Cor] Submersions are open maps

Pf] local normal form

Ex of submersions]

1) F, M mfds
$M \times F \to M$
$(m, f) \to f$ } trivial eg
$\hookrightarrow \mathbb{R}^n \xrightarrow{\pi_k} \mathbb{R}^k$ $\quad k \le n$

Defn A submersion $f: M \to N$ a fibre bundle with fibre F iff f is surjective and $\exists$ covering of N $\{V_\alpha\}$ st. $\forall \alpha \in I$,
$f^{-1}(V_\alpha) \underset{\varphi_\alpha}{\overset{\text{diffeo}}{=}} V_\alpha \times F$ $\to$ F is a manifold

$\hookrightarrow$
$$\begin{tikzcd} f^{-1}(V_\alpha) \arrow[rr, "\varphi"] \arrow[dr, "f"'] & & V_\alpha \times F \arrow[dl, "\mathrm{proj}_{V_\alpha}"] \\ & N & \end{tikzcd}$$
} Commutes

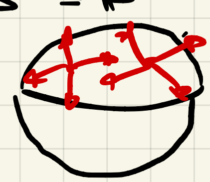
e.g.] 1) $N \times F \to N$ projection $\to$ trivial fibre bundle with fibre F and base N

2) Möbius Band. $F = (-1,1)$, $M =$ Mobius, $N = S^1$

Note: local q as Mobius not diffeo to $S^1 \times (-1,1)$

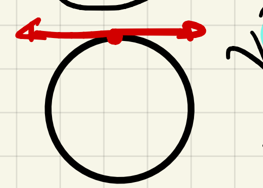
3) $N \subseteq \mathbb{R}^L$ embedded submanif.

$S(N) = \{ v \in T_p \mid p \in T_pN, \ \|v\| = 1 \} \to$ $\quad S^2 \subseteq \mathbb{R}^3$



Fact $S(S^2)$ is not trivial fibre bundle

HW $S(S^3)$ is a trivial fibre bundle
$\hookrightarrow$ its a grp (unit quaterions)



2 tang vect

$S(S^1) = S^1 \times \{\pm 1\}$

Def A vector bundle of a diff mfd Z.

$\Leftrightarrow$ It is a fibre bundle $V \xrightarrow{\pi} Z$ $\longrightarrow$ V mfd

$\rightarrow$ with fibre $= V$, a vector sp

satisfy | $\pi^{-1}(z_0)$ has struct of vect sp

s.t. local trivializations $\longrightarrow$ ie covering of Z by open sets $\{Z_\alpha\}$

$\varphi_\alpha|_{\pi^{-1}(z_0)} \underset{\text{linear iso}}{=} \{z_0\} \times V$

$$\begin{tikzcd} \pi^{-1}(Z_\alpha) \arrow[rr, "\varphi_\alpha", "\cong"'] \arrow[dr, "\pi"'] & & Z_\alpha \times V \arrow[dl, "\text{proj}_{Z_\alpha}"] \\ & Z_\alpha \arrow[u, phantom, "\circlearrowleft"] & \end{tikzcd}$$

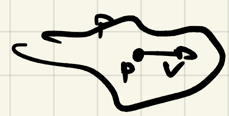
$$\begin{tikzcd} V \arrow[d, "\pi"] & \pi^{-1}(Z_\alpha) \arrow[l, phantom, "\supset"] \arrow[d, "\pi"] & \pi^{-1}(z_0) \arrow[l, phantom, "\supset"] \arrow[d] \arrow[r, bend left, "\varphi_\alpha"] \arrow[dr, bend right, "\varphi_\alpha|_{\pi^{-1}(z_0)}"'] & Z_\alpha \times V \\ Z & Z_\alpha \arrow[l, phantom, "\supset"] & z_0 \arrow[l, phantom, "\ni"] & \{z_0\} \times V \end{tikzcd}$$

Def | let M diffble mfd. The tangent bundle of M

$$TM = \bigsqcup_{p \in M} T_pM$$

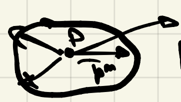
Claim: this is a fibre bundle / vector bundle

map $TM \xrightarrow{\pi} M$
$v \longmapsto \pi(v)$ $\longrightarrow$ $p = \pi(v)$ $\rightarrow$ footpt of v



$V = \mathbb{R}^n$ $\qquad n = \dim M$

e.g. | 1) $M \underset{\text{emb submfd}}{\subseteq} \mathbb{R}^l$



$TM = \{v$ based tangent vector in $\mathbb{R}^l$ at $p \in M$, tangent to $M\}$

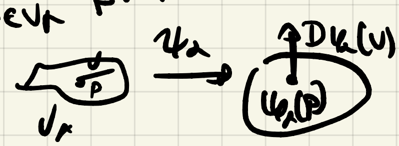
2) M is a general diff mfd data collection of open sets for M $\rightarrow$ just take same atlas

$V_\alpha \xrightarrow{\psi_\alpha} W_\alpha \underset{\text{open}}{\subset} \mathbb{R}^n$ $\longrightarrow$ chart

cover $\leftarrow \{(V_\alpha, \psi_\alpha)\}$

$\pi^{-1}(V_\alpha) = \bigsqcup_{p \in V_\alpha} T_pM$ $\qquad \pi : TM \longrightarrow M$

$D\psi_\alpha \downarrow$
$W_\alpha \times \mathbb{R}^n$



$D\psi_\alpha|_{T_pM} : T_pM \longrightarrow \mathbb{R}^n \cong T_{\psi_\alpha(p)}\mathbb{R}^n$

is trivial FB