Prop| (Lifting Crit) Given the solid diag

$$\begin{tikzcd}
 & (\tilde{X}, \tilde{x}_0) \arrow[d, "p\text{-covering sp}"] \\
(Y, y_0) \arrow[r, "f"] \arrow[ur, dotted, "\tilde{f}"] & (X, x_0)
\end{tikzcd}$$

w Y path conn, locally path conn

$\exists$ lift $\tilde{f} : (Y, y_0) \to (\tilde{X}, \tilde{x}_0)$

$\iff f_* \pi_1(Y, y_0) \subseteq p_* \pi_1(\tilde{X}, \tilde{x}_0)$

Pf| $\implies$ by comm of diagram as $f_* = (p \circ \tilde{f})_*$

$\impliedby$ $y \in Y$, choose path α in Y from y_0 to y

$\rightsquigarrow$ get path in X from x_0 to $f(y)$ by post-comp

$\exists !$ lift of $f \circ \alpha$ called $\widetilde{f \circ \alpha}$ to a path

$\tilde{X}$ st at $\tilde{x}_0$ defn $\tilde{f}(\alpha) := \widetilde{f \circ \alpha}(1)$

$\hookrightarrow$ by constr, if we show well def we are done!

Cl| doesn't depend choice of α. $\longrightarrow$ wts lift to $\tilde{X}$ has same endpt

Say α' is another path in Y from $y_0 \longrightarrow y$

$\rightsquigarrow f \circ \alpha'$ path in X from x_0 to $f(y)$

Let $\beta := (f \circ \alpha') \cdot \overline{(f \circ \alpha)}$ loop at x_0

$= f \circ (\alpha' \cdot \bar{\alpha}) \rightsquigarrow$ defn

$\in f_* \pi_1(Y, y_0) \subseteq p_* \pi_1(\tilde{X}, \tilde{x}_0)$ by assu

$\therefore$ the lift $\tilde{\beta}$ of β to $\tilde{X}$ starting at $\tilde{x}_0$ is a loop by the fact we are in the img of p_*.

by the uniqueness of lifts of paths

$\tilde{\beta}|_{[0,1/2]} = \widetilde{f \circ \alpha'}$ reparam to $[0, 1/2]$ similar for $\tilde{\beta}|_{[1/2,1]}$

$\implies \widetilde{f \circ \alpha'}(1) = \overline{\widetilde{f \circ \alpha}}(0) \implies$ same endpt!!

Now we need to show continuity of $\tilde{f}$

Pf) let $y \in Y$ choose

$$\begin{tikzcd} \tilde{f}(y) \in \tilde{U} \arrow[d, "p|_{\tilde{U}}"', "\cong"] & \subseteq \tilde{X} \arrow[d, "p"] \\ f(y) \in U & \subseteq X \end{tikzcd}$$

sts $\tilde{f}(V) \subseteq \tilde{U}$

Choose nbhd $y \in V \subseteq Y$ so V path conn & $V \subseteq f^{-1}(U)$ (by cont & loc path conn)

Choose α path in Y from y_0 to y

for $y' \in V$ choose path η from y to y' in V

$$\tilde{f}(y') := \widetilde{f \circ (\alpha \cdot \eta)}(1)$$

by defn

$$= (\widetilde{f \circ \alpha}) \cdot (\widetilde{f \circ \eta})(1)$$

tildes move outside by unique path lifting.

$$= (\widetilde{f \circ \alpha}) \cdot (p|_{\tilde{U}}^{-1} \circ f \circ \eta)(1)$$

lift of $f \circ \eta$ st at $\tilde{f}(y)$

$$\in \tilde{U}$$

So $\tilde{f}|_V = p|_{\tilde{U}}^{-1} \circ f|_V$ as $\tilde{f}(V) \subseteq \tilde{U}$ $\square$.

↳ cts locally

Lemma (Uniqueness of lifts): Given

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ Y \arrow[ur, shift left, "f_1"] \arrow[ur, shift right, "f_2"'] \arrow[r, "f"'] & X \end{tikzcd}$$

p cov sp

f_1, f_2 lifts of f i.e. diag commutes

If Y conn & f_1, f_2 agree at some pt in $Y \Rightarrow f_1 = f_2$

In particular $\tilde{f}$ in prev prop unique (agree at y_0).

Pf| let $Z = \{y \in Y \mid \tilde{f}_1(y) = \tilde{f}_2(y)\} \neq \emptyset$ by assum

STS Z clopen $\Rightarrow Z = Y$ by conn!

Fix $y \in Y$ w/ $f(y) \in U \subseteq X$ evenly cov nbhd

$p^{-1}(U) = \bigsqcup \tilde{U}_i$ $\quad U_i \xrightarrow[p]{\cong} U$

↳ sheets

let $\tilde{U}_1$, $\tilde{U}_2$ be sheets containing $\tilde{f}_1(y)$, $\tilde{f}_2(y)$

↳ by conn land in $\sqcup \tilde{U}_i$

Choose nbhd $y \in V \subseteq Y$ st by cts

$\tilde{f}_1(V) \subseteq \tilde{U}_1$ & $\tilde{f}_2(V) \subseteq \tilde{U}_2$

if $\tilde{f}_1(y) \neq \tilde{f}_2(y) \Rightarrow \tilde{U}_1$ & $\tilde{U}_2$ disj.

$\Rightarrow \tilde{f}_1(y') \neq \tilde{f}_2(y')$ for $y' \in V$

$\Rightarrow Z$ closed.

Similarly if $\tilde{f}_1(y) = \tilde{f}_2(y) \Rightarrow \tilde{U}_1 = \tilde{U}_2$

$\Rightarrow \tilde{f}_1(y') = \tilde{f}_2(y')$ by homeomorphism. $p|_{\tilde{U}_i}$

↳ $\tilde{f}_1|_V = \tilde{f}_2|_V$ as $p \circ \tilde{f}_1|_V = p \circ \tilde{f}_2|_V$

So Z open

Def| Given $p_i : \tilde{X}_i \to X$ $\quad i = 1, 2$

Covering sp.

A morphism of covering sp is a map $f : \tilde{X}_1 \to \tilde{X}_2$

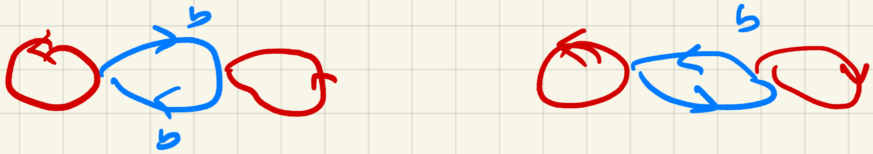
st $p_1 = p_2 \circ f$ → resp covering sp str

An iso is a morphism w/ inverse!

equiv f homeo!

↳ in addition.

e.g.)



Cover $S^1 \vee S^1$

Clearly $\cong$

want the homeo that flips middle & last circle.

Cov isom of covering sp.

Prop) X path conn loc path conn & $x_0 \in X$

& $p_1 : \tilde{X}_1 \to X$ $\quad p_2 : \tilde{X}_2 \to X$ cov sp.

w/ $\tilde{X}_1$ & $\tilde{X}_2$ p/c $\quad$ fix $\tilde{x}_1 \in p_1^{-1}(x_0)$ $\quad \tilde{x}_2 \in p_2^{-1}(x_0)$

$\exists$ isom of cov sp. $\tilde{X}_1 \cong \tilde{X}_2$ f with $f(\tilde{x}_1) = \tilde{x}_2$

$$\Updownarrow$$

$$p_* \pi_1(\tilde{X}_1, \tilde{x}_1) = p_* \pi_2(\tilde{X}_2, \tilde{x}_2)$$

Pf) $\Rightarrow$ ✓ apply functoriality

$\Leftarrow$ Lifting criterion gives f, g

$$\begin{tikzcd}
 & (\tilde{X}_2, \tilde{x}_2) \arrow[d, "p_2"] \arrow[dl, "g"] \\
(\tilde{X}_1, \tilde{x}_1) \arrow[r, "p_1"'] \arrow[ur, bend left, "f"] & (X, x_0)
\end{tikzcd}$$

So comm!

$$\begin{tikzcd}
\tilde{X}_1 \arrow[rr, "g \circ f"] \arrow[dr, "p_1"'] & & \tilde{X}_1 \arrow[dl, "p_1"] \\
 & X &
\end{tikzcd}$$

morph of cov sp st

$(g \circ f)(\tilde{x}_1) = \tilde{x}_1$ $\quad \therefore g \circ f = id_{\tilde{X}_1}$ by uniqueness of lifting