**Thm** Let $\Omega$ be a domain TFAE

(a) $\Omega$ is holo s.c (i.e. $\omega_\gamma(a) = 0$ $\forall$ cycle $\gamma$ in $\Omega$ $\forall$ $a \notin \Omega$)

(f) Every conn. comp of $\mathbb{C} \setminus \Omega$ is unbdd.

have left (a $\Rightarrow$ f)

Spse f not true. $\exists$ bdd conn comp of $\mathbb{C} \setminus \Omega$. Call it A.

Let, $B = \mathbb{C} \setminus (\Omega \cup A)$ so that $\mathbb{C} \setminus \Omega = A \cup B$ dsj union.

[B not nec conn]

Since $\mathbb{C} \setminus \Omega$ closed & A, B dsj conn

$\Rightarrow$ A, B are closed sets

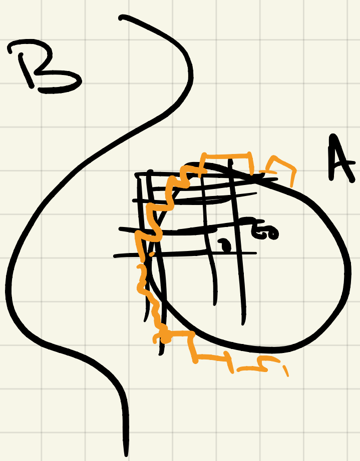
In part, A is cmpct. & B is closed.

$\Rightarrow$ dist $(A, B) > 0$.

will make small

Fix $z_0 \in A$ & cover $\mathbb{C}$ by a net of sq of side length $\delta$

s.t $z_0$ in the middle of a square.




Let $Q_j$, $j \in J$ be the closed solid sq that intersect A. ($J$ finite)

Define $\gamma = \sum_{j \in J} \partial Q_j$ b/c of cancellation $\gamma \cap A = \emptyset$.

if $\delta$ small enough (since dist $(A, B) > 0$)

$\Rightarrow$ so $\gamma \subset \Omega$ entirely (doesn't meet B).

Now $\omega_\gamma(z_0) = \sum_{j \in J} \omega_{\partial Q_j}(z_0) = 1$.

$\therefore$ $\Omega$ not holo sc. done by contra. $\square$

# 5.5 Homotopy deformation & topologically simply connected domains

**Def** (homotopic curves)

Two curves $\gamma_0(t)$, $\gamma_1(t)$ with $a \le t \le b$ in a domain with same starting & endpoints.

They are homotopic to each other if

$\exists\, \Gamma : [0,1]\times[a,b] \to \Omega$ cts — homotopy

so $\Gamma|_{0\times[a,b]} = \gamma_0$ & $F(s,a) = \gamma_0(a) = \gamma_1(a)$

$\Gamma|_{1\times[a,b]} = \gamma_1$ $\quad F(s,b) = \gamma_0(b) = \gamma_1(b)$

& $\boxed{\Gamma(s,t) = \gamma_s(t) \text{ a curve.}}$

**Thm** (homotopy invariance)

If $\gamma_0(t)$ & $\gamma_1(t)$ $a \le t \le b$ are htpic curves in $\Omega$ dom.

$\Longrightarrow \displaystyle\int_{\gamma_0} f(z)\,dz = \int_{\gamma_1} f(z)\,dz$ $\forall$ analytic $f$ on $\Omega$

**Pf)** Since $K = \Gamma([0,1]\times[a,b])$ is compact & contained in open $\Omega$

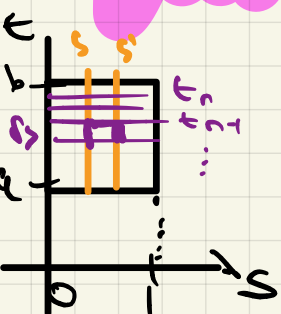
$\exists\, \varepsilon > 0$ so $D_\varepsilon(w) \subset \Omega$, $\forall w \in K$.

Since $\Gamma$ is unif cts, $|\Gamma(s,t) - \Gamma(s',t')| < \varepsilon$

$\forall (s,t), (s',t') \in [0,1]\times[a,b]$ satisfying:

$|s-s'| + |t-t'| < \delta$.

$L_\infty$ norm, but its equiv

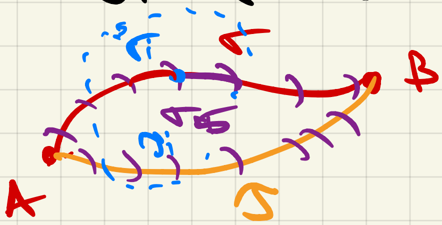



Fix 2 pts $0 \le s < s' \le 1$ so $s' - s < \delta$.

denote $\sigma = \gamma_s$ & $\tau = \gamma_{s'}$

Sts $\displaystyle\int_\sigma f\,dz = \int_\tau f\,dz$ (propagates upward) let $a = t_0 < t_1 < \dots < t_n = b$ be part so

$t_{i+1} - t_i < \delta$



let $\sigma_i = \text{im}\, \sigma|_{[t_{i-1}, t_i]}$

$\gamma_i$ similarly

$\sigma = \sigma_1 + \dots + \sigma_n$ & $\gamma = \gamma_1 + \dots + \gamma_n$

$$\int_\sigma f\,dz = \sum_i \int_{\sigma_i} f\,dz$$

let $D_i = D_\varepsilon(\sigma(t_i))$ by choice of $\varepsilon$,

$\Rightarrow \sigma_i \subseteq D_i$ & $\gamma_i \subseteq D_i$ (just break down choices of $\varepsilon, \delta$)

let $L_i$ be line seg from $\sigma(t_i) \to \gamma(t_i)$

$\Rightarrow L_{i-1}, L_i \subset D_i$

by the Cauchy thm for disc,

$$\int_{\sigma_i} f = \int_{L_{i-1}} f + \int_{\gamma_i} f - \int_{L_i} f$$

rev triangle

Adding up $i = 1 \dots n$ the sum telescope

noting $\sigma, \gamma$ same st & endpt $\Rightarrow \underline{\int_\sigma f = \int_\gamma f}$

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**e.g.1** $C_1 = C_{1000}(0)$, $C_2 = C_{2000}(0)$

$C_2 - C_1$ cycle

$$f(z) = \frac{z^{10} + 2z^7 + 6z^4 + 1}{z^{12} + 5z^6 + 1}$$

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

3 ways

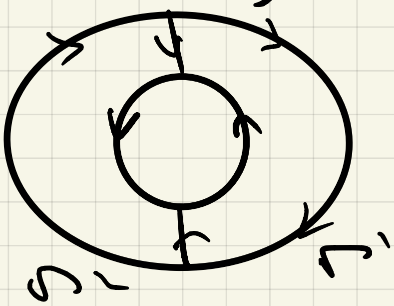
1) general Cauchy

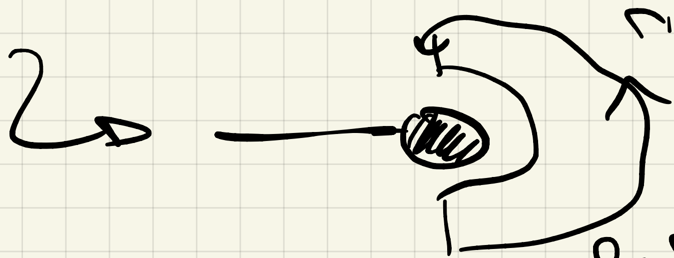
2) Cauchy thm for holo s.c domains + cut annulus down middle

3) homotopy inv

② Cauchy thm for holo sc. dom $\Omega = |z| > 10$

$C_2 - C_1 = \Gamma_1 + \Gamma_2$





$\Omega_1 = \Omega \setminus (-\infty, 0]$

Co Hs C

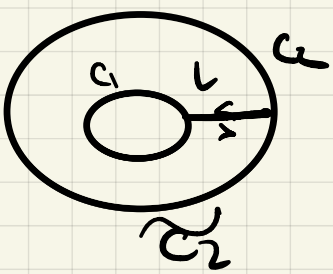
& $\Gamma'$ closed

$\Rightarrow \boxed{\int_{\Gamma'} f\, dz = 0}$

③ Homotopic invariance thm

$\Omega = \{z \mid |z| > 10\}$

$C_1 = C_{1000}$ & $\tilde{C}_2$ are homotopic in $\Omega$



Show $C_1$ htpc to $C_2 + L - L = \tilde{C}_2$

$\therefore \underline{\int_{C_1} f} = \int_{\tilde{C}_2} f = \underline{\int_{C_2} f}$.

**Def)** A domain $\Omega$ is topologically simply connected if every closed curve $\gamma(t)$, $a \le t \le b$ in $\Omega$ is htpic to the trivial const curve $z_0 = \gamma(a) = \gamma(b)$

**Thm)** (Top sc $\Rightarrow$ holo sc)

If $\Omega$ top sc $\Rightarrow$ holo sc

**Pf)** $\oint_\gamma f = 0$ sts for $\gamma$ closed curve

$\gamma \sim pt$ (top sc) $\rightarrow$ $\boxed{\oint_{pt} f = \oint_\gamma f}$ ($\oint_{pt} f = 0$) htpic invariance

Fact: $\boxed{\text{hsc} \Rightarrow \text{top sc}}$ // $\to$ $\boxed{\text{pf later}}$ Riemann mapping Thm.

## 5.6 Jordan Curve Thm

no self int except at end.

**Def** A jordan curve is a simple closed continuous curve

**Thm** (Jordan Curve Thm)

Let $\Gamma$ be a Jordan curve

$\Rightarrow$ $\mathbb{C} \setminus \Gamma$ consists of two connected components.

one of which is unbdd (exterior)

& the other is bdd (interior) & simply connected (top).

Furthermore, if $\Gamma$ is piecewise $C^1$, smooth, J curve

$$\Rightarrow W_\Gamma(a) = \begin{cases} 0 & a \in \text{ext}\,\Gamma \\ \pm 1 & a \in \text{int}\,\Gamma \end{cases}$$