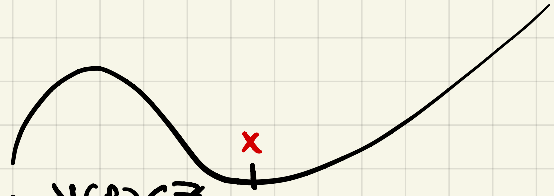
Consider

Ring A & $\mathrm{Spec}(A)$ is smooth, 1-dim
↳ Domain w/ $\mathrm{Frac}(A) = K$



given $f \in K^*$, f has some order of vanishing $V(f) \in \mathbb{Z}$ at x.

Assoc to x have map $\underline{v : K^* \longrightarrow \mathbb{Z}}$, valuation

Either f or f^{-1} doesn't have a pole at x.

I.e., $B \subset K$ be all the elts of K def'd at $x \Rightarrow f, f^{-1} \in B$

Def | A valuation ring B is an integral domain s.t. $\forall\, x \in \mathrm{Frac}(B)$, $x \neq 0$
either $x \in B$, $x^{-1} \in B$.

e.g. | • $B = \mathbb{Z}_{(p)}$ localize $\mathbb{Z}$ at $(p) \subset \mathbb{Z}$
$B =$ all rat'l #'s w no p in denom

B is a val ring b/c $\forall\, x \in \mathbb{Q}$ frac(B)
$x \in B$ or $x^{-1} \in B$ (look if p is in num or denom)

• $B = \mathbb{Z}_p$ p adics

• $B = \mathbb{C}[t]_{(t)}$ = rat'l funcs w/ no t in domain.

• $B = \mathbb{C}[[t]]$ power series $\mathrm{Frac}(B) = \mathbb{C}((t))$ laurent series.

Prop | B = val ring. $K = \mathrm{frac}(B)$

(a) B is local
(b) $B \subset B' \subset K \Rightarrow B'$ val ring [obvious]
(c) B is int closed.

Obs | If R is local w m, $m = \{\text{non units of } R\}$.

Pf | (a) let m = non units of B. STS. m ideal.

$a \in B$, $x \in m \Rightarrow ax \in m$ (otherwise $ax \in B^\times \Rightarrow x^{-1} = \underbrace{a}_{\in B}\underbrace{(ax)^{-1}}_{\in B}$
$\Rightarrow x^{-1} \in B$, so $x \in B^\times$) oops

let $x, y \in M \Rightarrow x+y \in M$ (Clear if $x=0$ or $y=0$, say not. $\frac{x}{y}$ or $\frac{y}{x} \in B$ say former. $x+y = \underbrace{(1+\frac{x}{y})}_{\in B}\underbrace{y}_{\in M} \Rightarrow \in M$ by prev)

(C) We want to show B is int cl. $/K$.

Say $x \in K$ is int $/B$ wts $x \in B$.

$\Rightarrow \exists\ x^n + b_{n-1}x^{n-1} + \dots + b_0$ mult by $(x^{-1})^{n-1}$

if $x \in B$ done, else $x^{-1} \in B$

So $x + b_{n-1} + b_{n-2}x^{-1} + \dots + b_0(x^{-1})^{n-1} = 0$

so $x \in B[x^{-1}] = B$

Now! Say B = val ring $K = \mathrm{Frac}(B)$

Def $\Gamma = K^\times / B^\times$. This is a comm grp (we will write additively)

$\exists$ natural grp homo $v : K^\times \longrightarrow \Gamma$

Γ is totally ordered $[a], [b] \in \Gamma$, $a, b \in K^\times$

Def $[a] \leq [b]$ if $b \in Ba$

Check total order

1. Reflexive: $[a] \leq [a]$ obv
2. Trans, $[a] \leq [b]$, $[b] \leq [c]$ obv
   $b = ya$, $c = xb$
   $\Rightarrow c = xya \in Ba$,
3. Anti Symmetry $[a] \leq [b]$, $[b] \leq [a]$ then $(=)$
   $b = xa$, $a = yb$ $\quad xy \in B$
   $\Rightarrow b = xyb$
   $\Rightarrow xy = 1$ so units $\in B$
   $\therefore$ same coset.
4. Totality $[a] \leq [b]$ or $[b] \leq [a]$ $\forall a, b$
   holds since valuation ring.

Cl. $v(x+y) \geq \min(v(x), v(y))$ (x, y, $x+y$ all nonzero)

Say $v(x) \geq v(y) \Rightarrow \frac{x}{y} \in B$

$x+y = (1+\frac{x}{y})y \Rightarrow v(x+y) \geq v(y)$

$\Gamma \Rightarrow$ $x \geq y \Rightarrow x+z \geq y+z \ \forall z$

Def) A valuation on a field K valued on a totally order ab grp Γ is a grp homo $v: K^\times \longrightarrow \Gamma$ s.t. $v(x+y) \geq \min(v(x), v(y))$

Conv: $v(0) = \infty$ & $\infty > \gamma \ \forall \gamma \in \Gamma$

Above shows: if $B \subset K$ valuation ring, get valuation on K w $\Gamma = K^\times / B^\times$

also rev,

given $v: K^\times \longrightarrow \Gamma$ is a valuation

define $B = \{x \in K^\times \mid v(x) \geq 0\} \cup \{0\}$

Cl. B is valuation ring

Pf) • Say $x, y \in B$ nonzero

$v(x+y) \geq \min(v(x), v(y)) \geq 0 \Rightarrow x+y \in B$

• $v(xy) = v(x)v(y) \geq \underbrace{0+0}_{\text{addition comp w ord}} \geq 0 \Rightarrow xy \in B$

Also, $1^2 = 1 \Rightarrow 2v(1) = v(1) \Rightarrow v(1) = 0$ so $1 \in B$

$(-1)^2 = 1 \Rightarrow \underbrace{2v(-1) = v(1)}$ $\Rightarrow v(-1) = 0$ so $-1 \in B$

So B closed under −

key fact: Γ total ordered $\Rightarrow$ torsion free.

$\therefore B$ subring of K

$x \in K$ either $v(x) \geq 0$ or $v(x) \leq 0 \Rightarrow v(x^{-1}) \geq 0$
$\Rightarrow x^{-1} \in B$

So $x \in B$ or $x^{-1} \in B$ (note units of B have valuation 0)

We have constr.

$$\{\text{val rings on } K\} \rightleftarrows \{\text{valuations } v: K^\times \to \Gamma\}$$

↳ note replacing Γ w/ im v gives same val ring trivially

So, to get invertible bijection

$$\{\text{val rings } B \subset K\} \rightleftarrows \{\text{val } v: K^\times \twoheadrightarrow \Gamma \text{ s.t. } v(K^\times) = \Gamma\} / \text{isom}$$

Isom of (v,Γ) & (v',Γ')

$$\begin{tikzcd} K^\times \arrow[r, "v"] \arrow[dr, "v'"'] & \Gamma \arrow[d, "i"] \\ & \Gamma' \end{tikzcd}$$

i isom of tot ord ab grps s.t. $v' = i \circ v$

Eg of val.

Defn A discrete val is one s.t. $v(K^\times) \cong \mathbb{Z}$

(a) (non disc) $K = \bigcup_{n \geq 1} \mathbb{C}(t^{1/n})$, $v: K^\times \to \mathbb{Q}$ is order of vanishing of 0

↳ $v(\sqrt{t}) = \frac{1}{2}$

(b) (non disc) $K = \bigcup_{n \geq 1} \mathbb{Q}(p^{1/n})$, $v: K^\times \to \mathbb{Q}$ p adic val

$v(p^{a/b}) = \frac{a}{b}$

(non disc).

(c) let $\Gamma = \mathbb{Z} \oplus \mathbb{Z}$ with lexicographic order.

$K = \mathbb{C}(x,y)$ rat'l $\exists$ val $v: K^\times \to \Gamma$ $\quad$ $v(\alpha) = 0$ for $\alpha \in \mathbb{C}^\times$

$(x^i y^j) \longmapsto (i,j)$

↳ called a rank 2 valuation.

given Γ consider $\mathbb{C}[\Gamma] = B$ → formal sums of Γ & coeff. → $\alpha_\gamma t^\gamma$

let $K = \mathrm{Frac}(B)$ $\quad v: K^\times \to \Gamma$

$t^\gamma \longmapsto \gamma$

(Related imag of Puiseux Series)

Constr. of val rings

k - field, Ω - alg closed field

Let $\Sigma = \{ (A,f) \mid A \subset K$ subfield, $f: A \to \Omega$ is a ring homo $\}$

empty iff different fin char
↳ Assume k char 0 or same fin char

Partially order Σ: $(A,f) \le (A',f')$ if $A \subset A'$ & $f'|_A = f$

Σ non-empty by $\cap$ $\Rightarrow$ Zorn gives max'l (B,g)

We'll show B is a val ring.

Lemma B is local w/ max ideal $m = \mathrm{Ker}(g)$.

Pf m is prime since g maps to a domain

g extends to a homo $g': B_m \longrightarrow \Omega$

$g'(b/s) = \dfrac{g(b)}{g(s)}$

$s \notin m \Rightarrow g(s) \neq 0$

$(B,g) \le (B_m, g') \Rightarrow B = B_m$ by maximality

$\Rightarrow$ $\boxed{\text{local}}$

Lemma $x \in K$, $B[x] \subset K$ $m(x) = m \cdot B[x]$

B subalg gen by x

Cl. either $m[x] \neq B[x]$ or $m[x^{-1}] \neq B[x^{-1}]$

Pf Say equality holds in both above,

$1 = a_0 + a_1 x + \dots + a_r x^r$ $a_i \in m$

$1 = b_0 + b_1 x^{-1} + \dots + b_s x^{-s}$ $b_i \in m$

choose r, s minimal

Say $r \ge s$, multiply $②$ by x^s

$\Rightarrow \underbrace{(1-b_0)}_{\text{unit b/c } b_0 \in m \text{ \& } B \text{ local}} x^s = b_1 x^{s-1} + \dots + b_s$

$\Rightarrow x^s = b_1' x^{s-1} + \dots + b_s'$ $b_i' \in m$

mult by x^{r-s} to get $x^r = b_1' x^{r-1} + \dots$ plug for contra

Thm B val ring $x \in K^\times$

Pf let $x \in K^\times$ be given. WTS, $x \in B$ or $x^{-1} \in B$

Wlog $m[x] \neq B[x] = B'$, $m[x] \subset \underbrace{m'}_{\text{max'l ideal}} \subset B'$

↳ note, $\underbrace{m' \cap B}_{\text{contraction}} = m$ b/c it contains m, not unit, m is max

$$\begin{array}{ccc} B/m = K & \longrightarrow & K' = B'/m' \\ \Big\downarrow g & & \\ \Omega & & \end{array}$$

K