Thm) (HW)

if M is a smooth mfd (paracompact) $\exists$ smooth Riemannian Metric (in fact many)

Pf) glue local ones together via partitions of unity.

Thm) Not every sm mfd has a Lorentz metric (integer)

i.e $\forall$ smooth fields X, Y on M

$$p \to \langle X(p), Y(p) \rangle_p \text{ is smooth } \curvearrowright \text{ smoothness of IP.}$$

want this to have signature $(n-1, 1)$

Thm) S^2 and any closed or surface of genus > 1 doesn't have Lorentz metric!

$\to T^2 = \mathbb{R}^2/\mathbb{Z}^2$ does as $x_1^2 - x_2^2$ on $\mathbb{R}^2$ is invariant under $\mathbb{Z}^2 \to$ thus $\langle\, \rangle$ descends to T^2.

$SL(2,\mathbb{R}) \to$ left inv $\langle\,,\rangle$ on $SL(2,\mathbb{R})$ (in fact bi-inv)

if $\mathfrak{g} \in X, Y$ where $\mathfrak{g} = \mathrm{Lie}\, SL(2,\mathbb{R}) = T_1 SL(2,\mathbb{R})$

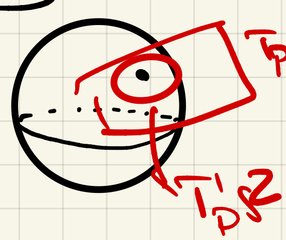
$$\langle X, Y \rangle_1 := \mathrm{tr}(\underset{\mathfrak{g}}{z} \to [X,[Y,z]])$$

Basic idea consider non-vanishing v.f on S^2 & then use fact that S^2 has no non-van v.f.

$\to$ close enough to construct a 1-dim'l distribution!

bad idea) look a $\{v \mid \langle v,v\rangle = 0\}$ but not a vec sp.

better)



T_pM have euclidean metric

look at unit sphere

$T_p^1 S^2$

$\to$ unit circle in T_pM wrt std euclian metric.

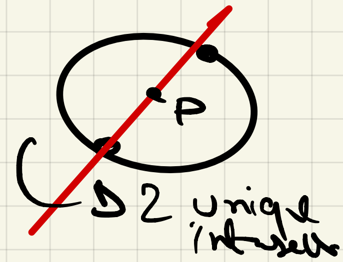
We assume $\langle\,,\rangle_p$ is a lorentz metric

$$\langle v,v\rangle_p \big|_{T_p^1 S^2} \quad \text{if } v_1, v_2 \text{ onb for lorentz metric} \to [1,1]$$

$$\Rightarrow \text{ for } \langle av_1 + bv_2, av_1 + bv_2 \rangle = a^2 - b^2 \to$$

Where is $\langle \, , \rangle_p |_{T_p S^2}$ maximum?

well where span v_1 intersects S^1



Aside on Lie Grp

fin dim.

Let G a Lie Grp and $T_1 G = \mathfrak{g}$ Lie Alg for $X, Y \in \mathfrak{g}$

define $\mathrm{ad}\, X : \mathfrak{g} \to \mathfrak{g}$

$Z \longmapsto [X, Z]$ → this is linear since $[\, ,]$ is bilinear

Thm

$B(X,Y) = \mathrm{tr}(\underbrace{\mathrm{ad}\,X \circ \mathrm{ad}\,Y}_{\text{linear}})$ → Cartan Killing form on G.

indep on basis

E.x $X, Y \in \mathfrak{gl}(n, \mathbb{R}) = \text{Lie } GL(n, \mathbb{R})$

$B(X,Y) = \ ? \ \mathrm{tr}(X \cdot Y)$

→ const to do with dim

→ B is non degen for $\mathfrak{sl}(n, \mathbb{R}) = \text{Lie } SL(n, \mathbb{R})$

→ rel to prev idea!

Defn) $\mathfrak{g}$ is semi-simple if $B : \mathfrak{g} \to \mathfrak{g}$ is non degen

$\Longrightarrow \mathfrak{sl}(n, \mathbb{R})$ is semi-simple!

Nos: if $\mathfrak{g}$ has center i.e. $\exists z \in \mathfrak{g}$

if $[x, z] = 0 \ \forall x \in \mathfrak{g} \Longrightarrow B$ is degen!

as $B(z, x) = \underline{0}$! z kills everything?.

Fact: G is compact no center $\Longrightarrow B$ is pos def!

& non degen!