Last time

SES of C.C $\rightsquigarrow$ LES on hom.

Cor $A \subseteq X \rightsquigarrow$ LES.

$$\cdots \to H_n(A) \to H_n(X) \to H_n(X,A) \xrightarrow{\delta} H_{n-1}(A) \to \cdots$$

Rmk $[\alpha] \in C_n(X,A) = \frac{C_n(X)}{C_n(A)}$ a cycle (relative cycle).

$$\alpha \in C_n(X), \quad \partial\alpha \in C_{n-1}(A) \subseteq C_{n-1}(X)$$

$$\delta([\alpha]) = [\partial\alpha] \in H_{n-1}(A)$$

Similar Cor the LES $\cdots \to \tilde{H}_n(A) \to \tilde{H}_n(X) \to H_n(X,A) \to \cdots$

Pf S.E.S of C.C.

$$0 \to \tilde{C}_\cdot(A) \to \tilde{C}_\cdot(X) \to C_\cdot(X,A) \to 0$$

$$\begin{tikzcd}
 & \vdots \arrow[d] & & & \\
0 \arrow[r] & C_1(A) \arrow[r] \arrow[d] & C_1(X) \arrow[r] \arrow[d, "\partial"] & C_1(X\backslash A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_0(A) \arrow[r] \arrow[d, "\varepsilon"] & C_0(X) \arrow[r] \arrow[d, "\varepsilon"] & C_0(X\backslash A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z} \arrow[r, "id"'] & \mathbb{Z} \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

✓ by earlier

To prove L.E.S for good pair (X,A). STS $H_n(X,A) \cong \tilde{H}_n(X\backslash A)$

E.g. $x_0 \in X$ $\quad H_n(X, x_0) \cong \tilde{H}_n(X)$ (use LES + $\tilde{H}_n(x_0) = 0$)

Functoriality $f: X \to Y$, $A \subset X$ w $f(A) \subset B \subset Y$

Notation $f: (X,A) \to (Y,B)$

$\rightsquigarrow f_\#: C_\cdot(X,A) \to C_\cdot(Y,B)$ ind by

$f_\#: C_\cdot(X) \to C_\cdot(Y)$

$\rightsquigarrow f_*: H_n(X,A) \to H_n(Y,B) \quad \forall n$

Thm (excision) Given $Z \subseteq A \subseteq X$ so $\overline{Z} \subset \text{int}(A)$

The incl $(X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$ induces isoms.

$$H_n(X \setminus Z, A \setminus Z) \xrightarrow{\sim} H_n(X, A) \quad \forall n$$

equiv given $A, B \subseteq X$ s.t. $X = \text{int} A \cup \text{int} B$ then,

$(B, A \cap B) \hookrightarrow (X, A)$ induces isom

$H_n(B, A \cap B) \cong H_n(X, A) \ \forall n$

$(B = X \setminus Z, \ Z = X \setminus \overset{\circ}{A})$

Pf sk let $X = \text{int}(A) \cup \text{int}(B)$

$C_n(A+B) = \{ \sum m_i \sigma_i \mid \sigma_i : \Delta^n \to X$ has img in A or $B \ \forall i \}$ pres by ∂

$\subseteq C_n(X)$.

get incl of c.c. $i : C_\bullet(A+B) \hookrightarrow C_n(X)$

Prop 2.21 $\exists$ chain map, $\rho : C_\bullet(X) \to C_\bullet(A+B)$ so

1) $\rho \circ i = \text{id}_{C_\bullet(A+B)}$

2) $i \circ \rho \sim \text{id}_{C_\bullet(X)}$ via chain htpy $D : C_n(X) \to C_{n+1}(X)$

$\partial \circ D + D \circ \partial = \text{id}_{C_\bullet(X)} - i \circ \rho$

3) i, ρ, D all pres chains that lie in A.

get

$\Rightarrow \bar{i} : \dfrac{C_\bullet(A+B)}{C_\bullet(A)} \to \dfrac{C_\bullet(X)}{C_\bullet(A)} = C_\bullet(X, A)$

$\bar{\rho} : \dfrac{C_\bullet(X)}{C_\bullet(A)} \to \dfrac{C_\bullet(X+A)}{C_\bullet(A)}$

$\bar{\rho} \circ \bar{i} = \text{id}$ & $\partial \circ \bar{D} + \bar{D} \circ \partial = \text{id} - \bar{i} \circ \bar{\rho}$

$\Rightarrow \bar{i}_* : H_n\left(\dfrac{C_\bullet(A+B)}{C_\bullet(A)}\right) \to H_n(X, A)$

$\bar{\rho}_* : H_n(X, A) \to H_n\left(\dfrac{C_\bullet(A+B)}{C_\bullet(A)}\right)$ are inv $\cong$'s

But $\dfrac{C_\bullet(B)}{C_\bullet(A\cap B)} \cong \dfrac{C_\bullet(A+B)}{C_\bullet(A)}$ (both sides are free ab grps on simpl $\sigma : \Delta^n \to X$ in B but not A)

$$\Longrightarrow H_n\left(\frac{C_\bullet(B)}{C_\bullet(B\cap A)}\right) \cong H_n\left(\frac{C_\bullet(A+B)}{C_\bullet(A)}\right) \cong H_n(X,A)$$

$$\| \quad H_n(B, A\cap B)$$

Prop) (X,A) good pair. Then quot map $q : (X,A) \longrightarrow (X/A, A/A)$ induces isom

$\forall n \quad q_* : H_n(X,A) \xrightarrow{\cong} H_n(X/A, A/A) \cong \tilde{H}_n(X/A)$, where $A/A = \{x_0\}$

Compl | pf of LES of good pair!