Last time → saw some module thing.

Wed: Oct.

Say, M, N, T are R-modules,

Defn An R bilinear map $\varphi: M \times N \to T$ is a map that is a module homo in each entry

Defn A tensor prod of M, N is a pair (T, φ) where T is a R-module & φ a bilinear map $\varphi: M \times N \to T$ that is universal, s.t. if (T', φ') is another pair $\exists!$ R-module homo $\psi: T \to T'$ so $\psi \circ \varphi = \varphi'$

$$\begin{tikzcd} M \times N \arrow[r, "\varphi"] \arrow[dr, "\varphi'"'] & T \arrow[d, "\psi"] \\ & T' \end{tikzcd}$$

Rmk, Any two tensor prods canonically iso!
So we say the tensor prod denoted $M \otimes_R N$

Warning! say $R \to S$ ring homo
M, N S modules $\Longrightarrow$ can regard, restr. of sc, as R mod
$M \otimes_R N \neq M \otimes_S N$ in gen!

Prop! A tensor prod of M, N exists

Pf sk) let F be a free R-module w basis $[x,y]$ $\forall x \in M$, $y \in N$
usually huge!

Have a map $\tilde{\varphi}: M \times N \to F$ — not bilinear
$(x,y) \longmapsto [x,y]$ — formal sym

Not bil as $\tilde{\varphi}(x_1 + x_2, y) = [x_1 + x_2, y] \neq [x_1, y] + [x_2, y]$

Def submod $F \supset K$ gen'd by all elts of the form
$\tilde{\varphi}(a_1 x_1 + a_2 x_2, y) - a_1 \tilde{\varphi}(x_1, y) - a_2 \tilde{\varphi}(x_2, y)$
similar in y var

Let $T = F/K$ $\quad M\times N \xrightarrow{\tilde{\varphi}} F \longrightarrow F/K$, with $\varphi: M\times N \to F/K$ the composite.

φ is bilin by def of K

Idea of univ: given (T', φ') $\quad \varphi' : M\times N \longrightarrow T'$ bilin

$\Rightarrow$ mapping prop of F (free mod)

$\Rightarrow \exists! \ \tilde{\psi} : F \longrightarrow T'$

$[x,y] \longrightarrow \varphi'(x,y)$

Key $\rightarrow$ as φ' is bilin $\Rightarrow$ $\tilde{\psi}(K) = 0$

$\Rightarrow$ $\tilde{\psi}$ induces R-homo $\psi : F/K \to T'$

$\Rightarrow$ this sats $\varphi' = \psi \circ \varphi$

By defn, there is a univ bilinear map $M\times N \longrightarrow M \otimes_R N$

The img of $(x,y) \in M\times N$ in $M\otimes N$ den $x \otimes y$ in F/K

These elts are pure tensor, in prev pf $x\otimes_R y$ $\leftarrow$ coset of $[x,y]$

$\hookrightarrow$ Pf of prop $\Rightarrow$ $M\otimes N$ gen'd (as R mod) by pure tensors!

Warning! Not all elts of $M\otimes N$ are pure tensors!

Recall! If $R = F$ is a field & V, W fin dim'l F-vs

w/ basis $v_1, \dots, v_n$ & $w_1, \dots, w_m$

$\hookrightarrow$ Then $\{v_i \otimes w_j\}_{i\in \mathbb{N}_n, \, j\in \mathbb{N}_m}$ basis for $V \otimes_F W$ as F-vs

$\Rightarrow \dim_F V\otimes_F W = \dim_F V \cdot \dim_F W$

E.g.| Over a ring R, it is possible $M \otimes_R N = 0$ & $M, N \neq 0$

$\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z} = 0$

sts all pure tensors $x \otimes y$ vanish $x \in \mathbb{Z}/2,\ y \in \mathbb{Z}/3$

Note $x = 3x$ as $3 = 1 \pmod 2$

so, $x \otimes y = (3x) \otimes y = 3(x \otimes y) = x \otimes (3y) = 0$

as $\boxed{3y = 0}$

Properties of tens $\otimes$ $\to$ Monoid $\to$ category of R Modules $\to$ Mod_R $\to$ Symm monoidal category

1) Associative, M_1, M_2, M_3 are R mods

$$(M_1 \otimes_R M_2) \otimes_R M_3 \cong M_1 \otimes_R (M_2 \otimes M_3)$$

Naturally iso

$$(x_1 \otimes x_2) \otimes x_3 \mapsto x_1 \otimes (x_2 \otimes x_3)$$

2) Commutative, $M_1 \otimes M_2 \cong M_2 \otimes M_1$ natural iso

$$x \otimes y \mapsto y \otimes x$$

3) Unit obj $R \otimes_R M \cong M$

$$a \otimes x \mapsto a \cdot x$$

4) Additive $(M_1 \oplus M_2) \otimes_R N \cong (M_1 \otimes N) \oplus (M_2 \otimes_R N)$

More gen $\left(\bigoplus_{i \in I} M_i\right) \otimes_R N \cong \bigoplus (M_i \otimes N)$

Obs| 3,4 $\Rightarrow$ tensor prod of free modules same as VS

$R^{\oplus n} \otimes_R R^{\oplus m} \cong R^{\oplus nm}$ also true in gen.

Adjunctions

let $\mathcal{C}, \mathcal{D}$ two categories w/ Ψ, Φ funct

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend right, "\Phi"'] & \mathcal{D} \arrow[l, bend right, "\Psi"'] \end{tikzcd}$$

An adjunction b/w Φ, Ψ is a natural isom

$$\mathrm{Hom}_{\mathcal{D}}(\Phi(X), Y) \cong \mathrm{Hom}_{\mathcal{C}}(X, \Psi(Y)) \qquad X \in \mathrm{Ob}(\mathcal{C}),\ Y \in \mathrm{Ob}(\mathcal{D})$$

say Φ is left adj of Ψ

Ψ is right adj. of Φ

both are functors in (X, Y).

isom of functors

taking $Y = \Phi(X)$ & moving the identity on $\Phi(X)$ to the right

$\Rightarrow$ get map $\eta_X : X \to \Psi(\Phi(X))$ unit

Symm $X = \Psi(Y)$, $\varepsilon_Y : \Phi(\Psi(Y)) \to Y$ co-unit

typically not isom.

Restr + extn of sc

Let $\varphi: R \to S$ be a ring homo

If N is S-mod, can regard N as R mod by $a \cdot x = \varphi(a) \cdot x$

$\hookrightarrow$ this defines a functor $\Psi : \mathrm{Mod}_S \to \mathrm{Mod}_R$ (restriction of sc)

We also have a functor $\Phi : \mathrm{Mod}_R \to \mathrm{Mod}_S$

extn of scalars

$\hookrightarrow$ dep on φ as in purple.

$$M \longmapsto S \otimes_R M$$

S is S mod so restr. of sc w/ φ shows up here

can regard $S \otimes_R M$ as S mod this way

$$\begin{cases} a, a' \in S & x \in M \\ a' \cdot (a \otimes m) = aa' \otimes x \end{cases}$$

Prop 1 M - R mod, N - S mod, $\varphi: R \to S$

natural isom

$$\mathrm{Hom}_S(\underbrace{S \otimes_R M}_{\Phi(M)}, N) \cong \mathrm{Hom}_R(M, \underbrace{N}_{\Psi(N)})$$

S mod via extn

R mod via restr.

$\to \Phi, \Psi$ adj $\to \Phi \dashv \Psi$

Pf ish We need to def mutually inv bij b/w

$$\mathrm{Hom}_S(S\otimes_R M, N) \underset{\alpha}{\overset{\beta}{\rightleftarrows}} \mathrm{Hom}_R(M,N)$$

Say $f: M \to N$ is a R-mod hom

Need to def $\beta(f): S\otimes_R M \to N$

want, $\beta(f)(a\otimes x) = \underset{\in S}{a}\cdot \underbrace{f(x)}_{\in N}$ N is a S mod ✓

Now, say $g: S\otimes_R M \to N$ is given

need $\alpha(g): M \to N$

$x \longmapsto g(1\otimes_R x)$ $1 \to \in S$, $\otimes_R \to$ pure tensor

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Unit: $X \to \Psi(\Phi(X))$

R-mod M $\exists$ natural map of R-mods

$$M \to S\otimes_R M$$
$$x \longmapsto 1\otimes x$$

counit: $\Phi(\Psi(Y)) \to Y$

S-mod N $\exists$ natural map of R-mod

$$S\otimes_R N \to N$$
$$a\otimes x \longmapsto ax$$

Naturality of unit, if $f: M \to M'$ is a map of R-mods

$$\begin{tikzcd}
x \arrow[r, mapsto] & 1\otimes x & \\
M \arrow[r, "\text{unit}"] \arrow[d, "f"'] \arrow[dr, phantom, "\text{comm}"] & S\otimes_R M = \Psi(\Phi(M)) \arrow[d, "{\Psi(\Phi(f))\ \text{nat funct}}"] & a\otimes x \arrow[d, mapsto] \\
M' \arrow[r, "\text{unit}"] & S\otimes_R M' = \Psi(\Phi(M')) & a\otimes f(x) \\
x' \arrow[r, mapsto] & 1\otimes x' &
\end{tikzcd}$$

Upshot, mapping prop extn of sc.

giving map from $S \otimes_R M \to N$ as S mod $\iff$ giving map $M \to N$ as R mod.

Prop) If $M_1, M_2, M_3 \to R$-mods

Rec: set of homom is R module.

$\exists$ natural isom

$$\mathrm{Hom}_R(M_1 \otimes_R M_2, M_3) \cong \mathrm{Hom}_R(M_1, \mathrm{Hom}(M_2, M_3))$$

This is an adjunction b/w,

$$\left.\begin{aligned} \Phi(N) &= N \otimes_R M_2 \\ \Psi(N) &= \mathrm{Hom}(M_2, N) \end{aligned}\right\} \text{Functors } \mathrm{Mod}_R \to \mathrm{Mod}_R$$

So, Φ, Ψ are adj by prop