Lemma if X is nonempty, p.c.

$\Rightarrow H_0(X) \cong \mathbb{Z}$

Pf) $H_0(X) = \dfrac{\ker(\partial_0)}{\operatorname{im} \partial_1} = \dfrac{C_0(X)}{\operatorname{im} \partial_1}$ → free abelian grp on 0-simpl ↪ points.

$\varepsilon : C_0(X) \longrightarrow \mathbb{Z}$ → Clearly surj as $X \neq \emptyset$

$\sum_{i=1}^{n} m_i \sigma_i \longmapsto \sum m_i$ ↪ unique homo takes $\sigma_i \mapsto 1 \ \forall i$

(σ_i: 0-simpl)

STS $\ker(\varepsilon) = \operatorname{im} \partial_1$

$\supseteq$) let σ a 1 simplex $\sigma : \Delta^1 \to X$

enough to check on gens

$\varepsilon(\partial_1(\sigma)) = \varepsilon(\sigma|_{[v_1]} - \sigma|_{[v_0]}) = 1 - 1 = 0$

$\Rightarrow \partial_1(\sigma) \in \ker \varepsilon$ ✓

$\subseteq$) let $\varepsilon(\sum m_i \sigma_i) = 0 \iff \sum m_i = 0$

realize

Choose $x_0 \in X$, α_i: path x_0 to σ_i (1 simpl) ↪ σ_i 0 simpl point

$\partial(\alpha_i) = \sigma_i - x_0$

$\partial(\sum m_i \alpha_i) = \sum m_i \sigma_i - \cancel{\sum m_i x_0}$ → 0 as $\sum m_i = 0$

$= \sum m_i \sigma_i$

so $\sum m_i \sigma_i \in \operatorname{im} \partial_1$ $\square$.

Lemma let X be a space & $X = \bigsqcup_\alpha X_\alpha$ decomp of X into p/c

$\Rightarrow H_n(X) \cong \bigoplus_\alpha H_n(X_\alpha) \quad \forall n$

Pf | exer | $C_\bullet(X) \cong \bigoplus_\alpha C_\bullet(X_\alpha)$ ↪ b/c $\sigma : \Delta^n \to X$ factors thru p/c of X — isom as chain comple

Cor $H_0(X) \cong \mathbb{Z}^{\oplus \pi_0(X)}$ ⟶ path comp of X

Goal descr relation b/w H_1 & π_1

let $\alpha : I \longrightarrow X$ loop at $x_0 \in X$ get

1) $[\alpha] \in \pi_1(X, x_0)$

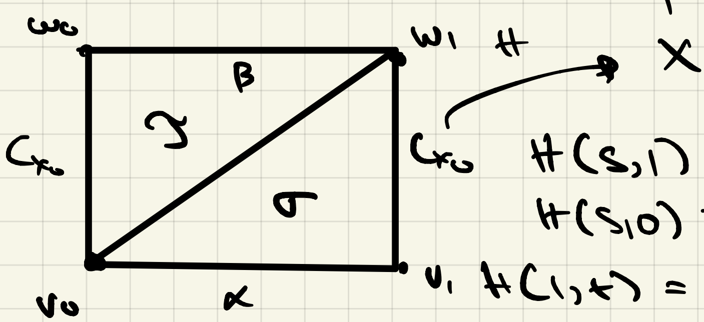
2) $\partial_1(\alpha) = x_0 - x_0 = 0$ i.e α is a 1 cycle i.e $\alpha \in \ker \partial_1$

$\rightsquigarrow [\alpha] \in H_1(X) = \dfrac{\ker(\partial_1)}{\operatorname{im}(\partial_2)}$

is a class

lemma β loop at $x_0 \in X$ $\alpha \underset{p}{\simeq} \beta$ $\Rightarrow [\alpha] = [\beta] \in H_1(X)$ } gives $\exists$ well def map $[\alpha] \mapsto [\alpha]$ $\varphi : \pi_1(X, x_0) \longmapsto H_1(X)$

Pf let $H : I \times I \longrightarrow X$ path htpy from $\alpha \to \beta$



$H(s,1) = \beta(s)$
$H(s,0) = \alpha(s)$
$H(1,t) = H(0,t) = x_0$

wts α, β differ by something in $\operatorname{im} \partial_2$

$I \times I = [v_0, v_1, w_1] \cup [v_0, w_0, w_1]$

$\sigma = H|_{[v_0, v_1, w_1]}$ $\quad \gamma = H|_{[v_0, w_0, w_1]}$

$\partial(\sigma) = \underbrace{H|_{[v_1, w_1]}}_{c_{x_0}} - H|_{[v_0, w_1]} + \underbrace{H|_{[v_0, v_1]}}_{\alpha}$

$\partial(\gamma) = \underbrace{H|_{[w_0, w_1]}}_{\beta} - H_{[v_0, w_1]} + \underbrace{H_{[v_0, w_0]}}_{c_{x_0}}$

$\Rightarrow \partial(\sigma - \gamma) = \alpha - \beta \Rightarrow \alpha - \beta \in \operatorname{im} \partial_2$

$\Rightarrow [\alpha] = [\beta] \in H_1(X)$ $\blacksquare$

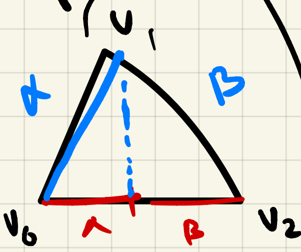
Lemma 2 $\varphi: \pi_1(X, x_0) \to H_1(X)$ is a <u>homo</u>.
$[\alpha] \mapsto [\alpha]$

More <u>generally</u>, α path from x_0 to x_1 in X
β ——— x_1 to x_0 in X

$$= \underbrace{\alpha + \beta - \alpha \cdot \beta}_{C_1(X)} \in \operatorname{im}(\partial_2)$$

↳ taking both to be loops $\Rightarrow \underset{\in H_1}{[\alpha+\beta]} = \underset{\in H_1}{[\alpha \cdot \beta]}$
↳ differ by $\operatorname{im} \partial_2$
$\Rightarrow \varphi$ well def

Pf $\sigma: \underset{[v_0, v_1, v_2]}{\Delta^2} =$



(σ is comp of everything)

when proj down only get α

$$\Delta^2 \xrightarrow{\text{proj}} \Delta^1 = [v_0, v_2] \xrightarrow{\alpha \cdot \beta} X$$

So, $\sigma|_{[v_0,v_2]} = \alpha \cdot \beta$, $\sigma|_{[v_0,v_1]} = \alpha$, $\sigma|_{[v_1,v_2]} = \beta$

$\operatorname{im}\partial_2 \ni \partial(\sigma) = \alpha + \beta - \alpha \cdot \beta$

So, <u>homo fact thru abelianization</u>.

$$\begin{tikzcd} \pi_1(X, x_0) \arrow[r, "\varphi"] \arrow[dr] & H_1(X) \\ & \pi_1(X, x_0)^{ab} \arrow[u, "\tilde{\varphi}"'] \end{tikzcd}$$

Thm X p.c $\Rightarrow \tilde{\varphi}$ is $\cong$ $\longrightarrow$ φ <u>is surj</u>

Pf (surj) Suppose $\sum m_i \sigma_i \in \ker \partial_1$

wts $\exists$ loop α at x_0 s.t.

$$[\sum m_i \sigma_i] = [\alpha] \in H_1(X)$$

1) May assume all $m_i = \pm 1$ (by relabeling i.e. $a + 2b = a + b + b$)

2) lemma 2 $\Rightarrow [\sigma_i] + [\bar{\sigma}_i] = [\sigma \cdot \bar{\sigma}_i] = 0$

$\Rightarrow [\sigma_i] = -[\bar{\sigma}_i]$ $\quad \in \dfrac{C_1(X)}{\operatorname{im} \partial_2}$ (might not be in $\ker \partial_1$)

$\therefore$ may assume all $m_i = +1$

3) $0 = \partial(\sum \sigma_i) \overset{\partial \text{ of } 1\text{-simpl}}{=} \sum (\sigma_i(1) - \sigma_i(0))$

$\therefore \exists$ some σ_i not a loop

$\Rightarrow \exists$ some σ_j so $\sigma_i(1) = \sigma_j(0)$ (get cancel)

$\therefore$ Combining $\sigma_i + \sigma_j$ to $\sigma_i \cdot \sigma_j$ by lemma 2

$\hookrightarrow$ by induction may assume α_i loop in X $\forall i$ at $x_i \in X$.

4) Choose γ_i path from x_0 to x_i in X

$\Rightarrow$ repl σ_i with $\gamma_i \sigma_i \bar{\gamma}_i$ (using lemma 2) gives same.

we may assume all σ_i loop at x_0.

5) $[\sum_{i=1}^{n} \sigma_i] \overset{\text{lem 2}}{=} [\sigma_1 \cdot \ldots \cdot \sigma_n]$