Cor) X path conn CW complex $X^2 \hookrightarrow X$ induces iso on fundamental grps $\longrightarrow$ $\pi_1(X^2) \xrightarrow{\sim} \pi_1(X)$

Pf) Okay if dim X is <u>finite</u> ✓

Lemma) (Hatcher A.1) A compact subsp of a CW complex is contained in finite dim subcomplex

Surj) $\alpha : I \longrightarrow X$ loop at x_0

$\alpha(I) \subseteq X$ cpct

$\Rightarrow \alpha(I) \subseteq Y \subseteq X$, $Y = X^n$ $\longrightarrow$ finite dim CW complex

$$\therefore [\alpha] \in \mathrm{Im}\left(
\begin{tikzcd}
\pi_1(X^2,x_0) \arrow[rr] \arrow[dr] & & \pi_1(X,x_0) \\
& \pi_1(Y,x_0) \arrow[ur] &
\end{tikzcd}
\right)$$

by surj in fin dim arg

Inj) α loop in x_0 in X

if $\alpha_p \sim c_{x_0}$ in X via $H : I \times I \longrightarrow X$

then as above H factors thru $Y \subseteq X$ fin dim'l CW complex

$\therefore [\alpha] = 0 \in \pi_1(X^2, x_0)$ by finite dim'l case $\square$

Cor) G grp $\Rightarrow$ $\exists$ CW complex (dim 2) X p.c s.t. $\pi_1(X) \cong G$

Pf) Choose a pres for $G = \langle S \mid R \rangle = \dfrac{\ast_{s\in S} \mathbb{Z}}{\langle R \rangle}$

$= \dfrac{\pi_1(\bigvee_{s\in S} S^1, *)}{\langle R \rangle}$ ($*$: natural base pt)

$r \in R \longleftrightarrow [\alpha_r] \in \pi_1(\bigvee_{s\in S} S^1)$

$\alpha_r : I \longrightarrow \bigvee_{s\in S} S^1$

I effectively S^1

$\varphi_r : S^1 \longrightarrow \bigvee_{s\in S} S^1$ $\longrightarrow$

$s_0 = (1,0) \longleftrightarrow \varphi_r(s_0) = *$

let X be the sp by obtaining D^2's to $\vee S^1$ along φ_r

$\Rightarrow \pi_1(X) \cong G$ $\square$.

Goal| To classify covering spaces via π_1

Lemma| $p: (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$ covering sp (takes $\tilde{x}_0 \mapsto x_0$)

$p_*: \pi_1(\tilde{X}, \tilde{x}_0) \longrightarrow \pi_1(X, x_0)$ injective

w/ $\operatorname{Im} p_* = \{[\alpha] \in \pi_1(X, x_0) \mid$ lift $\tilde{\alpha}$ of α starting at $\tilde{x}_0$ is a loop$\}$

doesn't dep on rep

Recall| • if α a path starting at x_0

$\Rightarrow \exists!$ lift $\tilde{\alpha}$ starting at $\tilde{x}_0$ & comm

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ I \arrow[ur, "\tilde{\alpha}"] \arrow[r, "\alpha"'] & X \end{tikzcd}$$

• α, β paths in X from x_0 to x

then $\alpha \sim_p \beta \Longrightarrow \tilde{\alpha} \sim_p \tilde{\beta}$ ($\Rightarrow \tilde{\alpha}, \tilde{\beta}$ have same endpt)

Pf| let $\gamma: I \rightarrow \tilde{X}$ loop at $\tilde{x}_0$

so $p_*([\gamma]) = 1 \in \pi_1(X, x_0)$

$\Rightarrow [\alpha] := [p \circ \gamma] \sim_p c_{x_0} \rightarrow$ const path

$\Rightarrow \tilde{\alpha} = \gamma \sim_p \tilde{c}_{x_0} = c_{\tilde{x}_0}$ ← unique lift

$\Rightarrow [\gamma] = 1 \in \pi_1(\tilde{X}, \tilde{x}_0) \Rightarrow$ triv ker

Img| $\operatorname{Im}(p_*) \subseteq A$ certainly

conversely if α has a lift loop $\tilde{\alpha}$ at $\tilde{x}_0$

$[\alpha] = p_*[\tilde{\alpha}]$ $\square$.

So $\pi_1(\tilde{X}, \tilde{x}_0)$ subgrp of $\pi_1(X, x_0)$ by embedding abv

Def | X is locally path conn if $\exists$ basis of open sets which are p/c.

Rmk | 1) If abv conn comp = p/c comp of X

2) If $\tilde{X} \to X$ covering sp

$\Rightarrow$ X locally pc $\Rightarrow$ $\tilde{X}$ is too!

3) X is locally path conn if X is mfd, X is CW compx

$\hookrightarrow$ $\boxed{\text{Hatcher}}$

Goal | For such an X constr, $\tilde{X} \xrightarrow{p} X$ w/ $\tilde{X}$ simply conn $\hookleftarrow$ $\pi_0(\tilde{X}), \pi_1(\tilde{X}) = 0$

$\hookrightarrow$ so get all subgrps

If $\tilde{X}$ exists $\Rightarrow$ $\forall x \in X$ $\exists$ $x \in \mathcal{U}^{\text{nbhd}}$

so $p^{-1}(\mathcal{U}) = \bigsqcup_\alpha \tilde{\mathcal{U}}_\alpha$

So choose one $\tilde{\mathcal{U}}_\alpha \xrightarrow[p]{\sim} \mathcal{U}$

$$\begin{tikzcd} \tilde{\mathcal{U}} \arrow[r, hook] \arrow[d, "p|_{\tilde{\mathcal{U}}}"', "\cong"] & \tilde{X} \arrow[d] \\ x \in \mathcal{U} \arrow[r, hook, "i"] & X \end{tikzcd} \xrightarrow{\text{functoriality}} \begin{tikzcd} \pi_1(\tilde{\mathcal{U}}) \arrow[r] \arrow[d, "\cong"] & \pi_1(\tilde{X}) = \{1\} \arrow[d, "p_*"] \\ \pi_1(\mathcal{U}) \arrow[r, "i_*"] & \pi_1(X) \end{tikzcd}$$

by comm $\Rightarrow$ $\pi_1(\mathcal{U}) \to \pi_1(X)$ is trivial

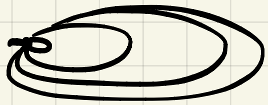
Def | X is semilocally simply conn if $\forall x \in X$ $\exists$ nbhd $x \in \mathcal{U} \subseteq X$ so $\underbrace{\pi_1(\mathcal{U}, x_0) \xrightarrow{i_*} \pi_1(X, x_0)}_{\text{trivial}}$

necessary for $\tilde{X}$ as abv to exist e.g | 1) Stronger: If X locally contractible (eg mfd or CW) then X is semi-locally simp conn)

2) $\exists$ non-semiloc simp. conn. sp

e.g "Shrinking wedge of circles" hatcher

$X = \left(\bigcup_{n \geq 1} \text{circle of rad } 1/n \text{ centered at } (1/n, 0) \right) \subseteq \mathbb{R}^2$

$X =$  any nbhd of base pt has a circle

↳ so non-trivial fundamental

↳ the inclusion nontrivial

Thm | If X is a locally path connected semi-locally simply connected sp

$\Rightarrow \exists\, p : \tilde{X} \to X$ covering sp w $\tilde{X}$ simply conn!

idea | if $\tilde{X}$ exists, let $\tilde{x}_0 \in p^{-1}(x_0)$

$\forall\, p, q \in \tilde{X}$ $\exists$ unique htpy class of paths from $p \to q$

↳ compose forward & backward

↳ must be triv $\tilde{X}$ simply

$\therefore\ \tilde{X} \xleftrightarrow{\text{bij}} \{ [\tilde{\alpha}] \mid \tilde{\alpha} \text{ is path in } \tilde{X} \text{ starting at } \tilde{x}_0 \}$

unique lift $\xleftrightarrow{\text{bij}} \{ [\alpha] \mid \alpha \text{ path in } X \text{ st at } x_0 \}$

↳ going to use this as def of $\tilde{X}$

↳ then topologize!