Recall.

(i) $|f_n(z)-1| \le c_n \ \forall z \in \Omega$, $\sum_{n=1}^{\infty} c_n < \infty$

$\Rightarrow \prod_{n=1}^{\infty} f_n(z)$ conv, analytic in Ω

(2) $E_k(z) := (1-z)\, e^{z+\frac{z^2}{2}+\cdots\frac{z^k}{k}}$

$\Rightarrow |E_k(z)-1| \le 6|z|^{k+1} \quad \forall |z| \le 1/2$

Thm (Weierstrass Product thm)

Let $\{a_n\}_{n=1}^{\infty}$ be a seq of cmplx # so $|a_n| \xrightarrow{n\to\infty} +\infty$

Then $\exists$ entire func f whose zeros are exactly a_ns.

Furthermore, ... uniqueness....] proved earlier!

Pf (existence).

Spse m of a_n are zeros, call the rest $b_1, b_2, \ldots$

(Note: $|b_n| \to +\infty$ as $n\to\infty$ as rem fin many)

Consider,

$$f(z) = z^m \prod_{n=1}^{\infty} E_n\left(\frac{z}{b_n}\right)$$

$$= z^m \prod_{n=1}^{\infty}\left(1-\frac{z}{b_n}\right) e^{\frac{z}{b_n}+\frac{z^2}{2b_n^2}+\cdots+\frac{z^n}{nb_n^n}}$$

Failed Att

$$f(z) = z^m \prod_{n=1}^{\infty}\left(1-\frac{z}{b_n}\right)$$

$\hookrightarrow$ if this is entire, it satisfies 0 prop.

But, does it conv?

need $\sum_{n=1}^{\infty} \frac{1}{|b_n|} < \infty$?

$\hookrightarrow$ not guaranteed!

Note, it has the req. zeros prop

let $R>0$ arb, we show f conv in $D_R(0)$

Since $|b_n| \to \infty$ as $n\to\infty$

$\exists N$ so $\frac{R}{|b_n|} \le \frac{1}{2} \ \forall n \ge N$ ↪ lemma

For $n \ge N$, $|E_n(\frac{z}{b_n})-1| \le 6\left|\frac{z}{b_n}\right|^{n+1} \le \boxed{\frac{6}{2^{n+1}}}\ c_n$

Since, $\sum_{n=N}^{\infty} \frac{6}{2^{n+1}} < \infty$, $\prod_{n=N}^{\infty} E_n(\frac{z}{b_n})$ conv to analy on $|z|<R$

$\hookrightarrow$ can repl w/ 1 as fin $\square$.

Thm) let $\{b_n\}_{n=1}^{\infty}$ be a sq of nonzero complex numbers satisfying $|b_n| \to \infty$ as $n \to \infty$

If $\sum_{n=1}^{\infty} \frac{1}{|b_n|^s}$ then $\forall\, k \geq [s]$,

$\prod_{n=1}^{\infty} E_k\left(\frac{z}{b_n}\right)$ converges & has req zeros!
$\hookrightarrow \{b_n\}_{n=1}^{\infty}$

Pf) Easy modification of prev pf! rep $\frac{6}{2^{kn}}$ w $\frac{6R^{k+1}}{|b_n|^{k+1}}$.

eg) $\prod_{n=1}^{\infty} E_1\left(\frac{z}{n}\right)$ $\quad \sum_{n=1}^{\infty} \frac{1}{n^{1+0.1}} < \infty$

So $s = 1.1$, $[s] = 1$ so take $k \geq 1$

So, $= \prod_{n=1}^{\infty}\left(1 - \frac{z}{n}\right) e^{z/n}$ conv to entire func!

$\hookrightarrow$ rel to gama fnc

eg) $\prod_{n=1}^{\infty} E_1\left(\frac{z}{n}\right) \prod_{n=1}^{\infty} E\left(\frac{z}{-n}\right) \cdot z$

is entire func with zeros at $z \in \mathbb{Z}$

$= z \prod_{n=1}^{\infty}\left(1 - \frac{z}{n}\right) \cancel{e^{z/n}} \left(1 + \frac{z}{n}\right) \cancel{e^{-z/n}}$

$= z \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right) \rightsquigarrow$ will show $\boxed{\frac{\sin(\pi z)}{\pi}}$.

$\hookrightarrow$ we know it is $\sin(\pi z)\, e^{(\text{entire } g)}$

9.4 Jensen's Formula.

Lemma 1 If f is analytic & non-vanishing in $\overline{D_R(0)}$

$$\Rightarrow \log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\,d\theta$$

real log!

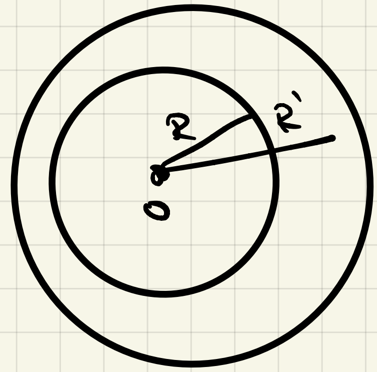
compactness

Pf. $\exists R' > R$ so f is analytic & nonvan on $D_{R'}(0)$

Since $D_{R'}(0)$ is h.s.c & f is nonv.

$\hookrightarrow \exists$ analytic br of $g(z) = \log(f(z))$ on $D_{R'}(0)$

By mean value prop



$$g(0) = \frac{1}{2\pi}\int_0^{2\pi} g(Re^{i\theta})\,d\theta$$

$$\left(\text{from } f(0) = \frac{1}{2\pi}\int_{|z|=R} \frac{g(z)}{z}\,dz\right)$$

$$\therefore \operatorname{Re}(g(0)) = \frac{1}{2\pi}\operatorname{Re}\left[\int_0^{2\pi} g(Re^{i\theta})\,d\theta\right]$$

$$= \int_0^{2\pi} \operatorname{Re} g(Re^{i\theta})\,d\theta$$

But $\operatorname{Re}(g(z)) = \log|g(z)|$.

So, we get the result! $\square$.

Thm (Jensen's formula)

Suppose f is analytic on $\overline{D_R(0)}$ & $f(0) \neq 0$.
Suppose f does not vanish on $C_R(0)$.
Let $a_1, \ldots, a_n$ be the zeros of f on $0<|z|<R$
with multiplicity. Then, $\rightarrow$

$$\log|f(0)| = \sum_{k=1}^{n} \log\left(\frac{|a_k|}{R}\right) + \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\,d\theta$$

Pf. Let $g(z) = \dfrac{f(z)}{\prod_{k=1}^{n}(z-a_k)}$ → Kill all multiplicity! → The a_ks singularity are removable!

extends to nonv analytic map.

By lemma,

$$\log|g(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|g(Re^{i\theta})|\,d\theta$$

$$\| \quad\quad Re^{i\theta}\left(1 - \frac{a_k}{Re^{i\theta}}\right)$$

$$\boxed{\log|f(0)| - \sum_{k=1}^{n}\log|a_k|}$$

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})| - \sum_{k=1}^{n} \underbrace{\frac{1}{2\pi}\int_0^{2\pi} \log|Re^{i\theta} - a_k|\,d\theta}$$

$$\frac{1}{2\pi}\int_0^{2\pi} \underbrace{\log|Re^{i\theta}|\,d\theta}_{\log R} + \frac{1}{2\pi}\int_0^{2\pi}\log\left|1 - \frac{a_k}{R}e^{-i\theta}\right|$$

$$= \log R + \underbrace{\frac{1}{2\pi}\int_0^{2\pi}\log\left|1 - \frac{a_k}{R}e^{i\theta}\right|}$$

note: $|\alpha|<1$ $h(z) = 1-\alpha z$ is analy nonv on $D_1(0)$.

$$\Rightarrow \log|h(0)| = \frac{1}{2\pi}\int_0^{2\pi}|\log - ae^{i\theta}|\,d\theta = 0$$

↳ can rewrite in terms cos, sin R int

$$\therefore \int_0^{2\pi} \log\left|1 - \frac{a_k}{R}e^{i\theta}\right|d\theta = \log|1| = 0.$$

$\therefore$ overall we have $n\log R$. get what we want ∎

So, $\displaystyle\sum_{k=1}^{m} \log|a_k| = \log|f(0)| - \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\, d\theta + m\log R$

Lemma Let $a_1, \dots, a_m$ be nonzero $\mathbb{C}$ nums (Repeats ok).
Let $N(r)$ be the number of those satisfying
$|a_k| < R$

So, $N(0) = 0$, $N(\text{large}) = m$

Then, $\displaystyle\int_0^R \frac{N(r)}{r}\, dr = \sum_{n=1}^{m} \log\left(\frac{R}{|a_k|}\right)$ $\forall\ R > \max_{k \in \mathbb{N}_m} |a_k|$

Pf For all $k = 1, 2, \dots, m$
Define indicator $I_k(r) = \begin{cases} 1 & |a_k| < r \\ 0 & \text{else.} \end{cases}$

$\displaystyle\Rightarrow \sum_{k=1}^{m} I_k(r) = N(r) \Rightarrow \int_0^R \frac{N(r)}{r}\, dr$

$\displaystyle= \sum_{k=1}^{m} \int_0^R \frac{I_k(r)}{r}\, dr$

$\displaystyle= \sum_{k=1}^{m} \int_{|a_k|}^{R} \frac{1}{r}\, dr$

$\displaystyle= \sum_{k=1}^{m} \log\left(\frac{R}{|a_k|}\right)$ $\square$

Cor (# of zeros of analytic map)
Let f be analytic func on $\overline{D_R(0)}$ so that $f(0) \neq 0$.
Let $N(r)$ be no of zeros of f in $|z| < r$ with multiplicities

if f doesn't vanish on $C_R(0)$

$\displaystyle\Rightarrow \int_0^R \frac{N(r)}{r}\, dr = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\, d\theta - \log|f(0)|$

9.5 Functions of finite order of growth

Def an entire function is of fin ord of growth if $\exists \rho \geq 0$, $\exists A, B > 0$ so $|f(z)| \leq Ae^{B|z|^\rho}$ $\forall z \in \mathbb{C}$

eg • e^{e^z} is not $\qquad |e^w| = |e^{\operatorname{Re} w}| \leq e^{|w|}$

• e^z is $\qquad$ ''

• e^{z^2} is as $\wedge \leq e^{|z|^2}$, $e^{P(z)}$ (poly) as $|e^{P(z)}| \leq e^{|P(z)|}$

• $\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$ is

• $(z^2+1)e^{z^3-z}$ is

Note: (a) $|f(z)| \leq Ae^{B|z|^\rho}$ for $|z| > R$

$\implies |f(z)| \leq (A +$