Mayer-Vietoris

Let $X = \operatorname{int} A \cup \operatorname{int} B$ w/ $A, B \subseteq X$.

recall $\Rightarrow$

$$i : H_n(C_\bullet(A+B)) \xrightarrow{\cong} H_n(C_\bullet(X)) = H_n(X)$$

Have a SES of chain complexes

$$0 \to C_\bullet(A \cap B) \to C_\bullet(A) \oplus C_\bullet(B) \to C_\bullet(A+B) \to 0$$

$\hookrightarrow$ need to defn, $0 \to C_n(A\cap B) \xrightarrow{\varphi} C_n(A) \oplus C_n(B) \xrightarrow{\psi} C_n(A+B) \to 0$

$$u \longmapsto (u, -u)$$

$$(u, v) \longmapsto u + v$$

i.e. if

$$\begin{tikzcd} & A \arrow[dr, "i_A"] & \\ A \cap B \arrow[ur, "j_A"] \arrow[dr, "j_B"'] & & X \\ & B \arrow[ur, "i_B"'] & \end{tikzcd}$$

$\varphi = (j_{A*}, -j_{B*})$

$\psi = i_{A*} - i_{B*}$

Is above short exact sequence.

1) $\operatorname{im}\varphi \subset \ker\psi$ trivially

$\ker\psi \subset \operatorname{im}\varphi$ let $(v, w) \in \ker\psi$

$v + w = 0$, $w = -v \in C_n(A\cap B)$ done!

2) φ is injective

3) ψ is surjective by def of $C_n(A+B)$

Finally these commute w/ ∂. (differential on $C_n(A) \oplus C_n(B)$ is $\partial^A \oplus \partial^B$)

Cor (Mayer Vietoris Seqnce)

check $H_n(C_\bullet(A) \oplus C_\bullet(A)) = H_n(A) \oplus H_n(B)$

Get LES of the following

$$H_n(A\cap B) \xrightarrow{(j_{A*}, -j_{B*})} H_n(A) \oplus H_n(B) \xrightarrow{i_{A*} - i_{B*}} H_n(C_\bullet(A+B)) \cong H_n(X) \xrightarrow{\delta} H_{n-1}(A\cap B)$$

Pf LES of prev SES!

Variants| (exer)

1) If $A\cap B \neq \emptyset$ reduced Mayer-Vietoris LES,

$$\cdots \to \tilde{H}_n(A\cap B) \to \tilde{H}_n(A)\oplus\tilde{H}_n(B) \to \tilde{H}_n(X) \to \tilde{H}_{n-1}(A\cap B) \to \cdots$$

SES $0 \to \tilde{C}_0(A\cap B) \to \tilde{C}_0(A)\oplus \tilde{C}_0(B) \to \cdots \to 0$

Pf| take seq

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & C_1(A\cap B) \arrow[r] \arrow[d] & C_1(A)\oplus C_1(B) \arrow[r] \arrow[d] & C_1(A+B) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_0(A\cap B) \arrow[r] \arrow[d, "\varepsilon"] & C_0(A)\oplus C_0(B) \arrow[r] \arrow[d, "(\varepsilon,\varepsilon)"] & C_0(A+B) \arrow[r] \arrow[d, "\varepsilon"] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

$u \mapsto (u,-u)$

$(u,v) \mapsto \underline{u+v}$.

2) Also have LES if $X = A\cup B$ where

- A defo retr of an open $U \subseteq X$
- B ——— $V \subseteq X$
- $A\cap B$ ——— $U\cap V \subseteq X$

e.g.| (Hatcher Appendix) $X = A\cup B$ w/ A, B subcomplexes!

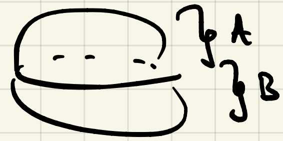
Pf) $H_n(A) \cong H_n(U)$, $H_n(B)\cong H_n(V)$, $H_n(A\cap B) \cong H_n(U\cap V)$

Rmk| Can think of M.V as analogue of V.K for π_1

Will show M.V $\Longrightarrow$ abelianized V.K).

e.g. $X = S^n$ let $A \subset S^n$ top hemisphere $B \subset S^n$ bottom hemi.

$A \cap B = S^{n-1}$



Note: $A, B \cong D^n$

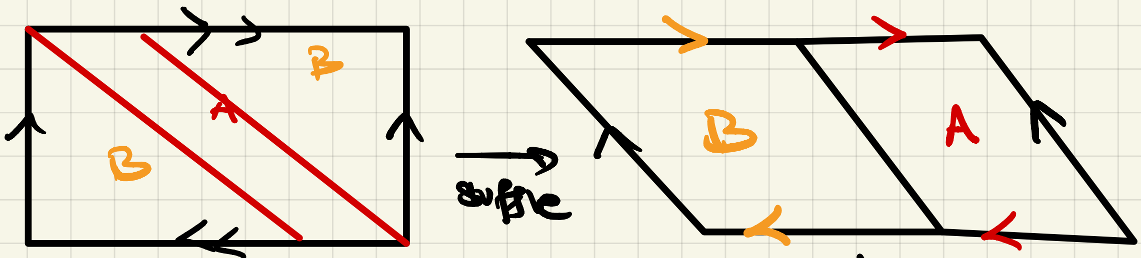
$$\to \tilde{H}_i(A\cap B) \to \cancel{\tilde{H}_i(A)}^{0} \oplus \cancel{\tilde{H}_i(B)}^{0} \to \tilde{H}_i(S^n)$$

$$\to \tilde{H}_{i-1}(A\cap B) \to$$

$$\Rightarrow \tilde{H}_i(S^n) \cong \tilde{H}_{i-1}(A\cap B) \cong \tilde{H}_{i-1}(S^{n-1})$$

$\Rightarrow$ old formula $\tilde{H}_i(S^n) = \begin{cases} \mathbb{Z} & i=n \\ 0 & \text{else} \end{cases}$

e.g. X = Klein



$A \cap B$ = boundary of möbius $\cong S^1$

Möbius $\leftarrow A \cup B \rightarrow$ möbius

MV

$$\rightsquigarrow \cdots \to \tilde{H}_n(A\cap B) \to \tilde{H}_n(A) \oplus \tilde{H}_n(B) \to \tilde{H}_n(X) \to$$

$$\to \tilde{H}_{n-1}(A\cap B) \to 0$$

Note, $A, B \sim S^1$ (defo retr) so most thing are S^1

- $\Rightarrow \tilde{H}_n(X) = 0$ for $n \geq 3$ as sandwiched b/w 3 0s

$$\tilde{H}_n(\cancel{A\cap B})^{0} \to \tilde{H}_n(\cancel{A})^{0} \oplus \tilde{H}_n(\cancel{B})^{0} \to \tilde{H}_n(X) \to \cancel{\tilde{H}_{n-1}(A\cap B)}^{0} \to \cdots^{0}$$

- $n=2$ get $0 \to \tilde{H}_2(X) \to \underset{\mathbb{Z}}{\underset{\parallel}{\tilde{H}_1(A\cap B)}} \to \underset{\mathbb{Z}\oplus\mathbb{Z}}{\tilde{H}_1(A) \oplus \tilde{H}_1(B)} \to \tilde{H}_1(X) \to 0$

inj $= \tilde{H}_n(X) = 0 \leftarrow 1 \longmapsto (2,-2) \rightarrow$

as $i_{A*}: H_1(A\cap B) \longrightarrow H_1(A)$ — n wraps twice

$j_{A*}: \pi_1(A\cap B)^{ab} \longrightarrow H_1(A)$ — $\boxed{HW}$,

$1 \longmapsto 2$

$\tilde{H}_1(X) = \operatorname{coker}(1 \mapsto (2,-2))$

$\tilde{H}_2(X) = 0$, $\tilde{H}_1(X) = \dfrac{\mathbb{Z}\oplus\mathbb{Z}}{\langle(2,-2)\rangle} = \dfrac{\langle(1,0),(1,-1)\rangle}{\langle(2,-2)\rangle} \cong \mathbb{Z}\oplus\mathbb{Z}/2$

$$\therefore \quad H_n(X) = \begin{cases} \mathbb{Z} & n=0 \\ \mathbb{Z}\oplus\mathbb{Z}/2 & n=1 \\ 0 & \text{else} \end{cases}$$