Goal: Recipe for π_1 of CW complex!

Def if X is a CW-cmpx then **dim X** is the **maximal dimension** of a cell in the CW-decomp of X.

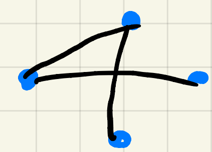
Today $\dim X = 1$

Thm, $$X = \frac{X^0 \sqcup D^1_\alpha}{x \sim \varphi_\alpha(x),\ x \in \partial D^1_\alpha} \qquad \varphi_\alpha : \partial D^1_\alpha \to X^0, \quad \Phi_\alpha : D^1_\alpha \to X^0$$

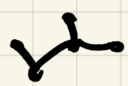
(X^0: disc set of pts)

$\underline{1 \text{ cell}} \to e_\alpha =$ (img of) interior of D^1_α under Φ_α

$\overline{e_\alpha} = \Phi_\alpha(D^1_\alpha)$



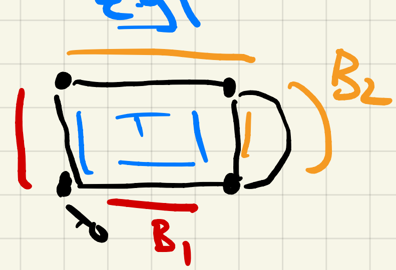
Terminology

Topology	=	Graph Thy
(conn) 1 dim'l CW complex		(conn) Graph
0-cell		vertex
1-cell		(open) edge
subcomplex		Subgraph
Contractible Subcmpx		tree 
Contractible Subcmplx containing all 0-cells		Maximal tree

Prop Let X be a connected 1 dim CW complex
$\Rightarrow \exists$ contractible subcomplex $C \subseteq X$ containing all the zero cells.

i.e. conn graphs have max'l trees

e.g.



Pf Let $x_0 \in X^0$

Step 1: Description of X as a union of subgps

Define $B_0 \subset B_1 \subset \cdots \subset X$ inductively

$B_0 = \{x_0\}$ $\qquad B_{i+1} = B_i \cup$ all $\overline{e_\alpha}$ with at least 1 endpt in B_i

Claim: $X = \bigcup_{i \geq 0} B_i$ → will show clopen & since X conn done

- RHS open in X, since if $p \in B_i \Rightarrow \exists$ open $U \subseteq X$ s.t. $p \in U \subset B_{i+1}$

  [use that B_{i+1} contains all edges w/ an endpt on p
  $U \subseteq X$ open $\iff$ preimage in each D_α^i is] → Quotient top

- RHS closed in X since its preim in $X \sqcup_\alpha D_\alpha^i$ consists of pts in X^0 and various D_α^i's

$\therefore X = \text{RHS}$. ▷

Step 2: Construction of subgraph in X containing all vertices of X & deformation retr to point x_0!

Define $T_0 \subset T_1 \subset \cdots \subset X$ inductively
$\cap \quad \cap \quad \cup$
$B_0 \subset B_1 \subset \cdots$

1) T_i subcmplx w/ all vertices of T_i ← can always find such an edge!
2) T_i defo retr to T_{i-1}

- $T_0 = \{x_0 = v_0\}$
- $T_{i+1} = T_i \cup e_{\alpha_v}$, vert $v \in B_{i+1}/B_i$ → chosen so it has v as one vertex & other one in T_i

$T = \bigcup_{i \geq 0} T_i$ → condition check

1) okay for T_{i+1} by constr
2) T_{i+1} def retr on T_i (defo each $\overline{e}_{\alpha_v}$ to vertex in T_i)

Ex. T defo retr to $\{x_0 = t_0\} = T_0$ → Then done as B has all vertices

idea: easier if V was only fin many edges!

⇒ $T = T_i$ for i large. ↳ in general using T_{i+1} to T_i on $[1/2^{i+1}, 1/2^i]$

Rmk) X is a CW cmplx (e.g. a graph)
X is conn $\Leftrightarrow$ it is path connected

Pf) Hatcher appendix A.

Cor) if X is a connected graph & $T \subseteq X$ max'll tree
$\pi_1(X)$ is a free group on the set of edges in $X \setminus T$

Pf) $X \sim X/T$ b/c (X,T) subcomplex & T contractible
$\sim \bigvee_{\text{edges in } X \setminus T} S^1$ $\rightarrow$ as T has all vertex
$\Rightarrow X/T$ only 1 vertex
$\sim \bigvee_{\substack{1\text{ cells in } X \\ -\, 0\text{ cells in } X \\ -1}} S^1$ $\rightarrow$ the tree has $n-1$ vertices!

Cor) if $T \subset X$ max'll tree & $T' \subset X$ tree st $T \subset T'$
$\Rightarrow T = T'$

Pf) if $T \subsetneq T'$, then $\pi_1(T')$ nontrivial by prev corollary

$\square$