Prop let $S \subset A$ mult set

$$\left\{\begin{matrix}\text{primes in} \\ S^{-1}A\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{primes of } A \text{ not} \\ \text{meeting } S\end{matrix}\right\}$$

$$q \longmapsto q^c$$

$$p^e \longleftarrow p$$

Pf • We know $q \subset S^{-1}A$ pr $\Rightarrow q^c \subset A$ prime

• **Claim** $\forall$ ideal $b \subset S^{-1}A$ $(b^c)^e = b$

We always have $(b^c)^e \subseteq b$

$(\supseteq)$ let some $\frac{x}{s} \in b$ w/ $x \in A$, $s \in S$

$\Rightarrow \frac{x}{1} \in b \Rightarrow x \in b^c$ by map!

So, $\frac{x}{1} \in (b^c)^e \Rightarrow \frac{x}{1} \cdot \frac{1}{s} \in (b^c)^e$
$= \frac{x}{s}$

(in part true for p pr) $\blacksquare$

• **Cl** i) $\Omega \subset A$ $\Omega^e = \{\frac{x}{s} \mid x \in \Omega\}$

b/c Ω^e is the smallest ideal containing $\frac{x}{1}$

& RHS is clearly this

Saturation of Ω wrt S

ii) $(\Omega^e)^c = \{x \in A \mid \exists s \in S \text{ so } sx \in \Omega\}$

$(\supseteq)$ if $x \in$ RHS $\exists s \in S$ so $sx = y \in \Omega$

$\Rightarrow \frac{x}{1} = \frac{y}{s} \in \Omega^e$

$\Rightarrow x \in (\Omega^e)^c$

Ex $6\mathbb{Z} \subset \mathbb{Z}$

$S = \{2^n\}$

S-sat of $6\mathbb{Z}$

is $3\mathbb{Z}$

$(\subseteq)$ let $x \in (\Omega^e)^c \Rightarrow \frac{x}{1} \in \Omega^e$

$\Rightarrow \frac{x}{1} = \frac{y}{s}$ $y \in \Omega$, $s \in S$

$\Rightarrow \exists t \in S \; t(xs - y) = 0$

$\underbrace{ts}_{\in S} x = ty \in \Omega$

- let $\mathfrak{p} \subset A$ prime not meeting S.

Cl. $(\mathfrak{p}^e)^c = \mathfrak{p}$ in gen have $(\supseteq)$

let $x \in (\mathfrak{p}^e)^c$ by abv $\exists\, s \in S$ so $sx \in \mathfrak{p}$ ↳ by earlier

$\Rightarrow \mathfrak{p}$ prime $\underline{s \in \mathfrak{p}}$ or $x \in \mathfrak{p}$

↳ no by assump

$\Rightarrow x \in \mathfrak{p}$!

- Finally need $\mathfrak{p}^e$ is prime

Say $\frac{x}{s}\frac{y}{t} \in \mathfrak{p}^e \underset{\text{ideal}}{\Longrightarrow} \frac{xy}{1} \in \mathfrak{p}^e \Rightarrow xy \in (\mathfrak{p}^e)^c = \mathfrak{p}$

So, $x \in \mathfrak{p}$ or $y \in \mathfrak{p} \Rightarrow \frac{x}{s}$ or $\frac{y}{s} \in \mathfrak{p}^e$

So, $\mathfrak{p}^e$ prime!

Cor 1 $\mathfrak{p} \subset A$ prime ideal. Prop says

$$\{\text{primes of } A_\mathfrak{p}\} \longleftrightarrow \{\text{primes } \mathfrak{p}' \text{ of } A \text{ s.t. } \mathfrak{p}' \subset \mathfrak{p}\}$$

localized at $A \setminus \mathfrak{p}$

Pf Prop

$$\{\text{primes of } A_\mathfrak{p}\} \longleftrightarrow \{\text{primes of } A \text{ doesn't meet } A \setminus \mathfrak{p}\}$$

$$= \{\text{primes of } A \text{ contained in } \mathfrak{p}\}$$

Rmk $\{\text{primes in } A/\mathfrak{p}\} \longleftrightarrow \{\text{pr of } A \text{ that contain } \mathfrak{p}\}$

In some sense reverse operation.

Cor] If $f \in A$ then

$$\left\{ \begin{matrix} \text{pr of} \\ A[\tfrac{1}{f}] \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{Pr of } A \text{ not} \\ \text{containing } f \end{matrix} \right\}$$

$$\| $$

$$D(f) \subseteq \operatorname{Spec}(A)$$

Let P be a property of module (takes module ret true or f)

We say P is a local property if

$$P(M) \iff \forall \text{ primes } p \;\; P(M_p)$$

Prop] $M = A$-mod TFAE:

(1) $M = 0$

(2) $M_p = 0 \;\; \forall$ pr $p \subset A$

(3) $M_m = 0 \;\; \forall$ max'l $m \subset A$

being 0 is a local prop

Pf] Clear $a \Rightarrow b \Rightarrow c$

Now $c \Rightarrow a$ say $x \in M$ is non-zero.

Let $R = \operatorname{Ann}(x) = \{a \in A \mid ax = 0\}$ ← always an ideal. In this case $R \neq (1)$ as $1x = x \neq 0$

So $\exists$ max'l m so $R \subset m \subsetneq R$

By assump $\frac{x}{1} = 0$ in M_m

$\Rightarrow \exists a \in A \setminus m$ so $ax = 0$ so $a \in R = \operatorname{Ann}(x)$

Oops! as $R \subset m \Rightarrow a \in m$ $\square$.

Prop | Say $f : M \to N$ map of A-mod. TFAE:

(a) f is inj

(b) $f_p : M_p \to N_p$ is inj $\forall$ prime ideals p

(c) $f_m : M_m \to N_m$ " " — max'l " m

Pf | (a) $\Rightarrow$ (b) localisation is exact so it pres injection

(b) $\Rightarrow$ (c) Tautology

(c) $\to$ (b) Spse $f_m : M_m \to N_m$ inj $\forall$ max'l m

let $K = \ker f$. Since localization exact

$K_m = \operatorname{Ker}(f_m) = 0$
$\quad\ \ \|$
localized kernel of f

By prev prop $\Rightarrow K = 0$.

$$0 \to K \to M \xrightarrow{f} N$$
$$\downarrow (\text{ex})$$
$$0 \to S^{-1}K \to S^{-1}M \xrightarrow{S^{-1}f} S^{-1}N$$
$$\Rightarrow$$
$$S^{-1}K = \ker S^{-1}f$$

Rmk | Φ is a func b/w mod | ct

TFAE:

(1) Φ pres SES

(2) Φ pres any exact seq

(2) $\Rightarrow$ (1) easy (1) $\Rightarrow$ (2) think.

Prop Flatness is local.

If M an A-mod

(a) M is flat A-mod

(b) M_p is a flat A_p mod $\forall$ pr p

(c) M_m is ——— A_m ——— max'l m

Pf (a) $\Rightarrow$ (b) let N be an A_p module

$$M_p \otimes_{A_p} N = (M \otimes_A A_p) \otimes_{A_p} (N)$$

$$\overset{hw}{=} (M \otimes_A N) \quad \text{as } A_p \text{ modules}$$

(b) $\Rightarrow$ (c) easy.

(c) $\Rightarrow$ (a) alr know tensoring is righ exact
just need to show pres exactng

Say $L \to N$ is inj of A modules

$\Rightarrow L_m \to N_m$ is inj (m - max'l ideal)

$\Rightarrow L_m \otimes_{A_m} M_m \to M_m \otimes_{A_m} N_m$ is inj (M_m is flat)

$$\begin{array}{ccc} \cong\ \text{Fact (think)} & & \cong \\ (M \otimes_A L)_m & \to & (M \otimes N)_m \end{array}$$

using loc and extn sc

prev prop

$\Rightarrow M \otimes_A L \to M \otimes_A N$ is inj! $\Rightarrow M$ flat

Prop If $R \to S$ is a ring homo.
Extn of scalars $\mathrm{Mod}_R \to \mathrm{Mod}_S$. Cl. This comp w/ $\otimes$

M, N - R mods

$$(S \otimes_R M) \otimes_S (S \otimes_R N) \overset{Hw}{=} M \otimes_R (S \otimes_R N)$$

$$= (M \otimes_R N) \otimes_R S$$

Noeth + Art Rings + Mod

asc chain con.

Def) An A module is **Noetherian** if the ACC holds for submod

↳ if $N_1 \subset N_2 \subset \cdots \subset M$
$\therefore \exists\, i$ so $N_i = N_{i+1} = \cdots$

Def) An A-module is **Artinian** if the DCC holds for submod

↳ if $\cdots \subset N_2 \subset N_1 = M$
if $\exists i$ so $N_i = N_{i+1} = \cdots$

Def) A is noetherian as a ring if it is noetherian as an A-mod (so ACC holds for ideals)

A is artinian similarly!

eg) (1) If M has finitely many elt
$\Rightarrow M$ is noetherian & Art

(2) V is a fin dim'l K-vec sp
$\Longleftrightarrow$ V is noeth + Art as K-mod

($\Rightarrow$) straight forward ($\Leftarrow$) contrapos with the basis

$\Longleftrightarrow V$ is noeth!

[Field is PID & art]

(3) PID are noeth $\mathbb{Z}$, $\mathbb{C}[x]$ are noeth not art.

(4) $\mathbb{Z}$, $\mathbb{C}[x]$ not art

↳ $(2) \supset (2^2) \supset (2^3) \supset \cdots$ ⟶ DC not stab.

(b) p prime # $M = \mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$

is an artinian $\mathbb{Z}$ module

$M_n = \mathbb{Z} \cdot \frac{1}{p^n} / \mathbb{Z} \subset M$ submod, w

There are (0), (1) all the submod.

see by hand ← TSC works.

$M_n \supset M_{n-1} \supset \dots$ → p non stab asc

Not noeth as $M_{n-1} \subset M_n \subset \dots$

(c) $\mathbb{C}[x_1, x_2, \dots]$ not noeth

$(x_1) \subsetneq (x_1, x_2) \subsetneq (x_1, x_2, x_3) \subsetneq \dots$

Prop 1 An A-mod M is noeth $\iff$ all submod fin gen!

Pf Say M is noeth, $N \subset M$. Spoze N is not f.g!

Pick $x_1 \in N$, $x_2 \in N \setminus (x_1)$, $x_3 \in N \setminus (x_1, x_2)$

$\hookrightarrow \exists$ as N not fg

submod gen

$\Rightarrow$ $(x_1) \subsetneq (x_1, x_2) \subsetneq \dots$ strict ascending chain

($\Leftarrow$) let $N_1 \subset N_2 \subset N_3 \subset \dots \subset M$

consider $N_\infty = \bigcup_{i \geq 1} N_i$ by assump fin gen

$(x_1, \dots, x_n)$

if i s.t. $x_1, \dots, x_n \in N_i$

$\therefore$ $N_i = N_\infty$ so the chain stab then!

Cor | A ring is noeth iff all ideals fin gen!

Prop | given a SES of A-mods

$$0 \to M_1 \to M_2 \xrightarrow{\pi} M_3 \to 0$$

(a) M_2 noeth $\iff$ M_1, M_3 are noeth

(b) M_2 art $\iff$ M_1, M_3 are

Pf | (a) ($\Rightarrow$) is easy as chain in M_1 is chain in M_2
chain in M_3 $\rightsquigarrow$ inv img is chain in M_2

($\Leftarrow$) say M_1, M_3 are noeth

$$N_1 \subset N_2 \subset \cdots \subset M_2$$

$$\Rightarrow N_1 \cap M_1 \subset N_2 \cap M_1 \subset \cdots \subset M_1$$

$\hookrightarrow$ will stab

stabilizes as M_1 noeth

$$\pi(N_1) \subset \pi(N_2) \subset \cdots \subset M_3$$

$\Rightarrow$ original chain stabilizes.

Cor | finite direct sum preserves these conditions
$\hookrightarrow$ look at splitting SES.