Let; V - fin dim'l $\mathbb{C}$-vs let $f : V \to \mathbb{C}$

Pick a basis $\leftarrow$ $\cong$ $\mathbb{C}^n \nearrow$ $\to$ whether or not this is a poly is ind of choice of basis (linear (ov))

Say f is a poly if green is for one or all choices of bases!

$\mathrm{Sym}(V^*) = \{\text{all poly on } V\}$

If $e_1, \ldots, e_n$ basis of $V \Rightarrow x_1, \ldots, x_n : V \to \mathbb{C}$ is the dual basis then

$\mathrm{Sym}(V^*) = \mathbb{C}[x_1, \ldots, x_n]$ literally

Modules

Def | If R is a ring, an R module is an abelian grp $(M, +)$ with an op $R \times M \to M$ s.t. $(a, x) \mapsto ax$

1) $1 \cdot x = x$

2) $(ab) \cdot x = (a(b \cdot x))$

3) $(a+b) \cdot x = ax + bx$

4) $a \cdot (x+y) = ax + ay$

E.g |
- R is an R-module (multiply in Ring)
- If $\mathfrak{a} \subset R$ ideal $\Rightarrow \mathfrak{a}$ is an R module as well
- If $f : R \to S$ ring homo. S is an R-module via

$$a \cdot x := \underbrace{f(a) \cdot x}_{\in S}$$

Def | If M, N are R-mods. An R mod homo is a map $f : M \to N$ Is a grp homo for $+$ & $f(a \cdot x) = a f(x)$

Defn M, N as earlier, $\text{Hom}_R(M,N) = \{$ set of R-mod homs $M \to N\}$

This is an R module naturally,

Addition, $f, g : M \to N \rightsquigarrow (f+g)(x) = f(x) + g(x)$

Scalar M $f : M \to N$, $a \in R \rightsquigarrow (af)(x) = a \cdot f(x)$

Warning: only works b/c R is commutative!

$$\underbrace{(af)(bx)}_{\text{need } af \text{ - a mod homo}} = a \cdot f(bx) = a \cdot b\, f(x)$$
$$= b \cdot (af)(x)$$

Subs + Quots

- If M is an R-module, then a submod of M is an additive subgrp closed under scalar mult.

  Eg| Ideal of R $\rightsquigarrow$ submod of R

- If $N \subset M$ is a submod $\Rightarrow$ M/N (as grp) has scalar mult
  $a(x+N) = ax + N$ $\rightsquigarrow$ makes M/N a module

- If $f : M \to N$ a module homo $\Rightarrow$
  1) $\ker(f)$ submod of M
  2) $\text{im}(f)$ is submod of N
  3) $\text{coker}(f) = \dfrac{N}{\text{im}(f)}$

Obs on Submod

- $\underbrace{N_1, N_2 \subset M}_{\text{submod}} \Rightarrow N_1 + N_2 = \{x + y \mid x \in N_1, y \in N_2\}$ a submod!

- More gen, if $\{N_i\}_{i \in I} \subset M$ submod $\sum_{i \in I} N_i$ submod.

- Same for intersection!
- If $\mathfrak{a} \subset R$ ideal & $N \subset M$ submod, def $\mathfrak{a}N = \{a_1x_1 + \dots + a_nx_n \mid a_i \in \mathfrak{a},\ x_i \in N\}$
  this is a submodule!
  e.g.: $M/\mathfrak{a}M$ constr.

- If M an R-mod $\mathrm{Ann}_R(M) = \{a \in R \mid ax = 0\ \forall x \in M\}$
  $\hookrightarrow$ ideal of R.

e.g. $\mathrm{Ann}_{\mathbb{Z}}(\mathbb{Z}/2 \oplus \mathbb{Z}/3) = (6)$

$\mathrm{Ann}_{\mathbb{Z}}(\mathbb{Z}/2 \oplus \mathbb{Z}) = 0$

Direct Sum / Product

- M, N are R-module, so is $M \oplus N$ in obv way!
- $\{M_i\}_{i \in I}$ family of R mods
  1) Dir product $\prod_{i \in I} M_i = \{(x_i) \mid x_i \in M_i\}$ compwise op
  2) Dir Sum $\bigoplus_{i \in I} M_i$ submod of $\prod_{i \in I} M_i$ a.a. $\neq 0$

mappin prop

$\Rightarrow M \to \prod_{i \in I} M_i \iff$ giving $N \to M_i\ \forall i \in I$ $\left(\begin{array}{c} \mathrm{Hom}(N, \prod M_i) \\ \shortparallel \\ \prod \mathrm{Hom}_R(N, M_i) \end{array}\right)$

giving $\bigoplus M_i \to N \iff$ giving from $M_i \to N$ $i \in I$ $\left(\begin{array}{c} \mathrm{Hom}(\bigoplus M_i, N) \\ \shortparallel \\ \prod \mathrm{Hom}(M_i, N) \end{array}\right)$

For any R-mod M $\exists$ natural isom $\mathrm{Hom}_R(R, M) = M$

$f \longmapsto f(1)$

$[a \mapsto ax] \longleftarrow x$

$\rightsquigarrow \mathrm{Hom}_R(R^{\oplus n}, M) \cong M^{\oplus n}$

std basis e_i — Rank n free mod

given $f: R^{\oplus n} \to M \rightsquigarrow (f(e_1), \dots, f(e_n))$

Def | If $M = R$-mod can look at $x_1, \dots, x_n \in M$.
The submod gen'd by $x_1, \dots, x_n$ is $\{a_1x_1 + \dots + a_nx_n \mid a_1, \dots, a_n \in R\}$
Also works with inf fam of elts with coeffs a.b.f.n.z.

Def | we say M is finely gen'd if it is gen'd by finite list of elts

E.g |
- $\prod_{i=1}^{\infty} \mathbb{Z}$ is not f.g as a $\mathbb{Z}$-mod
- $\mathbb{Z}[x] = \bigoplus \mathbb{Z}$ free mod of ∞ rank, not fg
- $\mathbb{Q}$ is a $\mathbb{Z}$ mod not fg

Obs |
- if $f : M \to N$ is a surj & M fin gen $\rightsquigarrow$ so is N (by the im's of M gens)

Warning | If $N \subset M$ & M is f.g. cannot conclude N is f.g!

E.g | $R = \mathbb{C}[x_1, \dots, x_n]$
$M = R \longrightarrow$ is f.g $/R$
$N = (x_1, x_2, \dots) \rightsquigarrow$ is not f.g] facts!

Prop | M is f.g $\iff \exists$ surj $f : R^{\oplus n} \to M$

Pf | $(\Leftarrow)$ by obs
$(\Rightarrow)$ say $(x_1, \dots, x_n)$ gen M consider $f : R^{\oplus n} \to M$, $e_i \mapsto x_i$ (mapping prop free mod)

Note: $\mathrm{im}(f)$ submod of M containing gns
$\Rightarrow \mathrm{im} f = M$.

Exact Sequence

A seq of module map $M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3$
is exact at M_2 if $\operatorname{im} f_1 = \ker f_2$

- Can apply to a longer seq of maps!
- A short exact seq $0 \to M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \to 0$
  ↳ exact everywhere that makes sense

↳ $\Rightarrow$ exact at $M_1 \Rightarrow f$ injective $\leadsto M_1 \subset M_2$
exact at $M_3 \Rightarrow g$ surjective
ex at $M_2 \Rightarrow \operatorname{im}(f) = \ker(g)$

If iso $M_3 \cong M_2 / \ker(g) \cong M_2 / \operatorname{im}(f)$

Prop) 1) If $M_1 \to M_2 \to M_3 \to 0$ is exact (left exact)
iff

$\forall N \quad 0 \to \operatorname{Hom}(M_3, N) \to \operatorname{Hom}(M_2, N) \to \operatorname{Hom}(M_1, N)$ exact.

contravariant

2) $0 \to N_1 \to N_2 \to N_3$ is exact
iff
$\forall M \quad 0 \to \operatorname{Hom}(M, N_1) \to \operatorname{Hom}(M, N_2) \to \operatorname{Hom}(M, N_2)$

functorial w/ composition $f: M_1 \to M_2$ & $\alpha \in \hom(M_2, N)$
$\Rightarrow \alpha \circ f \in \hom(M_1, N)$

If) $0 \to M_1 \xrightarrow{i} M_2 \xrightarrow{\pi} M_3 \to 0$ exact
$\Downarrow$?

$0 \to \operatorname{Hom}(M_3, N) \xrightarrow{\pi^*} \operatorname{Hom}(M_2, N) \xrightarrow{i^*} \operatorname{Hom}(M_1, N) \to 0$

exact

↳ not nec exact
here inj

So, π^* is <u>injective</u>

$$\begin{tikzcd} M_2 \arrow[r, "\pi"] \arrow[dr, dotted] & M_3 \arrow[d] \\ & N \end{tikzcd}$$

clear by <u>thinking</u>.

exactness at $\mathrm{Hom}(M_1, N)$ would mean i^* is <u>surj</u>

<u>i.e.</u> given any map $\alpha : M_1 \to N$

$\exists\ \beta : M_2 \to N$

so $i^*(\beta) = \beta \circ i = \alpha$

So maps from $M_1 \to N$ can be extended to $M_2 \to N$

thinking $M_1 \subset M_2$

<u>Can't always!</u>

e.g.| $2\mathbb{Z} \subset \mathbb{Z}$

$$\begin{tikzcd} 2\mathbb{Z} \arrow[r, "i"] \arrow[d, "x \mapsto \frac{x}{2}"'] & \mathbb{Z} \arrow[dl, dotted, "\nexists"'] \\ \mathbb{Z} & \end{tikzcd}$$

2) $$\begin{tikzcd} & & & M \arrow[d, "\alpha"] \arrow[dl, dotted, "\exists\beta?"'] & \\ 0 \arrow[r] & N_1 \arrow[r, "i"] & N_2 \arrow[r, "\pi"] & N_3 \arrow[r] & 0 \end{tikzcd}$$

$\rightsquigarrow 0 \to \mathrm{Hom}(M, N_1) \xrightarrow{i_*} \mathrm{Hom}(\quad) \xrightarrow{\pi_*} \mathrm{Hom}(N_3, M) \to 0$

exact

not exact

exactness on right would mean π_* is surj

$\Rightarrow$ given $\alpha : M \to N_3$ $\exists\ \beta : M \to N_2$

so $\pi_*(\beta) = \pi \circ \beta = \alpha$

e.g.| not

$$\begin{tikzcd} & \mathbb{Z}/2 \arrow[dl, "\nexists"'] \arrow[d, "id"] \\ \mathbb{Z} \arrow[r, "\pi"] & \mathbb{Z}/2 \end{tikzcd}$$

can only be zero by <u>torsion</u>

Rmk| M is free, the abv seq is exact <u>everywhere</u>!

Def| A Mod M is proj if this seq is exact – Hom(