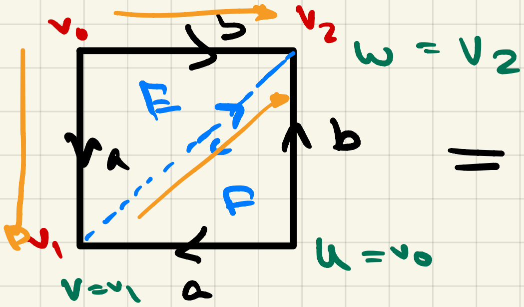
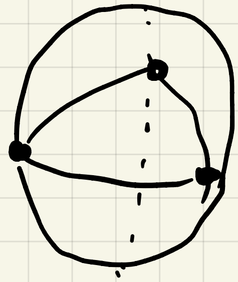
E.g Δ-cmplx.

S^2 =  = 

2-simplices E, F, map corr to $E \to \Delta^2 \longrightarrow S^2$
$\cong$ $[v_0, v_1, v_2]$
canonical iso

1-simplices a, b, c specified
$\Delta_1 \longrightarrow S^2$ spec by orange arrow

0-simplices u, v, w

In contrast to, CW complex struct on S^2  1 0-cell, 1 2-cell } not Δ cmplx struct

Let| X be a top sp w Δ-cmplx struct. → dep on chosen struct

$C_n^\Delta(X)$ = free abelian grp on the n-simplices of Δ struct

$$= \left\{ \begin{array}{l|l} \text{finite sums} & m_\alpha \in \mathbb{Z} \\ \sum m_\alpha \cdot \sigma_\alpha & \sigma_\alpha : \Delta^n \to X \text{ } n\text{-simplex} \end{array} \right\}$$

Co elts here are called simplicial n-chain

Idea| Construct $H_n(X)$ by considering n-chains that "form a cycle" → "close up"
& quotient by n-chains arising as bdry.

Eg| want to capture via H_2 that S^2 has 2-dim'l hole.

Bdry homo|

$$\partial_n : C_n^\Delta(X) \longrightarrow C_{n-1}^\Delta(X)$$

suffices to def on gens → extend linearly

$$\sigma_\alpha \longmapsto \sum_{i=0}^{n} (-1)^i \sigma_\alpha \big|_{\underbrace{[v_0, \dots, \hat{v}_i, \dots v_n]}_{(n-1)\text{-simplex in } X \text{ by } \Delta \text{ cmplx struct!}}}$$

i-th face

Eg| S^2, $\partial^2(E) = \overset{[v_1,v_2]}{c} - \overset{[v_0,v_2]}{b} + \overset{[v_0,v_1]}{a}$

$\partial^2(F) = c - b + a$

$\partial^2(E - F) = 0 \longrightarrow$ close up.

Lemma $\partial_{n-1} \circ \partial_n : C_n^{\Delta}(X) \to C_{n-2}^{\Delta}(X) = 0$

Pf $\partial_{n-1} \circ \partial_n (\sigma) = \partial_{n-1}\left(\underbrace{\sum_{i=0}^{n} (-1)^i \sigma}_{\text{lin}}\Big|_{[v_0, \dots \hat{v}_i, \dots v_n]}\right)$

$$= \sum_{i=0}^{n} (-1)^i \left(\begin{array}{l} \sum_{j<i} (-1)^j \sigma\Big|_{[v_0, \dots \hat{v}_j, \dots, \hat{v}_i, \dots v_n]} \\ + \sum_{j>i} (-1)^{j-1} \sigma\Big|_{[v_0, \dots \hat{v}_i, \dots \hat{v}_j \dots v_n]} \end{array} \right.$$

$$= \sum_{j<i} (-1)^{i+j} \sigma\Big|_{[v_0, \dots \hat{v}_j, \dots \hat{v}_i, \dots v_n]} + \sum_{j>i} (-1)^{i+j-1} \sigma\Big|_{[v_0, \dots, \hat{v}_i, \dots \hat{v}_j, \dots v_n]} = 0$$

Defn A chain complex (of abelian grps) is a sequence of homomorphisms (of ab grps) $n \in \mathbb{Z}$

$$C_\bullet \to \cdots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots$$

st $\partial_n \circ \partial_{n+1} = 0$ $\forall n \in \mathbb{Z}$ $\Rightarrow$ $\mathrm{im}(\partial_{n+1}) \subset \ker(\partial_n)$

Term C_n - grp of chains, ∂_n - differential

$\ker \partial_n \subset C_n \to$ grp of cycles

$\mathrm{im}\, \partial_{n+1} \subset C_n \to$ bdrys

Def $H_n(C_\bullet) := \dfrac{\ker \partial_n}{\mathrm{im}\, \partial_{n+1}}$

Def if X Δ-cmplx $\rightsquigarrow$ Simplicial chain cmplx

$$C_\bullet^{\Delta}(X) : \cdots \to C_{n+1}^{\Delta}(X) \to C_n^{\Delta}(X) \to C_{n-1}^{\Delta}(X) \to \cdots$$

(terms $= 0$ for $n < 0$)

Def) Simplicial homology grps Sub sure

$$H^\Delta_n(X) := H_n(C^\Delta_\bullet(X)) = \frac{\ker \partial_n}{\operatorname{im} \partial_n}$$

Thm) 1) $H_n^\Delta(X)$ doesn't dep ~~(up to $\cong$)~~ on Δ-cmplx struct on X

2) If X, Y sp w/ Δ-cmplx struct

& $X \underset{\text{hom equiv}}{\simeq} Y \Rightarrow H_n^\Delta(X) \cong H_n^\Delta(Y)$ $\forall n$

Pf) Later.

E.g.) $X = \{*\}$ pt 0-simplex v

$$C_n^\Delta(X) = \begin{cases} \mathbb{Z}\langle v \rangle & n=0 \\ 0 & \text{else} \end{cases}$$

$\mathbb{Z}\langle v\rangle$ → free ab grp w/ v.

$$\cdots \longrightarrow 0 \xrightarrow{\partial_1} \mathbb{Z} \xrightarrow{\partial_0} 0 \longrightarrow \cdots$$

$\ker \partial_0 = \mathbb{Z}$

$\operatorname{im} \partial_1 = 0$

$$H_n^\Delta(X) = \begin{cases} \mathbb{Z} & n=0 \\ 0 & \text{else} \end{cases}$$