(eg) $f(z) = \dfrac{1}{z^2(1-z)}$ $\qquad f(z) = \underbrace{\dfrac{1}{z^2} + \dfrac{1}{z}}_{\text{prin. part}} + 1 + z + \dots$ $\quad 0<|z|<1$

$\therefore$ the isolated singularity at 0 is a pole of order 2

Double pole!

$$f(z) = \underbrace{\frac{-1}{z-1}}_{\text{princ part.}} + a_0 + a_1(z-1) + \dots \qquad 0 < |z-1| < 1$$

$$\frac{1}{z^2} = b_0 + b_1(z-1) + \dots \quad \text{in power series exp}$$

$\therefore$ the isolated singularity at 1 is a single pole

(eg) $f(z) = \dfrac{1}{\sin^2 z}$

Closest singularity

$0 < |z| < \pi$

$$= \frac{1}{\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots\right)}$$

$$= \frac{1}{z^2\left(1 - \frac{z^2}{3!} + \dots\right)^2}$$

unique pow serie

$$= \frac{1}{z^2\left(1 - \frac{z^2}{3} + \dots\right)} \qquad 1 - z^2\left(-\tfrac{1}{3} + c\,z^2 \dots\right)$$

$$= \frac{1}{z^2}\left(1 + \left(\frac{z^2}{3} - \dots\right) + \left(\frac{z^2}{3} - \dots\right)^2 + (\ )^3\right)$$

$$= \frac{1}{z^2}\left(1 + \frac{z^2}{3} + \dots\right)$$

$$= \frac{1}{z^2} + \frac{1}{3} + a_2 z^2 + \dots$$

So, the singularity at 0 is a double pole!

eg) $f(z) = e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \cdots$ $\boxed{|z|>0}$

so the singularity at $z=0$ is essential!

eg) $f(z) = \frac{z^3-1}{z^2-1}$ singularity at $z = \pm 1$

$f(z) = \frac{(z-1)(z^2+z+1)}{(z-1)(z+1)}$ → analytic at 1 → $0 < |z-1| < 2$ power series

laurent at $z=1$ → $= \underset{(3/2)}{a_0} + a_1(z-1) + \cdots\cdots$

∴ the zero $z=1$ is removable!

at -1

$f(z) = \frac{(z^3-1)}{(z+1)(z-1)}$ → analytic $0 < |z+1| < 2$ → has power series

$= \frac{1}{(z+1)} + b_0 + b_1(z+1) + \cdots$

so $z=-1$ is a single pole!

6.4 Characterization of Isolated Singularity

Thm (Removable) Riemann's Thm on Removable Singularity.

Let z_0 be an isolated singularity of f. TFAE

(a) z_0 is a removable singularity

(b) $f(z)$ is bdd in a punctured disc $0<|z-z_0|<r$

(c) f can be extended to an analytic func in a disc $|z-z_0|<r$.

Pf) (a) $\Rightarrow$ (c).

The Laurent series of f at z_0 is of the form

$a_0 + a_1(z-z_0) + \cdots\cdots$ on $0<|z-z_0|<r$

Define, $F(z) = \sum a_n (z-z_0)^n$ on $|z-z_0|<r$

rem sing.

$\hookrightarrow$ this conv on $0<|z-z_0|<r$ as the Laurent is conv

$\hookrightarrow$ conv at $z=z_0$ trivially

$\therefore F(z)$ analy on disc $|z-z_0|<r$ & agree w/ f on $0<|z-z_0|<r$.

(c) $\Rightarrow$ (b)

clear?

(b) $\Rightarrow$ (a)

From (b) $\exists M>0$ so $|f(z)|<M$ $\forall z \in 0<|z-z_0|<r$

By the Laurent Series exp thm,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n \quad \text{for } 0<|z-z_0|<r$$

where $a_n = \dfrac{1}{2\pi i} \oint_{|z-z_0|=\rho} \dfrac{f(z)}{(z-z_0)^{n+1}}\,dz$ for $0<\rho<r$

For $k \geq 1$

$$|a_{-k}| = \frac{1}{2\pi} \left| \oint_{|z-z_0|=\rho} f(z) (z-z_0)^{k-1} \, dz \right|$$

ML

$$\leq \frac{1}{\cancel{2\pi}} \cancel{2\pi}\rho \cdot M \cdot \rho^{k-1}$$

$$= M \cdot \rho^k \qquad \text{true for all } 0<\rho<r$$

$\Rightarrow$ taking $\rho \to 0 \Rightarrow |a_{-k}| = 0 \rightarrow a_{-k} = 0$

$\therefore$ the negative exponent coefficients in the Laurent series are zero

$\Rightarrow$ z_0 is removable!

$\square$.

Thm (Char of poles)

Let z_0 be an isolated singularity of f. Then TFAE,

(a) z_0 is a pole of order N

(b) There is an analytic g in some $|z-z_0|<r$ so $g(z_0) \neq 0$ & $f(z) = \dfrac{g(z)}{(z-z_0)^N}$ in $0<|z-z_0|<r$

(c) $\dfrac{1}{f(z)}$ has a removable singularity at z_0 & it extends to an analytic function that has a zero of degree N at z_0.

Pf (a) $\Rightarrow$ (b)

$\Rightarrow$ $f(z) = \dfrac{a_{-N}}{(z-z_0)^N} + \dfrac{a_{-N+1}}{(z-z_0)^{N+1}} + \cdots + a_0 + a_1(z-z_0)$ on $0<|z-z_0|<r$ & $a_{-N} \neq 0$

$$= \frac{1}{(z-z_0)^N} \Big(\underbrace{a_{-N} + a_{-N+1}(z-z_0) + \cdots}_{} \Big)$$

$g(z) =$ power conv on $|z-z_0|<r$ (some justification)

and $g(0) = 0$ as $\boxed{a_{-N} \neq 0}$. $\lfloor$ hadamard to identity conv.

(b) $\Rightarrow$ (c)

by b $f(z) = \dfrac{g(z)}{(z-z_0)^N}$ & $g(z_0) \neq 0$
g anal on $|z-z_0|<r$

$\Rightarrow \dfrac{1}{f(z)} = \dfrac{(z-z_0)^N}{g(z)}$ in $0 < |z-z_0| < r$

$\dfrac{1}{g(z)}$ is anal in $|z-z_0| < r_2$ as $g(z_0) \neq 0$

so $g(z)$ doesn't vanish

fill in some details

D

(c) $\Rightarrow$ (a)

by (c) $\exists$ h analytic in $|z-z_0|<r$, $h(z_0) \neq 0$

so $\dfrac{1}{f(z)} = (z-z_0)^N h(z)$ in $0 < |z-z_0| < r$

$\Rightarrow f(z) = \dfrac{1}{(z-z_0)^N} \left(\dfrac{1}{h(z)}\right)$ $\rightarrow$ analytic in some nbd $|z-z_0| < r_2$ as $h(z_0) \neq 0$.

power law

$= \dfrac{1}{(z-z_0)^N} \left(b_0 + b_1 (z-z_0) + \cdots\right)$

$\rightarrow b_0 = \frac{1}{h(z_0)} \neq 0$

$\Rightarrow f(z)$ has a pole of degree N at z_0!

Thm (another characterization of poles)!

let z_0 be an isolated singularity of f.

Then z_0 is a pole $\Longleftrightarrow \lim_{z \to z_0} |f(z)| = +\infty$

Pf) ($\Rightarrow$)

Follows from (b) as earlier $\lim_{z \to z_0} \left| \dfrac{g(z_0)}{(z-z_0)^N} \right| = +\infty$ $\quad g(z_0) \neq 0$

($\Leftarrow$) say $\lim_{z \to z_0} |f(z)| = +\infty$

$\Rightarrow \exists\ r>0$ s.t. $|f(z)| \geq 1$ in $0<|z-z_0|<r$

$\Rightarrow \frac{1}{f(z)}$ has an isolated singularity at $z=z_0$

& $\left|\frac{1}{f(z)}\right| \leq 1$ in $0<|z-z_0|<r$

$\Rightarrow$ This is a removable singularity (at z_0 of $\frac{1}{f(z)}$)

$\Rightarrow \exists$ analytic extn of $\frac{1}{f(z)}$, call it $H(z)$ analy in $|z-z_0|<r$

Note: $H(z_0) = \lim_{z\to z_0} H(z)$

$= \lim_{z\to z_0} \frac{1}{f(z)} = 0$ (by assump).

Thus, h has a zero at $z=z_0$!

$\Rightarrow$ (by 3rd char of poles)

$\Rightarrow z_0$ is a pole of f of degree the order of the zero at z_0 $\square$

$\hookrightarrow$ note H is not $\equiv 0$!

Def | let Ω be a domain.

A func f is called meromorphic on Ω if it's analytic except possibly some poles!

Thm | (Casorati Weierstrass)

let z_0 be an isolated singularity of f.

Then z_0 is an essential singularity

$\Longleftrightarrow$

$\forall\ \delta>0$, the img of $D_\delta(z_0)\setminus\{z_0\}$ under f is dense in $\mathbb{C}$

Pf) ($\Rightarrow$)

spse $\delta > 0$ so that $D_\delta(z_0) \setminus \{z_0\}$ under f is not dense in $\mathbb{C}$

so $\exists\, b \in \mathbb{C}$, $\varepsilon > 0$

so $|f(z) - b| \geq \varepsilon \quad \forall \; 0 < |z - z_0| < \delta$

$\hookrightarrow$ not dense around b

let $h(z) = \dfrac{1}{f(z) - b}$

It is analytic in $0 < |z - z_0| < \delta$

& also bdd as $|h(z)| \leq \frac{1}{\varepsilon}$

$\Rightarrow$ So z_0 is a removable singularity of h!

$\Rightarrow \exists\, H$ analytic in $|z - z_0| < \delta$

so $H(z) = \dfrac{1}{f(z) - b}$ in $0 < |z - z_0| < \delta$

$\hookrightarrow$ b/c never 0

Case 1 Assume $H(z_0) \neq 0$

$\Rightarrow f(z) = b + \dfrac{1}{H(z)} \Rightarrow f$ has analytic extn in $D_\delta(z_0)$

$\Rightarrow z_0$ is a removable sing of f

~~cop~~ !

Case 2 Assume $H(z_0) = 0$

Noting $H(z) \neq 0$ in $D_\delta(z_0) \setminus \{z_0\}$

$\exists N$ so $H(z) = (z - z_0)^N G(z)$ for anal G w $G(z_0) \neq 0$

$\therefore$