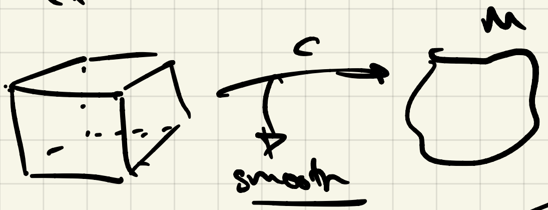
Singular Cubes

Review: Monday 4, Tues 4

Let M a mfd. Let $I^n = [0,1]^n$ be a cube

Let



Such a $c \to$ singular k-cube

want to consider

C_k formal sums of singular k-cubes

$\left\{ \sum_{i=1}^{m} a_i c_i \;\middle|\; a_i \in \mathbb{R},\ m \in \mathbb{N} \right\}$ c_i k-cubes $\to$ sing k

singular k-chains

$C_k =:$ free $\mathbb{R}$-module w/ basis $\{c \mid c\ k\text{-cube}\} =: \mathcal{C}^k$

∞ dim v sp

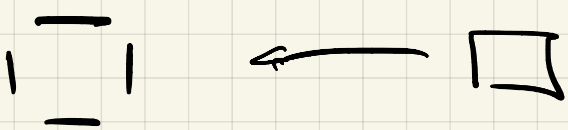
$$\mathcal{C}_0 \xleftarrow{\delta} \mathcal{C}_1 \cdots \leftarrow \mathcal{C}_k \xleftarrow{\delta} \mathcal{C}_{k+1}$$

$\mathcal{C}_0$: 0-cubes, $\mathcal{C}_1$: 1-cube

$\left\{ \sum a_i m_i \;\middle|\; m_i \in M \right\}$

want map

δ is just the "bdry operator"



Want $\delta^2 \equiv 0$

$I^n \to \{0,1\}^n$ $\qquad I^n = \{\partial : [0,1]^n \to\}$

$I^n_{(i,0)}(x) := (x_1, \dots, x_{i-1}, 0, x_i, \dots, x_n)$
$x = (x_1, \dots, x_{n-1})$

$I^n_{(i,1)}(x) := (x_1, \dots, x_{i-1}, 1, x_i, \dots, x_n)$

n-1 faces of I^n

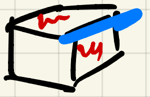
Ex $I^1_{(1,0)}() = (0)$

$I^2_{(1,0)}(x) = (0,x)$

$I^3 \to 6$ faces

$I^4 \to$

$\partial^2(I^3) = \partial\circ\partial(I^3) = \sum \pm$ edges



Crucial fact: each $(n-2)$ dim face of I^n the $(n-2)$ face of 2 $(n-1)$-face of I^n

$\hookrightarrow$ so, want to give each of the 2 opposite signs!

Formal Def $\partial I^n = \sum_{i=1}^{n} \sum_{\alpha=(0,1)} (-1)^{i+\alpha} I^n_{(i,\alpha)}$

Non formally $i \leq j$ $\quad \left(I^n_{(i,\alpha)}\right)_{(j,\beta)}(x)$ $\quad (x_1, \dots, x_{n-2})$

$$\left(I^n_{i,\alpha}\right)\left((x_1, \dots, x_{j-1}, \beta, x_j, \dots, x_{n-2}\right))$$

$$\hookrightarrow (x_1, \dots, x_{i-1}, \alpha, x_i, \dots x_{j-1}, \beta, x_j, \dots, x_{n-2})$$

$$\|$$

$$\left(I^n_{(j+1,\beta)}\right)_{(i,\alpha)} = (x_1, \dots, x_{i-1}, \alpha, x_i, \dots x_{j-1}, \beta, x_j, \dots, x_{n-2})$$

But signs are flipped

lhs $\quad (-1)^{i+\alpha} \cdot (-1)^{j+\beta} \quad$ vs $\quad (-1)^{j+\beta+1} (-1)^{i+\alpha}$

∂ for singular k-cubes $\quad c: I^k \to M$

$$\partial c = c \circ \partial I^k$$
$$= \sum_{\substack{i=1 \\ \alpha=0,1}}^{k} (-1)^{i+\alpha} \; c \circ I^k_{(i,\alpha)}$$

Next, extend ∂ linearly $\mathcal{C}_k \xrightarrow{\partial} \mathcal{C}_{k-1}$

clear $\partial^2 = 0$ as $\partial^2 = 0$ on basis I_k.

$\hookrightarrow \partial^2(c) = c \circ \partial^2(I^k) = c(0) = 0$.

Integrate / Singular k-Chain!

let c is a singular k-chain!

$c : I^k \to M$

$\omega \in \Omega^k M$ k-form

$$\int_c \omega = \int_{I^k} c^* \omega$$

$\to$ an k form on I^k!

isn $\to$ Technically I^k not open!

$\omega = f(x)\, \partial x_1 \wedge \dots \wedge \partial x_k$

$$\Rightarrow \int_{I^k} c^* \omega = \int_{I^k} f \quad \to \quad \int_{\sum \alpha_i c_i} \omega = \int \alpha_i$$

Stokes) RTC)

$\omega \in \Omega^{k-1}(A) \to A \subset \mathbb{R}^n$ open