Recall: going up thm

if $A \subset B$ is an integral extn & given chain of primes $\mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n \subset A$ & $\mathfrak{q}_1 \subset \cdots \subset \mathfrak{q}_m \subset B$ $m \le n$ w/ $\mathfrak{q}_i^c = \mathfrak{p}_i$

can complete $\mathfrak{q}_1 \subset \cdots \subset \mathfrak{q}_m \subset \overbrace{\mathfrak{q}_{m+1} \subset \cdots \subset \mathfrak{q}_n}^{\text{can find}} \subset B$ $\mathfrak{q}_i^c = \mathfrak{p}_i$

$\Rightarrow$ map $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ has dim 0 fibres.

Today: going down thm.

Prop: $A \subset B$ rings & $S \subset A$ multset. Let C be int cl. of $A \subset B$

$\Rightarrow S^{-1}C$ is int cl. of $S^{-1}A$ in $S^{-1}B$

"Int cl. comm w/ localization"

Pf: Localization pres integrality $\Rightarrow S^{-1}A \to S^{-1}C$ integral (all elt of $S^{-1}C$ int / $S^{-1}A$)

$S^{-1}C \subseteq$ int cl. of $S^{-1}A$

let $\frac{b}{s} \in$ int cl. of $S^{-1}A$, $b \in B$, $s \in S$

$\Rightarrow b$ satisfies monic poly in $S^{-1}A$ coef

$\Rightarrow \left(\frac{b}{s}\right)^n + \left(\frac{a_{n-1}}{s_{n-1}}\right)\left(\frac{b}{s}\right)^{n-1} + \cdots + \left(\frac{a_0}{s_0}\right) = 0$ $a_i \in A$, $s_i \in S$

Let $t = s_0 \cdot s_1 \cdot \cdots \cdot s_{n-1}$ multiply abv by st^n

$\Rightarrow (tb)^n + \left(\frac{st a_{n-1}}{s_{n-1}}\right)(tb)^{n-1} + \cdots + \frac{a_0 (st)^n}{s_0} = 0$ holds in $S^{-1}B$

$\Rightarrow \exists\, s' \in S$ so $s' \cdot (\text{abv}) = 0$ in B. Mult by $(s')^n$

$\Rightarrow (s'tb)^n + \underbrace{\left(\frac{ss'ta_{n-1}}{s_{n-1}}\right)}_{\text{in } A \text{ as } t \text{ mult of } s_i} (s'tb)^{n-1} + \cdots + \underbrace{\frac{a_0 (sts')^n}{s_0}}_{} = 0$ in B

$\Rightarrow s'tb$ int / A

$\Rightarrow s'tb \in C$

$\Rightarrow \frac{b}{s} \in S^{-1}C$ $\Rightarrow$ int cl. of $S^{-1}A \subset S^{-1}C$

Prop | A = int domain TFAE:

1) A int closed [A int closed in Frac(A)]
2) A_p ___ $\forall$ prime $p \subset A$
3) A_m ___ $\forall$ max'l $m \subset A$

} int cl. is local prop

Pf | let C be int closure of A in Frac(A)

equiv b/c map of A-mods isom iff localy an ism.

A int closed $\iff$ $A \to C$ is an isom

A_p ___ $\iff$ $A_p \to C_p$ is an ism

A_m ___ $\iff$ $A_m \to C_m$ is an ism

(prev prop $\Rightarrow$ C_p int cl of A_p)

Given, $A \subset B$ & ideal $\mathfrak{a} \subset A$. Say $x \in B$ int / ideal $\mathfrak{a}$

$\exists$ eqn $x^n + a_{n-1}x^{n-1} + \dots + a_0 = 0$ $\quad a_i \in \mathfrak{a}$

[$\Rightarrow x$ int /A]

Can defn integral cl of $\mathfrak{a}$ is all elts of B int / $\mathfrak{a}$

Prop | C = int cl. of A in B $\Rightarrow$ int cl. of $\mathfrak{a}$ in B = rad$(\mathfrak{a}^e)$ $\quad$ e ext to C

| **Cor |** int closure of $\mathfrak{a}$ closed under $(+)$ & $(\cdot)$ by itself.

Pf | Say $\underbrace{x \in B \text{ int}/\mathfrak{a}}$, $\quad x^n + a_{n-1}x^{n-1} + \dots + a_0 = 0$ $\quad \in C$

$\Rightarrow x$ int /A $\Rightarrow x \in C$ $\quad \Rightarrow \boxed{x^n} = -(a_{n-1}x^{n-1} + \dots + a_0)$

$\quad \cap$

$x \in \text{rad}(\mathfrak{a}^e) \Leftarrow$ $\quad \mathfrak{a}^e$ $\quad \mathfrak{a}^e$ to C

let $x \in \text{rad}(\mathfrak{a}^e) \Rightarrow x^n = \sum_{i=1}^{m} a_i y_i$ $\quad a_i \in \mathfrak{a}, y_i \in C$

$M = A[y_1, \dots, y_m]$ finite A-alg.

$\Rightarrow x^n M \subset \mathfrak{a}M$ choose gen for M as an A module.

Write matrix for x^n as op on M. x^n satisfies char poly of this mat
(entries in $\mathfrak{a}$ as $x^n M \subset \mathfrak{a}M$) $\quad$ entries in $\mathfrak{a}$

$\Rightarrow x$ satisfies $\dots\dots$ $\Rightarrow x$ int /A. $\quad$ D.

Prop Say $A \subset B$ extn of int domains. A int-closed $\hookrightarrow K = \mathrm{Frac}(A)$. $x \in B$ / ideal $\mathfrak{a} \subset A$

$\mathrm{Frac}(B) \ni x$
| field extn
$\mathrm{Frac}(A)$

Then x is an alg / K.

If its min poly is $\underbrace{t^n + a_{n-1}t^{n-1} + \dots + a_0}_{f(t) \in K[t]}$

$\Rightarrow$ $a_i \in \mathrm{rad}(\mathfrak{a})$.

Pf It is clear that if x int / A $\Rightarrow$ x is alg / K

Let L = splitting field of $f(t)$ / K $\rightarrow$ (subfield of $\bar{K}$ gen'd by roots of $f(t)$ in $\bar{K}$)

Say $x_1, \dots, x_m \in L$ are roots of f.

Each x_i conj x (fixed thing) $\Rightarrow$ $\boxed{x_i \text{ is integral } / \mathfrak{a}}$

$\hookrightarrow x_i \in \underbrace{\text{int cl. of } \mathfrak{a}}_{\text{Prop of A int cl.}} \subset B$

can find K emb $K(x) \xrightarrow{\sigma} \bar{K}$, $x \mapsto x_i$

Coeff of $f(t)$ are poly in the x_is

$\therefore$ coeff belong to $\boxed{\mathrm{rad}(\mathfrak{a}) \text{ too}}$

$\mathrm{rad}(\mathfrak{a})$.
$\cap$
A.

Thm (Going down)

$A \subset B$ int domains, A int closed, B int / A.

given $\mathfrak{p}_n \subset \dots \mathfrak{p}_m \subset \dots \subset \mathfrak{p}_1$ chain pr in A $(m \le n)$

$\mathfrak{q}_m \subset \dots \subset \mathfrak{q}_1$ ——— B $\mathfrak{q}_i^c = \mathfrak{p}_i$

$\exists$ ps $\mathfrak{q}_n \subset \dots \subset \mathfrak{q}_{m+1} \subset \mathfrak{q}_m$ st $\mathfrak{q}_i^c = \mathfrak{p}_i$

Pf Suffices to treat $\mathfrak{p}_2 \subset \mathfrak{p}_1$
$(\mathfrak{q}_2) \subset \mathfrak{q}_1$
$\hookrightarrow$ find them

ETS: $\mathfrak{p}_2$ is contr of a pr from $B_{\mathfrak{q}_1}$

ETS: $\mathfrak{q}_2 = \mathfrak{p}_2 B_{\mathfrak{q}_1} \cap A$

Gen fact: for any homo $A \to B$ & $\mathfrak{p} \subset A$ prime

$\mathfrak{p}$ = contr of pr of B iff $\mathfrak{p} = (\mathfrak{p}^e)^c$

Say $x \in P_2 B_{q_1}$ $\Rightarrow x = y/s$ $y \in P_2 B$, $s \in B \setminus q_1$

We know y is int $/P_2$

$\therefore \Rightarrow$ min poly for y (prev prop) over Frac(A) is

$$y^r + a_{r-1} y^{r-1} + \cdots + a_0 = 0$$
$a_i \in P_2$

eg $y = ab$ $a \in P_2$ $b \in B$

$b^n + \cdots + a_0 = 0$ $a_i \in A$

mult by a^n

$$(ab)^n + \underbrace{a a_{n-1}}(ab)^{n-1} + \cdots + \underbrace{a^n a_0} = 0$$
$\in P_2$

$\Rightarrow y$ int $/P_2$

Now say $x \in A$ too $y = sx$ plug & $\frac{a_i}{x^r}$

$$s^r + a_{r-1}' s^{r-1} + \cdots + a_0' = 0 \quad a_i' = \frac{a_i}{x^{r-i}} \Rightarrow x^{r-i} a_i' = a_i$$

$\in A$ by prev prop (using unit ideal)

$a_i \in P_2$

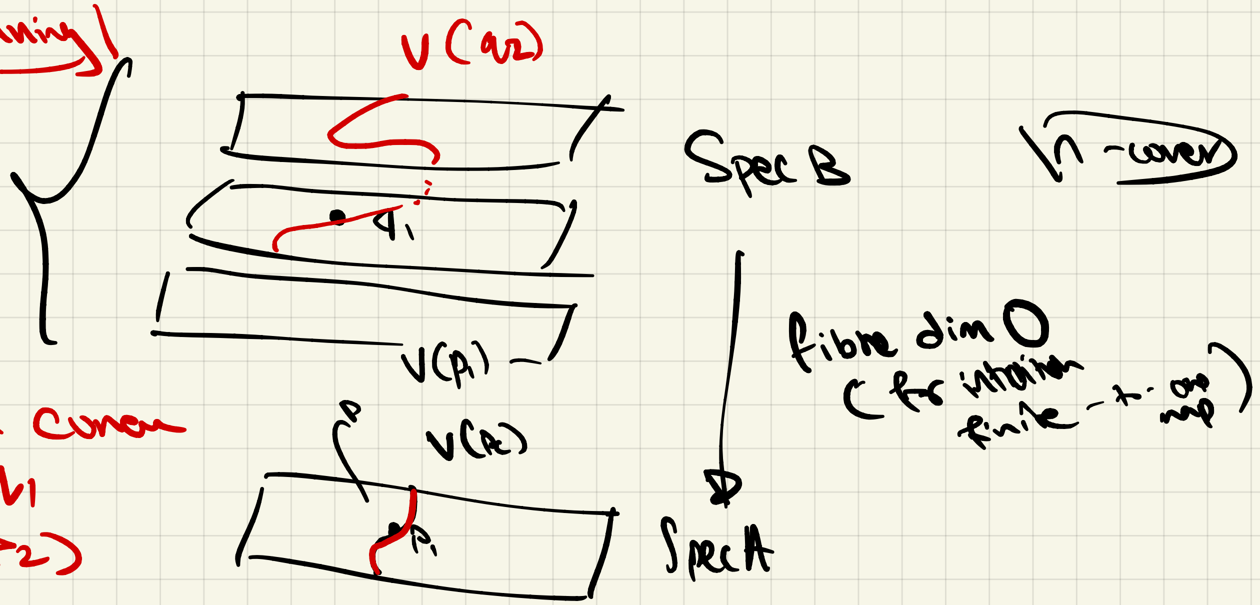
Say $\boxed{x \notin P_2} \Rightarrow a_i' \in P_2 \; \forall i$ $\boxed{\text{as } P_2 \text{ prime}}$

$$\Rightarrow s^r = -(a_{r-1}' s^{r-1} + \cdots + a_0')$$
$\in P_2 B \subset P_1 B \subset q_1$

$\Rightarrow s \in q_1$, $\boxed{\text{contradiction!}}$

$\Rightarrow x \in P_2$ i.e. $P_2 B_{q_1} \cap A = P_2$

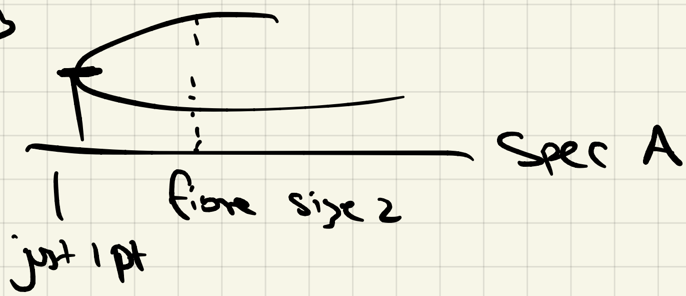
Geometric Meaning!



$V(q_2)$ is a curve through q_1 + lifts $V(P_2)$

Warning! if $A \to B$ is a finite ring map the sizes of the fibres $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ can have different sizes

(a) $\underbrace{\mathbb{C}[x]}_{A} \subset \underbrace{\mathbb{C}[x,y]/(y^2=x)}_{B}$



(b)