Recall: Homotopy Lifting Thm

e.g. 1 **Cor 1** Consider

$$\begin{tikzcd} & & \mathbb{R} \arrow[d] & x \arrow[d, mapsto] & \tilde{x}_0 = 0 \arrow[d, mapsto] \\ s \arrow[urr, mapsto, "ns"] & I \arrow[r, "w_n"] \arrow[ur, "\tilde{w}_n"] & S^1 & e^{2\pi i x} & x_0 = (1,0) \\ & s \arrow[r, mapsto] & e^{2\pi i n s} & & \end{tikzcd}$$

p lift $\tilde{\omega}_n$ of ω_n

Thm $\pi_1(S^1, x_0) \cong \mathbb{Z}$

wts $\mathbb{Z} \longrightarrow \pi_1(S^1, x_0)$

$n \longmapsto [\omega_n]$

wts isom!

Surj: let $\alpha : I \to S^1$ loop at x_0

Cor 1 $\overset{\text{lift}}{\Longrightarrow}$

$$\begin{tikzcd} & \mathbb{R} \arrow[d, "p"] & \tilde{x}_0 = 0 \\ I \arrow[ur, "\tilde{\alpha}"] \arrow[r, "\alpha"'] & S^1 & x_0 = (1,0) \end{tikzcd}$$

$\tilde{\alpha}(0) = 0$, $\tilde{\alpha}(1) \in p^{-1}(x_0) = \mathbb{Z} \subseteq \mathbb{R}$, $\tilde{\alpha}(1) = n$

$\tilde{\omega}_n$ also a path in $\mathbb{R}$ from 0 to n

$\tilde{\alpha} \simeq_p \tilde{\omega}_n$ (via linear htpy $(1-t)\tilde{\alpha} + t\tilde{\omega}_n$)

$\Rightarrow \alpha = p \circ \alpha \simeq_p p \circ \tilde{\omega}_n = \omega_n$

Injective: Suppose $\omega_n \simeq_p \tilde{\omega}_0$

Cor 2 $\Rightarrow \tilde{\omega}_n \simeq_p \tilde{\omega}_0$

$n = \tilde{\omega}_n(1) = \tilde{\omega}_0(1) = 0$ trivial kernel.

Pf of unique htpy lifting

$$\begin{tikzcd}
y \arrow[d, mapsto] & Y \arrow[r, "\tilde F_0"] \arrow[d, hook] & \tilde X \arrow[d, "p \text{ covering sp}"] \\
(y,0) & Y\times I \arrow[r, "F"'] \arrow[ur, dashed, "\exists!\,\tilde F"] & X
\end{tikzcd}$$

local existence

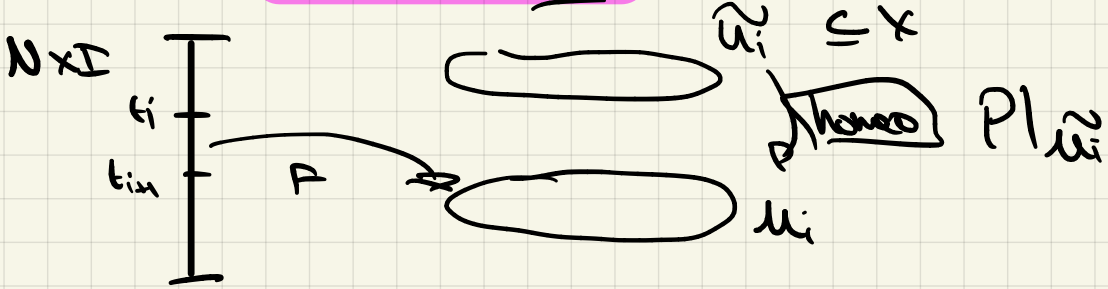
Step 1: Fix $y_0 \in Y$ $\exists$ N nbhd of $y_0 \in N \subseteq Y$ and $\tilde F_{N\times I} : N\times I \to \tilde X$ lifts $F|_{N\times I}$

$\forall\, t \in I$, as F is cts, $\exists\, y_0 \in N_t^{\text{nbhd}}$ $t \in (a_t, b_t)$ s.t. $F(N_t \times (a_t, b_t))$ is contained in evenly covered nbhd of $F(y_0, t)$

- I cpt $\implies$ can choose nbhd $y_0 \in N \subseteq Y$ & $0 = t_0 < t_1 < \dots < t_m = 1$ s.t.

$\forall i$ $F(N\times[t_i, t_{i+1}]) \subseteq U_i \to$ evenly covered open in X

↳ intersect all N_t's



- Assume inductively that we have consts $\tilde F_i = \tilde F_{N\times I}|_{N\times[0,t_i]}$ starting w/ $\tilde F_0$ at $i=0$.
- $\exists\, \tilde U_i \underset{\text{open}}{\subset} \tilde X$ so $p|_{\tilde U_i} : \tilde U_i \to U_i$ homeo

& by comm, $\tilde F(y_0, t_i) \in \tilde U_i$ — continuity of $\tilde F_i$

So, up to shrinking N, can assume $\tilde F(N\times\{t_i\}) \subseteq \tilde U_i$

$$\tilde F_{i+1}(y,t) := \begin{cases} \tilde F_i(y,t), & (y,t)\in N\times[0,t_i] \\ (p|_{\tilde U_i})^{-1}\circ F(y,t) & (y,t)\in N\times[t_i, t_{i+1}] \end{cases}$$

well def + cts by pasting!

Step 2 | if $Y = \{*\}$, $\tilde F$ is unique if it exists!

Pf)

$$\begin{tikzcd} * \arrow[r, mapsto, "\{x\}"] \arrow[d] & \tilde X \arrow[d, "p"] \\ 0 \quad I \arrow[r, "F"'] \arrow[ur, "\tilde F_1, \tilde F_2"] & X \end{tikzcd}$$

by comm $\tilde F_1(0) = \tilde F_2(0)$

wts $\tilde F_1 = \tilde F_2$

by earlier arg choose $0 = t_0 < t_1 < \dots < t_n = 1$

s.t. $F([t_i, t_{i+1}]) \subseteq U_i$ evenly covered open in X

Assume we've shown $\tilde F_1|_{[0,t_i]} = \tilde F_2|_{[0,t_i]}$

($t = 0$ by assump)

$\tilde F_1|_{[t_i,t_{i+1}]} \subseteq \tilde U_i$ → b/c $[t_i, t_{i+1}]$ is conn and open in $p^{-1}(U_i)$ s.t. $p|_{\tilde U_i} : \tilde U_i \xrightarrow[\text{ome}]{} U_i$

$\tilde F_2|_{[t_i,t_{i+1}]} \subset$ same $\tilde U_i$

↳ b/c $\tilde F_1(t_i) = \tilde F_2(t_i)$

$p \circ \tilde F_1 = p \circ \tilde F_2$ (both lift)

$\therefore$ $\tilde F_1 = \tilde F_2$ on $[t_i, t_{i+1}]$ ($= (p|_{\tilde U_i})^{-1} \circ F$)

Step 3 | $\tilde F_{N \times I}$'s from step 1 must agree on overlaps

See this by restricting to slices $\{y\} \times I \subseteq Y \times I$ & using step 2

$\therefore$ they glue to $\tilde F : Y \times I \to X$

↳ lift of F

↳ see $\tilde F$ is the unique lift by restr to slices $\{y\} \times I$

Lemma 1 $i: S^1 \hookrightarrow D^2$ does not admit a retraction.

i.e. $r: D^2 \to S^1$ s.t. $r \circ i = id_{S^1}$

Pf if we had r

$$\Rightarrow \underset{id_*}{\underset{\parallel}{(r\circ i)_*}} : \underset{\mathbb{Z}}{\underset{\parallel}{\pi_1(S^1, x_0)}} \xrightarrow{i_*} \underset{0}{\underset{\parallel}{\pi_1(D^2, x_0)}} \xrightarrow{r_*} \pi_1(S^1, x_0)$$

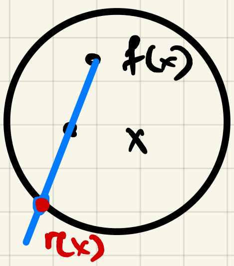
Oops but comp can't be injective! so not id_*

Thm 1 (Brouwer Fixed pt)

Any cts $f: D^2 \to D^2$ has a fixed pt.

Pf Spse no fixed pt $\forall x \in D^2$

Consider the following map $r: D^2 \to S^1$

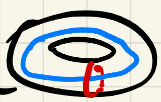


$x \longmapsto$ ray starting at $f(x)$ passing through x

Can see r cts!

But! r is a retraction Oops!

Hw 2 $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$

E.g. $\pi_1($  $, *) = \pi_1(S^1 \times S^1, *) = \mathbb{Z} \times \mathbb{Z}$

Thm $\pi_1(S^n, x_0) = 0 \quad \forall n \geq 2$