Thm $f: M \to N$ C^1 (with both M, N at least C^1) $q \in N$ reg value

i.e. $\forall p \in f^{-1}(q)$, Df_p surj.

Reg val q reg val then $f^{-1}(q)$ is a submanif of M

Pf $\dim M = n$, $\dim N = k$

1) Use coord charts to reduce to pb abt funcs

$F: \underset{\mathbb{R}^n}{\underset{\cap}{U_{open}}} \to \underset{\mathbb{R}^k}{\underset{\cap}{V_{open}}}$

2) $F: U \to V$, then $\hat{q} = \psi_\beta(q)$ is a reg val of F where ψ_β is a coord chart for N

3) let $\hat{p} \in F^{-1}(\hat{q}) \subseteq U \subseteq \mathbb{R}^n$ DF_p is surjective

if $F = (F_1, \dots, F_k)$ $\quad DF_p = \left(\frac{\partial F_i(p)}{\partial x_j} \right)_{\substack{j \in \mathbb{N}_n \\ i \in \mathbb{N}_k}}$ $\to k \times n$

$\hookrightarrow$ DF_p surj $\iff$ rank k $\iff$ k lin ind col vecs

WLOG $\to$ assume first k (coord perm)

4) Somehow apply IFT

want $\mathbb{R}^n \xrightarrow[G]{} \mathbb{R}^k \times \mathbb{R}^{n-k}$ extend $F \to G$

$$G(x_1, \dots, x_n) = (F(x_1, \dots, x_n), x_{k+1}, \dots, x_n)$$

$$DG_p = \begin{pmatrix} \frac{\partial F_1 \hat{p}}{\partial x_1} & \frac{\partial F_1 \hat{p}}{\partial x_2} & \cdots & \frac{\partial F_1 \hat{p}}{\partial x_n} & \\ \vdots & & \ddots & & * \\ \frac{\partial F_k \hat{p}}{\partial x_1} & \cdots & & \frac{\partial F_k \hat{p}}{\partial x_n} & \\ & 0 & & & \boxed{I_{k-n}} \end{pmatrix}$$

$\Rightarrow$ DG_p inv at $\hat{p}$

$\Rightarrow$ G is a local diffeo

4) $F^{-1}(\hat{q}) = G^{-1}(q_1, \dots, q_k, *)$ $\Rightarrow$ $\dim f^{-1}(q) = n-k$

$\hookrightarrow$ locally submanifold of $U \Rightarrow f(F^{-1}(\hat{q})) \to$

e.g. Canonical submersion

$\mathbb{R}^n \to \mathbb{R}^k$

$(x_1, \dots, x_n) \mapsto (x_1, \dots, x_k)$

Thm) if $f: M \to N$ C^∞ and q a reg value and $q \in f^{-1}(p)$
then $\exists$ coord charts (U, φ), (V, ψ) around p & q

Normal Form

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\varphi"'] & V \arrow[d] \\ \varphi(U) \arrow[r, dashed, "F"] & \psi(V) \end{tikzcd}$$

F is canonical submersion $\pi_{n,k}$

nice!

Engineering of IFT

↳ uses contraction mapping Thm → contractions in complete metric sp → fixed pt

↳ used in existence and uniqueness of ODE.

Application) $SO(n)$ is a (emb) submanif of $GL(n,\mathbb{R}) \underset{\text{open}}{\subseteq} \mathbb{R}^{n^2}$

How to prove

Defining $A \in O(n)$ $\quad A \cdot A^T = I$

$F: GL_n(\mathbb{R}) \to Sym(n^2) \to \{B = B^T\} = \mathbb{R}^k \xrightarrow{\text{submanif}} \mathbb{R}^{n^2}$ (linear)

$A \longmapsto A^T A$

$O(n) = F^{-1}(Id)$

$\Rightarrow$ suffices to show Id reg val (regular)

Calculate, Df_g for $g \in F^{-1}(I) = GL_n(\mathbb{R}) \Rightarrow g \cdot g^T = I$

$v \in \mathbb{R}^{n^2} \to$ use curves $Df_g(v)$

$F(g+tv) = (g+tv)(g+tv)^T = \underbrace{gg^T}_{I} + gtv^T + tvg^T + t^2 vv^T$

lim to $t \to 0$ $\xrightarrow{/t}$ t^2 term vanishes

$= I + gtv^T + vg^T$

$\Rightarrow Df_g = gv^T + vg^T \to$ Claim This is onto to $Sym(n \times n)$

any symmetric matrix has this form (fixed g)

↳ calculate $\ker(Df_g) = \{v \mid gv^T + vg^T = 0\}$

$\Rightarrow gv^T = -vg^T \Rightarrow vg^T$ is skew symm

$\Rightarrow \dim \ker Df_g = \frac{n(n-1)}{2} \Rightarrow \dim \text{im } Df_g = \frac{n(n+1)}{2}$ (rank nullity)

Cor) $SO(n) \subseteq O(n)$
$\quad\quad\quad\quad \{g \in O(n) \mid \det g = 1\}$

Note: $\det : O(n) \to \mathbb{R}$ → actually only $\{\pm 1\}$
↳ $SO(n)$ is just a connected comp of I_d in $O(n)$
⇒ submanifold → requires lemma $SO(n)$ is connected

Lem) L manif ⇒ any open subset is mfd
clopen