Lec 11: artinian $\iff$ noetherian + dim 0

Thm) (Struct of Art rings)

A artinian $\Rightarrow A = A_1 \times \cdots \times A_n$, A_i = local Art ring.

Crl) If $\mathfrak{a}_1, \ldots, \mathfrak{a}_n \in A$ ← any comm ring.

w/ $\mathfrak{a}_i + \mathfrak{a}_j = A \quad \forall i \neq j$

$\Rightarrow$ i) $\mathfrak{a}_1 \cdots \mathfrak{a}_n = \mathfrak{a}_1 \cap \ldots \cap \mathfrak{a}_n$

ii) $A/\mathfrak{a}_1 \cdots \mathfrak{a}_n \cong A/\mathfrak{a}_1 \times \cdots \times A/\mathfrak{a}_n$

Classic CRT

$\mathbb{Z}/nm \cong \mathbb{Z}/n \oplus \mathbb{Z}/m$

if n, m copr.

Pf) $n=2$ gen case by ind.

So, $\mathfrak{a}+\mathfrak{b} = (1)$ obs for any ideal

$(\mathfrak{a}+\mathfrak{b})(\mathfrak{a}\cap\mathfrak{b}) \subset \mathfrak{a}\mathfrak{b} \subset \mathfrak{a}\cap\mathfrak{b}$

if $(\mathfrak{a}+\mathfrak{b}) = 1 \Rightarrow \mathfrak{a}\mathfrak{b} = \mathfrak{a}\cap\mathfrak{b}$

$\exists$ ring homo, $\varphi : A \longrightarrow A/\mathfrak{a} \times A/\mathfrak{b}$

$x \longmapsto (x, x)$

$\ker\varphi = \mathfrak{a}\cap\mathfrak{b} \Rightarrow$ ind inj ring homo $\bar{\varphi} : A/\mathfrak{a}\cap\mathfrak{b} \to A/\mathfrak{a} \times A/\mathfrak{b}$

wts $\bar{\varphi}$ surj since $\mathfrak{a}+\mathfrak{b} = 1 \Rightarrow x+y = 1$ for $x \in \mathfrak{a}, y \in \mathfrak{b}$

$\Rightarrow x = 1-y = \begin{cases} 0 & \text{mod } \mathfrak{a} \\ 1 & \text{mod } \mathfrak{b} \end{cases} \Rightarrow \bar{\varphi}(x) = (0,1)$, $\bar{\varphi}(y) = (1,0)$ $\Rightarrow$ done! surj

$\bar{\varphi}(ax+by) = (b,a)$

Pf of Thm Let $m_1, \ldots, m_n$ be the ideals of A

Since radical is nilp $\Rightarrow \prod_{i=1}^{n} m_i^k = 0$ for suff large.

m_i^k, m_j^k coprime for $i \neq j$, say $m_i^k + m_j^k \subseteq m$ ← $\boxed{\text{max'l}}$

$\Rightarrow m_i^k \subset m$

$\Rightarrow m_i \subset m$ as pr

$\therefore m_i + m_j \subset m \Rightarrow (1) \subset m$ $\boxed{\text{oops}}$ $\Rightarrow m_i^k + m_j^k = (1)$

[CRT] $A/\prod_{i=1}^{n} m_i^k \cong A/m_1^k \times \cdots \times A/m_n^k$

$\quad\;\; \| $

$\quad\;\; A$

Each A_i artinian b/c its a quot of art ring.

A_i local $\to$ img of m_i in A_i is its uniq max ideal.

Rmk| The decomp of A is unique up to isom / perm.

Primary Decomp (Noether + Lasker)

Def| An ideal $q \subset A$ is called primary if $xy \in q \Rightarrow x \in q$ or $y^n \in q$

$\iff$ in A/q every zero divisor is nilp.

Prop| q primary $\Rightarrow$ rad(q) is prime

let $xy \in$ rad(q) $\Rightarrow (xy)^n \in q \Rightarrow x^n \in q$ or $y^{nm} \in q$

$\Rightarrow x$ or $y \in$ rad(q)

Def| For a pr ideal p, a p primary ideal is a primary ideal q s.t. rad$(q) = p$

Rmk| If q is p-primary $\Rightarrow p$ is the unique min elt in the set of prime cont q

eg| 1) $(p^n) \subset \mathbb{Z}$ is (p)-primary

2) $(x, y^2) \subset \mathbb{C}[x,y]$ is primary $\frac{\mathbb{C}[x,y]}{(x,y^2)} \cong \frac{\mathbb{C}[y]}{(y^2)}$

$\hookrightarrow$ rad$(x,y^2) = (x,y) = m$ $\quad$ $\hookrightarrow$ every zero div is nilp.

have, $m^2 \subsetneq (x,y^2) \subsetneq m$

$\Rightarrow (x,y^2)$ not a power of pr. ideal.

Warning! Not true that powers of prime are primary in gen.
↳ if radical of ideal $\mathfrak{r}$ prime $\not\Rightarrow$ $\mathfrak{r}$ is primary!

Prop) If $\mathrm{rad}(\mathfrak{r}) = \mathfrak{m}$ $\Rightarrow$ $\mathfrak{r}$ is primary

Cor) powers of max'l ideal primary!

Pf) $A/\mathfrak{r}$ is a local ring w/ $\bar{\mathfrak{m}}$ = img of $\mathfrak{m}$

Any elt of $A/\mathfrak{r}$ not in $\bar{\mathfrak{m}}$ is a unit.

Any elt of $\bar{\mathfrak{m}}$ is nilp. (as $\mathrm{rad}(\mathfrak{r}) = \mathfrak{m}$),

$\therefore$ any zero divisor of $A/\mathfrak{r}$ not unit $\Rightarrow$ in $\bar{\mathfrak{m}}$ $\Rightarrow$ $\boxed{\text{nilp}}$.

$\Rightarrow$ $\mathfrak{r}$ primary.

Prop) if $q_1, \dots, q_n$ are $\mathfrak{p}$-primary, so is $\bigcap q_i = q$

Pf) $\mathrm{rad}(q) = \bigcap_{i=1}^{n} \mathrm{rad}(q_i) = \mathfrak{p}$ ← $\boxed{\text{in gen true}}$

Say $xy \in q$ & $x \notin q$ $\Rightarrow$ $\exists$ some i so $x \notin q_i$
but $xy \in q_i$

$\Rightarrow$ as q_i primary $y^n \in q_i$

$\Rightarrow y \in \mathrm{rad}(q_i) = \mathfrak{p} = \mathrm{rad}(q)$

$\Rightarrow y^n \in q$ for some $n \geq 1$ $\Rightarrow$ $\boxed{q \text{ primary}}$.

Def) • A primary decomp of some ideal $\mathfrak{r}$ is an expr.

$\mathfrak{r} = \bigcap_{i=1}^{n} q_i$ where q_i is primary.

• We say $\mathfrak{r}$ is decomposable if abv exists

• We say the abv decomp minimal if (i) $\mathrm{rad}(q_i)$ dist
(ii) $q_i \not\supseteq \bigcap_{j \neq i} q_j$ $\forall i$

Comments 1) Main questions: $\exists$ & uniqueness

↳ existence: in general not all ideals decom'able

but always in Noetherian!

↳ uniqueness: basically never true in a strong sense, but some aspects are unique.

2) $A = \mathbb{Z}$ $\quad n = p_1^{e_1} \cdots p_n^{e_n}$ $\quad p_i$ dist & $e > 0$

$\Rightarrow$ $(n) = (p_1^{e_1})(p_2^{e_2}) \cdots (p_n^{e_n})$ is a min prim decomp.

3) If A decomposable $\Rightarrow$ exist minimal one,

↳ by previous prop can combine the p-primary over common p (common radical) (i)

+ throw out extra! (ii)

Def If $R \subset A$ & $x \in A$ $\quad (R : x) = \{ y \in A \mid xy \subset A \}$ $= \mathrm{ann}_A(x \in A/R)$

Not right

↳ can use ideal here!

Lemma Say q is p-primary & $x \in A$,

(a) If $x \in q \Rightarrow (q : x) = (1)$

(b) If $x \notin q \Rightarrow (q : x)$ is p primary

(c) If $x \notin p \Rightarrow (q : x) = q$

Pf (a) obvious, as $x \in q \Rightarrow yx \in q \Rightarrow y \in (q : x)$ $\forall y \in A$

(c) Say $y \in (q : x) \Rightarrow xy \in q$

As $x \notin \mathrm{rad}(q) \Rightarrow y \in q$ as primary.

So $(q : x) \subset q$ & reverse always true.

(b) Say $y \in (q:x) \Rightarrow xy \in q \xrightarrow[x \notin q]{} y^n \in q$

$\Rightarrow y \in \mathfrak{p}$

$q \subset (q:x) \subset \mathfrak{p}$ so rad sandwich $\Rightarrow$ $\text{rad}(q:x) = \mathfrak{p}$

Say $yz \in (q:x)$ & $y \notin \mathfrak{p}$ wts, $z \in q_x$

but, $xyz \in q$ as q is primary $xy \notin \text{rad}(q) = \mathfrak{p}$

$\Rightarrow xz \in q \Rightarrow z \in (q:x)$ $\Leftrightarrow y^n \notin q \; \forall n$

$\therefore (q:x)$ primary.

First Uniqueness Thm

Say $\mathfrak{a} = \bigcap_{i=1}^{n} q_i$ is a min pr decomp w $\mathfrak{p}_i = \text{rad}(q_i)$

$\Rightarrow \mathfrak{p}_i$ are exactly the prime ideals of the form,

$\text{rad}(\mathfrak{a}:x)$ w $x \in A$

(($\mathfrak{p}_i$ are pr radicals of annihilator of elts of $A/\mathfrak{a}$))

Cor $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ are unique (i.e, ind of choice of dec)

Pf $x \in A$ $\quad (\mathfrak{a}:x) = (\bigcap_{i=1}^{n} q_i : x) \overset{\text{check}}{=} \bigcap_{i=1}^{n} (q_i : x)$

$\Rightarrow \text{rad}(\mathfrak{a}:x) = \bigcap_{i=1}^{n} \text{rad}(q_i : x) \underset{\substack{\text{prop} \\ 3 \text{ cases} \\ (a)+(b)}}{=} \bigcap_{\substack{i=1 \\ x \notin q_i}}^{n} \mathfrak{p}_i$

rad comm w/ fin $\cap$

So, if $\text{rad}(\mathfrak{a}:x)$ prime, it must be one of $\mathfrak{p}_i$

$\Leftrightarrow$ if a prime is int of pr it is one of them in gen.

Given $1 \le i \le n$, $\exists x \notin q_i$ so $x \in \bigcap_{j \ne i} q_j$ by minimality

by above, $\text{rad}(\mathfrak{a}:x) = \mathfrak{p}_i$ $\Rightarrow$ all the pr $\text{rad}(\mathfrak{a}:x)$ app.

Defn | Let $\Omega = \bigcap_{i=1}^{n} q_i$ be min prime decomp w/ rad $q_i = p_i$

- $p_1, \dots, p_n$ are said to belong to Ω or be assoc to Ω
- Ω is primary $\iff n=1$, $\exists!$ assoc prim.
- Can have containment among p_i but not eq by minimality

↳ the minimal elts of $\{p_1, \dots, p_n\}$ are called the isolated or minimal pr assoc to Ω

↳ the non minimal elts are the emb. primes.

Eg | $\Omega = (x^2, xy) \subset \mathbb{C}[x,y] = A$

Ω not primary as y is 0 div but not nilp.

$\Omega = (x) \cap (x,y)^2$ → primary as power of max'l ideal

↳ is a minimal primary decomposition.

$\Longleftrightarrow$ primes assoc to Ω are (x) & (x,y)

w/ (x) isolated

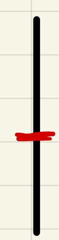
& (x,y) embedded

$V_m(\Omega) \subset \mathrm{MaxSpec}(\mathbb{C}[x,y]) \cong \mathbb{C}^2$

→ def by $x^2 = 0 \Rightarrow x = 0$ } vertical axis → lose infos

$xy = 0$ → redundant

Picture of $V_m(\Omega)$



→ emb part of x-axis.

Rmk | $\Omega = (x) \cap (x^2, y)$ is min primary decomp. → uniqueness fails.