Today $\mathfrak{a}$-adic top

Def The $\mathfrak{a}$-adic top on A is the top having a fund sys of open nbhds of 0 given by $\mathfrak{a}^n$, $n \geq 0$,

- $M = A$-mod $\rightsquigarrow$ $\mathfrak{a}$-adic top on M using $\mathfrak{a}^n M$, $n \geq 0$
- M is $\mathfrak{a}$-adically sep if $\mathfrak{a}$-adic is Haus $\iff \bigcap_{n\geq 1} \mathfrak{a}^n M = 0$
- The $\mathfrak{a}$-adic complet. of M is $\hat{M} = \varprojlim M/\mathfrak{a}^n M$
- M is $\mathfrak{a}$-adically complete is $M \to \hat{M}$ is isom !

1) $A = \mathbb{Z}$, $\mathfrak{a} = (p)$ p prime

- A is sep b/c $\bigcap_{n\geq 1} (p^n) = 0$

  not compl. $x_n = 1 + p + \dots + p^n$ this is a Cauchy seq for $\mathfrak{a}$-adic but doesn't conv.

  $\hat{A} = \mathbb{Z}_p$ p adic int.

- $M = \mathbb{Q}$ $\mathfrak{a}^n M$ is not sep as $\mathfrak{a}^n M = M$ $\forall n$ $\hat{M} = 0$

  $\mathfrak{a}$-adic top on M is indisc

- $M = \mathbb{Z}/p^m\mathbb{Z}$ $(p^n)M = 0$ if $n \geq m$ is sep + compl.

  $\Rightarrow$ $\mathfrak{a}$-adic top on M is disc. (all sets open)

2) $A = \mathbb{C}[x]$ $\mathfrak{a} = (x)$

- A is $\mathfrak{a}$-adically sep, not compl.

  ring of formal pow series

  $\hat{A} = \varprojlim \mathbb{C}[x]/(x^n) = \mathbb{C}[[x]]$

  $\hookrightarrow$ n^{th} term of this is $a_0 + \dots a_{n-1}x^{n-1}$ $a_i \in \mathbb{C}$

Rmk:

- $(1-x)^{-1} = \sum_{n\geq 0} x$ $\Rightarrow$ $1-x$ is unit in $\hat{A}$

- in gen $f(x) \in \mathbb{C}[[x]]$ unit if $f(0) \neq 0$

  $\sqrt{1+x} = \sum \binom{1/2}{n} x^n \Rightarrow 1+x$ a sq in $\mathbb{C}[[x]]$.

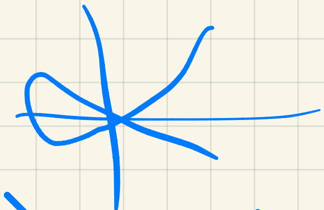
3) $A = \mathbb{C}[x_1,\dots,x_n]$ $\quad \mathfrak{m} = (x_1,\dots,x_n)$ $\quad \mathbb{C}$ $\cup$

∘ $A \longrightarrow \hat{A} = \mathbb{C}[[x_1,\dots,x_n]] \rightsquigarrow \sum_{i_1,\dots,i_n \geq 0} a_{i_1\dots i_n} x_1^{i_1}\cdots x_n^{i_n}$

Rmk| A is f.g red $\mathbb{C}$ alg
Spec(A) is an alg variety
$\mathfrak{m} \subset A$ max'l $\longleftrightarrow$ closed pt of Spec(A)

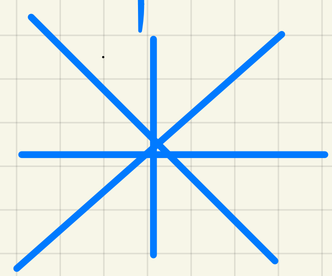
Fact: $\mathfrak{m}$ is a non sing pt of A $\iff$ $\mathfrak{m}$-adic compl. of A is formal pow series (# vars = dim of A @ $\mathfrak{m}$)

4) $A = \mathbb{C}[x,y,z]/(y^2 - x^3 - x^2)$ $\quad \mathfrak{m} = (x,y)$



$\hat{A} = \mathbb{C}[[x,y,z]]/(y^2 - x^3 - x^2)$ $\quad$ sq in $\mathbb{C}[[x,z]]$

$x^3 - x^2 = x^2(1+x) = \underbrace{u}^2$



$\Rightarrow \hat{A} \cong \mathbb{C}[[u,y]]/(y^2 - u^2)$ $\quad y = \pm u$

$M = A$ module $\overset{\text{compl.}}{\rightsquigarrow}$ $\hat{M} = \hat{A}$ mod

This is functorial, $f : M \to N$ is a map of A mods
it is automatically $\mathfrak{m}$-adically cts, $(f^{-1}(\mathfrak{m}^n N) \supseteq \mathfrak{m}^n M)$
$\Rightarrow$ get map $\hat{f} : \hat{M} \to \hat{N}$

$\Rightarrow$ Functor, $\mathrm{Mod}_A \to \mathrm{Mod}_{\hat{A}}$
$M \to \hat{M}$

we'll use this, can use negative n:

Def| A graded ring is a ring w/ a decomp $\underbrace{A = \bigoplus_{n \geq 0} A_n}_{\text{as } \mathbb{Z}\text{-mods ab gp}}$
so $A_n A_m \subset A_{n+m}$

Note: if A = graded ring $\Rightarrow$ A_0 is a subring of A
& each A_n is a A_0 submod

Prop| A graded ring A is noeth
$\iff$
A_0 is noeth + A is f.g A_0-alg.

Pf) $\Leftarrow$ Hilbert Basis Thm

$\Rightarrow A \longrightarrow A_0$ is a surj ring homo ($\ker = A_+ = \bigoplus_{n\geq 1} A_n$)

$\Longrightarrow A_0$ is noeth $A/\ker$.

Since A is noeth, A_+ is f.g. say $A_+ = (x_1, \ldots, x_m)$

Cl. A gen'd as A_0-alg by $x_1, \ldots, x_m$

Pf) Say $f \in A_n$ since, $f \in A_+$ $f = \sum_{i=1}^{m} g_i x_i$ $g_i \in A$

$\triangleright$ g_i's has smaller deg than f

$\xrightarrow{\text{ind}}$ g_i's gen'd by x_i!

Def) A = any ring $\mathfrak{a} \subset A$ ideal.

The Rees Algebra is $R_{\mathfrak{a}}(A) = \bigoplus_{n\geq 0} \mathfrak{a}^n$ ($\mathfrak{a}^0 = A$)

$\hookrightarrow$ graded ring. deg 0 part is A.

Practical issue. If $x \in \mathfrak{a} \subset A$ x can define 2 elts $(x, 0, 0, \ldots)$ & $(0, x, 0, \ldots)$

Soln: introduce dummy var, $R_{\mathfrak{a}}(A) = \bigoplus_{n\geq 0} \mathfrak{a}^n t^n$

$\hookrightarrow$ records gr.

So, x has deg 0 & xt has deg 1.

$\hookrightarrow$ t is not elt

M = A mod $R_{\mathfrak{a}}(M) = \bigoplus_{n\geq 0} \mathfrak{a}^n M \cdot t^n$ mod / $R_{\mathfrak{a}}(A)$.

e.g.) ① $A = \mathbb{C}[x]$, $\mathfrak{a} = (x)$ $R_{\mathfrak{a}}(A)$ contains $t^i x^j$ provided $x^j \in \mathfrak{a}^i$

i.e, $j \geq i$

the monomials are a basis for Rees alg as $\mathbb{C}$-vs!

$\Longrightarrow R_{\mathfrak{a}}(A)$ is a subring of $\mathbb{C}[x,t]$ sp by

If $j \geq i \Rightarrow t^i x^j = x^{j-i}(t^i x^i) \in R_{\mathfrak{a}}(A)$ as $x^i \in \mathfrak{a}^i$

$= x^{j-i}(tx)^i$

$\Rightarrow R_{\mathfrak{a}}(A) = \mathbb{C}[x, tx] \cong$ 2 var poly ring!

Say, $N \subset M$ are A-modules

Two top on N ① $\mathfrak{a}$ adic top, basis $\mathfrak{a}^n N$

② subsp top from $\mathfrak{a}$ adic on M gen'd by $\mathfrak{a}^n M \cap N$

Certainly $\mathfrak{a}^n N \subseteq \mathfrak{a}^n M \cap N$ (but do we have other)

$\Rightarrow \mathfrak{a}^n M \cap N$ is open in top (1)

Other way Is $\mathfrak{a}^n N$ open in top (2)

$\hookrightarrow$ to prove this need $\underbrace{\mathfrak{a}^m M \cap N \subset \mathfrak{a}^n N}_{\forall n\ \exists m}$

$\hookrightarrow$ might not happen $\to$ could be ① coarser!

$\hookrightarrow$ e.g.) $A = \mathbb{Z}$, $\mathfrak{a} = (p)$

$\mathbb{Z} \subseteq \mathbb{Q}$

p-adic on $\mathbb{Q}$ is indiscr. $\Rightarrow$ induced on $\mathbb{Z}$ as subsp is ind!

$\mathbb{Z}$ is not indiscr $\Rightarrow$ not same as

Thm (Artin-Rees Lemma)

A = noeth ring, M = f.g. A-mod, $N \subseteq M$ submod

$\underbrace{\exists k \text{ s.t. } \forall n \geq k,}$ $\mathfrak{a}^n M \cap N = \mathfrak{a}^{n-k}((\mathfrak{a}^k M) \cap N) \subseteq \mathfrak{a}^{n-k} M$

$\hookrightarrow$ uniform as above (in the earlier setting $m = n+k$)

Pf) $R_{\mathfrak{a}}(A)$ is gen'd as $\underset{\underset{R_{\mathfrak{a}}(A)_0}{\parallel}}{A}$-alg by $R_{\mathfrak{a}}(A)_1 = \mathfrak{a}$, which is fin gen as A-mod

$\Rightarrow R_{\mathfrak{a}}(A)$ is f.g. as A-alg $\Rightarrow$ noeth.

Sim, $R_{\mathfrak{a}}(M)$ is f.g as a $R_{\mathfrak{a}}(A)$-mod

↳ let $\tilde{N} = \bigoplus_{n \geq 0} (\mathfrak{a}^n M \cap N) \subset R_{\mathfrak{a}}(M)$

↳ $R_{\mathfrak{a}}(A)$-submod

↳ by Noeth $\tilde{N}$ is fg $R_{\mathfrak{a}}(A)$ mod.

let k = Max deg of gen of $\tilde{N}$

if $n \geq k \implies \boxed{\tilde{N}_n = R_{\mathfrak{a}}(A)_{n-k} \cdot \tilde{N}_k}$ ← equiv to statement.

□.

Cor) In the above setting the $\mathfrak{a}$-adic top on N coincides w/ subsp. top from M ($\mathfrak{a}$-adic top)

Cor) if A noeth & $0 \to M_1 \to M_2 \to M_3 \to 0$ SES of fg A-mod

$\rightsquigarrow 0 \to \widehat{M_1} \to \widehat{M_2} \to \widehat{M_3} \to 0$

Pf) $\mathfrak{a}$ adic top on M_1 ind from M_2 (above cor)

(obvious basis opens of induced top is img of $\mathfrak{a}^n M_2 = \mathfrak{a}^n M_1$)

Result for earlier $\implies \boxed{\text{SES}}$.

Warning) above cor false if drop