## E.g | Automorphism grp.

$$\begin{array}{ccc} S^1 & & z \\ {\scriptstyle p_n}\downarrow & & \downarrow \\ S^1 & & z^n \end{array}$$

**Cl.** $\mathbb{Z}/n \xrightarrow{\cong} \mathrm{Aut}(p_n)$

$1 \longmapsto \left( g: S^1 \longrightarrow S^1, \; z \longmapsto z \cdot e^{2\pi i/n} \right)$

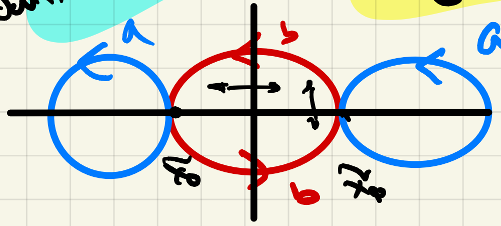
↳ this is a homomorphism → elt of order $n$ (mapping prop)

↳ inj by inspection

↳ surj as $f \in \mathrm{Aut}(p_n)$ $f(1)$ has to be $n^{th}$ root of unity $e^{2\pi i m/n}$ (by comm $S^1 \xrightarrow{f} S^1$, $p_n \searrow S^1 \swarrow p_n$)

$\therefore$ $f = g^m$ (since agree at some point)

if covering sp aut agree at some pt identical

**E.g |**  $\xrightarrow{p}$ $S^1 \vee S^1$ galois.

let $f$ = aut of $p$ given by 'flipping' along both axes

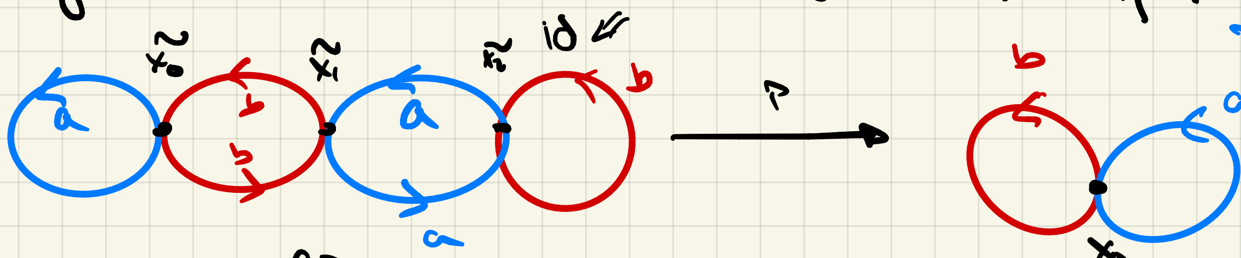
$f \in \mathrm{Aut}(p)$ & $f^2 = id$

have $\mathbb{Z}/2 \longrightarrow \mathrm{Aut}(p)$ inj homo

$\underline{1} \longrightarrow f$

It is also surjective by same arg as above..

any auto takes $\tilde{x}_0 \mapsto \tilde{x}_0$ (id), $\tilde{x}_0 \to \tilde{x}_1 \Rightarrow f$.

**E.g |**  not Galois

$\mathrm{Aut}(p) = \{1\}$, $1 = id$, $f$

Sts $f(\tilde{x}_0) = \tilde{x}_0$ b/c the lift of "a" loop in $S^1 \vee S^1$ starting at $\tilde{x}_0$ is a loop in $\tilde{X}$, but lift st at $\tilde{x}_1$ or $\tilde{x}_2$ is not a loop.

(auto will map loops to loops by bij)

**Defn** $p: \tilde{X} \to X$ cov sp is called Galois (AKA normal) if $\forall x_0 \in X$, $\mathrm{Aut}(p)$ acts transitively on $p^{-1}(x_0)$

i.e. $\forall y_1, y_2 \in p^{-1}(x_0)$
$\exists f \in \mathrm{Aut}(p)$ s.t $f(y_1) = y_2$

(it always acts on fibres by comm dg diag
i.e $f(p^{-1}(x_0)) \subseteq p^{-1}(x_0)$)

**e.x.1** if $X$ is path conn then $p$ galois iff
$\exists x_0$ s.t $\mathrm{Aut}(p) \curvearrowright p^{-1}(x_0)$ transitive

**e.g** $p: \mathbb{R} \to S^1$, $p_n: S^1 \to S^1$ Galois

**Lemma** let $p: (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ covering sp w/ $\tilde{X}, X$ pc, lpc

$H = p_* \pi_1(\tilde{X}, \tilde{x}_0) \subseteq \pi_1(X, x_0)$
$\Rightarrow$ $p$ Galois $\Leftrightarrow$ $H \subseteq \pi_1(X, x_0)$ normal subgp.

**Pf** Observation: $g = [\gamma] \in \pi_1(X, x_0)$
let $\tilde{\gamma}$ lift of $\gamma$ to $\tilde{X}$ st at $\tilde{x}_0$ end at $\tilde{\gamma}(1) =: \tilde{x}_1$

$gHg^{-1} = H$
$\Updownarrow$ (Monday).
$p_* \pi_1(\tilde{X}, \tilde{x}_1) = p_* \pi_1(\tilde{X}, \tilde{x}_0) = H$ — but actually lifting of fncs.
$\Updownarrow$ (corr of subgps and pointed cov sp)
$\exists! \cong$ of pointed covers $f: (\tilde{X}, \tilde{x}_0) \to (\tilde{X}, \tilde{x}_1)$
$\hookrightarrow$ uniqueness as it sends $\tilde{x}_0 \mapsto \tilde{x}_1$ (pointedness)

"$\Rightarrow$" $f_g$ exists for any $\tilde{x}_1 \in p^{-1}(x_0)$ — $\tilde{X}$ PC
"$\Leftarrow$" if inv under conj $\Rightarrow$ $\tilde{x}_1 \in p^{-1}(x_0)$, choose $\tilde{\gamma}$ from $\tilde{x}_1 \to \tilde{x}_0$, $\gamma = p \circ \tilde{\gamma}$
$\exists! f \in \mathrm{Aut}(p)$ $f(\tilde{x}_0) = f(\tilde{x}_1)$ apply obs to $gHg^{-1}$, $g = [\gamma]$

**Lemma** $p: (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ c.s w/ $\tilde{X}, X$ pc, lpc.

$H = p_* \pi_1(\tilde{X}, \tilde{x}_0) \subset \pi_1(X, x_0)$

$N(H)$ normalizer of $H$ $\quad N(H) = \{ g \in \pi_1(X, x_0) \mid gHg^{-1} = H \}$

**Thm** $\mathrm{Aut}(p) \cong N(H)/H \rightsquigarrow H \trianglelefteq N(H)$

**Pf)** $\varphi : N(H) \longrightarrow \mathrm{Aut}(p)$ (will show surj, $\ker \varphi = H$ & done by 1st iso)

$g \longmapsto f_g$

1) $\varphi$ is grp homo, let $g = [\gamma]$, $g' = [\gamma'] \in \pi_1(X, x_0)$

$\tilde{\gamma}'$ lift of $\gamma'$ st at $\tilde{x}_0$ end at $\tilde{x}_1' = \tilde{\gamma}'(1)$

$\rightsquigarrow f_{g'} : (\tilde{X}, \tilde{x}_0) \xrightarrow{\cong} (\tilde{X}, \tilde{x}_1)$

$gg' = [\gamma \cdot \gamma']$ & lift $\widetilde{\gamma \cdot \gamma'}$ st at $\tilde{x}_0$

$\widetilde{\gamma \cdot \gamma'} = \tilde{\gamma}$ concat w/ lift of $\gamma'$ st at $\tilde{x}_1$ — lifts to lifts

$= \tilde{\gamma}\,(\underbrace{f_g \circ \tilde{\gamma}'}_{\text{bas}})$ $f_g$ cov sp auto $\tilde{x}_0 \to \tilde{x}_1$

$\underline{f_{gg'}}(\tilde{x}_0) = \widetilde{\gamma\gamma'}(1)$

$= f_g(\tilde{\gamma}'(1))$ — two auto agree at point

$= f_g(\tilde{x}_1') = \underline{f_g f_{g'}}(\tilde{x}_0)$

$\Longleftrightarrow f_{gg'} = f_g \circ f_{g'}$

2) $\varphi$ is surj. let $f \in \mathrm{Aut}(p)$, $f(\tilde{x}_0) =: \tilde{x}_1 \in p^{-1}(x_0)$

$\Rightarrow f_g$ where $g = [p \circ \tilde{\gamma}]$ & $\tilde{\gamma}$ path fr $\tilde{x}_0 \to \tilde{x}_1$