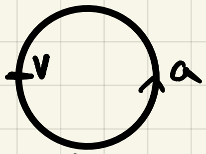
E.g. 1) $X = \{x\}$

$$H_n^\Delta(X) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & \text{else} \end{cases}$$

2) $X = S^1 =$ 

Δ-cmpx struct

free ab v gen.

$$C_\bullet^\Delta(X) : 0 \xrightarrow{\partial_2} \mathbb{Z}\langle a \rangle \xrightarrow{\partial_1} \mathbb{Z}\langle v \rangle \xrightarrow{\partial_0} 0$$

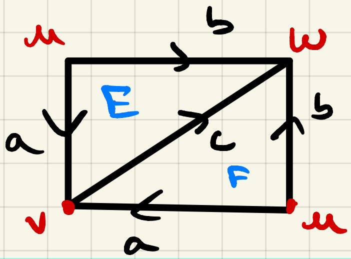
$$\mathbb{Z}\langle a \rangle = C_1^\Delta(X), \qquad \mathbb{Z}\langle v \rangle = C_0^\Delta(X)$$

$$\partial(a) = v - v = 0 \implies \partial_1 = 0$$

$$H_0^\Delta(S^1) = \frac{\ker(\partial_0)}{\operatorname{im}(\partial_1)} = \frac{\mathbb{Z}\langle v \rangle}{0} = \mathbb{Z}\langle v \rangle \cong \mathbb{Z}$$

$$H_1^\Delta(S^1) = \frac{\ker(\partial_1)}{\operatorname{im}(\partial_2)} = \frac{\mathbb{Z}\langle a \rangle}{0} = \mathbb{Z}\langle a \rangle \cong \mathbb{Z}$$

$$H_n^\Delta(S^1) \cong \begin{cases} \mathbb{Z} & n = 0,1 \\ 0 & \text{else} \end{cases}$$

3) $X = S^2$ 

E, F – 2-simplices
a, b, c – 1 simplices
u, v, w – 0 simplices

$E = \Delta^2_{[u,v,w]}$, $F = \Delta^2_{[u,v,w]}$, $a = \Delta^1_{[u,v]}$

$$C_\bullet^\Delta(S^2) : 0 \xrightarrow{\partial_3 = 0} \mathbb{Z}\langle E, F \rangle \xrightarrow{\partial_2} \mathbb{Z}\langle a,b,c \rangle \xrightarrow{\partial_1} \mathbb{Z}\langle u,v,w \rangle \xrightarrow{\partial_0 = 0} 0$$

$\partial_2 : \mathbb{Z}\langle E, F \rangle \to \mathbb{Z}\langle a,b,c \rangle$

$$\left.\begin{aligned} E &\longmapsto c - b + a \\ F &\longmapsto c - b + a \end{aligned}\right\} \text{spec by gen}$$

$\partial_1 : \mathbb{Z}\langle a,b,c \rangle \longmapsto \mathbb{Z}\langle u,v,w \rangle$

$$\begin{aligned} a &\longmapsto v - u \\ b &\longmapsto w - u \\ c &\longmapsto w - v \end{aligned}$$

So, $H_2^\Delta(S^2) = \dfrac{\ker(\partial_2)}{0} = \partial_2(xE + yF) = (x+y)(c-b+a)$

$\hookrightarrow x = -y$ to be ker

$\Rightarrow \ker \partial_2 = \langle E - F \rangle \cong \mathbb{Z}$

$H_1^\Delta(S^2) = \dfrac{\ker(\partial_1)}{\operatorname{im}(\partial_2)} = \dfrac{\langle c-b+a \rangle}{\langle c-b+a \rangle} = 0$

$\ker \partial_1 \Rightarrow 0 = \partial_1(xa + yb + zc) = (-x-y)u + (x-z)v + (y+z)w$

abv $0 \iff y = -x$ & $z = x$

$\Rightarrow \ker \partial_1 = \{xa - xb + xc \mid x \in \mathbb{Z}\} = \langle a - b + c \rangle = \langle c - b + a \rangle$

$H_0^\Delta(S^2) = \dfrac{\ker \partial_0}{\operatorname{im} \partial_1} = \dfrac{\mathbb{Z}\langle u, v, w \rangle}{\langle v-u, w-u \rangle} \overset{\text{COB}}{=} \dfrac{\langle v-u, w-u, u \rangle}{\langle v-u, w-u \rangle} \cong \mathbb{Z}\langle u \rangle$

$\hookrightarrow$ rearrange exp abv

$\operatorname{im} \partial_1 = \langle v-u, w-u, w-v \rangle = \langle v-u, w-u \rangle$

$\therefore\ H_n^\Delta(S^2) \cong \begin{cases} \mathbb{Z} & n = 0, 2 \\ 0 & \text{else} \end{cases}$.

Note $\Rightarrow$ S^2 is not contractible. (if we believe simplicial hom homotopy / Δ-cmplx struct inv)

Singular

$\longrightarrow$

Singular Homology

Let X be a space.

Def) A map $\sigma : \Delta^n \to X$ is called a (singular) n-simplex in X

Def) $C_n(X) :=$ free abelian grp on singular n simplices in X

$$= \left\{ \begin{array}{l} \text{finite formal} \\ \text{sums} \end{array} \;\middle|\; \begin{array}{l} m_\alpha \in \mathbb{Z} \\ \sigma_\alpha : \Delta^n \to X \end{array} \right\}$$

grp of singular n-chains

Def) Bdry homo

$$\partial_n : C_n(X) \longrightarrow C_{n-1}(X)$$

$$\left(\sigma : \underset{[v_0,\dots,v_n]}{\Delta^n} \to X\right) \longmapsto \sum_{i=0}^{n} (-1)^i \, \sigma|_{[v_0,\dots,\hat{v}_i,\dots,v_n]} \longrightarrow \text{gens map}$$

$$[v_0,\dots,\hat{v}_i,\dots,v_n] \cong \Delta^{n-1}$$

Lemma (Chain Complex) $\partial_n \circ \partial_{n+1} = 0$

Pf) Same for $C_\cdot^\Delta(X)$

So, get singular chain cmplx

$$C_\cdot(X) : \cdots \longrightarrow C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \to \cdots$$

(zero when $n < 0$)

$\Updownarrow$ terminology

(concentrated in degrees $n \geq 0$)

$\rightsquigarrow$ Singular hom grp $H_n(X) = H_n(C_\cdot(X))$

E.g.) $X = \{*\}$ $\sigma_n : \Delta^n \to X$ $\exists$ unique such map for each n

So call σ_n

$C_n(X) = \mathbb{Z}\langle \sigma_n \rangle$

$n = 0$ $\partial(\sigma_n) = 0$

$n > 0$ $\partial(\sigma_n) = \sum_{i=0}^{n} (-1)^i \sigma_{n-1} = \begin{cases} 0 & n \text{ odd} \\ \sigma_{n-1} & n \text{ even} \end{cases}$

so $\partial \neq 0$

So $C_\bullet(X): \cdots \to \mathbb{Z}\langle\sigma_3\rangle \xrightarrow{\partial_3=0} \mathbb{Z}\langle\sigma_2\rangle \xrightarrow{\partial_2 \cong} \mathbb{Z}\langle\sigma_1\rangle \xrightarrow{\partial_1=0} \mathbb{Z}\langle\sigma_0\rangle \xrightarrow{\partial_0} 0$

$$\Rightarrow H_n(C) = \begin{cases} \mathbb{Z} & n=0 \\ 0 & \text{else} \end{cases}$$