Week 2

Lie gps $\longleftrightarrow$ Lie Algebras

Thm G Lie gp & $H \subseteq G$ conn Lie subgrp
$\iff \exists\, \mathfrak{h} \subset \mathfrak{g} = \text{Lie } G$ so $\mathfrak{h} = \text{Lie } H$

Spse G_1, G_2 lie gp. Consider $\varphi : G_1 \to G_2$ diffble homomorphism (so assume C^∞ too)
$\varphi(ab) = \varphi(a)\varphi(b)$, $\varphi(a^{-1}) = \varphi(a)^{-1}$

There's trouble: $\exists\, \varphi : \mathbb{R} \to \mathbb{R}$ homo not diffble
by Galois Theory (transcendentals or something)

Then call φ a Lie gp homomorphism

Rmk good enough to assume φ is measurable (cts)
(wrt charts)

$\varphi : G_1 \to G_2$

$\Rightarrow D\varphi_1 : T_1 G_1 \to T_1 G_2$

$$\begin{array}{ccc} T_1G_1 & \longrightarrow & T_1G_2 \\ \| & & \| \\ \mathfrak{g}_1 & \longrightarrow & \mathfrak{g}_2 \end{array}$$

Think as left inv v.f.

Claim: $X_1, X_2 \in \mathfrak{g}_2$

$\varphi_*(X_1)$, $\varphi_*(X_2)$, v.f on G_1

assume φ is inj, $\Rightarrow D\varphi$ is inv

pullback

$$\varphi_*(X_1)(g) = (D\varphi_g)^{-1}(X_1(\varphi(g)))$$

fixed here by hom

$\varphi_*(X_2)$ is φ related to X_2

$\varphi_*(X_1)$ is φ related to X_1

$\Rightarrow \varphi_*[X_1, X_2] \xleftrightarrow[\varphi \text{ relo}]{} [X_1, X_2]$

$\Rightarrow D\varphi_1 : \mathfrak{g}_1 \to \mathfrak{g}_2$ is linear &
$D\varphi_1([X_1, X_2]) = [D\varphi_1(X_1), D\varphi_1(X_2)]$ $\to$ $\boxed{\text{Def}}$ Lie algebra homomorphism

Thm $G_1 \longrightarrow G_2$

$\mathfrak{g}_1 \longrightarrow \mathfrak{g}_2$

$\exists$ 1-1 corr betw lie grp homo & Lie alg homo!

Pf) $\Rightarrow$ we did above

$\Leftarrow$: trick $\mathfrak{g}_1 \xrightarrow{\psi} \mathfrak{g}_2$

look at graph $\psi = \{(x, \psi(x)) \mid x \in \mathfrak{g}_1\}$

Claim graph $\psi \to$ Lie subalgebra of $\mathfrak{g}_1 \times \mathfrak{g}_2$ $\to$ dir prod has involve bracket

$[(x_1,y_1),(x_2,y_2)] = ([x_1,x_2],[y_1,y_2])$

$\hookrightarrow$ consider $[(x_1, \psi(x_1)), (x_2, \psi(x_2))]$

$= ([x_1,x_2], [\psi(x_1), \psi(x_2)])$

as ψ Lie alg homo

grp ψ $\Rightarrow$ $= ([x_1,x_2], \psi[x_1,x_2])$ $\to$ so Lie subalgebra

$\Rightarrow$ $\exists$ corresponding connected subgrp of $G_1 \times G_2$

by result last

$\hookrightarrow$ call it H. Lie $G_1 \times G_2 = \mathfrak{g}_1 \times \mathfrak{g}_2$

$\hookrightarrow$ Lie H = graph ψ

$\hookrightarrow$ Claim: H is the grph of homo $\Psi : G_1 \to G_2$

$\Psi(g_1) = g_2$ if $(g_1, g_2) \in H$ $\to$ Check

$\hookrightarrow$ ① well def

② homomorphism

$\hookrightarrow$ what we want too !

Exponential map

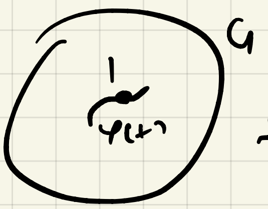
G Lie gp. $X \in \mathfrak{g}$ Lie algebra. $\{tX, t \in \mathbb{R}\}$ lie subalgebra

as $[tX, sX] = st[X,X] = 0$

$\Rightarrow$ $\exists$ subgrp of G corr to $X \in \mathfrak{g}$

$\hookrightarrow$ Down to earth: $X \in \mathfrak{g}$ think of left inv v.f.

$\hookrightarrow$ get local flow φ_t on G say at 1



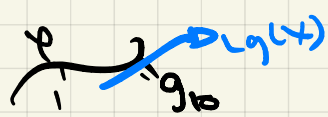
Call $\varphi_t(1) = g_t$

$g_t g_s = \varphi_t(1) \cdot \varphi_s(1) = \varphi_{s+t}(1) = g_{t+s}$ homomorphism

Claim: φ_t global (def for all time)

Pf ① appeal to subgrp stuff

② make argument like



local flow of X through g is L_g · local flow at 1 $= L_g \cdot [\varphi_t(1)]$

Thus if the local flow at 1 is def on $(-a, a)$ so it is around g !

$-\varepsilon$ ($\cdot$) a ($\cdot$)
1 $-\varepsilon$ g ε

e.g.) $G = GL_n(\mathbb{R})$ $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{R}) = M_{n\times n} \ni X$

What is φ_t corr to A (or $\{tA\}$ ...)

$e^{tX} = \sum \frac{(tX^n)}{n!}$

converges

take norm & triangle

op norm

$\left\| \sum \frac{(tX^n)}{n!} \right\|$

$\leq \sum \frac{\|t\| \, \|X\|^n}{n!}$

Note) $\frac{\partial}{\partial t}(e^{tX}) = X \cdot e^{tX}$

Thus, e^{tX} solves local flow ODE

one show $e^{tX} \in GL_n(\mathbb{R})$.

Also $e^{tX} \in GL_n(\mathbb{R})$ so invertible

e^{-tX} is defd

$e^{tX} \cdot e^{-tX} = e^{tX - tX} = e^0 = I = 1$