**Last time**: $\mathfrak{a} = \bigcap_{i=1}^{n} q_i$ primary ~~recomp~~ decomp.

where $q_i$ is primary ($xy \in q_i \Rightarrow x \in q_i$ or $y^n \in q_i$)

This is minimal if (i) $p_i = \mathrm{rad}(q_i)$ are dist

(ii) can't omit any $q_i$

**1st Uniqueness**: In a min primary decomp

$\underbrace{p_1, \ldots, p_n}$ are uniquely det by $\mathfrak{a}$

↳ primes assoc to $\mathfrak{a}$ — Isolated (embedded) / embedded

**Prop** Say $\mathfrak{a}$ has primary decomp, $p_1, \ldots, p_n$ are assoc pr.

$\mathfrak{a} \subset p \iff p_i \subset p$
(prime $p$) ↳ minimal

**Cor** Isolated primes of $\mathfrak{a}$ are exactly the min elts in set of pr $\supseteq \mathfrak{a}$ ($\longleftrightarrow$ min pr of $A/\mathfrak{a}$)

**Pf** If $\mathfrak{a} \subset p \Rightarrow \mathrm{rad}(\mathfrak{a}) \subset p \Rightarrow p_i \subset p$
$\mathrm{rad}(\mathfrak{a}) = \bigcap p_i$ ↳ for some $i$, assume $p_i$ min

$\mathfrak{a} \subset q_i \subset p_i$ so if $p_i \subset p \Rightarrow \underline{\mathfrak{a} \subset p}$.

**Prop** $\mathfrak{a} = \bigcap q_i$ min primal dec $p_i = \mathrm{rad}(q_i)$

$\bigcup_{i=1} p_i = \{x \in A \mid (\mathfrak{a} : x) \supsetneq \mathfrak{a}\}$

$= \{x \in A \mid x \text{ is a zero div in } A/\mathfrak{a}\}$

Special case $\mathfrak{a} = 0$

$\Rightarrow \bigcup p_i = \{\text{zero div of } A\}$

**Cor** A minimal pr of zero divisors!

**Pf)** Repl. $A$ w/ $A/\mathfrak{r}$ $\Rightarrow$ enough to treat $\mathfrak{r}=0$ case.

$$\{\text{zero div}\} = \bigcup_{x\neq 0} (0:x)$$

y zero div $\Longleftrightarrow$ $y^n$ 0

$$= \bigcup_{x\neq 0} \operatorname{rad}(0:x)$$

$$\operatorname{rad}(0:x) = \bigcap_{x\notin q_i} p_i \subset p_i \text{ for some } p_i$$

$$\Rightarrow \bigcup_{x\neq 0} \operatorname{rad}(0:x) \subset \bigcup_{i=1}^{n} p_i$$

so have 1 cont.

**1st Uniq thm:** each $p_i$ has form $\operatorname{rad}(0:x)$ for some $x$

$\Rightarrow$ each $p_i$ consists of zero divs.

**Prop)** $S\subset A$ mult set, $q \to p$-primary ideal

(a) $S\cap p \neq \emptyset \Rightarrow q^e = (1)$

(b) $S\cap p = \emptyset \Rightarrow S^{-1}q$ is $S^{-1}p$ primary

& $q = (S^{-1}q)^c$

$A \to S^{-1}A$ so can extend & contract

$\{\text{pr id } A \text{ not meeting } S\} \longleftrightarrow \{\text{pr of } S^{-1}A\}$ by contr & ext

**Def)** Setup as above $\operatorname{Sat}_S(\mathfrak{a}) = \{x\in A \mid \exists\, s\in S \text{ st } sx\in\mathfrak{a}\}$

Showed $\operatorname{Sat}_S(\mathfrak{a}) = (\mathfrak{a}^e)^c$

& $\operatorname{Sat}_S(\mathfrak{a}\cap\mathfrak{b}) = \operatorname{Sat}_S(\mathfrak{a}) \cap \operatorname{Sat}_S(\mathfrak{b})$

$\supseteq$: $x\in$ RHS so $sx\in\mathfrak{a}$, $s'x\in\mathfrak{b}$ $\;ss'x\in\mathfrak{a}\cap\mathfrak{b} \Rightarrow x\in\operatorname{Sat}_S(\mathfrak{a}\cap\mathfrak{b})$

**Pf)** a) $s\in S\cap p$ $\quad p = \operatorname{rad}(q) \to s^n\in q$ for some $n$

So $S^{-1}q$ contains a unit $\to S^{-1}q = (1)$

b) $\frac{x}{1}\cdot\frac{y}{1} \in S^{-1}q \Rightarrow \exists\, s\in S$ so $sxy\in q \Rightarrow xy\in q$

as $q$ is primary & $s\notin \operatorname{rad}(q) = p$

$\Rightarrow x\in q$ or $y^n\in q$

$\Rightarrow \frac{x}{1}\in S^{-1}q$ or $\frac{y^n}{1}\in S^{-1}q$.

$(S^{-1}q)^c = \mathrm{Sat}_S(q)$ $sx \in q \Rightarrow x \in q$, $s \in S$ $\Rightarrow \mathrm{Sat}_S(q) = q$

$\mathrm{rad}(S^{-1}q) = S^{-1}p$ → this is a pr in $S^{-1}A$ that contr to $p$ so it is $S^{-1}p$.

**Rmk)** Prop gives bijection

$\{p \text{ primary ideals in } A\} \longleftrightarrow \{S^{-1}p \text{ primary ideals in } S^{-1}A\}$

**Prop)** $\mathfrak{a} = \bigcap_{i=1}^{n} q_i$ minimal prim decomp $p_i = \mathrm{rad}(q_i)$ & $S$ = mult set.

Index ideals so $p_1 \cdots p_m$ don't meet $S$, rest do

$S^{-1}\mathfrak{a} = \bigcap_{i=1}^{m} S^{-1}q_i$ & $\mathrm{Sat}_S(\mathfrak{a}) = \bigcap_{i=1}^{m} q_i$

min primary decomp.

**Pf)** $S^{-1}\mathfrak{a} = \bigcap_{i=1}^{n} S^{-1}q_i = \bigcap_{i=1}^{m} S^{-1}q_i$ → $S^{-1}p_i$ primary

$i > m$

$\Rightarrow S^{-1}q_i = (1)$ can omit

So, this is prim decomp of $S^{-1}\mathfrak{a}$.

Let us show minimality, (1) $S^{-1}p_1, \ldots, S^{-1}p_m$ dist. as localization gives bijection on pr not meeting $S$

(2) If $S^{-1}q_i \supset \bigcap_{j \neq i} S^{-1}q_j$ contr to $A$

$\Rightarrow q_i \supset \bigcap_{j \neq i} q_j$ oops.

↳ Contr to $A$

$\mathrm{Sat}_S(\mathfrak{a}) = \bigcap_{i=1}^{m} q_i$

## 2nd Uniqueness thm

$\mathfrak{a} = \bigcap_{i=1}^{n} q_i$ minimal primary dec w/ $P_i = \text{rad}(q_i)$

let $J \subset \{1, \ldots, n\}$ so $\{P_j\}_{j \in J}$ is downwards cl in $\{P_1, \ldots, P_n\}$

meaning, if $P_i \subset P_j$ & $j \in J$ $\Rightarrow$ $i \in J$.

$\Rightarrow \bigcap_{j \in J} q_j$ only dep on $\mathfrak{a}, J$.

**Cor 1** if $P_j$ is an isolated pr, let $J = \{j\}$

shows $q_j$ dep only on $\mathfrak{a}$.

So so primary components at isolated primes are canonical! (can't do for emb).

**Pf)** let $S = A \setminus (\bigcup_{j \in J} P_j)$

S.t. $S\mathfrak{a} = \bigcap_{j \in J} q_j$ $\Rightarrow$ only dep on $\mathfrak{a}$ & $J$

$\therefore$ $P_j$'s survive and others don't by minimality.

I.e. $P \subset A$, $P$ is disj from $S \Leftrightarrow P \subset \bigcup_{j \in J} P_j$ pr avoid $\Leftrightarrow P \subset P_j$ for some $j$

So, if $P = P_i$

So $P_i$ disj from $S \Leftrightarrow i \in J$ by downwards closed.

## Prime avoidance lemma

let $P_1, \ldots, P_n$ in $A$ rng & $\mathfrak{a} \subset A$ ideal

$\Rightarrow \mathfrak{a} \subset \bigcup P_i \Rightarrow \exists i$ so $\mathfrak{a} \subset P_i$.

**Pf)** contrapos. if $\mathfrak{a} \not\subset P_i \ \forall i \Rightarrow \mathfrak{a} \not\subset \bigcup P_i$

Prove by ind on $n$. Clear if $n=1$

Assume true for $n-1$,

by induction for each $1 \leq i \leq n$ $\mathfrak{a} \subseteq \bigcup_{j \neq i} P_j$

$\Rightarrow$ pick $x_i \in \mathfrak{a}$ so $x_i \notin \bigcup_{j \neq i} P_j$

If some $x_i \notin P_i \Rightarrow x_i \notin \bigcup P_i \Rightarrow \mathfrak{a} \notin \bigcup_{i=1}^{n} P_i$

Now say $x_i \in P_i \; \forall i$ $\quad y = \sum_{i=1}^{n} x_1 \cdots \underbrace{\widehat{x_i}}_{\to \text{omit}} \cdots x_n \in \mathfrak{a}$

Cl. $y \notin P_i \; \forall i$

if $y \in P_i \Rightarrow x_1 \cdots \widehat{x_i} \cdots x_n \in P_i$ as all other terms in $P_i$

pr $\Rightarrow x_j \in P_i$ for some $j \neq i$ oops!

Existence

Def) A ideal $\mathfrak{a}$ is irred if $\mathfrak{a} = \mathfrak{b} \cap \mathfrak{c}$
$\Rightarrow \mathfrak{a} = \mathfrak{b}$ or $\mathfrak{a} = \mathfrak{c}$.

Obs) If $A$ noetn $\Rightarrow$ every ideal $\mathfrak{a}$ is a finite int of irred ideals.

Pf) by ACC $\exists$ ideal $\mathfrak{a}$ max'l among ideals for which stat is false.

If $\mathfrak{a}$ is irred $\mathfrak{a} = \mathfrak{a}$ irred decomp X

Else, $\mathfrak{a} = \mathfrak{b} \cap \mathfrak{c} \to$ both have decomp by max'l of $\mathfrak{a}$

$\to$ Put together to get irred decomp of $\mathfrak{a}$.

Prop) $A$ noetn $\Rightarrow$ any irred ideal is primary.

Pf) passing to quotient, sts $(0)$ irred $\Rightarrow (0)$ primary.
$\to$ every 0 div nilp.

Say $xy = 0$ & $y \neq 0$ wts $x^n = 0$
$\to \boxed{\text{nilp}}$.

**Cons** $\mathrm{Ann}(x) \subset \mathrm{Ann}(x^2) \subset \mathrm{Ann}(x^3) \subset \cdots$

by **ACC** $\exists N$ so $\mathrm{Ann}(x^n) = \mathrm{Ann}(x^{n+1}) = \cdots$

**C1.** $(0) = (x^n) \cap (y)$

**Reason** $a \in (x^n) \cap (y)$, $a \in (y) \Rightarrow ax = 0$

$a \in (x^n) \Rightarrow a = bx^n$

$\therefore$ either $(x^n) = 0$ or $(y) = 0$

$\Rightarrow bx^{n+1} = 0$

$\Rightarrow b \in \mathrm{Ann}(x^{n+1}) = \mathrm{Ann}(x^n)$

$\Rightarrow bx^n = 0$

$\boxed{a =}$

but $y \neq 0$ so $(y) \neq 0$

so $(x^n) = (0)$

$\Rightarrow x^n = 0 \Longrightarrow$ **x nilp!**

This prop + Prev $\Rightarrow$

**Thm)** In noeth ring, any ideal has prim decomp!

## Integral Dependence

$A \subset B$ subring.

**Def)** $x \in B$ is integral over $A$ if it satisfies a monic polynomial w/ coeff in $A$.

$x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0 \quad a_i \in A$

**e.g.)** $\mathbb{Z} \subset \mathbb{Q}$ $x \in \mathbb{Q}$ integral $/ \mathbb{Z}$ iff $x \in \mathbb{Z}$

$\hookrightarrow$ rational root test

$\boxed{\text{algebraic int}}$

$\mathbb{Z} \subset \mathbb{C}$ $\sqrt{-1}$ or $\sqrt{2}$ integral $/ \mathbb{Z}$

$\hookrightarrow$ $\frac{1+\sqrt{-3}}{2}$ is integral but $\frac{1+\sqrt{-3}}{-1}$ not.

$\hookrightarrow x^3 = 1$ $\qquad \hookrightarrow$ hard to see

Say $k$ is a number field $\to$ $K$ is a field of char 0
↳ fin dim'l as $\mathbb{Q}$ vs
$\to$ finite ext of $\mathbb{Q}$.

So $\mathbb{Z} \subset K$ so elt of $K$ int / $\mathbb{Z}$ are alg integers in $K$.

**Prop 1** $x \in B$ TFAE.

(a) $x$ is integral / $A$ $\to$ adjoin elt $x$.
(b) $A[x]$ is f.g as an $A$-mod
(c) $A[x] \subset C$ s.t $A \subset C \subset B$ $\to$ a subring that is fg as an $A$-mod
(d) $\exists$ faithful $A[x]$-mod that is fg as $A$ mod.

**Pf)** (a)$\Rightarrow$(b)

$x^n + a_{n-1}x^{n-1} + \dots + a_0 = 0$

$\Rightarrow x^n = -a_{n-1}x^{n-1} - \dots - a_0$

$\Rightarrow x^{n+1} = -a_{n-1}\underbrace{x^n}_{\to \text{ subst this}} - \dots a_0 x$

$\Rightarrow A[x]$ is gen'd by $1, \dots, x^{n-1}$

b$\Rightarrow$c trivial let $C = A[x]$

c$\Rightarrow$d Clear take $M = C$ $\to$ faithful.

d$\Rightarrow$a let $m_1, \dots, m_n$ gen $M$ as an $A$ mod

$M \ni xm_1 = a_{11}m_1 + \dots + a_{n1}m_n$
$\vdots$
$xm_n = a_{1n}m_1 + \dots + a_{nn}m_n$

mat $(a_{ij})_{1\le i,j\le n}$ $n\times n$ matrix w/ entries of $A$ (matrix of op mult by $x$)