Recall

Thm (General Cauchy)

If f is analytic in a domain Ω,
Then $\oint_\gamma f = 0$ $\forall$ cycles γ that is homologous to 0 in Ω

$w_\gamma(a) = 0$
winding number 0 $\forall\, a \notin \Omega$

Pf ① Assume Ω is bounded.

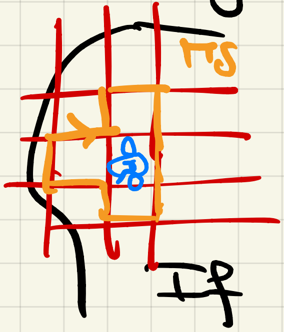
For all $\delta > 0$, $\exists$ net of squares of side length δ

Let Q_j, $j \in J$ be the closed solid squares cont. in Ω.

Denote $\Omega_0 = \operatorname{int}\left(\bigcup_{j\in J} Q_j\right)$ $\to$ so the edges of Q_j included is precisely those contained in 2 Q_j's

Cycle $\Gamma_\delta = \sum_{j\in J} \partial Q_j$ $\leftarrow$ oriented usually

$\to$ really $\partial\Omega_0 = \Gamma_\delta$.



If $w \in \Gamma_\delta \iff w \in Q_j$ for some j & $w \in Q$ for Q not a Q_j $\to$ it is slightly out of Ω.

$\to$ rest get killed by orientations

② Let γ cycle st $\gamma \sim 0$ in Ω
&
f analytic on Ω

take δ small enough so γ cont in Ω_0.

For $z \in \operatorname{Int}(Q_{j_0})$ $j_0 \in J$

then by CIF over ∂Q (hw2)

$$f(z) = \frac{1}{2\pi i}\oint_{\partial Q_{j_0}} \frac{f(w)}{w-z}\,dw$$

Also by Cauchy's thm for $\star$-conv dom.

$$0 = \oint_{\partial Q_j} \frac{f(w)}{w-z}\,dw \quad \text{for } j \neq j_0$$

since $z \notin Q_j$ so analytic always

So, $f(z) = \oint_{\Gamma_\delta} \frac{f(w)}{w-z}\, dz$ for $z \in \mathrm{Int}(Q_{j_0})$ for $j_0 \in J$

↳ sum of Q_j's → most 0 inc double ∂Q_j

By limiting argument, even for $\boxed{z \in \Omega_0}$ abv holds

↳ Both sides are cts func of z in Ω_0

Thus, $\oint_\gamma f(z)\, dz = \oint_\gamma \frac{1}{2\pi i} \oint_{\Gamma_\delta} \frac{f(w)}{w-z}\, dw\, dz$

↳ in Ω_0 → never 0 so cts in $(w,z) \in \Gamma_\delta \times \gamma$

$$= \frac{1}{2\pi i} \oint_{\Gamma_\delta} \oint_\gamma \frac{f(w)}{w-z}\, dz\, dw$$

$$= -\oint_{\Gamma_\delta} f(w)\, W_\gamma(w)\, dw$$

③ $W_\gamma(w)$ for $w \in \Gamma_\delta$. wts $\boxed{\text{this is } 0}$

→ for $a \in \mathbb{C}\setminus\Omega$, $W_\gamma(a) = 0$ by homologous. → can't just reason outside region not s.c.

In particular, points on Γ_δ will border a square not entirely in Ω

$Q \to a \in Q \cap \mathbb{C}\setminus\Omega$ ↳ so for this square winding num is 0

$\Rightarrow \boxed{W_\gamma(w) = 0}$ ← Q doesn't int γ

↳ in one conn comp det by γ

↳ W_γ const on Q

so abv integral is 0!!

④ Space Ω is unbdd.

let γ be a cycle in Ω & f analytic in Ω.

bnd

$\exists$ disc D so γ cont in D as cmpt

let $\Omega' = \Omega \cap D$ → bdd domain.

If we show $\gamma \sim 0$ in Ω' done by appy prev.

$\Omega'^c = \Omega^c \cup D^c$ if $a \in \Omega^c$ clear $W_\gamma(a) = 0$ by assump

if $a \in D^c$ then, outside

$\therefore \boxed{\gamma \sim 0 \text{ on } \Omega'}$

disc winding num is 0!

γ cont in disc. $\square$

Thm (General Cauchy Int formula)

If f analytic on domain Ω, $\forall \gamma \sim 0$ in Ω $\forall z \in \mathbb{C} \setminus \gamma$

$$\Rightarrow W_\gamma(z) f(z) = \frac{1}{2\pi i} \oint_\gamma \frac{f(w)}{w-z} dw$$

Pf) let $z_0 \in (\mathbb{C} \setminus \gamma) \cap \Omega$

$$g(w) = \begin{cases} \dfrac{f(w) - f(z_0)}{w - z_0} & w \in \Omega \setminus \{z_0\} \\ f'(z_0) & w = z_0 \end{cases}$$

$\Rightarrow g$ analytic on Ω (why check at z_0).

By general Cauchy Thm,

$$\oint_\gamma g(w) dw = 0$$

if $z_0 \in \mathbb{C} \setminus \gamma$ not in Ω

$\Rightarrow \oint \frac{f(w)}{w - z_0} dw = 0$ by Cauchy! (winding number 0 by $\gamma \sim 0$)

5.4 Holomorphically Simply Connected Domains

Recall holomorphic = analytic.

Def) A domain Ω is holomorphically simply connected (h.s.c) if every cycle, γ in Ω is homologous to 0 in Ω.

Thm1 (equiv of h.s.c-ness)

Let Ω be a domain. TFAE

(a) Ω is h.s.c (i.e $\forall$ cycle $\gamma \in \Omega$, $W_\gamma(a)=0$ $\forall a \in \mathbb{C}\setminus\Omega$)

(b) (Cauchy's thm for h.s.c domain)

a ⟹ b by gen CT

$\forall$ analytic f on Ω $\forall$ cycle γ in Ω

$\oint_\gamma f(z)\,dz = 0$

b ⟹ c by hw

(c) Every analytic function f on Ω has a primitive.

(d) Every nonvanishing analytic func f on Ω, there is an analytic branch of $\log f(z)$ on Ω.

(e) $\mathbb{C}^*\setminus\Omega$ is connected subset of $\mathbb{C}^*$

(f) every conn comp of $\mathbb{C}\setminus\Omega$ is unbdd.

Pf) a ⟹ b follows from gen Cauchy's thm

b ⟸ c was proved in HW 2.3

c ⟹ d the func $\dfrac{f'(z)}{f(z)}$ is analytic in Ω

by (c) $\exists$ prim $G(z)$ so $G' = \dfrac{f'}{f}$ in Ω

Thus $\dfrac{\partial}{\partial z}\left(e^{-G(z)} f(z)\right)$

$$= e^{-G(z)}\left(-G'(z) f(z) + f'(z)\right) = 0$$

$\Rightarrow e^{-G(z)} f(z)$ is const on Ω

Fix $z_0 \in \Omega$. Let $\log f(z_0)$ be any of the possible vals

Then $L(z) = G(z) - G(z_0) + \log f(z_0)$ is an analytic branch of $\log f(z)$ in Ω as $e^{-L(z)} f(z) = 1$

(d => a)

let γ be a cycle in Ω & $a \notin \Omega$ wts $W_\gamma(a) = 0$

the map $f(z) - \frac{1}{z-a}$ is non-v & analy on Ω

$\therefore$ by (d), $\exists$ analytic br of $\log f(z) = L(z)$ in Ω

$\Rightarrow L'(z) = \frac{1}{z-a}$ i.e. L is a prim of f

$\therefore 2\pi i \, W_\gamma(a) = \oint_\gamma \frac{1}{z-a} dz = 0$ has prim

So have $\boxed{a, \dots, d \text{ equiv}}$.

(e <=> f) purely topological. Hella true!

(f => a) let $\gamma \subset \Omega$ with comp of $\mathbb{C} \setminus \Omega$ unbdd.

If $a \in \mathbb{C} \setminus \Omega$, then a in the unbdd conn comp det by γ too.

$\therefore \boxed{W_\gamma(a) = 0}$

(a => f) later

(eg) Consider $\Omega = \mathbb{C} \setminus (-\infty, 0]$ → this is s.c.

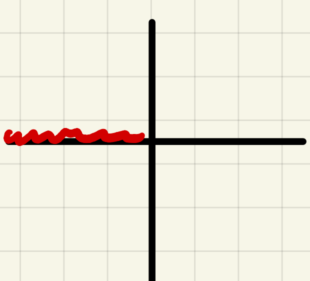
let $f(z) = z^2$ → analytic + nonv in Ω.

By thm $\exists$ analytic branch of

$L(z) = \log(z^2)$ on Ω

Not same as $\operatorname{Log}(z^2) \neq L(z)$

→ not good if $z^2 < 0$ ie $z \in i\mathbb{R}$



$2 \log z$ analy in Ω & good