8 Argument Principle & its Applications

8.1 The Arg principle.

Thm (Arg Princ.) let f be a meromorphic function on domain Ω with zeros a_j & poles b_k, listed with multiplicities.

isolated

Co for zero's usual sense for pole order.

$$\Rightarrow \frac{1}{2\pi i}\oint_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_j W_\gamma(a_j) - \sum_k W_\gamma(b_k)$$

Rmk: abtmz

$\forall$ cycle $\gamma \sim 0$ mod Ω. Does not pass through any zeros & poles

Cor let Γ be a piecewise smooth Jordan curve. & $\Omega = \text{int}\,\Gamma$. Spse f is meromorphic on $\Omega \cup \Gamma$ without zeros or poles on Γ

$$\Rightarrow \frac{1}{2\pi i}\oint_\Gamma \frac{f'(z)}{f(z)}\,dz = \sum_j W_\gamma(a_j) - \sum_k W_\gamma(b_k)$$

apply to $\hat{\Omega}$

$$= \#\text{zeros} - \#\text{poles} \text{ (with mult)}$$

(compact says finite

Γ, Ω, $\hat{\Omega}$

Pf This is a pf.

Pf of thm let $g(z) = \dfrac{f'(z)}{f(z)}$ meromorphic we want to compute it using residue thm. $\Rightarrow$ find si

We see singularities would be subset of singularities & zeros of f.

If α is a zero of f of order m

$\Rightarrow f(z) = (z-\alpha)^m h(z)$, h analytic near α $h(\alpha) \neq 0$

$$\Rightarrow \frac{f'(z)}{f(z)} = \frac{m}{z-\alpha} + \underbrace{\frac{h'(z)}{h(z)}}_{\text{power series exp.}}$$

analytic at $z=\alpha$

$$\Rightarrow \boxed{\operatorname{Res}_\alpha \frac{f'}{f} = m}$$ (order of 0) $\Rightarrow$ Multiplicity

If β a pole of order n

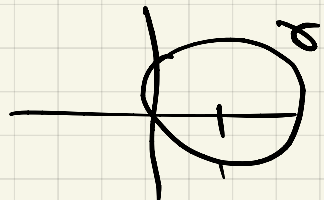
$\Longrightarrow f(z) = \dfrac{h(z)}{(z-\beta)^n} = (z-\beta)^{-n}\, h(z)$ h analytic at β, $h(\beta)\neq 0$

same comp $\Longrightarrow \operatorname{res}_{\beta} \dfrac{f'}{f} = -n \longrightarrow$ mult of pole

gen residue thm $\longrightarrow$ $\dfrac{1}{2\pi i}\oint_{\gamma} g(z)\,dz = \sum_{z_j} W_\gamma(z_j) \operatorname{Res}_{z_j} g$ $\longleftarrow$ considering above

$=$ result since we grab multiplicity by res. $\square$

eg) $f(z) = \dfrac{1}{(z-1)^2}$

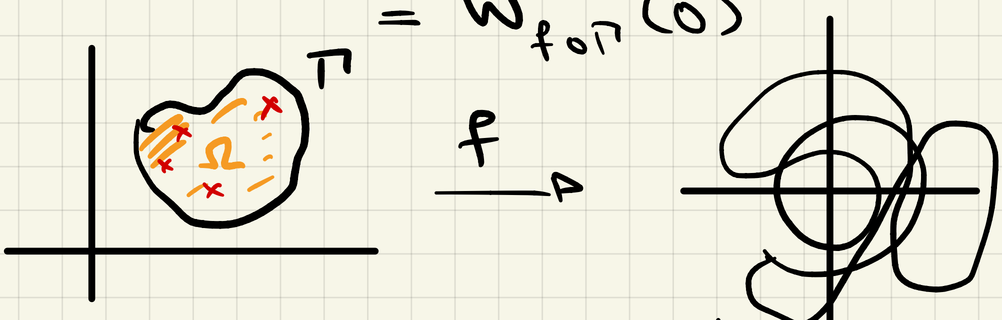


$\underline{W_\gamma(1)\cdot 2} \overset{\text{res}}{=} 2$ $\Big|$ $a_1 = 1,\ a_2 = 1$, $W_\gamma(1) + W_\gamma(1) = 2$

Cor) Assume same as prev cor w/ Jordan Curve Γ.

Then, $Z - P = \dfrac{1}{2\pi i}\oint_{\Gamma} \dfrac{f'(z)}{f(z)}\,dz$ by earlier

$= W_{f\circ\Gamma}(0)$



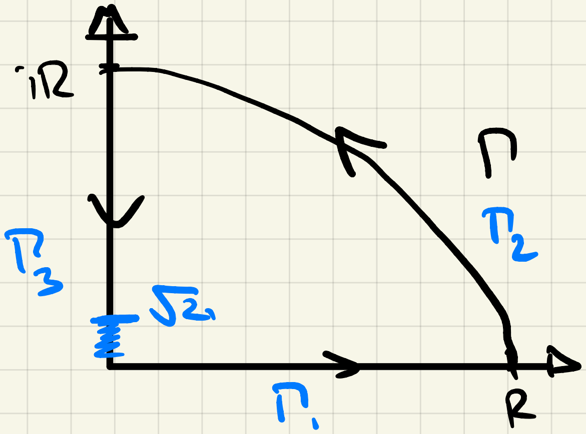
Pf) $\dfrac{1}{2\pi i}\oint_{\Gamma} \dfrac{f'(z)}{f(z)}\,dz \overset{\text{param}}{=} \dfrac{1}{2\pi i}\int_a^b \dfrac{f'(\Gamma(t))}{f(\Gamma(t))}\,\Gamma'(t)\,dt$

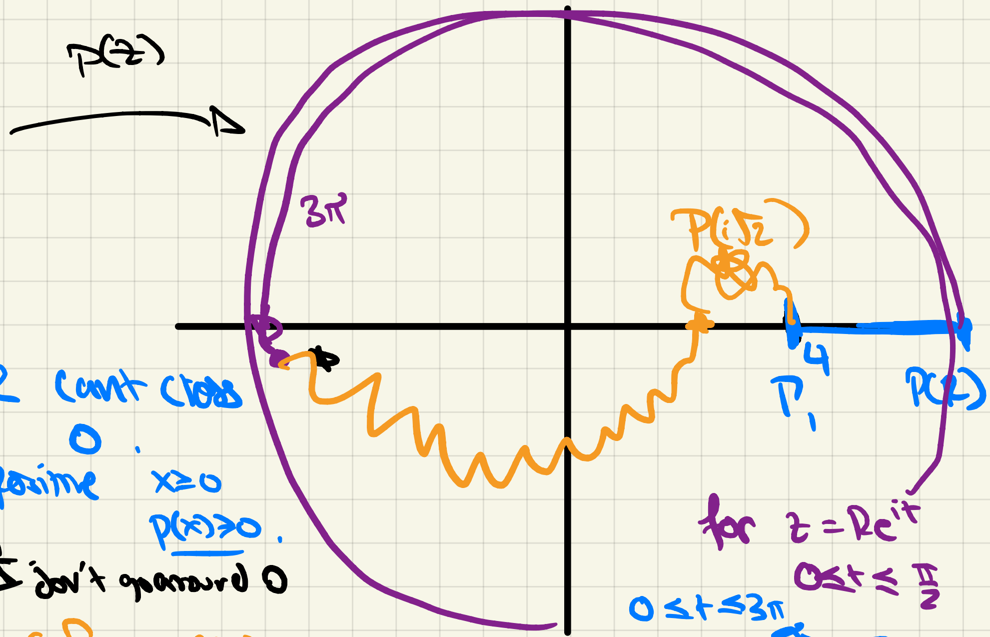
$= \dfrac{1}{2\pi i}\int_a^b \dfrac{\frac{d}{dt}(f\circ\Gamma(t))}{f(\Gamma(t))}\,dt$ call $f\circ\Gamma = \eta$

$= \dfrac{1}{2\pi i}\int_a^b \dfrac{\eta'(t)}{\eta(t)}\,dt = \dfrac{1}{2\pi i}\oint_{\eta} \dfrac{dw}{w} = W_\eta(0)$

e.g. Find the number of roots $z^6 + 9z^4 + z^3 + 2z + 4 = P(z)$ in the first quadrant.

$\leadsto \#z = W_{p\circ\Gamma}(0)$ as $\boxed{p=0}$





$\Gamma_1 \leadsto$ mapped to $\mathbb{R}$ can't cross 0. stays positive $x \geq 0$, $p(x) \geq 0$.

for $z = Re^{it}$, $0 \leq t \leq \frac{\pi}{2}$ → $0 \leq t \leq 3\pi$

$p(Re^{it}) \approx R^6 e^{6it}$ ⤳ dominant.

$z = iy \in \Gamma_3$ $p(iy)$

$= -y^6 + 9y^4 - iy^3 + 2iy + 4$

$= (-y^6 + 9y^4 + 4) + i(-y^3 + 2y)$

$\operatorname{im} p(iy) = -y^3 + 2y$

$-$ive for $\sqrt{2} < y \leq R$ → stay in bottom half, don't go around 0

when $y = R$ large ⤳ negative $\gg$ ☆

$p(i\sqrt{2}) = -8 + 9\cdot 4 + 4 > 0$

$\operatorname{Re} p(iy) = -y^6 + 9y^4 + 4 > 0$ when $0 \leq y < \sqrt{2}$ (so no wrap) stay on Right side

$\therefore W_{p\circ\Gamma}(0) = 2$ for large enough R.

Cor (Rouché's Thm)

Let Γ be a piecewise smooth Jordan curve & Ω its interior.
If f, g analytic on $\Gamma \cup \Omega$ & $|g(z)| < |f(z)|$ $\forall z \in \Gamma$ (so f has no 0 on Γ ⇒ $f+g$ doesn't.)
Then $Z_{f+g} = Z_f$

e.g.| How many zeros does the map $e^z - 1 - 2z$ have in $|z|<1$?

let $f(z) = -2z$, $g(z) = e^z - 1$

Check $|g(z)| < |f(z)|$ in $|z|=1$? If so ans to above is 1 as

$$|e^z - 1| \overset{?}{<} |2z| = 2$$

$$Z_{e^z - 1 - 2z} = Z_{-2z} = \boxed{1} \text{ in } |z|<1$$

$$|e^z - 1| = \left| z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \right|$$

on $|z|=1$

$$\leq |z| + \frac{|z^2|}{2!} + \frac{|z^3|}{3!} + \cdots$$

$$= 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots = e^1 - 1 \approx 1.71.. < 2$$

taylor series $e^1 = 1 + 1 + \frac{1}{2!} + \cdots$

$\therefore |e^z - 1| < 2$ on $|z|=1$ $\boxed{☺}$.

Pf of Rouche's |

Pf 1| $Z_{f+g} - Z_f = \frac{1}{2\pi i} \oint_\Gamma \frac{f'(z) + g'(z)}{f(z) + g(z)} dz$

(arg principle) as $f+g$, f don't vanish on Γ

note:

$$- \frac{1}{2\pi i} \oint_\Gamma \frac{f'(z)}{f(z)} dz$$

$$= \frac{1}{2\pi i} \oint_\Gamma \frac{\left(\frac{g(z)}{f(z)}\right)'}{1 + \frac{g(z)}{f(z)}} dz$$

$\frac{h'(z)}{h(z)}$ of form so by prev work

$$= Z_{1 + \frac{g}{f}}$$

modulus <1 on Γ

Can't get rid of real positive bit.

$$= \underbrace{\omega_{(1 + \frac{g}{f}) \circ \Gamma}}(0) = 0$$

(in right half plane)

Pf 2) for $0 \leq t \leq 1$ let

$$f_t(z) = f(z) + t\,g(z)$$

so that $f_0(z) = f(z)$, $f_1(z) = f(z) + g(z)$

$$Z(t) := \underbrace{Z_{f_t}}_{\substack{\text{check } f_t \\ \text{doesn't van on } \Gamma}} = \frac{1}{2\pi i} \oint_\Gamma \frac{f'(z) + t g'(t)}{f(z) + t g(z)}\, dz$$

Check $Z(t)$ is cts func of t

but $Z(t)$ is int valued $\Rightarrow$ Z is cts!

So, $\boxed{Z(1) = Z(0)}$ $\square$.