**Last time:** $K$ = field . $\Omega$ = alg closed

$\Sigma = \{(A, f) \mid A \subset K,\ f: A \to \Omega\}$

We showed if $(A, f) \in \Sigma$ is max $\Rightarrow$ $A$ is val ring.

**Prop)** $A \subset K$ subring $\Rightarrow$ $\underbrace{\text{int cl. of } A \text{ in } K}_{C} = \left.\bigcap_{\substack{A \subset B \\ B \text{ val ring}}} B \right\} C'$

**Pf)** $\overset{\text{val ring}}{A \subset B} \Rightarrow C \subset B$ as $B$ int cl.

$\Rightarrow$ $\boxed{\subseteq}$ . so $C \subset C'$

($\supseteq$) say $\underbrace{x \notin C}_{\substack{\text{i.e. } x \\ \text{not int / } A}}$ $\Rightarrow x \notin A[x^{-1}] = A'$

↳ if $x = a_0 x^{-n} + \cdots + a_n$ $a_i \in A$

↳ mult by $x^n$ $\Rightarrow x$ int in $A$.

$\Rightarrow$ in $A'$, $x'$ is not a unit so $x' \in m' \subset A'$

let $\Omega$ = alg cl of $A'/m'$ have a can $A' \xrightarrow{f'} \Omega$

$(A', f') \in \Sigma$. Choose a max elt $(B, g)$ so $(A', f) \leq (B, g)$

From last time, $B$ is a val ring, max ideal of $B = \ker(g)$

$x^{-1} \in m' \Rightarrow f'(x^{-1}) = 0$ so $g(x^{-1}) = 0$ as $g|_{A'} = f$

$\Rightarrow x^{-1} \in$ max'l ideal of $B$ $\Rightarrow$ $\boxed{x \notin B}$

$\Rightarrow x \notin C'$

**Defn)** A discrete val ring (DVR) is a val ring $B$ s.t

$\Gamma = K^\times / B^\times$ is infinite cyclic $\cong \mathbb{Z}$

↳ $K$ = Frac$(B)$

$\Longleftrightarrow$

$B$ is DVR is $\exists$ val $K^\times \to \mathbb{Z}$ s.t $B = \{x \in K \mid v(x) \geq 0\}$

Prop| $B$ = DVR $v : K^\times \twoheadrightarrow \mathbb{Z}$

Fix $\pi \in B$ st $v(\pi) = 1$

$\mathfrak{a}_n = \{x \in B \mid v(x) \geq n\}$ $(n \geq 0)$

(a) Every elt of $K^\times$ can be written uniquely as $u \cdot \pi^n$ where $u \in B^\times$ & $n \in \mathbb{Z}$

(b) $\mathfrak{a}_n = (\pi^n)$ $( \Rightarrow \mathfrak{a}_n$ is an ideal$)$

(c) The $\mathfrak{a}_n$'s are all the non-zero ideals of $B$

(d) $B$ is noetherian.

e.g| $K = \mathbb{Q}$ $v : K^\times \longrightarrow \mathbb{Z}$ is the p-adic val

i.e $v(p^n) = \mathbb{Z}$ for $n \in \mathbb{Z}$ $v(\frac{a}{b}) = 0$ if $p \nmid a, b$

$B = \{x \in \mathbb{Q} \mid v(x) \geq 0\} = \{\frac{a}{b},\ a, b \in \mathbb{Z} \mid p \nmid b\}$

Can take, $\pi = p$

(a) Says if $x \in \mathbb{Q}^\times$ can write $x = p^n \cdot \frac{a}{b}$ $n \in \mathbb{Z}$
$p \nmid a, b$ $\frac{a}{b} \in B^\times$

Pf| (a) let $x \in K^\times$ be given. let $n$ be its valuation.
let $u = \frac{x}{\pi^n}$ $\longrightarrow$ note $\mathrm{val}(u) = 0$, so $u \in B^\times$ a unit.

Thus, $x = u \cdot \pi^n$ where $u$ a unit of $B$.

(b) Clear $\pi^n \in \mathfrak{a}_n \Rightarrow (\pi^n) \in \mathfrak{a}_n$

let, $x \in \mathfrak{a}_n \Rightarrow v(x) \geq n$ so $y = \frac{x}{\pi^n}$ then $v(y) \geq 0$
$\Rightarrow y \in B \Rightarrow x = y \cdot \pi^n$
so $x \in (\pi^n)$

$\Rightarrow \mathfrak{a}_n = (\pi^n)$

(c) let $\mathfrak{a} \subset B$ a nonzero ideal. let $n = \min\{v(x) \mid x \in \mathfrak{a}\}$

Have $v(x) \geq n$ $\forall x \in \mathfrak{a}$ $\Rightarrow \mathfrak{a} \subset \mathfrak{a}_n$

OTOH, $\exists$ $y \in \mathfrak{a}$ s.t $v(y) = n \Rightarrow y = u \cdot \pi^n$
$\mathfrak{a}_n =$ $\Rightarrow (\pi^n) \subset \mathfrak{a}$

(2) $1 = \mathfrak{a}_0 \supset \mathfrak{a}_1 \supset \mathfrak{a}_2 \supset \dots \supset 0$

Complete lattice of ideals in $B$ clearly satisfies ACC.

Unique up to units

**Rmk:** An elt $\pi$ w/ $v(\pi) = 1$ is called a uniformizer

**Rmk)**
- uniformizer gens max'll ideal $(\pi) \to m$
- $\mathfrak{a}_n = m^n$ so all nonzero ideals of $B$ is a power of maximal ideal.

i.e. unit multiples of given unif.

**Prop)** let $B$ be a noetherian, local, domain of Krull dim 1 $\to$ all nonzero prime max'll $\to$ one such nonzero pr. exists.

TFAE:

a) $B$ is a DVR

b) $B$ is int closed

c) $m$ the max'll ideal is princ.

d) $\dim_K (m/m^2) = 1$ $\quad K = B/m \to$ residue field of local ring.

$\to B/m$

e) Every nonzero ideal of $B$ is a pow of $m$

f) $\exists\, \pi \in B$ s.t. every nonzero ideal is gen'd by pow of $\pi$

**Pf)** General Comments

$B$ has exactly 2 pr $(0), m$ as Kr. dim 1 & local

**(A)** If $\mathfrak{a} \subset B$ is non-zero then $\mathrm{rad}(\mathfrak{a}) = m$

$\Rightarrow \mathfrak{a}$ is $m$-primary (cool)

& $\exists\, n$ s.t. $m^n \subset \mathfrak{a}$ b/c $B$ is noeth $\to$ some power of rad contained in ideal

**(B)** $m^n \neq m^{n+1}$ if so $m^n = m \cdot m^n \to$ modules f.g. as Noeth.

$\Rightarrow m^n = 0$ by Nakayama $\to$ can't have in domain now equiv

(a ⇒ b) alr shown val rings are int closed

(b ⇒ c) pick $a \in m$ nonzero,

By rmk A, $m^n \subset (a)$ for some $n$, choose $n$ min.

pick $b \in m^{n-1} \setminus (a)$

let $x = \frac{a}{b} \in K$, know $x^{-1} \notin B$ as if $\frac{b}{a} \in B$ then $b \in Ba = (a)$ not true.

$\therefore x^{-1}$ not integral $/B$ as $\boxed{B \text{ int cl.}}$ (b).

$\Rightarrow x^{-1}m \not\subset m$ (Reason, if $x^{-1}m \subset m$ then $m$ would be a faithful $B[x^{-1}]$ that's f.g as a $B$-mod $\Rightarrow x^{-1}$ int / X)

$\Rightarrow x^{-1}m \subset B$

$m^n \supset b \cdot m \subset B \cdot a = (a)$

$\Rightarrow x^{-1}m$ is an ideal of $B$ not cont in $m$

so, $x^{-1}m = B$

$\Rightarrow m = (x)$ which pr c

(c ⇒ b) Say $m = (\pi)$. So $m/m^2$ is gen'd as a $B$ module by img of $\pi$ → of K-vs.

$\Rightarrow \dim_K (m/m^2) \leq 1$

OTOH, $m/m^2 \neq 0$ by (B)

(d ⇒ e) Nak lemma ⇒ $m = (\pi)$ for $\pi \in B$

Say, $\Omega \neq (0)$ is ideal of $B$

by rmk ⇒ $(\pi^n) \underset{\text{RMK A}}{\subset} \Omega \subset (\pi)$

$\Rightarrow \pi^{n-1} \subset \pi^{-1}\Omega \subset (1) = B$

By ind $\pi^{-1}\Omega = (\pi^r) \Rightarrow \Omega = (\pi^{r+1}) = m^{r+1}$

(e => f) By rmk B, $m \neq m^2$

$\therefore$ Choose $\pi \in m \setminus m^2$.

By (e), $(\pi) = m^r$, So since $\pi \notin m^2$

$\Rightarrow r = 1$

$\therefore m = (\pi)$

If $\mathfrak{a} \neq 0$ ideal of B, $\mathfrak{a} = m^r \overset{\text{assump}}{=} (\pi^r)$.

(f => a) let $\pi$ as in f be given.

let $0 \neq a \in B$ be given by f $(a) = (\pi^r)$ — r is unique

if $(\pi^r) = (\pi^s)$ $s \geq r$,

divide by $\pi^r$

$\Rightarrow (\pi^{s-r}) = (1) \Rightarrow r = s$

Def $v(a) = r$

More gen, $a, b \in B$ nonz. def $v(a/b) = v(a) - v(b)$.

Check: v is a valuation on K, & B is ring assoc to v

D.

Thm A = noeth domain of Krull dim 1

TFAE

a) A is int closed

b) every primary ideal is a prime power

c) All localizations at max'ls are DVR $A_p$, p max'l.

Pf (a => c) note localization will pres — by conv. noeth, domain, Krull dim 1

So $A_p$ is → 1 local.

$\therefore$ by prev, $A_p$ is DVR

($c \Rightarrow a$) Since $A_p$ is int closed $\forall p$ prime $\forall p$

(p as pr $\Rightarrow$ max'll nonzero & true for 0)

$\Rightarrow A$ is int cl

by prev prob abt int cl. being local

($c \Rightarrow b$) let $q$ be a $p$ primary ideal of $A$.

$q = (q^e)^c$ via $A \subset A_p$

So, $q^e = p^n A_p$ for some $n$ b/c $A_p$ is DVR

$q$ & $p^n$ are $p$ primary ideals of $A$ & have same extn to $A_p$ $\Rightarrow$ they're equal

($b \Rightarrow c$) let $\mathfrak{a} \neq 0$ be an ideal of $A_p$ & max'll ideal.

Earlier comment holds b/c rad$(\mathfrak{a}) = pA_p$

$\Rightarrow \mathfrak{a}$ is $pA_p$-primary

By corr thm for primary ideals

$\mathfrak{a} = q^e$ for $q \subset A$ $p$-prim

(b) $\Rightarrow q = p^n \Rightarrow \mathfrak{a} = p^n A_p \Rightarrow$ $\boxed{A_p \text{ DVR}}$

Def) Dedekind Dom.