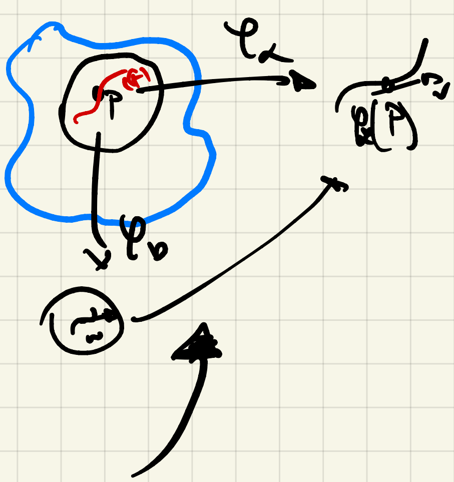
Lec 9

let $p \in M$ diff mfd. We want T_pM



Formally,

Def| $T_pM = \{[U_\alpha, \varphi_\alpha, v \in T_{\varphi_\alpha(p)}\mathbb{R}^n]\}$

equiv class of

$(U_\alpha, \varphi_\alpha, v) \sim (U_\beta, \varphi_\beta, w)$

if $DT_{\beta\alpha}(\varphi_\beta(p))(w) = (\partial T_{\alpha,\beta})_{(\varphi_\beta(p))}(w) = v$

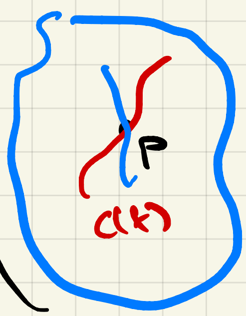
where $T_{\beta\alpha} = \varphi_\alpha \circ \varphi_\beta^{-1}$ (where def)

Def| Equiv Imgs

Call $c \sim d$

if "$c'(0) = d'(0)$"

ie for one (hence every) chart in the atlas $(U_\alpha, \varphi_\alpha)$



$c(t): (-\varepsilon, \varepsilon) \to M$

$c(0) = p$ c diff at 0

$d(t): (-\delta, \delta) \to M$

$d(0) = p$ c diff at 0

$\varphi_\alpha \circ c : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$

$\varphi_\alpha \circ d : (-\delta, \delta) \to \mathbb{R}^n$

$\Big\}\ (\varphi_\alpha \circ c)'(0) = (\varphi_\alpha \circ d)'(0)$

$$T_pM = \{[c] \mid c \text{ diffble at } 0,\ c:(-\varepsilon,\varepsilon) \to M,\ c(0) = p\}$$

Claim| T_pM has a vector space structure

$A, B \in T_pM$ if U_α, φ_α coord chart,

$A \leftrightarrow v \in T_{\varphi_\alpha(p)}\mathbb{R}^n$ $\quad B \leftrightarrow w \in T_{\varphi_\alpha(p)}\mathbb{R}^n$

$A + B := v + w \in T_{\varphi_\alpha(p)}\mathbb{R}^n$

competing coord chart, $\quad A \leftrightarrow v' \in T_{\varphi_\beta(p)}\mathbb{R}^n$

$[v' + w'] = [v + w]$? $\quad B \leftrightarrow w' \in T_{\varphi_\beta(p)}\mathbb{R}^n$

$$(\partial T_{\beta\alpha})_{\varphi_\beta(p)}(v' + w') = (\partial T_{\beta\alpha})_{\varphi_\beta(p)}(v') + (\partial T_{\beta\alpha})_{\varphi_\beta(p)}(w') = \underline{v + w}$$

Suppose M, N diffble. $p \in M$.

$f: M \to N$ diffble at p (by coord charts)

Def) $Df_p = D_p f = (\cancel{df_p} \; \cancel{d_p f}) = (f_*)_p$

$$Df_p : T_pM \longrightarrow T_pN$$

in terms of curves $c: (-\varepsilon, \varepsilon) \to M$ $\quad c(0) = p$ diff @ 0

then $f \circ c : (-\varepsilon, \varepsilon) \to N$ $\quad c(0) = f(p)$ diff @ 0

$Df_p : [c] \longrightarrow [f \circ c]$ $\to$ good conceptually
not for calculation

in terms of charts,

• p $(U_\alpha, \varphi_\alpha)$ chart $\quad A \in T_pM \leftrightarrow v \in T_{\varphi_\alpha(p)}\mathbb{R}^n$

$$Df_p(A) \longmapsto [\psi_\gamma, V_\gamma, D_{\varphi_\alpha(p)}(\psi_\gamma \circ f \circ \varphi_\alpha^{-1})(v)]$$

$$\begin{tikzcd} M \supseteq U_\alpha \ni p \arrow[r, "f"] \arrow[d, "\varphi_\alpha"'] & f(p) \in V_\gamma \arrow[d, "\psi_\gamma"] \\ \mathbb{R}^n \arrow[r, "\psi_\gamma \circ f \circ \varphi_\alpha^{-1}"'] & \mathbb{R}^\ell \end{tikzcd}$$

Thm) $f: U^{\subseteq \mathbb{R}^n} \to V^{\subset \mathbb{R}^\ell}$ diff at p

if Df_p is surjective.

$f^{-1}(p)$ is a manifold.