Last time: integral dep.

Recall: if $A \subset B$ subring. $x \in A$ is int $/A$ if it satisfies a monic poly with coeff in A.

Prop/ TFAE:

1) x is int $/A$
2) $A[x]$ is finite $/A$ (fin gen as A mod)
3) $A[x] \subset C$ where subring of B which is fin $/A$
4) $\exists$ faithful $A[x]$-mod that is f.g as A-mod.

Cor/ if $x_1, \ldots, x_n \in B$ int $/A \Rightarrow A[x_1, \ldots, x_n]$ is fin $/A$

Pf/
$$\underset{\text{(a fin}}{A \subset A[x_1]} \subset \underset{\text{fin as } x_2 \text{ int } / A[x_1]}{A[x_1, x_2]} \overset{\text{— fin}}{\subset} \cdots \subset A[x_1, x_2, \ldots, x_n]$$

$\Rightarrow A \subset A[x_1, \ldots, x_n]$ fin

Prop/ $A \xrightarrow{f} B \xrightarrow{g} C$

let $y_1, \ldots, y_n \in C$ gen as B mod

— $x_1, \ldots, x_m \in B$ — A-mod

Cl. $x_i y_j$ gen C as A-mod

Pf/ let $c \in C \Rightarrow c = \sum a_i y_i$, $a_i \in B \Rightarrow a_i = \sum e x_j$

Cor/ The set of elts of B int $/A$ forms a subring of B

Pf/ if $x, y \in B$ are int $/A \Rightarrow A[x,y]$ fin $/B$

This has $x \pm y$, $xy \Rightarrow x \pm y$, xy int $/B$

Def $A \subset B$

- The integral closure of A in B is the set of elts of B int $/A$. Its a subring of B containing A.
- A is integrally closed if its equal to its int closure.

Def A domain.

A is integrally closed if its int closed in $\mathrm{Frac}(A)$.

e.g. K = num field (fin ext of $\mathbb{Q}$)

The integral closure of $\mathbb{Z}$ in K is the ring of ints in K $\Rightarrow \mathcal{O}_K$

$K = \mathbb{Q}[i]$, $\mathcal{O}_K = \mathbb{Z}[i]$

$K = \mathbb{Q}[\sqrt{-5}]$, $\mathcal{O}_K = \mathbb{Z}[\sqrt{-5}]$

$K = \mathbb{Q}[\sqrt{-3}]$, $\mathcal{O}_K = \mathbb{Z}\left[\frac{1+\sqrt{-3}}{2}\right]$ $\Rightarrow$ root of unity

$\mathbb{Z}[\sqrt{-3}]$ not int closed as $\frac{1+\sqrt{-3}}{2}$ integral $/ \mathbb{Z}[\sqrt{-3}]$

$\Rightarrow \mathbb{Z}[\sqrt{-3}]$ not dedekind domain

Cor If $\underset{\text{integral}}{A \subset B} \underset{\text{integral}}{\subset C} \Rightarrow \underset{\text{integral}}{A/C \text{ is}}$

$\Rightarrow$ all C int $/B$

Cor $x \in C$ $\exists$ eqn

$$x^n + b_{n-1}x^{n-1} + \cdots + b_0 = 0 \qquad b_i \in B$$

Let $B' = A[b_0, \ldots, b_{n-1}]$. This is fin $/A$ as each b_i int $/A$

x is int $/B'$ $\Rightarrow B'[x]$ fin $/B'$

$\underbrace{A \subset B' \subset B'[x]}_{\text{fin}}$ $\Rightarrow B'[x]$ is fin $/A$

$\Rightarrow$ <u>x is int $/A$</u>.

Cor] $A \subset B$ B is fin $/A$ $\Longleftrightarrow$ B is integral + fin type $/A$
(f.g. as A mod)   (f.g. as A alg.)

Pf] If B fin $/A$ $\Rightarrow$ B is int & obv, f.g. as A alg

If B is fin type + int $\Rightarrow$ $B = A[x_1, \ldots, x_n]$ each x_i int $/A$

$\Rightarrow$ B is finite $/A$

eg] $\overline{\mathbb{Z}}$ = int cl. of $\mathbb{Z}$ in $\mathbb{C}$ (ring of alg ints)
$\mathbb{Z} \subset \overline{\mathbb{Z}}$ integral but not finite!

Cor] $A \subset B$, A' = integral cl. of A in B
$\Rightarrow$ A' is integrally closed in B.

Pf] A'' int cl of A' in B
$\Rightarrow$ have $\underbrace{A \subset A'}_{\text{int}} \subset A''$ $\Rightarrow$ $A \subset A''$ int
so $A'' \subset A'$

Prop] Say $A \subset B$ integral,
(a) $\mathfrak{b} \subset B$, $\mathfrak{a} = \mathfrak{b}^c$ $A/\mathfrak{a} \subset B/\mathfrak{b}$ is integral
(b) let $S \subset A$ mult sso A $\Rightarrow$ $S^{-1}A \subseteq S^{-1}B$ integral.

Pf] 1) let $\bar{x} \in B/\mathfrak{b}$ let $x \in B$ be a lift

$x = x^n + a_{n-1}x^{n-1} + \ldots + a_0 = 0$ in $a_i \in A$
reduce mod $\mathfrak{b}$ to get eqn.

2) given $\frac{x}{s} \in S^{-1}B$ wts $\frac{x}{s}$ int in $S^{-1}A$

have, $x^n + a_{n-1}x^{n-1} + \ldots + a_0 = 0$ $\Rightarrow$ monic for $\frac{x}{s}$ coeffs in $S^{-1}A$
divide by s^n $\Rightarrow$ $\left(\frac{x}{s}\right)^n + \frac{a_{n-1}}{s}\left(\frac{x^{n-1}}{s^{n-1}}\right) + \ldots + \frac{a_0}{s^n} = 0$

Behaviour of primes   going up + going down thms.

Prop 1 $A \subset B$ int ext of domains $\Rightarrow$ A a field $\Leftrightarrow$ B a field

Pf Say A a field, let $y \in B$ $(y \neq 0)$ wts, $y^{-1} \in B$

$$y^n + a_{n-1}y^{n-1} + \dots + a_0 = 0 \qquad a_i \in A$$

take n minimal $\Rightarrow a_0$ not 0 as $a_0 = 0 \rightarrow$ factor + divide by y as in dom

$$\Rightarrow \underbrace{-\frac{1}{a_0}}_{\in A \text{ as } A \text{ a field}}(y^n + \dots + a_1 y) = 1$$

$\Rightarrow zy = 1 \qquad z = -\frac{1}{a_0}(y^{n-1} + \dots + a_1) \qquad z = y^{-1} \in B.$

Now, say B is a field,

let $x \in A$ wts $x^{-1} \in A$.

Know, $x^{-1} \in B \Rightarrow x^{-1}$ int / A

$$\text{so} \quad x^{-n} + a_{n-1}x^{1-n} + \dots a_0 = 0 \quad a_i \in A$$

$$\text{mult by } x^{n-1} \quad x^{-1} + a_{n-1} + a_{n-2}x + \dots a_0 x^{n-1} = 0$$

solve for $x^{-1} \Rightarrow \boxed{x^{-1} \in A}$.

Cor $A \subset B$ int ext. $q \subset B$ pr & $p = q^c$

So q max'l $\Leftrightarrow$ p max'l

Pf $A/p \subseteq B/q$ $\underbrace{\text{integral ext}}_{\text{by prop earlier}}$ of $\overbrace{\text{domains}}^{\text{as prime}}$

$\Rightarrow$ by prev prop A/p a field iff B/q a field

$\therefore$ p max'l $\Rightarrow$ q max'l.

Cor) $A \subset B$ int ext. Say $q \subset q'$ pr of B s.t.

$$\underbrace{q^c = (q')^c}_{p} \implies q = q'$$

Pf) $A_p \subset B_p$ is integral by earlier prop

$n = qB_p$ & $n' = q'B_p$

$n^c = (n')^c = \underbrace{pA_p}_{\text{this is max'll}}$] $\to$ localization & contr work well

$\therefore$ n, n' are max'll but $n \subset n'$ (as $q \subset q'$) $\implies$ $n = n'$

$\Downarrow$

$q = q'$ as $q = n^c$, $q' = (n')^c$

eg) 1) $\exists$ $A \subset B$ integral s.t.

$\mathrm{Spec}(B) \xrightarrow{\text{contr}} \mathrm{Spec}(A)$ not inj (also says inj but w cont.)

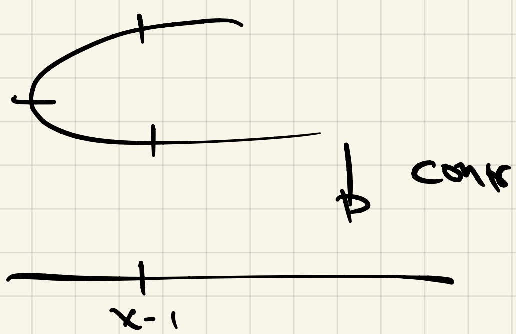
i) $A = \mathbb{Z}$, $B = \mathbb{Z}[i]$

$5 = (2+i)(2-i)$ so, $(2+i)^c = (2-i)^c = 5$.

ii $A = \mathbb{C}[x]$ $B = \mathbb{C}[x,y]/(y^2 - x)$

$A \to B$ finite b/c satisfies monic eqn.

Spec B:



$(y-1)^c = (y+1)^c = (x-1)$

2) $\exists$ $A \subset B$ non-integral & $q \not\subset q'$ s.t. $q^c = (q')^c$

$\mathbb{C} \subset \mathbb{C}[x]$

$q = (0)$ & $q' = (x)$ $\implies$ everything ctr to 0.

if have int ext $i: A \hookrightarrow B$ get $f^*: \mathrm{Spec}\, B \to \mathrm{Spec}\, A$

$\mathfrak{q} \mapsto \mathfrak{q}^c$

map says fibre has no nontrivial containments.

↳ fibres of map are Krull dim'll 0 → no containment.

Thm) Spec $A \subset B$ int ext, $\mathfrak{p} \subset A$ prime $\Rightarrow \exists\, \mathfrak{q} \subset B$ so $\mathfrak{q}^c = \mathfrak{p}$

Pf) $A_\mathfrak{p} \subset B_\mathfrak{p}$ is integral.

let $\mathfrak{n}$ = max'll ideal $B_\mathfrak{p}$

Not $\mathfrak{n}^c$ is max'll of $A_\mathfrak{p} \Rightarrow \mathfrak{n}^c$ is max'll ideal of $A_\mathfrak{p}$ (fact abt integral extn)

Since $A_\mathfrak{p}$ local, w max'll ideal $\mathfrak{p}A_\mathfrak{p} = \mathfrak{p}^e$

$\Rightarrow \mathfrak{n}^c = \mathfrak{p}A_\mathfrak{p}$

$$\begin{tikzcd}
\mathfrak{p} & A \arrow[r, phantom, "\subset"] \arrow[d] & B \arrow[d] & \mathfrak{q} \\
& A_\mathfrak{p} \arrow[r, phantom, "\subset"] & B_\mathfrak{p} & \\
& \mathfrak{p}A_\mathfrak{p} \arrow[uul, bend left, "\text{cont}"] & \mathfrak{n} \arrow[l, "\text{cont}"] \arrow[uur, bend right, "\text{cont}"'] &
\end{tikzcd}$$

let $\mathfrak{q}$ = cont of $\mathfrak{n}$ in B

The sq will commute

$\Rightarrow \mathfrak{q}^c = \mathfrak{p}$

& $\mathfrak{q}$ is pr (cont of prime)

Cor) $f: A \to B$ integral ring homo → B int / img (A).

$f^*: \mathrm{Spec}\, B \to \mathrm{Spec}\, A$ is surj

Going up thm)

let $A \subset B$ int extn $\mathfrak{p}_1 \subset \dots \subset \mathfrak{p}_n$ pr of A &

$\mathfrak{q}_1 \subset \dots \subset \mathfrak{q}_m$ $m \le n$ s.t. $\mathfrak{q}_i^c = \mathfrak{p}_i$ (pr).

$\Rightarrow \exists$ completion of chain so $\mathfrak{q}_i^c = \mathfrak{p}_i$ $\forall i \le n$

Pf| Suffices to show case $m=1,\ n=2$ & iterate

$p_1 \subset p_2$ & q_1 w $p_1 = q_1^c$, goal: q_2

$\overline{A} = A/p_1$, $\overline{B} = B/q_1$

& by earlier $\overline{A} \subset \overline{B}$ int ext of domains.

$\overline{p_2} \subset \overline{A}$ is prime ideal

by $\boxed{\text{thm}}$. $\exists\ \overline{q_2} \subset \overline{B}$ so $\overline{q_2}^c = \overline{p_2}$

let $\overline{q_2}$ corr to $q_2 \subset B$ by corr thm

& $q_2^c = p_2$

Cor| If $A \subset B \implies$ krull dim'll are equal.

Pf| ◦ Given $p_1 \subsetneq p_2 \subsetneq \cdots \subsetneq p_n$ in A

going up $\implies \exists$ similar chain in B

↳ also get proper cont of these

as $q_i^c = p_i \subsetneq p_{i+1} = q_{i+1}^c$

so $q_i \subsetneq q_{i+1}$

◦ given chain in B

$q_0 \subset \cdots \subset q$