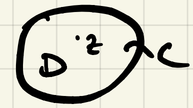
• Cauchy's Thm for star convex domains

↳ $\oint_\gamma f(z)\,dz = 0$ $\quad \forall$ closed γ in a star conv Ω, $\forall$ analytic f on Ω

• Cauchy integral formula over a circle

↳ $f(z) = \frac{1}{2\pi i}\oint_C \frac{f(w)}{w-z}\,dw$, $\forall z \in D$

f analytic on open nbhd of D



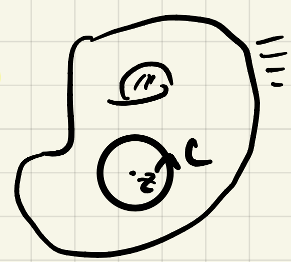
Thm (analytic $\Rightarrow C^\infty$)

If f is analytic in a domain Ω $\Rightarrow$ it is inf $\mathbb{C}$-diff'ble in Ω.

Furthermore, if C is a circle in Ω s.t. its interior D is also $\subseteq \Omega$

$$\forall z \in D \quad f^{(n)}(z) = \frac{n!}{2\pi i}\oint_C \frac{f(w)}{(w-z)^{n+1}}\,dw$$

↳ Cauchy integral formula of high ord (CIT)



Pf Induction on n

[$n=0$] Clear by CIT

Assume CIT of order $n-1$ holds,

$\Rightarrow f^{(n-1)}(z)$ exists & given by formula!

Must show $\mathbb{C}$-diff'ble 1 more time & given by for

$$\frac{f^{(n-1)}(z+h) - f^{(n-1)}(z)}{h} \overset{\text{(ind hyp)}}{=} \frac{(n-1)!}{2\pi i}\oint_C f(w)\,\frac{1}{h}\left[\frac{1}{(w-z-h)^n} - \frac{1}{(w-z)^n}\right]dw$$

note: $A^n - B^n = (A-B)(A^{n-1} + A^{n-2}B + \cdots + B^{n-1})$ Alg

$$= \frac{(n-1)!}{2\pi i}\oint_C f(w)\sum_{k=0}^{n-1}\frac{1}{(w-z-h)^{n-k}(w-z)^{k+1}}\,dw$$

integral & finite sum can exchange!

$\xrightarrow[n\to 0]{}$ $\frac{(n-1)!}{2\pi i} \underbrace{\sum_{k=0}^{n-1}}_{\text{factor to } n} \oint f(w) \frac{1}{(w-z)^{n+1}} dw$ if we can pull limit in

pulling the $h \to 0$ req $\frac{1}{(w-z-h)^{n-k}} \cdot \frac{1}{(w-z)^{k+1}} \xrightarrow[\text{in } w \text{ on } \bigcirc]{\text{unif}} \frac{1}{(w-z)^{n+1}}$

this is simlo $\lim_{h\to 0} \frac{1}{w-z-h} = \frac{1}{w-z}$ unif in $w \in C$ $\square$.

3.2 | Liouville's Thm

Thm | (Liouville) let f be an $\overbrace{\text{entire function}}^{\to \text{ analytic on } \mathbb{C}}$

If f is bdd on $\mathbb{C}$ (i.e $\exists B \in \mathbb{R}$ st $|f| \le B$)

$\Rightarrow f$ is constant!

Pf | let $z \in \mathbb{C}$ given! let $R > 0$ so $|z| < R$

so $z \in C_R$

by CIT order 1

$$f'(z) = \frac{1}{2\pi i} \oint_{|w|=R} \frac{f(w)}{(w-z)^2} dw$$ $\to$ lower bd deno

$$|f'(z)| \overset{(ML)}{\le} \frac{1}{\cancel{2\pi}} \frac{B}{(R-|z|)^2} \cancel{2\pi} R$$

since $\frac{R}{(R-|z|)^2} \to 0$ as $R \to \infty$

$\Rightarrow |f'(z)| = 0$ $\square$.

note: if take $|f(z)| \le B_1 |z| + B_2$

$\Rightarrow f''(z) \equiv 0 \Rightarrow$ the map is linear-esque!

So on and so forth holds

Also can have | FTA consequence of Liouville.

Spose $p(z)$ no zero $\Rightarrow \frac{1}{p}$ entire $\to$ try to bound & get contrad

3.3 Morera's Thm

Thm (Morera's Thm)

If f is cts in a domain Ω & $\oint_T f(z)\,dz = 0$ for all triangles T in Ω

(edges only)

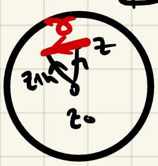
Then f is analytic in Ω.

Pf Let D be an arbitrary disc satisfying $\overline{D} \subset \Omega$.

It is enough to show that f is analytic in D

Let z_0 be the center of D → finding local prims

Define $F(z) = \oint_{\gamma_z} f(w)\,dw$ for $z \in D$



where γ_z is straight line seg $z_0 \to z$

If h is small enough so $z+h \in D$

$\Rightarrow F(z+h) - F(z) = \int_\gamma f(w)\,dw$ where γ is line seg $z \to z+h$

$$\Rightarrow \frac{F(z+h) - F(z)}{h} \xrightarrow[h \to 0]{?} f(z)$$

true by same argument in Goursat's thm

used $\boxed{2.6.3}$

we only used continuity of f

$\therefore F$ is diff'ble in Ω & $F'(z) = f(z)$

but! $F' = f$ is analytic by prev thm! ∎

Recall if $h:[a,b]\times[c,d]\to\mathbb{R}$ cts

$$\Rightarrow \int_a^b\int_c^d h(s,t)\,dt\,ds = \int_c^d\int_a^b h(s,t)\,ds\,dt$$

Fubini map generally

Let γ be a curve in $\mathbb{C}$. If $f:[a,b]\times\gamma\to\mathbb{C}$ cts

$$\Rightarrow \int_a^b\int_\gamma f(t,z)\,dz\,dt = \int_\gamma\int_a^b f(t,z)\,dt\,dz$$

by param

(even true if γ piecewise smooth) (Hw)

Thm (analytic func defd by integrals) cts by breaking

Let Ω be a domain in $\mathbb{C}$

Let $g:[a,b]\times\Omega\to\mathbb{C}$ be a cts map

s.t. for every $t\in[a,b]$, the map $z\longmapsto g(t,z)$ is analytic in Ω

Then $f(z)=\int_a^b g(t,z)\,dt$ is analytic in Ω.

Pf Let D be a disc st $\overline{D}\subset\Omega$

Since $[a,b]\times\overline{D}$ is cmpct $\Rightarrow g(t,z)$ is unif cts on $[a,b]\times\overline{D}$

Thus, f is cts on $\overline{D}$

For every $\Delta\, T\subset D$

$$\oint_T f(z)\,dz = \oint_T\int_a^b g(t,z)\,dt\,dz$$

g unif cts on $[a,b]\times T$

$$= \int_a^b \underbrace{\oint_T g(t,z)\,dz}_{\text{analytic on } \Omega\supseteq T \text{ \& } t \text{ frozen}}\,dt$$

$$= \int_a^b 0\,dt = 0$$

$\therefore$ by Morera $\Rightarrow f$ analytic! $\square$

e.g.) $J_n(z) = \frac{1}{\pi}\int_0^{2\pi} \cos(z\sin\theta - n\theta)\, d\theta$, $n = 0, 1, 2, \ldots$

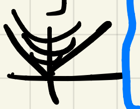
is entire! (continuity is clear & analyticity for fixed θ)

→ Bessel func!

3.4) Uniform conv of analytic func

Thm) (Weierstrass thm on the unif conv. of a seq of analytic func)

in $\mathbb{R}$ not true eg



If $(f_n(z))_{n=1,2,\ldots}$ is a seq of analytic func in a domain Ω such that $f_n \xrightarrow{\text{unif}} f$ on every compact sbs Ω

$\Rightarrow$ f is analytic in Ω

Pf) Morera!

e.g.) $\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$ for $\mathrm{Re}\, z > 1$ $\quad n^z = e^{z \ln n}$

Riemann Zeta Func

→ truncate the sum to get seq of entire func!

→ conv to $\zeta(z)$

→ to show this is unif happens when $\mathrm{Re}\, z > 1$.

Use weierstrass $\Rightarrow \zeta(z)$ is analytic $\mathrm{Re}\, z > 1$

e.g.) $\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1}\, dt$ → can't use integral idea as $[0, \infty)$ not compact

gamma func

$\Gamma(z) \overset{?}{=} \lim_{n\to\infty} \Gamma_n(z)$

$= \lim_{n\to\infty} \int_0^n e^{-t} t^{z-1}\, dt$ → integral thm shows this is analytic

Unif conv $\overset{\text{weier}}{\Longrightarrow}$ $\Gamma(z)$ analytic when $\mathrm{Re} > 0$

$\Gamma(n) = (n-1)!$ → "natural extn"