Lefschetz Contd.

Σ = set of simpl.

Step 2 | X is a finite simpl. complex.

Suppose now $f: X \to X$ fixed pt free (WTS, $\Lambda_f = 0$)

key input (Hatcher Simpl. Approx):

$\exists$ simpl compl structure Σ' refinement on X $(\Sigma' \supseteq \Sigma)$
and map $g: X \to X$ so,

a) $f \sim g$ htpic

b) g cellular wrt Σ'

c) $\forall \sigma \in \Sigma' \Rightarrow g(\sigma) \cap \sigma = \emptyset$

above: really main ing

$\hookrightarrow$ follows from no fixed pt's

will show $\Lambda_g = 0$ then done as $f \sim g$.

HW 11: g cellular $\Rightarrow$ g induces $g_\#: C_0^{cw}(X, \Sigma') \to C_0^{cw}(X, \Sigma')$
$\hookrightarrow$ cellular cc wrt Σ'

$\sigma \in \Sigma'$ a simplex, $g(\sigma) \cap \sigma = \emptyset \Rightarrow$ coeff of σ in $g_\#(\sigma)$ is 0

$\Rightarrow \mathrm{tr}(g_\#: C_0^{cw}(X, \Sigma') \to C_0^{cw}(X, \Sigma')) = 0$

as diagonal entries in mat rep for $g_\#$ are zero by
wrt the $\sigma \in \Sigma'$ basis!

$$0 = \sum (-1)^n \,\mathrm{tr}(g_\#: C_0^{cw}(X, \Sigma') \to C_0^{cw}(X, \Sigma'))$$

$$= \sum (-1) \,\mathrm{tr}(g_*: H_0^{cw}(X) \to H_0^{cw}(X))$$

$\hookrightarrow$ $\boxed{\text{by hw 12}}$ 2nd generalization of lemma abt Euler Char.

$\hookrightarrow$ $\boxed{\text{hom vs basis}}$

$\boxed{\text{by hw 11}}$ $= \Lambda_g = \Lambda_f$

Homology w/ Coefficients X = Top sp, G ab grp (if $G=\mathbb{Z}$ its old stuff.)

$$C_n(X;G) = \bigoplus_{\substack{n\text{ simpl.}\\ \sigma:\Delta^n\to X}} G \qquad \text{if } G=\mathbb{Z} \text{ then free ab grp } n\text{-simpl}$$

$\leftrightarrow$ {finite formal sum $\sum n_i\sigma_i$, $n_i \in G$.}

can define $\partial_n : C_n(X;G) \to C_{n-1}(X;G)$

$$\sum n_i\sigma_i \longmapsto \sum_{i,j} (-1)^j n_i \sigma_i|_{[v_0,\dots,\hat{v_j},\dots,v_n]}$$

$\partial_n \circ \partial_{n+1} = 0$ (same comp as before.)

get c.c, $C_\bullet(X;G)$: $\cdots \to C_n(X;G) \to C_{n-1}(X;G) \to \cdots$

singular cc w/ coeff in G

$C_\bullet(X;\mathbb{Z}) = C_\bullet(X)$.

Def) $H_n(X;G) = H_n(C_\bullet(X;G))$

$H_n(X;\mathbb{Z}) = H_n(X)$

Similarly, can define

- $\tilde{H}_n(X;G)$ reduced $H_n(\cdots \to C_0(X;G) \to G \to 0)$ — sum coeff
- $H_n(X,A;G)$ relative hom for $A \subseteq X$ subsp.
- $H_n^\Delta(X;G)$ Simpl. version
- $H_n^{cw}(X;G)$ cellular version.

All of the theorems for usual homology generalize directly!

1) LES assoc (X,A)
2) Mayer-Vietoris
3) Comparisons w/ simpl, cw homology (i.e equiv).

e.g. 1) $H_i(S^n; G) \simeq \begin{cases} G & i = 0, n \\ 0 & \text{else} \end{cases}$

Using normal CW-struct

$$C_\bullet(X; G) : \cdots \to 0 \to \overset{n}{G} \to 0 \cdots\cdots \overset{0}{G} \to 0$$

2) $\mathbb{R}P^n$, $G = k$ a field.

mult by e_{n+1} ... $\overset{0}{}$

$$C_\bullet^{CW}(X; G) : \cdots \to \overset{n}{k} \to k \to \cdots \xrightarrow{0} k \xrightarrow{2} k \xrightarrow{0} k \to 0$$

if char $k = 2 \Rightarrow$ all maps are 0

$$\Rightarrow H_i(\mathbb{R}P^n; k) = \begin{cases} k & i = 0, \dots, n \\ 0 & \text{else} \end{cases}$$

if char $k \neq 2 \Rightarrow k \xrightarrow{2} k$ is $\cong$

$$\Rightarrow H_i(\mathbb{R}P^n; k) = \begin{cases} k & i = 0, \; i = n \text{ odd} \\ 0 & \text{else} \end{cases}$$

e.g. X = attach D^4 to S^3 along

$\partial D^4 : S^3 \to S^2$ of degree $m > 0$.

$\pi : X \to X/S^2 = S^4$ (kinda $D^4 / \partial D^4$)

Is π htpic to const map? check if $\pi_* = 0 \;\forall n$

$$C_\bullet^{CW}(X) : 0 \to \underset{4}{\mathbb{Z}} \xrightarrow{m} \underset{3}{\mathbb{Z}} \to \cdots \to \underset{0}{\mathbb{Z}} \to 0$$

So $\tilde{H}_n(X) = \begin{cases} \mathbb{Z}/m & n = 3 \\ 0 & \text{else} \end{cases}$

$\Rightarrow \pi_* : \tilde{H}_n(X) \to \tilde{H}_n(X/S^2)$ is zero as one of the grps is zero $\forall n$

$\hookrightarrow$ concentrated in deg 3 $\qquad$ $\hookrightarrow$ conc. in deg 4

No info!

$C_\bullet^{cw}(X; \mathbb{Z}/m): 0 \to \mathbb{Z}/m \xrightarrow[m=0]{} \mathbb{Z}/m \to 0 \to 0 \to \mathbb{Z}/m \to 0$

$\Rightarrow \tilde{H}_n(X; \mathbb{Z}/m) = \begin{cases} \mathbb{Z}/m & n = 4, 3 \\ 0 & \underline{\text{else}}. \end{cases}$ 2nd mult by m is zero map

$\Rightarrow$ can have nontrivial map on 4th hom grp.

S^4

LES (X, S^3): $\underset{=0}{\tilde{H}_4(S^3; \mathbb{Z}/m)} \to \tilde{H}_4(X; \mathbb{Z}/m) \xrightarrow{\pi_*} \underbrace{\tilde{H}_4(X/S^3; \mathbb{Z}/m)}_{\text{since good pair}}$

$\therefore$ have injection $\tilde{H}_4(X; \mathbb{Z}/m) \xrightarrow{\pi_*} \tilde{H}_4(S^4; \mathbb{Z}/m)$
(actually isom by contity)

$\tilde{H}_4(X; \mathbb{Z}/m) = \mathbb{Z}/m$ as not zero like $\mathbb{Z}$, $\tilde{H}_4(S^4; \mathbb{Z}/m) = \mathbb{Z}/m$

So nonzero $\Rightarrow \pi \not\simeq$ const map