Defn $f, g : X \to Y$ cts maps of top sp.

Then a homotopy H from f to g is a cts map

$H : X \times [0,1] \to Y$ s.t.

$$H(x,0) = f(x) \quad \forall x \in X \quad =: H_0(x)$$
$$H(x,1) = g(x) \quad \forall x \in X \quad =: H_1(x)$$

general $H_t(x) = H(x,t)$

$H_t : X \to Y$ cts

Defn f, g are homotopic in the case above.

Short hand $\to$ $f \sim g$

Defn f is null homo if $f \sim$ (const map $X \to Y$)

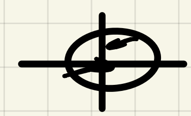
e.g. ① Any $f, g : X \to \mathbb{R}^n$ (homotopic)

via "straight line homotopy"

$H_t(x) = (1-t) f(x) + t g(x)$

$\therefore$ any $f : X \to \mathbb{R}^n$ is null homo!

Pic $f : S^1 \hookrightarrow \mathbb{R}^2$, $g = 0 : S^1 \hookrightarrow \mathbb{R}^2$
Co incl



② $\mathrm{Id} : S^1 \to S^1$ is not null homotopic (prove later).

exer 1) reln of homotopic is an equiv reln on $\{f : X \to Y\}$

2) let

$$\begin{tikzcd} X \arrow[r, shift left, "f_1"] \arrow[r, shift right, "f_2"'] & Y \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] & Z \end{tikzcd}$$

if $f_1 \sim f_2$ & $g_1 \sim g_2 \Rightarrow g_1 \circ f_1 \sim g_2 \circ f_2$

Defn $f : X \to Y$ is a homotopy equiv

if $\exists\, g : Y \to X$ s.t. $f \circ g \sim \mathrm{Id}_Y$ & $g \circ f \sim \mathrm{Id}_X$

- Say X, Y are homo equiv! in this case

denoted $X \simeq Y$

- X is contractible if it is homo equiv to $\{*\}$ ← 1 pt sp

e.x) above is equiv reln on top sp!

e.g) ① $\mathbb{R}^n$ is contractible

Pf) $f : \mathbb{R}^n \to \{x\}$, $\quad g : \{x\} \to \mathbb{R}^n$

$\qquad\qquad\qquad\qquad\qquad\qquad x \mapsto 0$

$g \circ f = 0 : \mathbb{R}^n \to \mathbb{R}^n$ $\qquad f \circ g = id_{\{x\}}$

$\wr$ homotopic to identity (showed any 2 maps into $\mathbb{R}^n$ homo)

② $D^n = \{x \in \mathbb{R}^n \mid |x| \le 1\} \subset \mathbb{R}^n$ unit disk

D^n is contractible! → kinda use the above idea

③ $S^n \subseteq \mathbb{R}^{n+1}$ is not contractible! → not obvious!

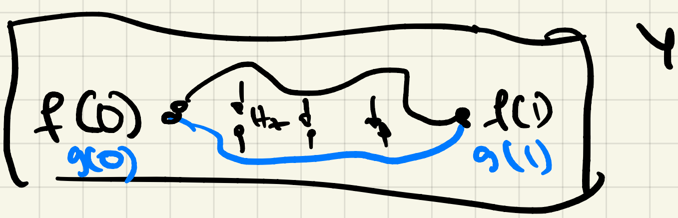
Thm) (topological Poincare conj) if M is a closed n-mfd (compact wo ∂) s.t. $M \simeq S^n \Rightarrow M \cong S^n$ (homeo)

Def) let $f, g : X \to Y$ let $B \subseteq X$ subsp s.t. $f|_B = g|_B$ (same map on B)

a homotopy rel to B from $f \to g$ is a homo H s.t.

$H_t(b) = H(b,t) = f(b) = g(b) \quad \forall\, b \in B$ → (fix those pts)

Picture) $X = [0,1] \to Y$ rel to $\{0,1\}$



Def let $B \subseteq X$ & $i : B \hookrightarrow X$ inct

then $r : X \to B$ is a retraction if $r \circ i = id_B$

(takes everything X to B fixing B)

Def A deformation retr for $X \to B$ is a homomotopy from id_X to $i \circ r$ for a retraction r

Def A strong def retr is a def retr H s.t.

$H_t(b) = b \quad \forall b \in B, t \in [0,1]$

(homotopy from id_X to r rel to B)

Warning: Hatcher calls ∂ a def retraction

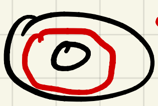
Lemma if X def retr to B

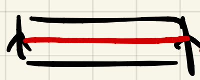
$\Rightarrow i : B \hookrightarrow X$ is a homo equiv

Pf $r \circ i = id_B$ $\quad i \circ r \sim id_X$ via H Cool!

e.g. 1) $\mathbb{R}^n$ strongly def retr to the origin

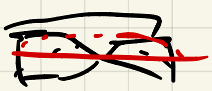
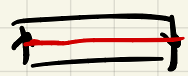
$H_t(x) = (1-t)x$

2) $X =$  S^1 strongly def retr onto S^1

$X \cong$  $y = 1/2$, S^1 $= [0,1] \times [0,1] / \sim$

$H_t(x,y) = (1-t)(x,y) + t(x, 1/2)$

$H : [0,1] \times [0,1] \times [0,1] \to X$ descends to def homo from $X \times I \to X$

3) $X =$  mobius $=$ 

str def retr to S^1 (same argument)

Inv 1

Def) if X top sp $\pi_0(X) := \{$ path conn comp of X $C\}$

$\hookrightarrow$ C is equiv class for equiv reln of points in X where $x \sim y$ if path conn

if $f : X \to Y$ induces a map

$\pi_0(f) : \pi_0(X) \to \pi_0(Y)$

$\quad\quad\quad\quad \in$

$\quad\quad\quad\quad C \longmapsto$ path comp of Y which contains $f(C)$

(f sends path comp to path conn subsp so well def)

Lemma) if $f \sim g : X \to Y$

$\Rightarrow \pi_0(f) = \pi$