1 Alg number thy

$K/\mathbb{Q}$ number field $\quad R =$ int closure of $\mathbb{Z}$ in K $\to$ elt of K that are integral $/\mathbb{Z}$

R is a Dedekind domain. $\to$ Investigate special prop of these
$\hookrightarrow$ Dirichlet thm unit grp
$\hookrightarrow R^\times$
$\hookrightarrow$ Finiteness of Cl. grp

Def) A Dedekind domain is a domain R s.t.,

1) R is Noetherian Ring
2) $\dim R = 1$ (non-zero primes are max)
3) R is integrally closed, i.e any elt of Frac(R) that is integral $/R$ is in R.

2 Grobner Bases

If $f, g \in K[x]$ div alg $\Rightarrow f = \overset{\text{quot}}{q} \cdot g + \overset{\text{rem}}{r}$ $\quad \deg(r) < \deg(f)$
$\hookrightarrow$ theoretical app $\to K[x]$ is PID.

Grobner Bases generalizes this to more variables.

$f \in K[x_1, \ldots, x_n] \xrightarrow[\text{initial term}]{} \mathrm{in}(f)$ (Monomial)

$I \subset K[x_1, \ldots, x_n] \xrightarrow[\text{initial ideal}]{} \mathrm{in}(I) = \langle \mathrm{in}(f) \mid f \in I \rangle$.

3 Non commutative Rings.

- Module theory / non-commutative rings
- Semi Simple rings (Wedderburn's Thm)
- Special Classes (eg group alg $\quad G$ - gp $\quad k$ - field
  $k[G] \to k$ basis on symbols G)
  $[g]\cdot[h] = [gh]$
- Generalise asp. of comm alg (prideal, jacobson rad)
- Localization.

4. Macaulay's Thm

$\subset$ as a $\mathbb{Z}$-mod.

Def) A graded ring is a ring R with a decomposition $R = \bigoplus_{n\in\mathbb{Z}} R_n$
so if $x \in R_n$, $y \in R_m \Rightarrow xy \in R_{n+m}$
Elts in R_n are homog in deg n.

e.g.) $R = \mathbb{C}[x_1, \ldots, x_n]$ can be given struct of gr ring.
by declaring x_i to be homog of deg d_i.
(usually 1) $\rightarrow$ Std grading

Fact if R is a fin gen, graded, k alg (k field).
$\Rightarrow R \cong \dfrac{k[x_1, \ldots, x_n]}{I}$ (I homog ideal (gen'd by homog elt))

R is a fin gen graded k-alg. The Hilb fnc of R $h_R : \mathbb{N} \to \mathbb{N}$, $n \mapsto \dim_k R_n$

$\hookrightarrow \exists$ poly $P_R \in \mathbb{Q}[x]$ so $h_R(n) = P_R(n)$ $\forall n \gg 0$

e.g.) 1) $R = \mathbb{C}[x]$ $\quad h_R(n) = 1$, $P_R(n) = 1$

2) $R = \dfrac{\mathbb{C}[x]}{(x^n)}$ $\quad h_R(k) = \begin{cases} 1 & k < n \\ 0 & \text{else} \end{cases}$

$P_R(k) = 0$.

3) $R = \mathbb{C}[x_1, \ldots, x_n]$, $S = R/(f)$ (f homog deg d elt.)
$S_n = R_n / f \cdot R_{n-d}$

$\Rightarrow h_S(n) = h_R(n) - h_R(n-d)$

Macaulay's Thm - Characterizes exactly the maps $\mathbb{N} \to \mathbb{N}$ of the form h_R for R as abv.

5 Equivariant Comm Alg

$G \curvearrowright R$ comm ring — can generalize a lot of comm alg to equivariant case.

A G-ideal of R is an ideal that's G-stable.

R is G-noeth if ACC holds for G-ideals.

A G-ideal $\mathfrak{p}$ is G-prime if $\mathfrak{a}\mathfrak{b} \subset \mathfrak{p} \Rightarrow \mathfrak{a} \subset \mathfrak{p}$ or $\mathfrak{b} \subset \mathfrak{p}$

e.g. $R = \mathbb{C}[x_1, \_\_]$

$G = S_\infty \to \infty$-symm grp

Cohen $\Rightarrow$ R is G-noeth.

$\hookrightarrow$ eg of a G-prime $\mathfrak{p} = (x_i^2)$

6 Toric Ideals

Say $\varphi: \mathbb{C}[x_1, \_, x_n] \to \mathbb{C}[y_1, \_, y_m]$ $\mathbb{C}$ alg homo so $\varphi(x_i) = \text{monomial}$.

$\hookrightarrow$ $\mathrm{im}(\varphi)$ is a toric ring (gen'd by monomial)

$\ker \varphi$ is a toric ideal

e.g. $R = \mathbb{C}[x_1, \_, x_n]$

Fix $d \geq 1$ $\quad \bigoplus_{d \mid n} R_n \subset R$ subring — $\boxed{\text{toric ring / Veronese Alg}}$.

Def Say M is an R-module.

Recall a pres for M is an exact seq

$$\cdots \cdots F_3 \to F_2 \to F_1 \to F_0 \to M \to 0$$

$\boxed{\text{free}}$ — F_i called Syzygy Mod of M

$\hookrightarrow$ A free resolution of M.

Thm (Hilbert Syzygy Thm) $\quad \cdots 0 \to F_n \to \cdots F_0 \to M \to 0$

If $R = \mathbb{C}[x_1, \_, x_n]$

$\Rightarrow$ Any R mod has a finite res of len at most n

$\mathbb{C}[x,y]/(x,y)$ has reso $\quad 0 \to R \to R^2 \to R \to \mathbb{C}[x,y]/(x,y) \to 0$

Ch helps understand Groth grp.

Prob: understand free reso of toric ring.

$\mathbb{C} \curvearrowright R$ toric ring $\Rightarrow$ a torus acts on $\mathrm{Spec}(R)$.

7. Descent thm

Say V is an $\mathbb{R}$ vs $\rightsquigarrow W = \mathbb{C} \otimes_{\mathbb{R}} V$ (ext of sc)

W admits "complex conj" def $\overline{a \otimes v} = \bar{a} \otimes v$ w/ $a \in \mathbb{C}$, $v \in V$

$\hookrightarrow V$ is what is fixed by cx conj!

Q1 What abt extn of scalars for $R \to S$?

Nice answer, if S is faithfully flat $/R$. (Fppf desc. flat desc.)

$\mathrm{Mod}_R \to \mathrm{Mod}_S$ functor by $\otimes_R$

8 intersectn thm

$f \in \mathbb{C}[x]$ of deg $d \Rightarrow f$ has d roots (w/ mult)

$f, g \in \mathbb{C}[x,y]$ how many common soln to $f = g = 0$?

Count the size of $\mathrm{Spec}\left(\frac{\mathbb{C}[x,y]}{(f,g)}\right)$

Ass $\gcd(f,g) = 1 \Rightarrow$ fin many solns

Bezout's thm: # soln $= \deg(f) \cdot \deg(g)$ provided

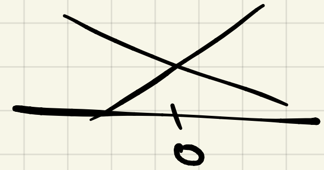
1) Count pts at $\infty \rightsquigarrow$ projectn

2) count w/ multiplicity.

9 Ramification Thry

Consider, $f: \mathbb{C} \longrightarrow \mathbb{C}$

$x \longmapsto x^2$



$\mathbb{C}$
$\downarrow f$
$\mathbb{C}$

for $a \neq 0$ $\# f^{-1}(a) = 2$

$\# f^{-1}(0) = 1$ — f is ramified at 0.

$\hookrightarrow \mathbb{C}[x] \subseteq \mathbb{C}[\sqrt{x}]$

$\mathbb{Z} \longrightarrow \mathbb{Z}[i]$ $p \neq 2$, p has a sq free factor in $\mathbb{Z}[i]$

but $p = 2 \Rightarrow 2 = \pm i (1+i)^2$

$\to$ so $(2) \subset \mathbb{Z}$ ramifies in $\mathbb{Z}[i]$.

General setup $\to$ discrete val ring $\longrightarrow$ local dedekind domain..

$R \subset S$ finite ext of DVR

10. Abelian Categories

If R is a ring $\Rightarrow$ Mod_R is a category.

an abelian category is a category having similar features to Mod_R

$\hookrightarrow$ eg $\hom(x,y)$ is a an ab grp $\forall x,y$, morphisms have ker + coker, ...