We say

$\Gamma(z)$ is a meromorphic map w/ poles $z = 0, -1, -2, -3, \dots$

$$\operatorname*{Res}_{-n} \Gamma = \frac{(-1)^n}{n!}$$

When $\operatorname{Re} z > 0$. $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt$

$\forall z$ where def'd $\Gamma(z+1) = z\Gamma(z)$

Lemma $\Gamma(z) = \displaystyle\sum_{n=0}^\infty \frac{(-1)^n}{n!\,(n+z)} + \int_1^\infty e^{-t} t^{z-1}\, dt$

$\forall z \neq 0, -1, -2, \dots$ → pole at $z = -n$ $\mathbb{C} \setminus \{0,-1,-2,\dots\}$

Pf Call RHS $H(z)$. second term is entire!

first term seq of analytic in ↑ it will converge

$\therefore$ H is meromorphic w/ poles $z = 0, -1, -2, \dots$

By uniqueness need only to check $\Gamma(z) = H(z)$ for $\operatorname{Re} z > 0$

Then $\Gamma(z) = \displaystyle\int_0^1 e^{-t} t^{z} \,dt + \int_1^\infty e^{-t} t^{z}\, dt$

→ complex pow series.

$= \displaystyle\int_0^1 \sum_{n=0}^\infty \frac{(-t)^n}{n!} t^{z-1}\, dt$ → conv unif in $+\operatorname{Re} z > 0$

$= \displaystyle\sum_{n=0}^\infty \frac{(-1)^n}{n!} \int_0^1 t^{n+z-1}\, dt$

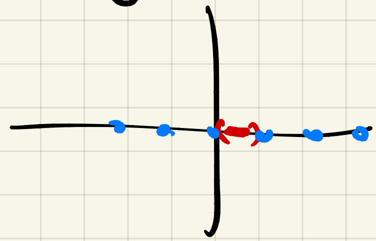
$\forall z \in \mathbb{C} \setminus \{0,-1,\dots\}$

$= \displaystyle\sum_{n=0}^\infty \frac{(-1)^n}{n} \cdot \frac{1}{n+z}$ Uniqueness $H(z) = \Gamma(z)$ ☺

Thm $\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin(\pi z)}$ $\quad \forall\, z \in \mathbb{C}\setminus\mathbb{Z}$

Pf Both sides are analytic on $\mathbb{C}\setminus\mathbb{Z}$

By uniqueness Thm, enough to check the identity for $z = x \in \mathbb{R}$, $0<x<1$.



$0<x<1$ & $0<1-x<1$

so integral formula is valid

$$\Gamma(x)\Gamma(1-x) = \int_0^\infty e^{-t}t^{x-1}\,dt \int_0^\infty e^{-s}s^{-x}\,ds$$

$$= \int_0^\infty e^{-t}t^{x-1}\int_0^\infty e^{-s}s^{-x}\,ds\,dt$$

$s = tu$.

$$= \int_0^\infty e^{-t}\cancel{t^{x-1}}\int_0^\infty e^{-tu}\underbrace{(tu)^{-x}\,t}_{u^{-x}\cancel{t^{-x+1}}}\,du\,dt$$

Fub.

$$= \int_0^\infty u^{-x}\underbrace{\int_0^\infty e^{-(1+u)t}\,dt}\,du$$

$$= \int_0^\infty u^{-x}\frac{1}{1+u}\,du = \int_0^\infty \frac{1}{u^x(1+u)}\,du$$

$$= \frac{\pi}{\sin(\pi x)} \quad \boxed{\text{midterm}} \quad x>0$$

∎

Cor $\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$

Cor $\Gamma(z)$ has no zeros!

Pf When $z \in \mathbb{Z}$ it is either a pole or $(z-1)!$ no zeros.

If $z \in \mathbb{C} \setminus \mathbb{Z}$ ⇒

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$ never 0

so $\Gamma(z) \neq 0$ ever.

Cor $\frac{1}{\Gamma(z)}$ is entire w/ simple zeros at $z = 0, -1, -2, \ldots$

noting $\sum_{n=1}^{\infty} \frac{1}{|-n|^{1+\varepsilon}} < \infty$

so, $\prod_{n=1}^{\infty} \underbrace{\left(1 + \frac{z}{n}\right) e^{-z/n}}_{E_1\left(\frac{z}{-n}\right)}$ conv.

So, $$\frac{1}{\Gamma(z)} = e^{g(z)} z \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}$$

$\boxed{\text{for entire } g}$.

→ If we know the order of $\frac{1}{\Gamma}$ then g is poly (if finite & can est)

Thm $\exists\, A, B > 0$ st

$$\left|\frac{1}{\Gamma(z)}\right| \leq A e^{B|z|\,|\log|z||} \quad \forall z.$$

$\boxed{\text{Even } z=0 \text{ all } g}$

Cor $\frac{1}{\Gamma(z)}$ is entire map of growth order ≤ 1

$A' e^{B'|z|^{1+\varepsilon}}$ → above $\log |z|^{\varepsilon}$ $\forall$ large enough.

Thm (Product Formula for Gamma Func) |

$$\frac{1}{\Gamma(z)} = e^{\gamma z} z \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}}$$

where, $\gamma = \lim_{N \to \infty} \left(\sum_{i=1}^{N} \frac{1}{n} - \log N \right)$

H.W show this exists

Pf| by Hadamard,

$$\frac{1}{\Gamma(z)} = e^{az+b} z \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n}$$

Recall $\operatorname{Res}_0 \Gamma = \frac{(-1)^0}{0!} = 1$ so $\Gamma(z) \sim \frac{1}{z} +$ analytic at $z=0$

$\Rightarrow \lim_{z \to 0} z\Gamma(z) = 1$

So in above divide by z & plug in 0

$1 = e^b \Rightarrow \boxed{b = 0}$.

Now determine a by $z=1$

$$1 = e^a \prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right) e^{-1/n}$$

So $e^a = \dfrac{1}{\prod (1+\frac{1}{n}) e^{-1/n}} = \dfrac{1}{\lim \prod_{n=1}^{N} (1+\frac{1}{n}) e^{-1/n}}$

cancelled

$= \dfrac{1}{\lim_{N \to \infty} \frac{N+1}{1} e^{-\sum_{n=1}^{N} \frac{1}{n}}} \Rightarrow a = \lim_{N \to \infty} \left(\sum_{n=1}^{N} \frac{1}{n} - \log(N+1) \right)$

γ

can replace $N+1$ w N in limit

$\blacksquare$

Pf) of order of growth.

$$\frac{1}{\Gamma(z)} = \Gamma(1-z)\ \frac{\sin \pi z}{\pi}$$

$S(z)$ series $\sum_{i=0}^{\infty} \frac{(-1)^n}{n!(n+1-z)}$

$$= \left(S(z) + \int_1^{\infty} e^{-t} t^{-z}\, dt \right) \frac{\sin \pi z}{\pi}$$

consider each step

$\frac{e^{iz}-e^{-iz}}{2i} \quad A+B$

① $\left|\frac{\sin \pi z}{\pi}\right| \leq \frac{e^{\pi|z|}}{\pi}$ smaller than req.

$|t^{-z}| = |t^{-x-iy}| = t^{-x}$

② $\int_1^{\infty} e^{-t} t^{-z}\, dt \leq \int_1^{\infty} e^{-t}\, t^{-x}\, dt$

i) for $x \geq -1$, $\int_1^{\infty} e^{-t} t^{-x}\, dt \leq \int_1^{\infty} e^{-t} t\, dt$

$\leq \int_0^{\infty} e^{-t} t\, dt = 1$

ii) for $x < -1$ let k be the pos int st

$-x < k \leq -x+1$

$$\int_1^{\infty} e^{-t} t^{-x}\, dt \leq \int_1^{\infty} e^{-t} t^{k}\, dt \leq \int_0^{\infty} e^{-t} t^{k}\, dt$$

$$= k! \leq k^k$$

$$\leq (-x+1)^{-x+1}$$

$$= e^{-(x+1)\log(-x+1)}$$

Cau

$$\leq e^{|-z+1|\,|\log|-z+1||}$$

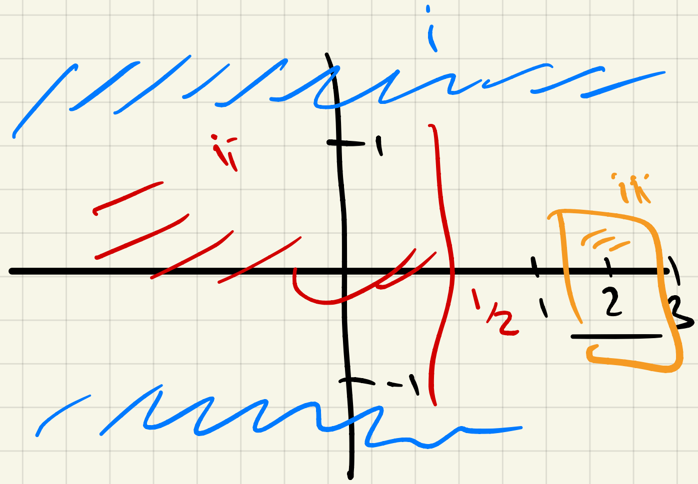
$$\leq A' e^{B'|z|\,|\log|z||}$$

③ $|S(z)| = \sum_{n=0}^{\infty} \frac{1}{n!\,|n+1-z|}$ $\qquad z = x+iy$

i) $|y| > 1 \Rightarrow |S(x+iy)| \le \sum_{n=0}^{\infty} \frac{1}{n!} = \underline{e}$.

ii. for $|y| \le 1$ & $x \le \frac{1}{2}$

$$|S(x+iy)| \le \sum_{n=0}^{\infty} \frac{2}{n!} = 2e$$



iii) for $|y| \le 1$ $\quad k \in \mathbb{N}$

$$k - \frac{1}{2} \le x \le k + \frac{1}{2}$$

$$S(z) = S_1(z) + S_2(z)$$

$$S_1(z) = \frac{(-1)^{k+1}}{(k-1)!\,(k-z)} \qquad S_2(z) = \sum_{\substack{n=0 \\ n \neq k-1}}^{\infty} \frac{(-1)^{k-1}}{n!\,(n+1-z)}$$

same arg $\hookrightarrow S_2(z) \le 2e$

$$|S_1(z) \sin(\pi z)| \le \left|\frac{\sin \pi z}{k-z}\right| = \left|\frac{\sin(\pi(k-z))}{k-z}\right|$$

$$\le C = \max_{\substack{w = x+iy \\ -\frac{1}{2} \le x \le \frac{1}{2} \\ |y|<1}} \left\{\left|\frac{\sin \pi w}{w}\right|\right\} \text{ bc compact}.$$

So all combined!

In $\mathbb{C}$ set result dep on where you are

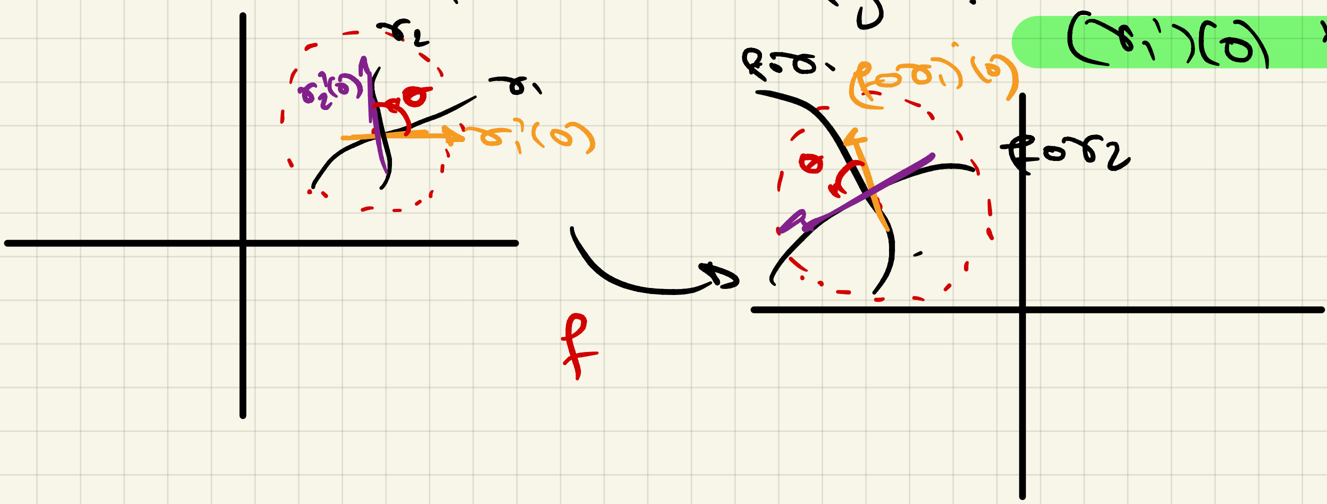
11 Conformal Mappings

11.1 Conformal Functions

Def (a) A C^1 function $f(x+iy) = u(x,y) + i\,v(x,y)$ is said to be conformal at $z_0 = x_0 + iy_0$ if for every pair of smooth curves $\gamma_1(t)$, $\gamma_2(t)$ satisfying $\gamma_1(0) = \gamma_2(0) = z_0$

$\mathbb{R}$ s.t. u, v are such

The angle from the vector $(f\circ\gamma_1)'(0)$ to $(f\circ\gamma_2)'(0)$ is equal to the angle from the vec $(\gamma_1')(0)$ to $(\gamma_2')(0)$



b) A func $f: U \to V$ (U, V open $\subset \mathbb{C}$) is a conformal map (mapping) if it is conformal at every point in U & it is bijective