Homology Groups $H_n(X)$, $n \geq 0$

will be a functor $H_n : hTop \longrightarrow Ab$.

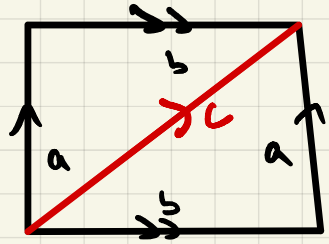
Very Comparable; but harder to define than π_n.

Plan 1) Δ-complexes.

2) Define the simplicial homology of a Δ-complex.

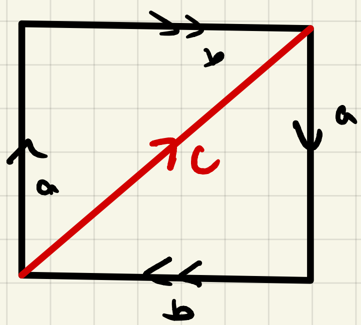
3) Singular homology of any sp.

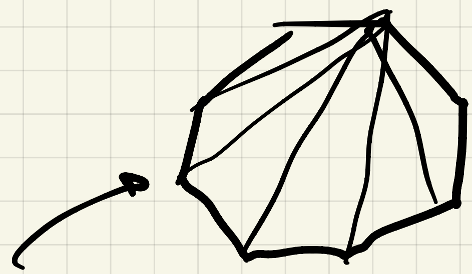
eg. 1) Torus.



built from triangles by cutting along & identifying c..

2) $\mathbb{RP}^2$





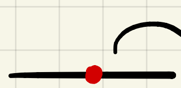
3) Can do for any genus g surface by trianglig an n-gon

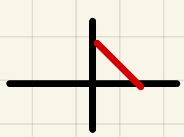
Rmk Abstracting the above $\rightarrow$ Δ complex built in a similar fashion.

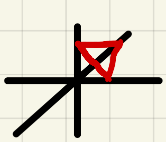
Standard N simplex

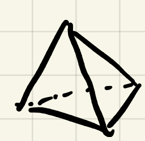
$$\Delta^n = \{(t_0, t_1, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^{n} t_i, \ t_i \geq 0\}$$

$$= \text{convex hull of } (e_0, \dots, e_n) \subseteq \mathbb{R}^{n+1}$$

$\Delta^0 =$  its a point

$\Delta^1 =$  $\cong [0,1]$

$\Delta^2 =$  $\cong$ 

$\Delta^3 =$  Solid tetrahedron.

Defn | Given $v_0, v_1, \dots, v_n \in \mathbb{R}^{n+1}$ which are affine lin ind.

i.e $v_1 - v_0, v_2 - v_0, \dots, v_n - v_0$ to be lin ind.

the n simplex on these vectors

$[v_0, \dots, v_n]$ = convex hull of $v_0, \dots, v_n$

e.g | Always homeo from $\Delta^n \xrightarrow{\cong} [v_0, \dots, v_n]$

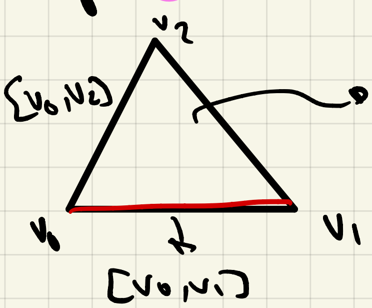
$$(t_0, \dots, t_n) \longmapsto \sum t_i v_i$$

Def | A face of $[v_0, \dots, v_n]$ is an $(n-1)$-simplex det by removing one of the v_i

i.e $[v_0, v_1, \dots, \hat{v_i}, \dots v_n]$

Rmk | Always use same ordering on vertices of a face.

e.g |



Def The boundary of a n simplex

$(\partial[v_0, \dots v_n])$ = union of all faces

$(\partial \Delta^n)$

Open simplex $[v_0, \dots v_n]^\circ = [v_0, \dots, v_n] \setminus (\partial[v_0, \dots, v_n])$

$(\mathring{\Delta}^n)$

Def | If X is a space then a Δ complex struct of X is a collection of maps $\sigma_\alpha : \Delta^n \to X$ (n dep on α)

s.t

1) $\sigma_\alpha|_{\mathring{\Delta}^n}$ injective and X is the dis union of img of $\sigma_\alpha|_{\mathring{\Delta}^n}$

$\hookrightarrow$ img of interior can't overlap

2) Each restriction of σ_α to a face of Δ^n coincides with a map $\sigma_\beta : \Delta^{n-1} \to X$ (where we identify the face w/ Δ^{n-1} via canonical iso)

given $\sigma_\alpha / \mathring{\Delta}^n$ homeo

3) $A \subset X$ is open iff $\sigma_\alpha^{-1}(A) \subseteq \Delta^n$ open $\forall \alpha$.

Rmk 1) $\displaystyle \bigsqcup_\alpha \Delta^n_\alpha \Big/ \left(\begin{array}{l}\text{face } \Delta^n_\alpha \text{ with} \\ \Delta^{n-1}_\beta \text{ via iso above}\end{array}\right) \overset{(\text{homeo})}{\cong} X \rightsquigarrow$ only glue along faces

2) Can also build X inductively like CW. cmplx

$$X^0 = \bigsqcup_{\substack{\alpha \text{ corr to} \\ 0\text{-simp}}} \Delta^0_\alpha$$

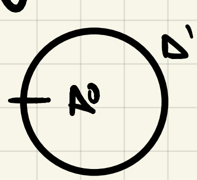
$$X^1 = X^0 \bigsqcup_{\substack{\alpha \text{ corr} \\ 1\text{-simpl}}} \Delta^1_\alpha \Big/ \begin{array}{l}\text{face } \Delta^n_\alpha \text{ with} \\ \text{corr vertices in } X^0.\end{array}$$

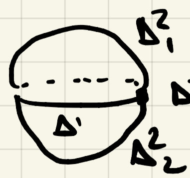
$\vdots$

3) A Δ-cmplx struct gives CW cmplx struct $\rightsquigarrow e^n_\alpha = \text{im } \sigma_\alpha|_{\mathring{\Delta}^n}$ + charmaps $\sigma_\alpha : \Delta^n \to X$, $\Delta^n \cong D^n$

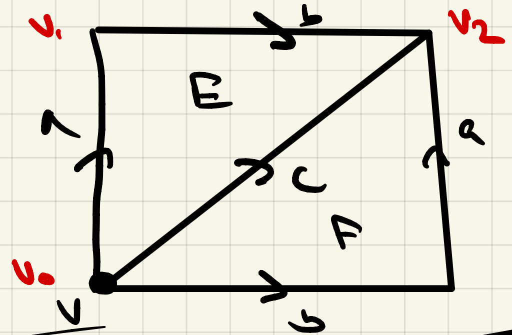
(see Hatcher Prop A.2 check same top)

4) Every closed (cmpct wOB) admits a Δ-cmplx struct.

eg 1) $S^1 =$  Δ-cmplx with 1 zero simplex & 1 1-simp, $\Delta^1 \to S^1 = \Delta^1/_{0\sim 1}$

$S^2 =$  Δ-cmplx w 1 0-simlx, 1 1-simpl, 2 2-simpl

2) Torus



good since all corners ident

Δ-cmplx struct

0 - simpl $\to$ v

1 - simpl $\to$ a, b, c

(map $\Delta^1 \to X$ is specified by line)

2 - simpl $\to$ E, F

$\Delta^2 \to X$ specified by picture.

$= [v_0, v_1, v_2]$