Thm | $\pi_1(S^n, x_0) = 0$ if $n \geq 2$

Pf) let $\alpha : I \to S^n$ loop at x_0

Choose $y_0 \neq x_0 \in S^n$

if $y_0 \notin \alpha(I)$, $\alpha : I \to S^n \setminus \{y_0\} \hookrightarrow S^n$

$$S^n \setminus \{y_0\} \cong \mathbb{R}^n$$

$\mathbb{R}^n$ contractible $\Rightarrow \pi_1(\mathbb{R}^n, x_0) = 0$

$\Rightarrow [\alpha] = 0 \in \pi_1(S^n, x_0)$

$\therefore$ STS $\alpha \underset{p}{\simeq} \alpha'$ where $y_0 \notin \alpha'(I)$

Choose $y_0 \in D \subseteq S^n$ is an open n disc containing y_0 not x_0.

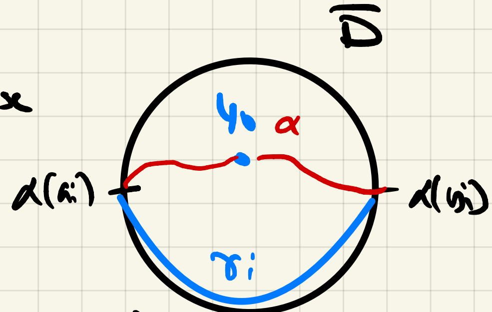
$\alpha^{-1}(D) \subset I$ union of open disjoing open int (cts)

$\alpha^{-1}(y_0) \subset I$ cpt subset $\Rightarrow$ contained in finitely many of such int

say, $(a_1, b_1), \dots, (a_n, b_n)$

$\alpha([a_i, b_i]) \underset{\text{cts}}{\subseteq} \overline{\alpha((a_i, b_i))} \subset \overline{D}$ $\hookrightarrow$ closed n disc

$\Rightarrow \alpha(a_i), \alpha(b_i) \in \partial \overline{D} = \partial D$



Ex | $\exists\, \gamma_i : [a_i, b_i] \to \overline{D}$ s.t.

$\alpha|_{[a_i, b_i]} \sim \gamma_i$ relative $\{a_i, b_i\}$ (path homo)

$y_0 \in \gamma_i([a_i, b_i])$ (uses $n \geq 2$)

Rmk: in general Y sp simply connected

$\Rightarrow \exists!$ path between any 2 points up to path homo!

$$\alpha' := \begin{cases} \alpha & \text{on } I \setminus \bigcup (a_i, b_i) \\ \gamma_i & \text{on } [a_i, b_i] \quad i = 1, \dots, n \end{cases}$$

cts well def by pasting & $\alpha \underset{p}{\simeq} \alpha'$ on each piecewise comp!

$\square$.

Def G_i grps indexed by some $i \in I$

Free produte $\underset{i \in I}{\ast} G_i$

(elts : reduced words in G_i $\Rightarrow$ sequence $g_1 g_2 \cdots g_m$, $m \geq 0$

where $1 \neq g_k \in G_{i_k}$ for some $i_k \in I$

where $i_k \neq i_{k+1}$ $\forall k$!

empty word, $m=0$, is 1

product is concat reduced word & then concat!

$(g_1 \cdots g_m)(h_1 \cdots h_n) = g_1 \cdots g_m h_1 \cdots h_n$ + reduce!

→ multiply adjacent elts of same grp

→ + cancelling identity!

e.g. $\mathbb{Z} * \mathbb{Z} \Rightarrow (aba^3b^2)(b^{-2}a^{-1}b) = aba^2b$

$\mathbb{Z} = \langle a \rangle$, $\mathbb{Z} = \langle b \rangle$

Exer 1) this defines a group!

2) this free product is the colimit (coproduct) of $G_i \in \mathrm{Grp}$!

$\Rightarrow$ abides universal prop $\Rightarrow$ to give homo out of $\ast G_i$ same as giving out of each G_i!

3) Colimits exist in grp.

Ex

$$\begin{tikzcd} H \arrow[r, "\varphi_2"] \arrow[d, "\varphi_1"'] & G_2 \arrow[d] \\ G_1 \arrow[r] & \, \end{tikzcd} \in \mathrm{Grp}$$

Colim is $G_1 *_H G_2 = \dfrac{G_1 * G_2}{\langle \varphi_1(h)\varphi_2(h)^{-1}, h \in H \rangle}$ → smallest normal subgrp

Let $x_0 \in X = \bigcup U_i$ open cov st $x_0 \in U_i \ \forall i \in I$

$$\begin{tikzcd} & U_i \arrow[dr, "f_i"] & \\ U_i \cap U_j \arrow[ur, "g_{ij}"] \arrow[dr, "g_{ji}"'] & & X \\ & U_j \arrow[ur, "f_j"'] & \end{tikzcd}$$

hit w/ π_1 functor

$$\begin{tikzcd} & \pi_1(U_i, x_0) \arrow[dr, "(f_i)_*"] & \\ \pi_1(U_i \cap U_j, x_0) \arrow[ur, "(g_{ij})_*"] \arrow[dr, "(g_{ji})_*"'] & & \pi_1(X, x_0) \\ & \pi_1(U_j, x_0) \arrow[ur, "(f_j)_*"'] & \end{tikzcd}$$

$$\Rightarrow \exists \varphi : \operatorname{colim}_{i,j} \left(\pi_1(U_i \cap U_j, x_0) \rightrightarrows \begin{matrix} \pi_1(U_i, x_0) \\ \pi_1(U_j, x_0) \end{matrix} \right) \to \pi_1(X, x_0)$$

by univ prop of colim!

$$\frac{\ast_i \, \pi_1(U_i, x_0)}{\langle g_{ij*}(\alpha)\, g_{ji*}(\alpha)^{-1}, \ \forall \alpha \in \pi_1(U_i \cap U_j, x_0) \ \forall i, j \rangle}$$

Thm (Van Kampen)

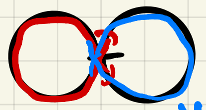
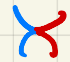
1) If U_i & $U_i \cap U_j$ are path connected $\forall i, j$
$\Rightarrow \varphi$ is surjective

2) If $U_i \cap U_j \cap U_k$ path conn $\forall i, j, k \Rightarrow \varphi$ isom!

Rmk $X = U \cup V$ (need $U, V, U \cap V$ path conn)

$$\Rightarrow \pi_1(X, x_0) = \frac{\pi_1(U, x_0) * \pi_1(V, x_0)}{\pi_1(U \cap V, x_0)}$$

Convention For path conn sp $\pi_1(X) = \pi_1(X, x_0)$ omit base pt!

eg 1 $\quad S^1 \wedge S^1 =$  

$U \sim S^1$ by deform ret $\qquad V \sim S^1 \qquad U \wedge V \sim \{x_0\}$

$$\Rightarrow \pi_1(S^1 \wedge S^1) = \frac{\pi_1(U) * \pi_1(V)}{\pi_1(U \wedge V)} = \underset{\substack{\langle a \rangle \\ \text{loop around } U}}{\mathbb{Z}} * \underset{\substack{\langle b \rangle \\ \text{loop around } V}}{\mathbb{Z}}$$

Mo