Hadamard Factorization Thm

Say f is entire, order $\le \rho_0$

f has a zero of order m at origin

$a_1, a_2, \ldots$ nonzero zeros of f listed w/ mult

$\Rightarrow f(z) = e^{P(z)} z^m E(z)$, $E(z) = \prod_n E_k\left(\frac{z}{a_n}\right)$

$P(z)$ = polynomial deg $\le k$, $k = \lfloor \rho_0 \rfloor$

Cor (Little Picard's Thm for Entire Maps of finite ord)

If f is entire & of finite order, then unless f is const,

$f(\mathbb{C})$ is either $\mathbb{C}$ or $\mathbb{C} \setminus$ one pt.

Pf Spse $f(\mathbb{C})$ omits a, b $(a \neq b)$

$\Rightarrow f(z) - a$ never vanishes!

$\Rightarrow \underbrace{f(z) - a}_{\text{finite order}} \xrightarrow{\text{hadamard}} f(z) - a = e^{P(z)}$ where P a poly.

$\Rightarrow \exists z_1$ so $e^{P(z_1)} = b - a$ as $b-a$ is nonzero,

↳ solve $\log(b-a) = P(z_1)$ FTA!

↳ any br.

Cor If f is entire fnc of fractional order

then it assumes every complex number inf many times!

↳ ie $\exists$ inf z set $\{f(z) = w\}$.

Pf let $w \in \mathbb{C}$ be given.

w/ mult

Say, $f(z) - w$ has only fin many zeros. $a_1, \ldots, a_n$

let $h(z) = \dfrac{f(z) - w}{(z-a_1)\cdots(z-a_n)}$ ⇝ entire & same order as f frac.

↳ (Thm)

But, Hadamard $\Rightarrow h(z) = e^{P(z)}$, polynomial order = deg P

so not frac oops.

Lemma Let $E_k(z) = (1-z)\, e^{z + \frac{z^2}{2} + \dots + \frac{z^k}{k}}$. Then,

(a) $|E_k(z)| \geq e^{-2|z|^{k+1}}$ when $|z| \leq \frac{1}{2}$

(b) $|E_k(z)| \geq |1-z|\, e^{-2^k |z|^k}$ for $|z| \geq \frac{1}{2}$

Pf (a) $\forall |z| < 1$, $E_k(z) = e^{\alpha(z)}$ where $\alpha(z) = \mathrm{Log}(1-z) + z + \frac{z^2}{2} + \dots + \frac{z^k}{k}$

$$= -\frac{z^{k+1}}{k+1} - \frac{z^{k+2}}{k+2}$$

$\Rightarrow$ for $|z| \leq \frac{1}{2}$, $|\alpha(z)| \leq |z|^{k+1} + |z|^{k+2} + \dots$

$$= \frac{|z|^{k+1}}{1-|z|} \leq 2|z|^{k+1}$$

$|z| < \frac{1}{2}$

$\Rightarrow$ $|E_k(z)| \geq e^{\mathrm{Re}\,\alpha(z)} \geq e^{-|\alpha(z)|}$

$$\geq e^{-2|z|^{k+1}}$$

(b) For $|z| \geq \frac{1}{2} \Rightarrow \frac{1}{|z|} \leq 2$

$$\left| z + \frac{z^2}{2} + \dots \frac{z^k}{k} \right| \leq |z| + |z|^2 + \dots + |z|^k$$

$$\leq \left(\frac{1}{|z|^{k-1}} + \frac{1}{|z|^{k-2}} + \dots + 1 \right) |z|^k$$

$$\leq \left(2^{k-1} + 2^{k-2} + \dots 1 \right) |z|^k$$

$$\leq 2^k |z|^k \qquad \square$$

Recall i) $\forall s > \rho_0$ $\sum_{n=1}^{\infty} \frac{1}{|a_n|^s} < \infty$

ii) $\forall s' > \rho_0$ $\exists$ $c_1', c_2' > 0$ so

\# zeros in $D_r(0)$ $\leftarrow$ $N(r) \leq c_1' r^{s'} + c_2' r^{s'}$

Pf | (Hadamard)

(nbhd of a_n

Step 1 $\quad \dfrac{1}{E(z)}$ is of order ρ_0 outside forbidden disc

$\forall\, s$ satisfying $s \in (\rho_0, k+1)$ $\quad k = \lfloor \rho_0 \rfloor$

$\exists\, c$ so that $\quad \left|\dfrac{1}{E(z)}\right| \le e^{c|z|^s} \quad \forall z \in \Omega$

$$\Omega = \mathbb{C} \setminus \bigcup_{n=1}^{\infty} D_{\frac{1}{|a_n|^{k+1}}}(a_n)$$

Pf st 1 |

let $\displaystyle \mathcal{E}_s = \sum_{n=1}^{\infty} \frac{1}{|a_n|^s} < \infty$ $\quad$ split prod.

Write $\displaystyle E(z) = \underbrace{\prod_{n:\,|a_n| \ge 2|z|} E_k\!\left(\frac{z}{a_n}\right)}_{a \text{ estimate}} \underbrace{\prod_{n:\,|a_n| < 2|z|} E_k\!\left(\frac{z}{a_n}\right)}_{b \text{ est}}$

$$\Longrightarrow |E(z)| \ge P_1(z) P_2(z) P_3(z)$$

$$P_1(z) = \prod_{n:\,|a_n| \ge 2|z|} e^{-2\left(\frac{|z|}{|a_n|}\right)^{k+1}}$$

first such

$$P_2(z) = \prod_{n:\,|a_n| < 2|z|} \left|1 - \frac{z}{a_n}\right|$$

$$P_3(z) = \prod_{n:\,|a_n| < 2|z|} e^{-2^k\left(\frac{|z|}{|a_n|}\right)^{k}}$$

i) If $|a_n| \ge 2|z|$ $\quad$ as $\boxed{k+1-s > 0}$

$$\frac{|z|^{k+1}}{|a_n|^{k+1}} = \frac{|z|^{k+1-s}}{|a_n|^{k+1-s}} \frac{|z|^s}{|a_n|^s} \le \left(\frac{1}{2}\right)^{k+1-s} \frac{|z|^s}{|a_n|^s}$$

ii) $\displaystyle P_1(z) \ge e^{-2\left(\frac{1}{2}\right)^{k+1-s} |z|^s \sum_{n=1}^{\infty} \frac{1}{|a_n|^s}}$ $\quad$ over est.

$$= \exp\left(-2^{s-k} \mathcal{E}_s |z|^s\right) \quad \forall z \in \mathbb{C}.$$

iii) if $|a_n| < 2|z|$, $\left(\frac{|z|}{|a_n|}\right)^k = \left(\frac{|a_n|}{|z|}\right)^{s-k}\left(\frac{|z|}{|a_n|}\right)^s$

$$\leq \left(\frac{1}{2}\right)^{s-k} \frac{|z|^s}{|a_n|^s}$$

So, $|P_3(z)| \geq e^{-2^k \frac{1}{2^{s-k}} |z|^s \sum_{n=1}^{\infty} \frac{1}{|a_n|^s}}$

$$\geq e^{-2^{2k-s}|z|^s \, \sigma_s} \quad \forall |z| \in \mathbb{C}$$

ii) let $z \in \Omega = \mathbb{C} \setminus \bigcup_{n=1}^{\infty} D_{\frac{1}{|a_n|^{k+1}}}(a_n)$

$$\Rightarrow \forall n \quad |z - a_n| \geq \frac{1}{|a_n|^{k+1}}$$

$$\Rightarrow \left|\frac{z}{a_n} - 1\right| \geq \frac{1}{|a_n|^{k+2}} \quad \text{div } \boxed{|a_n|}$$

Thus $P_2(z) \geq \prod_{n: |a_n| < 2|z|} \frac{1}{|a_n|^{k+2}}$ → This is finite prod as only finitely ~~many an in~~ $|a_n| \leq 2|z|$

$$\geq \left(\frac{1}{(2|z|)^{k+2}}\right)^{\# \text{zeros in } 2|z|\mathbb{D} \, = N(2|z|)}$$

→ min

$$= e^{-(k+2)\underbrace{N(2|z|)}_{\text{earlier bd}}\ln(2|z|)}$$

let $s' \in (\rho_0, s)$

$$\geq e^{-(k+2)(c_1'(2|z|)^{s'} + c_2')\ln(2|z|)}$$

$\exists B$ so $R^{s'}\ln(R) \leq BR^s \quad \forall R > 0$

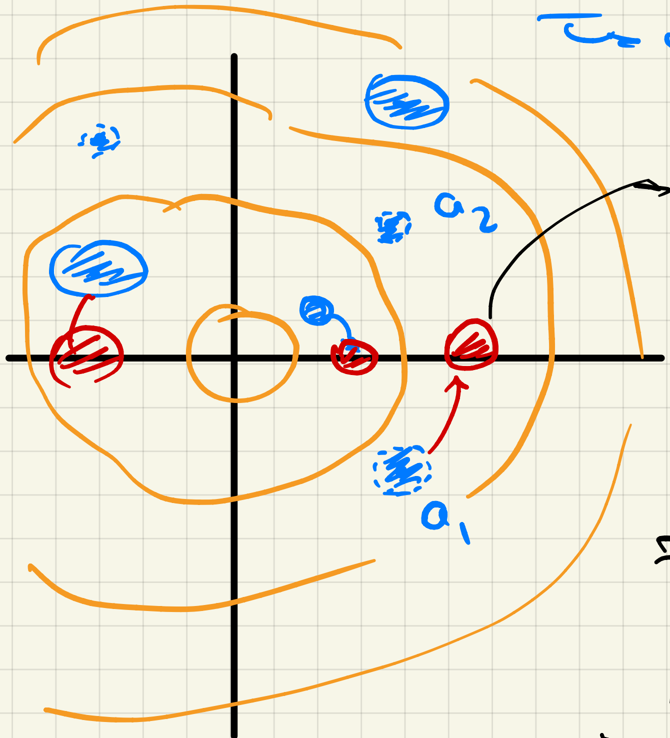
$$\geq e^{-B|z|^s} \quad \boxed{\forall z \in \Omega}$$

Step 2 (Forbidden Discs not too big)

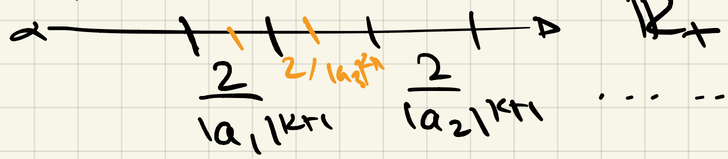
C1. $\exists\ 0 < r_1 < r_2 < \cdots$ tending to $+\infty$

s.t. $C_{r_m}(0) \subset \Omega \ \forall\ m = 1, 2, \ldots$

C = circle not disc



forbidden radii



Size of forbidden radii

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^{k+1}} < \infty \quad D.$$

$$\exists\ N \text{ s.t. } \underbrace{\sum_{n=N}^{\infty} \frac{2}{|a_n|^{k+1}} < \frac{1}{2}}_{(\text{cond})}$$

$\Rightarrow [L, L+1]$ will have some unforbidden sp $\boxed{L > N}$

$\Rightarrow r_1$ = okay in $\underline{[N, N+1]}$.

Step 3 By step 1 & 2

$\exists\ 0 < r_1 < r_2 \ \ldots \ \to +\infty$

& const C so $\left|\frac{1}{E(z)}\right| \le e^{C|z|^s} \ \forall\ z \in \bigcup_{m=1}^{\infty} C_{r_m}$

$\Rightarrow \exists\ A, B > 0$ s.t.

$$\left|\frac{f(z)}{z^m E(z)}\right| \le A e^{B|z|^s} \quad \text{for such}$$

$$\| \qquad\qquad$$

$e^{g(z)}$ for some g

$\Rightarrow \operatorname{Re}(g(z)) \le B r_m^s + \ln A$ for $|z| = r_m$