lec6

4.0 Power Series and zero

4.1 Elementary theory of power series

Def A power series about z_0 is $\sum_{n=0}^{\infty} a_n (z-z_0)^n$, $a_n \in \mathbb{C}$

$c = \limsup c_n = \lim_{k\to\infty} \sup_{n\geq k} c_n$

$\forall \varepsilon>0\ \exists N$ so $c_n \leq c+\varepsilon$ $\forall n \geq N$

Thm (Basis thm of pow series)

Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series

Let $\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n}$ → max subseq lim (Hadamard formula)

$\forall \varepsilon$ $\exists$ subseq $c_{n_k} \geq c - \varepsilon$

Then, (a) the power series converges absolutely for every $|z|<R$

(b) the power series does not conv $|z|>R$

for every $0<r<R$, the series conv unif in $|z|\leq r$

Terminology: $R \leftrightarrow$ radius of conv.

$D_R(0) \leftrightarrow$ disc of conv.

Here R can be 0 or $+\infty$

e.g. $\sum_{i=0}^{\infty} z^n \Rightarrow R=1$ $\qquad \sum_{i=0}^{\infty} n!\, z^n \Rightarrow R=0$

$\sum_{i=0}^{\infty} z^{n^2} \Rightarrow R=1$

$\sum_{i=0}^{\infty} \frac{z^n}{n!} \Rightarrow R=\infty$

Pf Spose $\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n}$ not 0 or $+\infty$ — argument later.

a) let $0<r<R$. By def of limsup $\exists n_0$, $\forall n \geq n_0$

$$|a_n|^{1/n} \leq \frac{1}{R} + \underbrace{\frac{R-r}{(R+r)R}}_{\varepsilon} = \frac{2}{R+r}$$

for $|z| \le r$, $n \ge n_0$

$$|a_n z^n| \le \left(\frac{2}{R+r}\right)^n r^n$$

$$= \left(\frac{2r}{R+r}\right)^n \quad \text{and} \quad \frac{2r}{R+r} < 1$$

By comparison thm, $\sum a_n z^n$ converges absolutely & uniformly (no dep on z) for $|z| \le r$

(b) let $|z| > R$ denote $|z| = r > R$

By the def of limsup $\exists$ subseq $|a_{n_k}|^{1/n_k} \ge \frac{1}{R} - \overbrace{\left(\frac{1}{R} - \frac{1}{r}\right)}^{\varepsilon} = \frac{1}{r}$

$$\Rightarrow |a_{n_k} z^{n_k}| \ge \left(\frac{1}{r}\right)^{n_k} r^{n_k} = 1$$

showing that $a_n z^n \not\to 0$ as $n \to \infty$ so no conv for.

Thm] (Power series defines an analytic func)

Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series w/ radius of conv R.

Then $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic in $|z| < R$.

and $f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$ in $|z| < R$

Moreover, the radius of conv of ______ is exactly R.

Pf] Use Weierstrass's Thm

"Moreover part" $\longrightarrow \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n$

check $\limsup_{n\to\infty} |(n+1) a_{n+1}|^{1/n}$ $\overset{(\text{check})}{=}$

$$\lim_{n\to\infty} (n+1)^{1/n} = \lim_{n\to\infty} e^{\frac{\ln(1+n)}{n}} = 1$$

$\limsup_{n\to\infty} |a_n|^{1/n} = R$

4.2 Power Series exp of analytic function.

Thm (analytic func in a disc has a power series repr)

If f is analytic in $D_\rho(z_0)$

$\Rightarrow f(z)$ is repr by a power series

$\sum_{n=0}^{\infty} a_n(z-z_0)^n$ for $|z-z_0| < \rho$

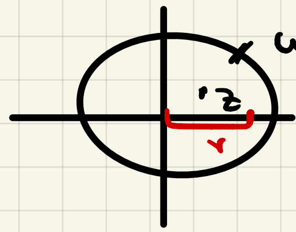
where $a_n = \frac{f^{(n)}(z_0)}{n!}$ and radius of conv of power series satisfies $R \geq \rho$

Pf WLOG, $z_0 = 0$.

let $0 < r < \rho$

From the CIF over a circle, $\forall z \in D_r(0)$

$$f(z) = \frac{1}{2\pi i} \oint_{C_r(0)} \frac{f(w)}{w-z}\, dw$$



For a given $|z| < r$

not 0 $\leftarrow$ $\frac{1}{w-z} = \frac{1}{w} \frac{1}{1-\frac{z}{w}} = \sum_{n=0}^{\infty} \frac{z^n}{w^{n+1}}$ unif in $|w| = r$ as $\left|\frac{z}{w}\right| = \frac{|z|}{r} < 1$

Due to uniform conv in w

$$f(z) = \frac{1}{2\pi i} \oint_{C_r(0)} f(w) \sum_{n=0}^{\infty} \frac{z^n}{w^{n+1}}\, dw$$

unif

$$= \sum_{n=0}^{\infty} \frac{z^n}{2\pi i} \oint_{C_r(0)} \frac{f(w)}{w^{n+1}}\, dw$$

high ord CIT

$$= \sum_{n=0}^{\infty} \frac{z^n f^{(n)}(0)}{n!}$$

$\square$

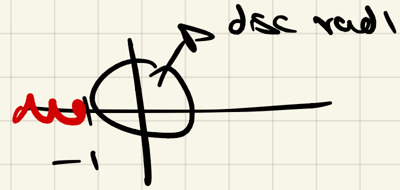
e.g. $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ $\rightsquigarrow$ power series conv everywhere as e^z is entire

$\hookrightarrow$ get conv for free

$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!}$ $\forall z \in \mathbb{C}$

$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots \quad \forall z \in \mathbb{C}$

eg) $\operatorname{Log}(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} \dots$ for $|z|<1$



Cor) spse f, g are analytic in $D_R(z_0)$

If $f^{(n)}(z_0) = g^{(n)}(z_0) \quad \forall n = 0, 1, 2, \dots$

$\Rightarrow f = g$ on $D_R(z_0)$

Pf) $f - g$ analytic on $D_R(z_0)$ → consider power series exp

$\Rightarrow f - g \equiv 0$ on $D_R(z_0)$

or just power series agree!

point where f not anal.

Cor) (radius of conv is the distance to the nearest singularity)

If f is analytic at z_0, the radius R of conv of the power series rep of $f(z)$ about z_0 is given by $R = \sup\{\rho \mid f(z) \text{ extends to be analytic in } D_\rho(z_0)\}$

Pf) let $R' = \sup\{\rho \mid \_\_\_\_\_\_\_\_ \}$

Then the extended function g is analytic in $D_{R'}(z_0)$

$\Rightarrow$ the power series of $\sum_{n=0}^{\infty} \frac{g^{(n)}(z_0)}{n!}(z-z_0)^n$

conv in $|z - z_0| < R$

but also, $g^{(n)}(z_0) = f^{(n)}(z_0)$ by extension!

$\Rightarrow$ The power series above is a power series for f.

$\Rightarrow R \geq R'$ on the otherhand since the powerseries def an analytic func $h(z) = \sum \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$

this is an analytic extn of f in $D_R(z_0)$ so $\boxed{R' \geq R}$.

eg. 1 let f be the map $f(1/2) = 5$

$f(z) = \frac{1}{1-z}$, $z \neq 1/2, 1$

→ power series of f about 0 is

$\sum_{n=0}^{\infty} z^n$ has radius of conv $R = 1$

more valid for more

→ it disagrees at the singularity at 1/2 but that is good.

→ the thm only gives radius of conv $\underline{R = 1/2}$.

→ at least

extn is $g(z) = \frac{1}{1-z}$, $z \neq 1$

→ can you go bigger than 1? perhaps 2

→ but the function will be analytic at 1.