Lec 7

G group $\quad G \curvearrowright M$ → want to understand M

Def Call action transitive if M is one G-orbit

i.e. $Gp = \{gp \mid g \in G\} = M$

e.g. $\mathbb{R}^{n+1} \supseteq S^n \curvearrowleft SO(n+1)$

Def $G \curvearrowright M$ $\quad p \in M$. $\quad G_p$ stabilizer p in G (isotropy group of p in G)

$= \{g \in G \mid gp = p\}$

Lemma this is a closed subgrp of G → topology

$G \times M \to M$ is cts
$(g, p) \mapsto gp$

$g_n \to g$, $g_n \in G_p$, $\quad g \in G_p$?

$gp = \lim_{n\to\infty} g_n p = \lim_{n\to\infty} p = p$ □

Future G Lie group, $H \leq G$ closed subgrp then H Lie Group!

e.g. $SO(n+1) \curvearrowright S^n$

$$SO(n+1)_{\left(\begin{smallmatrix}1\\ \vdots\\ 0\end{smallmatrix}\right)} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & * \end{pmatrix} \;\middle|\; * \in SO(n) \right\}$$

$$S^n = SO(n+1)/SO(n) = G/G_p$$

Lemma $G \curvearrowright M \ni p$ & G_p $\quad G/G_p \cong M$ (homeo)
(transitive)

top orbit stabilizer thm

Pf
$G \xrightarrow{f} M$ surj as transitive
$g \mapsto gp$

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d] & M \\ G/G_p \arrow[ur, dashed, "F"] & \end{tikzcd}$$

for $x \in G_p$ $\quad f(g \cdot x) = g \cdot x \cdot p = f(g)$
$= gp$

$F(g_1 G_p) = F(g_2 G_p)$
$\Rightarrow g_1 p = g_2 p$
$= g_2^{-1} g_1 p = p \Rightarrow g_2^{-1} g_1 \in G_p$

homeo guaranteed if G/G_p compact

ex | $GL(2,\mathbb{R}) \curvearrowright \mathbb{R}^2$

equiv for any vs single A

$(A,x) \mapsto A\cdot x$ Not transitive $A\cdot 0 = 0$ ⟶ trivial kernel

transitive $\mathbb{R}^2 - \{0\}$

what is

$$P = GL(2,\mathbb{R})_{\binom{1}{0}} = \left\{ \begin{pmatrix} 1 & c \\ 0 & d \end{pmatrix} \;\middle|\; c\in\mathbb{R},\ d\in\mathbb{R}\setminus\{0\} \right\}$$

$$GL(2,\mathbb{R})/P = \mathbb{R}^2\setminus\{0\}$$

e.g. | $\mathbb{RP}^n = S^n/\sim \; = \{\mathbb{R}\text{-lines through } 0 \text{ in } \mathbb{R}^{n+1}\}$

$GL(n+1,\mathbb{R}) \curvearrowright \mathbb{R}^{n+1}$ trans on $\mathbb{R}^{n+1}/\{0\}$

$\implies GL(n+1,\mathbb{R})$ transitive on lines

$GL(n+1,\mathbb{R}) \curvearrowright \mathbb{RP}^n$ transitively

$\ell_1 = \mathbb{R}\cdot e_1 = \mathbb{R}\cdot \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ ⟶ $Ae_1 = re_1$, $r\in\mathbb{R}\setminus\{0\}$

$G_{\ell_1} = GL(n+1,\mathbb{R})_{\ell_1}$

So $A\in G_{\ell_1} \implies \begin{pmatrix} a_1 & \\ \vdots & * \\ a_{n+1} & \end{pmatrix} = \begin{pmatrix} a_1 \\ \vdots \\ a_{n+1} \end{pmatrix}$ $a_1 = r$, $a_i = 0$ $\forall\, 2\le i\le n+1$

$$\implies G_{\ell_1} = \left\{ A\in GL(n+1,\mathbb{R}) \;\middle|\; A = \begin{pmatrix} r & x_2 \cdots x_{n+1} \\ 0 & \\ \vdots & \boxed{A_1} \\ 0 & \end{pmatrix} \right\}$$

ok (laplace)

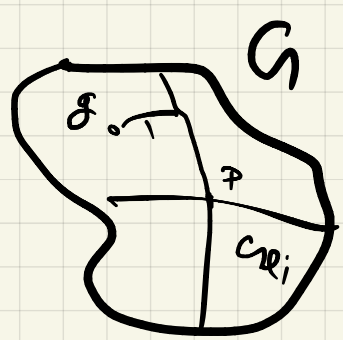
osc with $GL(n,\mathbb{R})$ $\det A_1 = r \det A$

$GL(n+1,\mathbb{R}) \underset{\text{open}}{\subseteq} \mathbb{R}^{(n+1)^2}$ ⟶ also $G/G_{\ell_1} \cong \mathbb{RP}^n$

Goal: | WTS G/G_{ℓ_1} is a diff mfd.

want $T \subseteq G$ "transversal" to G_{ℓ_1}

"dim" of T = $\mathbb{RP}^n$



in our case first colums zero elts!

$$T = \left\{ \begin{pmatrix} 1 & 0 \cdots 0 \\ a_1 & \\ \vdots & 1_n \\ a_n & \end{pmatrix} \;\middle|\; a_i\in\mathbb{R} \right\} \cong \mathbb{R}^n$$ use as chart

$T\ell_1 = \left\{ \begin{pmatrix} 1 & 0 & 0 \cdots 0 \\ a_1 & & \\ \vdots & & \mathrm{Id} \\ a_n & & \end{pmatrix} \right\} \cdot \ell_1 \cong \mathbb{R}^n$ since one to one!

claim

Recipe

① $G \to M$ transitive (Lie grp)

Endow M with diff struct ($C^1, C^r, \ldots C^\infty$)

$p \in M$ & take G_p = stab of p

② $G_p \subseteq G$ closed if you can find a "transversal subspace" of G

maybe locally $\mathbb{R}^n$

③ Try out coord chart of the form

$T \mapsto T_p$

$t \mapsto t_p$

Future Fact

If G is Lie. H closed subgrp. G/H is always C^∞ mfd