Prop | A is any ring, M f.g A-mod. $\hat{A} \otimes_A M \to \hat{M}$ (we have a map by adj.) is surj. If A is noeth its $\cong$.

Pf | a) $F \twoheadrightarrow M$ surj where F is fg free A mod.
↳ fg.

So,

$$\begin{tikzcd} \hat{A}\otimes_A F \arrow[r, two heads] \arrow[d, "\cong"] & \hat{A}\otimes_A M \arrow[d] \\ \hat{F} \arrow[r, two heads] & \hat{M} \end{tikzcd}$$

tensor prod right ex.

as F is free. $F = A \oplus \cdots \oplus A$

surj now by lifts

surj b/c comp. exact.

b) Now $F \to M$ as abv w ker N

$$\begin{tikzcd} \hat{A}\otimes_A N \arrow[r] \arrow[d, "\alpha"] & \hat{A}\otimes_A F \arrow[r] \arrow[d, "\beta", "\cong"'] & \hat{A}\otimes_A M \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & \hat{N} \arrow[r] & \hat{F} \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

surj by a)

↳ tensor right exact

↳ compl. exact as all are fin gen. + Artin Rees

Snake lemma says exact $\ker\beta \to \ker\gamma \to \operatorname{coker}\alpha$, with $\ker\beta = 0$ and $\operatorname{coker}\alpha = 0$.

$\Rightarrow \ker\gamma = 0$.

Cor | A noeth $\Rightarrow \hat{A}$ is flat $/A$
as $\hat{A}\otimes_A -$ is exact in f.g A mods $\Rightarrow \hat{A}$ is flat

flatness criterion with ideals

Prop | A is noeth
a) $\hat{\mathfrak{a}}$ is $\mathfrak{a}^e = \mathfrak{a}\hat{A}$
b) $\widehat{\mathfrak{a}^n}$ is $\hat{\mathfrak{a}}^n$
c) $\mathfrak{a}^n/\mathfrak{a}^{n+1} = \hat{\mathfrak{a}}^n/\hat{\mathfrak{a}}^{n+1}$
d) $\hat{\mathfrak{a}} \subset$ Jac rad $\hat{A}$

Pf | a) Any ideal $\mathfrak{b}$: $\hat{\mathfrak{b}} \cong \hat{A}\otimes_A \mathfrak{b} \cong \hat{A}\mathfrak{b}$

$\hat{A}$ is flat

b) $\widehat{\mathfrak{a}^n} = \hat{A}\mathfrak{a}^n = (\hat{A}\mathfrak{a})^n = (\hat{\mathfrak{a}})^n$

c) Alr shown $\hat{A}/\hat{\mathfrak{a}}^n \cong A/\mathfrak{a}^n$

have SES

$$\begin{array}{ccccccccc} 0 & \to & \mathfrak{a}^n/\mathfrak{a}^{n+1} & \to & A/\mathfrak{a}^{n+1} & \to & A/\mathfrak{a}^n & \to & 0 \\ & & \downarrow & & \downarrow\wr & & \downarrow\wr & & \\ 0 & \to & \hat{\mathfrak{a}}^n/\hat{\mathfrak{a}}^{n+1} & \to & \hat{A}/\hat{\mathfrak{a}}^{n+1} & \to & \hat{A}/\hat{\mathfrak{a}}^n & \to & 0 \end{array}$$

$\Rightarrow$ left is $\cong$

D) $x \in \hat{\mathfrak{a}} \Rightarrow 1-x$ is a unit

$(1-x)^{-1} = \underbrace{1+x+\cdots\cdots}_{\text{conv in } A \text{ as } x^n \in \mathfrak{a}^n} \in \hat{A}$

$\forall y \in A \Rightarrow xy \in \hat{\mathfrak{a}} \Rightarrow 1-xy$ a unit

$\Rightarrow x \in \mathrm{Jac}(\hat{A})$

Cor| A = noeth'n local $\mathfrak{a} = \mathfrak{m}$, $\hat{A}$ is local w/ max'll ideal $\hat{\mathfrak{m}}$

Pf| $\hat{A}/\hat{\mathfrak{m}} = A/\mathfrak{m}$ a field $\Rightarrow \hat{\mathfrak{m}}$ is max'll
by c

$\hat{\mathfrak{m}} \subset \mathrm{Jac}(\hat{A})$ $\to \cap$ all max'll ideals

$\Rightarrow \hat{\mathfrak{m}}$ unique max ideal of A.

Thm (Krull) |

If A noeth & $\mathfrak{a}$ ideal. M fg A mod.

$E = \ker(M \to \hat{M}) = \bigcap_{n \geq 1} \mathfrak{a}^n M$

$\hookrightarrow$ follows from inv lim def

Then $x \in E$ iff its killed by elt of $1+\mathfrak{a}$.

Pf| The induced top on E is trivial (indisc)

By Artin-Rees $\Longrightarrow$ $\mathfrak{a}$-adic top on E (indisc)

$\Rightarrow E = \mathfrak{a}E$

since E fin gen $\Rightarrow (1-\alpha)E = 0$ for some $\alpha \in \mathfrak{a}$.

↳ say $x_1 — x_n$ gen E

$\Rightarrow x_i = \sum_{j=1}^{n} a_{ij} x_j \quad a_{ij} \in \mathfrak{a}$

$$\Rightarrow \underbrace{\begin{pmatrix} 1-a_{11} & a_{12} & \cdots \\ a_{21} & 1-a_{22} & \\ \vdots & & \ddots \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = 0$$

$\Rightarrow \underbrace{A^{adj} A}_{\det(A) \cdot id} \vec{x} = 0 \Rightarrow \det A$ kills all E

$\det A \in 1 + \mathfrak{a}$

$\Leftarrow$ if $(1-\alpha)x = 0$ for some $\alpha \in \mathfrak{a}$

$\Rightarrow x = \alpha x \Rightarrow x = \alpha(\alpha x) \Rightarrow x = \alpha(\alpha(\alpha x))$

$\Rightarrow x = \alpha^n x \ \forall n \Rightarrow x \in E$

Obs $S = 1 + \mathfrak{a}$ mult set

Thm, $\underline{\ker(M \to \hat{M}) = \ker(M \to S^{-1}M)}$.

Cor A noeth domain, $\mathfrak{a} \neq (1) \Rightarrow \bigcap_{n \geq 1} \mathfrak{a}^n = 0$

↳ A is $\mathfrak{a}$-adically sep!

Pf By thm, $\bigcap \mathfrak{a}^n$ is killed by elt of $1 + \mathfrak{a}$

↳ since A is domain $\bigcap \mathfrak{a}^n = 0$ (as $0 \notin 1 + \mathfrak{a}$)

Cor A noeth ring, $\mathfrak{a} \subset Jac(A)$

$\Rightarrow$ fin gen A module M is $\overbrace{\bigcap \mathfrak{a}^n M = 0}^{\to\ \mathfrak{a}\text{-adic sep}}$

Pf By Thm $\bigcap \mathfrak{a}^n M$ killed by elt $\underline{1+\mathfrak{a}}$

↳ unit as $\mathfrak{a} \subset Jac(A)$

Cor A noeth + local, $(\mathfrak{a}) \neq 1$

thm says f.g A mod is $\mathfrak{a}$ adic sep $\left\{ \begin{array}{l} \Rightarrow A \text{ is } \mathfrak{a}\text{-adic} \\ \text{sep } \bigcap \mathfrak{a}^n = 0 \end{array} \right.$

↳ Krull, int thm

e.g.) $\mathbb{Z}/6 \cong \mathbb{Z}/2 \oplus \mathbb{Z}/3$

$\mathfrak{a} = (2) = 0 \times (1)$

$\Rightarrow \mathfrak{a}^n = \mathfrak{a} \quad \forall n$

So $\mathfrak{a}$ not unit ideal

but $\underline{\bigcap \mathfrak{a}^n = \mathfrak{a} \neq 0}$.

Def) A filtration on an abelian grp X is

$\cdots\cdots \subset F^1 X \subset F^0 X = X$ (gives top on X as fund open sets)

$\underline{\text{subgrp}}$

- A filtered ring is a ring A w/ a filter $F^\bullet A$ $\underline{\text{s.t.}}$ $F^n A \, F^m A \subset F^{n+m} A$

- If A is a filtered ring, a filtered Module M is a mod w/ a filter $F^n M \, F^m M \subset F^{n+m} M$

Def) If X is a filtered ab grp $\rightsquigarrow$ assoc graded

$$gr(X) = \bigoplus_{n \geq 0} F^n X / F^{n+1} X$$

If A filtered $\Rightarrow$ Assoc graded is graded ring!

If M - A-mod $\Rightarrow$ Assoc graded is $gr(A)$-mod!

Main ex) A ring $\mathfrak{a}$ ideal

$F^n A = \mathfrak{a}^n \quad \Rightarrow \quad gr(A) = \bigoplus \mathfrak{a}^n / \mathfrak{a}^{n+1}$

If M A-mod, define filter on M by

$\underline{F^n M = \mathfrak{a}^n M}$.

lemma) $\Phi : X \to Y$ map of filtered ab grp (Φ homom & $\Phi(F^n X) \subseteq F^n Y$)

a) If $gr(\Phi)$ inj $\Rightarrow \hat{\Phi}$ inj

b) ______ surj $\Rightarrow$ — surj

Φ induces

$gr(\Phi) : gr(X) \to gr(Y)$

$\hat{\Phi} : \hat{X} \to \hat{Y}$

Pf)

$$\begin{tikzcd} 0 \arrow[r] & F^n X / F^{n+1} X \arrow[r] \arrow[d, "\varphi^n"] & X / F^{n+1} X \arrow[r] \arrow[d, "\varphi_{n+1}"] & X / F^n X \arrow[r] \arrow[d, "\varphi_n"] & 0 \\ 0 \arrow[r] & F^n Y / F^{n+1} Y \arrow[r] & Y / F^{n+1} Y \arrow[r] & Y / F^n Y \arrow[r] & 0 \end{tikzcd}$$

a) Assume Φ^n inj $\forall n$

$\Rightarrow$ (by ind) Φ_n is inj $\forall n$

$\Rightarrow$ $\hat{\Phi}$ $(= \varprojlim \Phi_n)$ is inj $\longrightarrow$ inv lim left exact

b) Ass Φ^n surj $\forall n$

$\Rightarrow$ (ind) Φ_n is surj $\forall n$

Snake lemma, $\ker \Phi_{n+1} \to \ker \Phi_n$ is surj

$\Rightarrow$ $\ker(\Phi_\bullet)$ is a surj inv system

$\Rightarrow$ $\hat{\Phi}$ is surj

Eg inv lim not exact.

think $0 \to X_\bullet \to Y_\bullet \to Z_\bullet \to 0$

where $Y_n = \mathbb{Z}$ $\forall n$ $\quad Y_{n+1} \to Y_n$ is id

$Z_n = \mathbb{Z}/p^n\mathbb{Z}$ $\quad \mathbb{Z}_{n+1} \to \mathbb{Z}_n$ nat map

$X_n = \ker(Y_n \to Z_n) = p^n \mathbb{Z}$

$\varprojlim X_n = 0$ $\boxed{\text{intersection}}$, $\varprojlim Z_n$ = padic, $\varprojlim Y_n = \mathbb{Z}$

So, Y, Z surj sys by map on lim not surj!

Since X not a surj system

$\varprojlim^1 X_\bullet = \mathbb{Z}_p/\mathbb{Z}$ $\longrightarrow$ 5G term exact seq....

Prop) A ring, $\mathfrak{a}$ ideal, A $\mathfrak{a}$-adically comp.

M = filtered module that sep $\bigcap_{n \geq 1} F^n M = 0$

$gr(M)$ fin gen as $gr(A)$ mod $\Rightarrow$ M fg as A-mod.

Pf) Choose a finite set of homog gen for $gr(M)$

↳ $\overline{x_1}, \dots, \overline{x_n}$, $\deg(\overline{x_i}) = d_i$

let $x_i \in F^{d_i} M$ be a lift of $\overline{x_i}$

let P be the free A mod with n gens corr $x_1 \dots x_n$,

Filter P s.t. $gr(P)$ has gens of degs $d_1, \dots, d_n$

Map $\phi : P \to M$

Know $gr(\phi)$ surj

$\overset{\text{lem}}{\Rightarrow}$ $\hat{\Phi}$ surj

Ex of filter on A as A mod

$F^0 A = A$ $\qquad gr(A)_0 = 0$

$F^1 A = A$ $\qquad gr(A)_1 = A/\mathfrak{a}A$

$F^2 A = \mathfrak{a}A$

$\vdots$

$F^n A = \mathfrak{a}^{n-1} A$ $\qquad gr(A)_n = \mathfrak{a}^{n-1}/\mathfrak{a}^n$

$$\begin{tikzcd} \hat{P} \arrow[r, "\hat{\Phi}"] & \hat{M} \\ P \arrow[u, "\wr"] \arrow[r, "\phi"] & M \arrow[u] \end{tikzcd}$$

as P is free $P \to \hat{P}$ is $\cong$

$\hat{\Phi}$ surj $\qquad M \to \hat{M}$ inj

↳ $\ker(M \to \hat{M}) = \bigcap F^n M = 0$

$\Rightarrow$ Φ surj by lift! & $M \cong \hat{M}$ via map!

$\Rightarrow$ M fg.

Cor For A, M as abv, $gr(M)$ is a noeth $gr(A)$-mod
$\implies M$ is noeth A-mod.

Pf let $N \subset M$ be A submod.
Give N the induced filtration. $(F^n N = N \cap F^n M)$
Note N is sep

$\implies$ the map
$$F^n N / F^{n+1} N \longrightarrow F^n M / F^{n+1} M \text{ is inj } \forall n$$

$\implies gr(N) \to gr(M)$ inj of $gr(A)$ mods.

$gr(A)$ c/ noeth $\implies gr(N)$ is f.g $gr(A)$-mod

$\implies$ (prev) N is f.g A mod!

Thm If A is noeth $\implies \hat{A}$ noeth (for any choice $\mathfrak{a}$)

Pf $gr(A)$ is gen'd as an alg $/ gr(A)_0 = A/\mathfrak{a}$
by gens of $\mathfrak{a}$ thought of in $gr(A)_1 = \mathfrak{a}/\mathfrak{a}^2$

Since A noeth $\to A/\mathfrak{a} = gr(A)_0$ is noeth
as $gr(A)$ is gen'd as alg... $\implies gr(A)$ noeth.

Also, $gr(\hat{A}) = gr(A)$ bc, $\hat{\mathfrak{a}}^n / \hat{\mathfrak{a}}^{n+1} = \mathfrak{a}^n / \mathfrak{a}^{n+1}$

Abv Cor $\implies \hat{A}$ noeth $\hat{A}$-mod

Cor k field, $k[[x_1, \ldots, x_n]]$ is noeth