Today,

we observed that MaxSpec has flaws:
- Not functorial!
- Max'l ideals don't see everything

  ↳ e.g. $\left[\mathrm{rad}(\mathfrak{a}) = \bigcap_{\substack{\mathfrak{a} \subset \mathfrak{p} \\ \mathfrak{p} \text{ prime}}} \mathfrak{p} \right]$

  Not true in gen for max ideals

} Spec fixes

Def) The spectrum of a ring R, denoted $\mathrm{Spec}(R)$, is the set of all prime ideals.

e.g.)
$$\mathrm{Spec}(\mathbb{Z}) = \{(p) \mid p \text{ prime}\} \cup \{(0)\}$$
$$= \mathrm{MaxSpec}(\mathbb{Z}) \cup \{(0)\}$$
$$\mathrm{Spec}(\mathbb{C}[x]) = \{(x-\alpha) \mid \alpha \in \mathbb{C}\} \cup \{(0)\}$$
$$= \mathrm{MaxSpec}(\mathbb{C}[x]) \cup \{(0)\}$$

$$\mathrm{Spec}(0) = \emptyset$$
$$\mathrm{Spec}(\mathbb{F}) = \{(0)\}$$
(field) → only prime ideals (1), (0)

$$\mathrm{Spec}(\mathbb{Z}/6\mathbb{Z}) = \{(2),(3)\}$$ ↳ see this by ideal corr or

$$\mathrm{Spec}(R \times S) \cong \mathrm{Spec}(R) \sqcup \mathrm{Spec}(S)$$

$$\mathbb{Z}/6 \cong \underbrace{\mathbb{Z}/2}_{1 \text{pt}} \times \underbrace{\mathbb{Z}/3}_{1 \text{pt}}$$

For an ideal $\mathfrak{a} \subset R$ define $V(\mathfrak{a}) = \{\mathfrak{p} \mid \mathfrak{a} \subset \mathfrak{p}\} \subseteq \mathrm{Spec}(R)$ (prime)

subset $Z \subset \mathrm{Spec}(R)$ def $I(Z) = \bigcap_{\mathfrak{p} \in Z} \mathfrak{p}$

Obs: 1) $I(V(\mathfrak{a})) = \bigcap_{\substack{\mathfrak{a} \subset \mathfrak{p} \\ \mathfrak{p} \text{ prime}}} \mathfrak{p} = \mathrm{rad}(\mathfrak{p})$

2) If $\mathfrak{a} \subset \mathfrak{b} \implies V(\mathfrak{b}) \subseteq V(\mathfrak{a})$

- If $V(\mathfrak{b}) \subseteq V(\mathfrak{a}) \underset{\text{I ord rev}}{\implies} I(V(\mathfrak{a})) \subseteq I(V(\mathfrak{b}))$

$$\Updownarrow$$

$$\mathrm{rad}(\mathfrak{a}) \subseteq \mathrm{rad}(\mathfrak{b})$$

- $[V(\mathfrak{a}) = V(\mathrm{rad}(\mathfrak{a}))]$
- Abv imply, $V(\mathfrak{b}) \subset V(\mathfrak{a}) \iff \mathrm{rad}(\mathfrak{a}) \subseteq \mathrm{rad}(\mathfrak{b})$

3) $V((0)) = \mathrm{Spec}(R)$, $V((1)) = \emptyset$

4) $V(\mathfrak{a}+\mathfrak{b}) = V(\mathfrak{a}) \cap V(\mathfrak{b})$.

Infact $V(\sum_{i \in I} \mathfrak{a}_i) = \bigcap_{i \in I} V(\mathfrak{a}_i)$

5) $V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$

P: $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{b} \implies V(\mathfrak{b}) \subseteq V(\mathfrak{a}\mathfrak{b})$

Symm $\implies V(\mathfrak{a}) \subseteq V(\mathfrak{a}\mathfrak{b})$ $\quad \supseteq$ ✓

Now say $\mathfrak{p} \in V(\mathfrak{a}\mathfrak{b})$, i.e, $\mathfrak{a}\mathfrak{b} \subset \mathfrak{p}$ | wts $\mathfrak{a} \subset \mathfrak{p}$ or $\mathfrak{b} \subset \mathfrak{p}$

Spse $\mathfrak{a} \not\subset \mathfrak{p} \implies \exists x \in \mathfrak{a} \setminus \mathfrak{p}$

$\implies \forall y \in \mathfrak{b}$ $xy \in \mathfrak{a}\mathfrak{b} \subset \mathfrak{p} \underset{\substack{\mathfrak{p} \text{ prime} \\ x \notin \mathfrak{p}}}{\implies} y \in \mathfrak{p} \implies \mathfrak{b} \subset \mathfrak{p}$

6) $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$

P| $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{a} \cap \mathfrak{b} \subseteq \mathfrak{a}, \mathfrak{b} \rightsquigarrow V(\mathfrak{a}), V(\mathfrak{b}) \subset V(\mathfrak{a} \cap \mathfrak{b}) \subset V(\mathfrak{a}\mathfrak{b})$

$$\parallel$$

$$V(\mathfrak{a}) \cup V(\mathfrak{b})$$

Recall A topology on a set X is a coll of closed sets X s.t.

1) Empty set & X are closed $V((1)) = \emptyset$, $V((0)) = X$
2) Arb int of closed set is cl
3) finite union closed

Def The topology on $\mathrm{Spec}(R)$ in which $V(\mathfrak{r})$'s are the closed sets is called the Zariski Topology

Ex $X = \mathrm{Spec}(\mathbb{C}[x]) \cong \mathbb{C} \sqcup \{*\}$ $\quad * \mapsto (0)$

max ideals $\alpha \mapsto (x - \alpha)$

Say $\mathfrak{r} \subseteq \mathbb{C}[x]$ $\quad$ 2 cases

1) $\mathfrak{r} = (0)$ $\quad V(\mathfrak{r}) = X$
2) $\mathfrak{r} \neq (0) \Rightarrow \mathfrak{r}$ principal $\Rightarrow \mathfrak{r} = (f)$ fact

$$= \mathfrak{r} = \big(\underbrace{(x-\alpha_1) \cdots (x-\alpha_r)}\big)$$

contain corr fodiv in prime

$$\Rightarrow V(\mathfrak{r}) = \{\alpha_1, \ldots, \alpha_r\} \subset \mathbb{C} \subset X$$

So the closed sets of $X \to$ 1) X 2) finite subsets of $\mathbb{C}$

Rmk 1) $*$ is weird $\to \{*\}$ not closed (not T2 or whatever)

$\overline{\{*\}} = X$! So $*$ is dense, called generic pt

2) the pts in $\mathbb{C}$ are closed $\{\alpha\} = V((x-\alpha))$

3) Ordinary top on $\mathbb{C}$ irrelevant here

ex: $\mathrm{Spec}(\mathbb{C}[x,y]) = \mathbb{C}^2 \cup \{(f) \mid f \text{ is irred}\} \cup \{(0)\}$

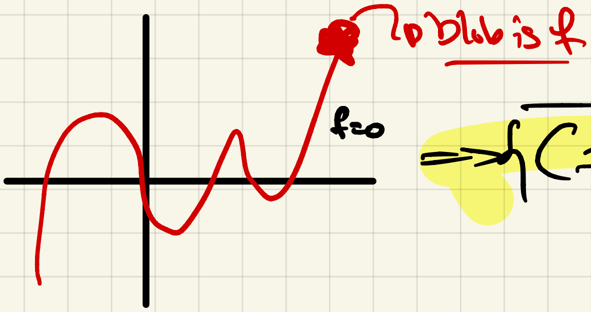
$(\alpha,\beta) \leftrightarrow (x-\alpha, x-\beta)$

These pts closed.

This is dense.

say f is irred. $V(f) = \{$the pts in $\mathbb{C}^2$ where $f=0\} \cup \{(f)\}$

this is generic for this set



$\Rightarrow \overline{\{(f)\}} = V(f)$

$V(f) \cong \mathrm{Spec}(\mathbb{C}[x])$

$\boxed{\text{homeo}}$

Def. $\varphi: R \to S$ be a ring homo.

- If $\mathfrak{a} \subset R$ is an ideal, its extension to S, $\mathfrak{a}^e$, is the ideal gen'd by $\varphi(\mathfrak{a})$
- If $\mathfrak{b} \subset S$ is an ideal, its contr. to R, $\mathfrak{b}^c$, is $\varphi^{-1}(\mathfrak{b})$ (alr an ideal)

((notation hides φ))

See

- $\mathfrak{a} \subset (\mathfrak{a}^e)^c$
- $(\mathfrak{b}^c)^e \subset \mathfrak{b}$

eg $\mathbb{Z} \hookrightarrow \mathbb{Q}$

$\mathfrak{a} = (2)$, $\mathfrak{a}^e = (1)$, $(\mathfrak{a}^e)^c = (1)$

so $\mathfrak{a} \subsetneq (\mathfrak{a}^e)^c$

eg $\mathbb{Z} \subseteq \mathbb{Z}[x]$

$\mathfrak{b} = (x)$ $\quad \mathfrak{b}^c = (0)$, $(\mathfrak{b}^c)^e = (0)$

so $(\mathfrak{b}^c)^e \subsetneq (0)$

Prop | if $\mathfrak{q} \subset S$ prime $\Rightarrow \mathfrak{q}^c \subset R$ is also prime!
$\varphi: R \to S$

Pf | Say $xy \in \mathfrak{q}^c \Rightarrow \varphi(xy) \in \mathfrak{q} \underset{\mathfrak{q}\text{ prime}}{\Rightarrow} \varphi(x)\varphi(y) \in \mathfrak{q}$

so $\varphi(x) \in \mathfrak{q}$ or $\varphi(y) \in \mathfrak{q}$

$\Rightarrow$ $x \in \mathfrak{q}^c$ or $y \in \mathfrak{q}^c$

Warning: doesn't work for max ideals, $\mathbb{Z} \to \mathbb{Q}$

let $\mathfrak{b} = (0)$ max'l.

but, $\mathfrak{b}^c = (0) \subseteq \mathbb{Z}$ not max'l.

Let $\varphi: R \to S$ ring homo, get map $\varphi^*: \operatorname{Spec}(S) \to \operatorname{Spec}(R)$

$\mathfrak{q} \longmapsto \mathfrak{q}^c$

Warning: in gen $\nexists$ $\operatorname{MaxSpec}(S) \to \operatorname{MaxSpec}(R)$

Prop | φ^* is cts map!

Pf | WTS $(\varphi^*)^{-1}(\text{closed set}) = \text{closed}$

wts: given $\mathfrak{a} \subset R$ find $\mathfrak{b} \subset S$ s.t

$$(\varphi^*)^{-1}(V(\mathfrak{a})) = V(\mathfrak{b}) \qquad \text{guess } \mathfrak{b} = \mathfrak{a}^e$$

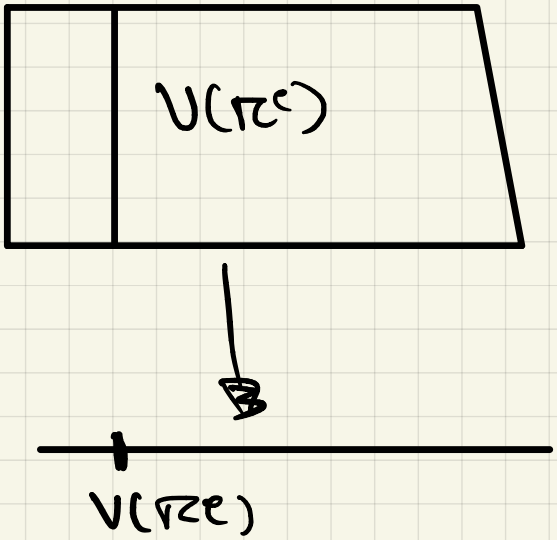
$$(\varphi^*)^{-1}(V(\mathfrak{a})) \overset{?}{=} V(\mathfrak{a}^e)$$

Say, $\mathfrak{q} \in \text{LHS}$ i.e $\varphi^*(\mathfrak{q}) \in V(\mathfrak{a})$

$\Rightarrow \mathfrak{a} \subseteq \mathfrak{q}^c \Rightarrow \mathfrak{a}^e \subset \mathfrak{q}$

so $\mathfrak{q} \subset \text{RHS}$

Reverse Similar.

Spec(S) 

fibres.

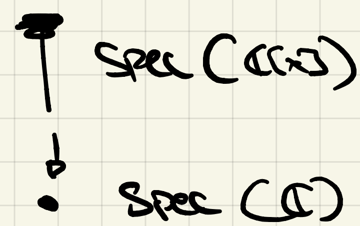
φ^*

Spec(R)

e.g. 1) $\varphi : \mathbb{C} \hookrightarrow \mathbb{C}[x]$

$$\varphi^* : \operatorname{Spec}(\mathbb{C}[x]) \longrightarrow \operatorname{Spec} \mathbb{C} = \{*\}$$

$$\mathbb{C} \sqcup \{*\}$$

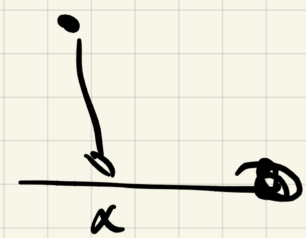


2) $\varphi : \mathbb{C}[x] \longrightarrow \mathbb{C}$

$x \longmapsto$

Spec S

Spec R



3) $\varphi : \mathbb{C}[x] \longrightarrow \mathbb{C}[y,z]$

$x \longmapsto F(y,z)$

$\leadsto$ in this case max'l ideals contr to max'l ideals.

Get