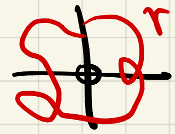
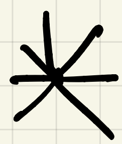
Recall • If f has a primitive in a domain Ω, then $\oint_\gamma f(z)dz = 0$ $\forall$ closed curve γ in Ω

• (ex) $\oint_\gamma \frac{1}{z^2} dz = 0$ where 

• Goursat. If f is analytic in Δ, then $\oint_T f(z)dz = 0$

2.6 Cauchy's Thm for Star Convex Domains

Defi: A set A in $\mathbb{R}^n$ is $\star$-convex if $\exists\, p_0 \in A$ s.t. $\forall\, p \in A$ the line set $\{pt + (1-t)p_0 \mid t \in [0,1]\} \subseteq A$

eg| • Convex sets •  • Slit disc

• Punctured disc is not (reflect it over whatever plane)

Thm| (Primitive in Star convex domains)

Every analytic function in a star convex domain has a primitive (in the same set)

Pf| Let Ω be a star conv domain by $z_0 \in \Omega$

Let $f: \Omega \to \mathbb{C}$ be analytic

Define $F(z) = \int_{L_z} f(s)\, ds$ where L_z is line from $z_0 \to z$

This is well-def by constr!

Let z_1 be a pt in Ω. Wts, $F'(z_1) = f(z_1)$. Let $\varepsilon > 0$

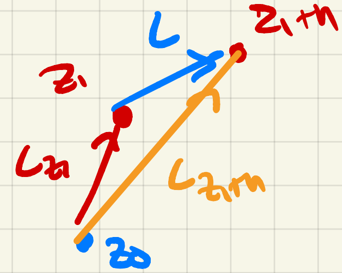
Wts $\left| \frac{F(z_1+h) + F(z_1)}{h} - f(z_1) \right| < \varepsilon$ for small enough h

As f cts, $\exists\, \delta > 0$ so $|f(z) - f(z_1)| < \varepsilon$ if $|z - z_1| < \delta$

we may choose δ so $D_\delta(z_1) \subseteq \Omega$

For $h \in \mathbb{C}$ st $z_1+h \in D_\delta(z_1) \subseteq \Omega$

$$F(z_1+h) - F(z_1) = \int_{L_{z_1+h}} f(s)\,ds - \int_{L_{z_1}} f(s)\,ds$$



Let L be the line segment from z_1 to z_1+h

The solid Δ L_{z_1}, L_{z_1+h}, L is contained in Ω (star conv)

Goursat $\Longrightarrow$ $F(z_1+h) - F(z_1) = \int_L f(s)\,ds$

↳ int over $L, -L_{z_1+h}, L_{z_1} = 0$

So, $\left|\frac{F(z_1+h) - F(z_1)}{h} - f(z_1)\right| = \frac{1}{h}\left|\int_L f(s) - f(z_0)\,ds\right|$

so comes this

$\frac{1}{h}\int_L f(z_0)\,\cancel{ds} = \frac{f(z_0)}{h}\int_L ds = f(z_0)$

$\boxed{\text{choice of } \varepsilon} \leftarrow ML \leq \frac{\varepsilon \cancel{|h|}}{\cancel{|h|}}$

$\therefore$ F is diff'ble at z_1 & $F'(z_1) = f(z_1)$

Cor (Cauchy's thm for star conv domain)

If f is analytic in a star-conv domain Ω

$\Longrightarrow \oint_\gamma f(z)\,dz = 0$ $\forall$ closed curves $\gamma \subset \Omega$

Cor (Cauchy's Thm over a circle)

Let C be a circle (pos oriented) let D be the open disc enclosed by C.

If f is analytic in $\underbrace{C \cup D}$ $\Longrightarrow \oint_C f = 0$

↳ closed $\rightarrow$ choose larger domain to be a a disc and apply prev thm.

eg) • $\int_0^{2\pi} e^{e^{2it}} e^{it}\, dt = \oint_{C_1(0)} e^{z^2} \frac{dz}{i} = 0$ $z = e^{it} \Rightarrow dz = ie^{it}$

analytic everywhere.

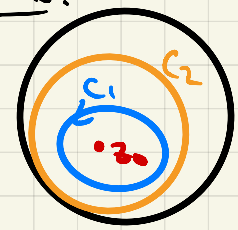
• 

→ slit disc

→ contours here keyhole

↳ any integral over this is 0 (analytic) as slit disc is star con.

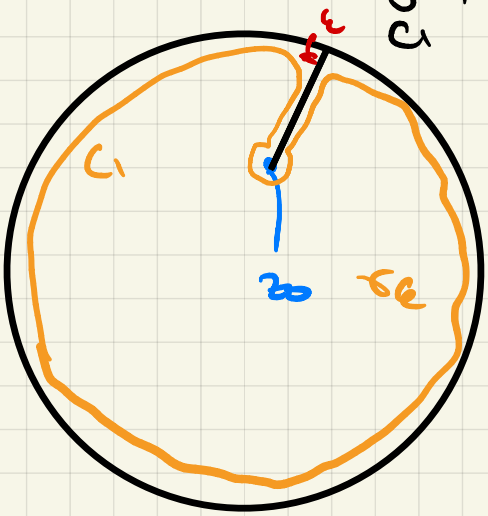
Cor) (Cauchy's thm for punct disc) → not ★ conn.



Let D be a disc & $z_0 \in D$ (so also C).
If f is analytic in $D \setminus \{z_0\}$, then

$\oint_{C_1} f = \oint_{C_2} f$ for all circles C_1, C_2 (in D) enclosing z_0

Pf) strat → show $\int_{C_1} f = \int_{\tilde{C}} f$ for $\tilde{C}$ tiny circle around z_0, interior to C_1

Img 

gap is ε. γ_ε has 4 sec

f is analytic in slit disc

$\Rightarrow \int_{\gamma_\varepsilon} f = 0$

$= \int_{\gamma_{1,\varepsilon}} f + \int_{\gamma_{2,\varepsilon}} f + \int_{\gamma_{3,\varepsilon}} f + \int_{\gamma_{4,\varepsilon}} f$

big  small ε

all conv due to cont.

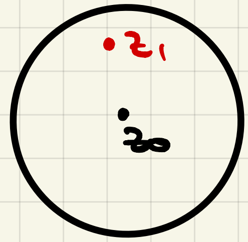
take $\varepsilon \to 0$: $\gamma_{1,\varepsilon} \to C_1$, $\gamma_{3,\varepsilon} \to -\tilde{C}$

$\gamma_{2,\varepsilon} \to -\gamma_{4,\varepsilon} \to$ int on slit

$0 = \int_{C_1} f + \int_{-\tilde{C}} f + \cancel{\int_{slit} f - \int_{slit} f}$ 0

$0 = \int_{C_1} f - \int_{\tilde{C}} f$ ∎.

e.g | Let $z_1 \in D_r(z_0)$



$$\oint_{C_r(z_0)} \frac{1}{z-z_1}\,dz$$

if $z_1 = z_0$

$$\int_0^{2\pi} \frac{rie^{it}}{re^{it}}\,dt = 2\pi i$$

$z = z_0 + re^{it}$

if $z_1 \neq z_0$

$$\int_0^{2\pi} \frac{rie^{it}}{z_0 + re^{it} - z_1} \longrightarrow \text{hard!}$$

but can take smaller center around z_1! → all good

$$\int_{C_r(z_0)} \frac{1}{z-z_1}\,dz = \int_{C_1(z_1)} \frac{1}{z-z_1}\,dz \underset{\text{param}}{=} 2\pi i$$

3. Cauchy Integral Formula Over a Circle

3.1 Cauchy integral formula over a circle

Thm | ( )

Let C be a circle & D be the open disc encl. by C.

If f is analytic in $C \cup D$,

then $f(z) = \dfrac{1}{2\pi i} \oint_C \dfrac{f(w)}{w-z}\,dw \quad \forall z \in D$

Pf | Let $z_0 \in D$ arb. Consider $g(w) = \dfrac{f(w)}{w - z_0}$

Analytic in $D \setminus \{z_0\} \cup C$, $\forall \varepsilon > 0$ small enough,

$$\to \oint_C g(w)\,dw \underset{\text{by prev}}{=} \oint_{C_\varepsilon(z_0)} g(w)\,dw = \oint_{C_\varepsilon(z_0)} \frac{f(z_0) + (f(w) - f(z_0))}{w - z_0}\,dw$$

$$= \boxed{f(z_0) \cdot 2\pi i} + \oint_{C_\varepsilon(z_0)} \frac{f(w) - f(z_0)}{w - z_0}\,dw \to \boxed{\text{Wts } 0}$$

let us take ($\varepsilon \to$

Since $\lim_{w\to z_0} \frac{f(w)-f(z_0)}{w-z_0}$ conv to $f'(z_0)$ ⇝ analytic

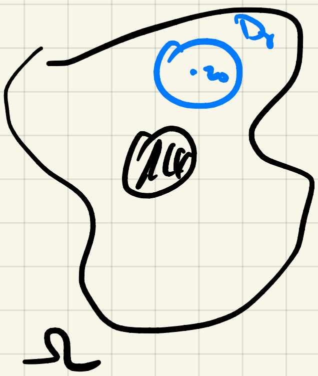
$\exists\, M>0$ & $\varepsilon_0>0$ so that $\left|\frac{f(w)-f(z_0)}{w-z_0}\right| \leq M$ $\forall\, w \in D_{\varepsilon_0}(z_0)$

$$\therefore \left|\oint_{C_\varepsilon(z_0)} \frac{f(w)-f(z_0)}{w-z_0}\,dw\right| \overset{ML}{\leq} M\cdot 2\pi\varepsilon \quad \forall\, 0<\varepsilon<\varepsilon_0$$

$\therefore$ can take $\varepsilon \to 0$ $\therefore$ $\boxed{\text{last term} = 0}$ $\square$

Cor) (Mean Value Thm)

Suppose f is analytic in a domain Ω.
Let $z_0 \in \Omega$. Then $\forall\, r>0$ so that $\overline{D_r(z_0)} \subseteq \Omega$



$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0+re^{it})\,dt$$

Pf) by Cauchy int formula

$$f(z_0) = \frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f(w)}{w-z_0}\,dw \qquad w=z_0+re^{it}$$

$$= \frac{1}{2\pi \cancel{i}}\int_0^{2\pi} \frac{f(z_0+re^{it})\cancel{ire^{it}}}{\cancel{re^{it}}}\,dt$$ $\square$

Thm (Analytic funct are C^∞)

If f is analytic in a domain Ω
$\Rightarrow$ it is inf differble in Ω &

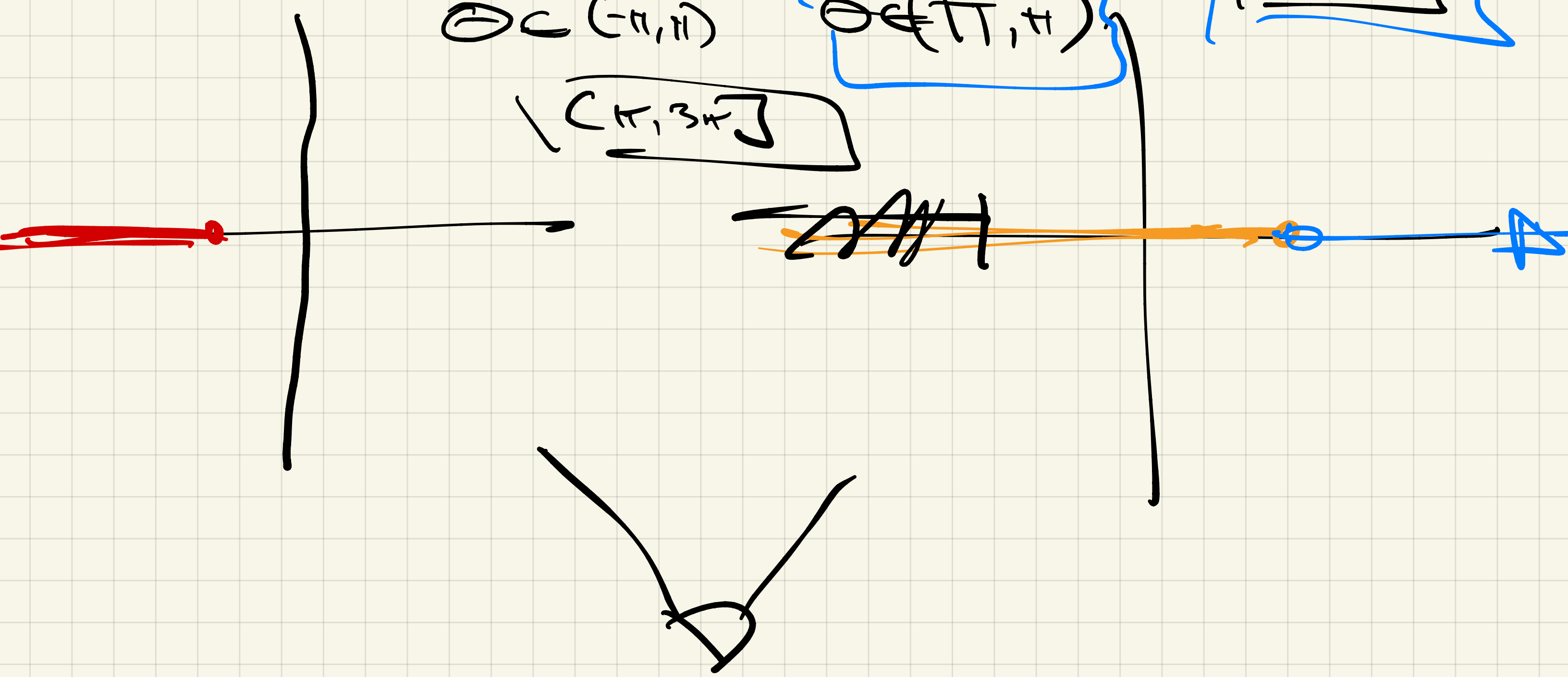
$$f^{(n)}(z) = \frac{n!}{2\pi i}\oint_C \frac{f(w)}{(w-z)^{n+1}}\,dw \quad \forall\, z\in\Omega$$

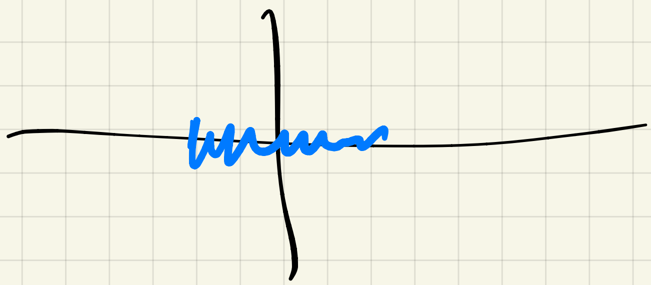
$\forall$ circle C contained in Ω containing z.

$\sqrt{z+1}\,\sqrt{z-1}$ $(0, 2\pi]$ $-3 = \sqrt{3e^{i\pi}}$

$\theta \in (-\pi, \pi)$ $\theta \in (\pi, \pi)$

$\setminus [\pi, 3\pi]$





$\sqrt{z+1}\,\sqrt{z-1}$

$\Downarrow$

$\sqrt{-2}\,\sqrt{-4}$

$(2e^{i\pi})^{1/2}(4e^{i\pi})^{1/2}$

$\sqrt{2}\,e^{i\pi/2}\,\sqrt{4}\,e^{i\pi/2} = \sqrt{8}\,e^{i\pi} = -\sqrt{8}$