Thm If $X^{\ni x_0}$ is path conn, locally p.c, semi-locally simply conn
$\Rightarrow \exists$ covering sp $p : \tilde{X} \to X$ w/ $\tilde{X}$ simply conn
$\Rightarrow \pi_0(\tilde{X}), \pi_1(\tilde{X}) = 0$

Recall as a set
$$\hat{\tilde{X}} := \{ [\alpha] \mid \alpha \text{ path in } X \text{ starting at } x_0 \} \xrightarrow{p} X$$
$$[\alpha] \longmapsto \alpha(1)$$

Note p is surj b/c X is path conn.
Now wanna topologize $\tilde{X}$.

$\mathbb{B} := \{$ collection of p.c open $U \subset X$ s.t $\pi_1(U) \xrightarrow{\text{trivial}} \pi_1(X) \}$
$\hookrightarrow$ natural basis of X from hypothesis ? shown below

Note: $U \in \mathbb{B}$ & $V \subseteq U$ path-conn open
$\Rightarrow V \in \mathbb{B}$

So as X loc path conn & semi-locally s.c $\Rightarrow \mathbb{B}$ basis!
((every open in X union of opens in $\mathbb{B}$))

Let $U \in \mathbb{B}$ & α path in X w/ $\alpha(0) = x_0$, $\alpha(1) \in U$
Let $\tilde{X} \supseteq U_{[\alpha]} := \{ [\alpha \cdot \eta] \mid \eta \text{ path in } U \text{ s.t } \alpha(1) = \eta(0) \}$

$p : U_{[\alpha]} \longrightarrow U$ is bijective.
$[\alpha \cdot \eta] \longmapsto [\alpha \cdot \eta](1)$

Surj as U is path connected (we appr η)
Inj say η, η' w/ same endpt
$\Rightarrow [\eta] = [\eta']$ in X b/c $\pi_1(U) \to \pi_1(X)$ triv
$\Updownarrow$
$[\eta \bar{\eta}'] = 1$
$\Rightarrow [\alpha \cdot \eta] = [\alpha \cdot \eta']$

Obs | $\mathcal{U}_{[\alpha]} = \mathcal{U}_{[\beta]}$ if $[\beta] \in \mathcal{U}_{[\alpha]}$

If $[\beta] \in \mathcal{U}_{[\alpha]} \Rightarrow [\beta] = [\alpha \cdot \eta]$ w/ η path in $\mathcal{U}$

$\mathcal{U}_{[\beta]} = \{ [\alpha \cdot \eta \cdot \mu] \mid \mu$ path in $\mathcal{U}$ so $\eta(1) = \mu(0) \} \subseteq \mathcal{U}_{[\alpha]}$

Similarly $[\alpha] = [\alpha \cdot \eta \cdot \bar{\eta}] = [\beta \cdot \bar{\eta}] \in \mathcal{U}_{[\beta]}$ gives $\supseteq$

$\square$

Claim | $\mathcal{U}_{[\alpha]}$'s form basis for top on $\tilde{X}$

Recall | Collection $\mathcal{B}$ of sso Y is a basis for a top on Y if
1) $\forall y \in Y$ $\exists B \in \mathcal{B}$ so $y \in B$
2) $\forall B, C \in \mathcal{B}$ if $B \cap C \ni y \Rightarrow \exists E \in \mathcal{B}$ so $y \in E$, $E \subseteq B \cap C$ $\square$

get a top on Y where $V \subseteq Y$ open $\Leftrightarrow$ V is union of elt in $\mathcal{B}$

Pf | 1) ✓ (find $\mathcal{U}$ containing given end pt)
2) $[\gamma] \in \mathcal{U}_{[\alpha]} \cap V_{[\beta]} \Rightarrow \left.\begin{array}{l} \mathcal{U}_{[\alpha]} = \mathcal{U}_{[\gamma]} \\ V_{[\beta]} = V_{[\gamma]} \end{array}\right\}$ by obs

$\Rightarrow W_{[\gamma]} \subset \mathcal{U}_{[\alpha]} \cap \mathcal{U}_{[\beta]}$ — as $\mathcal{B}$ is basis on X

if choose $W \in \mathcal{B}$ so $\gamma(1) \in W \subseteq U \cap V$

Topologize: $\tilde{X}$ using this basis.

Need to show $p: \tilde{X} \to X$ a covering sp.

1) $p: \mathcal{U}_{[\alpha]} \xrightarrow{\text{bij}} \mathcal{U}$ (as ptd out earlier) is a homeo!

b/c p induces a bij btween $V_{[\beta]} \subset \mathcal{U}_{[\alpha]}$ & $V \subseteq \mathcal{U}$

p in X st at x_0 ending in V — $\mathcal{B}$

$\Rightarrow \boxed{\text{open \& cts}}$

2) $p : \tilde{X} \to X$ is cts. (b/c rest'r to $U_{[\alpha]}$'s are cts)

3) $p : \tilde{X} \to X$ is covering sp:

b/c any $U \in \mathcal{B}$ is evenly covered

if $[\gamma] \in U_{[\alpha]} \cap U_{[\beta]} \Rightarrow U_{[\alpha]} = U_{[\beta]} = U_{[\gamma]}$ by obs

$\therefore$ $p^{-1}(U)$ = disjoint union of various $U_{[\alpha]}$'s

$\therefore$ U is evenly covered!

4) WTS| $\pi_0(\tilde{X}) = 1$ i.e $\tilde{X}$ p.c! let $[c_{x_0}] \in \tilde{X}$ be lift of $x_0 \in X$

let $[\alpha] \in \tilde{X}$ given (path starting at x_0)

let $s \in [0,1]$ let $\alpha_s := \begin{cases} \alpha & \text{on } [0,s] \\ \alpha(s) & \text{on } [s,1] \end{cases}$ path in X

$$\begin{aligned} I &\longrightarrow \tilde{X} \\ s &\longmapsto [\alpha_s] \end{aligned}$$ path from $[c_{x_0}]$ to $[\alpha]$ in $\tilde{X}$

exer check cts!

5) WTS| $\pi_1(\tilde{X}, [c_{x_0}]) = 1$

STS| $p_* \pi_1(\tilde{X}, [c_{x_0}]) = 1 \subset \pi_1(X, x_0)$

earlier inj lemma

let $[\alpha] \in p_* \pi_1(\tilde{X}, [c_{x_0}])$

$\iff$ the lift $\tilde{\alpha}$ of α starting at $[c_{x_0}]$ is a loop.

But, $\tilde{\alpha} : I \to \tilde{X}$ is this lift
$s \mapsto [\alpha_s]$

$\Updownarrow$

$\tilde{\alpha}(1)$

$[\alpha] := [\alpha_1]$

$= [c_{x_0}]$

$\iff [\alpha] = 1 \in \pi_1(X, x_0)$

■