Today: dim thy

Spse, $A = \bigoplus_{i=0}^{\infty} A_i$ be a gr. noeth ring &

$M = \bigoplus_{i=-\infty}^{\infty} M_i$ f.g graded A-mod

works w/ SES | λ = additive $\mathbb{Z}$-vald fn on fg A_0-mods $\to$ $\lambda : K_0(A_0) \to \mathbb{Z}$

Groth grp

Defn| The Poincaré series of M is $P_M(t) = \sum_{n=0}^{\infty} \lambda(M_n) t^n$

e.g|

- $A_0 = k$ a field, λ dimension

$$P_M(t) = \sum_{n=0}^{\infty} \dim_k(M_n) \cdot t^n$$

- A_0 = artinian ring, λ = length!
- $A = k[x]$ $\deg(x) = d \geq 1$ $P_A(t) = \frac{1}{1-t^d}$
- $A = k[x_1, \dots, x_m]$ $\deg(x_i) = d_i$ $P_A(t) = \prod_{i=1}^{m} \left(\frac{1}{1-t^{d_i}} \right)$

Say A, B graded k-alg $k \to$ field

$A \otimes_k B$ has natural grading

$$\deg d = \bigoplus_{i+j=d} A_i \otimes_k B_j$$

$$\text{coeff of } t^d \text{ in } P_{A\otimes B} = \sum_{i+j=d} \left(\text{coef } t^i \text{ in } P_A\right)\left(\text{coeff of } t^j \text{ in } P_B\right)$$

$$\Rightarrow P_{A\otimes B} = P_A \cdot P_B$$

e.g| • $A = \frac{k[x,y]}{(f)}$ x,y deg 1, $f \neq 0$ deg d

$$P_A(t) = \frac{1-t^d}{(1-t)^2}$$

Rmk| $0 \to M_1 \to M_2 \to M_3 \to 0$ is ex of gr A mods

$\Rightarrow$ $0 \to M_{1,n} \to M_{2,n} \to M_{3,n} \to 0$ exact $\forall n$

$\Rightarrow \lambda(M_{2,n}) = \lambda(M_{1,n}) + \lambda(M_{3,n})$ $\Rightarrow M \mapsto P_M$ is additive f of M

So, $B = k[x,y]$, $A = B/f$

$$0 \to f\cdot B \to B \to A \to 0$$

$\hookrightarrow$ isom to B as B-mod except shifted grading! to d

$$\Rightarrow P_A = P_B - P_{f\cdot B} \to t^d \cdot P_B$$

Thm | $P_M(t)$ is a rat'l func. Denom is prod of $(1-t^d)$ for various d

Pf | By induction on # gens of A as A_0-alg.

Say $x_1, \ldots, x_n$ gen A degs $d_1, \ldots, d_n$

$$0 \to \underset{\text{ker}}{K} \to M \xrightarrow{x_n} M \to \underset{\text{coker}}{C} \to 0$$

$\longrightarrow$ mult by x_n

$\hookrightarrow M[d_n]$

$\hookrightarrow$ M but gr shifted by d_n

$$\Rightarrow P_{M[d_n]} = t^d P_M.$$

Here, $x_n K = 0$ & $x_n C = 0$

$(\Rightarrow)$ K, C are fg mods / $A[x_1, \ldots x_{n-1}]$

(by ind) Know result for P_K, P_C] (rat'l)

$\Rightarrow$ additive $0 = P_K - P_{M[d_n]} + P_M - P_C$

$$\Rightarrow (1-t^d)P_M = P_K - P_C \Rightarrow P_M = \frac{P_K - P_C}{1-t^d}.$$

(actually denom is $\prod_{i=1}^{n} (1-t^{d_i})$

$\hookrightarrow$ $d_1, \ldots, d_n$ deg of gens of A as A_0 alg.

If A is gen'd over A_0 in deg 1

$\Rightarrow$ denom of P_M is a pow of $1-t$

Def $\partial(M)$ = order of Pole at $t=1$ of P_M

Note: $n \to \lambda(M_n)$ is a poly in n (for large n)
of deg $\partial(M)-1$

Prop $x \in A$ $\underbrace{\text{nonzero div on } M}_{xm=0 \implies m=0}$ $\implies \partial(M/xM) = \partial(M) - 1$

Pf $0 \to M[-\partial] \xrightarrow{x} M \to M/xM \to 0$ $\quad \partial = \deg(x)$
inj b/c x nonzero div
$P_{M/xM} = \underline{(1-t^{\partial})} \cdot P_M$
zero of order 1 at $t=1$

Def A ring $\mathfrak{a} \subset A$
An $\mathfrak{a}$-adic filter on M is a filt $F^\bullet M$ so $\mathfrak{a}F^nM \subset F^{n+1}M$
Say the filtration is stable if $\mathfrak{a}F^nM = F^{n+1}M$ large n

Obs if $F^\bullet M$ & $G^\bullet M$ are stable filt then $\exists\, m$ so that
$G^{n+m} \subset F^n$ & $F^{n+m} \subset F^n$ $\forall n$ → F & G have bdd differen
"Bdd diff" is equiv reln
→ so sts for $F^nM = \mathfrak{a}^nM$
$G^n \subset \mathfrak{a}G^{n-1} \subset \mathfrak{a}^2G^{n-2} \ldots\ldots \mathfrak{a}^nM = F^n$
Since G st. $\exists\, m$ so $G^n = \mathfrak{a}\cdot G^{n-1}$ $\forall n > m \implies G^{n+m} = \mathfrak{a}^nG^m$
$\subset \mathfrak{a}^nM = F^n$

Cons. any two st. filts def same top

Prop A = noeth local ring w max $\mathfrak{m}$. $\mathfrak{q}$ is $\mathfrak{m}$ primary
M = f.g A mod, $F^\bullet M$ = stable $\mathfrak{q}$ adic filt.
1) M/F^nM has finite length $\forall n \geq 0$
2) $n \mapsto \mathrm{len}_A(M/F^nM)$ is poly $n \gg 0$ deg $\leq$ min # gens of $\mathfrak{q}$
3) deg + leading coeff only dep on M & $\mathfrak{q}$ (not filtr).

PF) 1) A/q is artinian (max ideal is nilp as $m^n \subset q$)

$\Rightarrow$ any f.g A/q-mod has finite length.

$\hookrightarrow$ in particular, $F^nM/F^{n+1}M$ has fin length

$\hookrightarrow$ killed by q.

$$0 \to \underbrace{F^1M/F^2M}_{} \to M/F^2M \to \underbrace{M/F^1M}_{} \to 0$$

fin length $\Rightarrow$ M/F^2M fin len

$\rightsquigarrow$ similar arg $\rightsquigarrow$ M/F^nM fin len $\forall n$

2) $gr(A) = \bigoplus_{n=0}^{\infty} q^n/q^{n+1}$ $\qquad$ $gr(M) = \bigoplus_{n=0}^{\infty} F^nM/F^{n+1}M$

$\hookrightarrow$ graded ring
$gr_0(A) = A/q$ art
$gr(A)$ is gen'd as an alg over $gr_0(A)$ in deg 1 by set of gens of q.

$\hookrightarrow$ gr mod / $gr(A)$.
$\hookrightarrow$ its fg.

Consider, poincare series λ = length of - $gr_0(A)$-mod

$\Rightarrow n \mapsto \text{len}(F^n/F^{n+1})$ is poly of deg $\leq s-1$

$\text{len}(M/F^nM) = \text{len}(M/F^1M) + \text{len}(F^1/F^2) + \dots + \text{len}(F^{n-1}/F^n)$

$\Rightarrow n \mapsto \text{len}(M/F^nM)$ is poly of deg $\leq s$

3) $G^\bullet$ = second stable filt. let f, g poly s.t.

$\left.\begin{array}{l} f(n) = \text{len } M/F^n \\ g(n) = \text{len } M/G^n \end{array}\right\} n \gg 0$

$\exists\, m \gg \quad G^{n+m} \subset F^{n} \;\&\; F^{n+m} \subset G$

$M/G^{n+m} \twoheadrightarrow M/F^{n}$ ~~stuff~~

$g(n+m) \geq f(n) \quad \& \quad f(n+m) \geq g(n)$

$\Rightarrow$ f, g have same deg + leading coeff.

Def) χ_q^M = poly s.t. $\chi_q^M(n) = \mathrm{len}(M/q^n M)$ for large n

$\chi_q = \chi_q^A$

Note: $\deg \chi_q^M = d(\mathrm{gr}_q M)$ $\to$ assoc gr for q-adic filtr

$\hookrightarrow$ order of pole at $t=1$ of poincaré.

$\deg \chi_q \geq$ min # gens q

$\hookrightarrow \deg \chi_q = \chi_m$ (m is the max'l ideal in local ring)

Reason $m^r \subset q \subset m$

$\Rightarrow m^{rn} \subset q^n \subset m^n \Rightarrow \chi_m(rn) \geq \chi_q(n) \geq \chi_m(n)$

$\Rightarrow \deg \chi_m = \deg \chi_q$

Main thm) Fix noeth local ring A w/ max'l m

- $d(A)$ = common deg of χ_q for any m primary ideal q (deg of $\to n \mapsto \mathrm{len}(A/q^n)$)
- $\delta(A)$ = min # gen of some m-prim ideal
- $\dim A$ = Krull dim of A $\to$ max len of ch of primes

$d(A) = \delta(A) = \dim(A)$

Pf We'll show $\underbrace{\delta(A) \geq d(A)}_{\text{Alr showed}} \geq \dim A \geq \delta(A)$

Prop M f.g. k module, $x \in A$ nonzero div on M.
$M' = M/xM \implies \deg \chi_q^{M'} \leq \deg \chi_q^{M} - 1$

Pf let $N = xM$ → isom to M as A-mod b/c x is non-zero div $M \xrightarrow{\sim} N$

$F^n N = N \cap F^n M$ → Artin Rees says stable!

SES
↳ $0 \to N/F^n M \to M/q^n M \to M'/q^n M' \to 0$

$\implies \text{len}(M'/q^n M') = \text{len}(M/q^n M) - \underbrace{\text{len}(N/F^n M)}$

So the leading term cancels! & degree drops.

→ poly in n for n large & same degree & leading coeff as $\chi_q^N = \chi_q^M$

Cor if x nonzero div in ring
$d(A/x) \leq d(A) - 1$

Prop $\dim A \leq d(A)$

Say $d(A) = 0 \implies \chi_m$ deg 0, i.e., it's const
$\implies \text{len}(k/m^n)$ const for n large
$\implies m^n = m^{n+1}$ for n large
Nak → $m^n = 0$ → max'l nilp
$\implies$ Artinian $\implies$ $\dim A = 0$.

Now proceed by ind, $d = d(A)$

consider $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_r$ proper chain of pr in A.

Pick $x \in P_1 \setminus P_0$ let $A' = A/P_0$ ← domain

$x' =$ img of x → nonzero div

$\Rightarrow$ prev cor $\Rightarrow d(A'/xA') \leq d(A') - 1 \leq d - 1$

$\Rightarrow$ ind $\dim(A'/x'A') \leq d - 1$