Lec2

Let R a comm ring. Recall, $\text{rad}(R) = \{x \in R \mid x \text{ is nilp}\}$
↳ this is an ideal!

Prop 1 $\text{rad}(R) = \bigcap_{\mathfrak{p} \text{ prime ideal}} \mathfrak{p}$

Pf ($\subseteq$) show every nilp elt is in a prime ideal $\mathfrak{p}$

Pick n so $x^n = 0 \Rightarrow x^n \in \mathfrak{p}$ all ideals have 0
$x \cdot x^{n-1} \in \mathfrak{p}$ so either $x, x^{n-1} \in \mathfrak{p}$ by induction $x \in \mathfrak{p}$

($\supseteq$) contrapositive, if $x \in R$ not nilp $\Rightarrow \exists\, \mathfrak{p}$ prime ideal so $x \notin \mathfrak{p}$

Pf Let $\Sigma = \{\text{all ideals } \mathfrak{a} \text{ of } R \mid x^n \notin \mathfrak{a}\ \forall n \geq 1\}$

we not $(0) \in \Sigma$ & $x \neq 0$ as not nip

Consider a chain of ideals in $\{\mathfrak{a}\}_{i \in I}$ in Σ

let $\mathfrak{a} = \bigcup \mathfrak{a}_i$ This is ideal as Union of chain of id

If $x^n \in \mathfrak{a} \Rightarrow x^n \in \mathfrak{a}_i$ for some $\mathfrak{a}_i$. Not true

$\Rightarrow \mathfrak{a} \in \Sigma$

Zorn $\Rightarrow \exists$ max'l elt in $\mathfrak{p} \in \Sigma$

Cl. $\mathfrak{p}$ is prime.

Spose not & $ab \in \mathfrak{p}$ and $a \notin \mathfrak{p}$, $b \notin \mathfrak{p}$

$\mathfrak{p} \subsetneq \mathfrak{p} + (a)$ ↝ ideal gen'd by $\mathfrak{p}$ & a $\Rightarrow \mathfrak{p} + (a) \notin \Sigma$ by max'l

$\Rightarrow x^n \in \mathfrak{p} + (a)$ & symm $x^m \in \mathfrak{p} + (b)$

So $x^{n+m} \in (\mathfrak{p} + (a))(\mathfrak{p} + (b))$ prod ideal

$= \mathfrak{p}^2 + \mathfrak{p}(a) + \mathfrak{p}(b) + (ab) \subset \mathfrak{p}$ as each term does

So $\mathfrak{p}$ is prime!

Motivation for comm (Alg num thry)

Recall| An algebraic number $\alpha \in \mathbb{C}$ s.t $f(\alpha)=0$ for $f \in \mathbb{Q}[x]$ (nonzero)

algebraic int ________ $f \in \mathbb{Z}[x]$ monic.

eg| $\sqrt{2}$, $i = \sqrt{-1}$ alg ints $\qquad$ $1/2$ alg num not int

$x^2-2 \quad x^2+1 \qquad\qquad 4x^2+1$

Given $\alpha_1, \ldots, \alpha_n$ can study ring gen'd by these $\mathbb{Z}[\alpha_1,\ldots,\alpha_n] \subseteq \mathbb{C}$

ANT studies this

One important question: unique fact?

E.g|
- $\mathbb{Z}$ – yes (fund thm arith)
- $\mathbb{Z}[i]$ – yes
- $\mathbb{Z}[\sqrt{-5}]$ – no $\qquad 6 = 2 \cdot 3 = (1+\sqrt{-5})(1-\sqrt{-5})$

If R is the ring of all (algebraic) integers in a number field (finite ext of $\mathbb{Q}$)

Then, all nonzero ideals in R factors uniquely into a product of prime ideals

Technical pt| Consider $\mathbb{Z}[1+\sqrt{-3}]$, $\mathbb{Z}\left[\frac{1+\sqrt{-3}}{2}\right]$, $\mathbb{Z}\left[\frac{1+\sqrt{-3}}{4}\right]$

not "normal" $\qquad$ full ring of integers in $\mathbb{Q}\left(\frac{1+\sqrt{-3}}{2}\right)$ $\qquad$ not finite over $\mathbb{Z}$

In comm alg $\exists$ class of rings called "Dedekind domains"!

The full ring of integers in a # field is a dedekind domain

Many props of rings of ints works in arb ded doms (Eg unique fact of prime ideals)

Motivation for CA (alg geom)

Given $f_1, \ldots, f_r \in \mathbb{C}[x_1, \ldots, x_n]$

consider $\{\alpha \in \mathbb{C}^n \mid f_1(\alpha) = \cdots = f_r(\alpha) = 0\}$

$\hookrightarrow$ call $V_m(f_1, \ldots, f_r)$

E.g. | $n=2$, $f_1 = x^2+y^2-1$ $\rightsquigarrow$ in $\oplus$ unit circle

Sets of the form $V_m(f_1, \ldots, f_r)$ $\rightsquigarrow$ $\boxed{\text{algebraic set}}$

Obs | $V_m(f_1, \ldots, f_m)$ only dep on the ideal $(f_1, \ldots, f_m) \subset \mathbb{C}[x_1, \ldots, x_n]$

$\hookrightarrow$ $\boxed{\text{pair vanishes}}$

So, for an ideal $\mathfrak{a} \subset \mathbb{C}[x_1, \ldots, x_n]$

let $V_m(\mathfrak{a}) = \{\alpha \in \mathbb{C}^n \mid \forall f \in \mathfrak{a}\ f(\alpha) = 0\}$

So $V_m(f_1, \ldots, f_r) = V_m(\mathfrak{a})$ for $\mathfrak{a} = (f_1, \ldots, f_m)$

$$\{\text{ideals in } \mathbb{C}[x_1, \ldots, x_n]\} \xrightarrow{\quad} \{\text{alg set in } \mathbb{C}^n\}$$

$$\mathfrak{a} \longmapsto V_m(\mathfrak{a})$$

$\{f \in \mathbb{C}[x_1, \ldots, x_n] \mid f(\alpha) = 0\ \forall \alpha \in Z\}$ $\leftarrow$ $I(Z) \longleftarrow Z$

$\hookrightarrow$ $\boxed{\text{ideal}}$

$\mathfrak{a} \subset I(V_m(\mathfrak{a}))$ can be a proper incl.

None | take f and g with no common zero

$\hookrightarrow$ $\mathfrak{a} = (f, g) \implies V_m(\mathfrak{a}) = \emptyset$

$= (1) \implies I(V_m(\mathfrak{a})) = \mathbb{C}[x_1, \ldots, x_n]$

E.g. | $\mathfrak{a} = (x^2)$ $V_m(\mathfrak{a}) = \{0\}$ $I(V_m(\mathfrak{a})) = (x)$

$\boxed{(x^2) \subsetneq (x)}$

Def let $\mathfrak{r} \subset R$, $\text{rad}(\mathfrak{r}) = \{x \in R \mid x^n \in \mathfrak{r}\}$ for some n

$$\pi : R \longrightarrow R/\mathfrak{r}$$

$$\text{rad}(\mathfrak{r}) = \pi^{-1}(\text{rad}(R/\mathfrak{r}))$$

our description of $\text{rad}(R/\mathfrak{r})$ ideal corr

$$\Rightarrow \text{rad}(\mathfrak{r}) = \bigcap_{\mathfrak{r} \subset \mathfrak{p}} \mathfrak{p}$$

Obs $\text{rad}(\mathfrak{r}) \subset I(V_m(\mathfrak{r}))$

Why, let $f \in \text{rad}(\mathfrak{r}) \Rightarrow f^n \in \mathfrak{r}$ for some n

$\Rightarrow f^n \in I(V_m(\mathfrak{r}))$

$\Rightarrow f^n$ vanishes on all pts in $V_m(\mathfrak{r})$

$\Rightarrow f$ ______________

$\Rightarrow f \in I(V_m(\mathfrak{r}))$

Nullstellensatz $\text{rad}(\mathfrak{r}) = I(V_m(\mathfrak{r}))$

$\Rightarrow \{$ radical ideals (one that it rad) in $\mathbb{C}[x_1, \ldots, x_n]$ $\} \xleftrightarrow{\text{bij}} \{$ alg sets in $\mathbb{C}^n \}$

For $\alpha \in \mathbb{C}^n$ write $m_\alpha = (x_1 - \alpha_1, \ldots, x_n - \alpha_n) \subseteq \mathbb{C}[x_1, \ldots, x_n]$

Recall, m_α all the max'll ideals of $\mathbb{C}[x_1, \ldots, x_n]$

- have a ring homo $\Phi_\alpha : \mathbb{C}[x_1, \ldots, x_n] \longrightarrow \mathbb{C}$, $f \longmapsto f(\alpha)$

  & $m_\alpha = \ker \Phi_\alpha$

- $V_m(\mathfrak{r}) = \{\alpha \in \mathbb{C}^n \mid f(\alpha) = 0 \ \forall f \in \mathfrak{r}\}$

  $\Leftrightarrow \Phi_\alpha(f) = 0 \Rightarrow f \in m_\alpha$

  $\Updownarrow$

  $\mathfrak{r} \subset m_\alpha$

  $= \{\alpha \mid \mathfrak{r} \subseteq m_\alpha\}$

- $Z \subseteq \mathbb{C}^n$, $I(Z) = \{f \mid f(\alpha) = 0 \ \forall \alpha \in Z\}$

$$= \bigcap_{\alpha \in Z} m_\alpha$$

Nullstellensatz $\text{rad}(\mathfrak{a}) = \bigcap_{\substack{\mathfrak{a} \subset m \\ m \text{ maximal}}} m$

$\hookrightarrow$ warning specific to $\mathbb{C}[x_1, \ldots, x_n]$

Coin sen use prime ideals

Def • for a ring R, define $\text{MaxSpec}(R) = \{\text{max ideals of } R\}$

• For an ideal $\mathfrak{a} \subset R$, def $V_m(\mathfrak{a}) = \{m \text{ max ideal} \mid \mathfrak{a} \subset m\}$

$\subseteq \text{MaxSpec}(R)$

• For $Z \subset \text{MaxSpec}(R)$, def $I(Z) = \bigcap_{m \in Z} m$

E.g. $\text{MaxSpec}(\mathbb{C}[x_1, \ldots, x_n]) \cong \mathbb{C}^n$

$m_\alpha \longmapsto \alpha$

• $\mathfrak{a} \subseteq \mathbb{C}[x_1, \ldots, x_n]$ $\quad R = \dfrac{\mathbb{C}[x_1, \ldots, x_n]}{\mathfrak{a}}$ $\rightarrow$ finitely gen'd $\mathbb{C}$-alg

$$\text{MaxSpec}(R) \cong V_m(\mathfrak{a}) \subset \mathbb{C}^n$$

ideal corr $\leftarrow$ $\hookrightarrow$ bij

$\mathfrak{a} \subset R$ $\quad I(V_m(\mathfrak{a})) = \bigcap_{\substack{\mathfrak{a} \subset m \\ m \text{ max'l}}} m \neq \text{rad}(\mathfrak{a})$ in gen (for general R)

We can resolve this defect by using $\text{Spec}(R) = \{\text{prime ideals of } R\}$.

Recall if V, W 2 fin dim vs $\rightsquigarrow$ $V \otimes_{\mathbb{C}} W$ (tens. prod) is a f.d v.s. giving linear map outta this same as multi lin $V \times W$

$$V^{\otimes n} = \underbrace{V \otimes \cdots \otimes V}_{n \text{ times}}$$

$\mathrm{Sym}^n(V)$ = subs of $V^{\otimes n}$ inv under action S_n

$\mathrm{Sym}^n(V^*)$ = homog. degree n poly func on V

$\mathrm{Sym}(V^*) = \bigoplus_{n \geq 0} \mathrm{Sym}^n(V^*)$ = all poly fnc on V

Suppose, $e_1, \ldots, e_n$ is a basis of V.

let $x_1, \ldots, x_n$ be the dual basis of V^* $\quad x_i(e_j) = \begin{cases} 1 & j = i \\ 0 & \text{o/w} \end{cases}$

$\mathrm{Sym}(V^*) = \mathbb{C}[x_1, \ldots, x_n]$ (poly ring)

$\mathrm{MaxSpec}(\underbrace{\mathrm{Sym}\, V^*}_{\text{space of functions on MaxSpec}}) = V$ $\quad \begin{pmatrix} V^* & \text{fnc} \\ V & \text{space} \end{pmatrix}$

dwap dwisit

eg **let**, V, W be 2 fd. $\mathbb{C}$ vs

let $\mathrm{hom}(V, W)$ - all linear trans $V \to W$ (this is fd vs)

fix $r \geq 1$

let $Z \subset \mathrm{hom}(V, W)$ $\quad Z = \{T : V \to W \mid \mathrm{rank}(T) \leq r\}$

choose bases V, W identify $\mathrm{Hom}(V, W) = \mathrm{Mat}_{n,m}(\mathbb{C})$

$Z = V_m(\mathfrak{a})$ $\quad \mathfrak{a}$ = ideal of $\mathbb{C}[x_{ij}]$ ($= \mathrm{Sym}(\mathrm{Hom}(V, W)^*)$)

gen'd by all $(r+1) \times (r+1)$ minors.

Null | $I(Z) = \mathrm{rad}(\mathfrak{a})$

QA