Last time: Char DVR in many ways + Dedekind domains.

Thm A = noeth + dim 1. TFAE

1) A is int closed
2) Every primary ideal is a prime power
3) Every local A_p is a DVR

A is Dedekind Domain if noeth, dim 1 + (a) – (c).

Thm If A is a Dedekind domain
$\Rightarrow$ every non-zero ideal factors uniquely into prod of primes.

Pf Let $\mathfrak{a} \neq 0$ be a nonzero ideal.
Let $\mathfrak{a} = \bigcap_{i=1}^{n} q_i$ be a min primary decomp.
Let $p_i = \mathrm{rad}(q_i)$ is maximal cause dim 1.
Since $\mathrm{rad}(q_i)$, $\mathrm{rad}(q_j)$ are distinct max'l ideals
$\Rightarrow q_i, q_j$ coprime $q_i + q_j = 1$

Gen CRT $\Rightarrow \mathfrak{a} = \prod_{i=1}^{n} q_i$
Since A dedekind $\Rightarrow q_i = p_i^{m_i}$
$\Rightarrow \mathfrak{a} = \prod_{i=1}^{n} p_i^{m_i}$

Uniqueness, $\mathfrak{a} A_{p_i} = p_i^{m_i} A_{p_i}$ shows m_i unique
($\Rightarrow$ can recover).

let K = field A = fin dim'l K-alg

Given $x \in A$, $\mu_x : A \to A$, $y \mapsto xy$ } K-linear map.

def: $\mathrm{tr}\, x = \mathrm{tr}\, \mu_x$ so $\mathrm{tr} : A \to K$ is K-lin

Can use this to build sym bilin form on A

$A \times A \to K$, $(x,y) \mapsto \mathrm{tr}(xy)$ } trace pairing

$A \to A \otimes_K \bar{K}$ ($\bar{K}$: alg cl.), E_{alg}

Criterion for sep

If $L|K$ is a finite field ext.
Then $L|K$ is sep $\iff$ trace pairing non-degen on L

Sk of pf 1 The trace pairing on L rel to K is non degen

$\Updownarrow$

The ______ on $\overline{K}\otimes_K L$ rel to $\overline{K}$ ______

Say L is separable $\Rightarrow \overline{K}\otimes_K L \cong \prod_{i=1}^{n} \overline{K}$

$\hookrightarrow$ if $L \cong K[x]/(f(x))$ $\rightsquigarrow$ irred sep.

$\Rightarrow \overline{K}[x]/(f(x))$

$\quad \| $

$\prod_{i=1}^{n}(x-\alpha_i)$ $\rightsquigarrow$ α_i dist

$\overset{CRT}{\cong} \prod_{i=1}^{n} \overline{K}$

$\Rightarrow$ tr pairing non degen

Say L is insepar ble $\Rightarrow \overline{K}\otimes_K L$ non reduced

$\hookrightarrow$ a nonzero nilp elt will be in the kernel of tr pairing.

$\hookrightarrow$ eg. $A = K[x]/(x^n)$ basis $1, \dots, x^{n-1}$

$\mu_x = \begin{pmatrix} 0 & 0 & & 0 \\ 1 & 0 & \cdots & \vdots \\ & 1 & & 0 \end{pmatrix} \Rightarrow \operatorname{tr}\mu_x = 0$

Same reasoning $\Rightarrow \operatorname{tr}(\text{nilp}) = 0$

So if x nilp $\Rightarrow \operatorname{tr}(xy) = 0 \ \forall$ b/c xy nilp

Prop 1 A int cl. domain, $K = \operatorname{Frac}(A)$, $L|K$ = fin sep ext.

B = int cl. of A in L, $\exists\, v_1, \dots, v_n \in L$

st $\underbrace{B \subset \sum_{i=1}^{n} A v_i}$

$\hookrightarrow$ as A-mod, B sub of fin gen

$\Rightarrow$ if A noeth $\Rightarrow B$ fin gen! as A-mod

Pf given $x \in L$ $\exists$ eqn $a_n x^n + \cdots + a_0 = 0$ $a_i \in A$

$\Rightarrow$ mult by a_n^{n-1}, see $a_n x$ is int / A. ($\in B$)

$\exists$ K-basis of L $u_1, \ldots, u_n$ w/ $u_i \in B$

Trace pairing on L non-degen $\Rightarrow$ $\exists v_1, \ldots, v_n \in L$
$\hookrightarrow$ dual basis to u_i wrt trace pair

st $tr(u_i v_j) = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & \text{else} \end{cases}$

Say $x \in B$. Write $x = \sum_{i=1}^n y_i v_i$ for $y_i \in K$
& $y_i = tr(\underbrace{u_i x}_{B}) \boxed{\in A}$

$tr(B) \subset A$ b/c its $\subset K$ & if $x \in B$ $tr(x) = \underbrace{\text{sum of conj}}_{\text{int } / A}$

$\Rightarrow tr(x) \in K \cap \text{int } / A \Rightarrow \boxed{tr(x) \in A}$.

Prop 1 A = Dedekind Domain, K = Fr(A), L/K = fin sep,
B int cl. of A in L.

$\Rightarrow$ B is dedekind dom

Pf · B is fin. gen as A module by abv

$\Rightarrow$ B is noeth as ring (as it is as a A-module so ACC will hold)

· dim B = 1 by going up thm.

· B int closed since integral closure is always int closed.

Special Case 1 if K is a numberfield.
It's ring of int (= int cl. of $\mathbb{Z}$) is a
Dedekind domain (+ f.g. as $\mathbb{Z}$ mod $\cong \mathbb{Z}^n$ as no tors)

Say, A = int domain $K = \mathrm{Frac}(A)$

Def A frac ideal of A is a A submod $\mathfrak{a} \subset K$ s.t. $\exists\, x \neq 0 \in A$ $x\mathfrak{a} \subset A$ $\left(\iff \mathfrak{a} \in \frac{1}{x} A \right)$

eg| Given $u \in K$, $\mathfrak{a} = Au$ is a frac ideal denoted (u) called princ. frac ideal

A fin gen A-submod of K is a fr. ideal !

A frac ideal is integral if $\mathfrak{a} \subset A$

We say a frac ideal $\mathfrak{a}$ is invertible if $\exists$ frac ideal $\mathfrak{b}$ so $\mathfrak{a}\mathfrak{b} = (1)$

(Deg (u) inv in (u^{-1}) $u \neq 0$

Obs: if $\mathfrak{a}$ is frac inved + $\mathfrak{a}\mathfrak{b} = (1)$

$\Rightarrow \mathfrak{b} = (1 : \mathfrak{a}) = \{ x \in K \mid x\mathfrak{a} \subset A \}$

Reason| $\mathfrak{b} \subset (1 : \mathfrak{a}) = \underbrace{(1:\mathfrak{a}) \underbrace{\mathfrak{a}\mathfrak{b}}_{\text{unit ideal}}}_{\subseteq \mathfrak{b}} \subset \mathfrak{b}$

Consequence, $\mathfrak{a}$ inv $\iff \mathfrak{a}(1 : \mathfrak{a}) = (1)$

Obs: if $\mathfrak{a}$ inv $\Rightarrow$ it is fin gen.

Reason| $\mathfrak{a}(1:\mathfrak{a}) = (1) \Rightarrow 1 = \prod_{i=1}^{n} x_i y_i$ $\quad x_i \in \mathfrak{a}$, $y_i \in (1 : \mathfrak{a})$

Say $z \in \mathfrak{a} \Rightarrow \sum_{i=1}^{n} \underbrace{(z y_i)}_{\text{in } A} x_i$

$\Rightarrow \mathfrak{a}$ is gen'd by x_i (as a frac ideal)

Prop) $\mathfrak{a} \subset K$ frac ideal. TFAE.

(a) $\mathfrak{a}$ is inv

(b) $\mathfrak{a}$ is f.g + $\mathfrak{a}_p$ is inv. frac ideal of A_p $\forall$ pr p

(c) ——— $\mathfrak{a}_m$ ——— A_m $\forall$ max'l m

Pf) (a) $\Rightarrow$ (b)

So $\mathfrak{a}$ inv $\Rightarrow$ $\mathfrak{a}$ fin gen & $\underbrace{\mathfrak{a}(1:\mathfrak{a}) = (1)}_{\text{local at } p}$

Old facts $\Rightarrow$ $\mathfrak{a}_p(1:\mathfrak{a}_p) = (1)$

$\hookrightarrow (\mathfrak{a}\mathfrak{b})_p = \mathfrak{a}_p \mathfrak{b}_p$

$\hookrightarrow (1:\mathfrak{a})_p = (1:\mathfrak{a}_p)$ $\rightarrow$ if $\mathfrak{a}$ f.g

(b) $\Rightarrow$ (c) triv.

(c) $\Rightarrow$ (a) $\mathfrak{a}(1:\mathfrak{a}) \subset A$ $\rightarrow$ is eq after localizing at any m

$\Rightarrow$ it is equality to begin

Prop) A is a local domain

A is DVR $\Longleftrightarrow$ all nonzero frac ideals are inv.

Pf) Say A is DVR, let π = unif of A, $(\pi) = m$

Say $\mathfrak{a} \subset K$ is a non-z frac ideal.

By defn $\exists\, x \neq 0 \in A$ st $x\mathfrak{a} \subset A$.

$\Rightarrow x\mathfrak{a} = (\pi^n)$ for some n & $x = u \cdot \pi^m$ $u \in A^\times$

$\Rightarrow \mathfrak{a} = (\pi^r)$ for $r = n - m$ & $\mathfrak{a}^{-1} = (\pi^{-r})$

Say all nonzero frac ideals inv.

ETS, every nonzero ideal of A is pow of m (then a DVR).

A is noeth b/c every ideal inv $\Rightarrow$ f.g.

If not true $\exists$ max'l counterexample $\mathfrak{a} \subsetneq m$

$\Rightarrow$ $m^{-1}\mathfrak{a} \subset A$ so $\mathfrak{a} \subset m^{-1}\mathfrak{a}$ proper containment

Since if $\mathfrak{a} = m^{-1}\mathfrak{a}$ $\Rightarrow$ $m\mathfrak{a} = \mathfrak{a}$ $\Rightarrow$ $\mathfrak{a} = 0$ by Nakayama

So, by maxi'l, $m^{-1}\mathfrak{a} = m^{n}$ $\Rightarrow$ $\mathfrak{a} = m^{n+1}$

Prop A = int domain

A dedekind domain $\Leftrightarrow$ all nonzero frac ideal inv.

Pf Say A is Dedekind, $\mathfrak{a}$ - nonzero frac ideal.

Know, $\mathfrak{a}$ is fg as $\mathfrak{a} \subset \frac{1}{x}A$] fg as A mod

$\Rightarrow$ $\mathfrak{a}$ is fg b/c A noeth

By prev prop, $\mathfrak{a}_p$ is inv $\forall p^{\text{max}}$ $\Rightarrow$ $\mathfrak{a}$ is inv!

(A_p is DVR)

Conv, All ideals of A inv

$\Rightarrow$ all fg so A is noeth

ETS, all localizations are DVR A_p, $p \neq 0$

Let $b \subset A_p$ be ideal, $\mathfrak{a} = b \cap A$.

$\mathfrak{a}$ is inv + $b = \mathfrak{a}_p$ $\Rightarrow$ b is inv!

prev prop $\Rightarrow$ A_p is DVR.

Cor In a dedekind domain, the nonzero frac ideals form a group under $\cdot$

Fix dedekind domain A, let I be the abel grp!

Note, by unique fact, I is free ab grp on pr. ideals of A.

let $P \subset A$ be the princ. frac'l ideals (subgrp)