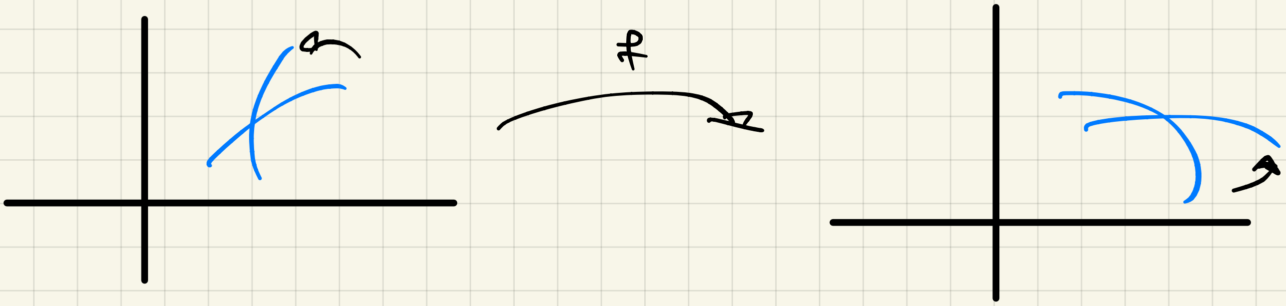
Recall

C^1-func

if $f(z) = u(x,y) + iv(x,y)$ is conformal at $z_0 = x_0 + iy_0$

if



$f: U \to V$ conformal map if f conv everywhere & bijective!

Thm if f is analytic at z_0 & $f'(z_0) \neq 0$ then f is conf at z_0.

Pf $f'(z_0) = \rho e^{i\varphi}$ w/ $\rho \neq 0$ as

For curve $\gamma(t)$ w/ $\gamma(0) = z_0$ The curve $\eta(t) = f \circ \gamma(t)$ satisfies $\eta'(0) = \gamma'(0) \cdot f'(z_0) = \rho e^{i\varphi} \gamma'(0)$

$\therefore$ f perturbs γ by rotation by φ $\to$ this doesn't dep on γ just f.

Thm let $f(x+iy) = u(x,y) + iv(x,y)$ be conf w/ C^1 in open set U

& $(u_x, u_y, v_x, v_y) \neq (0,0,0,0)$ always

If f is conf in U, then f is analytic in U.

& $f'(z_0) \neq 0$ $\forall z_0 \in U$

Pf Will use CR.

let $z_0 \in U$ given & θ consider the straight line

$\gamma(t) = z_0 + te^{i\theta} = (x_0 + t\cos\theta, y_0 + t\sin\theta)$

let $\eta(t) = f(\gamma(t)) = u(x_0 + t\cos\theta, y_0 + t\sin\theta) +$
$iv(\longrightarrow)$

$\gamma'(0) = e^{i\theta}$

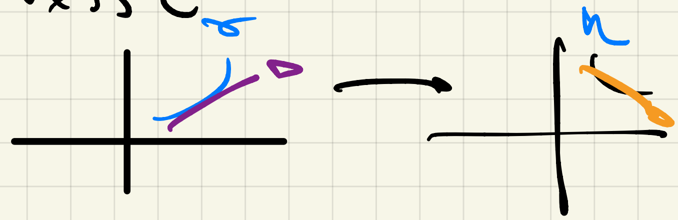
$\Rightarrow \eta'(0) = u_x(x_0, y_0)\cos\theta + u_y(x_0, y_0)\sin\theta$
$i(v_x(x_0, y_0)\cos\theta + v_y(x_0, y_0)\sin\theta)$

$\boxed{\text{alg}} \quad = \frac{1}{2}\left(u_x + v_y + i(-u_y + v_x)\right)e^{i\theta} \to$

cont. $+\frac{1}{2}\left(u_x - v_y + i(u_y+v_x)\right)e^{-i\theta}$

To ensure that conformal,

$\eta = \gamma e^{i\varphi_0}$



$$\Rightarrow \frac{\eta'(0)}{\gamma'(0)} = \frac{1}{2}\left(u_x + v_y + i(-u_y+v_x)\right) + \frac{1}{2}\left(u_x - v_y + i(u_y+v_x)\right)e^{-2i\theta}$$

↳ angle shouldn't dep on θ

$\Rightarrow$ req to be zero

$\Rightarrow \boxed{u_x = v_y \ \& \ u_y = -v_x}$ (CR)

$\Rightarrow$ f is analytic in U

S: f is conformal in U $\Leftrightarrow$ f is analytic w/ nonzero der in U

↳ local injectiveness → doesn't $\not\Rightarrow$ global inj

Conformal map from $U \to V$ $\Leftrightarrow$ $\begin{cases} \bullet \text{ analytic on } U & \bullet \text{ inj} \\ \bullet\ f' \neq 0 \text{ on } U & \bullet \text{ surj} \end{cases}$

e.g. $f(z) = z^2 : \mathbb{C}_+ \xrightarrow{\text{strict}} \mathbb{C}\setminus[0,\infty)$

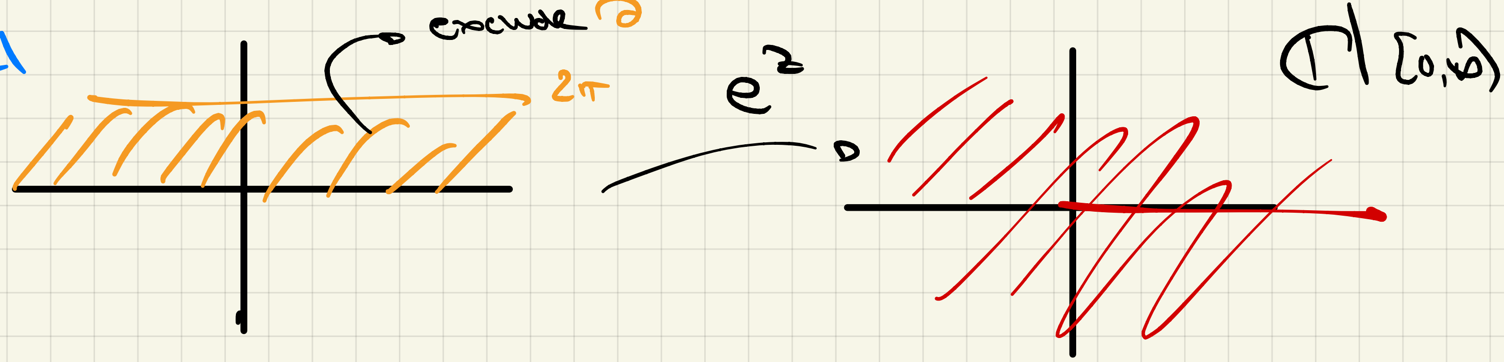
is a conformal map

BUT considering $\mathbb{C}\setminus\{0\} \to \mathbb{C}\setminus\{0\}$ it is

NOT a conformal map but it is conformal on $\mathbb{C}\setminus\{0\}$

confusing terminology

eg1



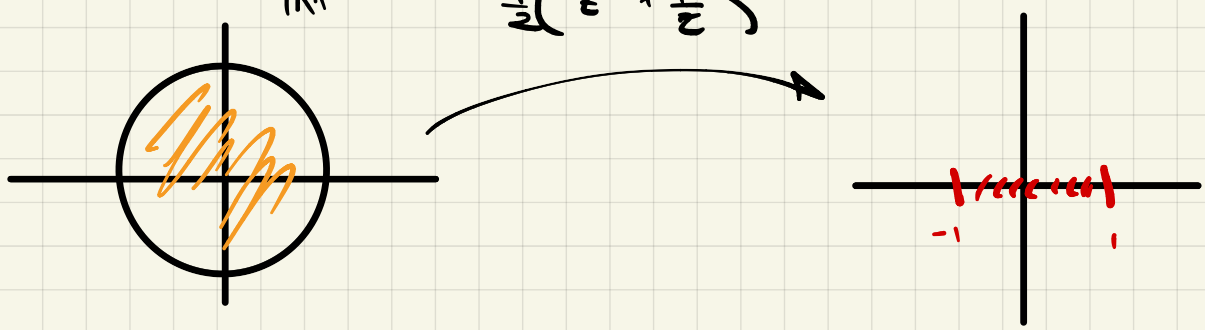
is a conformal map!

$e^z = e^x(\cos y + i\sin y)$

Joukowski map

int $\quad \frac{1}{2}\left(z + \frac{1}{z}\right)$

eg1



11.2 Fractional Linear Transformations

$f(z) = \dfrac{az+b}{cz+d}$ with $ad - bc \neq 0$

$f'(z) = \dfrac{ad-bc}{(cz+d)^2} \neq 0$ (conformal at all pts where def)

↳ conformal map on pts as global 1-1

e.g $\boxed{f(z) = \dfrac{z-i}{z+i}}$ conformal map

↳ sends $\mathbb{R}$ to unit circle bijectively,

↳ i to origin.

↳ sends $\mathbb{C}_+ \to \mathbb{D}$, interior

$f^{-1} = \dfrac{\alpha z + \beta}{\gamma z + \delta}$ where $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1}$

$\exists!$ frac linear transf $z_1, z_2, z_3 \mapsto w_1, w_2, w_3$

$$\frac{(w-w_1)(w_3-w_2)}{(w-w_2)(w_3-w_1)} = \frac{(z-z_1)(z_3-z_2)}{(z-z_2)(z_3-1)}$$

↳ set & solve for w

11.3 Automorphisms of unit disc

let $\mathbb{D} = D_1(0)$ open.

Schwarz lemma)

if $f: \mathbb{D} \to \mathbb{D}$ analytic & $f(0) = 0$

$\Rightarrow |f(z)| \leq |z| \quad \forall z \in \mathbb{D}$

& $|f'(0)| \leq 1$

equality in either of above holds $\iff$ $f(z)$ rotation!

Automorphism on Ω is a conformal map $\Omega \to \Omega$

$\mathrm{Aut}(\Omega)$ is a grp under comp!

→ Suppose that such an f was bijective (so conformal bijection $\mathbb{D} \to \mathbb{D}$)

$\Rightarrow |f(z)| = |z| \Rightarrow f$ is a rotation

↳ so the automorphisms of $\mathbb{C}$ fixing origin is rotation

If it doesn't do that (fix origin)

so f automorphism of $\mathbb{D}$ (consider hw prob)

$g(w) = \text{FLT} \Rightarrow (f \circ g)(z) = e^{i\phi} z$

$$\Rightarrow \mathrm{Aut}(\mathbb{D}) = \left\{ f(z) = e^{i\theta} \frac{z - \alpha}{1 - \bar{\alpha} z} \;\middle|\; \alpha \in \mathbb{D},\ \theta \in \mathbb{R} \right\}$$

let $\mathbb{H} = \mathbb{C}_+ = \{x + iy \mid y > 0\}$

$\mathrm{Aut}(\mathbb{H}) \ni f$

$$\begin{tikzcd} \mathbb{H} \arrow[r, "f"] & \mathbb{H} \arrow[d, "g"] \\ \mathbb{D} \arrow[u, "g^{-1}(z)"] & \mathbb{D} \end{tikzcd}$$

$g(z) = \frac{z - i}{z + i}$

So, $(g \circ f \circ g^{-1}) = e^{i\phi} \frac{z-\alpha}{1-\bar{\alpha}z}$

$\Rightarrow g \circ f(\omega) = e^{i\phi} \frac{g(\omega) - \alpha}{1 - \bar{\alpha} g(\omega)}$

↳ .... alg ...

$\text{Aut}(\mathbb{H}) = \left\{ f(z) = \frac{az+b}{cz+d} \;\middle|\; \text{w/ } a,b,c,d \in \mathbb{R} \text{ \& } ad-bc \neq 0 \right\}$

11.4 Riemann Mapping Thm

if we want this to be FLT quite restr.

↳ thinking about bij & FLT mapping $\partial \mathbb{D} \Rightarrow$ half plane of sorts

↳ outside circle

↳ inside circle

Thm 1 (Riemann Mapping Thm)

If Ω is holomorphically simple connected

& Ω is not $\mathbb{C}$

$\Rightarrow$ there is a conformal map $\Omega \to \mathbb{D}$

bianalytic bijection.

Rmk 1 $\Omega = \mathbb{C}$ $f: \mathbb{C} \to \mathbb{D} \Rightarrow |f(z)| \leq 1$

$\Rightarrow f$ const Liouville

So $\mathbb{C}$ not conf equiv to $\mathbb{D}$

Cor if Ω is h.s.c $\Leftarrow\Rightarrow$ Ω t.s.c

Pf $\Leftarrow$ had earlier 5.5.7

$\Rightarrow$ Riemann mapping thm $\Rightarrow \Omega$ homeo to $\mathbb{D}$ via conformal map. open mapping thm.

if Ω not $\mathbb{C}$ $\Rightarrow \Omega$ is t.s.c

$\Rightarrow$ if $\Omega = \mathbb{C}$ result is immediate.

Rmk II let $z_0 \in \Omega$, Ω hsc $\Omega \neq \mathbb{C}$

map from Riem

$\exists F: \Omega \to \mathbb{D}$ conformal map so $F(z_0) = 0$

↳ post compose w/ appr func in lect $\frac{z-\alpha}{1-\bar{\alpha}z}$ $\alpha = f(z_0)$

2) furthermore, F is unique if we impose

$F'(x_0) > 0$.