Recall curve is a map $\gamma : [a,b] \to \mathbb{C}$
↳ piecewise smooth
↳ each part want analytic + nonzero derivative!

OH this week W: 2-3, Th: 1:15-2

really look at equiv cl.

- end of curve, orientation → indep of param
- $-\gamma$ = reversal of γ, $-\gamma = \gamma \circ [\;]$ → param $[a,b]$ back
- γ is closed if $\gamma(a) = \gamma(b)$
- γ is simple if $\gamma(s) \neq \gamma(t)$ if $s \neq t$
- simple curve is closed if $\gamma(t) \neq \gamma(s)$ if $s<t$ unless $s=a,\ t=b$

$\square$ → simple, cl, piecewise sm

$C_r(a) = \{a + re^{it} \mid 0 \le t \le 2\pi\}$

2.2 Line Integrals (Contour Integrals)

- For a cts fnc $g : [a,b] \to \mathbb{C}$ define, if $g = u + iv$

$$\int_a^b g(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$$

Def | Let γ be a curve, let f a cts func on γ (meaning f is cts on the img of $\gamma([a,b])$) → $\boxed{\text{cmpct}}$

Define,
$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt$$

↳ if γ piecewise smooth. do each chunk

Check: ind of param of γ

Notation: $\oint_\gamma f(z)\,dz$ if γ is closed!

e.g. $\gamma = C_R(0)$ param by $re^{i\theta}$

$$\oint_C z^n \, dz = \int_0^{2\pi} (Re^{it})^n Rie^{it} \, dt = R^{n+1} i \int_0^{2\pi} e^{(n+1)it} \, dt$$

$$= R^{n+1} i \left[\int_0^{2\pi} \cos((n+1)t) \, dt + i \int_0^{2\pi} \sin((n+1)t) \, dt \right]$$

$$= \begin{cases} 0 & , n \neq -1 \\ 2\pi i & , n = -1 \end{cases}$$

Note | Spose $f = u + iv$, $\gamma = x + iy$

$$\int_\gamma f(t) \, dt = \int_a^b (u+iv)(x'(t) + iy'(t)) \, dt$$

$$= \int_a^b (ux' - vy') \, dt + i \int_a^b (vx' + uy') \, dt$$

$$= \int_\gamma (u \, dx - v \, dy) + i \int_\gamma (v \, dx + u \, dy)$$

Prop |

- $\int_\gamma \alpha f + \beta g = \alpha \int_\gamma f + \beta \int_\gamma g$
- $\int_{\gamma^-} f(z) \, dz = - \int_\gamma f(z) \, dz$

Def | (Line Integral)

$$\int_\gamma f(z) \, |dz| \overset{\text{def}}{=} \int_a^b f(\gamma(t)) \, |\gamma'(t)| \, dt$$

Note: if $f(z) = 1$ get length γ.

Lemma (ML Bound)

$$\left|\int_\gamma f(z)\,dz\right| \leq \int_\gamma |f(z)|\,|dz| \leq ML$$

where L = length of γ, M = max of $|f(z)|$ on γ

↳ compact so good!

Pf | Write in polar form

$$\int_\gamma f(z)\,dz = Re^{i\varphi}$$

$$\Rightarrow \left|\int_\gamma f(z)\,dz\right| = R \underset{\text{divide}}{=} e^{-i\varphi}\int_\gamma f(z)\,dz$$

$$= \int_\gamma e^{-i\varphi} f(z)\,dz$$

since it is real

$$= \mathrm{Re}\left(\int_\gamma e^{-i\varphi} f(z)\,dz\right)$$

$$= \mathrm{Re}\left(\int_a^b e^{-i\varphi} f(\gamma(t))\gamma'(t)\,dt\right)$$

Real pt of int is int of real

$$= \int_a^b \mathrm{Re}\left(e^{-i\varphi} f(\gamma(t))\gamma'(t)\right)dt$$

Comparing the real part of the func to modulus ← $\forall t$ true

$$\leq \int_a^b \underbrace{\left|e^{-i\varphi} f(\gamma(t))\gamma'(t)\right|}_{|f\circ\gamma(t)\gamma'(t)|}\,dt$$

$$= \int_\gamma |f(z)|\,|dz|$$

$$\leq \int_\gamma M\,|dz| = ML$$

Recall for cts $f: \mathbb{R} \to \mathbb{R}$ $\int_a^b f(t)\,dt = F(b) - F(a)$ if $F'(x) = f(x)\ \forall x$

always have antider? Yes if f cts $\to$ FTC pt 1.

$$F(x) = \int_0^x f(t)\,dt$$

$$\Rightarrow F'(x) = f(t) \to \text{FTC 2}$$

2.3 Primitives

Def let $\Omega \subset \mathbb{C}$ a domain (conn open)
let $f: \Omega \to \mathbb{C}$ be cts.
A func F in Ω is a primitive of f in Ω if F is $\mathbb{C}$ diff'ble & $F'(z) = f(z)\ \forall z \in \Omega$

e.g. For non-negative int n, $z^n \to \frac{1}{n+1} z^{n+1}$ in $\mathbb{C}$
for negative int $n \neq -1$ $z^n \to \frac{1}{n+1} z^{n+1}$ in $\mathbb{C} \setminus \{0\}$
for $n = -1$ $z^n \to \log z$ in $\mathbb{C} \setminus (-\infty, 0]$?

Thm (FTC for $\mathbb{C}$ funcs)
If a cts func f has a primitive F in a domain $\Omega \subseteq \mathbb{C}$
$\Rightarrow \forall$ curve $\gamma \subseteq \Omega$ have

$$\int_\gamma f(z)\,dz = F(b) - F(a)$$

for b, a endpts of curve!

Pf let $G(t) = F(\gamma(t))$ then $G'(t) = F'(\gamma(t))\gamma'(t)$
write $G(t) = u(t) + iv(t)$

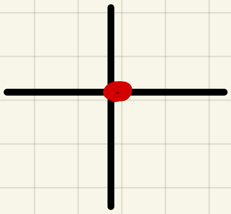
$$\Rightarrow \int_\gamma f(z)\,dz = \int_a^b (u'(t) + iv'(t))\,dt$$

$$= u(b) - u(a) + i(v(b) - v(a)) = G(b) - G(a) = F(b) - F(a)$$

Cor) if a cts func f has a primitive in a domain Ω then $\oint_\gamma f(z)\,dz = 0$ for every closed curve in Ω.

Ex) consider $f(z) = \frac{1}{z}$ & $\Omega = \mathbb{C}\setminus\{0\}$ for $\gamma = C_1(0)$

$\frac{1}{z} \longrightarrow$ analytic



$$\oint_\gamma \frac{1}{z}\,dz = \int_0^{2\pi} \frac{1}{\cancel{e^{it}}}\, i\cancel{e^{it}}\,dt = 2\pi i \neq 0$$

$\Rightarrow$ f doesn't have a primitive in this domain!

$\hookrightarrow$ see $\log z$ is a primitive but on $\mathbb{C}\setminus(-\infty, 0]$

$\hookrightarrow$ smaller

Cor) if f is analytic in Ω a domain & $f'(z) = 0$ $\forall z$
$\Rightarrow$ f is const!

Pf) For any two pts $\alpha, \beta \in \Omega$ $\quad f(\beta) - f(\alpha)$

$$f(\beta) - f(\alpha) = \int_\gamma f'(z)\,dz = 0 \quad \text{as } \boxed{f' = 0}$$

$\gamma \to$ path $\alpha \to \beta$

Is there always a primitive? Delicate...

Lemma) let f be a cts function in a domain Ω.
then f has a prim in Ω

$\Updownarrow$

$\oint_\gamma f(z)\,dz = 0$ $\forall$ closed curve in Ω

Pf) $\Rightarrow$ done $\quad \Leftarrow$ hw

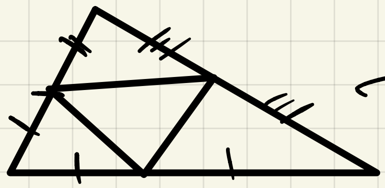
2.5 Goursat's Thm

Thm (Goursat) let Δ be a closed solid triangle let $T = \partial \Delta$ that is positively oriented.

If f is analytic on Δ, $\oint_T f(z)\,dz = 0$

Pf) let $\Delta_0 = \Delta$ let d_0 diam of Δ_0 (largest edge)

P_0 perim of Δ_0



to get 4 triangles $\Delta_1^{(1)}, \ldots \Delta_1^{(4)}$

note $\oint_{T_1^{(1)}} f\,dz + \cdots + \oint_{T_1^{(4)}} f\,dz = \oint_T f\,dz$

$\Rightarrow \exists\, k \in \{1,2,3,4\}$, due to Δ ineq, st.

$$\left| \oint_{T_0} f \right| \le 4 \left| \oint_{T_1^{(k)}} f \right|$$

let $\Delta_1 = \Delta_1^{(k)}$.

$d_1 = 1/2\, d_0$

$P_1 = 1/2\, P_0$

Repeat..... $\Delta_0 \supset \Delta_1 \supset \Delta_2 \ldots$ $d_n = \frac{1}{2^n} d_0$

$$\left| \oint_{T_0} f \right| \le 4^n \left| \oint_{T_n} f \right|$$

$P_n = \frac{1}{2^n} P_0$

$\Rightarrow \bigcap_{n \in \mathbb{N}} \Delta_n \ni \boxed{z_0}$

$z \neq z_0$

Now f is analytic at z_0. $g(z) = \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0)$

sat $\lim_{z \to z_0} g(z) = 0$

$$\oint_{T_n} f(z)\,dz = \oint_{T_n} \Big(\underbrace{f(z_0) + f'(z_0)(z-z_0)}_{\text{has primitives} \;\; \text{poly in } z} + g(z)(z-z_0) \Big)\,dz$$

$$= \oint_{T_n} g(z)(z-z_0)\,dz$$

$\Longrightarrow$