Lec 6

Two vector bundles over M & N respectively

$$\begin{tikzcd} V \arrow[r, "\Phi"] \arrow[d, "\pi"] & W \arrow[d, "\varrho"] \\ M \arrow[r, "\varphi"] & N \end{tikzcd}$$

Call W & V isomorphic as v-bundles if $\exists\, \Phi : V \to W$, and $\varphi : M \to N$ diffeo

s.t. Φ takes fibres to fibres

$\Phi(\pi^{-1}(p)) = \varrho^{-1}(\varphi(p))$

and $\forall\, p \in M$

$\Phi|_{\pi^{-1}(p)} : \pi^{-1}(p) \longrightarrow \varrho^{-1}(\varphi(p))$

is linear $\quad$ v sp

Define (Φ, φ) to be v bundle map.

Call (Φ, φ) v-bundle isom if $\exists\, (\Psi, \psi)$ v bd map which is inv of

Comments abt trivial vb

call W vec bundle on M if $V \overset{\text{isom}}{\cong} M \times \mathbb{R}^l$ and $\mathbb{R}^l \cong \pi^{-1}(p)$, $\forall p \in M$

if we have a trivial bundle $M \times \mathbb{R}^l \xrightarrow{\pi} M$

get section $\sigma_i(m) = (m, e_i)$

if $\{e_1, \dots, e_l\}$ basis for $\mathbb{R}^l$ then $\{\sigma_i(p)\}$ lin ind.

$\Rightarrow$ it is a basis for $\pi^{-1}(p)$ $\forall p \in M$

$\Rightarrow$ get l lin ind sections of $M \times \mathbb{R}^l \xrightarrow{\pi} M$

Conversely, if $V \xrightarrow{\pi} M$ is a v-bd &

$\exists\, l$ sections $\sigma_1, \dots, \sigma_l$ of π

and $l = \dim \pi^{-1}(p)$ then V is isomorphic to trivial bd

Pf) $\mathbb{R}^l \times M \xrightarrow{\Phi} V$

$((a_1, \dots, a_l), p) \longmapsto \sum_{i \in \mathbb{N}_l} a_i \sigma_i(p)$

Cor) The tgt bdl of a ∂ mfd M is trivial

$\Longleftrightarrow \exists \dim M$ many vector fields, lin ind @ each pt!

Ex) $S^2 = M$ $\quad X$ = non zero vector field (will be zero somewhere) anywhere

dim $\mathbb{R}X \subseteq \{\text{sections of } TM\}$

scalar null $\qquad$ $\underline{\dim \mathbb{R}X = 1}$

Q) dim $\left\{\text{section } \begin{matrix} TM \\ \downarrow \\ M \end{matrix}\right\} = \infty$ (unless dim $M = 0$)

Derivations

Def) M smooth mfd. The derivation @ $p \in M$

$S : C^\infty(M, \mathbb{R}) \to \mathbb{R}$

such that

$S(f \cdot g) = S(f) \cdot g + S(g) \cdot f$

e.g) $v \in T_pM \qquad v = [c(t)]$

$\delta_v(f) = \frac{\partial}{\partial t}\Big|_{t=0} f(c(t))$ = direct deriv of f at p in direct v

$= \delta_v(f)$

$\Rightarrow T_pM$ gives rise to derivations.

Can also look at a smooth vf X on M

$A : C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R})$ $\qquad X(p) \in T_pM$

$Af(p) := \delta_{X(p)}(f)$

Then $A(f \cdot g) = f \cdot Ag + Af \cdot g$ $\rightsquigarrow$ $\underline{\text{true pointwise}}$

e.g) $X = y \cdot \frac{\partial}{\partial x}$, $Y = x \frac{\partial}{\partial y}$ on $\mathbb{R}^2$

$(X \circ Y) = (y \frac{\partial}{\partial x})(x \frac{\partial}{\partial y}) \to$ should be $y \cdot (\frac{\partial}{\partial x} x) \frac{\partial}{\partial y} + \underline{yx \frac{\partial \partial}{\partial x \partial y}}$??

note: $(x \frac{\partial}{\partial y})(y \frac{\partial}{\partial x}) = x \frac{\partial}{\partial y} + yx \frac{\partial}{\partial x} \frac{\partial}{\partial y}$ $\rightsquigarrow$ 2nd der. term

Spose

$X : C^\infty(M) \to C^\infty(M)$, $Y : C^\infty(M) \to C^\infty(M)$

$\left.\begin{matrix} X \circ Y \\ Y \circ X \end{matrix}\right\} C^\infty M \to C^\infty M$

consider $X \circ Y - Y \circ X = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} \to$ vector field

Thm) for X, Y smooth C^∞ vect field on M

Then

$X \circ Y - Y \circ X$ is a derivation $\to [X, Y]$

Lie bracket of VF

Pf) Check prod rule

Thm) every derivation δ at p defines a unique tangent vector

ie. $\exists\, v \in T_pM$ so $\delta = \partial_v$

Cor) every derivation $\delta : C^\infty(M) \to C^\infty(M)$ defines a vector field (global derivation)

e.g.) $\mathbb{R}^2$ $X = \frac{\partial}{\partial x_1} = e_1$ $Y = \frac{\partial}{\partial x_2} = e_2$

$[X, Y] = 0$