Q What are the homomorphisms $\varphi : S^1 \to \mathbb{R}$?

Note: $\varphi(S^1)$ cmpct in $\mathbb{R}$ => bdd.

but for $a \in \varphi(S^1)$ => $an \in \varphi(S^1)$ $\forall n \in \mathbb{N}$

=> $a = 0$ so φ is trivial.

But

$$\begin{array}{ccc} \mathrm{Lie}\, S^1 & & \mathrm{Lie}\, \mathbb{R} \\ \| & & \| \\ \mathbb{R} & \xrightarrow[\Phi = \mathrm{id}]{} & \mathbb{R} \end{array}$$

So what homo from $S^1 \to \mathbb{R}$ corr to ?

DNE for this, thus statement from last time false.

But almost true. Spse $\mathfrak{g}_1 \xrightarrow{\phi} \mathfrak{g}_2$ fin dim Lie alg

Then,

$$\begin{array}{ccc} & \tilde{G}_1 & \\ \swarrow \pi & & \searrow \\ G_1 & & G_2 \\ \uparrow & & \uparrow \\ \mathrm{Lie}\, G_1 = \mathfrak{g}_1 & \xrightarrow{\phi} & \mathrm{Lie}\, G_2 = \mathfrak{g}_2 \end{array}$$

$\tilde{G}_1$ → univ cover of G_1

→ get

smooth mfd

Gen top) M reasonable sp (manifold, haus) => sp $\tilde{M}$ smooth mfd, π smooth

$$\begin{array}{c} \tilde{M} \\ \downarrow \pi \\ M \end{array}$$ → surjective

$\tilde{M}$ has prop $\pi_1(\tilde{M}) = 1$ $\{f : S_1 \to \tilde{M} \mid f(1) = p\}/\sim$ $p \in \tilde{M}$

cts

$f \sim g$ if $\exists$ $F : S^1 \times [0,1] \to \tilde{M}$ cts

s.t. $f = F|_{S^1 \times \{0\}}$, $g = F|_{S^1 \times \{1\}}$

$F(1,t) = p$ $\forall t \in [0,1]$!

Fact G Lie => $\tilde{G}$ Lie grp

Can do locally. π locally diffeo.

idea of pf of) $G_1 \times G_2$ $\quad \mathfrak{g}_1 \xrightarrow{\phi} \mathfrak{g}_2$ $\quad \mathcal{L} = \text{graph } \phi$

$\exists$ H subgp of $G_1 \times G_2$ corr $\mathcal{L}$.

$\dim H = \dim \mathfrak{h} = $ since graph $\dim \mathfrak{g}_1$

$\mathfrak{h} \cong T_1 H$

$$\begin{tikzcd}
H \arrow[dr, "\pi_1|_H"] & & G_1 \times G_2 \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
& G_1 & & G_2
\end{tikzcd}$$

Claim $\pi_1|_H$ is a local diffeo since $\dim H = \dim G_1$ &

Pf) $D\pi_1|_H = \pi_1|_H = \mathrm{id}|$ or smth

is $\pi_1|_H$ inj? no! $H \cap \pi_1^{-1}(1)$ could be disc.

Spse) G Lie grp. M smooth mfd.

Spse $G \curvearrowright M$ smoothly. let $\mathfrak{g}$ Lie alg.

Thm $\mathfrak{g} \xrightarrow[\text{Lie Alg hom}]{} \{\text{smooth vf on } M\}$

($\Longrightarrow$)

Pf) $X \in \mathfrak{g}$ i.e X is left inv v.f. on G

$g \cdot p =: E(g,p) =: E_p(g)$

$p \in M$

kinda want to show [illegible] smoothly

vec flo on orbit

push forward the vec flo X to M using E_p

Different pov) $X \longleftrightarrow$ one param subgrp $\exp(t \cdot X)$ $\subset g_t$

$\frac{\partial}{\partial t}\big|_{t=0} g_t \cdot p$.

$q = g_{t_0} \cdot p$ $\quad X(q) = \frac{\partial}{\partial t}\big|_{t=0} (g_t \cdot q)$

$= \frac{\partial}{\partial t}\big|_{t=0} g_t \cdot g_{t_0} \cdot p$

$= \frac{\partial}{\partial t}\big|_{t=0} g_{t+t_0} \cdot p$ $\longrightarrow$ push fw of vectors

converse holds too if $\mathfrak{g}$ fin dim Lie

& $\mathfrak{g} \longrightarrow$ smooth v.f. on M $\exists$ smooth local act of $G \curvearrowright M$.

Partitions of unity (compact case)

M is compact $\{U_\alpha\}_{\alpha\in A}$ open cover of M.

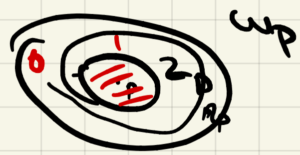
Then $\exists$ fin subcover $\{V_\beta\}_{\beta\in\mathbb{N}_\ell}$ $\forall \beta \ \exists \alpha$ so $V_\beta \subseteq U_\alpha$

& smooth function $\varphi_\beta : V_\beta \to [0,1]$

s.t. $\forall x\in M$ $\sum_\beta \varphi_\beta(x) = 1$

Pf) $\forall p\in M$, find W_p & smooth bump func. $\psi_p : W_p \to [0,1]$

$\psi_p \equiv 1$ on a nbhd Z_p, $p\in Z_p \subseteq W_p$



& $\exists R_p$ so $Z_p \subset R_p \subset W_p$ with

ψ_p is 0 outside R_p!

WLOG $W_p \subset V_\beta$ some.

As M is compact $W_{p_1}, \dots, W_{p_\ell}$ cover M

$$\left\{ \frac{\psi_{p_i}}{\sum_{i=0}^{\ell} \psi_{p_i}} \right\}$$