quick comment | in the pf of Frobenius

$\pi: \mathbb{R}^n \to \mathbb{R}^k$ & V is a k-dim dist

$\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k}$

idea: find $Y_1, \dots, Y_k$ tangent to V so $D\pi(Y_i) = \frac{\partial}{\partial x_i}$

$\Rightarrow Y_i, \frac{\partial}{\partial x_i}$ are π-related $\Rightarrow [Y_i, Y_j] = 0$

use integrability — $\Rightarrow$ know $[Y_i, Y_j]$ tangent to V

And $D\pi([Y_i, Y_j]) = [\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}] = 0$

$D\pi$ inj $\Rightarrow$ $[Y_i, Y_j] = 0$

G Lie grp

$\Rightarrow \mathfrak{g} = T_1 G$ Lie Algebra

If $H \subseteq G$ Lie subgrp

Lie $H \subseteq \mathfrak{h} \subset \mathfrak{g}$ as $T_1 G \supseteq T_1 H$

$T_1 H$

$v(1) \rightsquigarrow V$ left inv v.f on G

$\overset{\text{inner}}{\Longrightarrow} V$ left inv v.f. of H

$\Big\}$ Furthermore $[\,,\,]^H \cong [\,,\,]^G$

$\Rightarrow$ bracket of v.field so on inv.

upshot

Lie subgroup

$\downarrow$

Lie subalgebra

Def Given a Lie algebra $\mathfrak{g}$ then $\mathfrak{h} \subseteq \mathfrak{g}$ is a Lie subalgebra if $\mathfrak{h}$ is a v space &

$\forall H_1, H_2 \in \mathfrak{h} : [H_1, H_2] \in \mathfrak{h}$

$\uparrow$

$\mathfrak{g}$

Thm There is a 1-1 corr between connected Lie Subgrps of G & Lie subalgebras of $\mathfrak{g}$.

Pf ($\Rightarrow$) shown

($\Leftarrow$) Main application of Frobenius..

given $\mathfrak{h} \subseteq \mathfrak{g}$ Lie subalgebra

$\Rightarrow$ bunch of left inv v.f.

$\mathfrak{h} \subseteq T_1 G$

define dist $V(g) = DL_g(\mathfrak{h})$

corr to evaluat left inv v.f at g

Claim: V is integrable i.e. closed under brack

X_1, X_2 v.f tangent to V wts $[X_1, X_2] \in V$

let $Z_1, \dots, Z_k$ be a basis of $\mathfrak{h}$ ← $k = \dim \mathfrak{h}$

$\Rightarrow Z_i$ is a left inv v.f $\Rightarrow [Z_i, Z_j] \in \mathfrak{h} \to \boxed{\text{Lie subalg}}$

$$X_1 = \sum^k a_i Z_i \qquad X_2 = \sum^k b_j Z_j$$

$(a_i: G \to \mathbb{R})$

$$[X_1, X_2] = [\sum a_i Z_i, \sum b_j Z_j] = \sum \text{ functions } Z_i + \sum_{i,j} \text{ some func } [Z_i, Z_j]$$

V ✓!

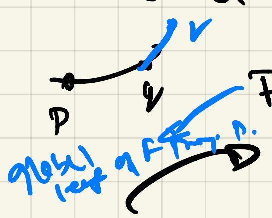
Lemma $[aF, bG]$
$= aF(bG) - bG(aF)$
$= aF(b)\cdot G + ab\,FG - bG(a)F - ba\,GF$
$= aF(b)\cdot G - bG(a)F + ab(FG - GF)$
$= aF(b)\cdot G - bG(a)F + ab[F,G]$

$\Rightarrow V$ is integrable, can use Frob.

let $\mathcal{F}$ be the foliation det by V. ↪ exist locally (in nbhd of each pt) → get global foliation $\mathcal{F}$ & global leaves $F(g)$.

Thm M mfd & V int dist on M $\exists$ foliation $\mathcal{F}$ globally

have local $\mathcal{F}_{loc}(p)$ $\forall p \in M$



$F(p) = \mathcal{F}_{loc}(p) \cup F_{loc}(q) \cup F_{loc}(r) \dots$

Lemma given $q \in F_{loc}(p)$ $\exists$ nbhd U of q s.t. $F_{loc}(p) \cap U = F_{loc}(q) \cap U$

Pf: both tgt to V

$\Rightarrow$ agree where overlap — valid

shows global leaf path conn → so conn $\boxed{\text{Lie subgp}}$

Note: $\mathcal{F}$ is left invariant. as dist left inv

G $F(gp) = L_g F(p)$

set $H = \mathcal{F}(1)$

Some $h \in H$ — h in leaf 1

$h \cdot H = L_h H = L_h(F(1))$
$= F(h \cdot 1) = F(h)$
$= F(1)$

$h \cdot H = H \to$ subgrp step 1

let $h \in H$, is $h^{-1} \in H$?

$h^{-1} = h^{-1}\cdot 1 \in h^{-1} F(h) = F(1) = H$

If not conn) $\mathbb{Z} \subseteq \mathbb{R}$ $\quad \text{Lie}(\mathbb{Z}) = \{0\}$

$= \text{Lie}(\langle 5 \rangle) \longrightarrow$ no $\underline{1 \text{ to } 1 \text{ corr}}$

or $\mathbb{Z} \times \mathbb{R} \subseteq \mathbb{R}^2$

$\text{Lie}(\mathbb{Z} \times \mathbb{R}) = \{0\} \times \mathbb{R}$

fact) Know given Lie gp $G \rightsquigarrow \mathfrak{g}$

Conversely given any Lie Algebra $\mathfrak{g}$ $\exists$ lie grp G w lie Algebra $\mathfrak{g}$?

how many?

Yes true :- could be Many!

$\hookrightarrow$ trick is to embed $\mathfrak{g}$ in lie algebra of $GL(n, \mathbb{R})$

$\text{Lie}(\mathbb{R}) = \mathbb{R}$ $\quad \text{Lie}(S^1) = \mathbb{R}$

e_1 $\quad e_2$