Properties of products of paths

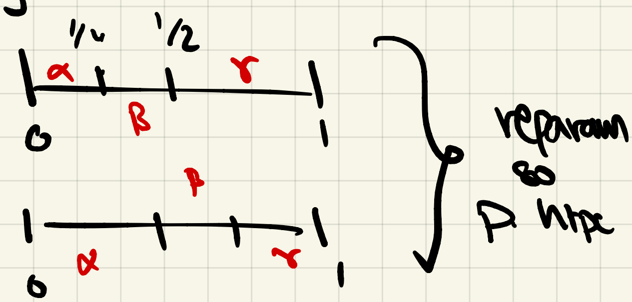
① Well def / path htpy.

$[\alpha]\cdot[\beta] = [\alpha'\cdot\beta']$ if $\alpha \underset{p}{\sim} \alpha'$, $\beta \sim \beta'$

② Assoc: if α, β, γ paths in X st, $\alpha(1) = \beta(0)$, $\beta(1) = \gamma(1)$

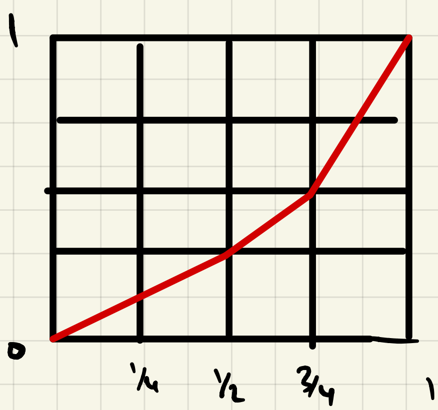
$[(\alpha\cdot\beta)\cdot\gamma] = [\alpha\cdot(\beta\cdot\gamma)]$

Pf) $(\alpha\cdot\beta)\cdot\gamma$ schematically is



Similarly $\alpha\cdot(\beta\cdot\gamma) \longrightarrow$

consider $\phi: I \longrightarrow I$ with graph



$\Rightarrow \alpha\cdot(\beta\cdot\gamma) = ((\alpha\cdot\beta)\cdot\gamma)\circ\phi$

is a reparam of $(\alpha\cdot\beta)\cdot\gamma$

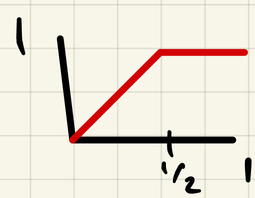
$\Rightarrow \alpha\cdot(\beta\cdot\gamma) \underset{p}{\sim} (\alpha\cdot\beta)\cdot\gamma$

③ (identity elt): For $x_0 \in X$ & $c_{x_0}: I \to X$ be const @ x_0

$[\alpha\cdot c_{x_1}] = [\alpha]$ if $x_1 = \alpha(1)$ so prod makes sense

$[c_{x_0}\cdot\alpha] = [\alpha]$ if $x_0 = \alpha(0)$

Pf) $\alpha\cdot c_{x_1} = \alpha\circ\phi$ for $\phi \longrightarrow$



$\Rightarrow \alpha\cdot c_{x_1}$ reparam of $\alpha \Rightarrow \alpha\cdot c_{x_1} \underset{p}{\sim} \alpha$

other is similar.

④ (Inverse): let $\alpha: I \to X$ path from x_0 to x_1

let $\bar{\alpha}: I \to X$ is the path x_1 to x_0

$s \mapsto \alpha(1-s)$

Claim: $[\alpha\cdot\bar{\alpha}] = [c_{x_0}]$ $[\bar{\alpha}\cdot\alpha] = [c_{x_1}]$

Pf| Step1 if $X = I$, $\alpha = \mathrm{Id}_X$

In fact, $f, g : I \to I$ paths from $x_0 \longrightarrow x_1$

$\Rightarrow f \underset{p}{\sim} g$ via homotopy (straight line)

$H : I \times I \longrightarrow I$

$(s,t) \longmapsto (1-t) f(s) + t g(s)$

$\therefore \ \mathrm{id}_I \cdot \overline{\mathrm{id}}_I \underset{p}{\sim} c_0^{=0}$ & $\overline{\mathrm{id}} \cdot \mathrm{id}_I \underset{p}{\sim} c_1^{=1}$

Step2| (General cases)

Say H is a path htpy from $\mathrm{id}_I \cdot \overline{\mathrm{id}}_I$ to c_0

$\Rightarrow \alpha \circ H$ is a path htpy from

$$\begin{array}{ccc} \alpha \circ (\mathrm{Id}_I \cdot \overline{\mathrm{Id}_I}) & \Longrightarrow & \alpha \circ c_0 \\ \| & & \| \\ \alpha \cdot \overline{\alpha} & & c_{x_0} \end{array}$$

$\Rightarrow [\alpha \cdot \overline{\alpha}] = [c_{x_0}]$ other dir: replace α w/ $\overline{\alpha}$

Def| Let X a space & $x_0 \in X$ "basept"

$\Rightarrow$ Fundamental grp of X based at x_0

$\pi_1(X, x_0) := \left\{ [\alpha] \;\middle|\; \begin{array}{l} \alpha \text{ loop at } x_0 \\ \text{Co path htpy cl} \end{array} \right\}$

Equip $\pi_1(X, x_0)$ with product op $[\alpha] \cdot [\beta] := [\alpha \cdot \beta]$

Prop 1 $\Rightarrow$ well def

Prop (2-4) $\Rightarrow$ this is a grp!

Def| Top_* category of pointed spaces

Objects (X, x_0) X top sp & x_0 base pt

Mor $f : (X, x_0) \longrightarrow (Y, y_0)$

cts so $f(x_0) = y_0$

Exer) Have a functor (HW2)

$$\pi_1 : Top_* \longrightarrow Grp$$
$$(X, x_0) \longmapsto \pi_1(X, x_0)$$
$$(f:(X,x_0)\to(Y,y_0)) \longmapsto f_* : \pi_1(X,x_0) \longrightarrow \pi_1(Y,y_0)$$
$$[\alpha] \longmapsto [f\circ\alpha]$$

In fact π_1 factors via

$Top_* \longrightarrow hTop_* \hookrightarrow$ Htpy cat of ptd top sp

Change of base pt

let $\gamma : I \to X$ is a path from x_0 to x_1

$\Rightarrow \exists\ \hat{\gamma} : \pi_1(X,x_0) \longrightarrow \pi_1(X,x_1)$

$$[\alpha] \longmapsto [\bar{\gamma}]\cdot[\alpha]\cdot[\gamma]$$

(go to x_0 do α go back)

Lemma) $\hat{\gamma}$ is a grp iso!

Pf) 1) homomorphism

$$\begin{aligned}\hat{\gamma}([\alpha]\cdot[\beta]) &= [\bar{\gamma}]\cdot[\alpha]\cdot[\beta]\cdot[\gamma]\\ &= [\bar{\gamma}]\cdot[\alpha]\cdot[c_{x_0}]\cdot[\beta]\cdot[\gamma]\\ &= [\bar{\gamma}]\cdot[\alpha]\cdot[\gamma]\cdot[\bar{\gamma}]\cdot[\beta]\cdot[\gamma]\\ &= \hat{\gamma}([\alpha])\cdot\hat{\gamma}([\beta])\end{aligned}$$

2) Isomorphism : $\hat{\bar{\gamma}} : \pi_1(X,x_1) \longrightarrow \pi_1(X,x_0)$ is inverse

$$\hat{\bar{\gamma}}\circ\hat{\gamma}([\alpha]) = [\bar{\bar{\gamma}}]([\bar{\gamma}][\alpha]\cdot[\gamma])\cdot[\bar{\gamma}]$$
$$= [\alpha]$$

$$\Rightarrow \pi_1(X,x_0) \cong \pi_1(X,x_1)$$

Cor) if X path conn $\Rightarrow$ $\pi_1(X, x_0) \cong \pi_1(X, x_1)$
$\forall\, x_0, x_1 \in X$

$\Rightarrow$ unambiguous to say $\underline{\pi_1(X)}$.

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let $f, g : \overset{x_0}{X} \longrightarrow Y$

$\Rightarrow$

$$\begin{tikzcd}
 & \pi_1(Y, f(x_0)) \arrow[dd, "\hat{\gamma}"] \\
\pi_1(X, x_0) \arrow[ur, "f_*"] \arrow[dr, "g_*"'] & \\
 & \pi_1(Y, g(x_0))
\end{tikzcd}$$

if $f_* \overset{H}{\sim} g_*$ for $H : X \times I \longrightarrow Y$

get a path $\gamma : I \longrightarrow Y$ from x_0 to $f(x_0)$

$t \longmapsto H(x_0, t)$

lemma) Diagram above commutes

i.e) $\hat{\gamma} \circ f_* = g_*$

Pf) WTS $[\alpha] \in \pi_1(X, x_0)$

LHS $\hat{\gamma} \circ f_*([\alpha]) = [\bar{\gamma}] \cdot [f \circ \alpha] \cdot [\gamma]$

RHS $= [g \circ \alpha]$

LHS = RHS $\Longleftrightarrow$ $[f \circ \alpha] = [\gamma] \cdot [g \circ \alpha] \cdot [\bar{\gamma}]$