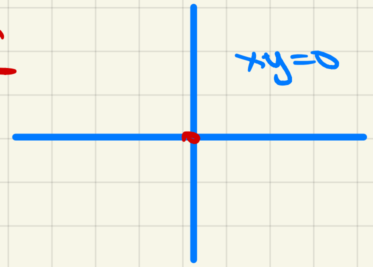
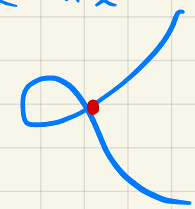
Consider



reducible variety (two irred comp)

$y^2 = x^3 + x^2$



irred var

$y = \pm x\underbrace{\sqrt{1+x}}$ ← x small this is 1.

$\sqrt{1+x} = \sum_{k=0}^{\infty} \binom{1/2}{n} x^n$

But! two pictures very similar near origin.

Say X, Y smooth curves.
We'd like it to be that X & Y are locally isomorphism.

let, G = top ab. gp → topology & ab grp, addition map, inv map } cts.
Assume G is first ctbl → ∃ fundamental system of nbhds at 0 that is ctbl
→ ∃ seq $U_1, \ldots$ nbhds of origin so if $0 \in V$ open then ∃i so that $U_i \subset V$

$a \in G$ if $t_a : G \to G$, $x \mapsto x + a$ is a homeomorphism $0 \to a$

So, the local struct at a same as 0 ⇒ $U_1 + a, U_2 + a$ local basis for a

Defn • a seq $x_\bullet \in G$ is Cauchy if
∀ open nbhd of U of 0 ∃ N so $x_n - x_m \in U$ $n, m > 0$

• Two Cauchy sequences, $x_\bullet$ & $y_\bullet$, are equiv if
$x_\bullet - y_\bullet \to 0$ i.e. ∀ open $0 \in U$ ∃ N, ∀ n > N $x_n - y_n \in U$

Defn The completion of G is the set of equiv classes of Cauchy seq

eg • Compl. of usual $\mathbb{Q}$ is $\mathbb{R}$
• Compl. of $\mathbb{Q}$ with p-adic top is $\mathbb{Q}_p$ ← p-adics
↳ p-adic abs $\left|p^n \frac{a}{b}\right|_p = p^{-n}$

Now, $0 \in G$ has a fundamental system of nbhds $G_1 \supset G_2 \supset \dots$ so G_i is a subgrp.

eg) If $G = \mathbb{Z}$ w/ p-adic top can take $G_n = p^n \cdot \mathbb{Z}$

Say, $x_\bullet$ is a Cauchy seq. Means, $\forall r$ $x_n - x_m \in G_r$ for $n,m \gg r$

$\iff x_n - x_m = 0$ in G/G_r

$\iff [x_n] = [x_m]$ in G/G_r

Put, ξ_r = img of x_n in G/G_r $n \gg r$

Note, $G/G_{r+1} \longrightarrow G/G_r$ as $G_{r+1} \subset G_r$
$\xi_{r+1} \longmapsto \xi_r$

$\rightarrow$ ξ_r are a "coherent system" of elts $G/G_\bullet$

Obs) ∘ if $x_\bullet$ & $y_\bullet$ are equiv Cauchy seq $\Rightarrow$ they define same coherent seq.

$\rightarrow$ [Reason: equiv means x_n & y_n have equal img in G/G_r $n \gg r$]

∘ Every coherent seq comes from a Cauchy seq.

[Reason: given $\xi_\bullet$, $\forall n$ choose $x_n \in G$ so $x_n \xrightarrow[\text{mod } G_n]{} \xi_n$

Say now $n > m$ $x_n - x_m \longmapsto 0$ in G/G_m $\Rightarrow$ x is Cauchy

$G/G_n \longrightarrow G/G_m$
$\xi_n \longmapsto \xi_m$

General Setup)

$\{A_n\}_{n \geq 1}$ seq of ab grps $\theta_n : A_n \to A_{n-1}$ gp homo (transition map)

$\cdots \to A_4 \xrightarrow{\theta_4} A_3 \to A_2 \to A_1$ inv. system.

Defn The inverse limit of $A_\bullet$ denoted $\varprojlim A_\bullet$ is the set of coherent system of elts

I.e an elt of $\varprojlim A_n$ is a seq $(x_n)_{n\geq 1}$ w/ $x_n \in A_n$ s.t. $\theta_{n+1}(x_{n+1}) = x_n$

Mapping property: giving a map $M \to \varprojlim A_n$ equiv to

$$\begin{tikzcd} & & & M \arrow[dll, bend right] \arrow[dl] \arrow[dr] & \\ \cdots \arrow[r] & A_3 \arrow[r] & A_n \arrow[rr] & & A_1 \end{tikzcd}$$

so it commutes.

So $\hat{G}$ (compl.) is $\varprojlim G/G_n$

Spse $A_\bullet$ & $B_\bullet$ are inverse systems

A map $f_\bullet : A_\bullet \to B_\bullet$ is a seq of grp homo $A_n \to B_n$ so diag comm.

$$\begin{tikzcd} \cdots \arrow[r] & A_3 \arrow[r] \arrow[d, "f_3"'] & A_2 \arrow[r] \arrow[d, "f_2"'] & A_1 \arrow[d, "f_1"] \\ \cdots \arrow[r] & B_3 \arrow[r] & B_2 \arrow[r] & B_1 \end{tikzcd}$$

$\Rightarrow$ $\exists$ category of inverse systems.

$\hookrightarrow$ inverse lim a funct from

$\{$category of inv system$\} \longrightarrow \{$cat of ab grp$\}$

Prop also is left exact.

Spse $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$ SES of inv systems

i.e $0 \to A_n \to B_n \to C_n \to 0$ exact

$\Rightarrow$ $0 \to \varprojlim A_\bullet \to \varprojlim B_\bullet \to \varprojlim C_\bullet$

Moreover if $A_\bullet$ is a surj sys then get 0 on right too.

$\hookrightarrow$ $\boxed{A_{n+1} \to A_n \text{ surj}}$

Pf) let $A = \prod_{n \geq 1} A_n$. By def $\varprojlim A_\bullet \subseteq A$ subgrp.

define, $d_A : A \longrightarrow A$ by $(d_A(a_\bullet))_n = a_n - \theta_{n+1}(a_{n+1})$

Note, $a_\bullet$ is coherent seq $\iff d_A(a_\bullet) = 0$

So, $\varprojlim A_\bullet = \ker d_A$

natural.

So,

$$\begin{tikzcd}
0 \arrow[r] & \ker d_A \arrow[r] \arrow[d] & \ker d_C \arrow[r] \arrow[d] & \ker d_C \arrow[d] \arrow[ddll, bend left, "\delta"] & \\
0 \arrow[r] & A \arrow[r] \arrow[d, "d_A"] & B \arrow[r] \arrow[d, "d_B"] & C \arrow[r] \arrow[d, "d_C"] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
& \operatorname{coker} d_A & \operatorname{coker} d_B & \operatorname{coker} d_C \arrow[r] & 0
\end{tikzcd}$$

prods

SES as prod of SES

So $\exists\ \delta$ so $0 \to \ker d_A \to \ker d_B \to \ker d_C \xrightarrow{\delta} \operatorname{coker} d_A \to \operatorname{coker} d_B \to \operatorname{coker} d_C \to 0$

is exact.

looking at $\boxed{}$ $\Rightarrow$ $0 \to \varprojlim A_\bullet \to \varprojlim B_\bullet \to \varprojlim C_\bullet$ exact.

If A is a surj system Sts, $\operatorname{coker} d_A = 0$
↳ for result.

Must show d_A surj if A surj system.

Spse $A_\bullet$ is surj system let $a_\bullet \in A$ be given

want $x_\bullet \in A$ so $d_A(x_\bullet) = a_\bullet \iff x_1 - \theta_2(x_2) = a_1$
$x_2 - \theta_3(x_3) = a_2$
$\vdots$

take $x_1 = 0$, x_2 s.t. $\theta(x_2) = -a_1$
x_3 s.t. $\theta(x_3) = x_2 - a_2 \cdots$

Poss as surj

Rmk: 1) $\mathrm{coker}(d_A)$ is called $\varprojlim^1 A$. This is the first right derived funct. $\varprojlim$

2) Can define $\varprojlim_{i \in I} A_i$ for any directed set I
↳ I partially ordered & $i, j \in I \to k \in I$, $k \geq i, j$

↳ in general if $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$ SES indexed by I, then A being a surj sys
$\not\Rightarrow$ $\varprojlim B_\bullet \to \varprojlim C_\bullet$ surj,

↳ also get higher derived functors $\varprojlim^k$ for $k = 1, 2, \ldots$

Prop) Consider SES

$$0 \to G' \to G \xrightarrow{\pi} G'' \to 0$$

of ab grps with G having top det by system of subgrp $G \supset G_1 \supset G_2 \supset \cdots$

Give G' the subsp. top $G' \supset G_1 \cap G \supset G_2 \cap G \supset \cdots$

Give G'' the quot top $G'' \supset \pi(G_1) \supset \pi(G_2) \cdots$

$$\Rightarrow 0 \to \widehat{G'} \to \widehat{G} \to \widehat{G''} \to 0 \text{ is \underline{exact}.}$$

Def) G is a ab grp with top det'd by $G \supset G_1 \supset G_2 \supset \cdots$ if $a + G_n$ are a fund syst of open nbhds of a

$\Longleftrightarrow$

A subset $U \subset G$ is open if $\forall a \in U$ $\exists n$ so $a + G_n \subset U$.

Pf) $0 \to G'/G' \cap G_n \to G/G_n \to G''/\pi(G_n) \to 0$ is SES

Now take $\varprojlim$ (each system is surj).

Cor Assume earlier situation.

$G_n \subset G$ subgrp.

$\hat{G}_n \subset \hat{G}$ subgrp with $\hat{G}/\hat{G}_n \cong G/G_n$

Pf Consider $0 \to G_n \to G \to G/G_n \to 0$

take compl. with induced top.

Comp of this is itself

$\hat{G}$ has a top def $\hat{G}_1 \supset \hat{G}_2 \supset \cdots$

Cor $\hat{\hat{G}} = \hat{G}$

$\hookrightarrow$ $\hat{\hat{G}}$ is inv lim of $\hat{G}/\hat{G}_n = \varprojlim G/G_n = \hat{G}$

Always a map $G \longrightarrow \hat{G}$

$a \longmapsto$ (const cauchy seq)

We say G is com