Homogeneous Space. G Lie grp and H closed subgrp. Interested in M looks like G/H

$\Longleftrightarrow$ $G \curvearrowright M$ $H = G_p$ for some $p \in M$

Warning. G_p dep on p Q1 what if not

E.g.1 Grassmannian of k-planes in n-space $(\mathbb{R}^n) \to Gr_{k,n}(\mathbb{R})$

Special case: $Gr_{1,n}(\mathbb{R}) = \mathbb{R}P^{n-1} = \text{homog sp} = GL(n,\mathbb{R}) \Big/ \begin{pmatrix} * & \cdots & * \\ 0 & & \\ \vdots & & \\ 0 & & \end{pmatrix}$

0 in top bottom to identify antipod

$= SO(n) / O(n-1)$

$= \left(SO(n) / SO(n-1) \right) \Big/ \mathbb{Z}_2$

general | If $e_1 \ldots e_n$ basis

$\langle e_1, \ldots, e_k \rangle \in Gr_{k,n}(\mathbb{R})$

$GL(n,\mathbb{R}) \curvearrowright Gr_{k,n}(\mathbb{R})$

$\quad\;\; \| \qquad\qquad\qquad \|$

$\quad\;\; A \qquad\qquad\;\; V$ - k dim subsp

$A\langle e_1, \ldots e_k \rangle = \langle v_1, \ldots v_k \rangle$

New subsp $AV = \{ A \cdot v \mid v \in V \}$

ext to basis.

transitive? $\langle v_1, \ldots v_k \rangle \to A = (v_1, \ldots, v_k, \ldots)$

$Gr_{k,n}(\mathbb{R}) = GL(n,\mathbb{R}) / G_p$ what is the isotropy grp

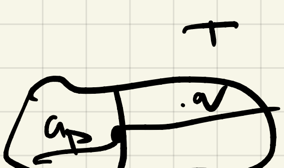
$A \in G_p$ if $A \cdot V = V$ ie $Ae_j \in V$

$$A = \begin{array}{c|cc} & k & n-k \\ \hline k & * & * \\ n-k & 0 & * \end{array}$$

If $Gr_{k,n}(\mathbb{R})$ find transversal make each 0 an a_i

Then $\dim Gr_{k,n}(\mathbb{R}) = (n-k)k$ $\Rightarrow$ diagonal is too

In general



T is transversal $T \to M$

$\longrightarrow tp$

$q = g_0 T$

Then $G_q = g_0 G_p^{-1} g_0$ PJ hw

$T_q \longrightarrow T_{g_0 p} = g_0 T_p g_0^{-1}$

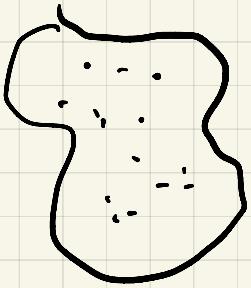
Properly Discontinuous Example

e.g. $S^1 = \mathbb{R}/\mathbb{Z}$, $T^n = S^1 \times \cdots \times S^1 = \mathbb{R}^n/\mathbb{Z}^n$

Bad eg | $\mathbb{Z} \curvearrowright S^1$ by irrational rotation wrt 2π n times → not Hausdorff

careful with disc grp → orbit pts have limit points

Γ discrete group (ctbl), $\Gamma \curvearrowright \tilde{M}$ (top mfd)



Properly disc → $\Gamma . x$ is separate

$\forall K \subseteq \tilde{M}$ compact!

1) $\forall p \in \tilde{M}$ $\Gamma \cdot (p \cap K)$ is finite

$\Gamma \times \tilde{M} \to \tilde{M}$ is proper (preimage of cmpct is cmpct)

Γ acts freely ← 2) $p \in \tilde{M}$, $\gamma \in \Gamma$ if $\gamma \in p = p \Rightarrow \gamma = 1$

↳ no fixed pts

nw | 1) if $\Gamma \curvearrowright \tilde{M}$ prop disc then $M = \tilde{M}/\Gamma$ is a top mfd.

↳ Hausdorffness should work

↳ for locally euc should be able to

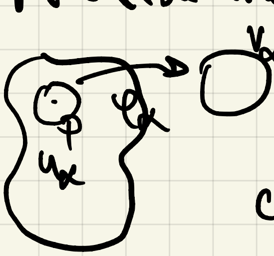
M

2) If $\tilde{M}$ C^k diffble $\Gamma \curvearrowright \tilde{M}$ by C^k maps, then M is C^k diffble

e.g. $SL(n,\mathbb{R})/SL(n,\mathbb{Z})$ → space of lattices in $\mathbb{R}^n$ vol 1

Tangent Vectors

M diffble mfd.



$V_\alpha \subset \mathbb{R}^n$ → Think of tangent vector to $\varphi_\alpha(p)$

what about competing coord chart?

$\varphi_\beta : U_\beta \to V_\beta$

Consider transition map

$\varphi_\beta(p) \xrightarrow{T_{\beta\alpha}} \varphi_\alpha(p)$

Def | equiv reln $\sim$ between tangent vec @ $V_\alpha(p)$, $V_\beta(p)$

$v \sim w$ if $\partial T_{\alpha,\beta}(v) = w$

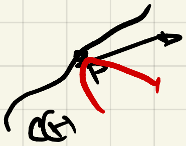
Def] $T_pM = \{[v] \mid [v]$ equiv cls of tangent vec$\}$

Nice interpretation:

$\to$ diffble

in $\mathbb{R}^n$, $p \in \mathbb{R}^n$ $\quad v \in T_pM \quad c(t) \quad t \in (-\varepsilon, \varepsilon)$

$c(0) = p$



$c'(t) \to c'(0)$

can talk about 2 diffble curves through p c_1, c_2 as equiv if $c_1'(0) = c_2'(0)$

M diff mfd $\quad p \in M$

we know what $c(t)$ being diffble means

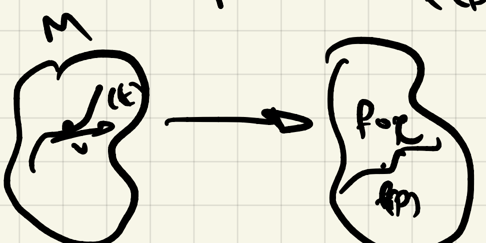
$[c]_p \to$ "deriv of c at p", "$c'(0)$"

$(-\varepsilon, \varepsilon)$

Def] $f: M \to N$ diff mfd

f diffble at $p \in M$ if f is diffble in charts

$df: T_pM \to T_{f(p)}M$



$[f \circ c]_{f(p)} \in T_pM$