Dimension thry A noeth local.

- $d(A) = \deg \chi_q(t)$ for any m-primary ideal q
  ↳ poly s.t. $\chi_q(n) = \text{len}(A/q^n)$ $\forall n \gg 0$.
- $\delta(A)$ = min # gens of m-primary ideal.
- $\dim A$ = Krull dim of A

Thm all abv equal. Will show $\delta(A) \le \dim(A) \le d(A) \le \delta(A)$
(1), (2) done

Recall Cor it that $\dim A < \infty$ → Krull.

$\Rightarrow$ B = noeth ring p = prime $\Rightarrow$ $ht(p) < \infty$
↳ max len of ch. $p_0 \subset \dots \subset p$ $= \dim B_p < \infty$

Prop A noeth local ring w/ $\dim A = d$
$\Rightarrow$ $\exists$ m-primary ideal gen'd by d elts
$\Rightarrow$ ($\dim A \ge \delta(A)$ done)

Pf Constr. $x_1, \dots, x_d$ s.t. any pr cont $(x_1, \dots, x_i)$ has ht $\ge i$

Do inductively. Say we've found $x_1, \dots, x_{i-1}$ w/ this prop.
Say $P_1, \dots, P_s$ are the min pr $|$ $(x_1, \dots, x_{i-1})$
[corr to min pr of $A/(x_1, \dots, x_{i-1})$ know fin many]
of height exactly $i-1$
Know $P_i \ne m$ $\forall i$ as m is max'l & P_i not max'l. If it were $\Rightarrow$ ht m = dim d & $ht(P_j) = i-1 < \delta$
$\Rightarrow$ $m \not\subset \bigcup_{j=1}^{s} P_j$ (prime avoidance)
Pick $x_i \in m \setminus \bigcup P_j$
Say q is a pr $\xrightarrow{\text{cont}}$ $(x_1, \dots, x_i)$ $\rightarrow$ ht P $\ge i-1$
+ contains some P min over $(x_1, \dots, x_{i-1})$

2 cases

1) $ht(p) = i-1$. In this case $p = p_j$ for some $1 \le j \le s$

$x_1 \in q$ & $x_i \notin p \Rightarrow p \subsetneq q \Rightarrow ht\, q \ge ht\, p + 1 \ge i$

② $ht\, p \ge i \rightarrow ht\, q \ge ht\, p \ge i$

let $q = (x_1, \ldots, x_d)$. We know any pr. cont q has $ht \ge d$

If $q \subset p \Rightarrow ht\, p \ge d \Rightarrow p = m$ ($p \subset m$ & $ht\, d$)

$\therefore$ So m is unique pr cont $q \Rightarrow q$ is m-primary (as radical is max'l so primary).

Noetherian

Cor 1 $\dim \hat{A} \le \dim_k (m/m^2)$ $k = A/m$, m = max'l ideal.

Pf) If $\overline{x_1}, \ldots \overline{x_r}$ be a k basis of m/m^2

& $x_i \in m$ be a lift of $\overline{x_i}$

$\boxed{\text{Naka}} \Rightarrow M = (x_1, \ldots, x_d) \rightarrow m$ is m primary

So $\delta(A) \le \delta$
$\overset{\shortparallel}{\dim A}$

doesn't have to drop

$\mathbb{C}[x,y]/(xy) = A$ — $(xy = 0)$

Codim A = codim $A/(x)$ $= 1$

imposing $x = 0$ no diff. makes sense.

Cor 1 B is noeth, & have $x_1, \ldots, x_r$ every minimal pr $\supset (x_1, \ldots, x_r)$ has height at most r.

Pf) let p be min pr $\supset (x_1, \ldots, x_r)$

imposing r eqn codim $\le r$

In B_p, $(x_1, \ldots, x_r)$ is p-Primary as p now max'l

$\rightarrow \dim B_p \le r \Rightarrow \delta(B_p)$.
$\overset{\shortparallel}{ht\, p}$

$\dim(A/p)$ is $\dim V(p)$

Intuition PCA $V(p) \subset \operatorname{Spec}(A)$

$\operatorname{depm}(p)$ $ht(p)$ is thought of as codim

Cor | (Krull's Prime Ideal – Hauptidealsatz)

B – noeth ring $x \in B$ that's not unit or zero-div

Then any min pr $/ (x)$ has $\underline{\text{ht } 1}$.

Pf | Let $\mathfrak{p}$ be min pr $/ (x) \Rightarrow ht(\mathfrak{p}) \leq 1$ by prev $\underline{\text{cor}}$.

If $ht(\mathfrak{p}) = 0 \to \mathfrak{p}$ min prime

↳ Elts of min pr are zero divs so this can't happen as $x \in (x) \subseteq \mathfrak{p}$.

Cor | A = noeth $\underline{\text{local}}$,

$x \in m$ nonzero div $\Rightarrow \dim A/(x) = \dim A - 1$

Pf | $\underline{\text{Know}}$, $d(A/xA) \leq \delta(A) - 1$ ① prev

If $\bar{y}_1, \dots, \bar{y}_d$ gen $m/(x)$ pr ideal in $A/(x)$

$\Rightarrow y_1, \dots, y_d, x$ gen m primary in A

$\Rightarrow \delta(A) \leq \delta(A/(x)) + 1$ ②

① + ② as $\delta = d = \dim$ $\underline{\text{done}}$!

Cor | B = noeth $\underline{\text{ring}}$. $\underline{\dim B[x] = \dim B + 1}$.

Pf | In local are clear as x not zero div $\times$ $\frac{B[x]}{(x)} = B$

$\Rightarrow$ $\underline{\text{Above cor}}$, $\dim B(x) = 1 + \dim B$

Gen rmk) Gen rmk $\dim B = \sup_{m \text{ max}} \dim B_m$ by corr (Gx)

$\underline{\text{Gen case}}$. $B[x]_{\mathfrak{p}} / xB[x]_{\mathfrak{p}} \overset{?}{=} B_{\mathfrak{q}}$

$\mathfrak{p} \subset B[x]$
$\mathfrak{q} = \mathfrak{p}^c \subset B$

$\Rightarrow \dim B[x]_{\mathfrak{q}} = 1 + \dim B_{\mathfrak{q}} \leq 1 + \dim B$

$\Rightarrow \dim B[x] \leq 1 + \dim B$

reverse ineq ez as $P_0 \subset \dots \subset P_r$ ch in B

$\Rightarrow P_0 \subset \dots \subset P_r \subset P_r[x]$ ch in $B[x]$

$\Rightarrow \dim B[x] \geq 1 + \dim B$

Also a nice way to see int var poly ring not noeth

Cor) $\dim k[x_1, \dots, x_n] = n$ (k = field)

more gen $\dim A[x_1, \dots, x_n] = n$ if A artinian.

$\dim A = 1$

Thm) If A noeth local. TFAE $\quad d = \dim A$ ($k = A/m$)

a) $gr_m(A) \underset{\text{dff}}{=} \bigoplus_{n \geq 0} m^n / m^{n+1} \cong k[x_1, \dots, x_d]$

(b) $\dim_k m/m^2 = d$

(c) m can be gen by d elts

A is regular if it sat the condit.

Pf) (a) $\Rightarrow$ (b) is clear as m/m^2 is the degree 1 summ in $gr_m(A)$

(b) $\Rightarrow$ (c) discussed, nakayama

(c) $\Rightarrow$ (a) A = noeth loc, $\dim A = d$

$x_1, \dots, x_d$ system of params if $(x_1, \dots, x_d)$ if they gen m-primary ideal

(main thm says this exists)

Prop] $x_1, \dots, x_d$ s.o.p. of param $\mathfrak{q} = (x_1, \dots, x_d)$ $\mathfrak{m}$-pri.

Say $f(t_1, \dots, t_d)$ is homog poly degree s w/ coeff in A s.t. $f(x_1, \dots, x_d) \in \mathfrak{q}^{s+1}$

$\Rightarrow$ coeff of f in $\underline{\mathfrak{m}}$!

Pf] Consider, $\alpha : (A/\mathfrak{q})[t_1, \dots, t_d] \twoheadrightarrow gr_{\mathfrak{q}}(A)$

$\alpha \rightsquigarrow$ surj $(A/\mathfrak{q})[t_1, \dots, t_d]/(\bar{f}) \to \varphi$ $\quad t_i \longmapsto x_i$ $\quad$ 0 in $\mathfrak{q}$'s deg

given $\alpha(\bar{f}) = 0$ where $\bar{f}$ im f in $(A/\mathfrak{q})[t_1, \dots, t_d]$

$(\bar{f}) \subseteq \ker$

Say some coeff of $\bar{f}$ is a unit

$\Rightarrow$ it is not a zero div (exer)

$d(gr_{\mathfrak{q}} A) \leq d\left((A/\mathfrak{q})[t_1, \dots, t_d]/(\bar{f})\right)$

$\leq d((A/\mathfrak{q})[t_1, \dots, t_d])$ non zero div.

$\leq d - 1$

$\boxed{\text{OOPS}}$ as $d(gr_{\mathfrak{q}}(A)) = d = \dim A$

So, coeff of $\bar{f}$ not unit are nilpotent

$\therefore$ they are in nilradical which is $\underline{\mathfrak{m}}$.

Pf] c $\Rightarrow$ a Say $(x_1, \dots, x_d)$ gen $\mathfrak{m}$

let $\alpha : (A/\mathfrak{m})[t_1, \dots, t_d] \longrightarrow gr_{\mathfrak{m}} A$

$t_i \longmapsto x_i$ $\quad$ & coeff as we

inj by prop (think of coeff vanishing)

So $\boxed{\text{b}}$

$\boxtimes$

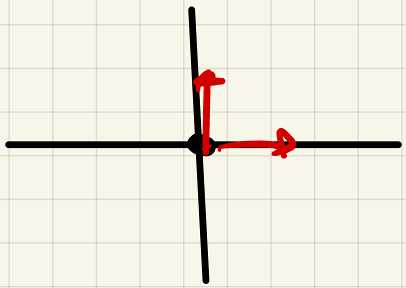
Def) A – noeth local

- m/m^2 ⟶ Zariski cotangent space.
- A is regular if its dimension is the dimension of cotangent sp.

So, B = fin gen k alg, $m \subset B$ max'll ideal.

⟹ m/m^2 is cotan sp to Spec(B) at m.
(dual sp is tangent sp)

e.g.)



Spec A where $A = \mathbb{C}[x,y]/(xy)$

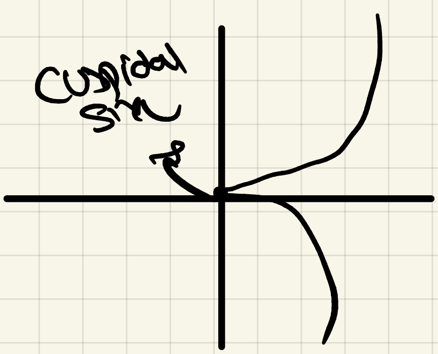
Cl $A_{(x,y)}$ not regular

m/m^2 is 2 dim'll with basis x, y

But $A_{(x,y)}$ has krull dim 1.

Intuition, tangent sp is dim $\boxed{2}$.

e.g.)



$A = \mathbb{C}[x,y]/(y^2-x^3)$

$A_{(x,y)}$ not regular.

so when we do m/m^2 (clr hmp eqn)

So m/m^2 has dim 2

& $A_{(x,y)}$ dim 1 oops.

General fact $f \in \mathbb{C}[x_1, \dots, x_n]$

$\mathbb{C}[x_1, \dots, x_n] / (f) = A$

if $f(\alpha_1, \dots, \alpha_n) = 0$ $\quad m = (x_1 - \alpha_1, \dots, x_n - \alpha_n) \in \mathrm{Spec}(A)$

A_m reg $\iff$ some partial deriv of f nonzero at $(\alpha_1, \dots, \alpha_n)$

[Can notate however, failure to be reg]

Fact) Regular local ring is a domain?

Follows from A = ring $\mathfrak{n} \subset A$ ideal so $\bigcap \mathfrak{n}^n = 0$

$gr_{\mathfrak{n}}(A)$ domain $\Rightarrow$ A domain.

[Note: if A local noeth reg $\Rightarrow \bigcap_{n \geq 0} m^n = 0$ by Krull int

$\Rightarrow gr_m(A) =$ poly ring $=$ domain]