Cellular Homology

E.g.) Σ_g genus g, or. cmct surface.

0-cell x_0, $2g$ 1-cells $a_1, b_1, \dots, a_g, b_g$, 1 2-cell D

$$C_\bullet^{cw}(\Sigma_g) : \quad \cdots\cdots \quad \mathbb{Z}\langle D\rangle \xrightarrow{0} \mathbb{Z}\langle a_1, b_1, \dots, a_g, b_g\rangle \xrightarrow{0} \mathbb{Z}\langle x_0\rangle \to \cdots$$

as 1 0-cell

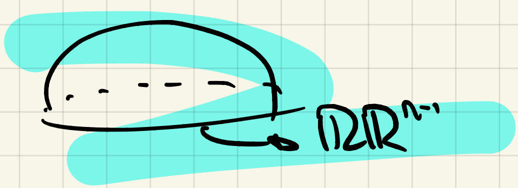
$$\partial(D) = \sum_{i=1}^{g} c_i a_i + d_i b_i \qquad c_i \in \mathbb{Z}_{a_1}, \; d_i \in \mathbb{Z}$$

$$S^1 - \left(\partial \bar{D} \longrightarrow \bigvee_{a_1, b_1, a_2} \xrightarrow{\mathbb{Z}} S^1_{a_1} \right) = 0$$

$$\Rightarrow H_n^{cw}(\Sigma_g) = \begin{cases} \mathbb{Z} & n = 0, 2 \\ \mathbb{Z}^{2g} & n = 1 \\ 0 & \text{else} \end{cases}$$

mod out hemisphere not eq

2) $\mathbb{RP}^n = \dfrac{\mathbb{R}^{n+1} \setminus \{0\}}{\mathbb{R}^\times} = \dfrac{S^n}{(x \sim -x)} = \dfrac{D^n}{\left(x \sim -x \text{ for } x \in \partial D^n = S^{n-1} \right)}$



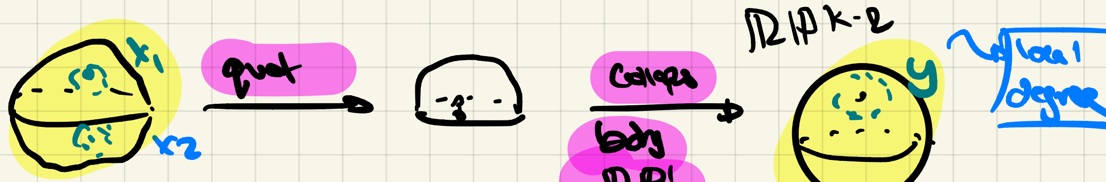
$= $ attach D^n to $\mathbb{RP}^{n-1}$ along $\partial D^n = S^{n-1} \xrightarrow{\text{quot}} \mathbb{RP}^{n-1} = \dfrac{S^{n-1}}{\{\pm x\}}$

$\therefore$ $\mathbb{RP}^n$ has a cw cmplx struct w/ cells $e^0, \dots, e^n$ — 1 cell per dim.

k-1 skeleton is $\mathbb{RP}^{k-1}$

$\boxed{k \geq 1}$ $\partial(e^k) = \deg$ of

$$f : S^{k-1} = \partial D^k \longrightarrow \mathbb{RP}^{k-1} \longrightarrow \frac{\mathbb{RP}^{k-1}}{\mathbb{RP}^{k-2}} \cong S^{k-1}$$



$$\deg f = \deg_{x_1} f + \deg_{x_2} f = \deg_{x_1} \mathrm{id}_x + \deg_{x_2}(\text{antipode}) = (-1)^k$$

$\therefore \deg f = \begin{cases} 2 & k \text{ odd} \\ 0 & k \text{ even} \end{cases}$

$\therefore C_\bullet^{CW}(\mathbb{RP}^n) =$

1 0-cell

(n even) $\mathbb{Z}\langle e^n\rangle \xrightarrow{2} \cdots \xrightarrow{2} \mathbb{Z}\langle e^2\rangle \xrightarrow{0} \mathbb{Z}\langle e^1\rangle \xrightarrow{0} \mathbb{Z}\langle e^0\rangle \to \cdots$

(n odd) $\mathbb{Z}\langle e^n\rangle \xrightarrow{0} \cdots \xrightarrow{2} \mathbb{Z}\langle e^2\rangle \xrightarrow{0} \mathbb{Z}\langle e^1\rangle \xrightarrow{0} \mathbb{Z}\langle e^0\rangle \to \cdots$

$$\Rightarrow H_k^{CW}(\mathbb{RP}^n) = \begin{cases} \mathbb{Z} & k=0 \quad k=n \text{ odd} \\ \mathbb{Z}/(2) & 0<k<n,\ k \text{ odd} \\ 0 & \text{else} \end{cases}$$

Goal: $H_n^{CW} = H_n$

Lemma: X CW cmplx, $H_k(X^n, X^{n-1}) \cong \begin{cases} \mathbb{Z}\langle n\text{-cells}\rangle = C_n^{CW}(X) & k=n \\ 0 & \text{else} \end{cases}$

Pf: $H_k(X^n, X^{n-1}) \overset{\text{good pair}}{\cong} \tilde{H}_k(X^n/X^{n-1})$

$X^n/X^{n-1} \cong \bigvee_\alpha D^n_\alpha/\partial D^n_\alpha \cong \bigvee_\alpha S^n$

attaching disc along bdry skel.

$\therefore \tilde{H}_k(X^n/X^{n-1}) = \tilde{H}_k(\bigvee_\alpha S^n) = \bigoplus_\alpha \tilde{H}_k(S^n)$ $\square$

Lemma: X CW cmplx

1) $H_k(X^n) = 0$ for $k > n$

2) $H_k(X^n) \to H_k(X)$ induced by incl.

- this is an iso if $k<n$
- this is surj for $k=n$

Pf: LES for pair (X^n, X^{n-1}):

$\longrightarrow$

$$\longrightarrow \cdots \underbrace{H_{k+1}(X^n, X^{n-1})}_{=0 \text{ if } k+1 \neq n} \longrightarrow H_k(X^{n-1}) \longrightarrow H_k(X^n) \longrightarrow \underbrace{H_k(X^n, X^{n-1})}_{=0 \text{ if } k \neq n} \longrightarrow$$

$\boxed{\text{D}}$

↳ so if $n < k$ $\Rightarrow$ $H_k(X^{n-1}) \xrightarrow{\cong} H_k(X^n)$

$\Rightarrow$ have $H_k(X^0) \xrightarrow{\cong} H_k(X^1) \xrightarrow{\cong} \cdots \xrightarrow{\cong} H_k(X^{k-1})$

↳ if $n = k$ get some inj & surj

$$H_k(X^{k-1}) \hookrightarrow H_k(X^k) \twoheadrightarrow H_k(X^{k+1})$$

↳ if $n > k+1$ $\Rightarrow$ $H_k(X^{n-1}) \xrightarrow{\cong} H_k(X^n)$

$\Rightarrow$ $H_k(X^{k+1}) \cong H_k(X^{k+2}) \cong \cdots$

$\Rightarrow$ 1) b/c $H_k(X^0) = 0$ for $k > 0$

2) if $\dim X < \infty$ as it is some skel

(ever $\dim X = \infty$)

⇝ exact seq

$$\begin{array}{ccccccc} H_{n+1}(X^{n+1}, X^n) & \xrightarrow{\partial_{n+1}} & H_n(X^n) & \longrightarrow & H_n(X^{n+1}) & \longrightarrow & \cancel{H_n(X^{n+1}, X^n)} \to 0 \text{ lem1} \\ \text{\rotatebox{90}{$\cong$}} & & & & \text{\rotatebox{90}{$\cong$}} \text{ lem2} & & \\ C^{cw}_{n+1}(X) & & & & H_n(X) & & \end{array}$$

$$\begin{array}{ccccccc} \cancel{H_n(X^{n-1})} & \longrightarrow & H_n(X^n) & \xrightarrow[j]{} & H_n(X^n, X^{n-1}) & \xrightarrow{\partial_n} & H_{n-1}(X^{n-1}) \\ \swarrow & & & & \text{\rotatebox{90}{$\cong$}} & & \\ 0 \text{ lem 2} & & & & C^{cw}_n(X) & & \end{array}$$

will splice exact seq,

$\longrightarrow$

$$\cdots \longrightarrow H_{n+1}(X^{n+1}, X^n) \xrightarrow[\partial_n = j_n \circ \partial_{n+1}]{d} H_n(X^n, X^{n-1}) \longrightarrow \cdots$$

Next time

- $\partial_n \circ \partial_{n+1} = 0$
- $H_n(CC) \cong H_n(X)$
- This CC $\cong C_\bullet^{CW}(X)$.