Defn | The projective dimension of an A-module M, denoted $pd_A(M)$ is the minimal n st $\exists$ proj res $0 \to P_0 \to P_n \to \cdots P_0 \to M \to 0$

eg |
- $pd_A(M) = 0 \iff M$ is proj.
- $A = k[x_1, \dots, x_n]$ then hilb syzygy $\Rightarrow pd_A(M) \le n \ \forall M$
  - Ex if $M = \frac{k[x_1, \dots, x_n]}{(x_1, \dots, x_n)}$, $pd_A(M) = n$

Defn | The global dim of A, gl.dim(A) is $\sup\{pd_A(M) \mid M/_A \text{ mod}\}$

- Syzygy $\Rightarrow$ gl-dim$(k[x_1, \dots, x_n]) = n$

e.g | 1) $A = k[x]/(x^2)$ $\quad M = A/(x) = k$

$$\cdots \to A \xrightarrow{x} A \xrightarrow{x} A \xrightarrow{x} A \to M \to 0$$

$0 \to (x) \to A$, $A \xrightarrow{x} (x) \to A$, $0 \to (x)$

$\to$ periodic: 1

$\swarrow$ compute tor

Apply $\otimes_A M$ all A turns to M : $M \xrightarrow{0} M \xrightarrow{0} M \to \cdots$
$\hookrightarrow$ all maps become 0

$\Rightarrow \mathrm{Tor}_i^A(M, M) = k \quad \forall i \ge 0$

$\Rightarrow pd_A(M) = \infty \Rightarrow$ gl-dim$(A) = \infty$

2) $A = \frac{k[x,y]}{(xy)}$ $\xrightarrow{\text{see}}$ $+$ $\quad k = A/(x)$

$$\cdots A \xrightarrow{x} A \xrightarrow{y} A \xrightarrow{x} A \to M \to 0$$

$A \xrightarrow{x} (x) \to A$, $0 \to (x)$

$\to$ periodic: 2.

can see $pd_A(M) = \infty = $ gl-dim(A)

Thm) (Auslander - Buchsbaum - Serre)

A noeth ring. $\mathrm{gl\,dim}(A) < \infty$ iff A is reg

if we have this $\mathrm{gl\,dim}(A) = \mathrm{Krull\,dim}\, A$

Thm) (Auslander - Buchsbaum)

A noeth - local ring

If M is a f.g. A mod s.t. $\mathrm{pd}_A(M) < \infty$

$\mathrm{depth}(A) = \mathrm{depth}(M) + \mathrm{pd}_A(M)$

$\to$ This is max length of reg seq on M

i.e. max n s.t. $\exists$ elts $x_1, \ldots, x_n \in M$

so x_i nonzero div in $M/(x_1, \ldots, x_{i-1})M$

$\mathrm{depth}(A) \leq \dim(A)$

A is Cohen - Macaulay if eq

Regular $\Rightarrow$ C-M

eg) $A = K[x_1, \ldots, x_m]_m$ $\to$ $m = (x_1, \ldots, x_m)$

$\to$ local $\Rightarrow$ C M

$\mathrm{depth}\, A = n$ reg seq $x_1, \ldots, x_n$

AB Formula $\Rightarrow$ $\mathrm{depth}(M) + \mathrm{pd}(M) = n$

- $M = A/(x_1, \ldots, x_m)$ depth $= 0$ $\Rightarrow$ $\mathrm{pd}(M) = n$

- $M = A/(f)$ $f \neq 0$ $\mathrm{pd}(M) = 1$

A-B $\Rightarrow$ $\mathrm{depth}\, M = n-1$ (same as depth of M (ring) as M mod $\Rightarrow$ it is Cohen Macaulay)

→ From now on, A is fg k-alg, $A_0 = k$, $A_n = 0$ for $n<0$
all modules graded!

Say M is a fg gr A mod. M/A_+M = fg gr k-vs.

A homo basis of M/A_+M lifts to homog gen set of M (Nakayama)

Such a gen set for M is called minimal

A surj $F \to M$ is called minimal if a basis for F maps to min gen set!
(F free gr.)

Equiv $F/A_+F \xrightarrow{\simeq} M/A_+M$

Obs| if $F \xrightarrow{\varphi} M$, $F' \xrightarrow{\psi} M$ are min surj
$\Rightarrow$ φ & ψ isoms
i.e. $\exists$ isom $F \xrightarrow{\theta} F'$ s.t. $\psi \circ \theta = \varphi$

$$\begin{tikzcd} & F' \arrow[d, "\psi"] \\ F \arrow[ur, dashed, "\exists\theta"] \arrow[r, "\varphi"'] & M \end{tikzcd}$$

θ exists b/c F is free + ψ surj

θ is isom mod A_+
$\Rightarrow$ isom!

warning| θ not canonical!

Defn| A free reso $F_\bullet \to M$ is minimal if
$F_{i+1} \to \ker(F_i \to F_{i-1})$ min $\forall i$|

Equiv: in the cx $F \otimes_A (A/A_+)$, all diffs are 0

Fact| Any two minimal free reso are isomorphic!

let $F_\bullet \longrightarrow M$ be minimal free res!

$$F_i = A[0]^{b_{0,i}} \oplus A[1]^{b_{1,i}} \oplus A[2]^{b_{2,i}} \oplus \cdots\cdots$$

$A[n]$ - free mod rank 1 gen has degree n

The $b_{j,i}$ are the Betti numbers of M

Obs! let $F \longrightarrow M$ min surj.

Say n minimal so $M_n \neq 0$

$\hookrightarrow M_n = K^d$

$\implies$ map here degree n is an isomorphism!

$\implies$ ker is empty in degree n $\to$ concentrated in higher degrees

Upshot: if $i=1$ $\quad b_{0,1} = 0$

$i=2$ $\quad b_{0,2} = b_{1,2} = 0$

$i=3$ $\quad b_{0,3} = b_{1,3} = b_{2,3} = 0$

$\vdots$

The graded Betti table of M is

$b_{0,0}$	$b_{1,1}$	$b_{2,2}$	$b_{3,3}$	$\cdots$
$b_{1,0}$	$b_{2,1}$	$b_{3,2}$	$b_{4,3}$	$\cdots$
$b_{2,0}$	$b_{3,1}$	$b_{4,2}$	$b_{5,3}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

$\to F_2$

$\hookrightarrow$ tells $F_0 \to$ min gens degs.

no of non zero cols

$\implies$ length of res

$=$ proj dim$(M) + 1$

nonzero rows,

$=$ (Castelnuovo – Mumford) regularity of M. $+1$

e.g. 1) $A = k[x_1, \ldots, x_n]$ $\quad M = A/A_+ = k$

Koszul compl. gives minimal free reso

$n=2$)

$$\begin{tikzcd}
 & & x_1e_2 - x_2e_1 \text{ gens ker} \arrow[d] & & & \\
0 \arrow[r] & A(-2) \arrow[r] & A(-1)^{\oplus 2} \arrow[r] \arrow[dr] & A \arrow[r] & k \arrow[r] & 0 \\
 & & & (x_1, x_2) \arrow[u] & & \\
 & & 0 \arrow[ur] & & &
\end{tikzcd}$$

$\Rightarrow b_{0,0} = 1 \quad \& \quad b_{i,0} = 0 \quad i \neq 0$

$b_{11} = 2, \quad b_{j,1} = 0 \quad j \neq 1$

$b_{22} = 1, \quad b_{j,2} = 0 \quad j \neq 2$

$\Rightarrow$

1	2	1	0	0	0	
⋮						

$pd = 2 \quad reg = 0$

In gen, $n \geq 1$, Betti table conc. in first row w/ binom coeffs.

Eg 2) V, W fd $\mathbb{C}$-v.s. $\quad Z \subseteq \text{Hom}(V,W)$ is the var of rk $\leq r$ maps

Pick bases $V \cong \mathbb{C}^n \quad W \cong \mathbb{C}^m \quad \text{Hom}(V,W) = M_{n,m}(\mathbb{C})$

Coordinate Ring for $\text{Hom}(V,W) \cong \underbrace{\mathbb{C}[x_{ij}]}_{= S}{}_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}}$

$\cong \text{Sym}(\text{Hom}(V,W)^*)$

x_{ij} is map on $\text{Mat}_{n,m}$ taking (i,j) entry.

Z is defd by the vanishing of all $(r+1)-(r+1)$ minors of (x_{ij}).

Call $I_r \subset \mathbb{C}[x_{ij}]$ the ideal gen'd by minors.

Coord ring of Z is S/I_r

Prob: det min free res of S/I_r as S-mod

General obs! F free A-mod $\to$ any ring

$$F = A[0]^{\oplus r_0} \oplus A[1]^{\oplus r_1} \oplus \cdots \cdots$$

$$= (A[0] \otimes_k V_0) \oplus (A[1] \otimes_k V_1) \oplus \cdots$$

$V_i \to$ k vs of dim r_i

$$= A \otimes_k V \qquad V = V_0[0] \oplus V_1[1] \oplus \cdots\cdots$$

$\hookrightarrow$ graded k-vs!

In our case,

we can find minimal free reso $F_\bullet \to S/I_r$

that is compact wrt $GL(V) \times GL(W)$ action.

$F_i = S \otimes_k V_i \longrightarrow V_i$ - rep of $GL(V) \times GL(W)$

$GL(V) = GL_n$ irred reps $\longleftrightarrow$ highest weights $(\lambda_1, \ldots, \lambda_n)$

$\lambda_1 \geq \lambda_2 \geq \cdots$

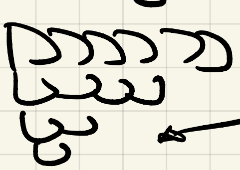
$\lambda_i \in \mathbb{Z}$

$\to$ we only need case $\lambda_i \geq 0$

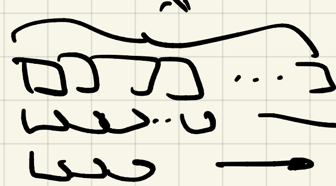
$\hookrightarrow$ can then think of λ_i as a partition.

$= \lambda_1 + \cdots + \lambda_n$

Young diagram of λ



$(6, 4, 2, 1)$



λ_1

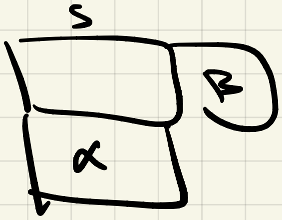
λ_2

λ_3

$\vdots$

Thm (Lascoux) $V_i = \bigoplus_{i = s^2 + |\alpha| + |\beta|} L_\lambda(V) \otimes L_\mu(W)$ $\to$ in deg $|\lambda| = s^2 + rs + |\alpha| + |\beta|$

$\lambda = r \times s$



s, β, α

$\mu = p/s$

s, α^t, β^t

- $s \in \mathbb{N}$ — # cols $\alpha \leq s$
- α, β pts — # rows $\beta \leq r - s$
- α^t transp along diag

$L_\lambda(V)$ is irred rep of $GL(V)$ w/ highest wt
λ provided # rows $\lambda \le n$
or 0 if $\lambda > n$.

$|\lambda| - i = rs$.

Comments

1) V_i is a multiplicity free rep
i.e. every irred shows up either 0 or 1 times!

2) If $r=0 \Rightarrow Z = \{0\} \subset \mathrm{hom}(V,$