Language of Category Theory

Def A category $\mathscr{C}$ consists of:

1) A collection of objects $(ob(\mathscr{C}))$
2) $\forall x, y \in ob(\mathscr{C})$, set of morphisms $Hom_{\mathscr{C}}(x,y)$
3) $\forall x, y, z \in ob(\mathscr{C})$, composition

$$Hom_{\mathscr{C}}(y,z) \times Hom_{\mathscr{C}}(x,y) \longrightarrow Hom_{\mathscr{C}}(x,z)$$
$$(g,f) \longmapsto g \circ f$$

4) $\forall x \in ob(\mathscr{C})$ $\exists\, id_x \in Hom_{\mathscr{C}}(x,x)$ s.t

$$g \circ id_x = g \quad , \quad id_x \circ f = f$$

when g, f morphisms s.t above are defd.

5) (Assoc of comp) $(h \circ g) \circ f = h \circ (g \circ f)$
for composable morphisms h, g, f

eg

$\mathscr{C}$	$ob(\mathscr{C})$	Morphisms
Set	set	maps of sets
Top	Topological sp.	Continuous maps
h Top	top sp	homotopy cl of cts maps
Grp	groups	homomorphisms of grps
Ab	Abelian grps	homomorphisms of grps
Modules / R	R modules	R module homomorphisms

hom cat of sp

Def A functor F from $F : \mathscr{C} \longrightarrow \mathscr{D}$ consists of:

cats

1) A map $ob(\mathscr{C}) \longrightarrow ob(\mathscr{D})$
$$x \longrightarrow F(x)$$

2) $\forall$ $X, Y \in ob(\mathcal{C})$, a map

$$\mathrm{Hom}_{\mathcal{C}}(X,Y) \longmapsto \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$$

$$f \longmapsto F(f)$$

s.t.

i) $F(id_X) = id_{F(X)}$ $\forall X \in ob(\mathcal{C})$

ii) $F(g \circ f) = F(g) \circ F(f)$

$\forall$ comp g, f

e.g.) ① Forgettable functor : $\mathrm{Top} \longrightarrow \mathrm{Sets}$

$X \longrightarrow$ underlying set

$\boxed{\text{forget Structure}}$ ← $(f: X \to Y) \longmapsto (f: X \to Y)$

② Free functor : $\mathrm{Sets} \longrightarrow R$ modules

$X \longmapsto R^{\oplus X}$ ⤳ free prod & the

$(f: X \to Y) \longmapsto R^{\oplus X} \longrightarrow R^{\oplus Y}$

formal finite sum $\sum_{x \in X} a_x \cdot x \longmapsto \sum_{x \in X} a_x \cdot f(x)$

③ $\pi_0 : \mathrm{Top} \longrightarrow \mathrm{Set}$

$X \longrightarrow \pi_0(X)$ as from last time

$f \longrightarrow \pi_0(f)$

In fact, this factors via a functor

natural functor $\to h\mathrm{Top} \longrightarrow \mathrm{Set}$

A rephrasing of the statement that $f \sim g \Rightarrow \pi_0(f) = \pi_0(g)$)

Alg top = study of category of $h\mathrm{Top}$ via functors to "alg cat"!

$\to$ (set, Grps, R-modules)

Given a diagram in the category $\mathscr{C}$

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d, "b"] \\ X \arrow[r, "a"'] & P \end{tikzcd}$$

a pushout is an object P with morphisms $a : X \to P$ & $b : Y \to P$ so that extended diagram commutes

And is universal w/ this property

i.e $\forall$ commutative diagram $\longrightarrow$

$\exists!\ c : P \to P'$ making the extended diagram commute $\longrightarrow$

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"'] & Y \arrow[dl, "b"'] \arrow[ddr, bend left, "b'"] & \\ X \arrow[r, "a"] \arrow[drr, bend right, "a'"'] & P \arrow[dr, dotted, "c"] & \\ & & P' \end{tikzcd}$$

↪ $\boxed{\text{colimit}}$

Rem · if the pushout exists, it is then unique up to (unique) isomorphism → map in both dir that comp to id

↪ get a $c : P \to P'$ & $c' : P' \to P$ $\quad c \circ c'$ map from $P' \to P'$

· pushout is a special case of the notion of a colimit!

e.x) Pushouts exist in the category of top sp!

$$P = \frac{X \sqcup Y}{(f(z) \sim g(z)\ \forall z \in Z)}$$

→ quotient top & natural maps $a : X \to P$, $b : Y \to P$. (inclu + proj)

quotienting forces commute

Def) A CW complex (cell complex) is a space X built inductively as follows:

- X^0 is a discrete set of pts
- $(n \geq 1)$ X^n is built from X^{n-1} by attaching n discs

i.e. $$X^n = \frac{X^{n-1} \sqcup_\alpha D^n_\alpha}{x \sim \varphi_\alpha(x)\ \forall x \in \partial D^n_\alpha}$$

where $D^n \cong D^n_\alpha$ n disc $\quad \varphi_\alpha : \partial D^n_\alpha = S^{n-1}_\alpha \longrightarrow X^{n-1}$ (cts) attaching map

• $X := \bigcup_n X^n$ w/ weak (CW) topology

$U \subseteq X$ open $\iff \forall n$, $U \cap X^n \subset X^n$ open

Terminology | $X^n \to$ n-skeleton

$e_n^\alpha :=$ interior of D_α^n, an open n-disc

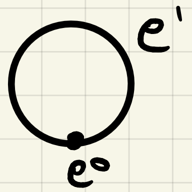
↳ often called n-cell

$\Phi_\alpha : D_\alpha^n \to X^n \longrightarrow X$ Characteristic map

natural maps or "inclusions" (so that)

$\Phi_\alpha|_{S_\alpha^{n-1}} = S_\alpha^{n-1} \xrightarrow[\varphi_\alpha]{} X^{n-1} \to X$

$\Phi_\alpha|_{e_\alpha^n}$ homeo to its image

e.g. ① $S^1 =$ 

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