9.8 Mittag-Leffler Thm

Lemma If f is analytic in a domain Ω & $a \in \Omega$ then $\forall N \in \mathbb{N}$ $\exists$ analytic map f_N on Ω s.t.

$$f(z) = \sum_{k=0}^{N-1} \frac{f^{(k)}(a)(z-a)^k}{k!} + f_N(z)(z-a)^N \quad \forall z \in \Omega$$

Pf Define $f_N(z) = \dfrac{f(z) - \sum_{k=0}^{N-1} f^{(k)}(a)\,(z-a)^k}{(z-a)^N}$ in $z \in \Omega \setminus \{a\}$

define, $f_N(a) = \dfrac{f^{(N)}(a)}{N!}$

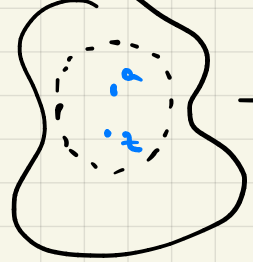
↳ this defn ensures that f_N is analytic on Ω

↳ map that is diff everywhere but a point where it is cts $\Rightarrow$ diff'ble at that point.

↳ Morera, defn of diff, pow series

Lemma The map f_n satisfies,

$$f_N(z) = \frac{1}{2\pi i} \oint_C \frac{f(w)}{(w-a)^N (w-z)}\, dw$$



Ω

$\forall$ circles C s.t. its interior disc D satisfies $a, z \in D$ & $\overline{D} \subset \Omega$

Pf

$$\oint_C \frac{f(w)}{(w-a)^N (w-z)} = \underbrace{\sum_{k=0}^{N-1} \frac{f^{(k)}(a)}{k!} \oint_C \frac{dw}{(w-a)^{N-k}(w-z)}}_{\to 0} + \underbrace{\oint \frac{f_n(w)\, dw}{w-z}}_{\text{Cauchy}}$$

So must show $\to 0$ $\qquad + 2\pi i\, f_n(z)$

$$\oint_C \frac{1}{(w-a)^m (w-z)} \text{ for } m \geq 1 \quad \boxed{\text{Cl is } 0}.$$

C is a circle cont. z, a

$\| 0$ → $\boxed{\text{increase the radius}}$!

Thm 1 (Mittag-Leffler)

Let $\{a_n\}_{n=1}^{\infty}$ be a seq of distinct complex numbers so $|a_n| \to \infty$ as $n \to \infty$

let $\{P_n(z)\}_{n=1}^{\infty}$ be a seq of polynomials.

Then $\exists$ a meromorphic function f which has poles at exactly a_n's & the principle part of the Laurent series at a_i being $P_i\left(\frac{1}{z-a_i}\right)$

Every such function is of the form.

$$f_n(z) = \sum_{n=1}^{\infty}\left(P_n\left(\frac{1}{z-a_n}\right) - h_n(z)\right) + g(z)$$

poly

entire.

Pf) It is enough to consider the case when a_n are non-zero!

Denote, $q_n(z) = P_n\left(\frac{1}{z-a_n}\right)$ is analytic in $\mathbb{C} \setminus \{a_n\} =: \Omega_n$

$\forall n$, let $M_n = \max\left\{ |q_n(z)| \,\middle|\, |z| = \frac{|a_n|}{2} \right\}$

and let N_n be a positive integer satisfying $2^{N_n} \geq M_n 2^n$

From the lemma, $\exists$ analytic r_n on $\Omega_n = \mathbb{C} \setminus \{a_n\}$

so that $q_n(z) = h_n(z) + r_n(z) z^{N_n}$ for $z \in \Omega_n$

where $h_n(z) = \sum_{k=0}^{N_n - 1} \frac{q_n^{(k)}(0)}{k!} z^k$ is a polynomial!

Define, $f(z) = \sum_{n=1}^{\infty} (q_n(z) - h_n(z))$

We will show that this conv to meromorphic funct w/ desired prop!

Let $R > 0$. Since $|a_n| \to \infty$ $\exists n_0$ so $|a_n| > 4R$ $\forall n \geq n_0$.

For $n \geq n_0$ & $|z| \leq R$

$$r_n(z) = \frac{1}{2\pi i} \oint_{|w| = \frac{|a_n|}{2}} \frac{q_n(w)}{w^{N_n}(w-z)} dw$$

Circle has $z, 0$ & in domain where map is analytic $\mathbb{C} \setminus \{a_n\}$

$\Rightarrow \quad n \ge n_0 \quad , \quad |z| \le R$

$$|q_n(z) - h_n(z)| = |r_n(z)\, z^{N_n}| \le \overbrace{\frac{1}{\cancel{2\pi}} \frac{M_n}{\left(\frac{a_n}{2}\right)^{N_n} R}\, 2\pi \frac{|a_n|}{2}}^{r_n(z)} \overbrace{R^{N_n}}^{|z^{N_n}|}$$

$$= \frac{M_n\, 2^{N_n-1}}{|a_n|^{N_n-1}} R^{N_n-1} = M_n \underbrace{\left(\frac{2R}{|a_n|}\right)^{N_n-1}}_{\to\ |a_n| > 4R}$$

$$\le \frac{M_n}{2^{N_n-1}} \underbrace{\le}_{\text{Choice of } N_n} \frac{2}{2^n}$$

For $|z| < R$ $\quad f(z) = \sum_{n=1}^{n_0-1} (q_n(z) - h_n(z))$

$$+ \sum_{n=n_0}^{\infty} (q_n(z) - h_n(z)) \Bigg] \text{ analytic } \underline{|z|<R}.$$

conv is unif

By Weierstrass, conv to anal in $|z|<R$

... (check notes) R arb always true ∎

10 two special functs!

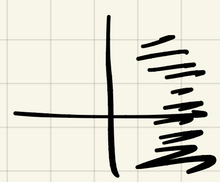
10.1 Gamma Funct

Def) The gamma func is

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt \quad \text{for } \operatorname{Re}(z) > 0 \quad \text{when conv?}$$

$$|e^{-t} t^{z-1}| = e^{-t}\, t^{\operatorname{Re} z - 1}$$

Lemma) $\Gamma(z)$ is analytic in $\operatorname{Re} z > 0$



Pf) hw 3 A7

Lemma 1 $\operatorname{Re} z > 0$ $\Gamma(z+1) = z\Gamma(z)$ for $n = 1, 2, \ldots$ $\Gamma(n) = (n-1)!$

Pf Int by parts!

Thm 1 The gamma fnc extends to a meromorphic fnc on $\mathbb{C}$ w/ simple poles $z = 0, -1, -2, \ldots$ & res $z = -n$ is $\frac{(-1)^n}{n!}$

Pf Note: $\Gamma(z) = \frac{\Gamma(z+1)}{z}$ for $\operatorname{Re}(z) > 0$

Define $F_1(z) = \frac{\Gamma(z+1)}{z}$ for $-1 < \operatorname{Re} z$, $z \neq 0$ (analytic)

For $\operatorname{Re} z > 0$ $\boxed{F_1(z) = \Gamma(z)}$

$\therefore$ F_1 is the analytic extn.

Def $F_2(z) = \frac{\Gamma(z+2)}{(z+1)(z)} = \frac{F_1(z+1)}{(z+1)} \ldots$

analytic on $\operatorname{Re} z > -2$

$F_n(z) = \frac{F_{n-1}(z+1)}{(z+n-1)} = \frac{\Gamma(z+n)}{(z+n-1)\cdots(z+1)}$ $\operatorname{Re} z > -m$, $z \neq -m+1, \ldots, 0$

Note F_n agrees w/ F_{n-1} on $\operatorname{Re} z > -m+1$

$\therefore$ can keep going.

$\operatorname{Res}_0 F_1 = \Gamma(1) = 1$ check rest of residues for $z = -1, \ldots,$

The prop $\Gamma(z+1) = z\Gamma(z+1)$ exist