9. Infinite Products.

9.1 Basis of infinite products.

Def) $\prod_{n=1}^{\infty}(1+a_n)$ ($a_n \in \mathbb{C}$) is said to be conv if $\lim_{k\to\infty} \underbrace{\prod_{n=1}^{N}(1+a_n)}_{P_N}$ conv.

$$\lim_{k\to\infty} \prod_{n=1}^{N}(1+a_n) = \prod_{n=1}^{\infty}(1+a_n)$$

e.g)

$$\prod_{n=1}^{\infty}\left(1+\frac{(-1)^{n+1}}{n}\right) = (1+1)\left(1-\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\cdots\cdots$$

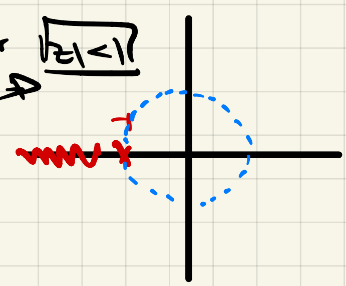
$$\prod_{n=1}^{N}\left(1+\frac{(-1)^{n+1}}{n}\right) = \begin{cases} 1 & N \text{ even} \\ \frac{N+1}{N} & N \text{ odd} \end{cases}$$

$$\Rightarrow \prod_{n=1}^{\infty}\left(1+\frac{(-1)^{n+1}}{n}\right) = 1$$

Lemma) $|\log(1+z)| < 2|z| \quad \forall\ |z| \le \frac{1}{2}$

Pf) $|\log(1+z)| = \left|z - \frac{z^2}{2} + \frac{z^3}{3} + \cdots\cdots\right|$ for $\boxed{|z|<1}$

power series at 0



$$\le |z| + \frac{|z|^2}{2} + \frac{|z|^3}{3} + \cdots$$

$$\le |z| + |z|^2 + |z|^3 + \cdots$$

$$= \frac{|z|}{1-|z|} \underset{|z|<\frac{1}{2}}{=} 2|z|$$

Thm (conv of inf prod)

If $\sum_{n=1}^{\infty}|a_n| < \infty$, then $\prod_{n=1}^{\infty}(1+a_n)$ conv.

Furthermore, in this case, the inf prod vanishes iff, $\exists\, k$ so $1+a_k=0$.

Pf | Since $\sum_{n=1}^{\infty} |a_n| < \infty$ $\lim_{n\to\infty} |a_n| = 0$

Thus, $\exists n_0$ so $|a_n| < 1/2 \quad \forall n \geq n_0$

$\Rightarrow \prod_{n=n_0}^{N} (1+a_n) = e^{\sum_{n=n_0}^{N} \mathrm{Log}(1+a_n)}$ → well def as

But, $\sum_{n=n_0}^{N} |\mathrm{Log}(1+a_n)| \leq 2\sum_{n=0}^{N} |a_n|$ conv as $N \to \infty$

↳ lemma

assump

$\therefore \prod_{n=n_0}^{\infty} \mathrm{Log}(1+a_n)$ conv

$\Rightarrow \left[\prod_{n=1}^{n_0-1} (1+a_n)\right] \cdot e^{\sum_{n=n_0}^{N} \log(1+a_n)}$ conv as $N \to \infty$

eg | $\prod_{n=1}^{\infty} \left(1 - \frac{1}{n^2}\right)$ conv. = we will compute!

9.2 Inf prod of analytic func

Thm | (Inf prod of analytic funcs)

Let $\{f_n\}_{n=1}^{\infty}$ be a seq of analytic funct on Ω

Suppose, $\exists$ consts $c_n \geq 0$ so that

① $|f_n(z) - 1| \leq c_n \quad \forall z \in \Omega \quad \forall n = 1, 2, \ldots$

② $\sum_{n=1}^{\infty} |c_n| < \infty$. Then,

(a) $\prod_{n=1}^{\infty} f_n(z)$ conv to analytic map on Ω → uniformly!

& it is 0 iff $f_n = 0$ for some n

(b) for $z_0 \in \Omega$ if $f_n(z_0) \neq 0 \quad \forall n$

$\Rightarrow$ writing $F(z) = \prod^{\infty} f_n(z)$

logarithmic derivate of F $\left[\frac{F'(z)}{F(z)} = \frac{f_1'(z)}{f_1(z)} + \frac{f_2'(z)}{f_2(z)} + \cdots \right.$

Pf) (a) let $g_n(z) = f_n(z) - 1$ so that $\prod_{n=1}^{\infty} (1+g_n(z))$

Since $|g_n(z)| \leq c_n$ $\forall z \in \Omega, \forall n$ & $\sum c_n < \infty$

$\exists n_0$ s.t. $|g_n(z)| < c_n \leq 1/2$ $\forall n \geq n_0, \forall z \in \Omega$

Thus, $\sum_{n=n_0}^{N} \operatorname{Log}(1+g_n(z))$ conv unif for $z \in \Omega$, as $N \to \infty$.

$\hookrightarrow |\log(1+g_n(z))| \leq 2\, |g_n(z)| \leq 2c_n$

By Weierstrass, this limit is analytic on Ω!

(b) let $F_N(z) = \prod_{n=1}^{N} f_n(z)$

$$\frac{F_N'(z)}{F_N(z)} = \sum_{n=1}^{N} \frac{f_n'(z)}{f_n(z)}$$

by Weis, $\boxed{F_N'(z) \to F'(z)}$

$N \to \infty$ makes LHS $\boxed{\frac{F'(z)}{F(z)}}$.

$\therefore$ right will conv too

$\hookrightarrow$ to what is expected!

e.g.) $F(z) = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$

let $R > 0$ show analytic in $D_R(0)$

For $\forall z \in D_R(0) \Rightarrow \left| -\frac{z^2}{n^2} \right| \leq \frac{R^2}{n^2}$

Note $\sum_{n=1}^{\infty} \frac{R^2}{n^2} < \infty$ $\to \frac{R^2 \pi^2}{6}$

So, the inf prod conv to an analytic func in $|z| < R$

As R is arb, conv everywhere! $\hookrightarrow$ entire.

Also $\frac{F'(z)}{F(z)} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{-\frac{2z}{n^2}}{1 - \frac{z^2}{n^2}} = \frac{1}{z} - 2\sum \frac{z}{n^2 - z^2}$

F is an entire func w/ simple zeros at $z = n \in \mathbb{Z}$

$\sin(\pi z)$ is also such a funcn.

$G(z) = \frac{\sin(\pi z)}{F(z)}$ meromorphic & singularities removable $\to$ cancel

$\Longrightarrow$ extends to entire map on $\mathbb{C}$

Furthermore, cancellation to remove removable singl.

$\Rightarrow G(z) \neq 0 \quad \forall z$.

(by prev work)

$\Longrightarrow \exists g$ entire s.t. $G(z) = e^{g(z)}$ ↪ $\log z$ well def.

$\therefore \sin(\pi z) = e^{g(z)} z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$

with entire fnct $\underline{g(z)}$!

9.3 Weierstrass Infinite product

Lemma $\forall \alpha, \beta \in \mathbb{C} \quad |e^{\alpha} - e^{\beta}| \leq |\alpha - \beta| e^{\max\{|\alpha|, |\beta|\}}$

Pf $(\alpha - \beta) \int_0^1 e^{t\alpha + (1-t)\beta} \, dt = e^{t\alpha + (1-t)\beta} \Big|_{t=0}^{1}$

$= e^{\alpha} - e^{\beta}$

$e^{\alpha} - e^{\beta} \leq |\alpha - \beta| \int_0^1 |e^{t\alpha + (1-t)\beta}| \, dt$ $\quad |e^{a+ib}| = e^a$

$\leq |\alpha - \beta| \int_0^1 e^{\mathrm{Re}(t\alpha + (1-t)\beta)} \, dt$ $\quad \Rightarrow |e^z| = e^{\mathrm{Re}\, z} \leq e^{|z|}$

$\leq |\alpha - \beta| \int_0^1 e^{t|\alpha| + (1-t)|\beta|} \, dt$

$\leq |\alpha - \beta| \int e^{(t + (1-t)) \max\{|\alpha|, |\beta|\}} \, dt$

$\leq |\alpha - \beta| e^{\max\{|\alpha|, |\beta|\}}$

freeb α int

lemma (Canonical Factors)

let $E_k(z) = (1 - z) e^{z + \frac{z^2}{2} + \dots \frac{z^k}{k}}$ for $k = 0, 1, 2, \dots$

Then, $|E_k(z) - 1| \leq 6|z|^{k+1}, \quad \forall |z| \leq 1/2$

→

Pf) For $|z| \le 1/2$ $\quad E_k(z) = e^{\alpha(z)}$ $\quad \alpha(z) = \text{Log}(1-z) + z + \frac{z^2}{2} + \dots z^k/k$

Expand α

$$\alpha(z) = -\frac{z^{k+1}}{k+1} - \frac{z^{k+2}}{k+2} - \dots$$

$$|\alpha(z)| \le |z|^{k+1} + |z|^{k+2} + \dots = \frac{|z|^{k+1}}{1-|z|} \le 2|z|^{k+1}$$

for $|z| \le \frac{1}{2}$

So, $|E_k(z) - 1| = |e^{\alpha(z)} - e^0|$

$$\underset{\text{lemma}}{\le} |\alpha(z) - 0| e^{\max\{\alpha(z), 0\}}$$

$$\le 2|z|^{k+1} e^{2|z|^{k+1}}$$

$$\le 2|z|^{k+1} e$$

$$\le \boxed{6|z|^{k+1}}$$

$|z| \le \frac{1}{2}$

Thm (Weierstrass Product Thm)

Let $\{a_n\}_{n=1}^{\infty}$ be a seq of complex numbers $\ne 0$ (repeats ok)

so that $|a_n| \to \infty$ as $n \to \infty$ $\quad$ not req $\sum \frac{1}{|a_n|} < \infty$

Then, there is an entire function f whose zeros are exactly a_n!

not nec as earlier

[Furthermore, there if there is another such fcn it is of the form $f(z) e^{g(z)}$ entire.