Recall

Prop) $0 \to M_1 \to M_2 \to M_3 \to 0$ SES of modules then

a) M_2 Noeth $\iff M_3, M_1$ noeth

b) M_2 Art $\iff M_3, M_1$ art

Cor) A finite $\oplus$ of noeth mods is noeth (same for art)

Prop) If A is a noeth ring (or Art) any fin gen A-mod is Noeth (or Art)

Pf) As M fin gen $\exists$ surj $A^{\oplus n} \xrightarrow{\varphi} M$

↳ noeth A-mod as A noeth

$\implies$ quotient $A^{\oplus n}/\ker\varphi \cong M$ is noeth

Prop) A quotient of a noeth ring (art) is noeth (art)

Pf) A = ring, consider $A/\mathfrak{a}$.

$A/\mathfrak{a}$ satisfies Acc on ideals by corr thm (as A sat acc on ideals)

Def) An A-mod is called simple if $M \neq 0$ & the only submods of M are 0 & M.

Prop) The simple mods of A are exactly $A/\mathfrak{m}$ ← max'l ideal

Pf) Say M—simple A mod. Let $x \in M$ nonzero.

Ax is nonzero submod of M $\implies M = Ax$

So, by $A \to M$, $a \mapsto ax$ surj, we have $M \cong A/I$

for $I = \mathrm{ann}_A(x)$.

As M is simple I max'l (else exists submod by corr thm) (corr thm via corr thm)

e.g. $A = \mathbb{C}[x,y]$. For $\alpha, \beta \in \mathbb{C}$, define A-mod $L_{\alpha,\beta}$ as

$L_{\alpha,\beta} = \mathbb{C} \ni 1 \quad x \cdot 1 = \alpha, \quad y \cdot 1 = \beta \quad (f(x,y) \cdot 1 = f(\alpha, \beta))$

$\lambda \in \mathbb{C} \subset A \quad \lambda \cdot 1 = \lambda$

$L_{\alpha,\beta}$ simple & $\mathbb{C}[x,y] / (x-\alpha, y-\beta)$

$\hookrightarrow$ A submod will be $\mathbb{C}$ submod but by dim cannot.

e.g. $A = \mathbb{Q}[x] \quad L = \mathbb{Q}^2$ basis e_1, e_2 Matrix for x on L $\longrightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$x e_1 = -e_2 \quad x e_2 = e_1$

$L \cong A / (x^2+1) \cong \mathbb{Q}(i)$

e.g. $\mathbb{Z}[x] \quad (x^2+1, 3)$ is max'l $\mathbb{Z}[x]/(x^2+1)$ is gaussian int need a prime in here which are primes 3 mod 4.

Say M is an A module.   len n $\longrightarrow$ conv.

Def A chain in M is $\overbrace{0 = M_n \subsetneq M_{n-1} \subsetneq \cdots \subsetneq M_0}= M$

A chain is a composition series is if M_i / M_{i+1} is simple

Prop Say M has a comp series of length n.
Then any comp series has length n.
Moreover, any chain in M extends to comp series.

Pf Put $\ell(M) = $ min len of comp series ($\ell(M) = \infty$ is dne)

i) If $N \subsetneq M$, $\ell(M) < \infty \implies \ell(N) < \ell(M)$

let $n = \ell(M)$ & $0 = M_n \subsetneq \cdots \subsetneq M_0 = M$

Put $N_i = N \cap M_i \quad \ker(N_i \xrightarrow[\text{incl}]{} M_i / M_{i+1}) = N_{i+1}$

So, get inj $N_i/N_{i+1} \hookrightarrow M_i/M_{i+1}$

Since M_i/M_{i+1} simple $N_i/N_{i+1} = 0$ or M_i/M_{i+1}

Co $\Rightarrow$ $N_i = N_{i+1}$.

So, taking the intersected "chain" & discard duplicates we get an actual chain and comp. series for N.

So $\ell(N) \leq \ell(M)$!

Equality $\Rightarrow$ never discard $\Rightarrow$ $N_i/N_{i+1} = M_i/M_{i+1}$ $\forall i$

$\Rightarrow N = M$ by $i = n-1 \Rightarrow N_{n-1} = M_{n-1}$

$i = n-2 \Rightarrow N_{n-2} = M_{n-2}$ by corr

$\vdots$

ii) Any chain in M has len $\leq \ell(M)$

$0 = M_n \subsetneq \cdots \subsetneq M_0 = M$

So, $0 = \ell(M_n) < \ell(M_{n+1}) < \cdots < \ell(M) = \ell(M)$

part i)

So, $\ell(M) \geq n$

iii) Any comp series has length n.

If $M_\bullet$ is a comp series of len n then

$\ell(M) \leq n$ by defn of $\ell(M)$ and

$\ell(M) \geq n$ by abv.

iv) Any chain extends to a comp series

$0 = M_n \subsetneq \cdots\cdots \subsetneq M_0 = M \overset{\text{add}}{\rightsquigarrow} 0 = M_n \subsetneq X_1 \subsetneq X_2 \subsetneq M_{n-1}$

$\subsetneq \cdots \subsetneq X_p \subsetneq \cdots$

- Say $M_\bullet$ a chain. If len$(M_\bullet)$ is $\ell(M)$ then cannot extend to longer chain $\Rightarrow$ M_i/M_{i+1} simple

$\Rightarrow M_\bullet$ comp series

- If len $M_\bullet < \ell(M) \Rightarrow$ not comp series (min) $\Rightarrow$ extend.

Def A module M has finite length if it admits a comp. series. $\ell(M)$ = common length of any comp series called length of M

e.g. If $A = K$ a field, an A-mod M is fin len $\iff$ fin dim K-vs
& len = dim

e.g. If $A = \mathbb{C}[x,y]$ $\quad A/(x^n, y^n)$ is a finite len A-mod
as $\mathbb{C}$ v.s has basis $x^i y^j$ $0 \le i,j \le n$
$\Rightarrow \dim_{\mathbb{C}} = n^2$

Simple A mods are f.d $\mathbb{C}$ vs $\Rightarrow \operatorname{len}_A(M) = \dim_{\mathbb{C}}(M)$

e.g. $A = \mathbb{Z}$, $M = \mathbb{Z}/p^n$ $\quad \operatorname{len}_A(M) = n$
(or some sub(?))

Def Suppose λ is a rule assigning to each A module M a quantity $\lambda(M) \in$ ab grp.

We say λ is additive iff
$$0 \to M_1 \to M_2 \to M_3 \to 0$$
$$\lambda(M_2) = \lambda(M_1) + \lambda(M_3)$$

Prop length is additive (on finite length modules)

Pf let $0 \to M_1 \xrightarrow{i} M_2 \xrightarrow{\pi} M_3 \to 0$

be a SES of fin len A-mod WTS $\ell(M_2) = \ell(M_1) + \ell(M_3)$

Pick $0 = X_n \subsetneq X_{n-1} \subsetneq \cdots \subsetneq X_0 = M_1$
$\quad\;\; 0 = Y_m \subsetneq Y_{m-1} \subsetneq \cdots \subsetneq Y_0 = M_3$

$\Rightarrow$ get comp series $\longrightarrow$ length $\boxed{m+n}$.

$0 = i(X_n) \subsetneq i(X_{n-1}) \subsetneq \cdots \subsetneq i(X_0) = M_1 \subsetneq \pi^{-1}(Y_{m-1}) \subsetneq \cdots \pi^{-1}(Y_0)$ $= M_2$

Prop) finite length $\iff$ Noeth + Art

Pf) $\Rightarrow$ clear b/c any chain has len $\le \ell(M)$

$\Leftarrow$ say M is noeth + art

Fact: any noeth (or f.g.) has a max'll proper submod

A max'll submod of M is a proper submod that's max'll among proper submod

Wrong: not always exist

eg) $\mathbb{Z}[1/p]/\mathbb{Z}$ has no max submod.

Pick, $M_0 = M$
$\cup\!\!\!\!\!\times$
M_1 = max submod of M (exists as M is noeth, it too is noeth)
$\cup\!\!\!\!\!\times$
M_2 = max submod of M_1
$\cup\!\!\!\!\!\times$
$\vdots$

by DCC (Art) this process stops $\Rightarrow$ gives comp series in M

as $N \subseteq M$ max'll

$\Rightarrow$ M/N simple!

Noeth ring

Hilbert Basis Thm: A is noeth $\Rightarrow$ $A[x]$ is a noeth ring.

Pf) if $f \in A[x]$ nonzero say $f = a_n x^n + \cdots + a_0$ ($a_n \neq 0$)

define $\text{in}(f) = a_n$ $\to$ init coeff

Given ideal $\mathfrak{a} \subset A[x]$. The init ideal is (ideal of A)

$$\text{in}(\mathfrak{a}) = \{\text{in}(f) \mid f \in \mathfrak{a}, f \neq 0\} \cup \{0\}$$

Since A is noeth $\Rightarrow$ $\text{in}(\mathfrak{a})$ is fin gen!

$\Rightarrow$ can pick $f_1, \_\_, f_r \in \mathfrak{a}$ so

$\text{in}(f_1), \_\_, \text{in}(f_r)$ gen $\text{in}(\mathfrak{a})$

let $d_i = \deg f_i$

Let $d = \max \deg f_1, \_\_, f_n$ $\longrightarrow$

Say $g \in \Omega$ has deg $\geq d$ $\quad g = a_n x^n + \cdots + a_0$

$n \geq d, a_n \neq 0$

So, $a_n = in(g) \in in(\Omega)$ $\quad a_n = b_1\, in(f_1) + \cdots + b_r\, in(f_r)$

$b_1, \ldots, b_r \in A$

Consider, $\quad b_1 x^{n-d_1} f_1 + \cdots + b_r x^{n-d_r} f_r$

has deg d & leading coeff a_n.

$\Rightarrow \underbrace{g - (b_1 x^{n-d_1} f_1 + \cdots + b_r x^{n-d_r} f_r)}$ has $\deg < n = \deg(g)$

belongs to $(f_1, \ldots, f_r) \subset \Omega$.

So, any $g \in \Omega$ can be written as $g_1 + g_2$

So $g_1 \in (f_1, \ldots, f_r)$

$g_2 \in \Omega$ & $\deg(g_2) < d$

Consider $\Omega_{<d} = \{ f \in \Omega \mid \deg(f) < d \}$ not ideal!

but it is an A-submod of $A[x]_{<d}$

$= A \oplus xA \oplus \cdots \oplus x^{d-1}A$

$\cong A^{\oplus d-1}$

Since A noeth

$\Rightarrow \Omega_{<d}$ is f.g as an A mod