Recall f is $\mathbb{C}$ diff'ble at z_0 provided

$$\lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0} \text{ exists}$$

→ Gamelin uses C'

functions are analytic at z_0 if it is diff'ble in a nbhd of z_0

Thm (Cauchy-Riemann eqns)

(a) If $f(z) = u(x,y) + i v(x,y)$ is $\mathbb{C}$-diff at z_0 then

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \text{ at } z_0$$

(b) Let $f = u + iv$ be a func on an open set Ω → partials u_x, u_y, v_x, v_y cts

if u, v are C^1 on Ω & satisfy the CR eqns

→ f analytic in Ω $\quad f' = u_x + i v_x$

Pf (a) $f'(z_0) = \lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ exists so approach doesn't exist

approach $\rightarrow \underset{z_0}{\cdot} \leftarrow$

$$= \lim_{x \to x_0} \frac{f(x+iy_0) - f(x_0+iy_0)}{(x+iy_0)-(x_0-iy_0)}$$

$$= \lim_{x \to x_0} \frac{u(x,y_0) + iv(x,y_0) - u(x_0,y_0) - iv(x,y_0)}{x-x_0}$$

$$= u_x(x_0,y_0) + i v_x(x_0,y_0)$$

approach ↕

$$= \lim_{y \to y_0} \frac{f(x_0+iy) - f(x_0+iy_0)}{i(y-y_0)}$$

$$= \frac{u_y(x_0,y_0) + i v_y(x_0,y_0)}{i}$$

D

(b) Let z_0 be an arb pt in Ω. Let $h = h_1 + ih_2$ small so $z_0 + h \in \Omega$ by open.

Let $R(h_1,h_2) = u(x_0+h_1, y_0+h_2) - u(x_0,y_0) - u_x(x_0,y_0)h_1$
$- u_y(x_0,y_0)h_2$

$S(h_1,h_2) = v(\quad\_\_\_\_\quad) - v(-) - v_x(-)h_1$
$- v_y(-)h_2$

Since u, v are $C^1 \Rightarrow \dfrac{|R(h_1,h_2)|}{\sqrt{h_1^2+h_2^2}} \to 0$ as $(h_1,h_2) \to (0,0)$

& $\Rightarrow \dfrac{|S(h_1,h_2)|}{\sqrt{h_1^2+h_2^2}}$ similarly.

eval (x_0, y_0)

CR

Now, $\dfrac{f(z_0+h) - f(z_0)}{h} = \dfrac{u_x h_1 + \overset{-v_x}{\cancel{u_y}} h_2 + R(h) + i(v_x h_1 + \overset{u_x}{\cancel{v_y}} h_2 + S)}{h}$

$\overset{CR}{=} \dfrac{(u_x + iv_x)h_1 + (-v_x + iu_x)h_2 + R+S}{h}$

factor i in second

$= \dfrac{(u_x+iv_x)\cancel{(h_1+ih_2)} + R+S}{\cancel{h}}$

$\therefore \left| \dfrac{f(z_0+h)-f(z_0)}{h} - (u_x(x_0,y_0) + iv_x(x_0,y_0)) \right| \leq \left| \dfrac{R(h)+S(h)}{h} \right|$

$\overset{\text{triang}}{\leq} \dfrac{|R(h)|}{|h|} + \dfrac{|S(h)|}{|h|} \to 0$ as $h \to 0$

E.g. $f(z) = \bar{z} = x - iy$

$u(x,y) = x \quad v(x,y) = -y$

$u_x = 1 \neq v_y = -1 \to$ not diff'ble anywhere despite u, v are best func!

Q: $\mathbb{C}$ as an $\mathbb{R}$ algebra gives this struct on $\mathbb{R}^2$ what do $\mathbb{R}^n$?

Def| $e^z = e^{x+iy} \overset{\text{split}}{=} e^x(\cos y + i \sin y)$

$e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos\theta - i\sin\theta$ odd even

$\Rightarrow \frac{e^{i\theta} - e^{-i\theta}}{2i} = \sin\theta$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\sinh z = \frac{e^z - e^{-z}}{2} \qquad \cosh z = \frac{e^z + e^{-z}}{2}$$

Check| $e^z = e^x(\cos y + i \sin y)$

$u = e^x \cos y$
$v = e^x \sin y$

CR shows $\Rightarrow$ all analytic

$(e^z)' = e^z \qquad (\sinh z)' = \cosh z$

$(\sin z)' = \cos z \qquad (\cosh z)' = \sinh z$

$(\cos z)' = -\sin z$

1.6 Log function

For $z \neq 0$ define $\log z = \ln|z| + i \arg(z)$ ← this is the set

↳ multivalued function.

want $\log(xy) = \log x + \log y$

so $\log(x+iy) = \log(re^{i\theta}) = \log(r) + i\theta$ (& others)

E.g.| $\log(1+i) = \ln\sqrt{2} + i\left(\frac{\pi}{4} + 2\pi n\right) \quad n \in \mathbb{Z}$

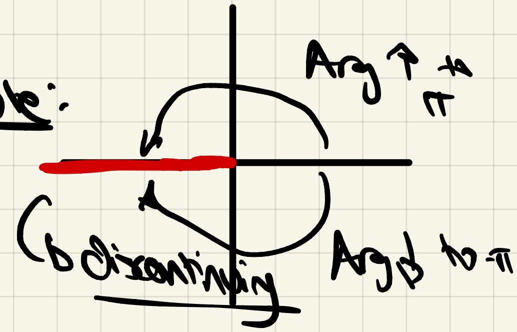
Note| $e^{\log(z)} = z$ as sin cos kills periodicity

Define principle branch of ω

$$\text{Log } z = \ln|z| + i \,\text{Arg}(z)$$

$\text{Arg}(z) \in (-\pi, \pi]$

Note:

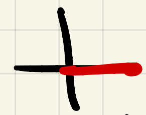


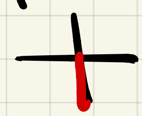
Log z is analytic in the slit plane $\mathbb{C}\setminus(-\infty, 0]$ with $(\text{Log } z)' = \frac{1}{z}$

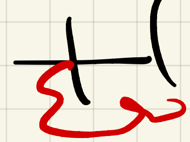
(check w/ CR) $u = \ln|z|$ $v = \text{Arg}(z)$

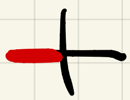
We may choose different branches (to move the discontinuity)

→ could pick $\log z = \ln|r| + i\theta$ $z = re^{i\theta}$

$\theta \in (0, 2\pi)$ 

$\theta \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$ 

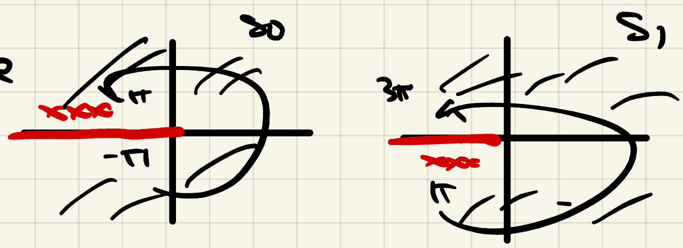
or | $\log z = \ln|z| + i\theta$ $\theta \in (\pi, 3\pi)$  for this if advantageous

 → same as Log but $\log z = \text{Log } z + 2\pi i$.

For each $m \in \mathbb{Z}$

let $f_m(z) = \text{Log}(z) + 2\pi i m$ which is analytic in $S_m = \mathbb{C}\setminus(-\infty, 0]$ usual split plane

Kinda this glue



. . . .

Now choose f_m based on which S_m are on!

→ defining Log on extended complex (Riemann surface) & analytic everywhere!

1.7 Square Root Function

Def | $z^{1/2} = (re^{i\theta})^{1/2} = r^{1/2} e^{i\theta/2}$

or $r^{1/2} e^{i(\theta + 2\pi m)/2}$

$= r^{1/2} e^{i\theta/2} e^{i\pi m}$

So $z^{1/2} = \pm r^{1/2} e^{i\,\mathrm{Arg}(z/2)}$

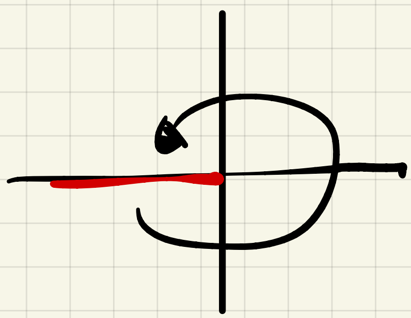
$(e^{\frac{1}{2}\log z} =)$

Principle Branch

$z^{1/2} = r^{1/2} e^{i\theta/2}$ $\quad -\pi < \theta \le \pi$

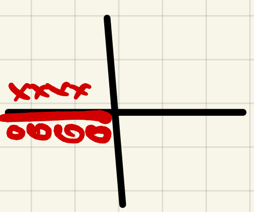
analytic in $\mathbb{C} \setminus (-\infty, 0]$

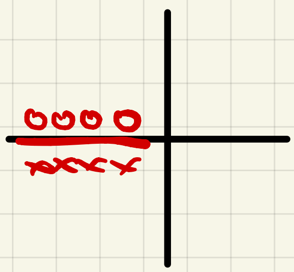
Now!



r same but θ diff by 2π

So there is a factor of $e^{i\pi} = -1$ (phase factor)





Therefore we end up at negative of value as we go around!

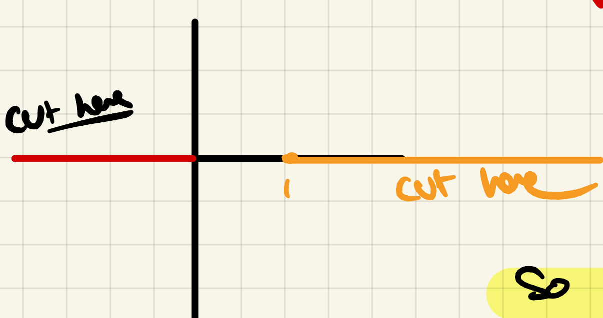
So! this time only req 2 sheets

↳ going around second time gives negative.

1.8 Power Func

For a complex number define $z^\alpha = e^{\alpha \log z}$

$f(z) = \sqrt{z(1-z)} = \underbrace{\sqrt{z}}\,\underbrace{\sqrt{1-z}}$



so $\sqrt{1-z}$ is well def $\mathbb{C} \setminus$

& $\sqrt{z}$ is $\mathbb{C} \setminus$

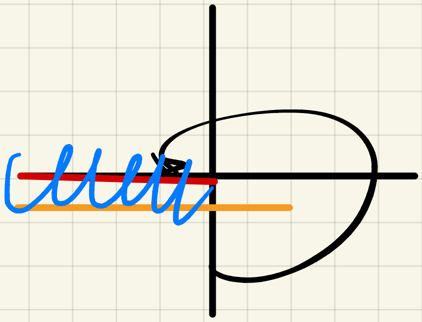
So all good in $\mathbb{C} \setminus$

$\alpha = 1/3$ need 3 slit pl

$\alpha = \frac{a}{b}$ req $2b$ slit at most

α irr keep flying!

for|



take (orange) for $\sqrt{1-z}$

(red) for $\sqrt{z}$ as usual

but now, as we go around, we gain a negative for each factor which cancel!

So, analytic on $\mathbb{C}\setminus[0,1]$

2. Complex Integrals

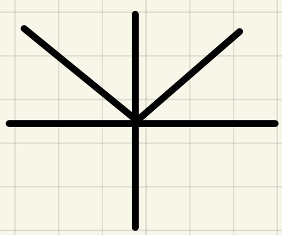
2.1 Curves

Def| A param curve (=path=arc) is a cts func
$$\gamma : [a,b] \longrightarrow \mathbb{C}$$

A param curve is smooth is γ is C^1 & $\gamma'(t) \neq 0$ $\forall t$

A param curve is piecewise smooth if $\exists$ a partition of $[a,b]$ $a=a_0 < a_1 < \cdots < a_k = b$ so γ is smooth on $[a_i, a_{i+1}]$ in the prior sense

eg| $\gamma(t) = t^3 + i\,|t|^3$, $-1 \le t \le 1$



$\gamma([-1,1])$ $\quad \gamma'(0) = \lim \dfrac{\gamma(t) - \gamma(0)}{t}$ exists

γ is C^1

This is why we want this $\neq$ — but $\gamma'(0) = 0$.

you can slow to zero & make sharp turns!

Def| Two param piecewise sm curves
$\gamma : [a,b] \longrightarrow \mathbb{C}$ & $\tilde{\gamma} : [c,d] \longrightarrow \mathbb{C}$
are reparams of each other if $\exists$ C^1 bij $s \longmapsto t(s)$
$t : [c,d] \longrightarrow [a,b]$ so $t'(s) = 0$, $\forall s$

$\gamma \circ t = \tilde{\gamma}$ $\rightsquigarrow$ reparam time

Def! A curve is an equiv cl. of param curves!

↳ check prior is equiv reln!

all curves will be at least pw smooth

will abuse terminology

curve (equiv class)

$\updownarrow$

param curve (rep)