Math 591 Lec 4

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1. Sketch of a proof of 1: M is a 1D connected, compact manifold. For all $x \in M$ there exists some $U_x \ni x$ so that U_x is homeo to (a, b). WE have a finitely many U_i so that $\bigcup_{i \in \mathbb{N}_N} U_i = M$ with each $U_i \equiv (a_i, b_i)$. Suppose that this is minimal in number (so there are no subsets). Lemma: $U_i \cup U_j$ homeo to an open interval or S^1

2. We now have topological manifolds and differentiable manifold. For differentialble manifolds, we have for overlapping open sets U_α, U_β with charts ϕ_α, ϕ_{beta}, then we want the transition functions to be differentiable.

3. Theorem, there exist topological manifold which do not admit any differentiable structure.

4. Piecewise mamifolds: (loally combinations of affine maps) don't admit differentiable structure

5. All differentiable manifolds are topological manifolds. So there is a function from diff mfd to top mfd. But there is no inverse. Milnor constructred exotic 7 spheres ($M_1, \dots, M_{28}$ such that M_i homeo to M_j but not diffeo unless $i = j$). Can do this for infinitely. many hihger dimensions. These are the exotic spheres.

6. Let us say that M, Ω is a manifold with a differentiable structure. Ω is an open cover with a bunch of charts and transition maps. Suppose that we have an Ω_1 and Ω_2 on M. Are called equivalent (the same) if they are compatible if $\Omega_1 \cup \Omega_2$ is a differentiable structure. Ω_1 is also called an atlas.

7. We can look at the Maximal Atlas $= \bigcup$ compatible atlases. (Lemma union of compatible atlases is an atlas). Bourbakian Method: use the maximal atlas. The hands on approach find your atlas and work with it.

8. e.g. S^2 consider the atlas 1. $S^2 - \{$ north pole $\}$ and $S^2 - \{$ south pole $\}$ using stereographic projection. But, any open subset that isn't the whole thing is an apt cover and produces a maximal atlas.

9. Construction. Suppose that M is a manifold and N are both diffble manifolds. Consider $M \times N$ with the product topology. Note that the basis is $\{U \times V\}$ for U, V open in M, N resp. $M \times N$ has a differentiable structure. Say $(U_\alpha, \phi_\alpha)_\alpha$ is an atlas for M and $(V_\beta, \psi_\beta)_\beta$ an atlas for N. Then $(U_\alpha \times V_\beta, (\phi_\alpha, \psi_\beta))_{\alpha,\beta}$ is a diffble structure on the product manifold.

   Must check,

   (a) $\{U_\alpha, U_\beta\}_{\alpha,\beta}$ is a cover

   (b) $(\phi_\alpha, \psi_\beta) \to \mathbb{R}^m \to \mathbb{R}^n$ and check that transition functions are differentiable.

10. Construction. We want to do quotients.
    (a) S^1 as subset of $\mathbb{C}$ then $x \sim y$ if $xy^{-1} = e^{in\alpha 2\pi}$ for $\alpha \in \mathbb{R}$ and $n \in \mathbb{Z}$. If $\alpha = \frac{2\pi}{3}$ then points $2\pi/3$ are glued and we see $S^1/\sim \equiv S^1$.
    (b) Consider $\alpha \notin \mathbb{Q}$ irrational so you don't have a cycle. $[x]$ is dense in S^1. Therefore, $S^1/\sim$ is not hausdorff.
11. Consider M is a manifold with $\sim$ and equiv reln. Suppose that $\sim$ is open. THat is, if $x \sim y$ and $y \in U$ which is open then there exists a notion that V open neighborhood of x such that $y' \in U$ is equiv to some $x' \in V$.
12. If X is second countable and $\sim$ and open equiv reln then $\pi : X \to X/\sim$ where $x \mapsto [x]$. Right so $\sim$ is open iff π is an open map. Therefore, we have that $X/\sim$ is second countable.