Last time: Hilbert Basis thm, if A noeth so is $A[x]$

Cor: A noeth $\Rightarrow$ any f.g. A-alg is noeth.

Prop: A noeth $\Rightarrow S^{-1}A$ noeth.

Pf: let $b \subset S^{-1}A \Rightarrow b = (b^c)^e$

b^c is fg b/c A is Noeth $\Rightarrow (b^c)^e = b$ is fg.

Cor: A noeth $\Rightarrow A_p$ is noeth $\forall p \subset A$

Warning: A noeth $\not\Rightarrow$ subrings of A are noeth.

e.g. $A = \mathbb{C}[x,y]$ $\quad B = \mathrm{span}\{x^i y^j \mid i < \alpha j\}$

$\alpha \in \mathbb{R}_{>0}$

if $\alpha \notin \mathbb{Q}$ $\quad B$ not fg / noeth

Recall: A top sp is irred if it is not the union of two proper cl. subsets.

Prop: X - any top sp:

(a) $Y \subset X$ irred $\Rightarrow \overline{Y}$ is irred

(b) any irred subset of X contained in a max'l.

Pf a) Say $\overline{Y} = A \cup B$ where $A, B \subset \overline{Y}$ are closed.

$Y = (A \cap Y) \cup (B \cap Y) \Rightarrow$ by irred $\Rightarrow Y = Y \cap A$ or $Y = Y \cap B$

Say first, i.e., $A \supset Y$.

Since Y dense in $\overline{Y}$ & A cl. in $\overline{Y} \Rightarrow A = \overline{Y}$

so $\overline{Y}$ irred.

b) Say $Y \subset X$ irred.

let $\Sigma = \{\text{irred } Z \subset X \mid Y \subset Z\}$ $\quad$ wts Σ has max'l

use Zorn.

Say $Z_1 \subset Z_2 \subset \cdots$ chain in Σ. Say $Z = \bigcup_{i \geq 1} Z_i$

<u>wts</u>: Z is irred

Say $Z = A \cup B$, $A, B \subset Z$ <u>closed</u>.

$A_i = Z_i \cap A$ & $B_i = Z_i \cap B$ (cl sso Z_i)

So, $Z_i = A_i \cup B_i$. As Z_i irred $A_i = Z_i$ or $B_i = Z_i$

One holds inf. often. Say first,

$\Rightarrow A \supset Z_i \ \forall i$. So $A = Z$ $\Rightarrow$ (Zorn $\Rightarrow$ result)

The max'l irred sso X are the irred components.

Obs) ∘ If Y is irred comp $\Rightarrow$ Y is closed ($Y \subset \overline{Y}$ irred $\Rightarrow_{\text{max}} Y = \overline{Y}$)

∘ Every irred sso X is contained in some irred comp

$\Rightarrow$ every pt contained in irred comp $\Rightarrow X = \bigcup_{Y \text{ irred compx}} Y$.

Prop) $X = \operatorname{Spec}(A)$

Irred components $\longleftrightarrow V(\mathfrak{p})$ $\mathfrak{p} \subset A$ is <u>min pr.</u>.

Pf) Say $Y \subset X$ is irred comp. $I \subset A$

as closed $\Rightarrow Y = V(I) = V(\operatorname{rad} I)$ irred by hw

$\cong \operatorname{Spec}(A/\operatorname{rad} I)$

$V(\mathfrak{p})$, $\mathfrak{p} = \operatorname{rad} I$ by hw $\operatorname{rad} I$ is prime ideal

It's minimal, b/c if $\mathfrak{q} \subsetneq \operatorname{rad} I \Rightarrow V(\operatorname{rad} I) \subsetneq V(\mathfrak{q})$

Contra maximality of $V(\mathfrak{p})$

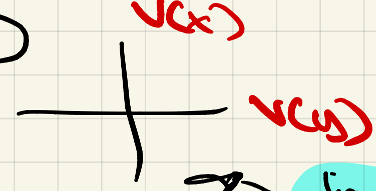
Rmk) Every prime contains a <u>min prime</u>.

eg) $A = \mathbb{C}[x,y]/(xy)$ → vanishing locus

$\mathrm{Spec}(A) = V(xy)$ → union of two axes.

$= V(x) \cup V(y)$

proper closed ss.



love it!

So spec A not irred w/ comps $V(x)$, $V(y)$.

Recall) A top sp X is Noeth is DCC holds for closed ss.

HW: If A noeth $\Rightarrow$ spec(A) noeth.

Prop) X noeth sp $\Rightarrow$ every closed ss of X is a finite union of irred sets

$\Rightarrow$ X has fin many irred comp.

Pf) Say $\Sigma = \{A \subset X \mid A \text{ cl.}, A \neq \text{finite union of irred}\}$

WTS $\Sigma = \emptyset$.

Suppose not, cl: it has a minimal elt $A \in \Sigma$ (by DCC)

If A irred, then contradiction

Otherwise $A = B \cup C$ B, C proper cl. in A $\Rightarrow B, C \notin \Sigma$ (minimality)

$\Rightarrow B, C$ fin union of irred $\Rightarrow$ so is A! oops! $\square$.

Noeth induction)

X - noeth sp. $\mathcal{P}$ is a prop of cl. sso X.

Sp:
- $\mathcal{P}(\emptyset)$ is true
- If $A \subset X$ is cl. $\mathcal{P}(B)$ holds $\forall B \subsetneq A$ closed $\Rightarrow \mathcal{P}(A)$ holds

$\Rightarrow \mathcal{P}$ holds $\forall$ closed $A \subset X$

$\square$.

Cor A noeth ring has finitely many min pr. ideals.

Reason Spec(A) has fin many irred comp (as Noeth sp.) which corr to min pr.

Artin Rings: Very restr. cl. of rings.

Ex $\mathbb{Z}/p^n\mathbb{Z}$ is art, $\mathbb{C}[x_1, \ldots, x_n]/(x_1, \ldots, x_n)^k$ art.

Prop Every pr. ideal in an Artinian ring is max'l!

Pf Sppose $\mathfrak{p} \subset A$ prime, $B = A/\mathfrak{p}$ wts B is a field

(know its domain

(know its art.

Let $x \in B$ given nonzero.

Consider $(x) \supset (x^2) \supset (x^3) \supset \cdots$

DCC $\Rightarrow$ Stab, $(x^n) = (x^{n+1})$

$\Rightarrow x^n \in (x^{n+1})$ so $x^n = y x^{n+1}$

$\xRightarrow{\text{dom}} 1 = xy$ $\therefore$ B field.

Def The Krull dim of a ring A is the supr. of n's st $\exists$ chain of primes $P_0 \subsetneq \cdots \subsetneq P_n \subset A$

Companion The Krull dim of a sp. A is the supr. of n's $\exists$ chain $x_0 \subsetneq \cdots \subsetneq x_n \subset A$ of irred cl. ss.

$\dim A = \dim \operatorname{Spec}(A)$

e.g. $A = \mathbb{C}[x_1, \ldots, x_n]$

$0 \subset (x_1) \subset (x_1, x_2) \subset \cdots \Rightarrow \dim A \geq n$

e.g. $\dim(\text{field}) = 0$ $A = \text{PID} + \text{not ring}$ $\dim A = 1$

$\dim(\text{Art ring}) = 0$ by $\boxed{\text{prop}}$.

Prop A artinian ring $\Rightarrow$ rad(A) is nilp

Pf Let $I = \mathrm{rad}(A)$

$I \supset I^2 \supset \ldots$

by DCC $\underbrace{I^n = I^{n+1}}_{\mathfrak{a}}$

will show $\mathfrak{a} = 0$

Say $\mathfrak{a} \neq 0$

$\Sigma = \{\text{all ideals } \mathfrak{b} \mid \mathfrak{a}\mathfrak{b} \neq 0\}$

$(1) \in \Sigma$

Since A is art, $\exists$ min $\mathfrak{c} \in \Sigma$.

Pick $x \in \mathfrak{a}\mathfrak{c}$ nonzero as $\mathfrak{a}\mathfrak{c} \neq 0$

$\Rightarrow (x) \in \Sigma \Rightarrow \mathfrak{c} = (x)$ by minimality

$(x\mathfrak{a})\mathfrak{a} = x\mathfrak{a}^2 = x\mathfrak{a} \neq 0$ as $\mathfrak{a}^2 = \mathfrak{a}$

$x\mathfrak{a} \in \Sigma \quad x\mathfrak{a} \subset (x) \xRightarrow[\text{min of } (x)]{} x\mathfrak{a} = (x)$

$\Rightarrow \exists\, y \in \mathfrak{a}$ st $x = xy$.

So, $x = xy^n$ iterate $\forall n$

But, y is in nilrad $\Rightarrow y^n = 0$ for some n

$\Rightarrow x = 0$ oops

$\therefore I^n = 0$ done!.

Rmk if A noeth $\Rightarrow$ rad(A) is nilp

since A noeth, $\mathrm{rad}(A) = (x_1, \ldots, x_n)$

x_i nilp $\Rightarrow x_i^{m_i} = 0$

$\mathrm{rad}(A)^{m_1 + \ldots + m_n} = 0$

In gen rad A doesn't have to be nilpotent

$A = \mathbb{C}[x_1, \ldots]/(x_1^2, \ldots)$

$\mathrm{rad}\,A = (x_1, \ldots)$

$\mathrm{rad}\,A^k \ni x_1 \cdots x_k \neq 0$

$\Rightarrow$ rad A not nilp.

Prop Artinian ring has fin many max'll ideals

Same as min prime for art ring!

Pf Spose not. $\exists$ distinct max'll ideals

$m_1, m_2, m_3, \ldots\ldots$

$m_1 \supset m_1 \cap m_2 \supset m_1 \cap m_2 \cap m_3 \supset \cdots\cdots$

desc chain.

Cl. inclusions are strict always

Say $m_1 \cap \cdots \cap m_n = m_1 \cap \cdots \cap m_n \cap m_{n+1}$

$\Rightarrow \underbrace{m_1 \cap \cdots \cap m_n}_{\text{contains } m_1 \cdots m_n} \subset m_{n+1}$

$\therefore m_i \subset m_{n+1}$ for is.t. — max'll $\Rightarrow$ pr

contradiction

$\therefore$ this contra DCC

Thm Artinian $\Longleftrightarrow$ Noeth + dim $= 0$

Lemma A is a ring $\exists$ max'll ideals $m_1, \ldots, m_n$ s.t. $m_1 \cdots m_n = 0$ (not nec dist)

A is noeth $\Longleftrightarrow$ A is Art

Pf $A \supset m_1 \supset m_1 m_2 \supset \cdots \supset m_1 \cdots m_n = 0$

let $\dfrac{m_1 \cdots m_i}{m_1 \cdots m_{i+1}} = A_i$ is an A module killed by m_{i+1}

$\Rightarrow$ it's an $\underbrace{A/m_{i+1}}_{\text{field}}$ - mod

$\therefore A_i$ as A mod is noeth $\Leftrightarrow$ art $\Leftrightarrow$ f d as A/m_{i+1} mod

A noeth $\Longrightarrow$ $\dfrac{m_1 \cdots\cdots m_i}{m_1 \cdots\cdots m_{i+1}}$ is noeth

$n=2$ & ind

$m_1 m_2 = 0$

A/m_1 art

$m_1/\underset{0}{m_1 m_2}$ art

$0 \to m_1 \to A \to A/m_1 \to 0$

ex & outside art so inside ∴

$\Longrightarrow$ $\dfrac{m_1 \cdots\cdots m_i}{m_1 \cdots\cdots m_{i+1}}$ is art

$\Longrightarrow$ A is noeth ($\Longleftarrow$ same)

$n=3$ A/m_1, $m_1/m_1 m_2$, $\dfrac{m_1 m_2}{m_1 m_2 m_3} = 0$

∴ $0 \to \underbrace{m_1 m_2}_{\text{art}} \to m_1 \to \underbrace{m_1/m_1 m_2}_{\text{Art}} \to 0$

∴ art

Pf of thm) $\Longrightarrow$

Say A is Art, wts Noeth & dim = 0 ✓

List max'l ideals,

$m_1, \text{——} \, m_n$ be the max'l ideals of A (These are the prime)

∴ $\mathrm{rad}(A) = m_1 \cap \ldots \cap m_n$ (rad & Jac rad same here)

$\exists\, k$ s.t. $\mathrm{rad}(A)^k = 0$

$\supseteq (m_1 \text{ — } m_n)^k \longrightarrow$ ∴ by lema Noeth

$\Longleftarrow$ A is noeth & dim A = 0 ✓ wts Art