Setup $p: \tilde{X} \to X$ a covering sp of degree 2

Given $\sigma: \Delta^n \to X$ n-simplex, can lift

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] & \tilde{x}_1, \tilde{x}_2 \arrow[d, mapsto] \\ \Delta^n \arrow[ur, "\tilde{\sigma}"] \arrow[r, "\sigma"'] & X & \\ \delta_0 \arrow[r, mapsto] & x_0 & \end{tikzcd}$$

b/c $\sigma_* \pi_1(\Delta^n, \delta_0) = 1 \subseteq p_* \pi_1(\tilde{X}, \tilde{x}_1)$

$\tilde{x}_1 \rightsquigarrow \tilde{\sigma}_1$, $\tilde{x}_2 \rightsquigarrow \tilde{\sigma}_2$

$\exists$ exactly 2 lifts of σ, $\tilde{\sigma}_1, \tilde{\sigma}_2$

Lemma $\exists$ SES of c.c.

$$0 \to C_\bullet(X;\mathbb{Z}/2) \xrightarrow{\tau} C_\bullet(\tilde{X}, \mathbb{Z}/2) \xrightarrow{p_\#} C_\bullet(X, \mathbb{Z}/2) \to 0$$
$$\sigma \mapsto \tilde{\sigma}_1 + \tilde{\sigma}_2$$

Pf • $p_\#$ is surjective b/c existence of lift. $p_\#(\tilde{\sigma}_1) = \sigma$

• τ is injective $\sum a_i \sigma_i \in C_\bullet(X, \mathbb{Z}/2)$ w/ $\tau(\sum a_i \sigma_i) = 0$

but, $\tau(\sum a_i \sigma_i) = \sum a_i \big((\tilde{\sigma}_i)_1 + (\tilde{\sigma}_i)_2 \big) = 0$

these are basis so distinct

$\Rightarrow a_i = 0 \ \forall i$ $\square$

• $\operatorname{im}(\tau) \subseteq \ker p_\#$ as $p_\#(\tilde{\sigma}_1 + \tilde{\sigma}_2) = \sigma + \sigma = 2\sigma = 0$ ($\mathbb{Z}/2$)

• $\ker p_\# \subseteq \operatorname{im}(\tau)$ let $\delta: \Delta^n \to \tilde{X}$ it is of the form $\tilde{\sigma}_1 + \tilde{\sigma}_2$ for some $\sigma: \Delta^n \to X$, $p \circ \delta$

Any elt in $C_n(\tilde{X}, \mathbb{Z}/2) \ni w$ is of form (by unique lift)

$u = \sum a_i (\tilde{\sigma}_i)_1 + b_i (\tilde{\sigma}_i)_2$

$0 = p_\#(u) = \sum (a_i + b_i) \sigma_i \Rightarrow a_i + b_i = 0$

basis $\Rightarrow a_i = b_i \ \forall i$ as in $\mathbb{Z}/2$

$\Rightarrow u = \sum a_i (\tilde{\sigma}_i)_1 + a_i (\tilde{\sigma}_i)_2 = \sum a_i \tau(\sigma_i) \in \operatorname{im}\tau$ $\square$

Cool

Cor| (Transfer LES) get LES

$$\cdots \longrightarrow H_n(X;\mathbb{Z}/2) \xrightarrow{\tau_*} H_n(\tilde{X};\mathbb{Z}/2) \xrightarrow{p_*} H_n(X;\mathbb{Z}/2) \longrightarrow H_{n-1}(X;\mathbb{Z}/2)$$

Rmk| this LES is natural: swap fiber.

$\mathrm{Aut}(\tilde{X}/X) \cong \mathbb{Z}/2$ let $\phi \in \mathrm{Aut}(\tilde{X}/X)$ gen

If $f:\tilde{X} \longrightarrow \tilde{X}$ so

If $f\circ\phi = \phi\circ f$ get $\bar{f}$ in comm diag.

$$\begin{tikzcd} \Delta^n \arrow[r, "\tilde{\sigma}_1, \tilde{\sigma}_2"] \arrow[d, "\sigma"'] & \tilde{X} \arrow[r, "f"] \arrow[d, "p"'] \arrow[dr, phantom, "\circlearrowright"] & \tilde{X} \arrow[d] \\ \tilde{X}/_{x\sim\phi(x)} \arrow[r, phantom, "="] & X \arrow[r, "\bar{f}"] & X \end{tikzcd}$$

b/c $\bar{f}\circ p = p\circ f$

Cl: This comm

$$\begin{tikzcd} 0 \arrow[r] & C_\bullet(X;\mathbb{Z}/2) \arrow[r, "\tau"] \arrow[d, "\bar{f}_\#"] \arrow[dr, phantom, "\circlearrowright"] & C_\bullet(\tilde{X};\mathbb{Z}/2) \arrow[r, "p_\#"] \arrow[d, "f_\#"] \arrow[dr, phantom, "\circlearrowright"] & C_\bullet(X;\mathbb{Z}/2) \arrow[r] \arrow[d, "\bar{f}_\#"] & 0 \\ 0 \arrow[r] & C_\bullet(X;\mathbb{Z}/2) \arrow[r, "\tau"'] & C_\bullet(\tilde{X};\mathbb{Z}/2) \arrow[r, "p_\#"'] & C_\bullet(X;\mathbb{Z}/2) \arrow[r] & 0 \end{tikzcd}$$

comm b/c $f_\#\circ\tau(\sigma) = f_\#(\tilde{\sigma}_1+\tilde{\sigma}_2)$

$$= \underbrace{f\circ\tilde{\sigma}_1 + f\circ\tilde{\sigma}_2}_{\text{exactly the 2 lifts of } \bar{f}\circ\sigma}$$

b/c red bit commutes abv

exer ensure $f\circ\tilde{\sigma}_1$ distinct $f\circ\tilde{\sigma}_2$ => abv $\tau\circ\bar{f}_\#(\sigma)$

Upshot: 2 SES w/ such morphisms $\Rightarrow$ LES is natural, i.e. have comm diag

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(X;\mathbb{Z}/2) \arrow[r] \arrow[d, "f_*"] & H_n(\tilde{X};\mathbb{Z}/2) \arrow[r] \arrow[d, "f_*"] & H_n(X;\mathbb{Z}/2) \arrow[r] \arrow[d, "f_*"] & \cdots \\
\cdots \arrow[r] & H_n(X;\mathbb{Z}/2) \arrow[r] & H_n(\tilde{X};\mathbb{Z}/2) \arrow[r] & H_n(X;\mathbb{Z}/2) \arrow[r] & \cdots
\end{tikzcd}$$

Thm If $f: S^n \to S^n$ odd i.e. $f(-x) = -f(x)$
$\Rightarrow$ degree of f is odd.

Pf exercise: $f_*: H_n(S^n;\mathbb{Z}/2) \xrightarrow{\cdot \deg(f)} H_n(S^n;\mathbb{Z}/2)$, with $H_n(S^n;\mathbb{Z}/2) \cong \mathbb{Z}/2$ on both sides.

derivable since this is how we defined $\deg(f)$ but $\mathbb{Z}$ coeff.

$\therefore$ wts $f_*: H_n(S^n;\mathbb{Z}/2) \xrightarrow{\cong} H_n(S^n;\mathbb{Z}/2)$
$\Rightarrow$ degree odd

have,

$$\begin{tikzcd}
S^n \arrow[r, "f"] \arrow[d, "p"] & S^n \arrow[d, "p"] \\
\mathbb{RP}^n \arrow[r, "\bar{f}"] & \mathbb{RP}^n = S^n/x\sim -x
\end{tikzcd}$$

f satisfies $f \circ \phi = \phi \circ f$ as f odd

by transfer LES:

$$0 \to H_n(\mathbb{RP}^n;\mathbb{Z}/2) \xrightarrow{\cong} H_n(S^n;\mathbb{Z}/2) \to H_n(\mathbb{RP}^n;\mathbb{Z}/2) \xrightarrow{\delta,\ \cong}$$
$$H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) \to H_{n-1}(S^n;\mathbb{Z}/2) \cdots$$
$$H_0(\mathbb{RP}^n;\mathbb{Z}/2) \to H_0(S^n;\mathbb{Z}/2) \xrightarrow[p_*]{\cong} H_0(\mathbb{RP}^n;\mathbb{Z}/2) \to 0$$

$H_n(\mathbb{RP}^n;\mathbb{Z}/2) \cong \mathbb{Z}/2$; injection $\mathbb{Z}/2 \to$ $\Rightarrow \cong$

$\mathbb{Z}/2 \,\|\!\!\, 2$

surj $\mathbb{Z}/2 \to$; $\cong$

$H_{n-1}(\mathbb{RP}^n;\mathbb{Z}/2) \cong \mathbb{Z}/2$; $H_{n-1}(S^n;\mathbb{Z}/2) = 0$

$\delta \cong$

as surj onto 0

Upshot!

$$H_n(\mathbb{RP}^n;\mathbb{Z}/2) \xrightarrow[\cong]{\partial} \cdots\cdots H_1(\mathbb{RP}^n;\mathbb{Z}/2) \xrightarrow{\delta} 0$$