Recall. Have CC as follows

$$\begin{tikzcd}
 & & & 0 \\
 & 0 \arrow[r] & H_n(X^n) \arrow[r] \arrow[dr, "j_n"] & H_n(X^{n+1}) \arrow[r] & 0 \; (H_n(X^{n+1}, X^n) = 0) \\
\cdots \; H_{n+1}(X^{n+1}, X^n) \arrow[urr, "\delta_{n+1}"] \arrow[rrr, "\partial_{n+1}"'] & & & H_n(X^n, X^{n-1}) \arrow[r, "\partial_n"] \arrow[dr, "\delta_n"'] & H_{n-1}(X^{n-1}, X^{n-2}) \\
 & & 0 \arrow[r] & & H_{n-1}(X^{n-1}) \arrow[u, "j_{n-1}"']
\end{tikzcd}$$

$\hookrightarrow$ this is just $H_n(X)$ by prev

$C_\bullet^{sk}(X)$ (middle row)

w/ diag exact

so $\partial_n = j_{n-1} \circ \delta_n$

Note $\partial_{n-1} \circ \partial_n = j_{n-2} \circ \delta_{n-1} \circ j_{n-1} \circ \delta_n$

$= 0$

$\Rightarrow$ $\boxed{\text{middle row c.c.}}$

Lemma $H_n(C_\bullet^{sk}(X)) \cong H_n(X)$

Pf obs: $H_n(X^n) \xrightarrow[\text{(exactness)}]{j_n,\ \cong} \ker(\delta_n) = \ker(\partial_n)$ → by injectivity

$\cup \qquad\qquad\qquad\qquad \cup$ so quotients same

$\delta_{n+1}(H_{n+1}(X^{n+1}, X^n)) \longrightarrow \operatorname{im}(\partial_{n+1})$

done b/c $\dfrac{H_n(X^n)}{\partial_{n+1}(H_{n+1}(X^{n+1}, X^n))} \cong \underline{H_n(X)}$. (now understand which is why)

Thm $C_\bullet^{sk}(X) \cong C_\bullet^{cw}(X)$ as chain complexes.

$\left(\Leftarrow\!\!\Rightarrow \; H_n(C_\bullet^{sk}(X)) \cong H_n^{cw}(X) \right)$

$\quad \cong$

$\quad H_n(X)$

Pf
$C_n^{sk}(X) = H_n(X^n, X^{n-1}) \cong C_n^{cw}(X)$

$\big\downarrow \partial_n$

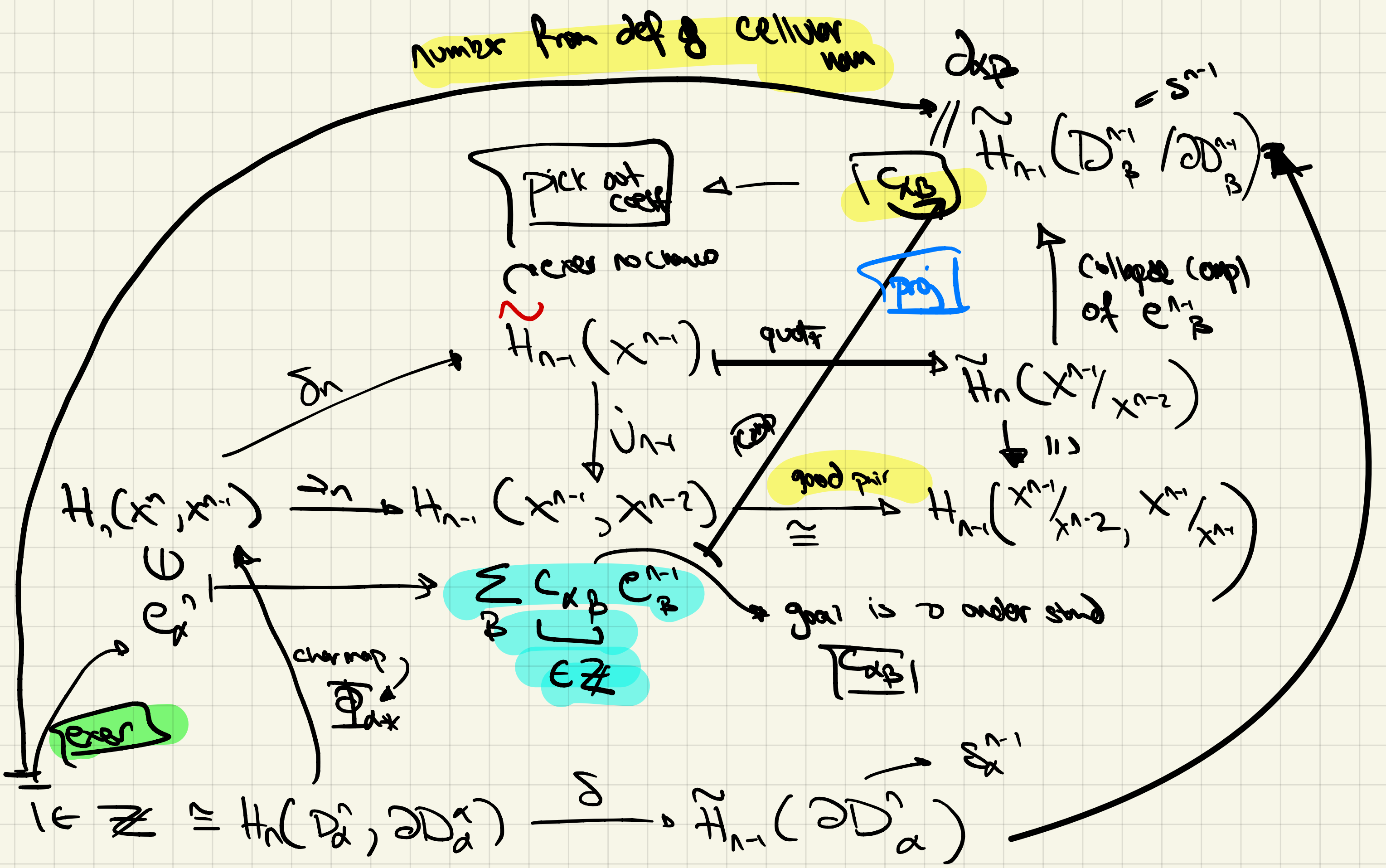
$C_{n-1}^{sk}(X) = H_{n-1}(X^{n-1}, X^{n-2}) \cong C_{n-1}^{cw}(X)$

} last line

have left to argue that ∂_n is given by $\longrightarrow$

Our formula for CW c.c. under isomorphism (for isom of CC need to show for maps)

have basis on RHS $\underline{e^n_\alpha}$ of n cells
$\hookrightarrow$ identify with e^n_α on lhs (above notation)



Everything commutes by naturality ..... so $\underline{d_{\alpha\beta} = c_{\alpha\beta}}$.

insert img 3/4/22:

Say X top sp

Assume, $H_n(X)$ is finitely gen'd $\forall n$ (abelian)

Betti number | $b_n(X) = \text{rank } H_n(X)$

(recall: f A fin gen ab grp $\Rightarrow A \cong \mathbb{Z}^r \oplus$ (tors)

$r = \text{rank}(A)$

If all the $H_n(X)$ are fin gen ab grp & all but finitely many vanish.

Euler Char | $\chi(X) := \sum (-1)^n b_n(X)$.

E.g. | 1) $\chi(S^n) = 1 + (-1)^n$ as $H_i(S^n) = \begin{cases} \mathbb{Z} & i = 0, n \\ 0 & \text{else} \end{cases}$

2) $\chi(\Sigma_g) = 2 - 2g$.

Important case where our assumptions are satisfied:

X finite CW cmplx.
(i.e fin many cells).

Prop | $c_n(X) := \#$ n-cells (depends on struct)

$\chi(X) = \sum (-1)^n c_n(X)$