Recall prop | (X,A) good pair $\Rightarrow$ quot map $q:(X,A)\to(X/A,\ A/A)$ induces isom

$$q_*: H_n(X,A)\xrightarrow{\ \cong\ } H_n(X/A,\ \underset{\{x_0\}}{\underset{\shortparallel}{A/A}}) \cong \tilde H_n(X/A)$$

Pf of prop | $A\subset U^{\text{open}}\subset X$ where U strongly defo retr to A.

$$\begin{tikzcd}
H_n(X,A) \arrow[r, "④"] \arrow[d, "q_*"'] & H_n(X,U) \arrow[d, "q_*"] & H_n(X\setminus A,\ U\setminus A) \arrow[l, "①"'] \arrow[d, "q_*\ ③"] \\
H_n(X/A,\ A/A) \arrow[r, "⑤"'] & H_n(X/A,\ U/A) & H_n((X/A)\setminus(A/A),\ (U/A)\setminus(A/A)) \arrow[l, "②"]
\end{tikzcd}$$

STS 1-5 are isoms

①,② are isos by excision $(\bar A = A\subset \operatorname{int}(U)=U)$

③ $\cong$ b/c $(X\setminus A,\ U\setminus A)\xrightarrow{\ \cong\ }((X/A)\setminus(A/A),\ (U/A)\setminus(A/A))$

④,⑤ $\cong$'s by wk 9 Idea: L.E.S of triple (X,U,A), $A\subset U\subset X$

$$\to \underset{0}{\underset{\shortparallel}{H_n(U,A)}} \to H_n(X,A)\to H_n(X,U)\to \underset{0}{H_{n-1}(U,A)}\to \cdots \quad \text{as } (U,A)\simeq(A,A)$$

(from ses of C_*)

Cor | Let (X_α, x_α) pted sp s.t. $(X_\alpha,\{x_\alpha\})$ good.

$$\Rightarrow \tilde H_n\Big(\bigvee_\alpha X_\alpha\Big)\cong \bigoplus \tilde H_n(X_\alpha)$$

wedge at x_α

Pf |
$$\tilde H_n\Big(\bigvee_\alpha X_\alpha\Big)\cong H_n\left(\frac{\bigsqcup_\alpha X_\alpha}{\bigsqcup_\alpha\{x_\alpha\}},\ \frac{\bigsqcup\{x_\alpha\}}{\bigsqcup\{x_\alpha\}}\right)$$

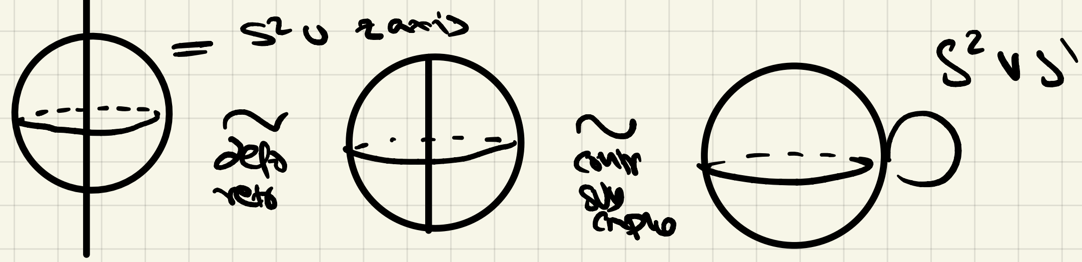
by prop on good pairs
$$\cong H_n\Big(\bigsqcup_\alpha X_\alpha,\ \bigsqcup_\alpha\{x_\alpha\}\Big)$$

$$\cong \bigoplus_\alpha H_n(X_\alpha,\{x_\alpha\})$$

(rel hom version of H_n comm w/ dir sum)

$$\cong \bigoplus_\alpha \tilde H_n(X_\alpha)$$

eg 1 $X :=$



So $\tilde{H}_n(X) \cong \tilde{H}_n(S^2) \vee \tilde{H}_n(S^1)$

$$\cong \begin{cases} \mathbb{Z} & n = 1,2 \\ 0 & \underline{\text{else}}. \end{cases}$$

Thm (Invariance of domain)

$U \subset \mathbb{R}^m$, $V \subseteq \mathbb{R}^n$ nonempty open, $U \cong V$ homeo. $\quad Z = \mathbb{R}^m \setminus U$

$\Rightarrow m = n$ $\quad$ excision for $(\mathbb{R}^m, \mathbb{R}^m \setminus \{x_0\})$

Pf) fix $x \in U$, $H_i(U, U \setminus \{x\}) \cong H_i(\mathbb{R}^m, \mathbb{R}^m \setminus \{x\})$

(from LES + $H_i(\mathbb{R}^m) = 0\ \forall i$) $\quad \cong \tilde{H}_{i-1}(\mathbb{R}^m \setminus \{x\})$

($\mathbb{R}^m \setminus \{x\} \sim S^{m-1}$) $\quad \cong \tilde{H}_{i-1}(S^{m-1})$

$$= \begin{cases} \mathbb{Z} & i = m \\ 0 & \text{else} \end{cases}$$

if $f: U \to V$ homeo, $H_i(U, U \setminus \{x\}) \cong H_i(V, V \setminus \{f(x)\})$

$\Rightarrow \boxed{m = n}$

Singular = Simplicial

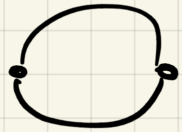
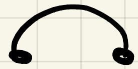
Let X - Δ cmplx, i.e. we have a collection $\Sigma = \{\sigma_\alpha : \Delta^n_\alpha \to X\}$ s.t.

1) $\sigma_\alpha|_{\mathring{\Delta}^n_\alpha}$ inj & $X = \bigsqcup_\alpha \sigma_\alpha(\mathring{\Delta}^n_\alpha)$ open interior

2) σ_α rest to any face in Σ

3) $A \subset X$ open $\Leftrightarrow \sigma_\alpha^{-1}(A) \subset \Delta^n_\alpha$ open $\forall \alpha$.

Def) $A \subset X$ is a sub Δ-cmpx if $A \subset X$ closed & A is disj union of $\sigma_\alpha(\mathring{\Delta}^n_\alpha)$ for $\alpha \in \Sigma'$ where $\Sigma' \subset \Sigma$ ($\Longrightarrow A$ Δ-cmpx too wrt Σ')

E.g.| $S^1 =$  $\supseteq A =$  say (X,A) Δ cmpx pair.

Rel simpl CC|

$$C^\Delta_\bullet(X,A) = \frac{C_\bullet(X)}{C_\bullet(A)}, \quad H^\Delta_n(X,A) = H_n\left(C^\Delta_\bullet(X,A)\right)$$

Have,

1) $H^\Delta_n(X,A) = H_n(X)$ if $A \neq \emptyset$

2) LES

$$\cdots \to H^\Delta_n(A) \to H^\Delta_n(X) \to H^\Delta_n(X,A) \to H^\Delta_{n-1}(A) \to \cdots$$

3) Chain map $C^\Delta_\bullet(X,A) \to C_\bullet(X,A) = \frac{C_\bullet(X)}{C_\bullet(A)}$

$$\rightsquigarrow H^\Delta_n(X,A) \xrightarrow{\varphi} H_n(X,A)$$

Thm| φ is an isom $\forall n$!