Lec 15 —

Fibre Bundle

$$\begin{tikzcd} F \arrow[r] & E \arrow[d, "\pi"] \\ & B \arrow[u, bend left, "\sigma"] \end{tikzcd}$$

fibre, total, base

$\sigma : B \to E$ is called a **section** if $\pi \circ \sigma = id_B$ → (cts, diff'ble, smooth)

e.g. $TM = E$ → tangent bundle

$$\begin{tikzcd} TM \arrow[d, "\pi"] \\ M \arrow[u, bend left, "\sigma"] \end{tikzcd}$$

section

π = footpt map

D: A **vector field** V or $X : M \to TM$ is a section
(ensures that you map to tangent on the base pt.)

e.g. $M = \mathbb{R}^n$, there are special v-field e_1, or e_2 or ....
if e_i then $\frac{\partial}{\partial x_i} = e_i$
if $X : \mathbb{R}^n \to \mathbb{R}^n$ is any v-field, we can write it as

$$X = a_1(p)\frac{\partial}{\partial x_1}(p) + \cdots + a_n(p)\frac{\partial}{\partial x_n}(p)$$

coefficient functions

$$X(x_1, x_2) = e^{x_1 + x_2}\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}$$

X cts $\iff \forall\, a_i$ cts
diff'ble diff'ble

differentiable structure

What does diff mean? $\sigma : M \to TM$ so σ is diff'ble

How to check in prac?

$$\begin{tikzcd} TM \arrow[d, "\pi"] \\ M \arrow[u, bend right, "X"'] \end{tikzcd}$$

TM C^∞ mfd

M gen C^∞ mfd

→ take a chart $(U_\alpha, \varphi_\alpha)$ for M

$U_\alpha \subseteq M$
$\downarrow \varphi_\alpha$
$\mathbb{R}^n$

$D\varphi_\alpha : TU_\alpha \to T(\varphi_\alpha(U_\alpha))$

$z \in \varphi_\alpha(U_\alpha)$

is $z \mapsto D\varphi_\alpha(X(\varphi_\alpha^{-1}(z)))$ C^∞?

Fact/Ex) if M, N mfd $f: M \to N$ C^∞

$Df: TM \to TN$ $\quad Df(\underset{\substack{\wedge \\ TM}}{p, v}) = (\underbrace{f(p), Df_p(v)}_{TN})$

$\Rightarrow$ Df is C^∞

$\hookrightarrow$ check using coord

$(D\psi_\beta) \circ Df \circ D\varphi_\alpha^{-1} : TU_\alpha \to T(\psi_\beta(V_\beta))$

$\quad \parallel$

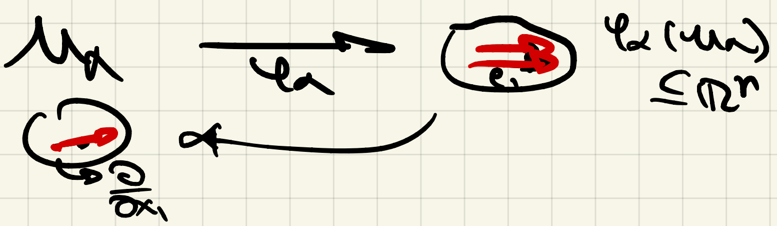
$D(\underbrace{\psi_\beta \circ f \circ \varphi_\alpha^{-1}}_{\to \text{smooth}})$ so this is too!

if C^k then C^{k-1}

back to vector field....

to check smoothness (or any calc)

Write X in a chart $(U_\alpha, \varphi_\alpha)$



write on U_α

$\frac{\partial}{\partial x_i}$ for the pullback of e_i

$$\frac{\partial}{\partial x_i}(p) = D(\varphi_\alpha^{-1})_{\varphi_\alpha(p)}(e_i)$$

$$X\big|_{U_\alpha} = \sum a_i(p) \frac{\partial}{\partial x_i}(p)$$

$a_i(p) \to$ i^{th} coord fune fr $(U_\alpha, \varphi_\alpha)$

Warning: $\frac{\partial}{\partial x_i}$ are vector fields.

There will also be $\partial x_i \to$ cotangent fields / diff 1-form

$(TM)^* =: T^*M = \Lambda^1 M \to$ alternating 1 form

Def) $\partial x_i \in (TM)^*$ at every $p \in U_\alpha$

$\{\partial x_i(p)\}_{1 \le i \le n}$ is the dual basis to $\{\frac{\partial}{\partial x_i}(p)\}$

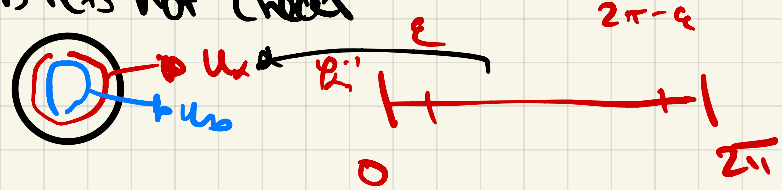
$\to V$ with basis $v_1, \ldots, v_n$ then $w_1, \ldots, w_n \in V^*$ is the dual basis $w_j(v_i) = \delta_i^j$

e.g. 1 $M = S^1$

$X : S^1 \longrightarrow TS^1$

$p = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \longmapsto \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$

→ lets not cheat



$\longrightarrow t \xrightarrow{\varphi_\alpha^{-1}} e^{it}$ on $(\varepsilon, 2\pi - \varepsilon)$

let some $X(\varphi_{\alpha_1}^{-1}(t)) = \sin t \frac{\partial}{\partial t}$

next chart $\varphi_{\alpha_2}^{-1} : t \longmapsto e^{it}$ $[0, \frac{\pi}{2} - \varepsilon) \cup (\frac{\pi}{2} + \varepsilon, 2\pi]$

find map that agrees

Note) if σ_1, σ_2 sections $\sigma_1, \sigma_2 \left(\begin{array}{c} E \\ \downarrow \pi \\ M \end{array} \right.$ v-bundle

$(f_1 \sigma_1 + f_2 \cdot \sigma_2)(p) = f_1(p)\sigma_1(p) + f_2(p)\sigma_2(p)$

$f_1, f_2 : M \to \mathbb{R}$

Note: C^k sections of v-bundle are a module over $C^k(M)$

Derivation

A map $\partial : C^\infty(M) \longrightarrow \mathbb{R}$ is a derivation at p, $\boxed{M \to \mathbb{R}}$

if $\forall f, g \in C^\infty(M) \longrightarrow \partial(fg) = f(p)\,\partial(g) + \partial(f$