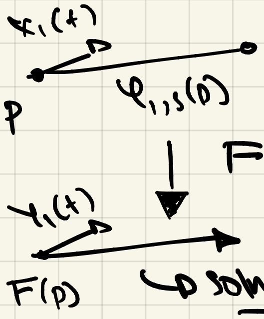
Spse $F: M \to N$ smooth. X is a v-field on M, Y vf on M.
X, Y are F-related if
$$T_pM \ni DF_p(X(p)) = Y(F(p)) \in T_{F(p)}N$$

Lemma let X_1, X_2 be F relt to Y_1, Y_2
$$\Rightarrow [X_1, X_2] \text{ ——— } [Y_1, Y_2]$$

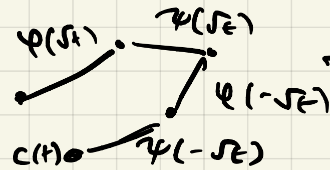
Pf $X_1 \leftrightsquigarrow \varphi_{1,t}$
$X_2 \leftrightsquigarrow \varphi_{2,s}$



$\hookrightarrow$ soln curve for Y_1

$F(\varphi_{1,s}(p))$ is soln curve ODE for Y_1

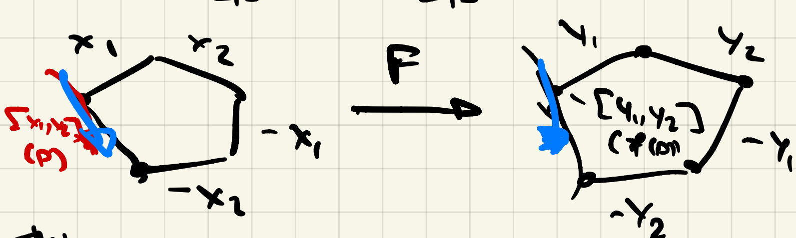
$[X, Y](p) = c'(0)$ where



$\to G(\sqrt{t}, \sqrt{t})$

$$G(s,t) = \psi_{-s} \circ \varphi_{-t} \circ \psi_s \circ \varphi_t(p)$$
wnt $\frac{\partial G}{\partial s} \frac{\partial G}{\partial t}$ at (p)
$$\frac{\partial}{\partial s}(G(s,0)) = \frac{\partial}{\partial s}\Big|_{s=c}(p) = c$$

$\Rightarrow F \circ \varphi_{1,s} = \psi_{1,s}$ & $F \circ \varphi_{2,s} = \psi_{2,s}$ use
$\hookrightarrow$ so,



Distributions $\to$ means many things

M mfd C^∞ $\dim M = m$ $\quad \forall p \in M$ have $D(p) \subseteq T_pM$
$\hookrightarrow$ subsp of dim $k \le m$

$$T_pM \rightsquigarrow Gr_k(T_pM)$$
$\to \{k\text{-dim v subsp of } T_pM\}$

make a fibre bundle out of TM $\to$ fibre is grass mannian of k planes

$$Gr_k(M) = \bigsqcup_{p \in M} Gr_k(T_pM)$$

Topologize using local product struct of TM

$$\begin{array}{ccc} Gr_{k,n} & \xrightarrow{\text{inj}} & Gr_k(M) \\ & & \big\downarrow \text{footpt} \\ & & M \end{array}$$

$n = \dim M$
in $\mathbb{R}^n$

D

$\to$ D is a smooth section of this!

In down to earth terms! $T_pM \supseteq D(p)$ k dim subsp with basis $(v_1,\dots,v_k)$ each in T_pM

↳ also want smoothness condition across points $p \in M$.

Locally $(v_1(q),\dots,v_2(q))$ $\forall q \in \text{nbhd}(p)$ $v_i(q) \in T_q(M)$

↳ smooth v.f on nbhd!

↳ i.e. locally looks like product of lin ind vector fields

Thus, smooth distributions D are given by following data

① $\forall p \in M$ $D(p) \subseteq T_pM$ of dim $k \leq \dim M = n$

② to define smoothness, suffices to do locally, i.e.

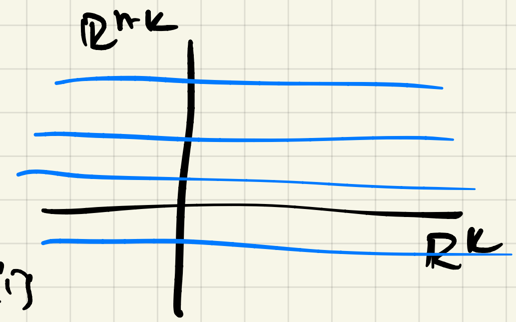
$\forall p \in M$ $\exists\, U$ nbhd of p and smooth v.f v_i on U.

s.t $\forall i$ $v_i(q) \in D(q)$ & $v_1(q),\dots v_k(q)$ lin ind!

↳ This is a generalization of non-zero vector fields

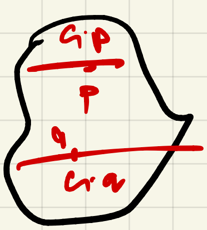
↳ a vf is a 1-distribution if it doesn't vanish.

e.g.| ① $\mathbb{R}^n = M$ $p \in M$ let $D(p) = \mathbb{R}^k = \{(x_1,\dots x_k, 0 \dots 0)\}$



② Suppose G is a Lie grp that acts on mfd M.

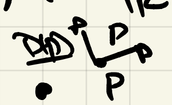
$\forall p \in M$ G_p is discrete $\to$ stab e.g. $G_p = \{1\}$



let $D(p) = T_p(G\cdot p)$

③ Let V v.f nonvanishing on $\mathbb{R}^n$ $D(p) = V(p)^\perp$ ↳ where we have inner prod sheet

3a $M = \mathbb{R}^n \setminus \{0\}$ $V(p)$ = radial vf ↳ v.f given by angles in polar coord



$V(p) = p$.

↳ tangent space to concentric spheres.

$\left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & & 1 \end{pmatrix} \right\} \to$ Heisenberg grp $\to$ Lie grp

$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ Tangent to identity

$\overline{A_1}, \overline{A_2}$ left inv v.f.

$D = \mathbb{R} \cdot \overline{A_1} + \mathbb{R} \cdot \overline{A_2}$