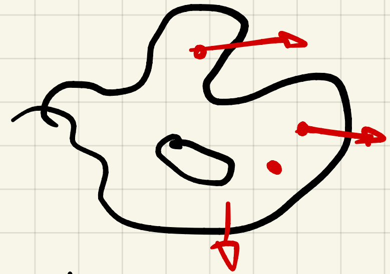
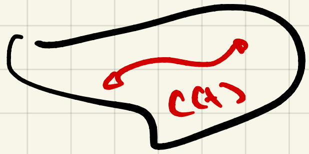
# Vector Fields

section of the tangent bundle. but think of how do we find a bunch of tangent vectors

M






$\dot{c}(t_0) = \frac{\partial}{\partial t}\Big|_{t=0} (t)$

↳ but only defines tangent vectors along curve

**Recipe**: Fill up M mfd with disjoint diff $(C^k)$ $c_p$ → so well def

$X(p) = \dot{c}_p(p)$

→ curve through p

↳ if a curve is $C^k$ → vec field along curve is $C^{k-1}$

↳ what happens when we swap curves?

↳ "transversality" to the curves, regularity is unclear

→ put if $p \to c_p$ is super diffable then good.

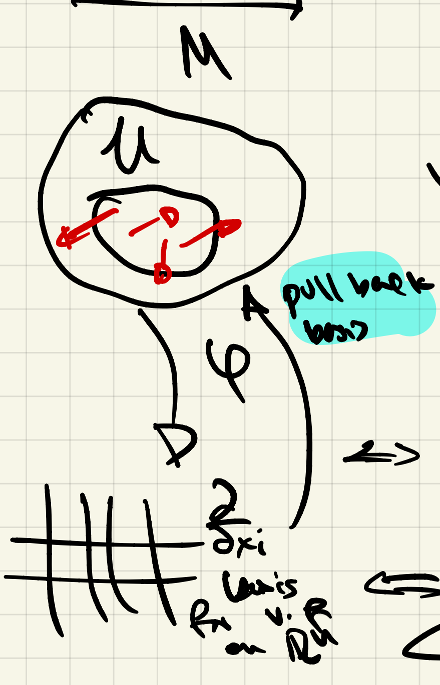
So we want curves → vector fields! what abt backwards

To go from v.f → curves we need to define.

**Def** $X$ is a v.f M. Then $c : (-\varepsilon, \varepsilon) \to M$ is a solution curve for $X$ if

$\forall t_0 \in B_\varepsilon(0)$ $\frac{\partial}{\partial t}\Big|_{t=t_0} c(t) = X(c(t_0))$ → coincides with vec field

in coord




$X|_U = \sum_{i=1}^n a_i(p) \frac{\partial}{\partial x_i}$

$X$ is $C^\infty \iff a_i$ $C^\infty$.

write $c(t) = (c_1(t), \dots, c_n(t))$ in the coord

$\dot{c}(t_0) = (c_1'(t_0), \dots, c_n'(t_0))$

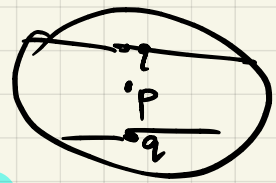
$X(c(t_0)) = \dot{c}(t_0)$

$\iff \sum_{i=1}^n a_i(c(t_0)) \frac{\partial}{\partial x_i} c(t_0) = \sum_{i=1}^n c_i'(t_0) \cdot \frac{\partial}{\partial x_i}(c(t_0))$

$\iff a_i(c(t_0)) = c_i'(t_0) \quad \forall i \in \mathbb{N}_n$

↳ basic ode, can solve uniquely is $a_i$ Lipschitz

More precisely, (since the former is only guaranteeing local sol)
so, for what $t$ do we get a sol$^n$
for $p$ get soln $c: (-\varepsilon(p), \varepsilon(p)) \to \mathbb{R}^n$ $(M)$
→ can't always get non-local solution → if we blow up
→ more precise $p \in \mathbb{R}^n$, $X$ v.f on $\mathbb{R}^n$
Then $\exists\, \varepsilon, \delta > 0$ so $\forall\, q \in B_\varepsilon(p)$ there exist soln to ode on the interval $(-\delta, \delta)$.



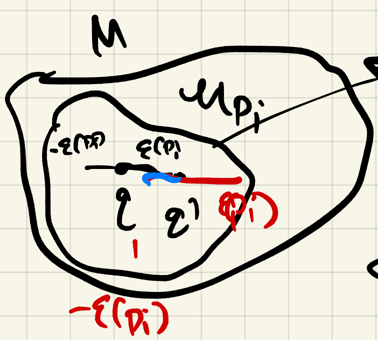

comes from contraction mapping thm → | existence & uniqueness of local soln

**Def** | If $M$ $C^\infty$ mfd. we call $X$ v.f on $M$ complete ⇒ solution curves exist for any point at all time.

**ad** | Its near impossible to calculate these solution curves for arbitrary ODEs.
→ explicit vs qualitative study of ODEs

**Lemma** | If $M$ is compact $C^\infty$ mfd then $X$ - $C^k$ mfd ($k \geq 1$) then $X$ is complete.

**Pf** | • true for short amt of time in a nbhd $U_p$ of $p$
• $p \in U_p$, $(-\varepsilon(p), \varepsilon(p))$
→ ∃ fin many $p_1, \dots, p_n$ so $\bigcup_{i=0}^{n} U_{p_i} = M$
$q \in M$, $q \in U_{p_i}$ for some $i$. Find solution curve for time $(-\varepsilon(p_i), \varepsilon(p_i))$




→ chain together curves for time $(-\varepsilon(p_i), \varepsilon(p_i))$
→ gluing is good ($C^2$) since 2 curve agree on overlap (blue) (can always move for min $\varepsilon(p_i)$)
→ even works between balls
→ if the path hits a zero for $X$ you stop for all time!