Prop ($\otimes$-Hom adj)

let M_1, M_2, M_3 R-mods. $\exists$ natural isom

$$\mathrm{Hom}_R(\underbrace{M_1 \otimes_R M_2}_{\text{apply } -\otimes_R M_2 \text{ to } M_1}, M_3) \cong \mathrm{Hom}_R(M_1, \underbrace{\mathrm{Hom}_R(M_2, M_3)}_{\text{apply } \mathrm{Hom}_R(M_2,-)})$$

in other words, $- \otimes_R M_2$ (left) & $\mathrm{Hom}_R(M_2, -)$ (right) are adjoint

Pf sk)

$$\mathrm{Hom}_R(M_1 \otimes_R M_2, M_3) \underset{\alpha}{\overset{\beta}{\rightleftarrows}} \mathrm{Hom}_R(M_1, \mathrm{Hom}_R(M_2, M_3))$$

Defn of α) given $f : M_1 \otimes_R M_2 \longrightarrow M_3$

want $\alpha(f) : M_1 \longrightarrow \mathrm{Hom}(M_2, M_3)$

given $x \in M_1$ need to define

$(\alpha f)(x) : M_2 \longrightarrow M_3$

For $y \in M_2$ $(\alpha f)(x)(y) = f(x \otimes_R y)$.

Defn of β) given $g : M_1 \longrightarrow \mathrm{Hom}_R(M_2, M_3)$

want $(\beta g) : M_1 \otimes_R M_2 \longrightarrow M_3$

given: $(\beta g)(x \otimes_R y) = \underbrace{g(x)}_{\mathrm{Hom}_R(M_2,M_3)}(y)$ $\longrightarrow$ this is what it does on the pure tns

to show this

Show $: M_1 \times M_2 \longrightarrow M_3$

$(x, y) \longmapsto g(x)(y)$ $\longrightarrow$ Check bilin & Make mapping prop!

Prop ($\otimes$ is right exact) Given an exact seq of R-mods

(1) $M_1 \to M_2 \to M_3 \to 0$

& an R-mod N the seq

(2) $N \otimes_R M_1 \to N \otimes_R M_2 \to N \otimes_R M_3 \to 0$

So, $N \otimes_R -$ is right exact

Pf) given (1) is exact. wts 2 exact ($\Leftarrow$ | Same iff statement after $\mathrm{Hom}(-,L)$)

To show 2 is exact, it sts exactness after applying $\mathrm{Hom}_R(-,L)$, $\forall$ arb $\boxed{R\text{-module}}$

apply $\mathrm{Hom}_R(-,L)$ into 2 $\to$ Contravariant & left exact

$$\Rightarrow (3)\quad 0 \to \mathrm{Hom}_R(N\otimes_R M_3, L) \to \mathrm{Hom}_R(N\otimes_R M_2, L) \to \mathrm{Hom}(N\otimes_R M_1, L)$$

$$\begin{array}{ccccccc} & & \wr\wr\ s & & \wr\wr\ s & & \wr\wr\ s \\ 0 & \to & \mathrm{Hom}_R(M_3, \mathrm{Hom}_R(N,L)) & \to & \mathrm{Hom}_R(M_2, \mathrm{Hom}_R(N,L)) & \to & \mathrm{Hom}(M_1, \mathrm{Hom}(N,L)) \end{array}$$

STS (3) is exact for all L

Two statement,

- (4) comes from (1) by applying $\mathrm{Hom}(-, \mathrm{Hom}(N,L))$
  $\Rightarrow$ (4) is exact
- (3) $\cong$ (4) via the adjunction isomorphism!

(as say all comm) $\Rightarrow$ (3) is exact!

Rmk) if (Φ, Ψ) are an adjoint pair of funct b/w Module category (covariant) (abelian cat)

$\Rightarrow$ Φ is right exact & Ψ is left exact.

e.g) let $\mathfrak{R} \subset R$ ideal $\quad 0 \to \mathfrak{R} \to R \to R/\mathfrak{R} \to 0$ SES

say M an R-module

$$\rightsquigarrow \mathfrak{R}\otimes_R M \xrightarrow{f} R\otimes_R M \xrightarrow{g} R/\mathfrak{R}\otimes_R M \to 0$$

($R\otimes_R M = M$; f is the effectively id)

$\hookrightarrow$ exact as tensor w/ M is right exact

$\mathrm{im}\, f = \mathfrak{R}M$ as $R\otimes_R M \cong M$ (this is the iso)

$(x\otimes y) \longmapsto xy$

$\ker g = \mathrm{im}\, f = \mathfrak{R}M \Rightarrow R/\mathfrak{R}\otimes M = M/\mathfrak{R}M$

Defn| A R-module F is flat if $F \otimes_R -$ is exact
↳ e.g. R as R-module.

Rmk| • F is flat $\iff$ $\forall$ inj of R mods $M_1 \hookrightarrow M_2$
that $F \otimes_R M_1 \hookrightarrow F \otimes_R M_2$

• F is flat $\iff$ $\forall$ ideals $\mathfrak{a} \subset R$, $\mathfrak{a} \hookrightarrow R$
$F \otimes_R \mathfrak{a} \hookrightarrow F \otimes_R R \overset{\shortparallel}{F}$
(in fact need only f.g ideals)

• If $\{F_i\}_{i \in I}$ are flat $\Rightarrow$ $\bigoplus_{i \in I} F_i$ flat (not true for dir prod see Chase's Thm)

• Summand of flat is flat
F flat & $F \cong M \oplus N \Rightarrow M, N$ flat

E.g| • R is a flat R-module
• Free modules are flat
• Projective modules (equiv to summand of free mod) flat
↳ $\mathrm{Hom}(P,-)$ is exact too
• $\mathbb{Q}$ is a flat $\mathbb{Z}$-mod

Pf| $\mathbb{Q} \otimes_{\mathbb{Z}} \mathfrak{a} \longrightarrow \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}$ is inj $\forall$ ideals $\mathfrak{a} \subset \mathbb{Z}$

$0 \neq \mathfrak{a} = (n)$

$$\begin{tikzcd}
\mathfrak{a} \arrow[r] & \mathbb{Z} \\
\mathbb{Z} \arrow[u, "\wr"] \arrow[r, "n"'] \arrow[ur, phantom, "\circlearrowright"] & \mathbb{Z} \arrow[u, equal]
\end{tikzcd}$$

(thinking of $\mathfrak{a} \subset \mathbb{Z}$ isom to $\mathbb{Z} \xrightarrow{n} \mathbb{Z}$)

$\mathbb{Q} \otimes$

$$\begin{tikzcd}
\mathbb{Q} \otimes_{\mathbb{Z}} \mathfrak{a} \arrow[r] \arrow[d, "\wr"] & \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z} \arrow[d, "\wr ?"] \\
\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z} \arrow[r] \arrow[d, equal] & \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z} \arrow[d, equal] \\
\mathbb{Q} \arrow[r, "\text{mult by}"', "n"] & \mathbb{Q}
\end{tikzcd}$$

injective.

- $\mathbb{Z}/3$ not flat

Reason

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "\times 3"] \arrow[d, phantom, ""] & \mathbb{Z} \arrow[d, squiggly] \end{tikzcd} \quad \text{inj of } \mathbb{Z}\text{-mod}$$

$$\begin{tikzcd} \mathbb{Z}/3 \otimes \mathbb{Z} \arrow[r, "0"'] \arrow[d, equal] & \mathbb{Z}/3 \otimes \mathbb{Z} \\ \mathbb{Z}/3 \arrow[r] & \mathbb{Z}/3 \end{tikzcd}$$

not injective

Generalize | R-domain.

- An elt $x \in R$ is torsion if $\exists\, a \neq 0 \in R$ s.t. $ax = 0$
- The set of M_{tors} of all torsion elts is a R-submod
- M is torsion if $M = M_{tors}$, M is tors free if $M_{tors} = 0$

Obs $\Rightarrow$ Flat $\Rightarrow$ tors free,

Pf) F flat R-module, $a \neq 0 \in R$

$R \xrightarrow{a} R$ is inj + F flat

$\Rightarrow F \otimes_R R \longrightarrow F \otimes_R R$ is inj

$$\begin{tikzcd} F \otimes_R R \arrow[r] \arrow[d, equal] & F \otimes_R R \arrow[d, equal] \\ F \arrow[r, "a"] & F \end{tikzcd} \quad \text{is inj}$$

$\Rightarrow$ no elt of F is killed by a except 0.

Defn | R = ring, an R-algebra is a ring S with a ring homo $R \to S$ ($\Rightarrow S$ naturally R mod)

(e.g) $\mathbb{C}[x_1, \_, x_n]$ is a $\mathbb{C}$ alg.

Def | If S_1 & S_2 R algs a R-alg-homo $S_1 \xrightarrow{f} S_2$ so

$$\begin{tikzcd} & S_1 \arrow[d, "f"] \\ R \arrow[ur, "i"] \arrow[r, "j"'] & S_2 \end{tikzcd} \quad \text{comm}$$

($\Leftrightarrow f$ is R-linear, i.e. map of R-mods)

Pf that R lin

$a \in R$, $x \in S$,

wts, $f(ax) = a f(x)$

$f(i(a)x) \qquad j(a) f(x)$

$f(i(a)x) = f(i(a)) f(x)$ ← f is a ring homo

$= j(a) f(x)$ ← $f \circ i = j$

Def | let $f: R \to S$ be a ring homo, i.e. S-R alg

- S is finite (as R-alg) or module finite if S is finitely gen as an R-mod

e.g. | $R = \mathbb{Z}$ • $\mathbb{Z}[i]$ is finite $\mathbb{Z}$-alg gen'd by $1, i$

• $\mathbb{Z}[x]$ is not finite $\mathbb{Z}$-alg basis $\to 1, x, x^2, \ldots$

- S is finite type (as R-alg) or finite gen'd if $\exists$ finitely many elts $a_1, \ldots, a_n \in S$ s.t. S is gen'd as a ring by $R \cup \{a_1, \ldots, a_n\}$

$\Longleftrightarrow \exists$ surj of R-alg $R[x_1, \ldots, x_n] \to S$

if is finite type

also flat

- S is flat as R-alg if it is flat as R mod

e.g. | • $\mathbb{Q}$ is flat as $\mathbb{Z}$-alg

• If R is a domain, $\mathfrak{a}$ non-zero ideal, $R/\mathfrak{a}$ not flat as R-alg.

Nakayama's lemma

Recall the radical of R is set of nilp elts $= \bigcap_{P \text{ prime}} P$

Def | the Jacobson radical of R is $J(R) = \bigcap_{M \text{ max'l}} M$

$\Rightarrow \operatorname{rad}(R) \subseteq J(R)$, but not equal in gen

Prop) $x \in J(R) \iff \forall y \in R$ $1-xy$ is a unit.

Pf) let $x \in J(R)$ $\forall y \in R$ know $xy \in J(R)$ $\to$ $J(R)$ ideal

$\Rightarrow xy \in M$ $\forall$ max ideals

$\Rightarrow 1-xy \notin M$ (as $1 \in M$ if so ... not max'l!)

$\Rightarrow (1-xy) = (1) \Rightarrow (1-xy)$ unit.

so not in any max'l ideals

Say $x \notin J(R)$. $\exists$ max'l ideal m so $x \notin M$

$\Rightarrow m + (x) = (1) \Rightarrow 1 = z + xy$ for $z \in m, y \in R$

$\Rightarrow 1 - xy = z \in M$

$\Rightarrow 1-xy$ not a unit

Defn) R is a local ring if it has a unique max'l ideal

e.g.) let $R \subset \mathbb{Q}$ be set of elts $\frac{a}{b}$ where b is odd

If p is an odd prime $\to$ p unit in R

$\Rightarrow (p) = pR = R$

can show $R/(2) \cong \mathbb{Z}/2\mathbb{Z}$

$\Rightarrow 2R$ is a maximal ideal $\to$ the unique one!

$\text{rad}(R) = 0$ but $J(R) = 2R$

b/c R - domain $\Rightarrow (0)$ prime

Nakayama Lemma) $M =$ fg R mod, $I \subset J(R)$ st $M = IM$

$\Rightarrow M = 0$ (or $M/IM = 0$)

Pf) let $x_1, \ldots, x_n$ be a min gen set of M