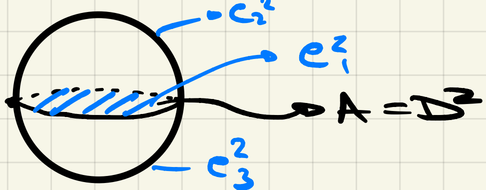
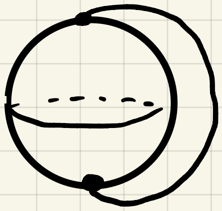
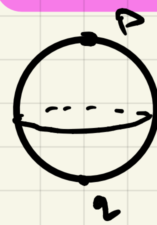
**Recall** if $(X,A)$ CW pair, $A$ contractible $\Rightarrow \pi: X \to X/A$ htpy equiv

**e.g.** $X =$ 


thm $\sim$ 

**e.g.** $X =$  $\sim$ 

$/p \sim q$

$\exists$ CW struct so $A$ subcomplex

## 2) Homotoping attaching maps!

if $(X,A)$ CW pair, $f,g: A \to Y$ homotopic.

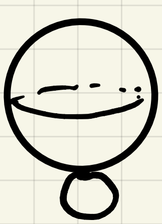
$X \cup_f Y :=$ pushout
$$\begin{array}{ccc} A & \xrightarrow{f} & Y \\ \downarrow & & \\ X & & \end{array}$$

$X \cup_g Y :=$ pushout
$$\begin{array}{ccc} A & \xrightarrow{g} & Y \\ \downarrow & & \\ X & & \end{array}$$

**Thm** (Hatcher 0.18) $X \cup_f Y \sim X \cup_g Y$

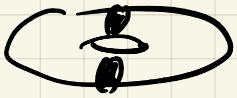
**e.g.** 

$/p \sim q$ $\overset{\text{earlier}}{\sim}$ 

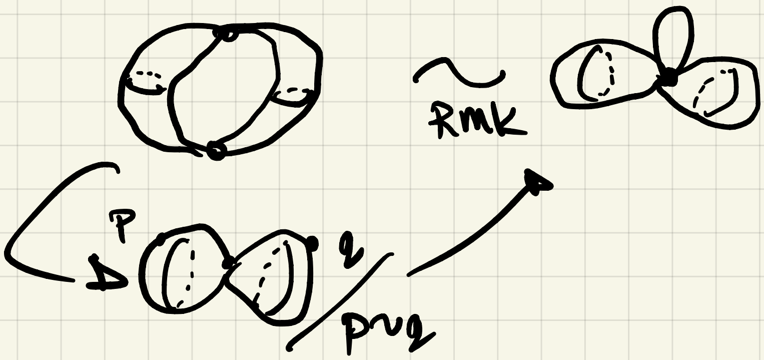
$\sim$  $S^2 \vee S^1$

b/c $\{0,1\} \to S^2$, $0 \mapsto p$, $1 \mapsto q$ htpc to $\{0,1\} \to S^2$, $0 \to q$, $0 \to q$ by path conn!

**More generally** if $Y$ path conn, $p,q \in Y$ distinct
$Y/p \sim q \sim Y \vee S^1$

Eg| $X =$  torus w/ 2 $D^2$'s glued in

2-thm 1 collapsing criteria




$\simeq_{\text{RMK}} \quad = S^2 \vee S^2 \vee S^1$

## Fundamental Grp

$\pi_0(X) = \{\text{path components of } X\}$

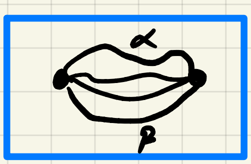
$= \mathrm{Hom}_{htop}(\{*\}, X) \rightsquigarrow$ fancy

Next, replace $\{*\}$ with $S^1$

**Def|** $X$ is a top sp. A path in $X$ from $x_0$ to $x_1$ is a map $I = [0,1]$ $\alpha : I \to X$
$\alpha(0) = x_0$, $\alpha(1) = x_1$

**Def|** A loop in $X$ with $x_0 \in X$ ——— $\alpha(0) = x_0 = \alpha(1)$

**Def|** $\alpha, \beta$ are path homotopic if
- $\alpha(0) = \beta(0)$, $\alpha(1) = \beta(1)$
- $\alpha$ is homotopic to $\beta$ relative to $\{0,1\} \subset I$




denote $\alpha \underset{p}{\simeq} \beta$

**exer|** (like Hw1 Q2) $\alpha \underset{p}{\simeq} \beta$ is an equiv reln

**Notation** $[\alpha]$ the path homotopy class of $\alpha$

**Lemma** let $\beta$ be a reparameterization of $\alpha : I \to X$

i.e $\beta = \alpha \circ \phi$ where $\phi : I \to I$ cts $\phi(0)=0$ $\phi(1)=1$

$\Rightarrow \alpha \underset{p}{\sim} \beta \Rightarrow [\alpha] = [\beta]$

**Pf** $id_I : I \to I$

$\hookrightarrow id_I \underset{p}{\sim} \phi$ via the homotopy

$$H : I \times I \longrightarrow I$$
$$(s,t) \longrightarrow (1-t)s + t\phi(s)$$

$H_0 = id_I$ , $H_1 = \phi$

$H(0,t) = 0$ $H(1,t) = 1$

Note $H$ has img in $I$

$\Rightarrow \alpha \underset{p}{\sim} \alpha \circ \phi = \beta$ via $\alpha \circ H : I \times I \xrightarrow{H} I \xrightarrow{\alpha} X$ ∎

**Def** Product of 2 paths $\alpha, \beta : I \to X$ w/ $\alpha(1) = \beta(0)$

is $\alpha \cdot \beta : I \longrightarrow X$

$$s \longmapsto \begin{cases} \alpha(2s) & s \in [0, 0.5] \\ \beta(2s-1) & s \in [0.5, 1] \end{cases}$$

well def : 1) as $\alpha(1) = \beta(0)$

2) $\alpha \cdot \beta$ is cts b/c restriction to each $[0, 1/2]$, $[1/2, 1]$ cts (pasting lemma)

**Properties of Product**

① (well-defined modulo path homotopy) :

$[\alpha] = [\alpha']$ , $[\beta] = [\beta']$ $\alpha(1) = \beta(0)$

$\Rightarrow [\alpha' \cdot \beta'] = [\alpha \cdot \beta]$

let $H$ be homotopy of $\alpha, \alpha'$ , $G$ similar $\beta \to \beta'$

let $H \cdot G : I \times I \longrightarrow X$ similar well def cts

$\alpha \cdot \beta \sim \alpha' \cdot \beta'$ as $(H \cdot G)_t = H_t$ $(s,t) \longmapsto \begin{cases} H(2s,t) & s \in [0, 1/2] \\ G(2s-1, t) & s \in [1/2, 1] \end{cases}$