Math 591 - Lec 3

Recall...

Lemma | A second ctbl, Hausdorff, locally compact space admits an exhaustion by cmpct sets.

Pf | ..... defined K_i Pf | Lee's book

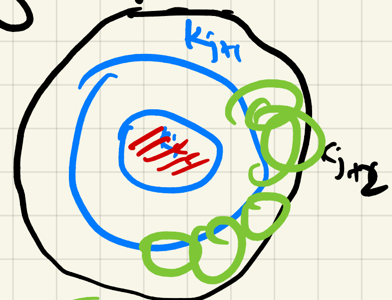
Thm | Topological Mfds M are paracompact! → open covers have locally finite refinements

Pf | By previous, find an exhaustion by cmpct sets K_j. → no repeats

Let $V_j = K_{j+1} \setminus (\text{Int } K_j)$ → closed → cmpct → rings; $\text{Int } K_j \to K_j^\circ$

$W_j = \text{int } K_{j+2} \setminus K_{j-1}$

V_j cmpct & W_j open. $V_j \subseteq W_j$



Given an open cover $\mathcal{X}$.

Given $x \in M$. Let $\mathcal{X}_x \in \mathcal{X}$ be a set in $\mathcal{X}$ containing x

Let $\mathcal{B}$ be a ctbl basis let B_x st $x \in B_x \subseteq \mathcal{X}_x$ (openness)

V_j cmpct $\Rightarrow \exists$ fin many B_{x_ℓ} which cover V_j

$\{B_{x_\ell}\}$ refinement of $\mathcal{X}$ → can arrange so $B_{x_\ell} \subseteq W_j$ → doesn't touch K_{j-1}

$\Rightarrow$ locally finite.

Deep Fact | $\varphi : \mathcal{U}^{\text{open}} \subseteq \mathbb{R}^n \to V^{\text{open}} \subseteq \mathbb{R}^\ell$ homeo $\Rightarrow n = \ell$ inv of domain

SD | $\varphi : (a,b) \to \mathcal{U} \subseteq \mathbb{R}^\ell$ open $\Rightarrow \ell = 1$. Can show by disc.

Cor | For M conn, dim is well def. dim $M = \ell$ s.t open nbhds of pts in M are homeo to $\mathbb{R}^\ell$.

Conv | Assume dim is const for all conn comp.

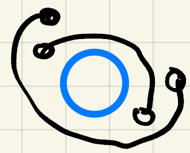
Prop | M top mfd. M is conn $\Leftrightarrow M$ is path conn

$\Leftarrow$ obv. $\Rightarrow$ idea $p \cdot \quad \cdot q$ $X = \{y \in M \mid \exists \text{ cts path } p \to y\}$

Claim: X is clopen. open is easy as balls exist. points of closure are in set $\overline{X}$!

Def) Let M a top mfd has a diff structure (C^∞) $\to 1, \dots, \infty, \omega$ i.e.
$\exists$ a cover by open sets U_i of M & homeos
$\varphi_i : U_i \to V_i \subseteq \mathbb{R}^n$
s.t. $V_j \xrightarrow{\varphi_j^{-1}} U_i \cap U_j \xrightarrow{\varphi_i} V_i$ with app rest.

$\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$ open to open!
where $\varphi_i \circ \varphi_j^{-1}$ is C^k.
This is an atlas!

e.g. $M = \mathbb{R}^7$, $U_1 = \mathbb{R}^7$, $\varphi_1 = 1_{\mathbb{R}^7}$
e.g. $S_1 = \{x \in \mathbb{R}^2 \mid |x| = 1\}$ need 2  2-sphere — stereographic proj