Prop | if X path conn, locally p.c, semi-locally simply conn

$\forall\ H \subseteq \pi_1(X, x_0)$ subgrp

$\exists\ p_H : (\tilde{X}_H, \tilde{x}_0) \to (X, x_0)$ covering sp

s.t. $p_{H*}(\pi_1(\tilde{X}_H, \tilde{x}_0)) = H$

Pf | from earlier $\tilde{X} = \left\{ [\alpha] \,\middle|\, \begin{matrix} \alpha \text{ path in} \\ X \text{ st at } x_0 \end{matrix} \right\} \xrightarrow{p} X$

$[\alpha] \longmapsto \alpha(1)$

has topology & is simply conn

Now $H \curvearrowright \tilde{X}$ by $H \times \tilde{X} \longrightarrow \tilde{X}$

$([h], [\alpha]) \longmapsto [h \cdot \alpha]$

h loop at x_0 then take α

e.g | This is a cts action of H on $\tilde{X}$

(action is by homeo of $\tilde{X}$)

→ factors!

$p : \tilde{X} \xrightarrow{\pi_H} \tilde{X}/H \xrightarrow{p_H} X$

↳ map induced by p

everything in orbit same end pt.

① $\pi_H : \tilde{X} \to \tilde{X}/H$ is a covering sp.

by HW3 sts if $1 \neq [h] \in H$

$[h] \cdot (\mathcal{U}_{[\alpha]}) \cap \mathcal{U}_{[\alpha]} = \emptyset$

$\left(\begin{matrix} \mathcal{U} \in \mathcal{B} \iff \mathcal{U} \subset X \text{ pc} \\ \pi_1(\mathcal{U}) \to \pi_1(X) \text{ tr} \\ \alpha \text{ in path } X,\ \alpha(1) \in \mathcal{U} \end{matrix} \right)$

Pf | if the above fails

$[h][\alpha \cdot \eta] = [\alpha \cdot \eta']$ where η, η' path in $\mathcal{U}$ st at $\alpha(1)$

$\Rightarrow [h] = [\alpha \cdot \underbrace{\eta \cdot \bar{\eta}'}_{} \cdot \bar{\alpha}]$

↳ in img of $\pi_1(\mathcal{U}) \to \pi_1(X)$ so triv

$= [\alpha \cdot \bar{\alpha}] = 1 \in \pi_1(X)$ ◻

② $\mathcal{U}^H_{[\alpha]} :=$ img of $\mathcal{U}_{[\alpha]}$ in $\tilde{X}_H$

$\Rightarrow$ have homeo

$$\begin{tikzcd} \mathcal{U}_{[\alpha]} \arrow[r, "\pi_H"', "\cong"] \arrow[d, "p", "\cong"'] & \mathcal{U}^H_{[\alpha]} \arrow[dl, "p_H"] \\ \mathcal{U} & \end{tikzcd}$$

homeo

bijective cts map in fact homeo (can check open by HW)

Rmk| covering map.

$\therefore$ p_H is homeo from $\mathcal{U}^H_{[\alpha]} \longrightarrow \mathcal{U}^H$

③ $p_H^{-1}(\mathcal{U})$ = disjoint union of various $\mathcal{U}^H_{[\alpha]}$'s

Pf| Already know this statement for p. take preimage all the way back & use surj of π_H

$\mathcal{U}^H_{[\alpha]} \cap \mathcal{U}^H_{[\alpha']} \neq \emptyset \underset{\text{think}}{\Longrightarrow} [h]\cdot[\alpha] = [\alpha']$

$\Rightarrow [h]\cdot[\alpha\cdot\eta] = [\alpha'\cdot\eta] \quad \forall\, \eta$ in $\mathcal{U}$ st at $\alpha(1)$

$\Rightarrow \mathcal{U}^H_{[\alpha]} = \mathcal{U}^H_{[\alpha']}$

$\therefore$ $p_H : \tilde{X}/_H \longrightarrow X$ a covering sp

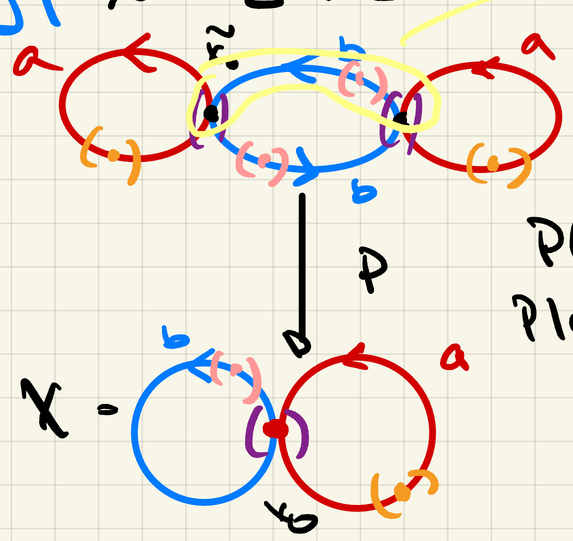
ex| $p_{H*}(\pi_1(\tilde{X}/H, \tilde{x}_0)) = H$

Use| This img consists of elts of $[\alpha] \in \pi_1(X, x_0)$

so lift $\tilde{\alpha}$ to $\tilde{X}_H$ is a loop at $\tilde{x}_0$.

E.g.| $X = S^1 \vee S^1$

max'll tree



Show cov covering!

$\longrightarrow$

$P(\text{vertex}) = x_0$

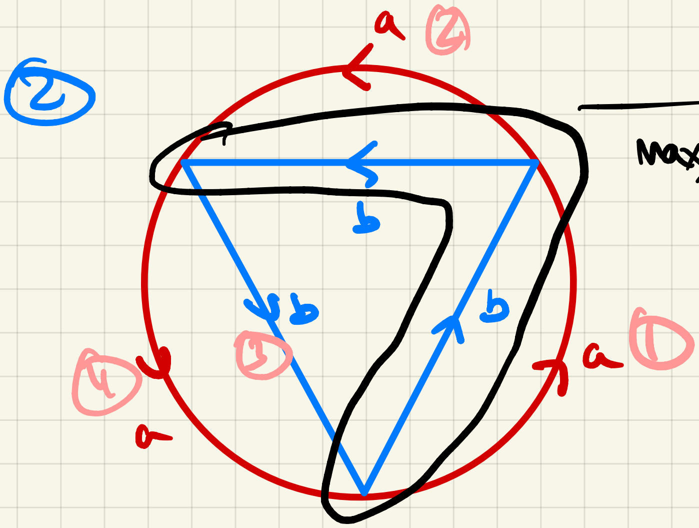
$P|\text{edge}$ = orient pres homeo to corr edge in X

$p_*(\pi_1(\tilde{X},\tilde{x}_0)) = \langle a, b^2, bab^{-1} \rangle_{\text{subgp}}$ img of gen!

↳ first 0 ↳ second 0 ↳ third 0 of $S^1 \vee S^1 \vee S^1$

$\cap$

$\pi_1(X, x_0) = \underset{a}{\mathbb{Z}} * \underset{b}{\mathbb{Z}}$



$\xrightarrow[\text{max'l tree}]{p} S^1 \vee S^1$ (degree 3 covering sp)

Corresponding subgp of $\pi_1(X, x_0)$ is

$\langle ab^{-1}, bab^{-1}b^{-1}, b^3, a^{-1}b^{-1}b^{-1} \rangle_{\underline{\text{subgp}}}$.

See Hatcher pg 58 for more eg.