If $x \in \mathbb{H}^2$

let ν_x be measure on $\partial\mathbb{H}^2$ induced by identifying w/ $T^1_x\mathbb{H}^2$

$\nu_0 =$ leb measure

$$\nu_{\gamma(x)} = \gamma_* \nu_x = e^{b_z(x, \gamma(x))} \nu_x \quad \sim (\gamma^{-1})'(x)$$

$T^1\mathbb{H}^2 = (\partial\mathbb{H}^2)^2 \times \mathbb{R}$ via Hopf param based at x_0

$$m_{x_0}(w, z, t) = e^{2G_{x_0}(w,z)} d\nu_{x_0}(w)\, d\nu_{x_0}(z)\, dt(s)$$

$$\gamma_* m_{x_0}(w,z,s) = e^{2G_{x_0}(\gamma^{-1}(w), \gamma^{-1}(z))} \cdot e^{b_w(x_0, \gamma(x_0))} e^{b_z(x_0, \gamma(x_0))} d\nu_{x_0}\, d\nu_{x_0}\, dt$$

$$\exp\left(2G_{x_0}(\gamma^{-1}(w), \gamma^{-1}(z)) + b_w(x_0, \gamma(x_0)) + b_z(x_0, \gamma(x_0))\right)$$

$y \in \overline{wz}$

$$b_{\gamma^{-1}(w)}(x_0, \gamma^{-1}(y)) + b_{\gamma^{-1}(z)}(x_0, \gamma^{-1}(y))$$
$$+ b_w(x_0, \gamma(x_0)) + b_z(x_0, \gamma(x_0))$$

busemann fn

$$b_w(\gamma(x_0), y) + b_z(\gamma(x_0), y)$$
$$+ \ldots$$

$$b_w(\gamma(x_0), y) + b_w(x_0, \gamma(x_0))$$
$$- b_w(x_0, y)$$

$$\therefore = 2G_{x_0}(w,z) \longrightarrow \underline{\text{leb measure}}$$

If $\Gamma \subseteq PSL(2,\mathbb{R})$ discrete & $x_0 \in \mathbb{H}^2$

<u>Poincaré Series</u>

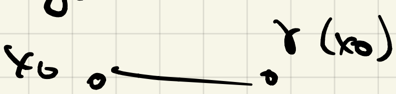
$$\mathcal{Q}_\Gamma(x_0,s) = \sum_{\gamma\in\Gamma} e^{-s\, d(x_0,\gamma(x_0))}$$

$\exists\, \delta(\Gamma)$ so if $s<\delta(\Gamma)$ then $\mathcal{Q}_\Gamma$ div

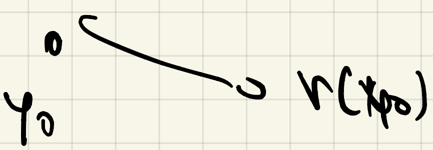
& $s>\delta(\Gamma)$ then $\mathcal{Q}_\Gamma$ conv

$$\delta(\Gamma)\in[0,\infty]$$

If $y_0\in\mathbb{H}^2$ then



$$e^{-2s\, d(x_0,y_0)}\,\mathcal{Q}_\Gamma(x_0,s) \le \mathcal{Q}_\Gamma(y_0,s) \le e^{2s\, d(x_0,y_0)}\,\mathcal{Q}_\Gamma(x_0,s)$$



Convergence pattern is same,

so $\delta(\Gamma)$ <u>well def</u>

let $\Gamma(x_0,R)=\{\gamma\in\Gamma \mid d(x_0,\gamma(x_0))\le R\}$

$N(x_0,R)=\#\Gamma(x_0,R)$

Lemma 1 $\quad \limsup_{R\to\infty} \dfrac{\log N(x_0,R)}{R} \le 1$

$X = \mathbb{H}^2/\Gamma$

Let $\varepsilon = \mathrm{inj}_X(\pi(x_0)) > 0$

$\Rightarrow$ all ε balls about pts in $\Gamma(x_0)$ are disj

The collection of ε-balls about points in $\Gamma(x_0,R)(x_0)$ all lie in ball of radius $R+\varepsilon$ about x_0

$$N(x_0,R)(2\pi\cosh\varepsilon - 2\pi) \le 2\pi\cosh(R+\varepsilon) - 2\pi$$

$$N(x_0,R) \le \frac{2\pi\cosh(R+\varepsilon)}{2\pi\cosh(\varepsilon - 2\pi)} \le k e^{R+\varepsilon}$$

So $\quad h \le \limsup \dfrac{\log k e^{R+\varepsilon}}{R} = 1$

Fact 1 If $X = \mathbb{H}^2/\Gamma$ is a closed surface, $h = 1$

Let $D = \mathrm{diam}(X)$

Then D-balls about orbit pts cover $\mathbb{H}^2$

$$N(x_0, R+D)(2\pi\cosh D - 2\pi) \ge 2\pi\cosh R - 2\pi$$

$$N(x_0, R+D) \ge \frac{2\pi\cosh R - 2\pi}{2\pi\cosh D - 2\pi} > Le^R \Rightarrow \underline{h \ge 1}.$$

If X is conv. co-compact,
Same arg shows h is the exp rate of growth of vol $(CH(L_\Gamma) \cap B(R, x_0))$

Upshot: "X is a compact hyperbolic mfd of dim $n+1$".

Lemma: $h(\Gamma) = \delta(\Gamma)$.

so if $s > h(\Gamma)$ then Poincare series converges;

if $s < h(\Gamma)$ then poin ser diverges!

$\Rightarrow h(\Gamma) = \delta(\Gamma)$ weak.

A measure μ on $\partial \mathbb{H}^2$ is a Patterson Sullivan measure for Γ is $\operatorname{supp}(\mu) = \Lambda(\Gamma)$ $\exists x_0$ s.t.
$\forall \gamma_* \mu(z) = e^{-\delta b_z(x_0, \gamma(x_0))} \mu(z)$

Fact: $\exists$ patterson sullivan measure.

Assume $\mathcal{Q}_\Gamma(x_0, O(\Gamma)) = +\infty$ (say Γ is of div type)

in $\mathrm{Isom}(\mathbb{H}^2)$

Γ fin gen $\Rightarrow \Gamma$ div type.

in $\mathrm{Isom}\ \mathbb{H}^3$

$\exists\, \Gamma$ fin gen but not div type.

Pf ......