Missed some stuff

Boij - Söderberg Thm

$S = K[x_1, \ldots, x_n]$, M, fg gr S-mod.

min res

$\cdots \to F_1 \to F_0 \to M \to 0$ $\qquad F_i = \bigoplus_{j \geq 0} S[j]^{\oplus B_{ij}}$

(b Syzygy $\Rightarrow$ stops after n step.

$B(M) = (B_{ij})$ is the Betti Table for M

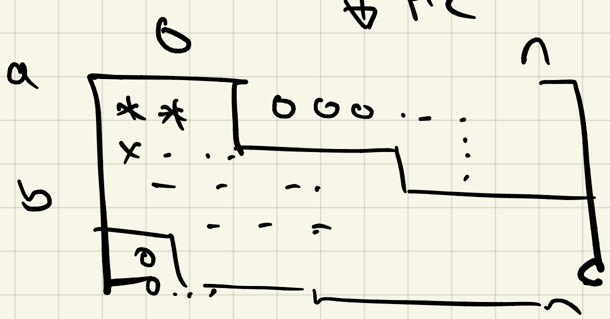
Pr: what are possible vals of B_{ij}?

Def) A degree seq is $0 \leq a_0 < a_1 < \cdots < a_n$ a_i's int.

Given deg seq $a < b$ ($a_i < b_i$)

let $D(a,b) = \{ (B_{ij}) \mid B_{ij} \in \mathbb{Q}$ $B_{ij} \neq 0$ only $a_i \leq j \leq b_i \}$

$\downarrow$ pic



$D(a,b)$ fg $\mathbb{Q}$ vs!

$L(a,b) \subset D(a,b)$ $\mathbb{Q}$ vs sp by $B(M)$'s where M is fin length module

$B(a,b) \subset L(a,b)$ this all the rays do form $+ B(M)$ (b $\mathbb{Q}_{\geq 0}$

$\hookrightarrow$ Boij - Söderberg describes $B(a,b)$.

Obs) $B(a,b)$ is a cone, $x, y \in B(a,b)$ $\alpha, \beta \in \mathbb{Q}_{\geq 0}$

$\to \alpha x + \beta y \in \mathbb{Q}_{\geq 0}$

Idea) $B(M \oplus M') = B(M) + B(M')$

$0 \to F_n \to \cdots \to F_0 \to M \to 0$

$H_M(t) = \sum_{i=0}^{n} (-1)^i H_{F_i}(t)$ $\quad H_S(t) = \dfrac{1}{(1-t)^n}$

$F_i = \bigoplus_{j\geq 0} S[j]^{\oplus B_{ij}}$ $\quad S_{S[j]}(t) = \dfrac{t}{(1-t)^n}$

$S_{S[d]}(t) = \dfrac{t^d}{(1-t)^n}$

$H_{F_i} = \dfrac{\sum_{j\geq 0} B_{ij} t^j}{(1-t)^n}$

$\Rightarrow (1-t)^n H_M(t) = \sum_{ij} (-1)^i B_{ij} t^j$ $\to$ polynomial as M has fin len!

$\to$ diff this p times $0 \leq p \leq n-1$

& plug $t=1$

$\to$ actually doing $t \cdot \frac{\partial}{\partial t}$

$\to$ $\boxed{0 = \sum_{ij} (-1)^i j^p B_{ij}}$ Herzog-Kühl eqns

Write $L^{HK} \subset \mathbb{D}(a,b)$ set eqns

$L(a,b) \subset L^{HK}$

Def) d-degree seq

A table, (B_{ij}) is pure (of type d) $B_{ij} \neq 0$ only for $j = d_i$

Ls eg Koszul.

Fixed d $\exists$ 1-d sp of solns for H-K eqn for pure tables of type d $\to \pi(d)$.

The soln is, $B_{i,d_i} = (-1)^i \prod_{k\neq i} \dfrac{1}{(d_k - d_i)}$ (this is >0)

Prop: If $a = d^1 < d^2 < \cdots < d^r = b$ is a mx chain

$\Rightarrow \pi(d^1), \ldots, \pi(d^r)$

$\mathbb{Q}$-basis of $L^{HK}(a,b)$

$\Rightarrow \dim L^{HK}(a,b) = 1 + \Sigma(b_i - a_i)$

Boij - Söderberg Conj 1

Some scalar mult of $\pi(d)$ occurs as $\beta(M)$ fin len M

Cor $L(a,b) = L^{HK}(a,b)$

B-S conj 2

The $\pi(d)$'s gen $B(a,b)$. Ie if M is fin length mod

$\Rightarrow \exists\ d^1, \_\_\_, d^r$ deg seq $c_1, \_\_\_, c_r \in \mathbb{Q}_{\geq 0}$

$\Rightarrow \beta(M) = c_1 \pi(d^1) + \_\_ + c_r \pi(d^r)$

Conj 1: Proved by Eisenbud - Fløystad - Weyman in 2007

pf uses rep thy of GL_n (in char 0).

Conj 2 Proved by Eisenbud - Schreyer in 2007

two key pts

- Give new constr of pure res, works in char p.
- Dual btw cohom tables & Betti tables.

What abt res over non poly ring?

In the graded case non-poly $\Rightarrow$ non-reg $\Rightarrow$ ∞ gl dim.

Let S be gr ring, M = fg gr S mod

$F_\bullet \to M$ min free res

Poincaré Series $\quad P_M(t) = \sum_{i \geq 0} \operatorname{rank}(F_i)\, t^i$

Serre: rat'll func $\longrightarrow$ This is false!

Disproven.

$S = k[x_1, \ldots, x_n] \qquad R = S/(f) \qquad f$ is homog deg d

$f \neq 0$

Thm (Eisenbud) Min free res of R mod is eventually periodic w/ period 2.

$\longrightarrow$ kick in after n steps.

$$F_{n+2} \xrightarrow{\partial_{n+2}} F_{n+1} \xrightarrow{\partial_{n+1}} F_n \cdots\cdots \to F_0 \to M \to 0$$

$\left.\begin{array}{l} F_i = F_{i+2} \\ \partial_i = \partial_{i+2} \end{array}\right\}$ for $i \geq n$

$$\begin{array}{ccccc} F_{n+2} & \xrightarrow{\partial_{n+2}} & F_{n+1} & \xrightarrow{\partial_{n+1}} & F_n \\ \| & & \| & & \| \\ R^{\oplus a_{n+2}} & & R^{\oplus a_{n+1}} & & R^{\oplus a_n} \end{array}$$

& $a_{n+2} = a_n$ $\qquad \partial_{n+2}$ is mat $a_{n+1} \times a_{n+2}$ w R entries

∂_{n+1} —— $a_n \times a_{n+1}$ ——

Say $\partial_{n+1} = A \in M_{a_{n+1}, a_n}(R)$

$\partial_{n+2} = B \in M_{a_{n+2}, a_{n+1}}(R)$

Chose lifts of $\tilde{A}$ & $\tilde{B}$ to S

as $\partial^2 = 0 \Rightarrow AB = BA = 0 \Rightarrow \tilde{A}\cdot\tilde{B}, \tilde{B}\tilde{A}$ entr divis by f

Eisenbud pt common choice of $\tilde{A}, \tilde{B}$ K

$$\left.\tilde{A}\cdot\tilde{B} = f\cdot \mathrm{id} \qquad \tilde{B}\cdot\tilde{A} = f\cdot \mathrm{id}\right\}$$ Matrix factorization.