Thm Let (X,A) a Δ cmplx pair.

The natural map $H_n^\Delta(X,A) \xrightarrow{\cong} H_n(X,A)$ is $\cong$ $\forall n$

$$\begin{array}{ccc} H_n^\Delta(X,A) & \xrightarrow{\cong} & H_n(X,A) \\ \| & & \| \\ H_n(C_\bullet^\Delta(X,A)) & & H_n(C_\bullet(X,A)) \end{array}$$

Lemma (Naturality of LES on homology) (hw9)

Given

$$\begin{tikzcd} 0 \arrow[r] & A_\bullet \arrow[r] \arrow[d, "\alpha"] & B_\bullet \arrow[r] \arrow[d, "\beta"] & C_\bullet \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & A_\bullet' \arrow[r] & B_\bullet' \arrow[r] & C_\bullet' \arrow[r] & 0 \end{tikzcd}$$

- rows are SES of C.C.
- α, β, γ are chain maps
- diag comm.

Then the diag on L.E.S commutes.

$$\begin{tikzcd} \cdots \arrow[r] & H_n(A_\bullet) \arrow[r] \arrow[d, "\alpha_*"] & H_n(B_\bullet) \arrow[r] \arrow[d, "\beta_*"] & H_n(C_\bullet) \arrow[r, "\delta"] \arrow[d, "\gamma_*"] & H_{n-1}(A_\bullet) \arrow[r] \arrow[d, "\alpha_*"] & \cdots \\ \cdots \arrow[r] & H_n(A_\bullet') \arrow[r] & H_n(B_\bullet') \arrow[r] & H_n(C_\bullet') \arrow[r, "\delta"'] & H_{n-1}(A_\bullet') \arrow[r] & \cdots \end{tikzcd}$$

Pf Main case: Assume X is finite dim'l (i.e. Δ cmplx struct on X involves Δ^n for bdd n)

and $A = \emptyset$

$X^k :=$ union of simplices of dim $\leq k$ in X (k skeleton analogue)

$\subseteq X^{k+1} \subseteq X$ sub Δ cmplxes.

(X^k, X^{k-1}) is a Δ-cmplx pair

Lem $\Rightarrow$ comm diagram.

$$\begin{tikzcd} \cdots \arrow[r] & H_{n+1}^\Delta(X^k, X^{k-1}) \arrow[r] \arrow[d, "1"] & H_n^\Delta(X^{k-1}) \arrow[r] \arrow[d, "2"] & H_n^\Delta(X^k) \arrow[r] \arrow[d, "3"] & H_n^\Delta(X^k, X^{k-1}) \arrow[r] \arrow[d, "4"] & H_{n-1}^\Delta(X^{k-1}) \arrow[d, "5"] \\ \cdots \arrow[r] & H_{n+1}(X^k, X^{k-1}) \arrow[r] & H_n(X^{k-1}) \arrow[r] & H_n(X^k) \arrow[r] & H_n(X^k, X^{k-1}) \arrow[r] & H_{n-1}(X^{k-1}) \end{tikzcd}$$

WTS1 ③ $\cong$ (prove thm b/c $X = X^k$ where $k >> 0$)

By 5 lemma. STS ①②④⑤ isom.

it'll pick $\mathbb{Z}$ pts

By induction on K. Assume 2 & 5 are isoms (base case $k=0$ is easy ever)

WTS ①④ $\cong$ | focus on 4 since they are similar.

$\to C_n^\Delta(X) = C_n^\Delta(X^n)$

Simplicial! $C_n^\Delta(X^k, X^{k-1}) = \dfrac{C_n^\Delta(X^k)}{C_n^\Delta(X^{k-1})}$

$$= \begin{cases} C_n^\Delta(X) & n=k \\ 0 & \text{else} \end{cases}$$

$\Rightarrow$ think if $n>k$ then non exist

if $n<k$ then it'll show up below

$$H_n^\Delta(X^k, X^{k-1}) = \begin{cases} C_n^\Delta(X) & n=k \\ 0 & \text{else} \end{cases}$$

Singular $\Phi : \underset{\substack{k\text{-siml in} \\ \Delta\text{-cmplx struct on} \\ X}}{\bigsqcup} (\Delta_\alpha^k, \partial\Delta_\alpha^k) \longrightarrow (X^k, X^{k-1})$ } map well def.

$$\rightsquigarrow \begin{array}{ccc} H_n(\bigsqcup \Delta_\alpha^k, \bigsqcup \partial\Delta_\alpha^k) & \xrightarrow{\Phi_*} & (X^k, X^{k-1}) \\ \text{S}\!\wr\!\wr & \xleftarrow{\text{good pair}} & \wr\!\wr \\ \tilde{H}_n(\bigsqcup \Delta_\alpha^k / \bigsqcup \partial\Delta_\alpha^k) & \xrightarrow{\cong} & H_n(X^k/X^{k-1}) \end{array}$$

The map $\bigsqcup \Delta_\alpha^k / \bigsqcup \partial\Delta_\alpha^k \xrightarrow{\cong} X^k/X^{k-1}$ homeo $\therefore$

$\to \Phi^*$ isom by comm!

Recall (+) k simp $H_n(\Delta_\alpha^k, \Delta_\alpha^{k-1}) \cong H_n(\bigsqcup \Delta_\alpha^k, \bigsqcup \partial\Delta_\alpha^k)$

But see $H_n(\Delta_\alpha^k, \Delta_\alpha^{k-1}) \cong \tilde{H}_n(\underbrace{\Delta_\alpha^k / \Delta_\alpha^{k-1}}_{S^k})$

$$\cong \begin{cases} \mathbb{Z} & n=k \\ 0 & \text{else} \end{cases}$$

$$\Rightarrow H_n(X^k, X^{k-1}) \cong \begin{cases} \bigoplus_{k \text{ simpl}} \mathbb{Z} = C_n^\Delta(X) & n=k \\ 0 & \text{else} \end{cases}$$

Shown $H_n(X^k, X^{k-1}) \cong H_n^\Delta(X^k, X^{k-1})$ also

need to show ④ is an isom!

(hw9)
$$\begin{array}{ccc} H_n^\Delta(X^k, X^{k-1}) & \xrightarrow{④} & H_n(X^k, X^{k-1}) \\ \cong & & \cong \\ \begin{cases} C_n^\Delta(X) & n=k \\ 0 & \text{else} \end{cases} & & \begin{cases} C_n^\Delta(X) & n=k \\ 0 & \text{else} \end{cases} \end{array}$$

④ identified w/ iden map under these $\cong$'s $\blacksquare$.

Case 2 | dim X is ∞ & $A = \emptyset$

Can reduce to the case where dim X is finite.
(we use $\Delta^k \to X$ factors thru some finite X^n skeleton)

Case 3 | X, A arbitrary:

$$\begin{array}{ccccccc} \cdots \to H_n^\Delta(A) & \to & H_n^\Delta(X) & \to & H_n^\Delta(X,A) & \to & \cdots \\ \cong & & \cong & & \downarrow & & \cong \quad \cong \\ \cdots \to H_n(A) & \to & H_n(X) & \to & H_n(X,A) & \to & \end{array}$$

from earlier

by 5-lemma is $\cong$