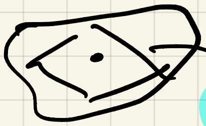
Last time

distributions (k-plane fields)



$V(p) \subset T_pM$

V subsp

$\dim(V(p)) = k$

V is C^∞ if locally $\exists$ k lin ind smooth v.f. $X_1, \dots, X_k$

so $V(p) = \sum_{i=1}^{k} \mathbb{R}X_i = \langle X_1(p), \dots, X_k(p) \rangle$

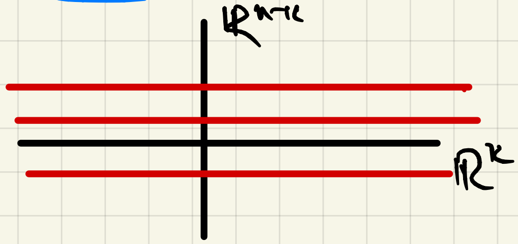
two kinds of distributions.

1) Integrable $\to$ we call V integrable if $\forall p \in M$ $\exists$ coord chart (U, φ) s.t. $D\varphi(V(x)) \subseteq T_{\varphi(x)}\mathbb{R}^n$

& Non

$$D\varphi(V(x)) = (x_1, \dots, x_k, \dots 0) \to \text{canonically embedded}$$

e.g. 1 $\quad M = \mathbb{R}^n \quad V(p) = \langle \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k} \rangle$



e.g. of foliation (cut $\mathbb{R}^n$ into k dim surfaces)

Def | M is a smooth mfd $\mathcal{F}$ = partition of M by immersed sub

$\forall x \in M$ F_x = immersed submfd of M

s.t. $F_x \cap F_y \neq \emptyset \Leftrightarrow F_x = F_y$ i.e. equiv reln

and $\forall p \in M$ $\exists$ coord chart (U, φ) s.t.

U

look at $(F_x \cap U)^\circ$ $\to$ conn comp could loop like the 8 is one guy!

This is a foliation

Given a foliation $\mathcal{F}$, F_x — 'leaf' of $\mathcal{F}$ through x

Then can define $V(p) = T_pF_p$ k-dim dist, C^∞ by

Consider X, Y v.f on M s.t $\forall p \in M$ $X(p), Y(p) \in V(p) = T_p\mathcal{F}_p$

By hw 7.3) $[X,Y](p) \in V(p)$

S.)

If $V(p)$ is a tangent distribution to a foliation $\tilde{\mathcal{F}}$

i.e $V(p) = T_p\mathcal{F}_p$

Then any v.f. X, Y s.t $X(p), Y(p) \in V(p) \Rightarrow [X,Y](p) \in V(p)$

Def Given any smooth k-dim dist V on C^∞-mfd M

call V integrable if

Frobenius Thm | Both notions of integrability agree (in the sense of charts and holds)

Pf) $(\Rightarrow)$ showed

$(\Leftarrow)$ hard, Lee 490

e.g) $\mathbb{R}^3$ and $V(p) = p + \mathbb{R}^2$

Clear it is integrable (or by both def)

Now quotient by $\mathbb{Z}^3$

$\mathbb{R}^3/\mathbb{Z}^3 = T^3$

$\mathbb{R}^3 \xrightarrow{\pi}$

$\bar{V}(\bar{p}) = D\pi(V(p))$ for $\bar{p} = \pi(p)$

Get 2-Tori foliating T^3

Mess it up $V(p)$ = angle irration with $\mathbb{R}^2$

$= \mathbb{R}v_1 + \mathbb{R}v_2$ when v_1, v_2 irr wrt $\mathbb{Z}^3$

Then $\mathbb{R}^3/\mathbb{Z}^3$ get $\bar{V}(p) = D\pi(V(p))$

push down

T^3 is foliated by "planes" that wrap around densely

like  dimension above.

Pf ($\Leftarrow$) Frobenius

by Friday

can find chart so

$X_i = \frac{\partial}{\partial x_i}$

Special case: Span $\forall p \ \exists\ \underset{\text{nbhd}}{U} \ni p$ w/ v.f. $X_1, \ldots X_k$ v.f. on U so

since so don't vanish

$\forall q \in U \ \langle X_1(q), \ldots X_k(q) \rangle = V(q)$

$[X_i, X_j] = 0 \ \forall i, j$