Goal $H_n(X)$ is a htpy invariant. (i.e. a functor $h\mathrm{Top} \to \mathrm{ab}$

Let $f: X \to Y$ cts map of top sp $\rightsquigarrow$

$f_\#: C_n(X) \to C_n(Y)$ $\longrightarrow$ n - ommited in notation

$(\sigma: \Delta^n \to X) \longmapsto (f \circ \sigma: \Delta^n \to X)$ (extend linearly)

Note $(\partial \circ f_\#)(\sigma) = \sum_{i=0}^{n} (-1)^n (f_\# \circ \sigma)|_{[v_0, \ldots, \hat{v}_i, \ldots v_n]}$

$= f_\#(\partial(\sigma))$ this is the $n-1$ $f_\#$

$\therefore f_\#: C_n(X) \to C_n(X)$ is an eg of

Def $C_\bullet, D_\bullet$ are chain complexes, a morphism of chain complexes (chain map) $\varphi: C_\bullet \to D_\bullet$ is a collection of homo $\varphi_n: C_n \to D_n \ \forall n$ so the following commutes:

$$\begin{tikzcd}
C_\bullet: \cdots \arrow[r] & C_{n+1} \arrow[r, "\partial^C_{n+1}"] \arrow[d, "\varphi_{n+1}"] & C_n \arrow[r, "\partial^C_n"] \arrow[d, "\varphi_n"] & C_{n-1} \arrow[r, "\partial^C_{n-1}"] \arrow[d, "\varphi_{n-1}"] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "\partial^D_{n+1}"] & D_n \arrow[r, "\partial^D_n"] & D_{n-1} \arrow[r, "\partial^D_{n-1}"] & \cdots
\end{tikzcd}$$

"$\partial \circ \varphi = \varphi \circ \partial$" with appr inter.

Note $\varphi_n(\ker \partial_n^C) \subseteq \ker \partial_n^D$ (by comm of diag)

$\varphi_n(\operatorname{im} \partial_{n+1}^C) \subseteq \operatorname{im} \partial_{n+1}^D$

$\rightsquigarrow$ induced map from $H_n(C_\bullet) \to H_n(D_\bullet)$ by φ_n but den $\varphi_*, H_n(\varphi)$

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back in case, $f: X \to Y$

$f_\#: C_\bullet(X) \to C_\bullet(Y)$

write, $f_*: H_n(X) \to H_n(Y)$ for induced map.

Obs| have functor $H_n: \mathrm{Top} \to \mathrm{Ab}$ → wts this factors thru hTop

$$X \longmapsto H_n(X)$$

$$(f: X \to Y) \longmapsto f_*: H_n(X) \to H_n(Y)$$

(concretely have left $(f \circ g)_* = f_* \circ g_*$

$(\mathrm{id}_X)_* = \mathrm{id}_{H_n(X)}$)

Thm| if $f, g: X \to Y$ htpic $\Rightarrow f_* = g_*: H_n(X) \to H_n(Y)$ $\forall n$

Cor| H_n is a functor $H_n: \mathrm{hTop} \to \mathrm{Ab}$ (factors)

In part if $f: X \to Y$ is a htpy equiv

$\Rightarrow f_*: H_n(X) \to H_n(Y)$ is an isom!

Def| let $\varphi, \psi: C_\bullet \to D_\bullet$ morphisms of chain complexes.
A chain htpy $h: C_\bullet \to D_{\bullet+1}$ from φ to ψ cons of homom $h_n: C_n \to D_{n+1}$ $\forall n$

s.t. $\psi_n - \varphi_n = \partial^D_{n+1} \circ h_n + h_{n-1} \circ \partial^C_n$

$$\begin{tikzcd}
C_\bullet & \cdots \arrow[r] & C_{n+1} \arrow[r] \arrow[d, shift right, "\varphi_{n+1}"'] \arrow[d, shift left, "\psi_{n+1}"] & C_n \arrow[r] \arrow[d, shift right, "\varphi_n"'] \arrow[d, shift left, "\psi_n"] \arrow[dl, "h_n"'] & C_{n-1} \arrow[r] \arrow[d, shift right, "\varphi_{n-1}"'] \arrow[d, shift left, "\psi_{n-1}"] \arrow[dl, "h_{n-1}"'] & \cdots \\
D_\bullet & \cdots \arrow[r] & D_{n+1} \arrow[r] & D_n \arrow[r] & D_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Say in this case, say that φ, ψ are chain htpic den $\varphi \simeq \psi$

Lemma | $\varphi \sim \psi \implies H_n(\varphi) = H_n(\psi) \quad \forall n$

Pf) $\sigma \in \ker(\partial_n^C)$ (rep of $H_n(C_\bullet)$)

$$(\psi_n - \varphi_n)(\sigma) = \partial_{n+1}^D h_n(\sigma) + \cancel{h_{n-1}\,\partial_n^C(\sigma)}^{\,0}$$

$$= \partial_{n+1}^D h_n(\sigma)$$

$$\in \operatorname{im} \partial_{n+1}^D$$

D $\dfrac{\ker \partial_n^D}{\operatorname{im} \partial_{n+1}^D}$

$\Rightarrow \psi_n(\sigma), \varphi_n(\sigma)$ same cl in $H_n(D)$

Pf Thm | given $H: X \times I \to Y$ htpy from $f \to g$

S1 | get chain htpy

$h: C_\bullet(X) \to C_{\bullet+1}(Y)$ from $f_\# \to g_\#$

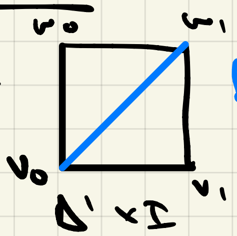
idea | given $\sigma: \Delta^n \to X$, try to define h by.

$$\underbrace{\Delta^n \times I}\xrightarrow{\sigma \times id_I} X \times I \xrightarrow{H} Y$$

↳ but this isn't an $(n+1)$ simplex ... so chop!

Subdivide $\Delta^n \times I$ into $(n+1)$ simplices:

$n=1$



Subdiv

$$\Delta^1 \times I = [v_0, v_1, w_1] \cup [v_0, w_0, w_1]$$

(then restr...)

→ (n simpl)

in gen $\Delta^n \times I$ let $\Delta^n \times \{0\} = [v_0, \dots, v_n]$

$\Delta^n \times \{0\} = [w_0, \dots, w_n]$

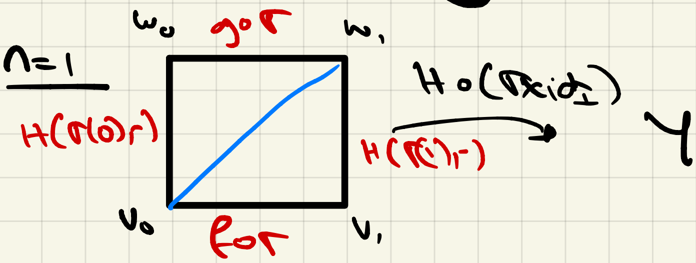
$$\Delta^n \times I = \bigcup_{i=0}^{n} [v_0, \dots, v_i, w_i, w_{i+1}, \dots, w_n]$$

$h_n: C_n(X) \to C_{n+1}(Y)$ → subdivide then alt.

$$(\sigma: \Delta^n \to X) \longmapsto \sum_{i=0}^{n} (-1)^i \left(H \circ (\sigma \times id_I)\right)\Big|_{[v_0, \dots, v_i, w_i, \dots, w_n]}$$

C1. \ h is a chain htpy, $f_\#$ to $g_\#$

$\underline{n=1}$



$$h_1(\sigma) = H\circ(\sigma\times id)\big|_{[v_0,w_0,w_1]} - H\circ(\sigma\circ id)\big)\big|_{[v_0,v_1,w_1]}$$

$$\partial(h_1(\sigma)) = H\circ(\sigma\times id)\big|_{[w_0,w_1]} - \cancel{H\circ(\sigma\times id)}\big|_{[v_0,w_1]} + H\circ(\sigma\times id)\big|_{[v_0,w_0]}$$

$$-\Big(H\circ(\sigma\circ id)\big|_{[v_1,w_1]} - \cancel{H\circ(\sigma\times id)}\big|_{[v_0,w_1]} + H\circ(\sigma\circ id)\big|_{[v_0,v_1]}\Big)$$

Note $H\circ(\sigma\times id)\big|_{[v_0,v_1]} = f\circ\sigma$ (setting time in htpy to be 0)

$H\circ(\sigma\times id)\big|_{[w_0,w_1]} = g\circ\sigma$ 1

$$= g_\#(\sigma) - f_\#(\sigma) + H(\sigma(0),-) - H(\sigma(1),-)$$

$\Delta^0\times I \longrightarrow Y$

$$h_0\partial(\sigma) = h_0(\sigma(1) - \sigma(0))$$

$$= H(\sigma(1),-) - H(\sigma(0),-)$$

$$\therefore\ \partial h_1(\sigma) + h_0\partial(\sigma) = g_\#(\sigma) - f_\#(\sigma)$$

general n works similarly   Hatcher thm 2.10 . $\square$