

Thm $f: M \rightarrow N$ (' (with both M, N at least C^1) \hat{q} reg value
i.e. $Hpt f^{-1}(\hat{q})$, Df_p surj.
Reg val \hat{q} reg val then $f^{-1}(\hat{q})$ is a submanif of M

Prf $\dim M = n$, $\dim N = k$

1) Use coord charts to reduce to prob abt func

$$F: U \rightarrow V$$

\uparrow open \uparrow open
 \mathbb{R}^n \mathbb{R}^k

2) $F: U \rightarrow V$, then $\hat{q} = \gamma_p(q)$ is a reg val of F
where γ_p is a coord chart for V

3) (if $\hat{p} \in F^{-1}(\hat{q}) \subseteq U \subseteq \mathbb{R}^n$) $Df_{\hat{p}}$ is surjective

if $F = (F_1, \dots, F_n)$ $Df_{\hat{p}} = \left(\frac{\partial F_i(\hat{q})}{\partial x_j} \right)_{\substack{i \in \mathbb{N}_n \\ j \in \mathbb{N}_k}} \xrightarrow{k \times n$

$\Leftrightarrow Df_{\hat{p}}$ surj \Leftrightarrow rank $k \Leftrightarrow$ k lin ind col vect

whole \rightarrow assume first k (coord form)

4) Somehow apply IFT

want $\mathbb{R}^n \xrightarrow{G} \mathbb{R}^k \times \mathbb{R}^{n-k}$ extend $F \rightarrow G$

$$G(x_1, \dots, x_n) = (F(x_1, \dots, x_n), x_{k+1}, \dots, x_n)$$

$$DG_{\hat{p}} = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} \hat{p} & \frac{\partial F_1}{\partial x_2} \hat{p} & \dots & \frac{\partial F_1}{\partial x_n} \hat{p} \\ \vdots & \ddots & & \end{pmatrix}_{\hat{p}} \xrightarrow{*} \begin{pmatrix} \frac{\partial F_k}{\partial x_1} \hat{p} & \dots & \frac{\partial F_k}{\partial x_n} \hat{p} \\ 0 & \ddots & \xrightarrow{\text{Id}_{n-k}} \end{pmatrix} \Rightarrow DG_{\hat{p}} \text{ inv at } \hat{p}$$

$\Rightarrow G$ is a local diff

$$4) F^{-1}(\hat{q}) = G^{-1}(q_1, \dots, q_k, *) \Rightarrow \dim f^{-1}(\hat{q}) = n-k$$

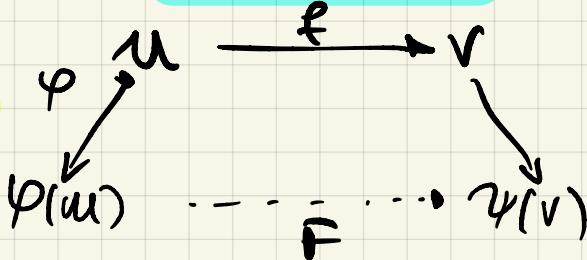
\Leftrightarrow locally submanif of $U \Rightarrow f(F^{-1}(\hat{q})) \rightarrow$

e.g. Canonical Submersion

$$\mathbb{R}^n \longrightarrow \mathbb{R}^k$$

$$(x_1, \dots, x_n) \mapsto (x_1, \dots, x_k)$$

Then) if $f: M \rightarrow N$ C' and p a reg value and $q = f^{-1}(p)$
 Then \exists coord charts $(U, \varphi), (V, \psi)$ around $p \in q$



F is canonical submersion
 $\pi_{n,k}$
Nice!

Engineering of MT

- (a) Uses contraction mapping then \rightarrow contractions in complete metric sp \rightarrow fixed pt
- (b) used in existence and uniqueness of ODE.

Application $SO(n)$ is a ^{sub}manif of $O(n, \mathbb{R})$ open \mathbb{R}^{n^2}

How to prove

Defining $A \in O(n)$ $A \cdot A^T = I$

$$F: GL_n(\mathbb{R}) \rightarrow \text{Symm } (\mathbb{R}^n)$$

$$A \mapsto A^T A$$

$$O(n) = F^{-1}(I)$$

\Rightarrow suffices to show I reg val

Calculate, Df_g for $g \in F^{-1}(I) = GL_n(\mathbb{R}) \Rightarrow g \cdot g^T = I$
 $\forall v \in \mathbb{R}^{n^2} \rightarrow$ use curve $Df_{g+t}(v)$

$$F(g+tv) = (g + tv)(g + tv)^T = \underbrace{gg^T}_{I} + \underbrace{gtv^T + tvg^T}_{\cancel{t^2 \text{ term vanishes}}} + \cancel{t^2 vvv^T}$$

$$= I + gtv^T + vgt^T$$

$$\Rightarrow Df_g = gtv^T + vgt^T \rightarrow \text{Claim this is onto to Symm } (n \times n)$$

A calculate $\ker(Df_g) = \{v | gtv^T + vgt^T = 0\}$
 $\Rightarrow gtv^T = -vgt^T \Rightarrow vgt^T$ is skew symm

$$\Rightarrow \dim \ker Df_g = \frac{n(n-1)}{2} \rightarrow \dim \ker Df_g = \frac{n(n+1)}{2}$$

Cor | $SO(n) \subseteq O(n)$

$\{g \in O(n) \mid \det g = 1\}$

Note: $\det: O(n) \rightarrow \mathbb{R} \rightarrow$ actually only $\{1, -1\}$

$(\Rightarrow SO(n)$ is just a connected comp of I_d in $O(n)$)

~~=> symmetric~~

\hookrightarrow requires some

~~$SO(n)$ is connected~~

Lemma) L norm \Rightarrow any open subset is ~~neither~~ \hookrightarrow clopen