Lec 7

1. Power series, within domain of conv is an analytic func
2. If f is analytic in $D_\rho(z_0)$ then the power series $\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)(z-z_0)^n}{n!}$ has the radius of con $R \geq \rho$ & is equal to $f(z)$.

4.3 Zeros of analytic func

Thm 1 (no zeros of infinite order)

Spse f is analytic in domain Ω & is not $\equiv 0$

If $z_0 \in \Omega$ is a zero of $f \Rightarrow \exists$ positive in $\mathbb{N}$ so that $f^{(n)}(z_0) \neq 0$

z_0 altitr $\perp$ $\mathbb{R}$ C^∞ not C^ω $e^{-1/x}$

Pf

Let $\mathcal{U} = \{a \in \Omega \mid f^{(n)}(a) = 0 \ \forall n = 0, 1, 2, \ldots\}$

Let $V = \Omega \setminus \mathcal{U}$ will show $\mathcal{U}$ clopen

Since Ω conn $\Rightarrow \mathcal{U} = \emptyset$ or $V = \emptyset$ $\to$ not true as $f \not\equiv 0$

Let some $a \in \mathcal{U}$ be given.

As Ω open $\exists \rho$ so $D_\rho(a) \subseteq \mathcal{U}$, $z \in D_\rho(a)$

By (2) we see that $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n = 0$

$\Rightarrow f^{(n)}(z) = 0 \ \forall n$

$\therefore D_\rho(a) \subseteq \mathcal{U}$ $\therefore \mathcal{U}$ open!

Now, let us show V open. $a \in V$

$\Rightarrow \exists k \in \{0, 1, 2, 3 \ldots\}$ so $f^{(k)}(a) \neq 0$

as $f^{(k)}(z)$ is cts (analytic has analytic der)

$\exists$ open disc D about a so $f^{(k)}(z) \neq 0 \quad \forall z \in D$

$\therefore D \subseteq V$ $\therefore V$ is open!

Def] (Order of zero)

Spse f is analytic in domain Ω & is not identically 0 we say f has a zero of order N at z_0 if

$$f(z_0) = f'(z_0) = \dots f^{(N-1)}(z_0) = 0 \;, \quad f^{N}(z_0) \neq 0$$

Simple zero = zero of ord 1

Double " = " " " 2

Triple " = " " " 3

Lemma] Spse f is analytic in a domain Ω & $f \not\equiv 0$. Let $z_0 \in \Omega$. The the following are equiv.

(a) z_0 is a zero of f of order N

(b) $\exists$ open disc D about z_0 (so that $D \subset \Omega$) and $\exists$ an analytic func g on D so $g(z_0) \neq 0$ and $f(z) = (z-z_0)^N g(z) \;\; \forall z \in D$

(c) $\exists$ analytic func h in Ω so $h(z_0) \neq 0$ & $f(z) = (z-z_0)^N h(z) \;\; \forall z \in \Omega$

Pf] (a) $\Rightarrow$ (b)

$\exists D_\rho(z_0) \subseteq \Omega$ so $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z-z_0)^n$ in $D_\rho(z_0)$

first N are zero

$$= \sum_{n=N}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z-z_0)^n$$

$$= (z-z_0)^N \underbrace{\sum_{k=0}^{\infty} \frac{f^{(N+k)}(z_0)}{(N+k)!} (z-z_0)^k}_{\text{def } g(z)}.$$

original power series conv in $D_\rho(z_0)$ so will this power series!

$$g(z_0) = \frac{f^{(N)}(z_0)}{N!} \neq 0$$

$\square$

(b) $\Rightarrow$ (a)

$f(z) = (z-z_0)^N g(z)$, $\quad f'(z) = N(z-z_0) + \dots$

$f'(z_0) = 0$ if $N > 1$

$f^{(N)}(z_0) = N!\, g(z_0) \neq 0$ $\quad\square$

(c) $\Rightarrow$ (b) Clear

(b) $\Rightarrow$ (c) clear

Define, $h(z) = \begin{cases} g(z) & \text{in } D \\ \dfrac{f(z)}{(z-z_0)^n} & \text{in } \Omega \setminus \{z_0\} \end{cases}$ agree on intersection!

In the overlapping dom $D \setminus \{z_0\}$ the def is well-def! ☒

e.g.| $\cos(z)$ has a simple zero at $z = \pi/2$

$\sin(z) - z$ has a triple zero at $z = 0$ $\leadsto$ $\cos z - 1$, $-\sin z$, $-\cos z$

Thm| (zeros of analytic func are isolated)

If f is analytic on domain Ω & $f \not\equiv 0$

$\Rightarrow$ the zeros

Pf| By the last lemma, $\exists$ disc D ($z_0 \in D \subset \Omega$)

& $\exists$ analytic func g in D st $g(z_0) \neq 0$

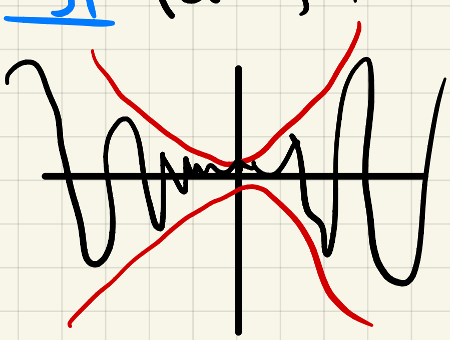
& $f(z) = (z-z_0)^n g(z)$ $\forall z \in D$

Since g is cts, $\exists$ (possibly smaller) disc $\tilde{D}_{z_0} \subseteq \Omega$

So $g(z) \neq 0$

So, z_0 is only zero of f in $\tilde{D}$

e.g.| for $f: \mathbb{R} \to \mathbb{R}$ $\quad x \longmapsto \begin{cases} x^2 \sin(x) & x \neq 0 \\ 0 & x = 0 \end{cases}$ diff on $\mathbb{R}$



$f(z) = \begin{cases} z^2 \sin(\frac{1}{z}) & z \neq 0 \\ 0 & z = 0 \end{cases}$

not analytic at $\boxed{z = 0}$.

4.4 Uniqueness Principle

not $\partial\Omega$

Thm (Uniqueness (Identity Principle))

Let f, g analytic on Ω. If $f(z_n) = g(z_n)$ for a seq of distinct points z_n in Ω that conv to $z_0 \in \Omega$ then $f \equiv g$

Pf) The map $h(z) = f(z) - g(z)$ on Ω

& has zeros at z_n.

By assumption $\exists z_0$ in Ω s.t $z_n \to z_0$ as $n \to \infty$

Since h is cts $h(z_0) = 0$

as z_n are not eventually const!, this implies z_0 is a non-isolated zero. $\therefore \boxed{h \equiv 0}$.

e.g) $f(z) = \sin^2(z) + \cos^2(z)$

$g(z) = 1$

$f(x) = g(x) \ \forall x \in \mathbb{R} \implies f(z) = g(z) \ \forall z \in \mathbb{C}$

So $\boxed{\sin^2 z + \cos^2 z = 1}$

eg) $\sin(z+w) = \sin z \cos w + \cos z \sin w$

take $w = y$ $\square$.

Cor (Unique analytic extn prop)

Let f be analytic on a domain Ω_1, let Ω_2 s.t $\Omega_1 \subset \Omega_2$.

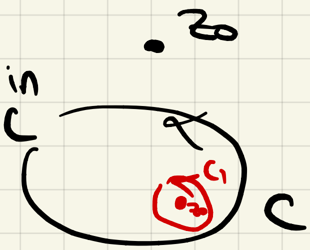
If g & h are 2 analytic extn of f on $\Omega_2 \implies f \equiv g$

5. General Cauchy's Thm (Ahlfors)

5.1 Winding Numbers

Note that $z \longrightarrow \frac{1}{z-z_0}$ analytic everywhere but z_0

eg) for a circle C,



$$\frac{1}{2\pi i}\oint_C \frac{1}{z-z_0}\,dz = \begin{cases} 0 & \text{if } z_0 \text{ is in ext of } C \\ 1 & \text{if } z_0 \text{ in interior} \end{cases}$$

winding num

$\longrightarrow$ $f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(w)}{z-z_0}\,dw$

Lemma) for every closed curve γ that doesn't pass through z_0

$\frac{1}{2\pi i}\oint_\gamma \frac{1}{z-z_0}\,dz$ is an integer.

Pf) Using a param of γ

$$\oint_\gamma \frac{1}{z-z_0}\,dz = \int_a^b \frac{\gamma'(s)}{\gamma(s)-z_0}\,ds$$

def $g(t) = \int_a^t \frac{\gamma'(s)}{\gamma(s)-z_0}\,ds$ for $t\in[a,b]$.

Note: $g'(t) = \gamma'(s)$