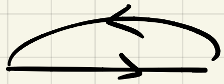
Residue

① $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx$



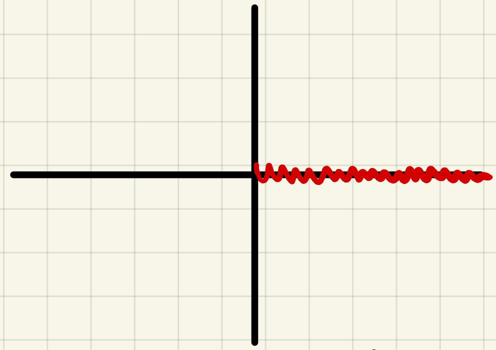
② $\displaystyle\int_{0}^{\infty} \frac{1}{x^4+1}\,dx$

③ $\displaystyle\int_{0}^{2\pi} \frac{1}{a+\cos\theta}\,d\theta,\ a>1$

$\hookrightarrow d\theta = \dfrac{dz}{iz}$ on unit ⊖

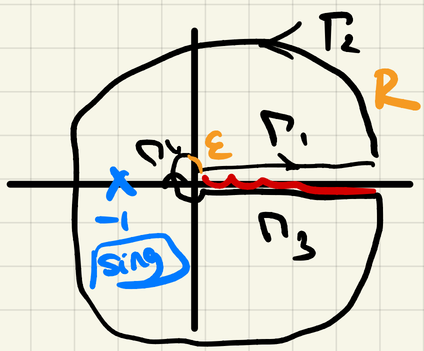
④ $0<\alpha<1$

$\displaystyle\int_0^{\infty} \frac{x^{\alpha}}{(x+1)^2}\,dx \qquad f(z) = \frac{z^{\alpha}}{(z+1)^2}$ where we take branch of z^{α} so that



so $z^{\alpha} = r^{\alpha} e^{i\theta\alpha}$, $0<\theta<2\pi$.

Contour



$$\int_{\Gamma_1} f(z)\,dz \xrightarrow{z=x+i\delta} \int_0^{\infty} \frac{x^{\alpha}}{(x+1)^2}\,dx \quad \text{as } \varepsilon \to 0$$

$$\int_{\Gamma_3} f(z)\,dz = -\int_{-\Gamma_3} f(z)\,dz \xrightarrow{z=x-i\delta} -\int_0^{\infty} \frac{x^{\alpha} e^{2\pi i\alpha}}{(x+1)^2}\,dx$$

$$\left|\int_{\Gamma_2} f(z)\,dz\right| \le \frac{R^{\alpha}}{(R-1)^2} R\pi 2 \longrightarrow 0 \quad \text{as } R\to\infty \text{ as } 0<\alpha<1.$$

$$\left|\int_{\Gamma_4} f(z)\,dz\right| \le \underbrace{\frac{\varepsilon^{\alpha}}{1-\varepsilon}}_{\text{rev tri}} \cdot 2\pi\varepsilon \longrightarrow 0 \quad \text{as } \varepsilon\to 0 \text{ as } 0<\alpha<1.$$

Thus, $\int_0^\infty \frac{x^\alpha}{(x+1)^2}\,dx - e^{2\pi i\alpha}\int_0^\infty \frac{x^\alpha}{(x+1)^2}\,dx = 2\pi i \operatorname{Res}_{-1} f$

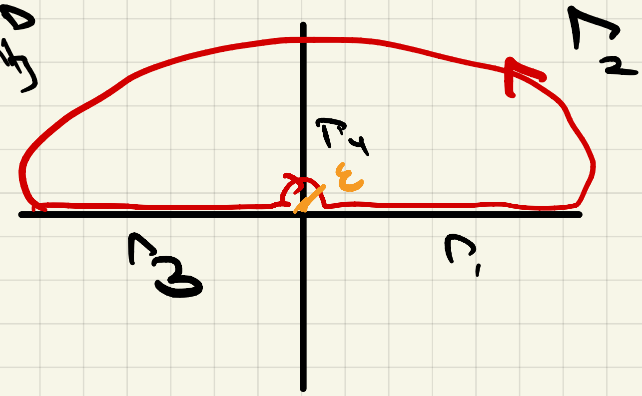
$\operatorname{Res}_{-1} f = \alpha\left(e^{\pi i}\right)^{\alpha-1}$ as double pole

So, $\int_0^\infty \frac{x^\alpha}{(x+1)^2}\,dx = \frac{2\pi i\alpha\, e^{\pi i\alpha}(-1)}{1-e^{2\pi i\alpha}} = \frac{-2\pi i\alpha}{e^{-\pi i\alpha}-e^{\pi i\alpha}}$

$= \frac{\pi\alpha}{\sin(\pi\alpha)}$.

5) $\int_0^\infty \frac{\sin x}{x}\,dx = \lim_{R\to\infty}\int_0^R \frac{\sin x}{x}\,dx$ (converges why?)

$f(z) = \frac{e^{iz}}{z}$ → no singularity ∴ residue 0



$\int_{\Gamma_1} f(z)\,dz \xrightarrow{z=x} \int_\varepsilon^R \frac{e^{ix}}{x}\,dx$

$\int_{\Gamma_3} f(z)\,dz \xrightarrow{z=-x} \int_{-\Gamma_3} f(x)\,dz \overset{T}{=} -\int_\varepsilon^R \frac{e^{-ix}}{x}\,dx$

$\Longrightarrow$ adding both $2i\int_\varepsilon^R \frac{\sin(x)}{x}\,dx + \int_{\Gamma_2} f(z)\,dz + \int_{\Gamma_4} f(z)\,dz = 0$

$\int_{\Gamma_4} f(z)\,dz = -\int_{-\Gamma_4}\frac{e^{iz}}{z}\,dz = -\int_{-\Gamma_4}\frac{e^{iz}-1}{z}\,dz - \int_{-\Gamma_4}\frac{1}{z}\,dz$

($\varepsilon \to 0$) this is removable sing. at $z=0$ so? $\qquad = \int_0^\pi \frac{1}{\varepsilon e^{it}}\, i\varepsilon e^{it}\,dt$

$\to 0 \qquad \boxed{-\pi i} \qquad$ '' πi

$$\left|\int_{\Gamma_2} \frac{e^{iz}}{z}\,dz\right| = \left|\int_0^{\pi} \frac{e^{iRe^{it}}}{\cancel{Re^{it}}}\, i\cancel{Re^{it}}\,dt\right|$$

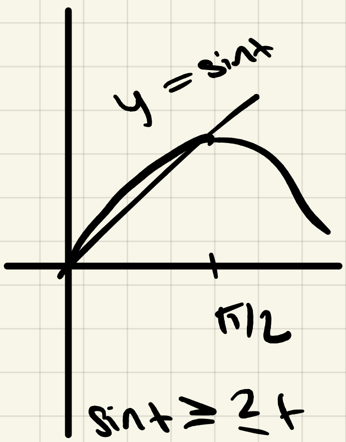
→ ML won't work
doesn't decay
→ gives π.

$$= \left|\int_0^{\pi} e^{iR(\cos t + i\sin t)}\,dt\right|$$

$$\leq \int_0^{\pi} \left|e^{iR\cos t - R\sin t}\right|\,dt$$

$$= \int_0^{\pi} e^{-R\sin t}\,dt$$

$$= 2\int_0^{\pi/2} e^{-R\sin t}\,dt$$



$\sin t \geq \frac{2}{\pi} t$

for $0 \leq t \leq \frac{\pi}{2}$

$$\leq 2\int_0^{\pi/2} e^{-\frac{2R}{\pi}t}\,dt$$

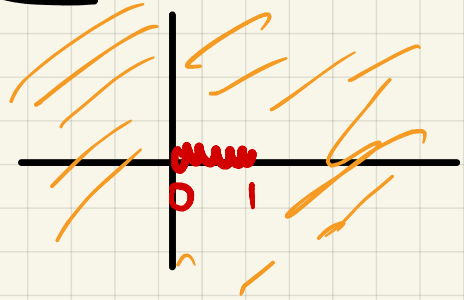
$$= -2\,\frac{\pi}{2R}\, e^{-\frac{2\pi}{\pi}t}\Big|_0^{\pi/2}$$

$$= -\frac{\pi}{\boxed{R}}\left(e^{-R} - 1\right) \longrightarrow R \to \infty \quad \boxed{\to 0}$$

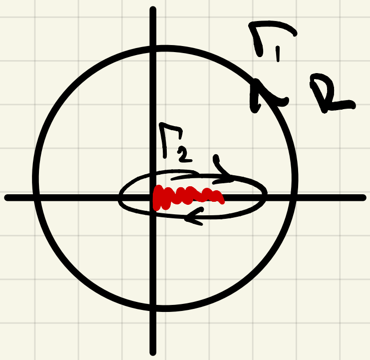
$$\partial 0 \longrightarrow \int_0^{\infty} \frac{\sin x}{x}\,dx = \frac{\pi i}{2i} = \frac{\pi}{2} \quad \text{as} \; \boxed{\text{residue is } 0}.$$

⑥ $\int_0^1 \frac{1}{\sqrt{x(1-x)}}\,dx$

let $f(z) = \frac{1}{\sqrt{z(1-z)}}$ with branch that is analytic in



Then, we contour



$\int_{\Gamma_1} + \int_{\Gamma_2} = 0$ by generalized Cauchy thm as $\Gamma_1 + \Gamma_2 \sim 0$.

$\int_{\Gamma_2} f(z)\,dz \to \int_0^1 f_+(x)\,dx - \int_0^1 f_-(x)\,dx$

$\lim_{\epsilon \to 0} f(x - i\epsilon)$

similar but plus.

$\boxed{f_+(x) = -f_-(x)}$ for $0<x<1$ by branching idea

So $\int_{\Gamma_2} f(z)\,dz \to 2\int_0^1 f_+(x)\,dx$

$\int_{\Gamma_1} f(z)\,dz = \underbrace{\int_{\Gamma_1}\left(\frac{1}{\sqrt{z(1-z)}} - \frac{1}{z}\right)dz}_{0 \text{ as } \boxed{R\to\infty}} + \overbrace{\int_{\Gamma_1} \frac{1}{z}\,dz}^{2\pi i}$

kee $\frac{z''}{z^{12}}$ is

$= \boxed{2\pi i}$ check

$$\Rightarrow \boxed{2\int_0^1 f_+(x)\,dx + 2\pi i = 0}$$

$\Downarrow$

$\frac{-i}{\sqrt{(x)(1-x)}}$ check

So, ans is 2π

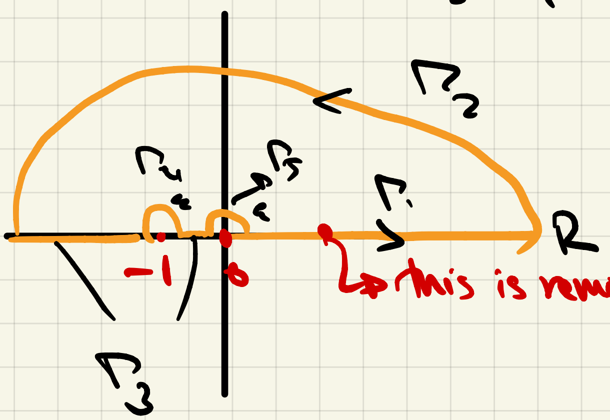
7 $\displaystyle\int_0^\infty \frac{\log x}{x^2-1}\,dx$ $\rightarrow$ note $\displaystyle\lim_{x\to 1}\frac{\log x}{x-1} = \lim \frac{\frac{1}{x}}{1} = 1$

so, 1 not a singularity

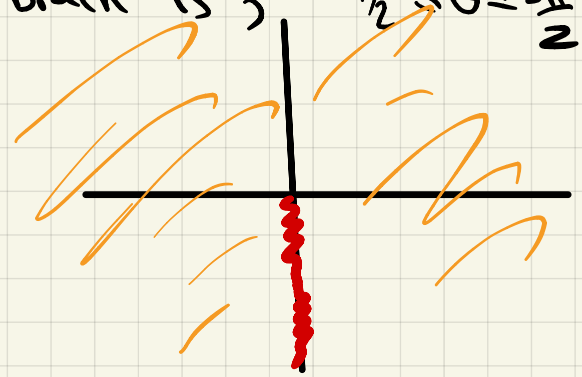
↳ converge at 0 & ∞ mainly at 0 eg

$$\int_0^{\delta} \log x = \underbrace{x\log x}_{\to 0} - x \Big|_0^{\delta} = \ldots \quad (x \to 0)$$

↳ $f(z) = \dfrac{\log z}{z^2-1}$ where branch is, $-\frac{\pi}{2} < \theta < \frac{3\pi}{2}$



↳ this is removable so g



$|\log z| = |\ln R + i\theta|$

$$\left|\int_{\Gamma_2} f(z)\,dz\right| \leq \frac{\ln R + \pi}{R^2-1}\,\pi R \longrightarrow 0 \text{ as } R \to \infty$$

$$\left|\int_{\Gamma_5} f(z)\,dz\right| \leq \frac{|\ln \varepsilon| + \pi}{1-\varepsilon^2}\,\pi\varepsilon \longrightarrow 0 \text{ as } \varepsilon \to 0$$

$$\int_{\Gamma_4} f(z)\,dz =$$