Relative Homology

Goal, Given $A \subset X$ understand reln b/w the following

$$H_n(X),\ H_n(A),\ H_n(X/A)$$

Reduced Singular C.C

$$\tilde{C}_\bullet(X): \cdots \to C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

$$\sum m_i \sigma_i \longmapsto \sum m_i$$

Note: $\varepsilon \circ \partial_1(\sigma) = \varepsilon(\sigma(1) - \sigma(0)) = 1 - 1 = 0$

$\Rightarrow \varepsilon \circ \partial_1 \equiv 0 \Rightarrow \tilde{C}_\bullet(X)$ is a c.c !

Reduced Singular Hom

$$\tilde{H}_n(X) := H_n(\tilde{C}_\bullet(X))$$

Prop

1) $\tilde{H}_n(X) = H_n(X) \quad \forall n > 0$
2) $H_0(X) \cong \tilde{H}_0(X) \oplus \mathbb{Z}$ when $X \neq \emptyset$
3) $\tilde{H}_n(X)$ def a functor $hTop \to Ab$
4) X p.c (nonempty) $\Rightarrow \tilde{H}_0(X) = 0$
5) X contr $\Rightarrow \tilde{H}_n(X) = 0 \quad \forall n$

Defn A seq of homos & ab gps (possibly (in)finite)

$$\cdots \to C_{n+1} \xrightarrow{f_{n+1}} C_n \xrightarrow{f_n} C_{n-1} \to \cdots$$

is exact if $\ker f_n = \operatorname{im} f_{n+1} \ \forall n \in \mathbb{N}$

Ex

$0 \to A \xrightarrow{f} B$ exact $\Leftrightarrow f$ inj

$A \xrightarrow{f} B \to 0$ exact $\Leftrightarrow f$ surj

$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ ex $\Leftrightarrow f$ inj, g surj, $\ker g = \operatorname{im} f$

$\Rightarrow B/A \cong C$.

$C_\bullet : \to C_{n+1} \to C_n \to C_{n-1}$

exact $\Leftrightarrow H_n(C_\bullet) = 0 \ \forall n$

Defn A good pair (X,A) is a sp X w/ a closed nonempty subsp $A \subseteq X$ s.t. $\exists$ open $U \subset X$ w/ $A \subseteq U$ & U str. defo retr to A.

Lemma (Hatcher Prop A.5) (X,A) CW pair $\Rightarrow$ (X,A) good pair

Thm (Long exact seq for good pair)

(X,A) good pair $\Rightarrow$ $A \xhookrightarrow{i} X \xrightarrow{j} X/A$

$$\cdots \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \xrightarrow{j_*} \tilde{H}_n(X/A) \xrightarrow{\delta} \tilde{H}_{n-1}(A) \xrightarrow{i_*} \tilde{H}_{n-1}(X) \xrightarrow{j_*} \tilde{H}_{n-1}(X/A) \xrightarrow{\delta} \cdots$$

$$\to \tilde{H}_0(A) \xrightarrow{i_*} \tilde{H}_0(X) \xrightarrow{j_*} \tilde{H}_0(X/A) \to 0$$

is exact! (define δ later)

Pf later

Appl $\tilde{H}_i(S^n) = \begin{cases} \mathbb{Z} & i=n \\ 0 & \text{else} \end{cases}$

Pf induct on n.

$\underline{n=0}$: $H_i(S^0) \cong H_i(\{*\})^{\oplus 2} = \begin{cases} \mathbb{Z}^{\oplus 2} & i=0 \\ 0 & \text{else} \end{cases}$

$\Rightarrow \tilde{H}_i(S^0) \cong \begin{cases} \mathbb{Z} & i=0 \\ 0 & \text{else} \end{cases}$

$\underline{n>0}$: $(D^n, \partial D^n) = (D^n/S^{n-1})$ good pair

$D^n/\partial D^n \cong S^n$ from prev. Hw

$\to \tilde{H}_i(S^{n-1}) \to \cancel{\tilde{H}_i(D^n)}^{0} \to \tilde{H}_i(S^n) \to$ $\quad \tilde{H}_i(D^n) = 0 \ \forall i$ as contr.

$\to \tilde{H}_{i-1}(S^{n-1}) \to \cancel{\tilde{H}_{i-1}(D^n)}^{0} \to \tilde{H}_{i-1}(S^n)$

C_0 $\quad \tilde{H}_0(S^{n-1}) \to \cancel{\tilde{H}_0(D^{n-1})}^{0} \to \tilde{H}_0(S^n) \to 0$

$\Rightarrow \tilde{H}_i(S^n) = \tilde{H}_{i-1}(S^{n-1}) \quad \forall i > 0$

for $i = 0$ know $\tilde{H}_0(S^n) = 0$ (p.c.) $\Big\}$ done by induction

<u>Cor</u>| ∂D^n not a retract of D^n

$\Rightarrow$ any $D^n \to D^n$ has a <u>fixed point</u>!

<u>Relative CC</u>| $C_\cdot(X,A)$ with <u>X & subsp A</u>.

$i : A \to X$ incl get $\quad i_\# : C_n(A) \to C_n(X)$

$\sigma \mapsto i \circ \sigma$

$$C_n(X,A) =: \frac{C_n(X)}{C_n(A)} \quad \&$$

induced by $\partial_n : C_n(X) \to C_{n-1}(X)$

$$C_n(X,A) = \frac{C_n(X)}{C_n(A)} \xrightarrow{\partial_n} C_{n-1}(X,A) = \frac{C_{n-1}(X)}{C_{n-1}(A)}$$

<u>Relative hom</u>| $H_n(X,A) := H_n(C_\cdot(X,A))$

have morphisms of C.C

$$C_\cdot(A) \xrightarrow{i_\#} C_\cdot(X) \xrightarrow{p} C_\cdot(X,A)$$

eg of

<u>Defn</u>| A S.E.S of C.C. $\quad 0 \to A_\cdot \xrightarrow{\alpha} B_\cdot \xrightarrow{\beta} C \to 0$

Cons. of C.C. maps s.t

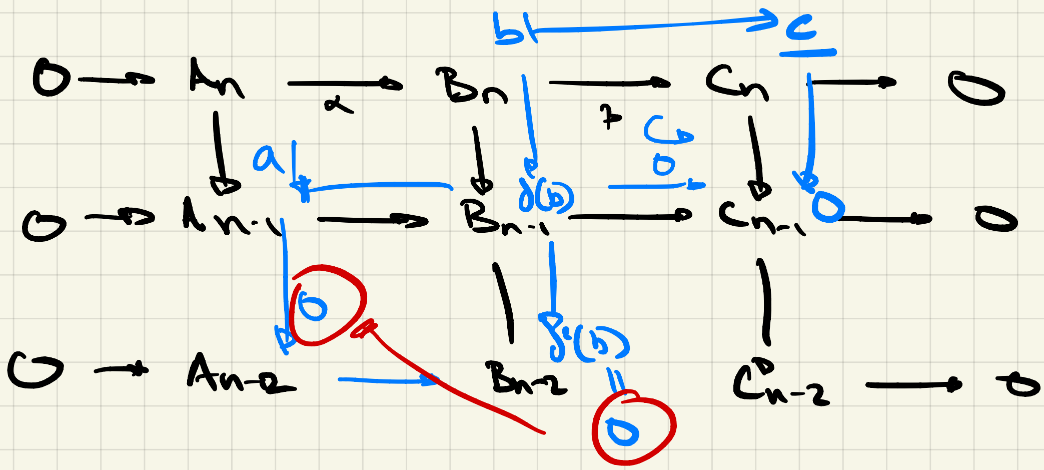
$0 \to A_n \xrightarrow{\alpha_n} B_n \xrightarrow{\beta_n} C_n \to 0$ exact $\forall n$

Prop Given SES of CC as earlier, $\exists$ induces LES

$$\cdots \to H_n(A_\bullet) \xrightarrow{\alpha_*} H_n(B_\bullet) \xrightarrow{\beta_*} H_n(C_\bullet) \xrightarrow{\delta} H_{n-1}(A_\bullet) \to \cdots$$

Pf Defn of δ: let $c \in C_n$ n-cycle

$$\delta([c]) = [a]$$



maps n-cycy to n-cycle!