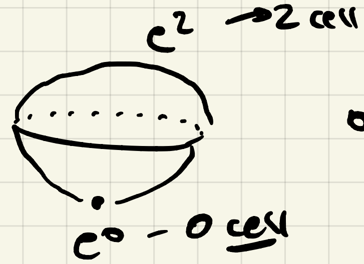
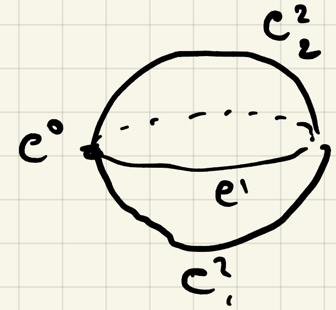
CW - Complexes Cont

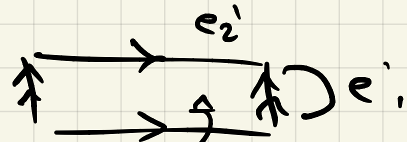
Eg| 2) S^2 =



or

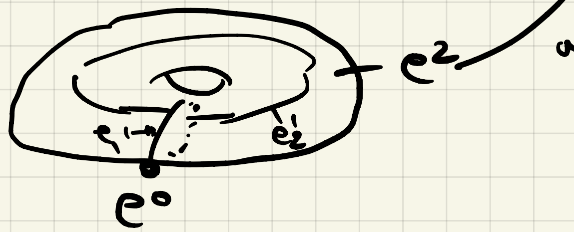


3) $X = S^1 \times S^1$ =



as $D^2 \cong [-1,1]^2$

=



Thm| Every manifold is homotopy equivalent to a CW complex.

Properties| If X is a CW complex ⇒ (Hatcher Appen)

1) X is hausdorff
2) X is locally contractible, i.e. basis for top which are contr.

Operations on CW

1) Products: X, Y CW compl ⇒ X × Y CW compl.

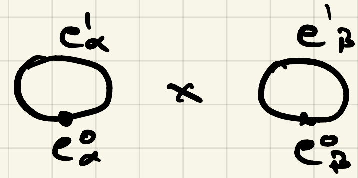
Pf Idea| X is built from $D^n_\alpha \xrightarrow{\Phi_\alpha} X$ char maps

Y .... $D^n_\beta \xrightarrow{\Phi_\beta} Y$

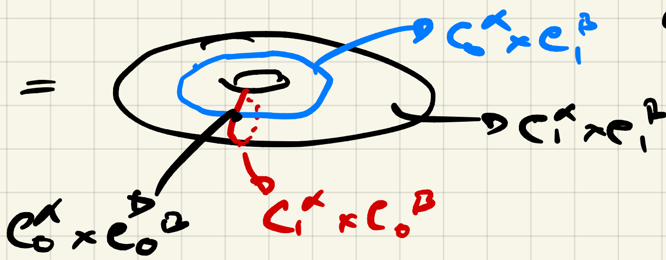
⇒ X × Y

$$\begin{tikzcd} D^m_\alpha \times D^n_\beta \arrow[r, "\Phi_\alpha \times \Phi_\beta"] \arrow[d, phantom, "\cong"] & X \times Y \\ D^{m+n}_{\alpha,\beta} \arrow[ur, dashed, "\Phi_{\alpha,\beta}"'] & \end{tikzcd}$$

E.g| Torus = $S^1 \times S^1$ =



using prod cons ⇝ =



Subtlety/warning (Hatcher Thm A.6)

In gen, the weak top on the CW complex $X \times Y$ is finer than the product topology, but it agrees if X or Y cmpct or if X & Y have ctbly many cells.

2) Quotients: let X a CW compl.

Def | a) $A \subseteq X$ is a subcompl. if A is the union of the img of some collection of cells $\Phi_\alpha(e_\alpha^n)$ in X. Each of whose closure $(\Phi_\alpha(D_\alpha^n))$ is in A

$e_\alpha^n \subseteq D_\alpha^n$ open disc

$\Rightarrow$ A itself is a CW compl

$\hookrightarrow (X, A)$ is a CW pair!

b) if (X, A) CW pair $\Rightarrow$ X/A is a CW compl ⇝ collapsing A to a pt

$\hookrightarrow$ pushout (figure out)

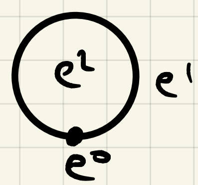
furthermore X/A has zero cells

• $(X \setminus A)^0 = X^0 \setminus A^0 \sqcup \{*\}$ pt $\hookrightarrow$ img of A in X/A

zero-skeleton

∘ Attaching maps $\partial D_\alpha^n \xrightarrow{\varphi_\alpha} X^{n-1} \longrightarrow X^{n-1}/A^{n-1}$

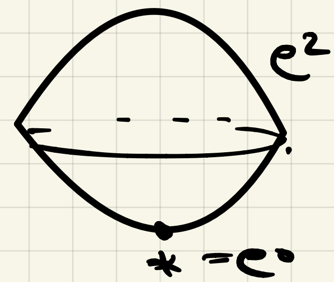
$D_\alpha^n \xrightarrow{\Phi_\alpha} X \xrightarrow{\ \ } X/A$

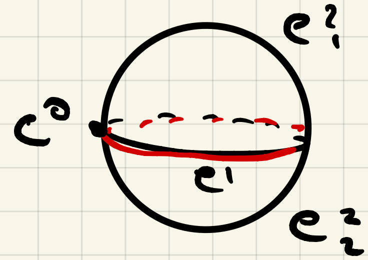
e.g. 1) $X = D^2 =$ 

$A = S^1 = \partial X = \partial D^2$

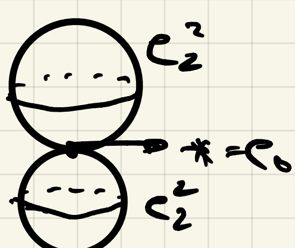
$X/A = D^2/S^1 \cong S^2$

and the quot CW comp



2) $X = S^2$ 

$A = S^1$

$X/A =$ 

3) <u>Wedge Sum</u>) let X, Y spaces w/ base pts $x_0 \in X$, $y_0 \in Y$

Wedge sum $X \vee Y := \dfrac{X \sqcup Y}{x_0 \sim y_0}$

↪ so $S^2/S^1 \cong S^2 \vee S^2$

if X, Y CW comp with $x_0 \in X$, $y_0 \in Y$ 0-cells

$\Rightarrow X \vee Y$ CW comp

Pf) Clearly $X \sqcup Y$ CW compl.

$\Rightarrow$ then quotient by subcompl $\{x_0, y_0\}$

Rmk) $X \vee Y$ doesn't dep on base pts up to homotopy <u>equiv</u>! (for X, Y CW compl.)

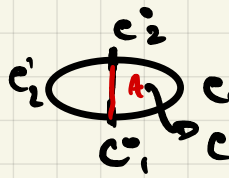
<u>Making homotopy equiv</u> with CW <u>compl.</u>

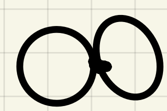
① Quotients:

<u>Thm (Hatcher 5.0)</u>

if (X, A) is a CW pair & A <u>contr.</u>

$\Rightarrow X \xrightarrow{\pi} X/A$ is a homo <u>equiv</u>!

E.g.) $X =$ 

$\simeq X/A \rightarrow$ 

(X, A) CW pair