Lec 6

Fix from last time ...

X is hausdorff, second ctbl, $\sim$ open equiv reln with $[x]$ compact (cmpct)
$\Rightarrow X/\sim$ is ~~haus dorff~~ hausdorff

Pf **Prop 1:** X top space, $\sim$ open equiv reln graph $\sim = \Gamma := \{(x,y) \mid x \sim y\}$

$X/\sim$ hausdorff $\iff \Gamma$ is closed subset of $X \times X$

Pf $\Rightarrow$ ex

$\Leftarrow$ Spse $(x,y) \in X$, $x \not\sim y \Rightarrow [x] \neq [y] \Rightarrow (x,y) \notin \Gamma$

Since $X \times X \setminus \Gamma$ is open, $(x,y) \notin \Gamma$, $\exists$ basis set $U \times V \subseteq X \times X$ so $(x,y) \in U \times V$, $U \times V \cap \Gamma = \emptyset$

$\Rightarrow \pi(U) \cap \pi(V) = \emptyset$ otherwise if $\exists [z] \in \pi(U), \pi(V)$

$\Rightarrow \exists a \in U, b \in V$ so $a \sim b \Rightarrow U \times V \cap \Gamma \neq \emptyset$

Last time

G top grp $\to$ G lie grp $\to$ top grp with Lie(?) structure

E.g. Lie grps $\to$ $G = \mathbb{R}$, $G = S^1$, $G = S^1 \times \dots \times S^1$, $G = GL(n, \mathbb{R}) \subseteq \mathbb{R}^{n^2}$ (open) so chart is inclusion

Note: S^2 not a Lie group any way.

Notation G grp. X space.

G acts on X if there exists a map $G \times X \to X$, $(g,x) \mapsto g \cdot x$

$1 \cdot x = x \quad \forall x$ $\quad G \curvearrowright X$

$(g_1 \cdot g_2) x = g_1 (g_2 x)$

E.g. S^1 on S^1 by complex multiplication

$\mathbb{R}^n$ on $\mathbb{R}^n$ $\quad (a,b) \mapsto (a+b)$

$GL(n,\mathbb{R}) \curvearrowright \mathbb{R}^n \quad A \cdot v = A$ matrix mult v

Def Call an action cts (X top sp, G top grp)
if $G \times X \to X$ mult is cts.

Def | We call a grp action diff'ble if G top gp $\curvearrowright X$ diff mfd. if $G \times X \to X$ cts and for each $g \in G$ $x \mapsto gx$ is diff'ble.
If G is a Lie Group call the action jointly diff'ble (C^1, C^k, C^∞) if $G \times X \to X$ is diffe

Thm | G compact top grp, cts $G \curvearrowright X$, X compact Hausdorff
$\Rightarrow X/G$ cmpct hausdorf

Pf | let $\sim$ be the equiv reln.
Check ~~1)~~ $\sim$ open
2) Γ closed

let $U \subset X$ open wts $\tilde{U} = \{y \in X \mid y \sim x\} \supseteq \overline{\text{orbit}}$
$= \{g \cdot u \mid g \in G, u \in U\}$

spse $g_0 u_0 \in X$ consider $\tilde{U} \supseteq U_0 \ni u_0$ open consider gU_0

2) Γ is closed $\xrightarrow{}$ graph $\sim$

$\varphi : G \times X \to (X, X)$
$(g, x) \mapsto (x, gx) \in \Gamma \quad x \sim gx$

Claim: $\operatorname{im} \varphi = \Gamma$

Since G, X compact & $X \times X$ cmpct Hausdorff. If φ cts then $\varphi(G \times X)$ is compact. And compact sso hausdorff closed.
φ trivially cts.

eg | $X = S^n$, $G = \mathbb{Z}/2\mathbb{Z} = \{1, A\}$

$G \curvearrowright S^n \quad x \in S^n \quad Ax = -x$

$S^n = \mathbb{R}P^n$

$X = S^{2n-1} \subseteq \mathbb{C}^n \quad S^1 \curvearrowright \mathbb{C}^n$
take $e^{i\alpha} \in S^1 \quad z = (z_1, \dots z_n) \in \mathbb{C}^n$
$e^{i\alpha} z = (e^{i\alpha} z_1, \dots, e^{i\alpha} z_n) \longrightarrow$ check action

S^{2n-1}/S^1 is Hausdorff $= \mathbb{C}P^n$

e.g let H top grp (Hausdorff). $G \subseteq H$ compacts subgrp. → homogeneous sp

$G \curvearrowright H$ by $(g,h) \to gh$ $\quad H/G$ compact iff H cmpct

→ homog spaces important cuz
(1) you can calculate stuff
(2) systems with symmetry are typically homog.

e.g homog space → $\mathbb{R}$, $\mathbb{Z} \subseteq \mathbb{R}$ $\quad \mathbb{R}/\mathbb{Z}$ is compact as you can $\cong S^1$

$\mathbb{Z} \curvearrowright \mathbb{R}$ $\quad (z,a) \mapsto (z+a)$

graph of $\sim$ is closed