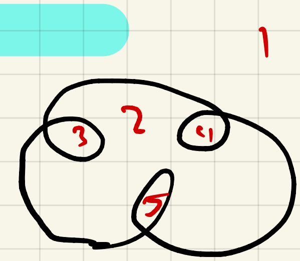
Recall

For $a \in \mathbb{C}$ and closed curve γ (not passing a)

$$W_\gamma(a) = \frac{1}{2\pi i}\oint_\gamma \frac{1}{z-a}\,dz \longleftarrow \text{integer!}$$

is the winding num of γ around a



Lemma | (basic prop of $W_\gamma(\cdot)$)

(a) $W_{-\gamma}(a) = -W_\gamma(a)$ trivially

(b) If γ is contained in a disc D (image of γ compact $\subset$ cont in disc)

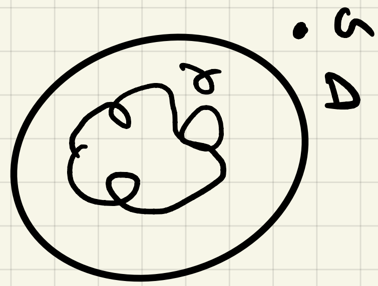
$W_\gamma(a) = 0 \quad \forall a \notin D$

(c) The func $a \longmapsto W_\gamma(a)$ is constant func on each connected component of $\mathbb{C} \setminus \operatorname{im}\gamma$ ($\subset \mathbb{C}$ def by γ) $\boxed{5}$

(d) With (b), (c) considering the unbdd comp determined by γ call it U.
Then $\forall a \in U$ $W_\gamma(a) = 0$.

Pf (a) clear.

(b)+(c) $\Rightarrow$ (d)



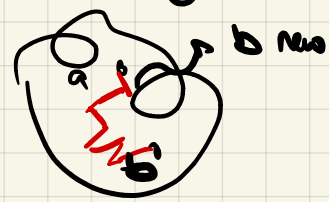
(b) The map $g(w) = \frac{1}{z-a}$ is analytic in D (a is outside)

$\therefore$ by Cauchy's thm of disc,

$$\Rightarrow \oint_\gamma \frac{1}{z-a}\,dz = 0 \quad \therefore \; W_\gamma(a) = 0$$

(c) Pick $a, b \in$ two pts in same conn comp def by γ

Since it is open, conn set



$\exists$ polygonal path $a \longrightarrow b$ in the same conn comp!

enough to show endpts of 1 line segment have same winding number $\longrightarrow$ propagates down the chain!

Spse a, b joined by line seg $L \to$ contained in conn comp $\to$ doesn't int γ

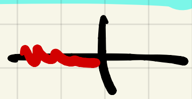
WTS $\oint_\gamma \frac{1}{z-a}\,dz - \oint_\gamma \frac{1}{z-b}\,dz = 0$

Consider the map $g(z) = \text{Log}\left(\frac{z-a}{z-b}\right)$

Cl. g well def on $\mathbb{C} \setminus L$ & is analytic & derivative is $\frac{1}{z-a} - \frac{1}{z-b}$

γ doesn't intersect L $\to$ in anal domain

So we have primitive. $\therefore$ integral over closed curve (as ab) is 0



Well def g well def (and analytic) as $\frac{z-a}{z-b}$ is not negative real number or 0

$t \in [0,\infty)$ $\frac{z-a}{z-b} = -t$ so $z = \frac{1}{1+t}a + \frac{t}{1+t}b$ for $t \in [0,\infty)$

non neg, sum to 1 $\to$ conv comb!

$\Rightarrow \boxed{z \in L} \to$ convex line

$\therefore$ anal in $\mathbb{C} \setminus L$

Der $g'(z) = \frac{1}{z-a} - \frac{1}{z-b}$

As γ contained in $\mathbb{C} \setminus L$ the ftc

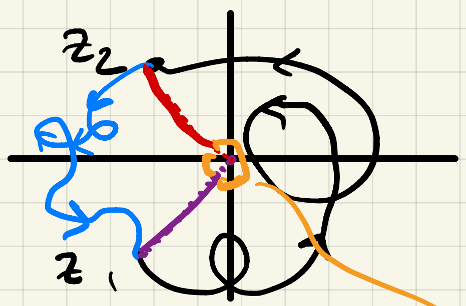
$\Rightarrow \oint_\gamma \frac{1}{z-a} - \frac{1}{z-b} = g(z)\Big|_{\text{start}}^{\text{end}} = 0 \to$ closed

Lemma (sufficient condit. so winding num is 1)

Let γ be a closed curve not passing 0 (for convinience). Strict $\neq$

Spse that there are 2 points $z_1 \in \gamma \cap \mathbb{C}^-$ $\quad z_2 \in \gamma \cap \mathbb{C}^+$ so that the path of γ from z_1 to z_2 in the dir of orientation doesn't pass $\mathbb{R}^-$ and the path from z_2 to z_1 doesn't intersect $\mathbb{R}^+$

$$\Rightarrow W_\gamma(0) = 1$$



So black $\cap \mathbb{R}^- = \emptyset$

blue $\cap \mathbb{R}^+ = \emptyset$

Pf) By picture.

$\to$ can find small nbhd of 0 doesn't int γ by compact

Γ_1 start $z_1 \to z_2 \to$ down around $\circlearrowleft$ $\to z_1$

Γ_1 start $z_2 \to z_1 \to$ up around $\circlearrowleft$ $\to z_1$

$W_{\Gamma_1}(0) = 0$ $\leftarrow$ origin in unbdd comp det by Γ_1 as no int $\mathbb{R}^-$

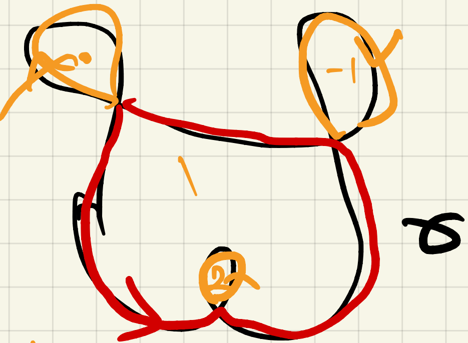
$\Gamma_1 + \Gamma_2 = \gamma - c$ $\to$ similar

$$\text{So, } \oint_\gamma \frac{1}{z} = \underbrace{\oint_{\Gamma_1} \frac{1}{z}}_{\text{(Lemma 1} \Rightarrow \text{winding) } 0} + \underbrace{\oint_{\Gamma_2} \frac{1}{z}}_{0} + \underbrace{\oint_c \frac{1}{z}}_{\text{CIF over circle}}$$

$$2\pi i\, W_\gamma(0) = 0 + 0 + 2\pi i$$

$$\Rightarrow W_\gamma(0) = 1$$

So,



gets -1 from the obv component

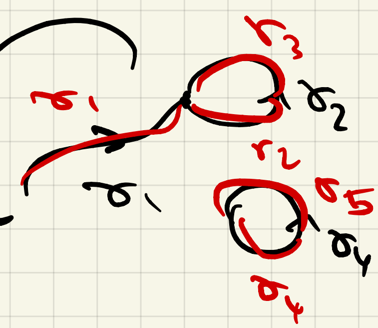
0 else!

int over γ is actually int over 4 closed curves

5.2 Chains & Cycles

Def) A chain is a formal sum of curves $\gamma_1 + \cdots + \gamma_n$

More precisely, it is an equivalence cl. using the following operations



(i) permutations of two curves.

(ii) reparameterizations of a curve

(iii) subdivisions of a curve.

(iv) fusion of 2 curves.

(v) cancellation of curve ($\gamma_7 - \gamma_7 = 0$)

Two chains can be added by juxtapositions! If identical curves are added multiple times (naïvely) write $\gamma = k_1\gamma_1 + \cdots + k_n\gamma_n$ $\quad k_i \in \mathbb{Z}$

we define an integrals over a chain by

$$\int_\gamma f(z)\,dz = \sum_{i=1}^{n} k_i \int_{\gamma_i} f\,dz$$

This defn is well def? i.e. doesn't dep on choice of rep.

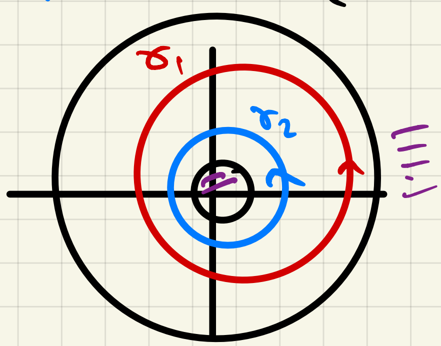
Def | A cycle is a chain that can be repr as a sum of closed curves eg

A cycle in Ω is a cycle that can be repr by sum of closed curves in $\underline{\Omega}$!

Note: $W_{k_1\gamma_1+\dots+k_n\gamma_n}(a) = \sum_{i=1}^{n} k_i W_{\gamma_i}(a)$

Def | A cycle γ in an open set Ω is said to be homologous to 0 in Ω (write $\gamma \sim 0$) if the winding number $W_\gamma(a) = 0 \ \forall a \in \mathbb{C} \setminus \Omega$

e.g | $\Omega = \{z \in \mathbb{C} \mid 1 < |z| < 100\}$



let $\gamma = \gamma_2 - \gamma_1$ where $\gamma_1 = C_{10}(2)$

$\gamma_2 = C_3(1)$

to see if γ homol. check $W_\gamma(a)$ in

For a so $|a| \geq 100$

$\Rightarrow W_\gamma(a) = W_{\gamma_2}(a) - W_{\gamma_2}(a)$

$= 0 - 0$ as unbdd comp!

For a so $|a| < 1$

$\Rightarrow W_\gamma(a) = W_{\gamma_2}(a) - W_{\gamma_2}(a)$

$= 1 - 1 = 0$

So, γ homologous to 0 in $\underline{\Omega}$. $\boxed{\text{comp}}$

5.3 General Cauchy's Thm

very powerful $\oint_\gamma \frac{z^{10} + ze^2}{z^9 - \frac{1}{2}} = 0$ in prev eg

Thm | (general Cauchy thm)

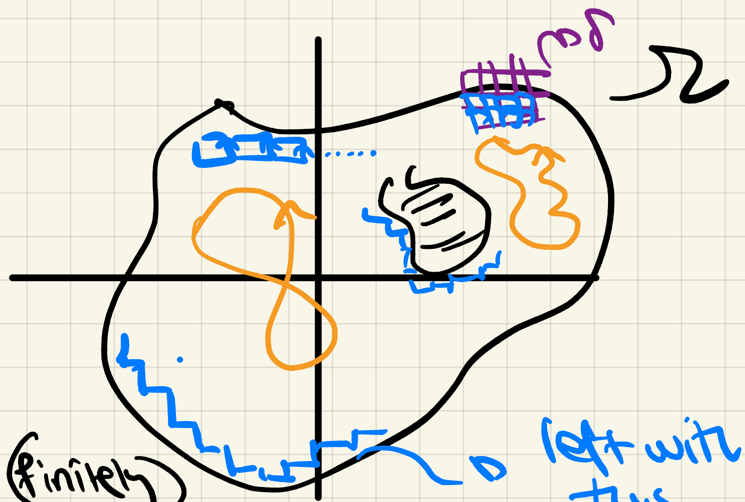
If f is analytic in a domain Ω, then $\oint_\gamma f(z)\,dz = 0$ $\forall$ cycle γ in Ω that is homologous to 0 in Ω

$W_\gamma(a) = 0 \quad \forall a \in \mathbb{C} \setminus \Omega$

Pf | (Ahlfors pg 142)

① If Ω is bdd.

For any $\delta > 0$, cover $\mathbb{C}$ by a net of sq of side length δ



Let Q_j, $j \in J$ be the (finitely) closed sq cont in Ω $\to$ (by bdd)

Also $J \neq \emptyset$ for small enough δ.

Let $\Omega_0 = \text{int}\left(\bigcup_{j \in J} Q_j\right)$ — border matters!

Consider the cycle $\sum_{j \in J} \partial Q_j = \Gamma_\delta$ (edges b/w sq will cancel)

$\longrightarrow$ after cancel

left $\leftarrow$ with bdry approx

Cauchy Cor from 1!

Γ_δ is the sum oriented line seg which are sides of exactly 1 Q_j

② Let γ be a cycle ~ 0 in Ω in Ω. Since γ is a compact set in an open set Ω can take δ small enough so $\gamma \subseteq \Omega_0$