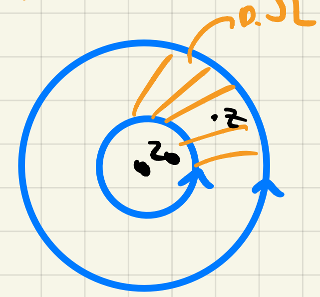
6. Laurent Series & isolated singularities

6.1 Laurent Series

→ $r_1 < r_2$ assume ..

domain containing z



f is analytic in $r_1 \le |z-z_0| \le r_2$ closed

$\gamma_1 = C_{r_2}(z_0) - C_{r_1}(z_0)$

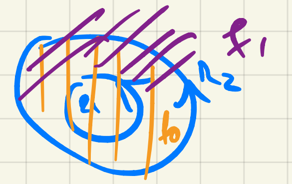
$\gamma \sim 0$ in Ω 1 2 CIF

$$\frac{1}{2\pi i}\oint_\gamma \frac{f(w)}{w-z}\,dw = f(z)\overbrace{W_\gamma(z)}^{1} = f(z)$$

So, $f(z) = \dfrac{1}{2\pi i}\displaystyle\oint_{C_{r_2}(z_0)} \frac{f(w)}{w-z}\,dw - \frac{1}{2\pi i}\oint_{C_{r_1}(z_0)} \frac{f(w)}{w-z}\,dw$

for $r_1 < |z-z_0| < r_2$ (if $r_1 = 0$ get CIF for disc)

→ CIF over annulus.



Thm (Laurent Decomp)

Let $0 \le R_1 < R_2 \le +\infty$ if f analytic in (open) annulus $R_1 < |z-z_0| < R_2$

⟹ f can be decomp $f(z) = f_0(z) + f_1(z)$ in annulus where

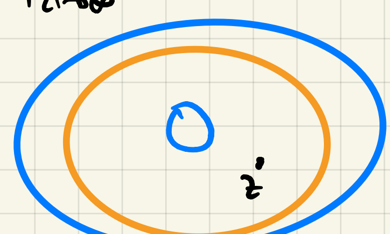
f_0 is analytic in $|z-z_0| < R_2$ & f_1 analytic in $|z-z_0| > R_1$

Here we can choose the decomp so $f_1(\infty) = 0$

i.e. $\lim_{|z|\to\infty} f_1(z) = 0$

& under the decomp is unique!

This is called the Laurent Decomposition.



Pf For every $R_1 < r < R_2$ defn

$$f_0(z;r) = \frac{1}{2\pi i}\oint_{C_r(z_0)} \frac{f(w)}{w-r}\,dw \quad \text{for } z \in D_r(z_0)$$

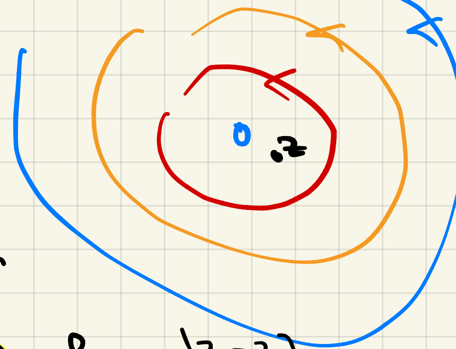
HW5 part A2
HW2

$f_0(z;r)$ analytic in $D_r(z_0)$ as it is a Cauchy type int

If $R_1 < r < s < R_2$

$\Rightarrow f_0(z,r) = f_0(z,s)$ for $z \in D_r(z_0)$

(why? Cauchy's thm).



So, for $|z-z_0| < R_2$ $\quad f_0(z) = f_0(z, r)$ for $\max\{|z-z_0|, R_1\} < r < R_2$

So, f_0 analytic in $|z-z_0| < R_2$

Similarly def $f_1 = -\frac{1}{2\pi i}\oint_{C_r(z_0)} \frac{f(w)}{w-z}\,dw$ for $|z-z_0| > R_1$

using any $R_1 < r < \min(|z-z_0|, R_2)$

f_1 analytic in $|z-z_0| > R_1$

$\Rightarrow$ for $R_1 < |z-z_0| < R_2$ by CIF over annulus,

$f(z) = f_0(z) + f_1(z)$

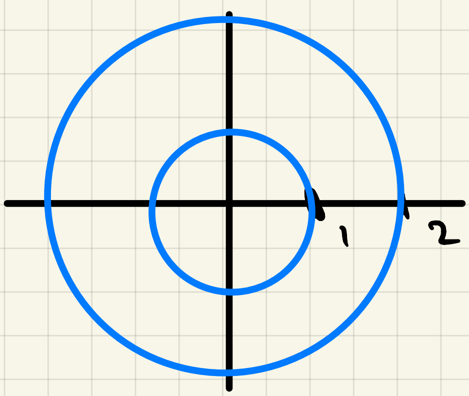
Uniqueness, The given map satisfies $f_1(\infty) = 0$ as it look like $\frac{c}{z}$

Suppose, $f_0 + f_1 = g_0 + g_1$

$\Rightarrow f_0 - g_0 = g_1 - f_1$ in annulus $\quad h = \begin{cases} f_0 - g_0 & \text{in } D_{R_2} \\ g_1 - f_1 & \text{in } \mathbb{C} \setminus D_{R_1} \end{cases}$

∴.... Liouville .... agree ... zero!

e.g.) $f(z) = \frac{1}{(z-1)(z-2)}$



① $0 \le |z| < 1$

② $1 < |z| < 2 \to f(z) = \overbrace{\frac{1}{z-2}}^{f_0} + \overbrace{\frac{1}{z-1}}^{f_1}$

③ $2 < |z| < +\infty \to f(z) = \overbrace{0}^{f_0} + \overbrace{\frac{1}{(z-1)(z-2)}}^{f_1}$

6.2 Laurent Series

Def A Laurent Series about a point z_0 is $\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$

we say it converges if both $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ ↳ conv on $|z-z_0|<R_2$

& $\sum_{n=-\infty}^{-1} a_n(z-z_0)^n$ converges $\rightarrow$ ↳ converges on $R_2>|z-z_0|$

↳ conv & holo

By the power series basis thm, for a given Laurent series

$\exists!\ 0\le R_1\le R_2\le+\infty$ s.t. the series conv absolutely on

$R_1<|z-z_0|<R_2$

& converges unif in every cmpct sbs of the open annulus.

Thm (Laurent expansion)

no nonemp

If f is analy in $R_1<|z-z_0|<R_2$

$\Rightarrow$ it has Laurent series exp

$f(z)=\sum_{-\infty}^{\infty} a_n(z-z_0)^n$ in $R_1<|z-z_0|<R_2$

Here coeff are unique!

& given by $a_n=\frac{1}{2\pi i}\oint_{|z-z_0|=r}\frac{f(z)}{(z-z_0)^{n+1}}dz$ $\quad \forall n\in\mathbb{Z}$, $R_1<r<R_2$

Pf $f=f_0+f_1$ Laurent decomp

↳ analy in $|z-z_0|<R_2$ $\rightarrow$ has a power series here!

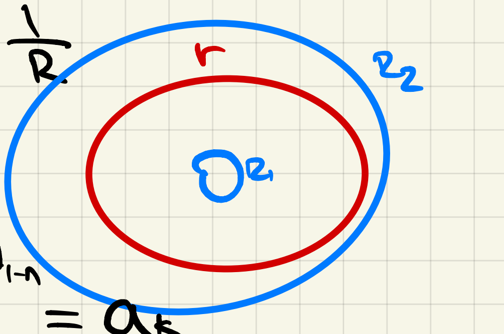
↳ analy in $|z-z_0|>R_2$ $\rightarrow$ $g(w)=f_1\left(z_0+\frac{1}{w}\right)$ for $0<w<\frac{1}{R_1}$ & $0,\ w=0$ $\quad w=\frac{1}{z-z_0}$

Check g is analytic in $|w|<\frac{1}{R_1}$ (Morera).

$\Rightarrow$ g has power series in $|w|<\frac{1}{R_1}$

& then combine these together ....

$$\oint\frac{f(z)}{(z-z_0)^{k+1}}dz=\oint\sum_{n=-\infty}^{\infty}\frac{a_n(z-z_0)^n}{(z-z_0)^{k+1}}\overset{\text{unif}}{=}\sum a_n\oint_{|z-z_0|=r}\frac{a_n}{(z-z_0)^{k+1-n}}=a_k$$



$$\oint_{|w|=r} w^n \, dw = 0 \quad \text{if } n \neq -1$$
$$\underbrace{\hookrightarrow}_{\text{C.O.V.}} 2\pi i \quad \text{if } n = -1$$

$\Rightarrow a_k$ for prev sum $\quad a_k = \frac{1}{2\pi i} \oint \frac{f(z)}{(z-z_0)^{k+1}} \, dz.$

e.g. in $1 < |z| < 2$ $\quad f(z) = \frac{1}{(z-1)(z-2)}$

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1}$$

geom $|z|<2$

$$= \frac{1}{-2(1-\frac{z}{2})}$$

$|z|>1$

$$\frac{1}{z-1} = \frac{1}{z(1-\frac{1}{z})} \quad \text{for } \left|\frac{1}{z}\right| < 1$$

$$= \frac{1}{z}\left(1 + \frac{1}{z} + \cdots\right)$$

geom

$$= -\frac{1}{2}\left(1 + \frac{z}{2} + \frac{z^2}{4} + \cdots\right) + \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \cdots\right)$$

for $|z|<2$

$$f(z) = \sum_{n=-\infty}^{-1} z^n - \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} z^n \quad \text{in } 1<|z|<2.$$

6.3 Isolated Singularities.

Def A point z_0 is an isolated singularity of f if f is analytic in a punctured disc $0 < |z-z_0| < \rho$ $\quad \boxed{\rho > 0}$

e.g. $f(z) = \frac{1}{(z-1)(z-2)}$ isolated singularities at $z=1$, $z=2$

$f(z) = \frac{1}{\sin(\pi z)}$ isol singularity at $z=n$, $n \in \mathbb{Z}$

$f(z) = e^{1/z}$ isol singularity at $z=0$.

$f(z) = \frac{z^3-1}{z^2-1}$ isol sing at $z = 1, -1$.

$f(z) = \text{Log } z$ $\quad z=0$ not isolated sing

Def (type of isol singularity)

Spse z_0 is an isolated singularity.

let $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$ be the Laurent series exp

about z_0 (since punctured disc at z_0 is degenerate annulus)

(a) z_0 is a removable singularity if $a_n = 0$, $\forall n<0$.

(b) z_0 is a pole of order N if $a_n = 0$, $\forall n \leq N-1$
$a_N \neq 0$

(c) z_0 is an essential singularity otherwise

i.e $\forall N \in \mathbb{N}$ $\exists M > N$ so $a_{-M} \neq 0$

$\Leftrightarrow$ inf many nonzero coeff for neg powers!

Def Spse f has a pole of order N at z_0

The principle part of f at z_0 is the sum

$$\sum_{n=-N}^{-1} a_n (z-z_0)^n = \frac{a_{-N}}{(z-z_0)^N} + \cdots + \frac{a_{-1}}{(z-z_0)}$$

eg $f(z) = \dfrac{1}{(z^2)(1-z)}$

about $\underline{z=0}$ $f(z) = \frac{1}{z^2}\left(1 + z + z^2 + \cdots\right)$ $\quad 0<|z|<1$

$$= \underbrace{\frac{1}{z^2} + \frac{1}{z}}_{\text{Princ part}} + 1 + \cdots \quad 0<|z|<1$$

$\therefore$ $\underline{z=0 \text{ is pole of ord } 2}$,