Recall $f, g : X \to Y$ we have

$$\begin{tikzcd} & \pi_1(Y, f(x_0)) \arrow[d, "\hat{\gamma}"] \\ \pi_1(X, x_0) \arrow[ur, bend left, "f_*"] \arrow[r, bend right, "g_*"'] & \pi_1(Y, g(x_0)) \end{tikzcd}$$

If $f \sim g$ via $H : X \times I \to Y$

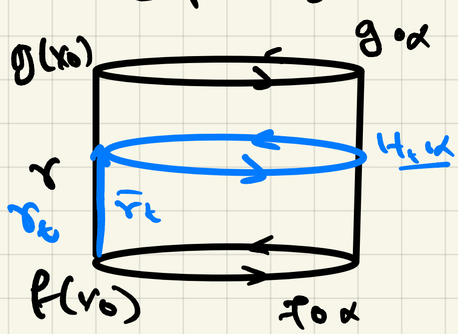
$\exists$ a path $\gamma : I \to Y$
$\qquad s \longmapsto H(x_0, s)$

from $f(x_0)$ to $g(x_0)$

Lemma $\hat{\gamma} \circ f_* = g_* : \pi_1(X, x_0) \to \pi_1(Y, g(x_0))$

Pf Suffices $[\alpha] \in \pi_1(X, x_0)$

$[f \circ \alpha] = [\gamma] \cdot [g \circ \alpha] \cdot [\bar{\gamma}]$



pic of

$$\begin{tikzcd} I \times I \arrow[rr] \arrow[dr, "\alpha \,\, id"'] & & Y \\ & X \times I \arrow[ur, "H"'] & \end{tikzcd}$$

$\gamma_t : I \to Y \qquad t \in [0,1]$
$\qquad s \mapsto \gamma(st)$

when $t=0$ get $f \circ \alpha$, when $t=1$ get $[\gamma] \cdot [g \circ \alpha] \cdot [\bar{\gamma}]$

Check: $I \times I \to Y$
$\qquad (s,t) \longmapsto (\gamma_t \cdot (H_t \circ \alpha) \cdot \bar{\gamma}_t)(s)$

is a path htpy of $f \circ \alpha$ to $\gamma \cdot (g \circ \alpha) \cdot \bar{\gamma}$

Prop $f : X \to Y$ htpy equiv

$\Rightarrow f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is $\cong$ $\forall x_0 \in X$

Pf let g be htpy inv of f

$$X \xrightarrow{f} Y \xrightarrow{g} X \xrightarrow{f} Y$$
$$x_0 \longmapsto y_0 \longmapsto x_1 \longmapsto y_1$$

1) $g \circ f \sim id \implies (g \circ f)_* = \hat{\gamma}(x, x_0) : \pi_1(X, x_0) \to \pi_1(X, x_1)$

$g_* \circ f_* =$

for γ path from x_0 to x_1 by earlier (this isom by last class)

$\implies g_* : \pi_1(Y, y_0) \to \pi_1(X, x_1)$

is surjective.

2) Similarly $(f \circ g)_* = f_* \circ g_* : \pi_1(Y, y_0) \to \pi_1(Y, y_1)$ is an $\cong$

$\implies g_* : \pi_1(Y, y_0) \to \pi_1(X, x_1)$ is injective

$\therefore$ g_* is injective

3) $\implies f_*$ is an $\cong$ as $g_* \circ f_*$ is an $\cong$.

E.g. 1 X contractible

$\implies$ 0) $\pi_0(X) = \{*\}$

1) $\pi_1(X, x_0) = 0 \quad \forall x_0 \in X.$

b/c X contractive $\implies X \sim \{*\}$ and 0, 1 true for $\{*\}$.

Defn In general any sp satisfying 0) 1) is called simply connected.

$\pi_1(S^1)$.

let $\omega : I \to S^1 \subseteq \mathbb{C}$ path

$s \longmapsto e^{2\pi i s} = (\cos(2\pi i s), \sin(2\pi i s))$

loop at $x_0 = (1, 0)$

$[\omega] \in \pi_1(S^1, x_0)$

get homo $\phi : \mathbb{Z} \to \pi_1(S^1, x_0)$

$n \longmapsto [\omega]^n = [\omega_n]$

where $w_n : I \longrightarrow S^1$
$\qquad s \longmapsto e^{2\pi i n s}$

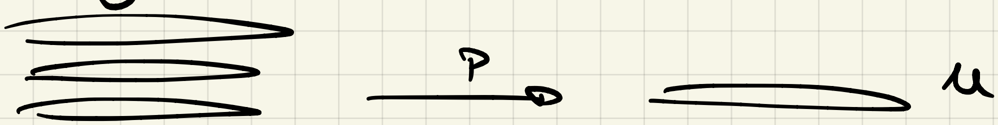
Goal: $\Phi : \mathbb{Z} \longrightarrow \pi_1(S^1, x_0)$ is an isom!

Detour: Covering Maps

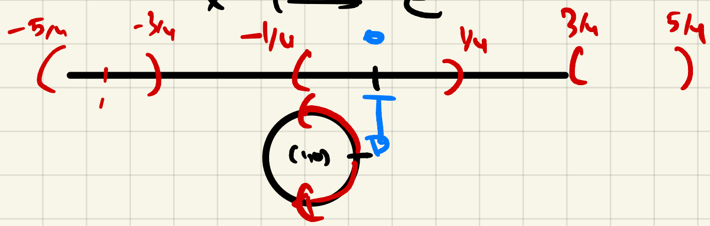
Defn| A covering sp of a top sp X is a (cts) map $p : \tilde{X} \longrightarrow X$
s.t. $\forall x \in X$ $\exists x \in U^{open} \subseteq X$
so, $p^{-1}(U) = \bigsqcup_{i \in I} \tilde{U}_i$ disjoint union of $\tilde{U}_i \subseteq \tilde{X}$ open

$I = \emptyset$

such that $p|_{\tilde{U}_i} : \tilde{U}_i \longrightarrow U$ is a homeomorphism
Say such an open $U \subset X$ is evenly covered!



E.g.| $p : \mathbb{R} \longrightarrow S^1$
$\qquad x \longmapsto e^{2\pi i x}$ is a covering sp



Thm| A covering sp $p : \tilde{X} \longrightarrow X$
satisfies the unique homotopy lifting property!
i.e. given any solid com diagram

$$\begin{tikzcd} Y \arrow[r, "\hat{F}_0"] \arrow[d, hook] & \tilde{X} \arrow[d, "p"] \\ Y \times I \arrow[r, "F"'] \arrow[ur, dashed, "\hat{F}"] & X \end{tikzcd}$$

$\exists! \, \hat{F}$ so diagram commutes

Cor1 $p: \tilde{X} \to X$ covering sp

1) $\alpha : I \to X$ path starting at $x_0 = \alpha(0)$

$\tilde{x}_0 \in p^{-1}(x_0)$

$\Rightarrow \exists!\ \tilde{\alpha} : I \to \tilde{X}$ which is a path st at $\tilde{x}_0$ & lifts α

i.e $p \circ \tilde{\alpha} = \alpha$

$$\begin{tikzcd} & \tilde{X} \ni \tilde{x}_0 \arrow[d, "p"] \\ I \arrow[ur, dashed, "\exists!\, \tilde{\alpha}"] \arrow[r] & X \ni x_0 \end{tikzcd}$$

$\to$ Pf $Y = \{*\}$ and apply earlier

2) $\alpha, \beta : I \to X$ paths from x_0 to x_1

$H : I \times I \to X$ path htpy $\alpha \to \beta$

$\tilde{x}_0 \in p^{-1}(x_0)$, $\tilde{\alpha}, \tilde{\beta}$ lifts guaranteed by 1)

$\Rightarrow \exists!\ \tilde{H} : I \times I \to \tilde{X}$ path htpy of $\tilde{\alpha} \to \tilde{\beta}$

That lifts H

$\hookrightarrow$ i.e $p \circ \tilde{H} = H$

In particular $\tilde{\alpha}(1) = \beta$