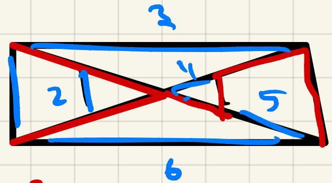
Van Kampen

e.g. 1) $\pi_1\Big($  $\Big)$?

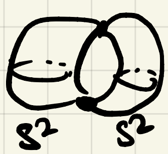
A

$A \subseteq X$ is a subcomplex & contractible
$\{*\}$

$\Rightarrow X \sim X/A$ ⟶ all line seg become circles

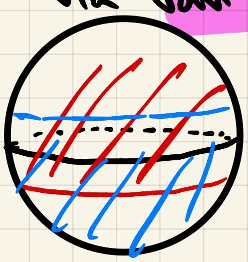
$\Rightarrow X/A \sim \bigvee_{i=1}^{6} S^1$ $\Rightarrow \pi_1(X) \cong \ast_{i=1}^{6} \mathbb{Z}$

2) $X =$  = torus with 2 D^2 glued in

$X \underset{\text{contract } D^2}{\sim}$  $\underset{\text{earlier lec}}{\sim}$  $= S^2 \vee S^2 \vee S^1$

$\Rightarrow \pi_1(X) = \mathbb{Z}$

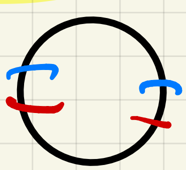
3) S^2 via van Kampen



$V = D^2 \sim \{*\} \sim U$
$U \cap V \sim S^1 \times I \sim S^1$

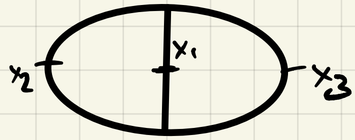
$$VK \Rightarrow \pi_1(S^2) \cong \frac{\overset{0}{\pi_1(U)} * \overset{0}{\pi_1(V)}}{\pi_1(U \cap V)} \cong 0$$

Similarly $\pi_1(S^n) = 0$ $n \geq 2$

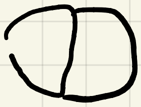
Non eg $S^1 =$  but $U \cap V$ not conn!

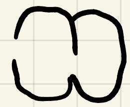
$$0 = \frac{\pi_1(U) * \pi_1(V)}{\pi_1(U \cap V)} \longrightarrow \pi_1(S^1) = \mathbb{Z}$$

VK part 1 (surj) fails if $U_i \cap U_j$ not path conn.

Non eg2| $X =$ 

Let $U_i = X \setminus \{x_i\}$ open cover

$U_i =$  $\sim S^1$ via deform ret

$U_1 \cap U_2 = X \setminus \{x_1, x_2\} =$  $\sim \{*\}$

$$\underbrace{\operatorname*{colim}_{ij}\left(\pi_1(U_i \cap U_j) \rightrightarrows \begin{matrix}\pi_1(U_j)\\ \pi_1(U_i)\end{matrix} \right)}_{\mathbb{Z} * \mathbb{Z} * \mathbb{Z}} \longrightarrow \pi_1(X)$$

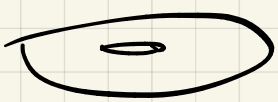
$\pi_1(U_i \cap U_j) = 0 \longrightarrow$ just free prod

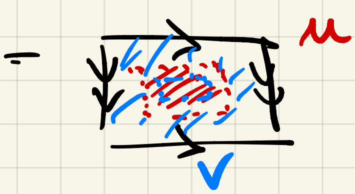
$$\mathbb{Z} * \mathbb{Z} * \mathbb{Z} \longrightarrow \mathbb{Z} * \mathbb{Z}$$

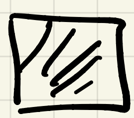
surj
not isom

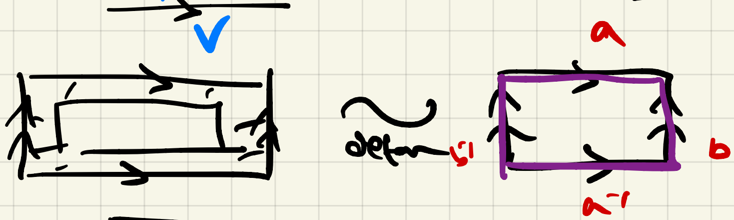
$X \sim S^1 \vee S^1$

VK crit 2 fails for the above (triple int must be path conn)

e.g.| $X =$  torus

$=$ 

$U =$  $\sim \{*\}$

$V =$  $\sim S^1 \vee S^1$

$U \cap V =$  $\sim S^1$

$$\pi_1(X) = \frac{\pi_1(U) * \pi_1(V)}{\pi_1(U \cap V)} = \frac{\{0\} * (\mathbb{Z} * \mathbb{Z})}{\mathbb{Z}}$$

$f \uparrow \ \mathbb{Z} \ \nearrow g$

$g : \mathbb{Z} \longrightarrow \mathbb{Z} * \mathbb{Z}$

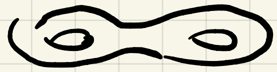
$1 \longrightarrow aba^{-1}b^{-1}$

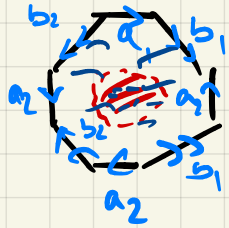
push loop fw purple

$$= \frac{\mathbb{Z} * \mathbb{Z}}{\langle g(x) \underbrace{f(x)^{-1}}_{\text{triv}} \mid x \in \mathbb{Z} \rangle}$$

$$= \frac{\mathbb{Z} * \mathbb{Z}}{\langle g(x) \mid x \in \mathbb{Z} \rangle}, \quad \mathbb{Z} * \mathbb{Z} \rightsquigarrow \langle a, b \rangle, \quad = \frac{\langle a, b \rangle}{\langle aba^{-1}b^{-1} \rangle} \cong \mathbb{Z} \oplus \mathbb{Z}$$

HW3

E.g.) $X =$ 

$=$  $V \sim S^1 \vee S^1 \vee S^1 \vee S^1$

$U \sim \{*\}$

$U \cap V \sim S^1$

$$\pi_1(X) \cong \frac{\overset{\langle a_1 \rangle}{\mathbb{Z}} * \overset{\langle b_1 \rangle}{\mathbb{Z}} * \overset{\langle a_2 \rangle}{\mathbb{Z}} * \overset{\langle b_2 \rangle}{\mathbb{Z}}}{\langle a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \rangle}$$

go around

If X genus g cmpct orientable surface

$\pi_1(X)$ similar w/ 2 repl w/ g.

Van Kampen

1) Surjectivity!

$x_0 \in X \qquad X = \bigcup_{i \in J} U_i$ open $\qquad x_0 \in U_i \ \forall i \in J$

$U_i \cap U_j$ pc $\forall i, j \in J$

equiv $\quad \mathop{\ast}_{i \in J} \pi_1(U_i) \xrightarrow[\text{surj}]{} \pi_1(X)$

↳ don't care abt quotient here!

Pf) $\alpha : I \to X$ a loop at x_0

WTS $[\alpha] = [\beta_1] \cdot \cdots \cdot [\beta_n]$

where β_k loop at x_0 in some U_{i_k}