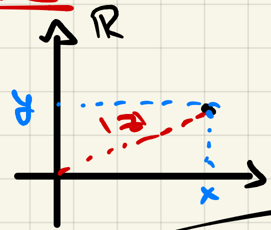
Lec 1 HW Thurs 10pm.

Texts: Ahlfors, Gamelin, Stein

1. Complex Numbers & Func

1.1 Complex numbers

Consider $\mathbb{R}^2$ as a $\mathbb{R}$-vs!



Usually in $\mathbb{R}^n$ we have 2 notions of products.

But in $\mathbb{R}^2$ we have a cool product!

$$(x_1,y_1)(x_2,y_2) := (x_1x_2 - y_1y_2,\ x_1y_2 + x_2y_1)$$

$\mathbb{R}^2$ becomes a field with this structure! → can add coords wise!

↪ but $(x,y) = x\underset{1}{(1,0)} + y\underset{i}{(0,1)} \longleftrightarrow x+iy$

Now: $(x_1+iy_1)(x_2+iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)$

Note: $(0,1)(0,1) = (-1,0)$! $\Rightarrow i^2 = -1$

Call this $\mathbb{C}$

Def The modulus / abs val of complex no $z = x+iy$ is

$$z = \sqrt{x^2+y^2}$$

Check $|z+w| \le |z| + |w|$

$|z|^2 = |z||\bar{z}|$ → complex conjugate

$$\frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{|z|^2}$$

$$Re(z) = \frac{z+\bar{z}}{2} \qquad Im(z) = \frac{z-\bar{z}}{2i}$$

Polar Coords

Def $e^{i\theta} = \cos\theta + i\sin\theta$ for $\theta \in \mathbb{R}$

→

Check $e^{i(\theta_1+\theta_2)} = e^{i\theta_1} + e^{i\theta_2}$

$|e^{i\theta}| = 1$

$e^{2\pi i n} = 1 \quad \forall n \in \mathbb{Z}$

Polar Form the polar form of a complex no $z \neq 0$ is
$z = re^{i\theta}$, $r > 0$, $\theta \in \mathbb{R}$

$\Rightarrow |z| = r$

θ is not unique up to perturb by $2\pi n$ $n \in \mathbb{Z}$

$\mathrm{Arg}(z)$ = angle taken to be $(-\pi, \pi]$

$\arg(z) = \{\mathrm{Arg}(z) + 2\pi n \mid n \in \mathbb{Z}\}$

Note $\mathrm{Arg}(z_1) + \mathrm{Arg}(z_2) \neq \mathrm{Arg}(z_1 z_2)$ by $z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$

$\arg(z_1) + \arg(z_2) = \arg(z_1 z_2)$

1.2 Convergence

$\mathbb{C}$ is a metric sp with $|\cdot|$ euclidean metric on $\mathbb{R}^2$

Def $\lim_{n\to\infty} z_n = w$ if $\forall \varepsilon > 0 \ \exists N \in \mathbb{N}$ so $\forall n > N$ we have $|z_n - w| < \varepsilon$

or $\lim_{n\to\infty} |z_n - w| = 0$

$\mathbb{C}$ is a complete metric sp $\rightarrow$ all Cauchy seq conv!

1.3 Sets in $\mathbb{C}$

- $\mathbb{C}^+ := \{z \in \mathbb{C} \mid \operatorname{im} z > 0\}$
- $\mathbb{C}^- := \{z \in \mathbb{C} \mid \operatorname{im} z < 0\}$
- $D_r(z_0) := \{z \in \mathbb{C} \mid |z - z_0| < r\}$
- $\overline{D_r}(z_0) := \overline{D_r(z_0)}$
- $C_r(z_0) = \partial D_r(z_0)$

Def| Connectedness

A set A is connected if dne open $C_1, C_2 \subset A$ (→ rel open)
so that $A = C_1 \cup C_2$ & $C_1 \cap C_2 = \emptyset$ & $C_1, C_2 \neq \emptyset$

Def| A **domain** (region)
is an open, connected sso $\mathbb{C}$

Thm| Polygonal Path connectedness (piecewise linear) → holds in $\mathbb{R}^n$

Let Ω be an open set in $\mathbb{C}$, $\Omega \neq \emptyset$.

Then it is conn $\iff$ any 2 pts in Ω can be joint by a polygonal path in Ω

We can choose this path to contain just horizontal & vertical segm

Pf| ($\Longrightarrow$) fix a point a in Ω

Let Ω_1 be the set of points in Ω that can be joined with A using a polygonal path.

Let $\Omega_2 = \Omega \setminus \Omega_1$

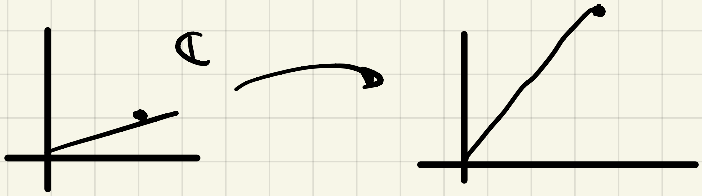
wts, Ω_1, Ω_2 open (rel open $\iff$ open as Ω open)

easy → so, as $\Omega_1 \neq \emptyset$ & Ω conn $\Longrightarrow \Omega_2 = \emptyset$

($\Longleftarrow$) have poly path $\Longrightarrow$ path conn $\Longrightarrow$ conn $\square$

1.4 Continuous Func

$f : A \subseteq \mathbb{C} \to \mathbb{C}$, $f(z) = z^2$ $f(x+iy) = x^2 - y^2 + 2xyi$



or $f(z) = \bar{z}^2 = x^2 - y^2 - 2xyi$

Can param

$$f(z) = \underbrace{u(z)}_{\mathrm{Re}\, f(z)} + i\, \underbrace{v(z)}_{\mathrm{Im}\, f(z)}$$

Def Let f be a complex func on $A \subseteq \mathbb{C}$
Then $f(z)$ has a limit $z \to z_0$ (here z_0 not necess in A could be ∂A
$\underset{w_0}{\overset{\|}{}}$ if $\forall \varepsilon > 0$ $\exists \delta > 0$ $\hookrightarrow z_0 \in \bar{A}$)
s.t $|f(z) - w_0| < \varepsilon$ when $|z - z_0| < \delta$

Note: $\lim\limits_{z \to z_0} f(z) = w_0 \iff \lim\limits_{\substack{|z-z_0| \to 0 \\ z \in A}} |f(z) - w_0| = 0$

$\iff \lim\limits_{n \to \infty} f(z_n) = w_0$
$\forall$ seq $\{z_n\}_{n \in \mathbb{N}} \subset A$
so $\lim\limits_{n \to \infty} z_n = z_0$

Def If f is a func A of $\mathbb{C}$ we say that
f is cts $z_0 \in A$ is $\lim\limits_{z \to z_0} f(z) = f(z_0)$

Note: $f(z)$ is cts $\iff$ $\operatorname{Re} f(z)$, $\operatorname{Im} f(z)$ are cts
$f(z)$ is cts $\implies$ $|f(z)|$ cts
$f(z)$ is cts on cmpt set K $\implies$ f is bdd in K
$\hookrightarrow \exists M$ so $|f(z)| < M$ $\forall z \in K$
$\hookrightarrow$ max modulus attained too

1.5 Analytic functions

Def Let f be a cmplx func on A in $\mathbb{C}$ let $z_0 \in A$
The complex derivate of f at z_0 is the limit
$f'(z) = \lim\limits_{z \to z_0} \dfrac{f(z_0) - f(z)}{z - z_0}$ if it exists.
$\hookrightarrow$ complex mult

Def A function f is analytic at z_0 if it is complex diffble in an open nbhd of z_0.

Def f is analytic (holomorphic) on an open set Ω if it is diff'ble at every point of Ω

f is analytic on a closed set K → (func actually def'd on slightly larger domain) if it is diff'ble on an open set that contains K

Def An entire function is analytic on the whole $\mathbb{C}$-plane

Check f, g diff'ble $\Rightarrow$ so is $c_1 f + c_2 g$, fg)
& $\frac{f(z)}{g(z)}$ if $g(z)$ not 0 at the nbhd pt!

e.g. $f(z) = z^2 \xrightarrow{\text{limit}} \frac{z^2 - z_0^2}{z - z_0} = z + z_0 \overset{\lim}{=\!=} 2z_0$

$\therefore$ monomial $\mathbb{C}$-diff'ble $\Rightarrow$ poly $\mathbb{C}$-diff'ble everywhere
entire

- Rtl func analytic where sensible!

Thm (C-R eqn)

a) if $f(z) = u(x,y) + i\,v(x,y)$ is $\mathbb{C}$ diff'ble at z_0

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{at } z_0$$

written
$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \quad \text{at } z_0$$

b) let $f(z) = u + iv$ be a func def in $\Omega^{\text{open}} \subseteq \mathbb{C}$.
If u, v are C^1 in Ω & satisfy CR-eqn

$\Rightarrow f$ analytic in Ω & $f' = u_x + i v_x$

e.g.) $f(z) = z^2 = \underbrace{x^2 - y^2}_{u} + \underbrace{2xy}_{v} i$ → can check sat CR ✓

$f(z) = \overline{z}^2 = \underbrace{x^2 - y^2}_{u} - \underbrace{2xy}_{v} i$ → not satis CR ✗