Last time: algebras & Nakayama

$R \to S \quad , \quad R \to S'$

Spse | R -- rings & S and S' are R-algs

$\Rightarrow S \otimes_R S'$ is naturally R-alg — so how do we mult pure tens

$(x \otimes x')\times(y \otimes y') = (xy \otimes x'y')$

R-ing hom: $R \to S \otimes_R S' \qquad a \longmapsto \varphi(a) \otimes 1 = 1 \otimes \varphi'(a)$

$\qquad\qquad (a\cdot 1)\otimes 1 \qquad 1 \otimes (a \cdot 1)$

Restr. of scalar: S as R mod, $a \in R$, $x \in S$, $ax = \varphi(a)x$

Universal property of $S \otimes_R S'$ — this is coproduct in category of R-alg

Giving an R-alg homo $S \otimes_R S' \xrightarrow{\varphi} T$ (T R alg)

$\Leftrightarrow$ giving R alg homo $S \xrightarrow{\psi} T$, $S' \xrightarrow{\psi'} T$

given $\varphi \to$ note $S \to S \otimes_R S'$ is R alg hom

$\qquad s \longmapsto s \otimes 1$

so $S \to S \otimes_R S' \xrightarrow{\varphi} T$

given $\psi, \psi' \to \varphi(a \otimes b) := \psi(a) \cdot \psi'(b)$

Nakayama | If M is an f.g. R-mod & $\mathfrak{r} \subset J(R)$ s.t. $M = \mathfrak{r}M \Rightarrow M = 0$

$M/\mathfrak{r}M = 0$

Cor | let R be a local ring, unique max'l ideal $\mathfrak{m}$

$\Rightarrow \mathrm{Jac}(R) = \mathfrak{m}$

let M be fin gen R module spse $x_1, \ldots, x_n \in M$

s.t. $\bar{x}_1, \ldots, \bar{x}_n \in M/\mathfrak{m}M$ span — $M/\mathfrak{m}M$ is a $R/\mathfrak{m}$ mod $\Rightarrow M/\mathfrak{m}M$ is a vec sp

$\Rightarrow x_1, \ldots, x_n$ generate M

Pf) let N = submod of M gen by $x_1, \dots, x_n$ WTS $M/N = 0$ $\Leftrightarrow M = N$

Call $\overline{M} = M/N$ $\longrightarrow$ f.in gen

Clear that $\overline{M}/m\overline{M}$ $\xRightarrow{\text{Nakayama}}$ $\overline{M} = 0$ $\square$

$\hookrightarrow M \twoheadrightarrow \overline{M} \Rightarrow M/mM \twoheadrightarrow \overline{M}/m\overline{M}$

$\text{sp}(\bar{x}_1, \dots, \bar{x}_n) \longrightarrow \boxed{0}$

Localization

let $\mathbb{Q}$ by considering formal expr $\frac{a}{b}$ where $a, b \in \mathbb{Z}$, $b \neq 0$

subject to $\frac{a}{b} \sim \frac{a'}{b'} \Leftrightarrow ab' = a'b$

More gen if R is an integral domain, can build $\text{Frac}(R)$ in same way.

Even more gen.

let R be a commutative ring.

S be a mult subset $\{1 \in S,\ x, y \in S \Rightarrow xy \in S\}$

Can build $S^{-1}R$, elts are formal symbols $\frac{a}{b}$, $a \in R$, $b \in S$

Consider $S \times R$. Define equiv reln,

$$(s, a) \sim (s', a') \Longleftrightarrow (s'a - sa') \cdot t = 0 \quad \exists t \in S$$

need to get equiv reln

in domain, can just cancel t as nonzero!

Why equiv reln?

trans.

$(a_1, s_1) \sim (a_2, s_2)$ & $(a_2, s_2) \sim (a_3, s_3)$

$\exists t, t' \in S$ $\quad t(s_2 a_1 - s_1 a_2) = 0 = t'(s_3 a_2 - a_3 s_2)$

mult by $t's_3$ $\qquad$ ts_2

take diff $\underbrace{tt's_2}_{\in S}(a_3 s_1 - s_3 a_2) = 0 \Rightarrow \boxed{\sim}$

Formally define $S^{-1}R = (S \times R)/\sim$ & write $\frac{a}{s}$ equiv cl. (s,a)

note $\frac{a}{1} = \frac{0}{1}$ $\exists\, t \in S$ so $ta = 0$

$S^{-1}R$ is a comm ring using usual frac rules!

$\exists$ natural ring homo $R \xrightarrow{\varphi} S^{-1}R$, $x \longmapsto \frac{x}{1}$ → not always injective

if $s \in S \Rightarrow \frac{s}{1}$ unit in $S^{-1}R$ inv $\frac{1}{s}$

$\ker\varphi = \{x \in R \mid \exists\, t \in S \text{ so } tx = 0\}$

This $S^{-1}R$ has univ prop.

giving homo $\psi' : S^{-1}R \to T$

$\Updownarrow$

ring homo $\psi : R \to T$ s.t. $\psi = \psi' \circ \varphi$

$\psi(s)$ is a unit of T $\forall\, s \in S$

Eg 9.3.1

i) if p prime ideal $\Rightarrow R \setminus \{p\} = \{x \in R \mid x \notin p\}$ is mult set

if $x \notin p$, $y \notin p \Rightarrow xy \notin p$

Write R_p in pl of $S^{-1}R$ "localization of R at p"

always a local ring $m = p^e$

only sees local behav of R at $p \in \operatorname{Spec}(R)$

2) if $f \in R$ $S = \{f^n \mid n \geq 0\}$

write $R_f = R[\frac{1}{f}]$ as notation for $S^{-1}R$

$\cong R[x]/(fx-1)$ exer.

3) If $0 \in S \implies S^{-1}R = 0$ by defn $\frac{a}{1} \sim \frac{0}{1}$ as $t a = 0$ $(t = 0)$

4) If $\mathfrak{p} \subset R$ is an ideal $S = 1 + \mathfrak{p}$
$= \{1 + x \mid x \in \mathfrak{p}\}$ is mult set.

Can also do this constr for modules!

$M = A$-mod $\rightsquigarrow S^{-1}M = S^{-1}A$ mod

pairs $(s, m) \in S \times M$

$(s, m) \sim (s', m')$ if $\exists t \in S$

$\underline{st(sm' - s'm) = 0}$.

$S^{-1}M = (S \times M)/\sim$

$\hookrightarrow$ write $\frac{m}{s}$ for cl. $\underline{(s, m)}$.

If $S = R \setminus \mathfrak{p}$ write $M_{\mathfrak{p}} = S^{-1}M$

$S = \{f^n\}$ — $M[\frac{1}{f}]$ or M_f

We have a functor $\mathrm{Mod}_A \xrightarrow{S^{-1}} \mathrm{Mod}_{S^{-1}A}$

$M \longmapsto S^{-1}M$

If $f: M \to N$ a A-mod homo, get $S^{-1}A$ mod homo

$S^{-1}f : S^{-1}M \longrightarrow S^{-1}N$

$\frac{m}{s} \longmapsto \frac{f(m)}{s}$

Prop This functor is exact. $\quad \to \ker g = \operatorname{im} f \implies g \circ f = 0$

Let $M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3$ is exact seq of A-M

wts

$S^{-1}M_1 \xrightarrow{S^{-1}f} S^{-1}M_2 \xrightarrow{S^{-1}g} S^{-1}M_3$ exact!

by functoriality $S^{-1}g \circ S^{-1}f = S^{-1}(g \circ f) = 0$

$\implies \operatorname{im} S^{-1}f \subset \ker S^{-1}g$

say $\frac{m}{s} \in \ker S^{-1}g \quad m \in M_2, \; s \in S$

$\Rightarrow 0 = (S^{-1}g)\left(\frac{m}{s}\right) = \frac{g(m)}{s}$

holds in $S^{-1}M_3 \Rightarrow \exists\, t \in S$ so $t\,g(m) = 0$ in M_3

$\Rightarrow g(tm) = 0$

$\Rightarrow tm \in \ker g = \operatorname{im} f$

So $\exists\; tm = f(m')$

divide by unit

$\frac{tm}{1} = \frac{f(m')}{1}$ in $S^{-1}M_2$

so $\frac{m}{s} = \frac{f(m')}{ts} = (S^{-1}f)\left(\frac{m'}{ts}\right)$

$\Rightarrow \frac{m}{s} \in \operatorname{im} S^{-1}f$

Prop) $S^{-1}M \cong S^{-1}A \otimes_A M$, extn of scalars of M along hom $A \to S^{-1}A$

natural homo of $S^{-1}A$ mod

Pf) $\exists$ natural map of $S^{-1}A$ mod (check details)

$f : S^{-1}A \otimes_A M \longrightarrow S^{-1}M$

$\frac{a}{s} \otimes m \longmapsto \frac{am}{s}$

WTS, isom!

Surjective is clear! $\frac{m}{s} \xleftarrow{f} \frac{1}{s} \otimes m$

injectivity: consider gen elt of $S^{-1}A \otimes_A M$

$x = \sum_{i=1}^{n} \frac{a_i}{s_i} \otimes m_i \qquad a_i \in A, \; s_i \in S, \; m_i \in M$

sum of pure tensors!

Put $s = s_1 \cdots s_n$ $\quad \frac{1}{s_i} = \frac{t_i}{s}$ $\quad t_i = s_1 \cdots s_{i-1} s_{i+1} \cdots s_n$

$$x = \sum_{i=1}^{n} \frac{a_i t_i}{s} \otimes m_i$$ as we tensor over A can move these scalars across $\otimes$

$$= \sum_{i=1}^{n} \frac{1}{s} \otimes a_i t_i m_i$$ bilin

$$= \frac{1}{s} \otimes \Big(\underbrace{\sum_{i=1}^{n} a_i t_i m_i}_{= y \in M}\Big)$$

Shown all $x \in S^{-1}A \otimes_A M$ has form $\frac{1}{s} \otimes y$ $\quad s \in S$, $y \in M$

Say $0 = f(x) = \frac{y}{s}$ in $S^{-1}M$

so $\exists t \in S$ so $yt = 0$ in M

$x = \frac{1}{s} \otimes y = \frac{t}{ts} \otimes y$ can mult & div by $t \in S$

$= \frac{1}{ts} \otimes (ty)$ rnk

$= \frac{1}{ts} \otimes 0 = 0 \Rightarrow \boxed{\ker f = 0}$

Cor $S^{-1}A$ is a flat A mod

(or $A \to S^{-1}A$ is flat ring map.)

Prop there is biject corr

$$\left\{ \begin{array}{c} \text{prime ideals of} \\ S^{-1}A \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{prime ideals of} \\ A \text{ disj from } S \end{array} \right\}$$

$$q \longmapsto q^c \quad (\text{contract})$$

$$p^e \longleftarrow p$$

warn
not true in gen that we extend to prime

Special case $S = \{f^n\}$

Primes $A[\frac{1}{f}]$ $\xleftrightarrow{\text{bij}}$ primes of A don't have f

$D(f) \subset \operatorname{spec}(A)$.