Def Const rank

let $f: M \to N$ C^k with M, N C^k. Consider $p \mapsto \operatorname{rk} Df_p$
$:= \dim \operatorname{im} Df_p$

$\Rightarrow$ $f: M \to N \iff$ rank of $Df_p = \dim N$ $\forall p$

f has constant rank if $p \mapsto \operatorname{rk} Df_p$ is const in p

Const Rnk Thm

if f has const rank then $f^{-1}(q)$ is const rank.

Idea pf | $T_{f(p)} N \longrightarrow$

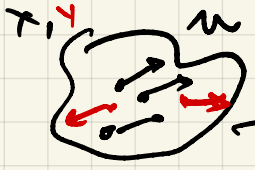
g = local proj $f(M)$ to $\operatorname{im} Df_p(T_pM)$
$\hookrightarrow$ use dimension Thm locally

Vector Fields

M is a manifold, X, Y are C^∞ vector fields.

$[X, Y] = \underbrace{X \circ Y - Y \circ X}_{\text{is no derivations}} \longrightarrow$ defines a vector field

Special Case: Suppose X, Y nonvanishing on M (or some open $U \subseteq M$)



Assume $[X, Y] = 0$

Then, $\exists$ coordinates on a chart s.t. $X = \frac{\partial}{\partial x}$ $Y = \frac{\partial}{\partial y}$

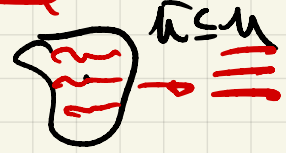
$\hookrightarrow$ need $X_1 \dots X_k$ lin ind

$\hookrightarrow$ so we commute

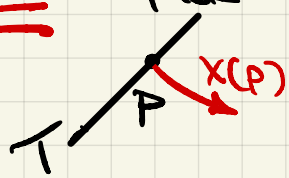
More generally $X_1, \dots, X_k$ don't vanish on U

$[X_i, X_j] = 0$ $\exists$ coord charts so $X_i = \frac{\partial}{\partial x_i}$

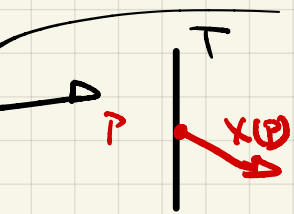
Pf idea | ① $k=1$ given $X \neq 0$ $X \rightsquigarrow \varphi_t \rightsquigarrow$ flow of X



Pick $T \subseteq U$ submfd of dimension $n-1$



$T = \{(z_1, \dots, z_n) \mid z_j = 0\}$
$\hookrightarrow$ choose this chart



Can make coor $y_1, \dots, y_n$ so $T = \{(0, y_2, \dots, y_n)\}$

Look $(t, y_2, \dots, y_n) \xmapsto{\phi} \varphi_t(0, y_2, \dots, y_n)$
$\hookrightarrow$ flow it

Check $d\phi_{(0,0,\dots,0)}$ inv $\Rightarrow$ ϕ local diffeo $\longrightarrow$ coord chart

$$\partial \Phi_{(0,\dots,0)} = \begin{pmatrix} 1 & & \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$ since $X(p)$ transversal to T this is invertible.

② for $\underline{k=2}$ will use $[X,Y]=0$

<u>Wlog</u> by above in suitable coords $X = \frac{\partial}{\partial x_1}$

<u>Check</u>: $\underline{n=2}$

$$X = \frac{\partial}{\partial x}, \quad Y = a(x,y)\frac{\partial}{\partial x} + b(x,y)\frac{\partial}{\partial y}$$

$$[X,Y] = 0$$

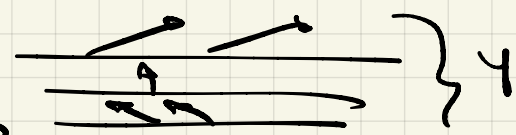
$$= \frac{\partial}{\partial x}\left(a(x,y)\frac{\partial}{\partial x} + b(x,y)\frac{\partial}{\partial y}\right) - \left(a(x,y)\frac{\partial}{\partial x} + b(x,y)\frac{\partial}{\partial y}\right)\frac{\partial}{\partial x}$$

$$= \underbrace{\frac{\partial a(x,y)}{\partial x}}_{0}\frac{\partial}{\partial x} + \underbrace{\frac{\partial b(x,y)}{\partial x}}_{0}\frac{\partial}{\partial y}$$ ... stuff cancels

$0 \longrightarrow$ const in x.

$\Rightarrow a(x,y) = a(y) \qquad b(x,y) = b(y)$

$$Y = a(y)\frac{\partial}{\partial x} + b(y)\frac{\partial}{\partial y}$$

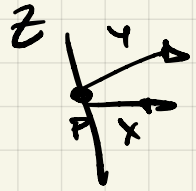


ψ_t 4-local flow of Y. multiloral

$\psi(x,t) = \psi_t(x,0)$, $\psi : \mathbb{R}^2 \longrightarrow M$

$\hookrightarrow Y = \frac{\partial}{\partial T}$.

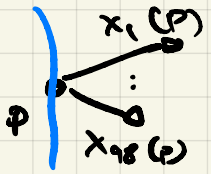
if $\dim M > 2 \longrightarrow$ find submfd through p transversal to X, Y. $Z \longrightarrow$ local



$\hookrightarrow$ rinse & repeat from earlier

$$\psi(s+t, z) = \psi_t \circ \varphi_s(z)$$ $\hookrightarrow$ <u>use</u> the flows commute so order doesn't matter

k, $\dim M$ arbitrary.



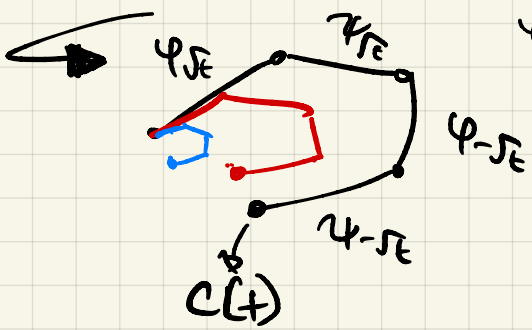
$$(t_1, \dots, t_{10^{44}}, z) \longmapsto \varphi^{10^{44}}_{t_{10^{44}}} \circ \cdots \circ \varphi^1_{t_1}(z).$$

7 **Cor** $[X,Y]=0 \Rightarrow$ their <u>flows commute</u>!

$\hookrightarrow$ don't req lin independence

When $K=2$ Any X, Y C^k v-fields

$X = \varphi_t$, $Y = \psi_s$



For $t<0$ do it backwards

Thm $c(t)$ is diff C^{k-1}

$c'(0) = [X,Y](p)$