Recall

Montel's Thm — 11.6

$\mathcal{F}$ is a family of analytic fns on Ω, a domain.
Sppse $\mathcal{F}$ is unif bdd, i.e. $\exists B$ s.t. $|f(z)| < B$ $\forall f \; \forall z$.

Then, every seq of fns in $\mathcal{F}$ has a subseq that conv unif on all cmpcts sbsts Ω.

Note: limit is not nec in $\mathcal{F}$.

Thm (Arzela - Ascoli)

Let K be a compact set in $\mathbb{R}^d$. Let $\mathcal{S}$ be a family of cts. fns on K.
If $\mathcal{S}$ is unif bdd, then the following are equiv
i) $\mathcal{S}$ is equicontinuous $\rightarrow$ unif cts uniformly lmao
$\Leftrightarrow$ $\forall \varepsilon > 0 \; \exists \delta > 0$ $|f(x) - f(y)| < \varepsilon$ if $|x - y| < \delta$, $x, y \in K$ $\forall f \in \mathcal{S}$

ii) Every sequence of fns in $\mathcal{S}$ has subseq that conv unif on K.

Pf of (i $\Rightarrow$ ii)

Let $(g_n)_{n \in \mathbb{N}} \in \mathcal{S}$ seq of maps.
Let $\{p_i\}_{i=1}^{\infty}$ be a ctbl dense set in K $(K \cap \mathbb{Q}^d)$
Consider, seq of numbers $\{g_n(p_1)\}_{n=1}^{\infty} \subseteq \mathbb{R}^{d'}$ or $\mathbb{C}$
$\overset{\text{unif bd}}{\Longrightarrow}$ $\{g_n(p_1)\}_{n=1}^{\infty}$ bdd by B
$\Rightarrow$ by Bolzano Weierstrass, $\exists$ conv subseq of

$\exists$ subseq of $(g_n)_{n=1}^{\infty}$ given by $g_{1,1}, g_{1,2}, g_{1,3} \ldots$ that conv if eval at p_1

Similarly, $\exists$ subseq of $\longrightarrow$ that conv at $(p_2 \,\&\, p_1)$
Call it $g_{2,1}, g_{2,2}, g_{2,3}, \ldots$

Keep it going!

$$\begin{matrix} g_{11} & g_{12} & g_{13} & \cdots \\ g_{21} & g_{22} & g_{23} & \cdots \\ g_{31} & g_{32} & g_{33} & \cdots \\ \vdots & \vdots & \vdots & \end{matrix}$$

diagonalization

Choose $\{g_{nn} = h_n\}_{n \in \mathbb{N}}$ seq.

this converges at every $p_1, p_2 \ldots$

extend conv to K & uniformly uses Equicontinuity.

let $\boxed{\varepsilon > 0}$ be given. $\exists \delta > 0$ for equicontinuity corr $\frac{\varepsilon}{3}$

Since $\{p_i\}_{i=1}^{\infty}$ is dense in K & K is cmpct.

$\exists$ finitely many $D_\delta(p_i)$ that cover K, $i = 1, \ldots, i_0$
Cd (dense gives cover, cmpct gives finite)

$\forall i \; \exists N(i)$ s.t. $|h_n(p_i) - h_{n'}(p_i)| < \frac{\varepsilon}{3}$ $\forall n, n' \geq N(i)$

Cauchy.

let $\boxed{N} = \max\{N(1), \ldots, N(i_0)\}$

$\boxed{\forall p \in K}$ $\exists i \leq i_0$ so $|p - p_i| < \delta$ (cover by $D_\delta(p_i)$)

equicontinuity $\Rightarrow$ $\boxed{\forall n, n_1 \geq N}$

$$\boxed{|h_n(p) - h_{n'}(p)|} \leq \underbrace{|h_n(p) - h_n(p_i)|}_{\text{equicont}} + \underbrace{|h_n(p_i) - h_{n'}(p_i)|}_{\text{cauchy seq}} + \underbrace{|h_{n'}(p_i) - h_{n'}(p)|}_{\text{equicont}}$$

$< 3\frac{\varepsilon}{3} = \boxed{\varepsilon}$

tracing h unif Cauchy!
$\Rightarrow$ unif conv

eg| if $\exists M$ s.t. $|g(x)-g(y)| \le M|x-y|$ $\forall x,y \in K$, $\forall g \in \mathcal{F}$

then $\mathcal{F}$ is equicont. $\rightsquigarrow$ Lipschitz

given ε choose $\delta = \frac{\varepsilon}{M}$ Uniformly

Thm | Let $\mathcal{A}$ be a family of analytic funct in domain Ω.

If $\mathcal{A}$ is unif bdd in Ω

$\Rightarrow$ $\mathcal{A}$ is equicontinuous on every compact sso of Ω

Meta.

① Unif bdd
② equicont.
$\Rightarrow$ ③ Every seq of fnc has conv subseq (compact w.r.t. norm)*

① $\xRightarrow{\text{thm above}}$ ② if analytic

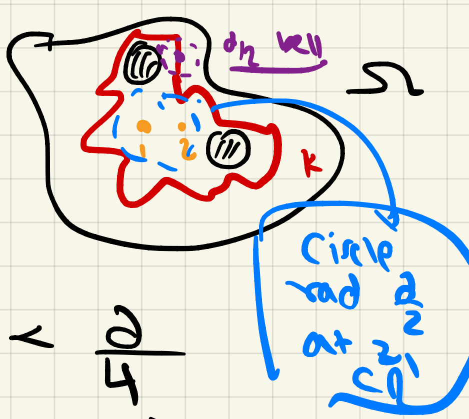
$\therefore$ ① $\Rightarrow$ ② $\Rightarrow$ ③ Montel ! & done $\square$

Pf | Bg assum $\exists B$ so $|g(z)| < B$ $\forall z \in \Omega$, $g \in \mathcal{A}$

Let, K be a compct sso Ω.

let $\partial = \operatorname{dist}(K, \mathbb{C} \setminus \Omega) > 0$

Note, if $z \in K$ $\overline{D_{\frac{\partial}{2}}(z)} \subseteq \Omega$



for $z_1, z_2 \in K$ satisfying $|z_1 - z_2| < \frac{\partial}{4}$

$$g(z_1) - g(z_2) = \frac{1}{2\pi i} \oint_{C_{\frac{\partial}{2}}(z_1)} \left(\frac{g(w)}{w - z_1} - \frac{g(w)}{w - z_2} \right) dz$$

$$= \frac{(z_1 - z_2)}{2\pi i} \oint_{C_{\frac{d}{2}}(z_1)} \frac{g(w)}{(w - z_1)(w - z_2)} \, dw$$

$$\underline{ML} \leq \frac{|z_1 - z_2|}{\cancel{2\pi}} \frac{B}{\underbrace{\cancel{\tfrac{d}{2}}}_{\text{exact}} \cdot \underbrace{\tfrac{d}{4}}_{\text{lower bd}}} \cdot \cancel{2\pi} \, \cancel{\frac{d}{2}} = \frac{4B|z_1 - z_2|}{d}$$

So $|g(z_1) - g(z_2)| \leq \frac{4B}{d} |z_1 - z_2|$ get Lipschitz.
when $|z_1 - z_2| < \frac{d}{4}$

Cor (Montel's Thm)
Let $\{g_n\}_{n=1}^{\infty}$ be a seq of maps from $\mathcal{A}$

Pf Almost here, now we have subseq dep on choice of $K \to$ we want ind of this!
↳ look at notes compact exhaustion blah, blah.
↳ keep going .... + diagonalization $\Box$.

Recall

$$f(z) = \frac{z - i}{z + i}$$ conformal map

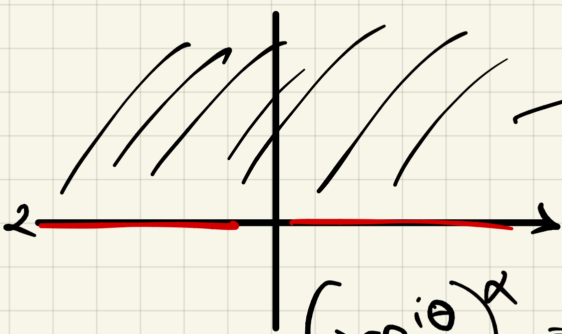
in this case, ctsly maps $\partial \to \partial$ too
as $\frac{z-i}{z+i}$ analytic on larger domain.

The map doesn't always ctsly extend to bdry!
↳ worth thinking about

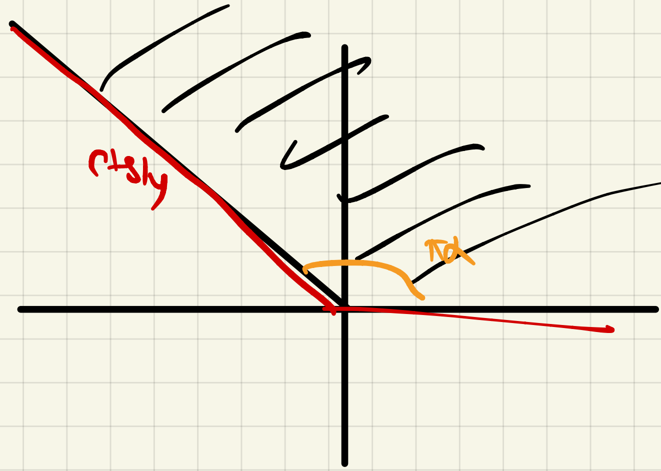
11.7 Conformal maps onto polygons → not ness convex.

eg 1 let $0<\alpha<2$

let $f(z) = z^{\alpha} = e^{\alpha \operatorname{Log} z}$ → princ.



$\xrightarrow[\text{Conformal}]{f}$



$(re^{i\theta})^{\alpha} = re^{i\theta\alpha}$

$0 \le \alpha\theta \le \pi\alpha$

cts up to bdry!

eg 1 let $\sqrt{1-w^2}$ be the analytic br st. it is anal in $\mathbb{C} \setminus (-\infty,-1] \cup [1,\infty)$

& takes +ive value when $w = x \in (-1,1)$

We may take, $e^{\frac{1}{2}\operatorname{Log}(1-w) + \frac{1}{2}\operatorname{Log}(1+w)}$ ⟹ $\boxed{\text{hw 2}}$.

↳ br when $1-w<0 \Rightarrow w>1$

$1+w<0 \Rightarrow w<-1$

let $f(z) = \displaystyle\int_0^z \frac{1}{\sqrt{1-w^2}}\, dw$ for $z \in \overline{\mathbb{H}}$

↳ upper half pl.

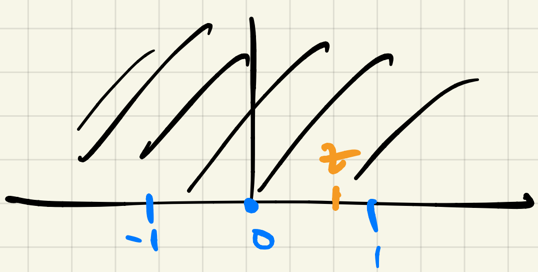
↳ arb path $0 \to z$ lies in $\overline{\mathbb{H}}$

↳ htpy equiv by straight line $\boxed{\text{htpy}}$.

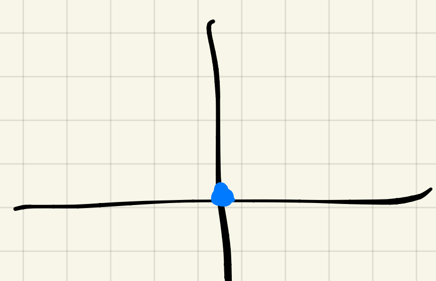
↳ Cauchy thm $\mathbb{H}$ simply conn.

↳ doesn't dep on path

new notation is good!



$\xrightarrow{f}$



⟹

for $w = t \in \mathbb{R}$

$$\sqrt{1-w^2} = \begin{cases} i\sqrt{t^2-1} & t<-1 \\ \sqrt{1-t^2} & -1 \le t \le 1 \\ -i\sqrt{t^2-1} & t>1 \end{cases}$$

anal een

For $z = x \in \mathbb{R}$ $\quad 0<x<1$