eg) $\sqrt{1-w^2}$ analytic away from $(-1,1)$

$$= e^{\frac{1}{2}\log(1-w)+\frac{1}{2}\log(1+w)}$$

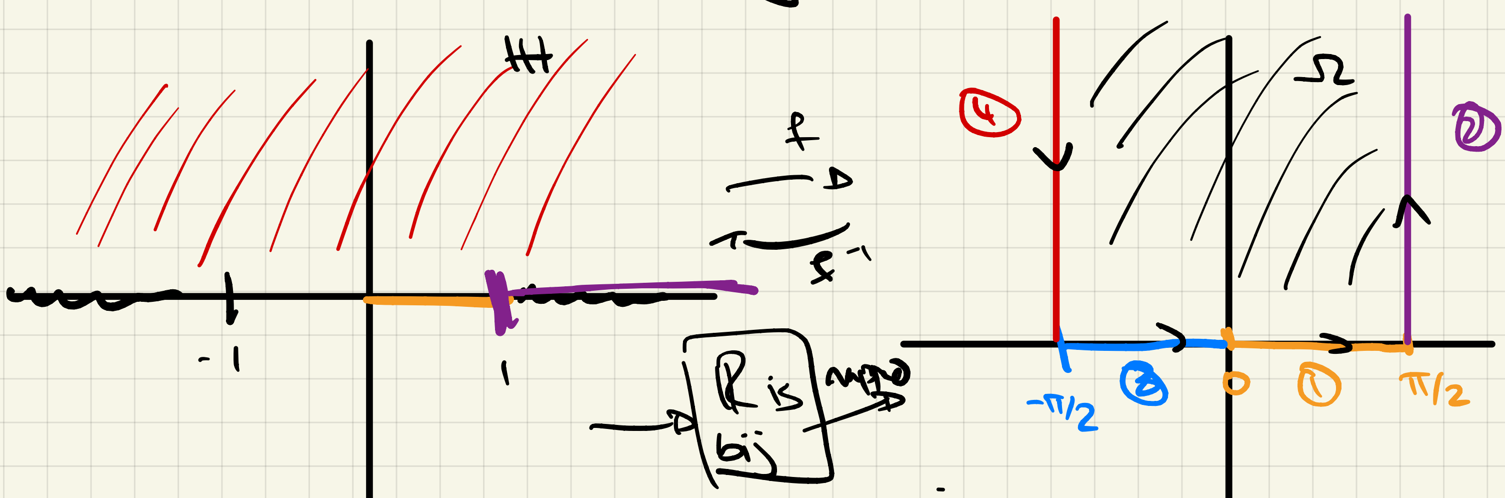
when $w = t \in \mathbb{R}+i0$ $\quad \log(1+t) = \begin{cases} \ln(1+t) & t>-1 \\ \ln|1+t|+\pi i, & t<-1 \end{cases}$

$$\log(1-t) = \begin{cases} \ln|1-t|-\pi i & t>1 \\ \ln(1-t) & t<1 \end{cases}$$

$$\Rightarrow \sqrt{1-w^2} = \begin{cases} i\sqrt{t^2-1} & t<-1 \\ \sqrt{1-t^2} & -1<t<1 \\ -i\sqrt{t^2-1} & t>1 \end{cases}$$

let $f(z) = \int_0^z \frac{1}{\sqrt{1-w^2}}\,dw$ for $z \in \mathbb{H}$ $\to$ upper half.

where the contour is any curve in $\mathbb{H}$ from $0 \to z$



① too map bij in this region

let $z = x \in \mathbb{R}$

1) $0<x<1$ $\quad f(x) = \int_0^x \frac{1}{\sqrt{1-t^2}}\,dt$ increasing fn.

$f(0) = 0 \qquad f(1) = \pi/2$

② $x > 1$

$$f(x) = \int_0^1 \frac{1}{\sqrt{1-t^2}}\,dt + \int_1^x \frac{1}{-i\sqrt{t^2-1}}\,dt$$

$$\pi/2 + i\int_1^x \frac{1}{\sqrt{t^2-1}}$$

Same increasing <u>rationale</u>

$$f(1) = \pi/2 \qquad f(\infty) = \pi/2 + i\infty$$

③ $-1 < x < 0$

$$f(x) = \int_0^x \frac{dt}{\sqrt{1-t^2}} = -\int_x^0 \frac{dt}{\sqrt{1-t^2}} \quad \text{(check)}$$

↳ <u>rationale</u>

④ $x < -1$

$$f(x) = \int_0^{-1} \frac{dt}{\sqrt{1-t^2}} + (-i)\int_{-i}^{x} \frac{dt}{\sqrt{t^2-1}} \;\ldots$$

We can <u>check</u>

$$\boxed{\begin{aligned} f^{-1} : \overline{\Omega} &\longrightarrow \overline{\mathbb{H}} \\ z &\longmapsto \sin z \end{aligned}}$$

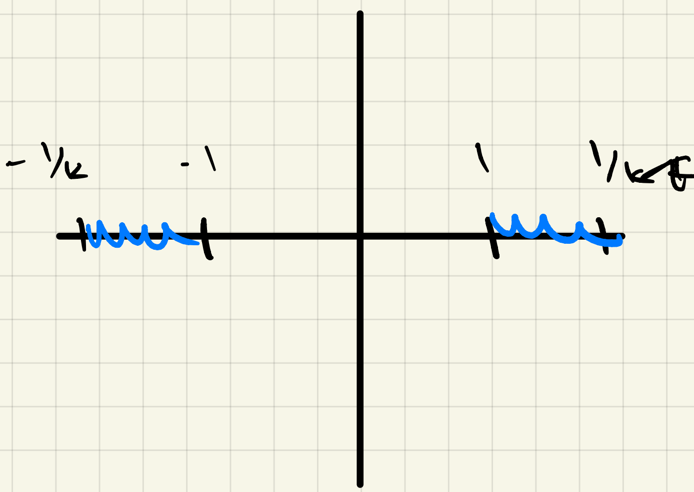
$$\mathbb{D} \xleftarrow{\frac{z-i}{z+i}} \mathbb{H} \xleftarrow{\sin z} \Omega$$

$\longrightarrow$ $\boxed{\text{conformal mapping}}$

eg) $0 < k < 1$

$$\sqrt{(1-w^2)(1-k^2w^2)} = e^{\frac{1}{2}\log(1-w) + \frac{1}{2}\log(1+w) + \frac{1}{2}\log(1-kw) + \frac{1}{2}\log(1+kw)}$$

$w = t \in \mathbb{R}$



analytic outside here.

$t > 1$

$w < -1$

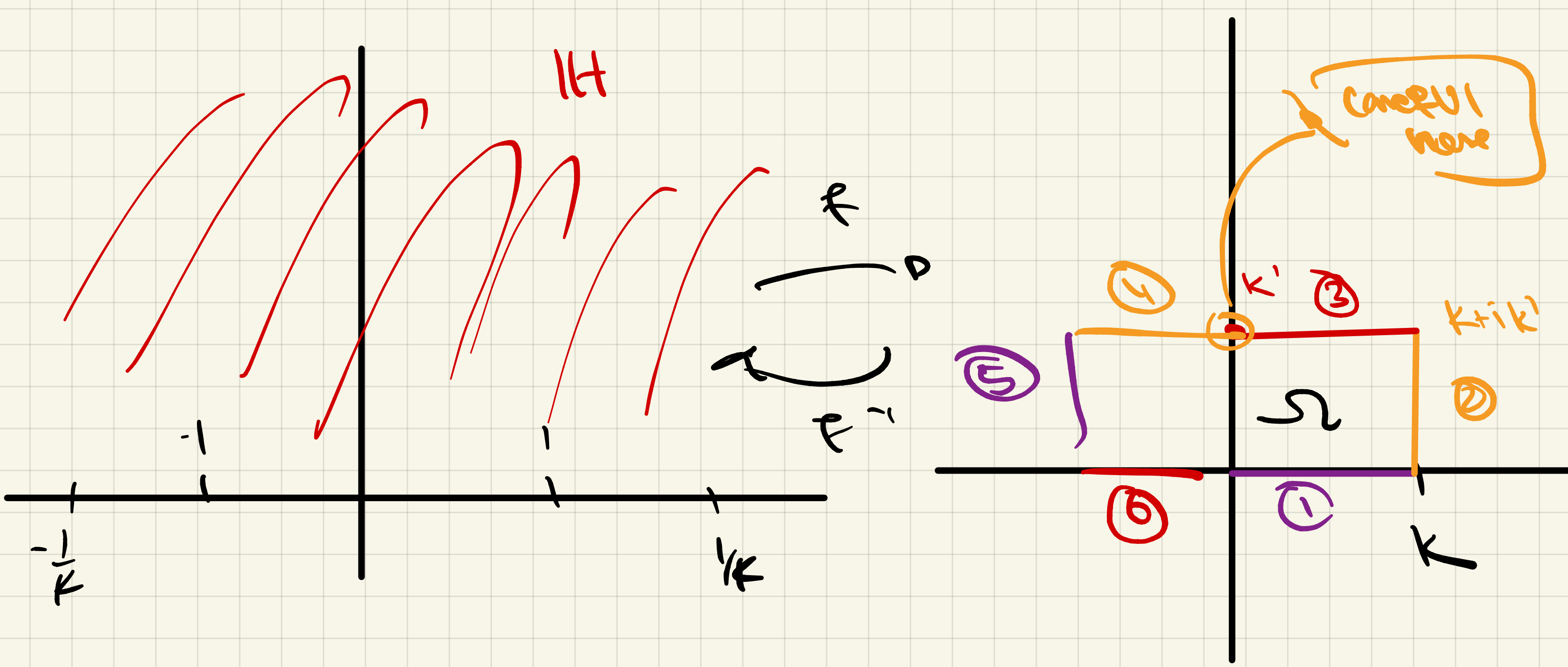
$w > \frac{1}{k}$

$w < -\frac{1}{k}$

for $w = t \in \mathbb{R} + i0$

$$\sqrt{(1-w^2)(1-k^2w^2)} = \begin{cases} -\sqrt{(t^2-1)(k^2t^2-1)} & t < -\frac{1}{k} \\ i\sqrt{(t^2-1)(1-k^2t^2)} & -\frac{1}{k} < t < -1 \\ \sqrt{(1-t^2)(1-k^2t^2)} & -1 < t < 1 \\ -i\sqrt{(t^2-1)(1-k^2t^2)} & 1 < t < \frac{1}{k} \\ -\sqrt{(t^2-1)(k^2t^2-1)} & \frac{1}{k} < t \end{cases}$$

Define $\quad f(z) = \int_0^z \frac{1}{\sqrt{(1-w^2)(1-k^2w^2)}}\, dw$



let $z = x \in \mathbb{R}$

① $0 < x < 1$ $\quad f(x) = \int_0^x \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$

$f(0) = 0$ $\quad$ denote $K(k) = \underline{f(1)} = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$

elliptic integral of first kind.

② $1 < x < \frac{1}{k}$

$$f(x) = K(k) + \int_1^x$$

$$= K(k) + i \int_1^x \frac{1}{\sqrt{(t^2-1)(1-k^2t^2)}} dt$$

not deriv

increasing remark to $\int_1^{1/k} \frac{1}{\sqrt{(t^2-1)(1-k^2t^2)}} dt = K'(k) < \infty$

check $K'(k) = K(\sqrt{1-k^2})$ cov.

③ for $x > \frac{1}{k}$

$$f(x) = K + iK' - \int_{1/k}^x \frac{1}{\sqrt{(t^2-1)(t^2k^2-1)}}$$

let $x \to \infty$ $\quad \to K$ $\quad$ by cov

④⑤⑥ symmetric

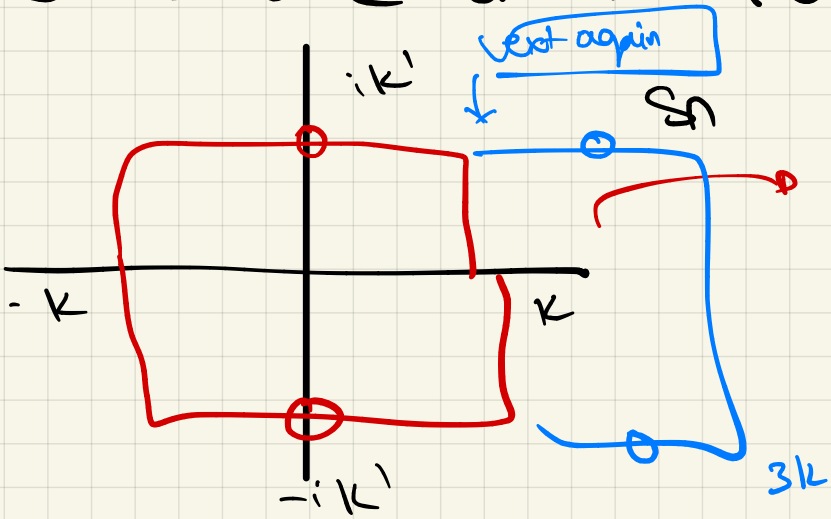
f is a conformal $\mathbb{H} \to \Omega$

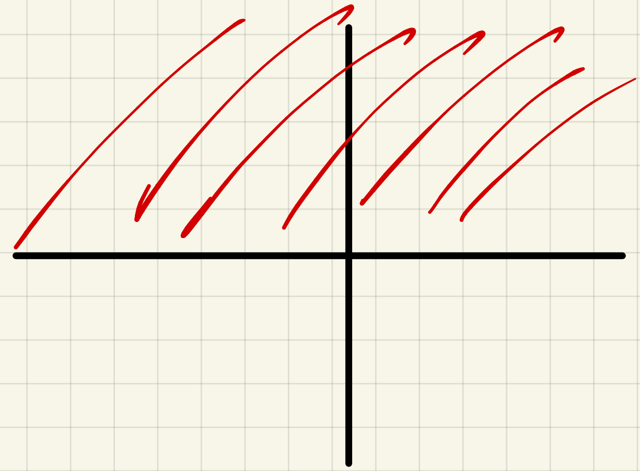
$f^{-1} : \Omega \to \mathbb{H}$ $\quad \therefore \frac{z-i}{z+i}$ comp w/ f^{-1}

$\hookrightarrow f^{-1}(z) = \boxed{sn(z)}$. $\quad \Omega \to D$

Jacobi elliptic map

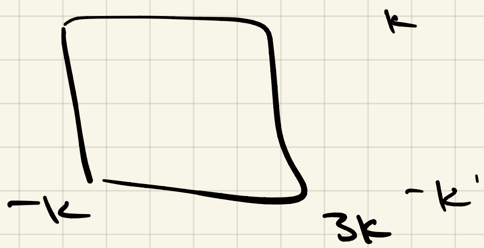
Can extend (Schwarz reflection)





$\therefore$ sn can be extended (keep reflect)

$\text{sn} : \mathbb{C} \setminus \{\text{dots}\} \to \mathbb{H}$



periodic

$$\boxed{\begin{aligned} \text{sn}(z+4K) &= \text{sn}(z) \\ \text{sn}(z+2iK') &= \text{sn}(z) \end{aligned}}$$

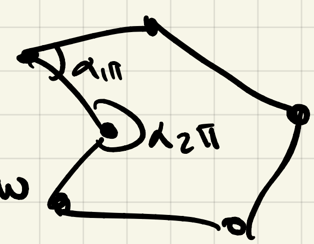
doubly periodic map

Liouville $\Rightarrow$ no nonconst doubly period entire func (must have singularities)

Thm (Schwarz-Christoffel formula)

Let Ω be a polygonal region w/ vertices $w_1, \ldots, w_n$ & inner angles $\alpha_1\pi, \ldots, \alpha_n\pi$ $(0 < \alpha_i < 2)$

Then $\exists$ conformal map from $\mathbb{H} \to \Omega$ of the form,



$$g(z) = A + B \int_0^z (w-x_1)^{\alpha_1 - 1} \cdots (w-x_n)^{\alpha_n - 1}\, dw$$

some $x_1, \ldots, x_n \in \mathbb{R}$ $\quad A, B \in \mathbb{C}$ $\quad$ where $g(x_i) = w_i$