Defn $f: S^n \to S^n$ $(n>0)$

Assume, $\exists\, y \in S^n$ s.t. $f^{-1}(y) = \{x_1, \dots, x_n\}$ is finite

Choose nbhd $V \ni y$ and disj nbhd $U_1, U_2, \dots, U_n$ so $x_i \in U_i$ s.t. $f(U_i) \subseteq V$

$f: (U_i, U_i \setminus \{x_i\}) \longrightarrow (V, V \setminus \{y\})$ map of pairs as $f(x_i) = y$

$$f_*: H_n(U_i, U_i\setminus\{x_i\}) \longrightarrow H_n(V, V\setminus\{y\})$$

$$\begin{array}{ccc} \cong & \text{nfd} & \cong \\ \mathbb{Z} & \text{—— monday ——} & \mathbb{Z} \\ & \text{(so) as nfd} & \end{array}$$

is mult by an integer : local degree of f at x_i

(so) $\deg_{x_i}(f) \in \mathbb{Z}$

Exer doesn't dep on choice of V, x_i.

Prop $\deg(f) = \sum_{i=1}^{n} \deg_{x_i}(f)$

Pf We have comm diag as following. $\longrightarrow$ it does com

exer

$$\begin{tikzcd}
1 \arrow[r, mapsto] & \deg(f) \\
H_n(S^n) \arrow[r, "f_*"] \arrow[d] \arrow[dr] & H_n(S^n) \arrow[d] \\
H_n(S^n, S^n\setminus\{x_1,\dots,x_n\}) \arrow[r, "f_*"] \arrow[ur, phantom, ""] & H_n(S^n, S^n\setminus\{y\}) \\
H_n(\sqcup U_i, \sqcup U_i\setminus\{x_i\}) \arrow[u, "\cong\ \text{(excision)}"] \arrow[r, dashed] & H_n(V, V\setminus\{y\}) \arrow[u, "\text{(excision)}\ \cong"'] \\
\bigoplus_{i=1}^{n} H_n(U_i, U_i\setminus\{x_i\}) \arrow[u, "\cong\ \text{(disj union)}"] \arrow[ur] & \sum_{i=1}^{n} \deg_{x_i}(f) \\
(1,1,\dots,1) \arrow[ur, mapsto] &
\end{tikzcd}$$

$H_n(S^n, S^n\setminus\{x_i\})$

iso $\sum \deg_{x_i}(f)$

idea exer

blue comm

& upper arrow iso for LES of pair so maps $1 \to 1$

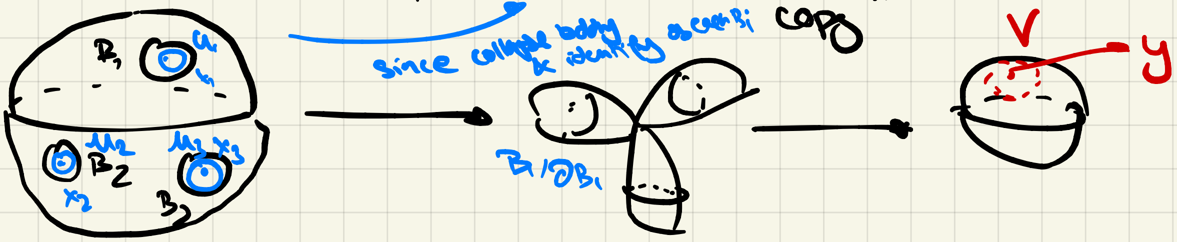
&

$H_n(S^n, S^n\setminus\{x_i\})$
$\cong$ $\boxed{\text{nfd}}$
$H_n(U_i, U_i\setminus\{x_i\})$

Eg | Choose $B_1, \dots, B_k \subset S^n$ disj union of open discs

$$f: S^n \xrightarrow[S^n \setminus \cup B_i]{\text{collapse}} \bigvee_{i=1}^{k} S^n \xrightarrow[\text{on each copy}]{\text{identity}} S^n$$

since collapse bdry & identity on each B_i



local degree of $\deg_{x_i}(f) = 1$

as each $U_i \xrightarrow{\text{identity}} V$.

$\Rightarrow \deg f = \sum_{i=1}^{n} \deg_{x_i}(f) = \underline{k}$.

If r refl $\deg(r \circ f) = -k$

$\therefore \forall k \in \mathbb{Z}$ $\exists f: S^n \to S^n$ w $\deg(f) = k$

$\hookrightarrow$ deg map surj.

Cellular Homology |

Say X CW complex. $X = \bigcup_{n \geq 0} X^n$ (n-skeleton), X^n built from X^{n-1} by attaching D^n_α along $\varphi_\alpha : \partial D^n_\alpha \to X^{n-1}$

$n \geq 0$ | $C_n^{CW}(X) :=$ free ab grp on n-cells. ($e^\alpha_n = \operatorname{int} D^n_\alpha$ n-cell) ($e^0 = pt$) $\boxed{\text{conv}}$.

$:= \left\{ \text{formal finite sums of } \sum m_\alpha e^n_\alpha \;\middle|\; m_\alpha \in \mathbb{Z},\ e^n_\alpha \text{ n-cells} \right\}$

Want $\partial_n : C_n^{CW}(X) \longrightarrow C_{n-1}^{CW}(X)$

$n=1$: $\partial_1(e^1_\alpha) = \varphi_\alpha(1) - \varphi_\alpha(0)$ → [difference of endpts]

$n>1$: $\partial_n(e^n_\alpha) = \sum_\beta d_{\alpha,\beta}\, e^{n-1}_\beta$ where

$d_{\alpha\beta}$ is degree $S^{n-1}_\alpha \xrightarrow{\varphi_\alpha} X^{n-1} \xrightarrow[X^{n-1} \setminus e^{n-1}_\beta]{\text{coll}} D^{n-1}_\beta / \partial D^{n-1}_\beta \cong S^{n-1}$

Σ is finite as S^{n-1}_α is compact

Thm This gives a chain cmplx $C_\bullet^{cw}(X)$

& $H_n(C_\bullet^{cw}(X)) \cong H_n(X)$

$\quad\quad \| $

$H_n^{cw}(X)$ **Pf** later.

cellular hom

ex S^n CW complex w/ 1 n-cell e^n & 1 zero cell e^0

$C_\bullet^{cw}(S^n): \cdots 0 \to \mathbb{Z}\langle e^n\rangle \longrightarrow \cdots \to \mathbb{Z}\langle e^0\rangle \to 0 \to$

$\Rightarrow H_i^{cw}(S^n) = \begin{cases} \mathbb{Z} & i=0,n \\ 0 & \text{else.} \end{cases}$ if $n>1$

(if $n=1$, need to show $\partial = 0$ w $0 \to \mathbb{Z}\langle e^1\rangle \to \mathbb{Z}\langle e^0\rangle \to$

but $\partial(e^1) = e^0 - e^0 = 0$ so $\partial = 0$)