Recall |

Cor | (of Jensen's Formula)

f is analytic in $\overline{D_R(0)}$ & $f(0)\neq 0$

$\Rightarrow N(r) = \#$ zeros of f in $|z|<r$ w/ mult.

If $f(z)\neq 0$ $\forall z\in C_R(0)$, then

$$\int_0^R \frac{N(r)}{r}\,dr = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\,d\theta - \log|f(0)|$$

6.5 Func. of finite order of growth

Def | entire map f is of finite order if

$\exists \rho\geq 0$, $\exists A,B>0$ so $|f(z)|\leq Ae^{B|z|^\rho}$ $\forall z\in\mathbb{C}$

$\longrightarrow \log|f(e^{i\theta})|\leq BR^\rho + \log A$ $\forall \theta\in[0,2\pi)$

The order of growth of an entire map $\rho_0 = \inf\limits_{\text{possible } \rho} \rho$

$\hookrightarrow$ note: inf may not be achievable.

In practice will say,

"the order of f is at most ρ_0"

$\Longleftrightarrow$

$\forall \rho>\rho_0$ $\exists A,B$ so $|f(z)|\leq Ae^{B|z|^\rho}$ $\forall z\in\mathbb{C}$

eg | 1) e^{z^2}, e^{-3z^2}, $\sin(z^2)$, $z^4e^{2z^2+4z+1}$ have order ≤ 2.

$e^{z^2} \leq e^{|z|^2}$

Recall
$|e^w| = e^{\operatorname{Re} w}\leq e^{|w|}$

2) $\cos(z^{1/2}) = 1-\frac{z}{2!}+\frac{z^2}{4!}-\cdots$

$\hookrightarrow \frac{e^{iz/2}+e^{-iz/2}}{2}$ $\Rightarrow$ order $\leq\frac{1}{2}$

3) polynomial $P(z)$ has order ≤ 0

$|z^5|\leq e^{5|\log z|}$

4) $f(z) = e^{P(z)} \Rightarrow$ order $f \leq \deg P$.

Check! orders are also exact!

note bound condit → s.t.

if $\exists R>0, \exists A', B' > 0$ s.t.

$|f(z)| \leq A' e^{B'|z|^\rho}, \ \forall |z| > R$ — so it suffices for large enough $|z|$.

$\Rightarrow \exists A, B > 0$ s.t. $|f(z)| \leq A e^{B|z|^\rho}$

Lemma) f, g entire, order $f \leq \rho_1$, ord $g \leq \rho_2$

$\Rightarrow fg$ order $\leq \rho_1 + \rho_2$, $f+g$ order $\leq \max\{\rho_1, \rho_2\}$

Pf) follows from def!

Lemma) f is entire order $\leq \rho_0$

f has a zero of order n at $z=0$

$\Rightarrow g(z) = \frac{f(z)}{z^n}$ extends to analytic at origin — so define entire map

$\rightarrow$ $\boxed{f \text{ is order} \leq \rho_0}$.

Pf) let $\rho > \rho_0$ by assumpt of f $\exists A, B$ so $|f(z)| \leq Ae^{B|z|^\rho}$

$\leadsto \forall |z| \geq 1$ have $|g(z)| = \left|\frac{f(z)}{z^n}\right| < f(z)$ "

Cor) $\boxed{\text{ok}}$.

$\Rightarrow \boxed{\text{ord } g \leq \rho_0}$.

D.

Thm) (growth of entire fn vs. # zeros) → $\neq 0$

let f be a nontrivial entire func

let $N(r) = \#$ zeros of f in $|z| < r$ w/ mult

If f has ord $\leq \rho_0 \Rightarrow \forall \rho > \rho_0 \ \exists$ const $c_1 > 0, c_2 > 0$

So, $N(r) \leq c_1 r^\rho + c_2 \quad \forall r > 0$

eg $f(z) = \sin z$ → zeros πn → $N(r) \leq \frac{2r}{\pi} + 3$ ish

$g(z) = \sin(z^2)$ → zeros $\pm\sqrt{\pi}\sqrt{n}$ $n \in \mathbb{Z}$ → $N(r) \leq 4\frac{r^2}{\pi} + 9$

Pf) (a) Spse $f(0) \neq 0$ let $\rho > \rho_0$

By assump $\exists A, B > 0$ so $|f(z)| \leq Ae^{B|z|^\rho}$ $\forall z$

From corollary of Jensen's, true $\forall R$ so use $2r$

$$N(r)\int_r^{2r} \frac{1}{x}dx \leq \int_r^{2r} \frac{N(x)}{x}dx \leq \int_0^{2r} \frac{N(x)}{x}dx = \frac{1}{2\pi}\int_0^{2\pi} \log|f(2re^{i\theta})|d\theta - \log|f(0)|$$

non-neg

$\hookrightarrow N$ is increasing

$N(r)\log 2$

ML + our bd

$$\leq B2^\rho r^\rho + \log A - \log|f(0)|$$

$\Rightarrow$ divide by $\log 2$ $\square$.

(b) let $g(z) = \frac{f(z)}{z^n}$ $\rightarrow$ order of 0 at 0 $\rightarrow$ differ in zeros by n from f.

apply pu to g

Lemma (number bd $\Rightarrow$ sum converge)

let $\{a_n\}_{n=1}^\infty$ be a seq of nonzero complex num repititions allow but $|a_n| \to +\infty$ as $n \to \infty$

let $N(r) = \#\ a_n$ in $|z| < r$

If $\exists c_1 > 0$, $\exists c_2 \geq 0$ so $N(r) \leq c_1 r^\rho + c_2$ $\forall r > 0$

$\Rightarrow \sum_{n=1}^\infty \frac{1}{|a_n|^s} < \infty$, $\forall s > \rho$

Pf) let $s > \rho$ break down into Annulus.

inside is only finite

$$\sum_{n:|a_n|\geq 1} \frac{1}{|a_n|^s} = \sum_{k=0}^\infty \sum_{n:\ 2^k \leq |a_n| \leq 2^{k+1}} \frac{1}{|a_n|^s}$$

$$\leq \sum_{k=0}^\infty \frac{N(2^{k+1})}{(2^k)^s} \leq \sum_{k=0}^\infty \frac{c_1(2^{k+1})^\rho + c_2}{(2^k)^s}$$

$$= c_1 2^\rho \sum_{k=0}^\infty \left(\frac{1}{2^{(s-\rho)}}\right)^k + c_2 \sum_{k=0}^\infty \left(\frac{1}{2^s}\right)^k$$

$\underbrace{s-\rho}_{>0}$

$< \infty$ as $s-\rho, s > 0$

Cor | Let f be non-trivial function, order $\le \rho_0$

Let $a_1, a_2, \ldots$ be the nonzero zeros of f listed w/ mult

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{|a_n|^s} < \infty \quad \forall s > \rho_0$

(b) $\displaystyle E(z) := \prod_{n} E_k\left(\frac{z}{a_n}\right) = \prod_n \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n} + \frac{z^2}{2a_n^2} + \cdots + \frac{z^k}{k a_n^k}}$

because finite so we don't add indices

conv. to entire fnc, $\forall$ int $k \ge [\rho_0]$ (int part of ρ_0),

eg | 1) $f(z) = e^z - 1 = e^{g(z)} z \underbrace{\prod_{n \neq 0} E_1\left(\frac{z}{2\pi i n}\right)}_{\prod_{n=1}^{\infty}\left(1 + \frac{z^2}{2\pi^2 n^2}\right)}$ comb $+$ & $-$

zeros, $z = 2\pi i n$.

2) $f(z) = e^z - z = e^{g(z)} \prod_{n=1}^{\infty} E_1\left(\frac{z}{a_n}\right)$ $a_n \to$ zeros.

9.6 Hadamard Factorization Thm.

Thm | (Hadamard's Factorization thm)

Say, f is entire, order $\le \rho_0$

Say f has a zero of ord $\underbrace{m}_{m=0 \text{ okay}}$ at $z = 0$

Let $a_1, a_2, \ldots$ be nonzero zeros of f

$$\Rightarrow f(z) = e^{p(z)} z^m \prod_n E_k\left(\frac{z}{a_n}\right) = e^{p(z)} z^m \prod_n \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n} + \cdots + \frac{z^k}{k a_n^k}}$$

where $k = [\rho_0]$

AND, p is a poly of degree $\le k$

e.g. $\sin(\pi z) = e^{az+b} z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$

Hadamard — combine the -ive zeros.

let $z \to 0$ in $\frac{\sin(\pi z)}{z}$

LHS

$\Rightarrow \pi = e^b \cdot 1 \Rightarrow b = \ln \pi$

$\Rightarrow \sin(\pi z) = e^{az} \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$

$\frac{d}{dz}$ $\xrightarrow{\text{log deriv}}$ $\frac{\pi \cos(\pi z)}{\sin(\pi z)} = a + \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{n^2 - z^2}$

$\lim_{z \to 0} \left(\frac{\pi \cos(\pi z)}{\sin(\pi z)} - \frac{1}{z} \right) = a$

L'hopital or powerseries

$$\frac{\pi\left(1 - \frac{\pi^2 z^2}{2} + \cdots\right)}{\pi z - \frac{\pi^3 z^3}{3!} + \cdots} - 1$$