Recall z_0 is an essential singularity of f — **Casorati Weierstrass**

$\iff \forall\, \delta > 0$ the img of $D_\delta(z_0)$ under f is dense in $\mathbb{C}$

e.g.1 $f(z) = e^{1/z} \overset{?}{=} w\, \rho e^{i\varphi} \implies \frac{1}{z} = \ln\rho + i\varphi + i2\pi n$

$f(D_\delta(0)\setminus\{0\}) = \boxed{\mathbb{C}\setminus\{0\}}$ $\qquad z = \dfrac{1}{\ln\rho + i\varphi + i2\pi n}$

Picard Thm (Little)

7 Residue Calculus

7.1 Residues

Def Let f have an isolated singularity at z_0.
The residue of f at z_0 is $\operatorname{Res}_{z_0} f = a_{-1}$
Where $\sum a_n (z-z_0)^n$ is the Laurent series of f in $0 < |z-z_0| < \rho$

Note $\operatorname{Res}_{z_0} f = \dfrac{1}{2\pi i} \displaystyle\oint_{|z_0-z|<r} f(z)\, dz \;, \quad 0 < r < \rho$

eg ① $f(z) = \dfrac{1}{z^2+1}$ has iso sing at $\pm i$

$\operatorname{Res}_i f$? $\to$ Laurent series $= \dfrac{1}{z-i}\,\dfrac{1}{z+i}$ → pow abt i

$\implies \operatorname{Res}_i f = b_0 = \dfrac{1}{2i}$ $\qquad = \boxed{\dfrac{1}{z-i}\,(b_0} + b_1(z-i) + \cdots)$
(eval $\frac{1}{z+i}$ at i)

② $\dfrac{1}{(z-1)^2} = f(z) \to$ alr Laurent at 1

$\implies \operatorname{Res}_1 f = 0$

③ $f(z) = e^{1/z} = 1 + \dfrac{1}{z} + \dfrac{1}{2!\,z^2} + \cdots\cdots$

$\operatorname{Res}_0 f = 1$

Lemma

(a) If z_0 is a simple pole of f → $\boxed{\text{order } 1}$

$$\operatorname{Res}_{z_0} f = \lim_{z \to z_0} (z-z_0) f(z)$$

→ $f(z) = \frac{a_{-1}}{z-z_0} + a_0 + \cdots$

$(z-z_0)f(z) = a_{-1} + a_0(z-z_0)$

→ $f(z) = \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)} + a_0$

(b) If z_0 is a double pole of f $\Rightarrow (z-z_0)^2 f(z) = a_{-2} + a_{-1}(z-z_0) + \cdots$

$$\operatorname{Res}_{z_0} f = \lim_{z \to z_0} \frac{\partial}{\partial z}\left[(z-z_0) f(z)\right]$$

(c) If f, g analytic at z_0, $f(z_0) \neq 0$ & g has simple zero at z_0

$$\operatorname{Res}_{z_0} \frac{f}{g} = \frac{f(z_0)}{g'(z_0)}$$

$$\frac{f}{g} = \frac{c_0 + c_1(z-z_0)\cdots}{b_1(z-z_0) + b_2(z-z_0)^2}$$

→ not zero as simple

$$= \frac{1}{z-z_0} \cdot \underbrace{\frac{c_0 + c_1(z-z_0) + \cdots}{b_1 + b_2(z-z_0) + \cdots}}_{\text{analytic at } z_0}$$

eg 1 $f(z) = \frac{z^3}{z^2+1}$

$$\operatorname{Res}_{i} f = \frac{i^3}{2i} = \frac{i^2}{2} = -\frac{1}{2}$$

$f(z) = \frac{z^2+1}{\sin z}$ $\quad \operatorname{Res}_0 f = \frac{1}{\cos(0)} = 1$

Thm (Residue)

Spse f is analytic on Ω except for finitely many iso sing. $a_1, \ldots, a_n$

Then, $$\oint_\gamma f(z)\,dz = 2\pi i \sum_{j=1}^{n} W_\gamma(a_j) \operatorname{Res}_{a_j} f$$

cycle γ ↩ $\forall \gamma \sim 0$ in Ω, doesn't pass through $a_1, \ldots, a_n$

PF 

Let $\Omega_0 = \Omega \setminus \{a_1, \ldots, a_m\}$ f analytic

$\forall j$ $\delta_j > 0$ & $D_{\delta_j}(a_j) \setminus \{a_j\} \subset \Omega_0 \setminus \gamma$

let $\gamma_j = C_{\delta_j/2}(a_j)$ Consider new cycle

$$\gamma - \sum_j W_\gamma(a_j) \cdot \gamma_j$$

$\gamma' \sim 0$ in Ω_0 as $W_{\gamma'}(a_j) = 0$ by constr.

as $W_{\gamma_j}(a_i) = \begin{cases} 1 & i=j \\ 0 & \text{else} \end{cases}$

if $z \notin \Omega \Rightarrow W_\gamma(z) = 0$ & $W_{\gamma_j}(z) = 0$ too by cons

$\Rightarrow \boxed{W_{\gamma'}(z) = 0}$

So, by Cauchy

Thus by general Cauchy,

$$\oint_{\gamma'} f(z)\,dz = 0$$

$$\| $$

$$\oint_\gamma f(z)\,dz - \sum_{j=1}^{n} W_\gamma(a_j) \overbrace{\oint_{\gamma_j} f(z)\,dz}^{2\pi i \operatorname{Res}_{a_j} f}$$

$$\Rightarrow \int_\gamma f(z)\,dz = 2\pi i \sum_{j=1}^{n} W_\gamma(a_j) \operatorname{Res}_{a_j} f$$

Cor Let Γ be a piecewise smooth Jordan curve.

Let Ω be the interior of Γ.

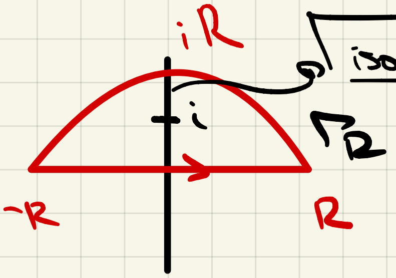
If f is analytic on $\Gamma \cup \Omega$ except for iso sing $a_1, \ldots, a_m$ (Ω)

$$\Rightarrow \oint_\Gamma f(z)\,dz = 2\pi i \sum_{j=1}^{n} \operatorname{Res}_{a_j} f$$

7.2 Evaluating of integrals [Gamelin Ch VII]

1. $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx =$

$f(z) = \dfrac{1}{z^2+1}$



iso sing

earlier

$\forall R > 1$ by thm, $\displaystyle\oint_{\Gamma_R} \frac{1}{z^2+1}\,dz = 2\pi i \operatorname{Res}_i f = \frac{2\pi i}{2i} = \boxed{\pi}$

$$\int_{\Gamma_R} \frac{1}{1+z^2}\,dz = \int_{\rightarrow} + \int_{\curvearrowleft} = \int_{-R}^{R} \frac{1}{x^2+1}\,dx$$

param

as true $\forall R > 1$

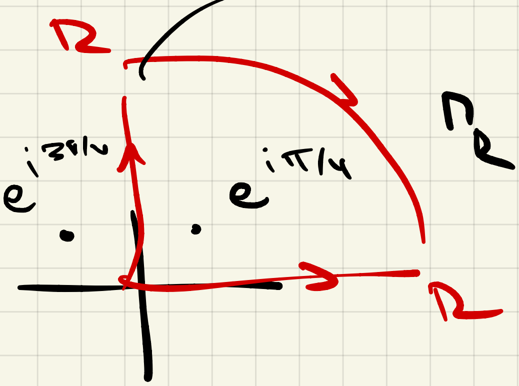
$\therefore$ letting $R \to \infty \Rightarrow$ $\boxed{\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx = \pi - \int_{\curvearrowleft} \frac{1}{z^2+1}\,dz}$

$$\Rightarrow \left| \int_{\curvearrowleft} \frac{1}{z^2+1}\,dx \right| < \frac{1}{R^2-1} \cdot \pi R \longrightarrow 0 \text{ as } R \to \infty$$

as $\frac{1}{|z^2+1|} \underset{\text{REV }\Delta}{<} \frac{1}{|z|^2-1} = \frac{1}{R^2-1}$

$\therefore$ $\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx = \pi$

② $\int_0^{\infty} \frac{1}{x^4+1}\,dx$



Reverse orientation $z \to iy$

$\forall R > 1$ $\int_0^R \frac{1}{x^4+1}\,dx + \int_{\curvearrowleft} \frac{1}{z^4+1}\,dz - \int_0^R \frac{1}{(iy)^4+1}\,i\,dy$

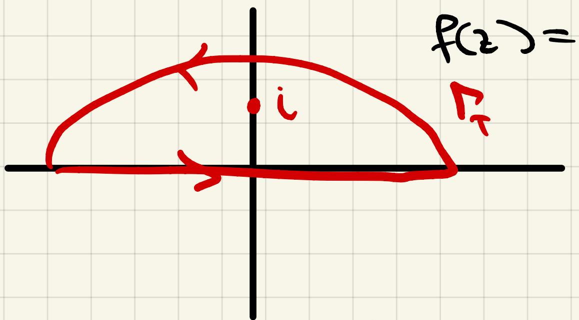
$$= 2\pi i \operatorname{Res}_{e^{i\pi/4}} \frac{1}{x^4+1} = 2\pi i \frac{1}{4e^{i3\pi/4}}$$

let $R \to \infty$ (middle $\to 0$)

$$= \underbrace{\int_0^{\infty} \frac{1}{x^4+1} - i\int_0^{\infty} \frac{1}{y^4+1}\,dy}_{\text{same}} \Rightarrow \int_0^{\infty} \frac{1}{x^2+1}\,dx = \frac{\pi i}{2(1-i)e^{3\pi i/4}}$$

(3) $t \in \mathbb{R}_{>0}$ $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(x^2+1)} \cos(tx)\, dx$

$$= \underbrace{\mathrm{Re} \int_{-\infty}^{\infty} \frac{1}{x^2+1} e^{itx}\, dx}$$



$f(z) = \frac{1}{z^2+1} e^{itz}$

$$\operatorname*{Res}_{i} \frac{e^{itz}}{z^2+1} = \boxed{\frac{e^{-t}}{2i}}$$

wts $\displaystyle\int_{\frown} \frac{e^{itz}}{z^2+1}\, dz$ goes to 0

$$\underline{ML} \to \left| \int \right| < \pi R \max_{\frown} \frac{|e^{itz}|}{|z^2+1|}$$

$$< \frac{\pi R}{R^2-1} \max |e^{itz}|$$

$\underbrace{it(x+iy)}_{\leq 0}$; $y>0$, $t>0$

$|e^{x+iy}| = |e^x \cdot e^{iy}| = |e^x|$

$\boxed{|e^w| = e^{\mathrm{Re}\, w}}$

$$< \frac{\pi R}{R^2-1} \max e^{\boxed{-t\,\mathrm{Im}(z)}}$$

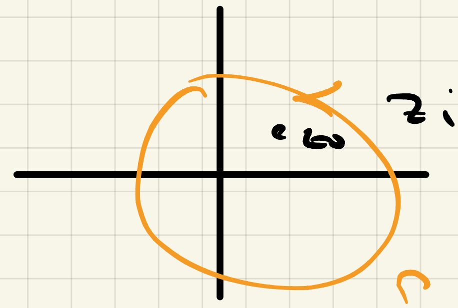
$$\leq \frac{\pi R}{R^2-1}$$

$$\therefore \int_{-\infty}^{\infty} \frac{1}{(x^2+1)} \cos(tx)\, dx = \mathrm{Re}\left(\frac{e^{-t}}{2i} 2\pi i\right) = \pi e^{-t}$$

4 $a>1$ $\displaystyle\int_0^{2\pi} \frac{1}{a+\cos\theta}\, d\theta$

$z=e^{i\theta}$

$$= \oint_{|z|=1} \frac{1}{a+\frac{z+1/z}{2}} \, \frac{1}{z}\, dz$$



$z=e^{i\theta}$

$dz = ie^{i\theta}d\theta$

$\Rightarrow d\theta = \frac{dz}{iz}$

$$= \frac{2}{i} \oint_{|z|=1} \frac{1}{\underbrace{2az+z^2+1}} \, dz$$

$$= \frac{2}{i} \oint$$

we have, if z_0, z_1 root

$z_0 z_1 = 1$

$\Rightarrow |z_0||z_1| = 1$

$\Rightarrow$ either $|z_0|, |z_1| = 1$

or

$|z_0|>1$, $|z_1|<1$

but $a>1 \Rightarrow z_0+z_1>1$

so latter case