Riemann Mapping Thm

If Ω s.c & $\Omega \neq \mathbb{C}$ $\Rightarrow \exists$ conformal map $\Omega \to \mathbb{D}$

Let $z_0 \in \Omega$ can ensure map $f : \Omega \to \mathbb{D}$ conf by post comp with aut($\mathbb{D}$)

$z_0 \longrightarrow 0$

& $f'(z_0)$ real pos

$\Rightarrow f$ is unique!

Pf 11.5 pf assuming Montel's Thm

Def univalent = analytic & injective! (surj univalent is conformal)

$\boxed{\text{Motiv}}$ Schwartz, $f : \mathbb{D} \to \mathbb{D}$ fixing origin

$\Rightarrow f$ surj $\Leftrightarrow |f'(0)| = 1$ else $f'(0) < 1$

So f surj maps that maximize $f'(0)$

Strat: fix $z_0 \in \Omega$

Define, $\mathcal{F} = \{g \mid g : \Omega \to \mathbb{D}$ univ, $g(z_0) = 0\}$

Showing $\varphi : \mathcal{F} \to \mathbb{R}_{\geq 0}$ achieves max

$g \mapsto |g'(0)|$

(i) $\mathcal{F} \neq \emptyset$

(ii) $\exists f \in \mathcal{F}$ s.t $\sup_{g \in \mathcal{F}} |g'(z_0)| = |f'(z_0)|$

(iii) f is surjective $\Rightarrow$ it is conformal $\Omega \to \mathbb{D}$

done!

Lemma (Step 1) $\mathcal{F} \neq \emptyset$

Pf note if Ω bdd easy (divide by max norm & shift)

if $\Omega \subseteq \mathbb{C}^+$ easy $\left(\frac{z-i}{z+i} : \mathbb{C}^+ \to \mathbb{D}\right)$

if $\Omega \subseteq$ slit easy (slit $\to \mathbb{C}^+$ via $\sqrt{z}$)

Since $\Omega \neq \mathbb{C}$ $\exists$ $a \in \mathbb{C} \setminus \Omega$

Since, $z \overset{h}{\longmapsto} z-a$ is nonvanishing on Ω

$\exists$ analytic branch of $h(z) = (z-a)^{1/2} = e^{\frac{1}{2}\log(z-a)}$ on Ω

by Thm 5.4.3. As Ω hsc!

Note: if $h(z_1) = \pm h(z_2)$ where $z_1, z_2 \in \Omega$

$$\Rightarrow h(z_1)^2 = h(z_2)^2$$

$$z_1 - a = z_2 - a \Rightarrow z_1 = z_2 \quad \&$$

$\therefore$ h is injective! (i.e univalent as analytic)

Furthermore since $h(z_1) = -h(z_2) \Rightarrow z_1 = z_2$

$\Rightarrow h(\Omega)$ and $-h(\Omega)$ are disjoint! (& h nonzero)

So denote $w_0 = h(z_0)$.

Since analytic maps are open, $h(\Omega)$ is open, so is $-h(\Omega)$

$\Rightarrow \exists\ \varepsilon > 0$ st $D_\varepsilon(-w_0) \subseteq -h(\Omega)$

i.e $|h(z) + w_0| > \varepsilon \quad \forall z \in \Omega$ $\quad (-(-w_0))$

$\Rightarrow g(z) = \dfrac{\varepsilon}{h(z) + w_0}$ Then g univ & inj

$g(\Omega) \subseteq \mathbb{D}$

$\Rightarrow$ g post comp with $\mathrm{aut}(\mathbb{D}) \in \mathcal{F}$

$\therefore \boxed{\mathcal{F} \neq \phi}$.

Thm (Montel) let $\mathcal{A}$ be a family of analytic func on U^{domain}.

Spse, $\mathcal{A}$ is uniformly bdd

(i.e $\exists\ B > 0$ st $|g(z)| \leq B \ \forall g \in \mathcal{A} \ \forall z \in U$)

$\mathcal{A}$ is a normal family. $\Big\{$ Then, every seq of func from $\mathcal{A}$ has a subseq. That converges uniformly in all compact sso Ω

Note: The limit funct is not necessarily in $\mathcal{F}$

Pf (ii) Assuming Montel

Note by Step 1 $\mathcal{F}$ not empty.

let $s = \sup_{g\in\mathcal{F}} |g'(z_0)|$

$\Rightarrow \exists$ seq $f_1, f_2, f_3, \ldots$ so $\lim_{n\to\infty} |f_n'(z_0)| = s$

Note $\mathcal{F}$ is unif bdd w/ const $B=1$ as $\to \mathbb{D}$

So, hypothesis of Montel ✓.

$\exists$ ssq $f_{k_1}, f_{k_2}, \ldots$ that conv. to f unif on every compact subset of Ω.

Clear $f(z_0)=0$ as $f_{k_\ell}(z_0)=0 \ \forall \ell$

Can check $f:\Omega\to\mathbb{D}$ (not just $\overline{\mathbb{D}}$ as f open)

By Weierstrass, f is analytic!

Injective, this was homework! (Hurwitz Thm HW 8.2)

So, $f\in\mathcal{F}$

Also $|f'(z_0)| = \lim_{n\to\infty}|f_{k_n}'(z_0)|$ (by unif converg on cmpct weier)

$\Rightarrow$

$\therefore$ $\boxed{\text{found required } f}$.

Pf (iii) f as abv is surj!

$\Omega \xrightarrow{f} \mathbb{D}$

$f(\Omega)\subseteq\mathbb{D}$

if f doesn't fill $\mathbb{D}$ will find $g\in\mathcal{F}$ so $|g'(z_0)| > |f'(z_0)|$ oops as sup

lemma 1 Let U be hsc domain st $U \subseteq \mathbb{D}$ & $0 \in U$. If $U \neq \mathbb{D} \Rightarrow \exists$ univ $\psi : U \to \mathbb{D}$ s.t. $|\psi'(0)| > 1$, $\psi(0) = 0$.

Rmk: This gives key result by composition & chain rule

Cor 1 f is surj,

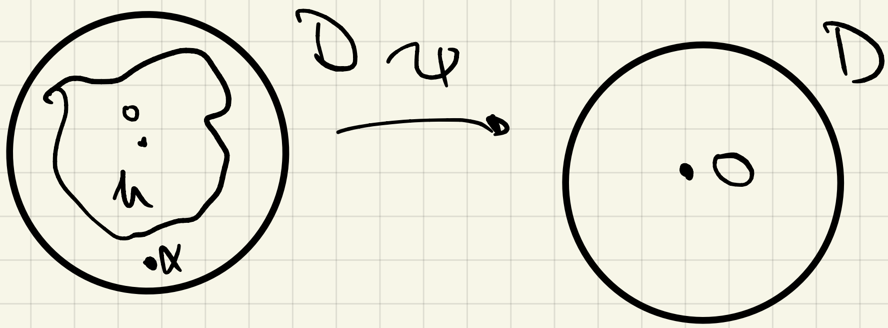
Pf If not find ψ on $f(\Omega)$

$\Rightarrow \psi \circ f : \Omega \to \mathbb{D}$ & $\psi \circ f \in \mathcal{F}$

& $|(\psi \circ f)'(z_0)| = |f'(z_0)||\psi'(0)| > |f'(z_0)|$ oops!

as f is sup of such

Pf of lemma 1



as $U \neq \mathbb{D}$ $\exists\, \alpha \in \mathbb{D} \setminus U$

let $g(z) = \dfrac{z - \alpha}{1 - \bar{\alpha} z}$. Let $V = g(U)$ img.

note, $\alpha \notin U \Rightarrow 0 \in V$

V is hsc as U is & g is conformal & $0 \in V$

So, $z \longmapsto z$ is nonvanishing on V

So can get br of $\log z$

$\exists$ analytic br of $h(z) = z^{1/2} = e^{\frac{1}{2}\log z}$ on V

let $\beta = h \circ g(0) = h(-\alpha) = (-\alpha)^{1/2}$

let $f(z) = \dfrac{z - \beta}{1 - \bar{\beta} z}$ so $f(\beta) = 0$

let $\psi = f \circ h \circ g$ Note $\psi(0) = 0$

as f, g, h all anal $\Rightarrow \psi$ anal.

g, f are inj & h as shown earlier is inj

$\Rightarrow \psi$ is inj $\Rightarrow$ ψ is univ.

Next, $\psi'(0) = g'(0) \cdot h'(g(0)) \cdot f'(\beta)$ ($g(0) = -\alpha$)

$g'(z) = \frac{1 - |\alpha|^2}{(1 - \bar{\alpha} z)^2} = (1 - |\alpha|^2) \left(\frac{1}{2h(-\alpha)} \right)$ · check

by prev reln $= (1 - |\beta|^4) \cdot \left(\frac{1}{2\beta} \right) \cdot \frac{1 - |\beta|^2}{1 -}$