Last time!

if M C^{∞} mfd, ν n-form volume.

Then $\int_M f\nu \to$ well def (but dep on choice of vol form)

difference in $\mathbb{R}^n \to$ No prefered volume form!

On $\mathbb{R}^n$ it's $dx_1 \wedge \dots \wedge dx_n$

Some other cases! M = Lie grp take $X_1, \dots, X_n$ basis of Lie $G = \mathfrak{g}$
i.e. a basis of left inv v.f.
$\hookrightarrow$ let $\eta_1, \dots \eta_n$ be a dual basis at $1 \in G$
make η_i left inv then

$$\eta_i(x_j) = \begin{cases} 1 & i=j \\ 0 & \text{else} \end{cases}$$

Then take $\eta_1 \wedge \dots \wedge \eta_n$ left inv

This ν_L

(Def: if G Lie Then $\exists!$ left inv vol form up to sc mult)

Also exists unique (up to sc) right inv vol form
$\hookrightarrow$ call it ν_R.

When is $\nu_L = \nu_R$? It is Not always.

e.g. $G = \left\{ \begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix} \middle| a \neq 0, b \right\}$

$\hookrightarrow$ $\boxed{ax+b \text{ grp}}$

Lie $\mathfrak{g} = \left\{ \begin{pmatrix} A & B \\ 0 & -A \end{pmatrix} \right\}$ (can check $\nu_L \neq \nu_R$ here)

This is called solvable

But for abelian it's obviously true
nilpotent
$SL(n, \mathbb{R})$

Call such grps unimodular

Turns out compact G
Also if $\exists \Gamma < G$ so G/Γ compact then

2. If M is a Riemannian Mfd & oriented

$\Rightarrow$ Riemannian Metric $\overset{\text{induces}}{\hookrightarrow}$ volume form!

3. Spse $\vec{M}$ has a special vol form ν from V.

Spse $\Gamma \curvearrowright M$ prop disc.

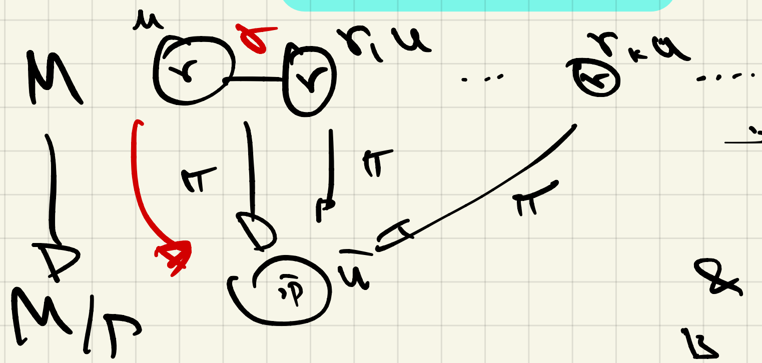
at M/Γ is a mfd (C^∞)

$\Rightarrow$ if ν is Γ inv then ν desc to M/Γ

i.e on M/Γ we get $\bar{\nu}$ = proj of ν

$\pi : M \to M/\Gamma$ submersion so locally can push forward $\nu \to \bar{\nu}$

$\Rightarrow$ i.e. $\nu = \pi^*(\bar{\nu})$



if push down from u using local diffeo & note ν from γ, u is same as ν from u

$\Rightarrow$ ν on u is $(\gamma_1^{-1})^*(\nu)$

e.g. 9.4) If $M^{2n \text{ dim}}$ has a 2-form α so that

$\underbrace{\alpha \wedge \alpha \cdots \wedge \alpha}_{n\text{-times}}$ $\to$ Does not vanish anywhere!

ν

do this!

More Gen?

k dim simplex

C is a smooth map. $\to$ must discuss smoothing on edges (bdry)

let α – k form on M $\Rightarrow$ $c^*(\alpha)$ – k form on Δ

Then $\int_\Delta c^*(\alpha) =: \int_c \alpha$

Exterior Der

$\alpha \to$ k-form on M

$\hookrightarrow$ we want assoc $d\alpha$ $(k+1)$ form on M

How: 0-form is scalar val func $f: M \to \mathbb{R}$

df - 1 form

T_pM $\quad df(v) =$ direct der of f along v

$\Lambda^k M \to$ k-forms on M

$\forall k$ $\quad d: \Lambda^k M \to \Lambda^{k+1} M$ $\quad$ linear / $\mathbb{R}$

$$\Lambda^0 M \xrightarrow{d} \Lambda^1 M \xrightarrow{d} \Lambda^2 M \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^n M \xrightarrow{d} 0$$

Want) 1) d agrees with d of Λ^0

2) $d^2 = 0$

3) $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$ $\quad$ k is dim of α

$\hookrightarrow$ Leibniz rule.

Thm) $\exists!$ d satisfy the prop ①-③

Goal) Poincaré lemma!

On $\mathbb{R}^n$ if α has $d\alpha = 0$ (closed form)

$\Rightarrow \exists \beta$ so that $\alpha = d\beta$

$\hookrightarrow$ converse always holds i.e if $\alpha = d\beta$ α is closed!!

$\operatorname{Im} d_{\Lambda^{k-1}M} < \ker(d_{\Lambda^k})$ $\quad$ Consider $\ker(d_{\Lambda^k}) / \operatorname{Im} d_{\Lambda^{k-1}M}$

note

$\| $

$H^k M$

de Rham