Thm) M smooth mfd. Then TFAE:

1) M is orientable

2) M has an atlas $\{U_\alpha, \varphi_\alpha\}$ s.t. $\forall \alpha, \beta$ we have $\det DT_{\alpha,\beta} > 0$ where defined

$\hookrightarrow$ transition map.

lemma) let V some fin dim v.s dim n / $\mathbb{R}$.

let $\mu \in \Lambda^n V$ $\longrightarrow$ let $e_1, \dots, e_n$ basis of V (essentially $\mathbb{R}^n$)

let $A \to n \times n$ matrix. (A_{ij})

let $f_i = \sum_{j=1}^{n} A_{ij} e_j \in V$

Then $\mu(f_1, \dots, f_n) = \det A \, \mu(e_1, \dots, e_n)$.

Pf)

$$\mu(f_1, \dots, f_n) = \mu\left(\sum_{j=1}^{n} a_{1j} e_j, \dots, \sum_{j=1}^{n} a_{nj} e_j\right)$$

$$= \sum_{J} a_{1j_1} \cdots a_{nj_n} \, \mu(e_{j_1}, \dots, e_{j_n})$$

$\to$ all possible non repeat perm

$\hookrightarrow$ using alt can remove terms with double e_j.

$$= \sum_{\sigma \in S(n)} a_{1\sigma(1)} \cdots a_{n\sigma(n)} \, \mu(e_{\sigma(1)}, \dots, e_{\sigma(n)})$$

$$= \sum_{\sigma \in S_n} (-1)^{\text{ord}\,\sigma} \prod_{i=1}^{n} a_{i\sigma(i)} \, \mu(e_1, \dots, e_n)$$

$$= \det A \, \mu(e_1, \dots, e_n)$$

Formal) Spse W, V & $F : W \to V$ linear.

Get pull back

$$\Lambda^k V \xrightarrow{F^*} \Lambda^k W \qquad \alpha \in \Lambda^k V$$

$$(F^*\alpha)(w_1, \dots, w_k) = \alpha \circ F(w_1, \dots, w_k)$$

Lemma) Spse $\dim V = \dim W = n$

$F : W \longrightarrow V$ linear, $e_1, \dots, e_n$ basis of V

$f_1, \dots, f_n$ basis of W

Let $\varepsilon_1, \dots, \varepsilon_n$ dual basis of V^*

$\phi_1, \dots, \phi_n$ dual basis of W^*

Let A be the matrix of F wrt $e_1, \dots, e_n$, $f_1, \dots, f_n$

i.e. $F(f_i) = \sum_{j=1}^{n} a_{ij} e_j$

Then, (Claim) $F^*(\underbrace{\varepsilon_1 \wedge \dots \wedge \varepsilon_n}_{\in \Lambda^n V}) = \det A \, (\phi_1 \wedge \dots \wedge \phi_n)$

Pf) Outline.

A: Repeat prev argument.

B: V, W are isomorphic by linear map L

$V \underset{L}{\cong} W$

Then pull everything back by L into V.

So, we do entire calculation in 1 vs.

or even $(F^*(\varepsilon_1 \wedge \dots \wedge \varepsilon_n))(f_1, \dots f_n)$

$= (\varepsilon_1 \wedge \dots \wedge \varepsilon_n)(F(f_1), \dots F(f_n))$

$= (\varepsilon_1 \wedge \dots \wedge \varepsilon_n)(\sum a_{1j} e_j, \dots \sum a_{nj} e_j)$

prev $= \det A \, \underbrace{(\varepsilon_1 \wedge \dots \wedge \varepsilon_n)(e_1, \dots, e_n)}_{1}$

$= \underline{\det A}$!

$\therefore$ since $\Lambda^n V, \Lambda^n W$ are 1-dim then done!

Defⁿ M, N mfd $T: M \to N$ smooth.
$\Phi^k N \; (= \Lambda^k T^* N)$
$\Rightarrow (\Phi^* \alpha)_p = (D\Phi)^*_p (\alpha)_{\Phi(p)}$

Let apply this lin alg

Let $\hat{u} \xrightarrow{T} \hat{v}$, $\hat{u}, \hat{v}$ open $\subseteq \mathbb{R}^n$.

secretly $\varphi_\alpha(W_\alpha)$ $\quad T_{\alpha,\beta} \quad$ $\varphi_\beta(U_\beta)$ $\quad$ T diffeo.

$\hat{u}$ has coords $y_1, \dots, y_n$
$\hat{v}$ has coords $x_1, \dots, x_n$

$dx_1 \wedge \dots \wedge dx_n$, $(dx_i)_q$ are dual to $\left(\frac{\partial}{\partial x_i}\right)_q$

we

Q: $\left(T^*(dx_1 \wedge \dots \wedge dx_n)\right)_p = \det DT_p \, (dy_1 \wedge \dots \wedge dy_n)_p$

as $DT_p : T_p\hat{u} \to T_q\hat{v}$

$T_p\hat{u} \cong \mathbb{R}^n$, $T_q\hat{v} \cong \mathbb{R}^n$ with diff basis should follow from previous home

$\left(\frac{\partial}{\partial y_i}\right)$ basis $\qquad \left(\frac{\partial}{\partial x_j}\right)$ basis

dy_i $\qquad dx_j$ dual basis

$\to E_i$ from prev $\qquad \to \Theta_j$ from prev