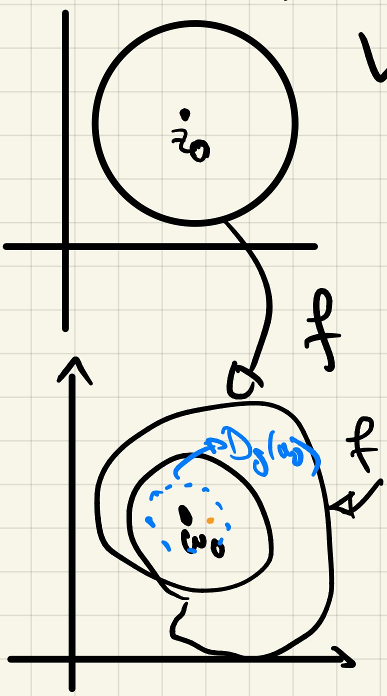
Recall Rouché.

small perturbation of f doesn't change # of zeros.

8.2 Open mapping Thm

Lemma 1 Suppose f is analytic in $\overline{D_r(z_0)}$, $f(z_0)=w_0$, $f(z)\neq w_0$ $\forall z\in\overline{D_r(z_0)}\setminus\{z_0\}$



Let $\delta = \operatorname{dist}(w_0, f(C_r(z_0)))>0$ by cmpct.

If $f(z)-w_0$ has zero of ord m at z_0

$\Longrightarrow$ $\forall w\in D_\delta(w_0)$, the eqn $f(z)=w$ has exactly m roots counted w/ multiplicity in $D_r(z_0)$

Pf Let $w\in D_\delta(w_0)$

let $F(z)=f(z)-w_0$ &

$G(z)=w_0-w$

For $z\in C_r(z_0)$, by choice of δ & $z\in C_r(z_0)$

$|F(z)|=|f(z)-w_0|\geq\delta>|w-w_0|=|G(z)|$ as $w\in D_\delta(w_0)$

By Rouché's thm,

zeros of $F(z)+G(z)=f(z)-w$

$\|$

zeros of $F(z)=m$ only 1 zero at z_0

Thm | (Open mapping Thm)

If f is analytic & nonconstant in Ω domain

$\Longrightarrow f(U)$ is open $\forall\, U$ open in Ω.

Pf| Let U be an open set in Ω & let $w_0 \in f(U)$ $z_0 \in U$ so $f(z_0) = w_0$

Since the zeros of analytic functions are isolated

$\exists\, r > 0$ s.t. $f(z) = w_0$ $\forall z$ s.t. $0 < |z - z_0| \leq r$

by lemma, & $D_r(z_0) \subseteq U$ by isolated zero of $f - w_0$.

$\exists\, \delta > 0$ s.t.,

$$D_\delta(w_0) \subseteq f(D_r(z_0)) \subseteq f(U)$$

$\therefore$ $f(U)$ is open

HW: prove MM princ using this (should be easy)

Thm | (Inverse func Thm).

Spse f is analytic on $\overline{D_r(z_0)}$, $f(z_0) = w_0$,

& $f'(z_0) \neq 0$, $f(z) \neq w_0$ $\forall z \in \overline{D_r(z_0)} \setminus \{z_0\}$

(so $f - w_0$ has simple zero at z_0)

let δ as before $\text{dist}(w_0, C_r(z_0)) > \delta$

let $U = f^{-1}(D_\delta(w_0)) \cap D_r(z_0)$ (open set contain z_0).

① $f|_U$ is a bijection from $U \to D_\delta(w_0)$.

② $f^{-1}(w) = \dfrac{1}{2\pi i} \displaystyle\oint_{C_r(z_0)} \dfrac{s f'(s)}{f(s) - w}\, ds$ $\forall w \in D_\delta(w_0)$

③ f^{-1} is analytic on $D_\delta(w_0)$.

Pf| Lemma $\Rightarrow$ f is a bijection for $U \to D_r(w_0)$

(1) $\hookrightarrow$ simple zero means only 1 preimage for each pt (w/ multiplicity)

(2) The map $s \longmapsto \dfrac{s f'(s)}{f(s) - w}$ has a simple pole at $s = f^{-1}(w)$

with residue,

$$\operatorname{Res}\left(\frac{s f'(s)}{f(s)-w}, s\right) = \frac{f^{-1}(w)\, f'(f^{-1}(w))}{f'(f^{-1}(w))}$$

derivative of bottom recall $\frac{f}{g} \to \frac{f}{g'}$

$$\Rightarrow \operatorname{Res}\left(\frac{s f'(s)}{f(s) - w}, s\right) = f^{-1}(w)$$

$\therefore$ Residue thm $\Rightarrow$ integral is $f^{-1}(w)$!

(3) Check. (Cauchy like or straight up).

Cor| If f is a bijective analytic function b/w open sets $A \to B$

$\Rightarrow f^{-1}$ is analytic.

Pf| Check bij $\Rightarrow f'(z) \neq 0 \ \forall z$ use prev.

HW

Recall • Ω is holomorphically simply conn

$\Longleftrightarrow$

$\forall$ domain $f: \Omega \to \mathbb{C}$ analytic have a analytic br of $\log f(z)$ on Ω

eg. $\Omega = \mathbb{C} \setminus \mathbb{R}$ hsc

subtle

let $f(z) = z^2$ analytic on Ω. $\to$ not well def in Ω.

use $\log(z^2) = 2\,\mathrm{Log}(z)$ not $\mathrm{Log}(z^2)$ doesn't work

• Discs are HSC!

Thm (Local Mapping)

Spse f is analytic in $\overline{D_r(z_0)}$, $f(z_0) = w_0$ & $f(z) \neq w_0$ for $\overline{D_r(z_0)} \setminus \{z_0\}$

Let $\delta = \mathrm{dist}(w_0, f(C_r(z_0))) > 0$

If $f(z) - w_0$ has a zero of ord M at z_0.

$\Rightarrow \forall\, w$ in $D_\delta(w_0) \setminus \{w_0\}$, the eqn $f(z) = w$ has exactly $\boxed{m \text{ distinct roots}}$. in $D_r(z_0) \setminus \{z_0\}$

Furthermore, choose these soln $z_1(w), \ldots, z_m(w)$ as analytic maps in $D_\delta(w_0) \setminus (w_0 - \delta, w_0]$

Pf $\exists\, h$ analytic in $D_r(z_0)$ s.t

$f(z) - w_0 = (z - z_0)^m h(z)$ & $h(z_0) \neq 0$

Note, $h(z) \neq 0$ $\forall\, z$ in $D_r(z_0)$ as $f(z)$ has no solution away from z_0.

Since h nonvanishing on disc, by Recall,

$h(z)^{1/m} = e^{1/m \log h(z)}$ can be chosen as analytic map on $D_r(z_0)$

For $w \in D_\delta(w_0) \setminus \{w_0\}$ the eqn $f(z) = w$ is the same as

$w - w_0 = (z - z_0)^n h(z) = g(z)^n$

Let $g(z) = (z - z_0) h(z)^{1/m}$

The soln are $z_k(w) = g^{-1}\left(e^{1/m \operatorname{Log}(w - w_0) + \frac{2\pi i k}{m}}\right)$

from inv func thm is bijective $\quad k = 0, 1, \dots, m-1$

so g^{-1} exists & is analytic ?, (Check)

D.

Def (Crit pt)

z_0 is a critical pt of an analytic func f if $f'(z_0) = 0$

It is called crit pt of ord n if $f'(z_0) = \dots = f^{(n)}(z_0) = 0$

& $f^{(n+1)}(z_0) \neq 0$

Let f be analytic, if z_0 is a non critical pt $\Rightarrow$ f is locally bijective w analytic inv (near z_0)

If z_0 is critical pt of order n

$\Rightarrow$ f is locally, $(n+1)$ to 1 map near z_0.