Cor| If $(\tilde{X},\tilde{x}_0) \xrightarrow{p} (X,x_0)$ a Galois cover

$\quad\quad G$ p.c $\quad\quad\quad\quad G$ pc, lpc, slsc

$$\longleftrightarrow H := p_* \pi_1(\tilde{X},\tilde{x}_0) \subset \pi_1(X,x_0)$$

$$\Longrightarrow \mathrm{Aut}(\tilde{X}/X) \cong \frac{\pi_1(X,x_0)}{H}$$

In particular, if $\tilde{X}$ univ cover of X

$$\Rightarrow \mathrm{Aut}(\tilde{X}/X) \cong \pi_1(X,x_0) \text{ as } \underline{H = \{1\}}$$

e.g.| Recovers our earlier computations

$\mathrm{Aut}(\mathbb{R}/S^1) \cong \mathbb{Z}$ by direct arg. But above shows this

$$\mathrm{Aut}\Big(\begin{array}{c} p_n : S^1 \to S^1 \\ z \mapsto z^n \end{array}\Big) \cong \mathbb{Z}/n\mathbb{Z}$$

e.g.| X pc, lpc, slsc

PC cover sp $Y \xrightarrow{p} X$

let $\tilde{X} \xrightarrow{q} X$ univ cover.

Have a factorization of q

$$\begin{tikzcd} \tilde{X} \arrow[r, "f"] \arrow[rr, bend left, "q"] & Y \arrow[r, "p"] & X \end{tikzcd}$$

$$\tilde{x}_0 \longmapsto y_0 \longmapsto x_0$$

with $\tilde{X} \xrightarrow{f} Y$ is a covering sp!

As $\tilde{X}$ s.c, it is the univ cover of Y

$$\overset{}{\underset{\text{cor}}{\Longrightarrow}} \mathrm{Aut}(\tilde{X}/Y) \cong \pi_1(Y,y_0)$$

$$\begin{tikzcd} \mathrm{Aut}(\tilde{X}/Y) \arrow[d, phantom, "\cap"] \arrow[r, phantom, "\cong"] & \pi_1(Y,y_0) \arrow[d, "p_* \rightsquigarrow \boxed{\text{check}}"] \\ \mathrm{Aut}(\tilde{X}/X) \arrow[r, phantom, "\cong"] & \pi_1(X,x_0) \end{tikzcd}$$

Covers from group actions

$G \curvearrowright X$ top sp. Assume the grp action is cts on X, and $\forall\, x \in X\ \exists$ nbhd $x \in U$ so $gU \cap U = \emptyset$ if $1 \neq g \in G$

$\underline{\text{On hw3}}$| $p : X \to X/G$ is a covering sp

Defn| This is a covering sp action.

Thm 6 if X is Hausdorff, G finite, $G \curvearrowright X$ freely
$\Rightarrow$ covering sp.

Note: A covering sp action $G \curvearrowright X$ is free
$(1 \neq g,\ x \in X,\ gx \neq x)$

Prop Say $G \curvearrowright X$ is a cov sp action

1) $p : X \to X/G$ is galois. So this looks like some sss aut $(X/(x_G))$

Pf for $g \in G$ $g\cdot : X \xrightarrow{\cong} X$ (left mult)
is a covering sp $\cong$ {deck aut}
And g acts transitively on $p^{-1}(p(x)) = G\cdot x$
$\forall\ x \in X$
This is the fibre

2) $\mathrm{Aut}(p) \cong G$ as long as X connected.

Pf have $G \to \mathrm{Aut}(p)$ as above
$g \mapsto L_g$

inj G acts freely on X

surj $f \in \mathrm{Aut}(p)$ $x \in X$ $f(x) \in p^{-1}(p(x))$ as f cov sp aut
$\Rightarrow x, f(x) \in G\cdot x$ g trans on fibre
& $\exists\ g \in G$ so $f(x) = g\cdot x$
$\Rightarrow f = L_g$ by matching p.t. (unique lift prop).

3) X PC + LPC
$$\Rightarrow G \cong \frac{\pi_1(X/G)}{p_*(\pi_1(X))}$$

Pf $G \overset{②}{\cong} \mathrm{Aut}(p) \overset{①+\text{cor at start}}{\cong} \dfrac{\pi_1(X/G)}{p_*(\pi_1(X))}$

Cor if X is SC, PC, LPC. $G \curvearrowright X$ cov sp act.

$$③ \Rightarrow G = \frac{\pi_1(X/G)}{p_*(\pi_1(X)) \to \{1\}} = \pi_1(X/G)$$

e.g.) 1) $\mathbb{Z} \curvearrowright \mathbb{R}$ — covering sp action (exer)

$n \longmapsto \left(\begin{aligned} g : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto x+n \end{aligned} \right)$

$S^1 = \mathbb{R}/\mathbb{Z} \implies \pi_1(S^1) \cong \mathbb{Z}$ as $\mathbb{R}$ sc.

2) $\mathbb{Z}/2 \curvearrowright S^n$ via antipodal map. covering sp act $\to$ hw 6

$S^n/\mathbb{Z}/2\mathbb{Z}$

$\parallel$

$\pi_1(\mathbb{RP}^n) \cong \begin{cases} \ldots\ldots & n=1 \\ \mathbb{Z}/2\mathbb{Z} & n>1 \to S^n \text{ simply conn} \end{cases}$

lem) if $\tilde{X} \xrightarrow{p} X$ cov sp. (conn)

$G = \mathrm{Aut}(\tilde{X}/X) \curvearrowright \tilde{X}$ is a covering sp action.

Pf) let $\tilde{x} \in \tilde{X}$ let $x = p(\tilde{x})$

$\exists\, x \in U \subset X$ so U evenly covered.

let $\tilde{U}$ be the sheet containing $\tilde{x}$.

Spse $g \in G$ so $g \cdot \tilde{U} \cap \tilde{U} \neq \emptyset$ wts $g = 1$.

$\implies g \cdot \tilde{y}_1 = \tilde{y}_2$ for some $\tilde{y}_1, \tilde{y}_2$ in $\tilde{U}$.

as $g \in \mathrm{Aut}(\tilde{X}/X) \implies p(\tilde{y}_1) = p(\tilde{y}_2)$ as cov sp auto pres fibres,

$\implies \tilde{y}_1 = \tilde{y}_2$ as $p|_{\tilde{U}} : \tilde{U} \xrightarrow{\cong} U$

$\implies g(\tilde{y}_1) = \tilde{y}_1$

$\implies g = \mathrm{id}_{\tilde{X}}$ by matching pt argument of conn cov sp.

Rmk) $\tilde{X} \longrightarrow \tilde{X}/G$ cov sp.

$\tilde{X}/G \neq X$ in general.

But is true iff $\tilde{X} \to X$ Galois

So far....

Invariants: π_0, π_1

Next: Can define homotopy grp π_n $\forall$ $n \geq 1$.

$$\pi_n(X, x_0) = \mathrm{Hom}_{hTop_*}((S^n, s_0), (X, x_0))$$ (has natural grp struct)

But, really hard to compute.