Attaching 2-cells

$x_0 \in X$ pathconn

$\varphi_\alpha : S^1_\alpha = \partial D^2_\alpha \to X$ attaching maps

$\rightsquigarrow Y = \dfrac{X \sqcup_\alpha D^2_\alpha}{x \sim \varphi_\alpha(x),\ x \in \partial D^2_\alpha}$

Think of φ_α as a loop $I \to X$ based at $\varphi_\alpha(s_0)$, $s_0 = (1,0)$

i.e param S^1 & compose $S^1_\alpha \to$

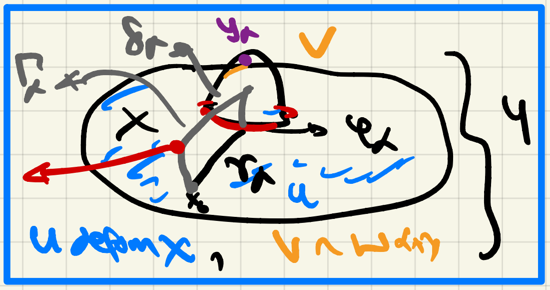
Choose $\gamma_\alpha : I \to X$ path from x_0 to $\varphi_\alpha(s_0) \in X$

$\Rightarrow \gamma_\alpha \cdot \varphi_\alpha \cdot \overline{\gamma_\alpha}$ loop at $x_0 \in X$

$X \xhookrightarrow{i} Y$ induces

$\pi(X, x_0) \xrightarrow{i_*} \pi(Y, x_0)$

$[\gamma_\alpha \cdot \varphi_\alpha \cdot \overline{\gamma_\alpha}] \longmapsto 1$



b/c $[\varphi_\alpha]$ in Y is in img of map from $\pi_1(D^2_\alpha) \to$ trivial

$\therefore i_*$ factors via $\omega : \dfrac{\pi_1(X, x_0)}{\langle [\gamma_\alpha \varphi_\alpha \overline{\gamma_\alpha}], \forall \alpha \rangle} \longrightarrow \pi_1(Y, x_0)$

Prop) ω is isom (suffices to show $\langle [\gamma_\alpha \varphi_\alpha \overline{\gamma_\alpha}], \alpha \rangle = \ker i_*$ $\Rightarrow$ 1st isom)

Pf) Construct space $Z \supset Y$ by

1) Gluing strips $\Gamma_\alpha = I \times I$ to Y along

$f_\alpha : I \times \{0\} \cup \{1\} \times I \to Y$

$f_\alpha |_{I \times \{0\}} = \gamma_\alpha$

$f_\alpha |_{\{1\} \times I} = \delta_\alpha$ (a path from $\varphi_\alpha(s_0)$ going radially along e^2_α)

Rmk: Z deform ret to Y by shrinking Γ_α to $I \times \{0\}$ (path γ_α)

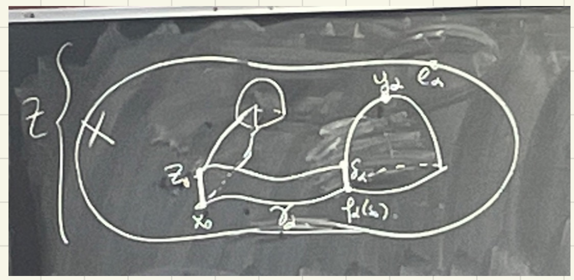
let $z_0 :=$ img of $(0,1) \in I \times I$ in Z

$\forall \alpha$, choose $y_\alpha \in e_\alpha^2$ not in img δ_α

$\mathcal{U} := Z \setminus \bigcup_\alpha y_\alpha \underset{\text{open}}{\subseteq} Z$

$\sim X$ (via defo retr of each Γ_α onto $I \times \{a\}$ and $D^2_\alpha \setminus \{y_\alpha\}$ to S^1_α)

$z_0 \in V := Z \setminus X$ is contractible



( $\sim \{*\}$)

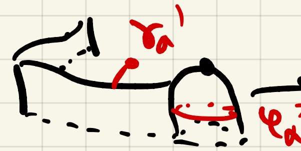
$U \cap V =$ (might have many domes)

$$\pi_1(Z, z_0) \underset{VK}{\cong} \frac{\pi_1(\mathcal{U}, z_0)}{\langle \mathrm{img}(\pi_1(U \cap V, z_0) \to \pi_1(\mathcal{U}, z_0)) \rangle}$$

$\wr\wr$

$\pi_1(Y)$

$U \cap V = \bigcup W_\alpha$, $W_\alpha = \left(U \cap V \setminus \bigcup_{\beta \neq \alpha} e^2_\beta\right) \sim S^1$

( ret to disc \ pt $\simeq S^1$)

$W_\alpha \cap W_\beta$ path conn $\forall \alpha, \beta$

$\mathbb{Z} \cong \pi_1(W_\alpha, z_0) \longrightarrow \pi_1(U \cap V, z_0)$

$1 \longmapsto [\gamma_\alpha' \varphi_\alpha' \overline{\gamma_\alpha'}]$

$\therefore$ VK pt 1 $\Rightarrow$ $\pi_1(U \cap V, z_0) = \langle [\gamma_\alpha' \varphi_\alpha' \overline{\gamma_\alpha'}] \ \forall \alpha \rangle \xleftarrow{VK\,1} \underset{\alpha}{*} \underbrace{\pi_1(W_\alpha, z_0)}_{\mathbb{Z}}$

$$\Rightarrow \pi_1(Z, z_0) \cong \frac{\pi_1(\mathcal{U}, z_0)}{\langle [\gamma_\alpha' \varphi_\alpha' \overline{\gamma_\alpha'}], \forall \alpha \rangle}$$

$\cong \pi_1(Z, x_0) \cong \pi_1(Y, x_0)$

Ex) Under $\pi_1(U, z_0) \cong \pi_1(X, x_0)$ (change of base pt $\cong$)

$\gamma_\alpha' \varphi_\alpha' \bar{\gamma}_\alpha' \longleftrightarrow \gamma_\alpha \varphi_\alpha \bar{\gamma}_\alpha$

Attaching n-cells, $n > 0$

$x_0 \in X$ path conn

$\varphi_\alpha : S_\alpha^{n-1} \to X$ attaching map

$\leadsto Y$ = attaching D_α^n along φ_α

Prop) $X \overset{i}{\hookrightarrow} Y$ induces isom

$\pi_1(X, x_0) \xrightarrow{i_*} \pi_1(Y, x_0)$

Pf) Similar to $n=2$ case this time $W_\alpha \sim S^{n-1}$

$\overset{n>2}{\Longrightarrow} \pi_1(W_\alpha) = 1$

$\Rightarrow$ thing in denominator (modded out) trivial

$\Rightarrow \pi_1(U \cap V) = 1$

$\Rightarrow \pi_1(Z) \cong \pi_1(U) \cong \pi_1(X)$

$\quad \wr\|$

$\pi_1(Y)$

$\square$

Cor) if X path conn CW complex

2skel $\leftarrow X^2 \overset{i}{\hookrightarrow} X$ induces isom

$\pi_1(X^2, x_0) \xrightarrow{\sim} \pi_1(X, x_0)$

X^2 disc $\Rightarrow X$ disc

so X conn $\Rightarrow X^2$ conn

gluing into one conn comp of X^2 as S^n is conn $n \geq 1$

Pf) OK when $\dim X < \infty$