1) Say, M is an A-module, presentation of M is an exact seq

(free) $\quad F_1 \to F_0 \to M \to 0$

Probl In the graded cases, how do we compute Hilbert series $H_M(t)$ from pres?

↳ this is hard!

But, if we have a longer exact seq like,

$$\underbrace{0 \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0}_{\text{Free Resolution of } M}$$

$$\Rightarrow \quad H_M(t) = \sum_{i=0}^{n} (-1)^i \underbrace{H_{F_i}(t)}_{\text{easy to compute}}.$$

2) Say $0 \to M_1 \to M_2 \to M_3 \to 0$ SES of A-mods.

$\Rightarrow \quad M_1 \otimes N \to M_2 \otimes N \to M_3 \otimes N \to 0$ right exact, not typically left ↳ lose injectivity.

Can we fix this?

↳ understand failure of inj.

Def) for A mod K, $K[f] = \{x \in K \mid fx = 0\}$ ← of torsion.

Simple exp: $N = A/fA$ → principle ideal.

$$\begin{tikzcd}
 & M_1[f] \arrow[d] & & & \\
0 \arrow[r] & M_1 \arrow[r] \arrow[d, "f"] & M_2 \arrow[r] \arrow[d, "f"] & M_3 \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & M_1 \arrow[r] \arrow[d] & M_2 \arrow[r] & M_3 \arrow[r] & 0 \\
 & M_1/fM_1 & & &
\end{tikzcd}$$

$\mathrm{Tor}_1(M_i, A/fA)$ → failure to be inj exp'd by this!

Snake: $0 \to M_1[f] \to M_2[f] \to M_3[f] \to M_1/fM_1 \to M_2/fM_2 \to M_3/fM_3 \to 0$

$M_1 \otimes N$

Tor gives smth similar

A fixed ring

Defn| A free reso of an A-mod M is a <u>LES</u>,

$$\cdots \longrightarrow F_3 \xrightarrow{\partial_3} F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

where F_i is free

Rmk| Can consider non-free reso's where F_i's arb or reso from other classes of mod
↳ i.e proj reso, flat reso.

Obs| M has a free resol.:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & 0 & & & & \\
 & & & K_1 \arrow[ur] \arrow[dr] & & & & & \\
\cdots \arrow[r] & & F_2 \arrow[ur] \arrow[rr] & & F_1 \arrow[rr, dotted] \arrow[dr] & & F_0 \arrow[r] & M \arrow[r] & 0 \\
 & K_2 \arrow[ur] & & & & K_0 \arrow[ur] \arrow[dr] & & & \\
0 \arrow[ur] & & & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

Consider| $f: M \to N$ map of A-mods

$F_\bullet \to M$ free reso, $G_\bullet \to N$ some reso

$\Rightarrow$ can lift f to a map of resos $f: F_\bullet \to G_\bullet$

$$\begin{tikzcd}
\cdots \arrow[r] & F_3 \arrow[r] \arrow[d, "f_3"] & F_2 \arrow[r] \arrow[d, "f_2"] & F_1 \arrow[r] \arrow[d, "f_1"] & F_0 \arrow[r] \arrow[d, "f_0"] & M \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r] & G_3 \arrow[r] & G_2 \arrow[r] & G_1 \arrow[r] & G_0 \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

Pf| f_0 exist b/c F_0 is free & $G_0 \to N$ is <u>surj</u>.

f_1 exist b/c F_1 is free & $F_1 \to G_0$ but partic $\ker(G_0 \to N)$ & $G_1 \to \ker(G_0 \to N)$ surj,

<u>Rinse</u>, & repeat!.

Q: How unique is the lift?

To answer this consider lifts of zero map $M \xrightarrow{0} N$

$$\begin{tikzcd}
\cdots \arrow[r] & F_3 \arrow[r] \arrow[d, "f_3"] \arrow[dl, "s_2"'] & F_2 \arrow[r] \arrow[d, "f_2"] \arrow[dl, "s_1"'] & F_1 \arrow[r, "\partial_1"] \arrow[d, "f_1"] \arrow[dl, "s_0"'] & F_0 \arrow[r] \arrow[d, "f_0"] & M \arrow[r] \arrow[d, "0"] & 0 \\
\cdots & G_3 \arrow[r] & G_2 \arrow[r] & G_1 \arrow[r] & G_0 \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

- Comm $\Rightarrow$ F_0 maps into $\ker(G_0 \to G_N)$
  & G_1 maps onto
  $\Rightarrow$ f_0 factors thru G_1

- Consider $f_1 - s_0 \circ \partial_1$. We have $\partial(f_1 - s_0\partial)$
$$= f_0 \partial - \partial s_0 \partial$$
$$= (f_0 - \partial s_0)\partial = 0$$
$\Rightarrow$ $f_1 - s_0\partial$ map into $\ker(G_1 \to G_0)$
& G_2 map onto this
$\Rightarrow$ $f_1 - s_0\partial$ lifts to a map G_2

i.e. $f_1 - s_0\partial = \partial s_1$ $\exists$ this.
$\Rightarrow$ $f_1 = \partial s_1 + s_0 \partial$.

Defn
- A complex is a diag $C_n \xrightarrow{\partial} C_{n-1} \to C_{n-2} \to \cdots$ st $\partial^2 = 0$
- A map of compl. $f_\bullet : C_\bullet \to D_\bullet$ cons of

$$\begin{tikzcd}
C_i \arrow[r, "\partial"] \arrow[d, "f_i"'] \arrow[dr, phantom, "\circlearrowright"] & C_{i-1} \arrow[d, "f_{i-1}"] \\
C_i \arrow[r, "\partial"] & C_{i-1}
\end{tikzcd} \quad \forall i$$

- A map $f : C_\bullet \to D_\bullet$ is null htpic if $\exists s_i : C_i \to D_{i+1}$

$$\begin{tikzcd}
C_{i+1} \arrow[r] \arrow[d] & C_i \arrow[r] \arrow[d] \arrow[dl, "s_i"'] & C_{i-1} \arrow[d] \arrow[dl, "s_{i-1}"'] \\
D_{i+1} \arrow[r] & D_i \arrow[r] & D_{i-1}
\end{tikzcd} \quad \exists s_i \text{ st } f_i = s_{i-1}\partial + \partial s_i \ \forall i$$

- two maps $f, f' : C_\bullet \to B_\bullet$ are htpic if $f - f'$ null htpic

We showed given $f : M \to N$, free reso $F_\bullet \to M$
reso $G_\bullet \to N$

$\Rightarrow$ f lifts to map of res $F_\bullet \to G_\bullet$ & any two lifts htpic!

Rmk $F_\bullet : \cdots \to F_2 \to F_1 \to F_0 \to 0 \to 0 \cdots$

Now say $F_\bullet \to M$ & $G_\bullet \to M$ are 2 free reso

so id lifts

$$\Rightarrow \begin{tikzcd} F_0 \arrow[r] \arrow[d, "f"'] & M \arrow[d, "id"] \\ G_0 \arrow[r] & M \end{tikzcd} \quad \& \quad \begin{tikzcd} G_0 \arrow[r] \arrow[d, "\exists g"'] & M \arrow[d, "id"] \\ F_0 \arrow[r] & M \end{tikzcd}$$

So
$$\begin{tikzcd} F_0 \arrow[r] \arrow[d, shift right, "gf"'] \arrow[d, shift left, "id"] & M \arrow[d, "id"] \\ F_0 \arrow[r] & M \end{tikzcd}$$
so, $gf \overset{\text{htpic}}{\sim} id_{F_0}$

$fg \sim id_{G_0}$

Let M, N = two A-modules, Pick free res $F_\bullet \to M$

Consider, $F_\bullet \otimes_A N$

General set $C_\bullet$ is a cx

$$C_{i+1} \xrightarrow{d_{i+1}} C_i \xrightarrow{d_i} C_{i-1}$$

$d^2 = 0 \Rightarrow \operatorname{im} d_{i+1} \subseteq \ker d_i$

Def $H_i(C) = \ker d_i / \operatorname{im} d_{i+1}$

cycles / bdrys

$H_i(C) \Leftrightarrow$ Cmplx exact at C_i

Now $F_\bullet \otimes N$ will be cmplx. $\curvearrowright$ This will be well def soon.

$$\mathrm{Tor}_i^A(M,N) = H_i(F_\bullet \otimes_A N)$$

Note! • $\mathrm{Tor}_i^A(M,N) = 0$ for $i<0$ (the chains don't exist for $i<0$)

• $F_1 \to F_0 \to M \to 0$ exact

$\Rightarrow F_1 \otimes N \to F_0 \otimes N \to N \otimes M \to 0$ exact

So looking at low deg.

$$F_\bullet \otimes N : \quad F_1 \otimes N \to F_0 \otimes_A N \longrightarrow 0 \to 0 \to \cdots$$

$$\Rightarrow \mathrm{Tor}_0^A(M,N) = M \otimes_A N$$

Cl. 1 Tor is well def (i.e. ind of choic)

Lemma 1 (a) A map $f: C_\bullet \to D_\bullet$ induces

$$H_i(f): H_i(C) \to H_i(D)$$

(b) If $g: C_\bullet \to D_\bullet$ is htpic to $f \Rightarrow H_i(f) = H_i(g)$

Pf (a)

$$\begin{tikzcd}
C_{i+1} \arrow[r, "\partial_{i+1}"] \arrow[d, "f_{i+1}"] & C_i \arrow[r, "\partial_i"] \arrow[d, "f_i"] \arrow[dl, dashed, "s_i"'] & C_{i-1} \arrow[d, "f_{i-1}"] \arrow[dl, dashed, "s_{i-1}"'] \\
D_{i+1} \arrow[r, "\partial'_{i+1}"] & D_i \arrow[r, "\partial'_i"] & D_{i-1}
\end{tikzcd}$$

$$\left.\begin{aligned} f_i(\ker \partial_i) &\subseteq \ker(\partial_i') \\ f_i(\mathrm{im}\, \partial_{i+1}) &\subseteq \ker(\partial_{i+1}') \end{aligned}\right\} \quad f_i \text{ induces } \frac{\ker \partial_i}{\mathrm{im}\, \partial_{i+1}} \to \frac{\ker \partial_i'}{\mathrm{im}\, \partial_{i+1}'}$$

(b) Say f is null htpic, wts $H_i(f) = 0$

$$f_i = s_{i-1}\partial_i + \partial_{i+1}' s_i$$

Say $x \in \ker(\partial_i) \Rightarrow f_i(x) = \overbrace{s_{i-1}\partial_i(x)} + \partial_{i+1}' s_i(x) \in \mathrm{im}\, \partial_{i+1}'$
$\Rightarrow 0$ in $H_i(D_\bullet)$

Pf of Cl.

Say $F_\bullet \to M$, $G_\bullet \to M$ two free res of M

$\exists$ maps, $f : F_\bullet \to G_\bullet$ & $g : G_\bullet \to F_\bullet$ st

$gf \sim id_F$ & $fg \sim id_G$

$$\begin{tikzcd} F_\bullet \otimes N \arrow[r, bend left, "f \otimes id"] & G_\bullet \otimes N \arrow[l, bend left, "g \otimes id"] \end{tikzcd}$$

Still, by functor: both comp $\sim$ identities.

$\Rightarrow$ so on homology, induces same map

↳ inverse bijections to one another

$$\begin{tikzcd} H_i(F_\bullet \otimes N) \arrow[r, bend left, "H_i(f \otimes id)"] & H_i(G_\bullet \otimes N) \arrow[l, bend left, "H_i(g \otimes id)"] \end{tikzcd}$$

both comp id $\Rightarrow$ each map isom.

A bit more, if $f' : F_\bullet \to G_\bullet$ is a second lift of id_M

$\Rightarrow f \sim f' \Rightarrow$ they induce same map!

$H_i(F \otimes_A N) \to H_i(G \otimes_A N)$

↳ So the isom is canonical!

Obs $Tor_i^A(M,N)$ is functorial on M & N

(a) Say $N \to N'$ map & $F_\bullet \to M$ free res.

$\Rightarrow F_\bullet \otimes N \xrightarrow[\text{map}]{\text{comple}} F_\bullet \otimes N'$

$\rightsquigarrow$ induces map on homology → this is the for grp!

$H_i(F_\bullet \otimes N) \to H_i(F_\bullet \otimes N')$

(b) let $M \to M'$ map of mods

w/ free res $F_\bullet \to M$, $G_\bullet \to M$

know, f lifts to map $F_\bullet \to G_\bullet$

$\Rightarrow$ get map $F_\bullet \otimes_A N \to G_\bullet \otimes_A N$

↳ take homol to get map on Tor

Fact) $\exists$ natural isom $\mathrm{Tor}_i^A(M,N) \cong \mathrm{Tor}_i^A(N,M)$

↳ not obvious at all

↳ relies on general, symmetric defn

General Defn

Say $C_\bullet$, $D_\bullet$ are cxs

$$\begin{array}{ccccccc}
\cdots \to & C_2 \otimes D_2 & \to & C_1 \otimes D_1 & \to & C_0 \otimes D_0 \\
& \downarrow & & \downarrow & & \downarrow \\
\cdots \to & C_2 \otimes D_1 & \to & C_1 \otimes D_1 & \to & C_0 \otimes D_0 \\
& \downarrow & & \downarrow & & \downarrow \\
\cdots \to & C_2 \otimes D_0 & \to & C_1 \otimes D_0 & \to & C_0 \otimes D_0
\end{array}$$

$\boxed{\text{Double Complex}}$ → complex in row + col + commute!

Totalization $\mathrm{Tot}_\bullet(C_\bullet \otimes D_\bullet)$

$$\cdots \to \bigoplus_{i+j=1} C_i \otimes D_j \to \bigoplus_{i+j=0} C_i \otimes D_j \longrightarrow$$

$$\mathrm{Tot}_1(C_\bullet \otimes D_\bullet) \qquad \mathrm{Tot}_0(C_\bullet \otimes D_\bullet)$$

Careful abt ∂ map. (throw in sign)

↳ say no neg terms

$$C_1 \otimes D_0 \oplus C_0 \otimes D_1 \longrightarrow C_0 \otimes D_0$$

Rmk) diff in Tot has signs

Say $F_\bullet \to M$ & $G_\bullet \to M$ two free res

$\operatorname{Tor}_\bullet^A(M,N)$ computed by $\operatorname{Tot}_\bullet(F_\bullet \otimes G_\bullet)$ → sym in M,N

$$\operatorname{Tor}_i^A(M,N) = H_i(\operatorname{Tot}_\bullet(F_\bullet \otimes G_\bullet))$$