Lec 5

Recall X top space, $\sim$ equiv on X. $X/\sim$ is top sp where
V open $\iff \pi^{-1}(V)$ open $\quad \pi : X \to X/\sim$, $x \mapsto [x]$

e.g. $X = S^1$ $\quad \underset{\alpha}{\sim}$ $\quad x \underset{\alpha}{\sim} y$ if $y^{-1}x = e^{2\pi i n \alpha}$ for $n \in \mathbb{Z}$

if $x \notin \mathbb{Q}$ $\forall x \in S^1$ $[x]$ is dense in S^1

What is open? $\emptyset$ & $S^1 \to$ indiscrete top

P spose $U \subseteq S^1$ non empty $\pi^{-1}(U) = V$ open is S^1.

let z in $S^1 \Rightarrow [z]$ is dense $\Rightarrow [z] \cap V \neq \emptyset$
$\Rightarrow [z] \subset U \Rightarrow U = S^1$

Cor Consider $e^{2\pi i [a,b]}$. Is $\pi : S^1 \to S^1/\sim$ open? Yes!

Def Call $\sim$ open if $\pi : X \to X/\sim$ open.

Equiv. if $U \subseteq X$ is open then $\{x \in X \mid \exists\, y \in U \ x \sim y\}$ is open in X
(is $\pi^{-1}(\pi(U))$)

Prop If X is compact, Hausdorff & $[x]$ is closed $\forall x \in X$
$\Rightarrow X/\sim$ is compact & Hausdorff.

Pf ① $X/\sim$ hausdorff. Spose not and $[a]$, $[b]$ not separate
we see that $[a]$ is compact. $\forall x \in [a]$ $\exists\, U_x \ni x$, $V_x \ni b$
so $U_x \not\ni b$ and $V_x \not\ni x$. Take finite subcover U_{x_i} & consider
$\bigcup U_{x_i}$ & $\bigcap V_{x_i}$ to separate $[a]$ from b.
② Compact ✓

Prop $\sim$ is open, X is 2nd ctbl Hausdorff $\Rightarrow X/\sim$ second ctbl

Pf let $\{B_i\}_{i \in \mathbb{N}}$ ctbl basis of X. Consider $\{\pi(B_i)\}_{i \in \mathbb{N}}$.

Aside: Zariski Top is an interesting non Hausdorff space.
Closed sets of $\mathbb{R}^n$, $\mathbb{C}^n$, ... is the zero set of polynomials
↳ compact! non-Hausdorff!

Def Let G a group. Suppose G has a topology s.t. $\begin{array}{l} G\times G \to G \\ (x,y) \to y \end{array}$ is cts (prod top) & $\begin{array}{l} G \to G \\ x \mapsto x^{-1} \end{array}$ is cts

Call G a top grp

e.g. $\{1\}$, $\mathbb{R}^n$, $\mathbb{C}^n$, any discrete grp, $GL_n(\mathbb{R})$ → det $\neq 0$

$O(n) = \{A_{n\times n} \in GL_n(\mathbb{R}) \mid A\cdot A^t = 1\}$ → rotations

$SO(n) = \{A \in O(n) \mid \det A = 1\}$

$SL_n(\mathbb{R}) = \{A \in GL_n(\mathbb{R}) \mid \det A = 1\}$

$S^1 \xrightarrow{\text{rotation}} SO(2)$

① $T^n = \underbrace{S^1 \times \dots \times S^1}_{\text{prod top}}$ then T^n → compact (Tychonoff)

② $\mathbb{Q}_p$ p-adic #'s → completion of $\mathbb{Q}$ wrt p-adic norm $a = p^k \cdot a'$ (coprime), $b = p^l \cdot b'$

$\|\frac{a}{b}\|_p = p^{l-k}$ → top field actually → so top grp also $SL(n, \mathbb{Q}_p)$ ....

Def Lie grp: G is a topological grp is a Lie grp if it has a diff structure s.t. $\begin{array}{l} G\times G \to G \\ (x,y) \to xy \end{array}$ $\begin{array}{l} G \to G \\ x \to y \end{array}$ are C^∞

Not quite def yet

Def M, N are 2 diff mflds & $f: M \to N$ is diffble (C^1, C^k, C^∞) if given $x \in M$ & coord chart $(U_\alpha, \varphi_\alpha)$ around x. let (V_β, ψ_β) coord chart around $f(x)$. Now

$x \in \underbrace{U_\alpha \cap f^{-1}(V_\beta)}_{\text{open}}$ now → domain rest

$$\underbrace{\mathbb{R}^n \supset \varphi_\alpha(U_\alpha \cap f^{-1}(V_\beta)) \xrightarrow[\varphi_\alpha^{-1}]{} \cdot \xrightarrow[f]{} \cdot \longrightarrow \cdot \xrightarrow[V_\beta]{} A \subseteq \mathbb{R}^l}_{f_{\alpha,\beta}}$$

f diff if $f_{\alpha,\beta}$ diff

same for C^1, C^k, C^∞

Lemma This is independent of compatible charts (dep on diffeo of mfd)