\beta \in \Omega^0$

So now, $S = [0,1]/0\sim1$

let α be a 1 form $\alpha = f\,dx$ ← makes sense on S^1 as translation inv

Let $\beta(x) = \int_0^x f(t)\,dt$

we need to know $\beta(0) = \beta(1)$ for well def! → also diff arising

wlog (diff const) $\beta(1) = \int_0^1 f(t)\,dt = 0$

Thus α is exact $= d\beta$ iff $\int_0^1 f(t)\,dt = 0$

→ each α exact $\Rightarrow$ $A = \int_0^1 f(t)\,dt$ then $\alpha - A\,dt$ is a closed 1 form

$\Rightarrow \alpha - A\,dt = (f-A)\,dt$ & $\int_0^1 (f-A)(t)\,dt = 0$

$\Rightarrow$ $\alpha - A$ exact!

Thus given α closed can subtract $\int_0^1 \alpha$ & get exact form

$\Rightarrow \beta \in \Omega^0(S^1)$ $\alpha - A\,dt = d\beta$

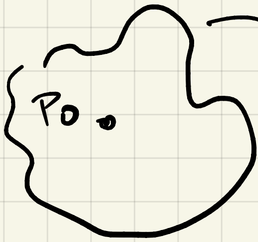
$\Rightarrow [\alpha] = [A \cdot dt] \in \mathbb{R}$

$\Rightarrow H^1(S^1) = \mathbb{R}$

Moral The way you put to gether the charts determines the deRham cohomology

Poincaré for $A \subseteq \mathbb{R}^n$ open, starshaped then any closed k-form is exact!

Motivations in dim 1: $\int_0^x \alpha \, dt = f(x)$ & $df = \alpha$



→ Star-convex

wlog $\boxed{P_0 = 0}$ *

2) idea: integrate radially

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1 M \to \dots \to \Omega^n(M)$$

we want a map!

$$0 \to \Omega^0 M \to \Omega^1 M \to \dots \to \Omega^{k-1} M \to \Omega^k M \to \Omega^{k+1} M \to \dots$$

$$I_{k-1}, \quad I_k, \quad I_{k+1}$$

$$\dots \to \Omega^{k-1} M \to \Omega^k M \to \Omega^{k+1} M$$

want $\partial \circ I + I \circ \partial = \partial_{k-1} \circ I_k + I_{k+1} \partial_k = Id$

consequence: if $\partial \alpha = 0 \Rightarrow \alpha = \partial \circ I \alpha + I \underbrace{\partial \alpha}_{0} = \partial \circ I \alpha \Rightarrow$ exact

defn $I : \Omega^\ell \to \Omega^{\ell-1}$ → smooth fn on A

$$\omega = \sum_{i_1 < \dots < i_\ell} \omega_{i_1, \dots, i_\ell} \, dx_{i_1} \wedge \dots \wedge dx_{i_\ell}$$

set

$$(I\omega)(x) = \sum_{i_1 < i_2 < \dots i_\ell} \sum_{\alpha=1}^{\ell} (-1)^{\alpha-1} \left(\int_0^1 t^{\ell-1} \omega_{i_1, \dots, i_\ell}(tx) \, dt \right) \circ x_{i_\alpha} dx_{i_1} \wedge \dots \wedge \widehat{dx_{i_\alpha}} \wedge \dots \wedge dx_{i_\ell}$$

not there

Pf $\partial \circ I + I \circ \partial = Id$

write it out → diff wrt x_{i_α}

$$\partial I \omega = \ell \sum_{i_1 < \dots < i_\ell} \left(\int_0^1 t^{\ell-1} \omega_{i_1, \dots, i_\ell}(tx) \, dx \right) dx_{i_1} \wedge \dots \wedge dx_{i_\ell}$$

$$+ \sum_{i_1 < \dots < i_\ell} \sum_{\alpha=1}^{\ell} \sum_{j=1}^{n} (-1)^{\alpha-1} \int_0^1 t^\ell \frac{\partial}{\partial x_j} \left(\omega_{i_1 \dots i_\ell}(tx) \right) dt \; x_{i_\alpha} \wedge dx_j \wedge dx_{i_1} \wedge \dots \wedge \widehat{dx_{i_\alpha}} \wedge \dots \wedge dx_{i_\ell}$$

It works