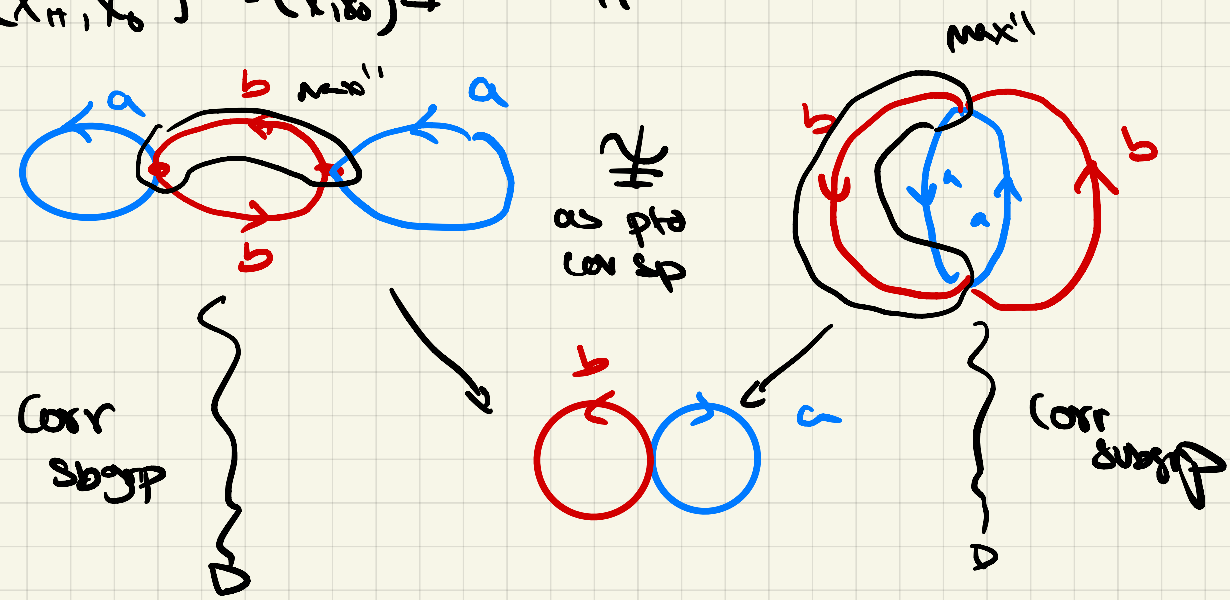
Thm) $x_0 \in X$ pc, lpc, slsc

$$\left\{ \begin{array}{c} (\tilde{X}, \tilde{x}_0) \text{ p.c cov sp} \\ \downarrow p \\ (X, x_0) \end{array} \right\} \Big/ \text{basept cov sp } \cong\text{'s} \quad \xleftrightarrow{\text{bij}} \quad \left\{ \begin{array}{c} \text{Subgrp} \\ H \subseteq \pi_1(X, x_0) \end{array} \right\}$$

$p : (\tilde{X}, x_0) \to (X, x_0) \longmapsto p_* \pi_1(\tilde{X}, x_0)$

$p_H : (\tilde{X}_H, \tilde{x}_0) \to (X, x_0) \longleftarrow H$

e.g.)



$\langle a, b^2, b^{-1}ab \rangle \neq \langle a^{-1}b, ab, b^2 \rangle$

note equality not iso.

$\Big\updownarrow ab \qquad\qquad \Big\updownarrow ab$

$\langle a, 2b, a \rangle \neq \langle b-a, a+b, 2b \rangle \subset \mathbb{Z}^2 = \pi_1(S^1 \vee S^1)^{ab}$

$a \in \qquad\qquad a \notin$

Changing base p.t

Spse $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ cover as abv

$\longleftrightarrow H = p_* \pi_1(\tilde{X}, \tilde{x}_0) < \pi_1(X, x_0)$

1) If $\tilde{x}_1 \in p^{-1}(x_0)$, choose $\tilde{\gamma}$ in $\tilde{X}$ from $\tilde{x}_0$ to $\tilde{x}_1$

$\rightsquigarrow \pi_1(\tilde{X}, \tilde{x}_1) \xrightarrow{\cong} \pi_1(\tilde{X}, \tilde{x}_0)$

$[\alpha] \longmapsto [\tilde{\gamma}] \cdot [\alpha] \cdot [\bar{\tilde{\gamma}}]$

$\therefore P_* \pi_1(\tilde{X}, \tilde{x}_1) = gHg^{-1}$, where $g = [\gamma] \in \pi_1(X, x_0)$

$\gamma = p \circ \tilde{\gamma}$

2) If $g = [\gamma] \in \pi_1(X, x_0)$

Choose ! lift $\tilde{\gamma}$ of γ to $\tilde{X}$ st at $\tilde{x}_0$

let $\tilde{x}_1 := \tilde{\gamma}(1)$

$\rightsquigarrow P_* \pi_1(\tilde{X}, \tilde{x}_1) = gHg^{-1}$

Both work well!

So, Modified Thm.

$x_0 \in X$ pc, lpc, slsc,

$$\left\{ \begin{array}{c} \tilde{X} \\ \downarrow p \\ (X, x_0) \end{array} \text{ pc cov sp} \right\} \Big/ (\cong \text{ cov spaces}) \xleftrightarrow{\text{bij}} \left\{ \begin{array}{c} \text{subgrp} \\ H \subseteq \pi_1(X, x_0) \end{array} \right\} \Big/ \left(H \sim gHg^{-1} \text{ for } g \in \pi_1(X, x_0) \right)$$

(not based pres).

$$= \left\{ \begin{array}{c} \text{conj classes of subgrp} \\ \text{of } \pi_1(X, x_0) \end{array} \right\}$$

Consequences

1) Simply connected cover $\tilde{X} \to X$ is thus unique up to isom of cov sp. (Universal Cover)

$\longrightarrow$

2) $(\tilde{X}_1, \tilde{x}_1) \xrightarrow{p_1} (X, x_0)$ pc cov sp.
$(\tilde{X}_2, \tilde{x}_2) \xrightarrow{p_2} (X, x_0)$

$H_1 \subseteq \pi_1(X, x_0)$

$H_2 \subseteq \pi_1(X, x_0)$

$\exists$ basept pres morphism of cov sp

$$\begin{tikzcd} (\tilde{X}_1, \tilde{x}_1) \arrow[rr, "f"] \arrow[dr, "p_1"'] & & (\tilde{X}_2, \tilde{x}_2) \arrow[dl, "p_2"] \\ & (X, x_0) & \end{tikzcd}$$

$\longleftrightarrow$ $H_1 \subset H_2$

f is a cov sp by HW6

as $\tilde{X}_1 = \tilde{X}_{H_1} = \tilde{X}/_{H_1} \longrightarrow \tilde{X}_2 = \tilde{X}/_{H_2}$

by grp act quotient H_2 bigger orbit than H_1.

$\exists$ morph of covers (basept forgetting)

$$\begin{tikzcd} \tilde{X}_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & \tilde{X}_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

$\longleftrightarrow$ $H_1 \subset gH_2g^{-1}$ for some $g \in \pi_1(X, x_0)$

Eg 1) up to $\cong$ path connected covering sp of S^1 are:
(as $\mathbb{Z}$ abelian no need to care about conj)

Universal cov $p: \mathbb{R} \longrightarrow S^1$
$x \longmapsto e^{2\pi i x}$ $\longleftrightarrow$ $0 \subset \pi_1(S^1) \cong \mathbb{Z}$

$p_n: S^1 \longrightarrow S^1 \subset \mathbb{C}$
$z \longmapsto z^n$ $\longleftrightarrow$ $\underbrace{n\mathbb{Z} \subset \pi_1(S^1)}_{\text{all subgrps of } \mathbb{Z}}$

Ex 2) $\pi_1(\mathbb{RP}^2 \times \mathbb{RP}^2) = \mathbb{Z}/2 \times \mathbb{Z}/2$ (5 subgps)

$0 \subset \mathbb{Z}/2 \times \mathbb{Z}/2$

$0 \times \mathbb{Z}/2 \subset \mathbb{Z}/2 \times \mathbb{Z}/2$

$\mathbb{Z}/2 \times 0 \subset \mathbb{Z}/2 \times \mathbb{Z}/2$

$\langle (1,1) \rangle \subset \mathbb{Z}/2 \times \mathbb{Z}/2$

$\mathbb{Z}/2 \times \mathbb{Z}/2 \subseteq \mathbb{Z}/2 \times \mathbb{Z}/2$

$\therefore$ 5 covering sp of $\mathbb{RP}^2 \times \mathbb{RP}^2$ up to $\cong$.

Defn $p : \tilde{X} \to X$ covering sp

$$\mathrm{Aut}(p) := \left\{ \text{isom of covering sp } \begin{tikzcd} \tilde{X} \arrow[rr, "f"] \arrow[dr, "p"'] & & \tilde{X} \arrow[dl, "p"] \\ & X & \end{tikzcd} \right\}$$

i.e. f that commute

$$\mathrm{Aut}(p) = \mathrm{Aut}(\tilde{X}/X)$$

$f \in$ is an automorphism of p, or deck transformation

Eg

$$\begin{tikzcd} \mathbb{R} \arrow[d, "p"] & x \arrow[d, mapsto] \\ S^1 & e^{2\pi i x} \end{tikzcd}$$

$n \in \mathbb{Z}$, $f_n : \mathbb{R} \to \mathbb{R}$, $x \mapsto x+n$ $\in \mathrm{Aut}(p)$ (doesn't chng lattice)

$\mathbb{Z} \hookrightarrow \mathrm{Aut}(p)$ (injective grp homo as none are same)

Rmk In general, $f \in \mathrm{Aut}(p)$ $p : \tilde{X} \to X$ cov sp, $\tilde{X}$ conn

$\Longrightarrow$ f determined by where it sends a single pt.

unique lifting prop

$\circ^{\circ}\circ$ Z