**Def)** $f: S^n \to S^n$ homo $(n>0)$

$f_*: \underset{\substack{\cong \\ \mathbb{Z}}}{H_n(S^n)} \longrightarrow \underset{\substack{\cong \\ \mathbb{Z}}}{H_n(S^n)}$ homo $\Rightarrow f_*(\alpha) = d\alpha$ for $d \in \mathbb{Z}$

$\longleftrightarrow$ degree of $f$

$\deg(f) := d$

**Prop)** 1) $\deg(id_{S^n}) = 1$ as $(id_{S^n})_* = id_{\mathbb{Z}}$

2) $f$ not surj $\Rightarrow \deg f = 0$

**Pf)** Say $x_0 \in S^n$ not in $\operatorname{im} f \Rightarrow f$ factors $f: S^n \to S^n \setminus \{x_0\} \hookrightarrow S^n$

$\Rightarrow f_*$ factors $f_*: H_n(S^n) \longrightarrow H_n(S^n \setminus \{x_0\}) \longrightarrow H_n(S^n)$

$H_n(S^n \setminus \{x_0\}) \cong H_n(\mathbb{R}^n) = 0$

3) $f \sim g \Rightarrow \deg f = \deg g$

**Pf)** $f_* = g_*$ (converse true but won't pv)

4) $\deg(f \circ g) = \deg(f) \cdot \deg(g)$

**Pf)** $(f \circ g)_* = f_* \circ g_*$

5) $f$ is htpy equiv $\Rightarrow \deg f = \pm 1$

**Pf)** $\exists g$ s.t. $f \circ g \sim id$ & $g \circ f \sim id$

$\overset{3,4}{\Longrightarrow} \deg(f) \cdot \deg(g) = 1 \Rightarrow \deg f = \pm 1$

**Exer)** 1) $S^n$ has $\Delta$-complex struct with 2 simplices $\sigma_1: \Delta_1^n \to S^n$, $\sigma_2: \Delta_2^n \to S^n$ glued along their $\partial \cong S^{n-1}$

2) $\sigma_1 - \sigma_2$ $n$-cycle (can think in now)

$[\sigma_1 - \sigma_2] \in H_n(S^n) \cong \mathbb{Z}$ gens it

6) If $f$ is a reflection of $S^n$ fixing an equator $\cong S^{n-1}$ and exchanging compl. hemispheres.

(i.e $S^n$  $\to (x_1, \dots, x_{n+1}) \longmapsto (x_1, \dots, x_n, -x_{n+1})$ eg $\to$ )

$\Rightarrow$ $\deg f = -1$

Pf) Should apply $f_*([\sigma_1 - \sigma_2])$ $\longrightarrow$ gens $H_n(S^n)$
↪ $f_*$ det by where it sends gen

$= [f \circ \sigma_1 - f \circ \sigma_2]$

$= [\sigma_2 - \sigma_1]$

$= -[\sigma_1 - \sigma_2]$

7) The antipodal map $a : S^n \longrightarrow S^n$, $x \longmapsto -x$

$\deg(a) = (-1)^{n+1}$

Pf) $a$ composes $n+1$ of the reflections as above (along each coord $S^n$)

i.e $a = r_1 \circ \cdots \circ r_{n+1}$

$r_i : (x_1, \dots x_i, \dots, x_{n+1}) \longmapsto (x_1, \dots, -x_i, \dots x_{n+1})$

(6) $\Rightarrow$ $\deg r_i = -1$ $\Rightarrow$ $\deg a = (-1)^{n+1}$

8) $f : S^n \to S^n$ fixed pt free. (i.e $f(x) \neq x \ \forall x \in S^n$)

$\Rightarrow$ $\deg f = (-1)^{n+1}$

Pf/ wts $f \sim a$ $\to$ antipodal.

via/ $H : S^n \times I \longrightarrow S^n$

$$(x, t) \longmapsto \frac{(1-t)f(x) + t\,a(x)}{\|(1-t)f(x) + t\,a(x)\|}$$

normalize $\longleftarrow$

Just need to show denom $\neq 0$

This holds as line from $p \in S^n$ to $q \in S^n$ passes thru origin $\iff p = -q$ apply when $p = f(x)$ $q = -x$

by assump $f(x) \neq -(-x) = x$

**Thm (Hairy Ball Thm)** $\boxed{\text{cts}}$

$S^n$ has a nonvanishing vector field $\iff$ $n$ odd.

↳ Co

**Df |** Cts vec field. $\longleftrightarrow$


Cts map $v : S^n \to \mathbb{R}^{n+1}$ so that $x, v(x)$ orth $\iff \langle x, v(x) \rangle = 0$

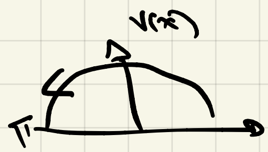
$v$ non-vanishing $\implies v : S^n \to S^n$ by repl. $v(x)$ w/ $\frac{v(x)}{|v(x)|}$ $\to$ normalize

**Check |** Given nonvanishing $v$ we show

$\implies \boxed{\text{id} \sim a}$ as $H : S^n \times I \to S^n$

$(x, t) \longmapsto \cos(\pi t)\, x + \sin(\pi t) \cdot v(x)$

i.e. Fix $x$




$x$ use $v$ in perp to $\boxed{\text{rotate}}$

↳ this is why $v$ nonvaning needed)

$$(-1)^{n+1}$$
$$\|$$
$$\implies \deg a = \deg \text{id}_{S^n} = 1$$

$\implies n$ is odd (pf $\implies$)

($\impliedby$) with $n$ odd, $n = 2k-1$, will expl. write down $v$)

$v(x_1, \dots x_{2k}) = (-x_2, x_1, -x_4, x_3, \dots, -x_{2k}, x_{2k-1})$

$\in S^n$ so not $0$ $\square$

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Let $X$ be a space (assume singletons are closed, e.g. $X$ hausdorff)

**Local hom |** at $x \in X$ is $H_i(X, X \setminus \{x\}) \overset{\text{excision}}{\cong} H_i(U, U \setminus \{x\})$

$Z = X \setminus U$ $\forall U$ nbhd