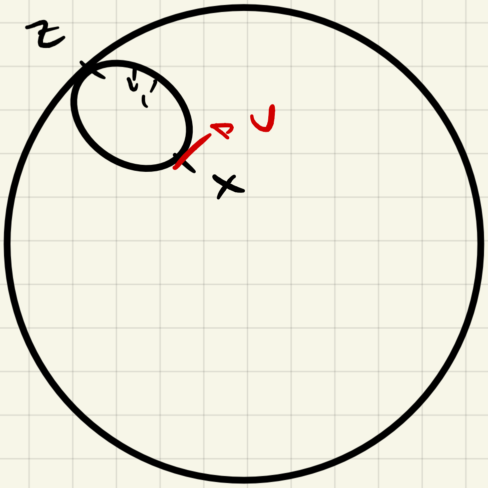
The horocycle flow $\{h_s\}$

want to def $h_s(x,\vec{v}) \longrightarrow (w, z, t)$



$\exists$ horocycle $H_{z,t}$ or $H_{z,x}$

based at z thru x

Let $\beta_{x,z} : \mathbb{R} \to H_{z,x}$

unit speed param of horocycle

$\beta_{x,z}(0) = x$ $\{\beta_{x,z}'(0), \vec{v}\}$ are pos or basis.

$h_s(\vec{x}, v) = (\beta_{x,z}(s),$ unit vec $\perp$ $\beta_{x,z}'(s)$ towards $z)$

By def, $h_s \circ \gamma = \gamma \circ h_s$ $\forall s \in \mathbb{R}$, $\gamma \in PSL(2,\mathbb{R})$

$h_s(i, [\begin{smallmatrix}0\\1\end{smallmatrix}]) = (s+i, [\begin{smallmatrix}0\\1\end{smallmatrix}])$

in $\mathbb{H}^2$ model. $\beta_{i,\infty}(s) = s+i$ as horocycle is $\longrightarrow$ euc line.

$\phi_t(i, [\begin{smallmatrix}0\\1\end{smallmatrix}]) = (e^t i, [\begin{smallmatrix}0\\e^t\end{smallmatrix}])$

$\phi_t(h_s(i), [\begin{smallmatrix}0\\1\end{smallmatrix}]) = \phi_t(s+i, [\begin{smallmatrix}0\\1\end{smallmatrix}]) = (e^t i + s, [\begin{smallmatrix}0\\e^t\end{smallmatrix}])$

$h_s(\phi_t(i, [\begin{smallmatrix}0\\1\end{smallmatrix}])) = h_s(e^t i, [\begin{smallmatrix}0\\e^t\end{smallmatrix}]) = (e^t i + se^t, [\begin{smallmatrix}0\\e^t\end{smallmatrix}])$

$\to$ don't quite commute

$h_{se^{-t}}(\phi_t(i, [\begin{smallmatrix}0\\1\end{smallmatrix}])) = (e^t i + s, [\begin{smallmatrix}0\\e^t\end{smallmatrix}])$

$= \phi_t(h_s(i), [\begin{smallmatrix}0\\1\end{smallmatrix}])$

(so commute if careful.

We know $\phi_t \circ h_s = h_{se^{-t}} \circ \phi_t$ $\forall s, t$

Space,

$$\phi_t(i,[0;1]) \qquad \phi_t(h_s(i,[0;1]))$$

let $b: T^1\mathbb{H}^2 \to \mathbb{H}^2$ be base pt

$$d\left(\phi_t(i,[0;1]),\ \phi_t(h_s(i,[0;1]))\right) \to 0$$

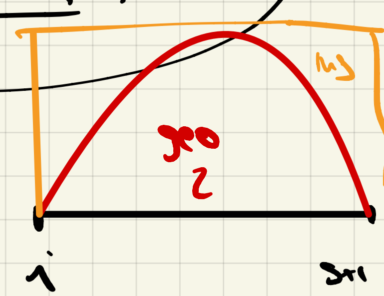
as $t \to \infty$

s fixed

also

$$d\left(\phi_t(x,\vec{v}),\ \phi_t h_s(x,\vec{v})\right) \to 0 \quad \text{as } t \to \infty$$

$\forall (x,v) \in T^1\mathbb{H}^2$ and any $s \in \mathbb{R}$.



$$d(i, s+i) = \sinh^{-1}\left(\frac{|s|}{2}\right)$$

if $s \sim 0$, $2\sinh^{-1}\left(\frac{s}{2}\right) \sim s$

if $s \sim \infty$, $2\sinh^{-1}\left(\frac{s}{2}\right) \sim 2\log|s|$

Consider,

$$d((x,\vec{v}),(y,\vec{u})) = \int_{-\infty}^{\infty} e^{-|t|}\, d(b(\phi_t(x,v)), b(\phi_t(y,\vec{u})))\, dt$$

$< d(x,y) + 2t$.

$\hookrightarrow \Gamma$-invariant & smooth!

$$d((w,z,r),(w,z,s)) = \int_{-\infty}^{\infty} e^{-|t|}\, |s-r|\, dt = 2|s-r|$$

$\phi_t(w,z,r) = (w,z,r+t)$. $\quad \phi_t(w,z,s) = (w,z,s+t)$.

$\to$ same flow line

eg of earlier

$(i, [\,^0_1\,])$ $\quad (e^r i, [\,^0_{e^r}\,])$ distance now is $|r|$

ϕ_t $\quad (e^t i, [\,^0_{e^t}\,])$ $\quad (e^{r+t} i, [\,^0_{e^{r+t}}\,])$ ϕ_t

$$\partial(e^t i, e^{r+t} i) = \log e^{r+t} - \log e^t = |r|$$

flow is dist pres over flow lines.

K

$$\partial\left((i, [\,^0_1\,]), (s+i, [\,^0_1\,])\right) = 4 \log\left(\frac{|s| + \sqrt{s^2+4}}{2}\right)$$

as $\quad s \to 0 \quad \boxed{\partial \sim 2s}$,

As $\quad s \to \infty \quad \partial \sim 4 \log s$

$$\partial(\phi_t(x,\vec{v}), \phi_t(h_s(x,\vec{v})))$$

$$= \partial(\phi_t(x,\vec{v}), h_{e^{-t}s}(\phi_t(x,\vec{v})))$$

$$= 4 \log\left(\frac{|e^{-t}s| + \sqrt{e^{-2t}s^2 + 4}}{2}\right)$$

$$\to 0 \quad \text{as } t \to \infty$$

$\phi : \mathbb{R} \times Y \to Y$ a flow

Def The stable set of $Y \ni y$

$$\mathrm{Stable}(\phi, y) = \{x \in Y \mid \lim_{t\to\infty} \partial(\phi_t(x), \phi_t(y)) = 0\}$$

$$\text{Stable}(\Phi, (x, v)) = \{ u_s(x, \vec{v}) \mid \forall s \in \mathbb{R} \}$$