Thm) (Van Kampen) Part 1

$X = \bigcup_{i \in J} U_i^{\text{open}}$, $x_0 \in X$ & $x_0 \in U_i \ \forall i \in J$

$U_i \cap U_j$ path conn $\forall i,j$

$$\Longrightarrow \mathop{\ast}_{i \in J} \pi_1(U_i, x_0) \xrightarrow[\text{surj}]{} \pi_1(X, x_0)$$

Pf) Let $\alpha : I \to X$ a path

Choose $0 = s_0 < s_1 < \dots < s_m = 1$ — uses continuity & compact

$\forall \ 1 \le k \le m$ $\exists \ i_k \in J$ s.t. $\alpha([s_{k-1}, s_k]) \subseteq U_{i_k}$

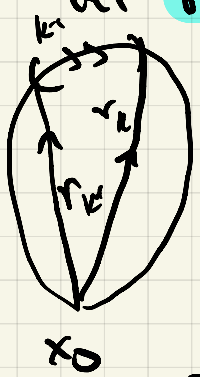
$[\alpha] = [\alpha_1] \cdots [\alpha_m]$ $\alpha_k : I \to X$ is

$s \longmapsto \alpha((1-s)s_{k-1} + s\, s_k)$

$\alpha|_{[s_{k-1}, s_k]}$ reparam to I

Choose $\gamma_k : I \to U_{i_k} \cap U_{i_{k+1}}$ path $x_0 \to x_k = \alpha(s_k)$ for $1 \le k \le m-1$

Let $\gamma_0 = c_{x_0} = c_{x_m} = \gamma_m$



$$[\alpha] = \underbrace{[\gamma_0 \cdot \alpha_1 \cdot \bar{\gamma}_1]}_{\substack{\text{loop in } U_{i_1} \\ \text{at } x_0}} \cdot [\gamma_1 \cdot \underbrace{\alpha_2 \cdot \bar{\gamma}_2]}_{\substack{\text{loop in } U_{i_2} \text{ at} \\ x_0}} \cdots [\gamma_{m-1} \cdot \alpha_m \cdot \gamma_m]$$

$\Rightarrow$ written α as product of paths in even U_{i_k} $\square$

Thm | Van Kampen 2

If also $U_i \cap U_j \cap U_k$ path conn $\forall i, j, k$
$\Rightarrow$ given a solid comm diagram (for any i, j)

$$\begin{tikzcd}
 & \pi_1(U_i) \arrow[dr, "f_{i*}"] \arrow[drr, bend left, "\psi_i"] & & \\
\pi_1(U_i \cap U_j) \arrow[ur, "g_{ij*}"] \arrow[dr, "g_{ji*}"'] & & \pi_1(X) \arrow[r, dotted, "\exists!\,\psi"] & H \\
 & \pi_1(U_j) \arrow[ur, "f_{j*}"'] \arrow[urr, bend right, "\psi_j"'] & &
\end{tikzcd}$$

Pf | let $\alpha : I \to X$ loop at x_0

Choose a presentation,

$[\alpha] = [\gamma_0 \alpha_1 \bar{\gamma}_1] \cdots [\gamma_{m-1} \alpha_m \bar{\gamma}_m]$

where $0 = s_0 < s_1 \cdots < s_m = 1$, $i_k \in J$ so $\alpha([s_{k-1}, s_k]) \subseteq U_{i_k}$

& α_k is $\alpha|_{[s_{k-1}, s_k]}$ reparam to I

∘ γ_k in $U_{i_k} \cap U_{i_{k+1}}$ for $1 \le k \le m-1$ from x_0 to $x_k = \alpha(s_k)$

$\gamma_0 = c_{x_0} = c_{x_m} = \gamma_m$

Set $h_k := \psi_{i_k}([\gamma_{k-1} \cdot \alpha_k \cdot \bar{\gamma}_k]) \in H$

$h_\alpha := h_1 \cdots h_m \in H$

Claim: h_α indep of all choices & rep of path homo cls of α.

Assume Cl $\Rightarrow \exists$ well def $\psi : \pi_1(X) \to H$
$[\alpha] \mapsto h_\alpha$

let $[\beta] \in \pi_1(X)$ $\beta = [\delta_0 \beta_1 \bar{\delta}_1] \cdots [\delta_{n-1} \beta_n \bar{\delta}_n]$
pres for β

$\Rightarrow [\alpha \cdot \beta] = [\alpha][\beta] = [\gamma_0 \alpha_1 \bar{\gamma}_1] \cdots [\gamma_{m-1} \alpha_m \bar{\gamma}_m][\delta_0 \beta_1 \bar{\delta}_1] \cdots$
is a pres for $[\alpha \cdot \beta]$ $\psi([\alpha \cdot \beta]) = \psi([\alpha]) \cdot \psi([\beta])$

$\Rightarrow \Psi$ is a grp homo.

<u>WTS</u> following commutes $f_i : U_i \hookrightarrow X$

$$\begin{tikzcd}
\pi_1(U_i) \arrow[rr, "\Psi_i"] \arrow[dr, "f_{i*}"'] & & H \\
& \pi_1(X) \arrow[ur, "\Psi"'] &
\end{tikzcd}$$

$\Psi(f_{i*}([\alpha])) \longrightarrow$ take $0 = s_0 < s_m = 1$

$\Psi([f_i \circ \alpha]) = \Psi_i([\alpha])$

only in 1 thing

this is a pf

By <u>cons</u> Ψ is the unique grp homo making diag comm $\forall i,j$

<u>Pf Thm 1</u> Road map

① Indep of γ_k

② Indep of indexing set J

③

④ Indep of rep of $[\alpha]$

① Indep of γ_k's

γ_k' path in $U_{i_k} \cap U_{i_{k+1}}$ from x_0 to $x_k = \alpha(s_k)$

$$h_k' = \Psi_{i_k}([\gamma_{k-1}' \alpha_k \bar{\gamma}_k'])$$

$$h_k = \Psi_{i_k}([\gamma_{k-1}\underbrace{\bar{\gamma}'_{k-1}][\gamma'_{k-1}}_{\text{cancel}}\alpha_k \underbrace{\bar{\gamma}'_k][\gamma'_k}\bar{\gamma}_k])$$

so true $\boxed{\text{think}}$

$$= \Psi_{i_k}([\gamma_{k-1}\bar{\gamma}'_{k-1}]) \cdot h_k' \cdot \Psi_{i_k}([\gamma_k \bar{\gamma}'_k])^{-1}$$

$\Psi_{i_k} = \Psi_{i_{k+1}}$ on <u>overlap</u>

$\Rightarrow h_1 \cdots h_m = h_1' \cdots h_m'$ cause cancel & endpts are const maps!

② Indep of choice of $i_k \in J$

Pf| Fix k say $\alpha([s_{k-1}, s_k]) \subset U_{i_{k'}}$

so $i_{k'} \in J$ new indexing point now

By step 1 we may compute h_α by using a path φ_k in $U_{i_{k'}} \cap U_{i_k} \cap U_{i_{k+1}}$

& φ_{k-1} in $U_{i_{k-1}} \cap U_{i_k} \cap U_{i_{k'}}$

$\Rightarrow$ can choose same pres of α as a prod that works with i_k & $i_{k'}$

contained in both

$\Rightarrow$ get same h_α both ways!

③