Recall if X a finite CW cmplx.

$c_n(X) = \#$ of n cells in CW struct for X (dep on struct)

Prop $\chi(X) = \sum_{n=0}^{\ell} (-1)^n c_n(X)$

Rmk LHS is homotopy invariant. A priori RHS not obv htpy inv

Pf Prop follows from, finite c.c

Lemma if $C_\bullet : 0 \to C_m \xrightarrow{\partial_m} C_{m-1} \to \cdots \to C_1 \xrightarrow{\partial_1} C_0 \to 0$

cc w all C_n finitely gen'd ab grp

e.g. $C_\bullet = C_\bullet^{CW}(X)$, X finite CW cmplx

$$\implies \sum (-1)^n \operatorname{rk} H_n(C_\bullet) = \sum (-1)^n \operatorname{rk} C_n$$

(e.g LHS euler char, RHS heuristic for CW struct)

Pf $Z_n = \ker \partial_n$ n-cycles $\qquad H_n = \frac{Z_n}{B_n}$

$B_n = \operatorname{im} \partial_{n+1}$ n-bdrys

$\implies$ SES, $0 \to B_n \to Z_n \to H_n(C_\bullet) \to 0$

w/ prop $\qquad 0 \to Z_n \to C_n \to B_{n-1} \to 0$

exer Rank is additive in SES.

$$\implies \operatorname{rk} Z_n = \operatorname{rk} B_n + \operatorname{rk} H_n(C_\bullet)$$

$$\operatorname{rk} C_n = \operatorname{rk} Z_n + \operatorname{rk} B_{n-1}$$

$$\implies \operatorname{rk} C_n = \operatorname{rk} B_n + \operatorname{rk} B_{n-1} + \operatorname{rk} H_n(C_\bullet)$$

$\implies$ taking alternating sum $\sum_{n=0}^{m} (-1)^n \operatorname{rk} C_n = \sum_{n=0}^{m} (-1)^n \operatorname{rk} C(C_\bullet)$

$$\cancel{+ (-1)^n \operatorname{rk} B_n}$$

$$\cancel{+ (-1)^n \operatorname{rk} B_{n-1}}$$

(tele to 0)

$\square$

Defn| let $f: X \to X \rightsquigarrow f_*: H_n(X) \to H_n(X)$

if $H_n(X)$ is fin gen $\rightsquigarrow$ get number w/ trace of f_*

$\rightsquigarrow \operatorname{tr}(f_*) \in \mathbb{Z}$

Recall| if $\varphi: \mathbb{Z}^r \to \mathbb{Z}^r$ homo, repr by matrix $\varphi = (a_{ij})$ $r \times r$ matrix val in $\mathbb{Z}$

$\Rightarrow \operatorname{tr}\varphi = \sum_{i=1}^{r} a_{ii}$ (independent of choice of basis for $\mathbb{Z}^r$)

If $\varphi: A \to A$ is a homo w/ A fin gen ab grp.

$A = \mathbb{Z}^r \oplus A_{tors}$ let $A_{tf} = \frac{A}{A_{tors}} \cong \mathbb{Z}^r$

φ sends tors $\to$ tors

$\varphi(A_{tors}) \subseteq A_{tors} \xRightarrow{\text{quot}} \bar{\varphi}: A_{tf} \to A_{tf}$

$\operatorname{tr}\varphi := \operatorname{tr}\bar{\varphi} \in \mathbb{Z}$

Assume, $H_n(X)$ fin gen & Abfmz.

$\Rightarrow$ lefschetz num f $\Lambda_f := \sum (-1)^n \operatorname{tr}(f_*: H_n(X) \to H_n(X))$

e.g.| if $f \simeq \mathrm{id}_X \Rightarrow \Lambda_f = \chi(X)$

$f \simeq \mathrm{id}_X \Rightarrow f_* = \mathrm{id}_*$ & $\operatorname{tr} \mathrm{id}_* = \operatorname{rk} H_n(X)$

Obs| Λ_f only dep on f up to htpy b/c

Thm| (Lefschetz fixed pt)

X is a finite CW cmplx (so lefschetz well def) $\to$ stronger

$f: X \to X$

$\Lambda(f) \neq 0$

$\Rightarrow$ f has a fixed pt.

Rmk| Converse not true **e.g.|** $f = \mathrm{id}_{S^n}$

$\Lambda_f = \chi(S^n) = 1 + (-1)^n = 0$ n odd but f has fp.

Rmk 2| finiteness on X is needed

eg | $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto x+a$ $a \neq 0 \in \mathbb{R}$

f has no fixed pt but $\Lambda(f) = 1$

$\mathbb{R}$ has no finite CW cmplx.

Cor | X is contractible & finite CW cmplx.

$\Rightarrow f: X \to X$ has a fixed pt. only $H_0(X) \neq 0$

Pf | $\Lambda(f) = tr(f_*: H_0(X) \to H_0(X)) = 1$

$\mathbb{Z} \xrightarrow{id} \mathbb{Z}$ (both $\cong$)

conn. gen'd by class of any pt.

as $[p] \mapsto [f(p)]$

true for any p/c sp

$\therefore$ by thm done!

Rmk | if $X = D^n \Rightarrow$ Brouwer fixed pt thm.

Eg | $f: S^n \to S^n$

$\Lambda(f) = tr(f_*: H_0(S^n) \to H_0(S^n)) + (-1)^n tr(f_*: H_n(S^n) \to H_n(S^n))$

by earlier conn

$= 1 + (-1)^n \deg f$.

$\therefore$ if $\deg f \neq (-1)^{n+1}$, then $\Lambda(f) \neq 0$

$\Rightarrow f$ has fixed pt

contra

earlier showed f fixed pt free

$\Rightarrow \deg f = (-1)^{n+1}$ as htpic antipodal map

Thm Later.

Def | A simplicial complex is a special Δ complex X whose collection Σ of simplices satisfies

1) $\forall \sigma: \Delta^n \to X) \to \Sigma \Rightarrow \sigma$ injective (not just on 1 At)

2) $\sigma: \Delta^n \to X$, $\tau: \Delta^m \to X \in \Sigma \Longrightarrow$

$[v_0, \dots, v_n]$ $[v_0, \dots, v_m]$

$\Rightarrow I = \sigma(\Delta^n) \cap \tau(\Delta^m)$ is empty or a subsimplex of σ or τ

i.e. $I = \sigma([v_{i_1}, \dots, v_{i_p}])$ for some $i_1, \dots, i_p$

$= \tau([w_{j_1}, \dots, w_{j_p}])$ $j_1, \dots, j_p$

Rmk) A simplicial cmplx struct sometimes called triangulation.

$$\left\{\begin{matrix}\text{simplicial}\\\text{cmplx}\end{matrix}\right\} \subsetneq \{\Delta\text{-cmpl}\} \subsetneq \{\text{CW-compl}\} \subsetneq \text{top sp}$$

e.g)