Lec 3

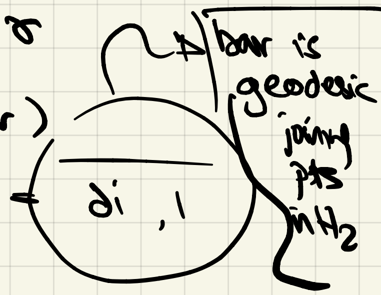
M	3-4
Tu	2-3
F	1-2

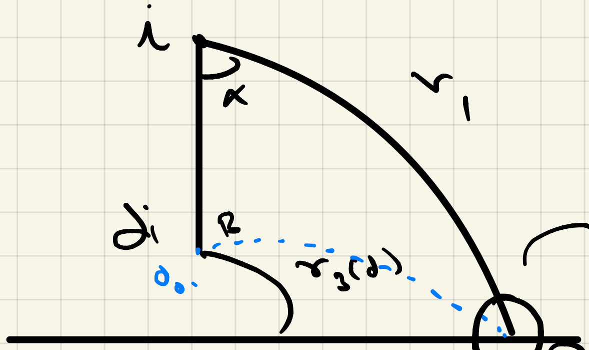
Fact 1 If $\alpha, \beta, \gamma \geq 0$, $\alpha+\beta+\gamma < \pi$

$\exists !$ up to $\cong$ Δ in $\mathbb{H}^2$ w/ angles α, β, γ

WLOG $\alpha, \beta > 0$ (handled other case earlier)

↳ put 1 endpt at i, $\lambda \in (0,1)$, $s_2(\lambda) =$



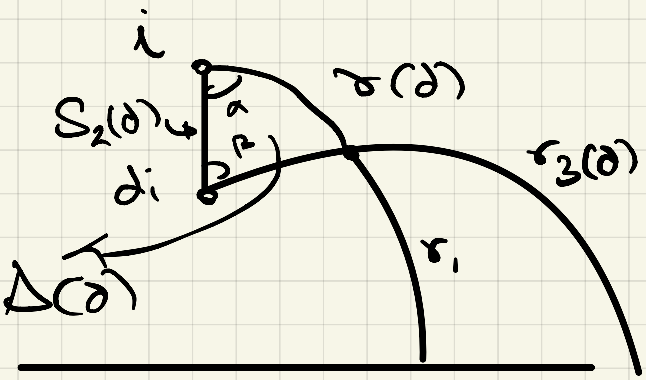


$\exists\, \lambda_0$ s.t. $\gamma_3(\lambda_0)$ is tang to γ_1 at $\partial\mathbb{H}^2$

→ this is Δ w angles $\alpha, \beta, 0$

→ angle 0 since tangent at $\partial\mathbb{H}^2$.

If $\lambda \in (\lambda_0, 1)$



$\gamma(\lambda)$ varies ctsly in λ,

$$\lim_{\lambda \to \lambda_0} \gamma(\lambda) = \gamma(\lambda_0) = 0$$

$$\lim_{\lambda \to \lambda} \gamma(\lambda) = \pi - (\alpha+\beta)$$

So, all values of γ b/w 0 & $\pi - (\alpha+\beta)$ occurs (IVT).

Spse, $\lambda_1 > \lambda_2 \Rightarrow \Delta(\lambda_1) \subsetneq \Delta(\lambda_2)$

$\Rightarrow \text{area}(\Delta(\lambda_1)) < \text{area}(\Delta(\lambda_2))$

$\Rightarrow \pi - (\alpha+\beta+\gamma(\lambda_1)) < \pi - (\alpha+\beta+\gamma(\lambda_2))$

$\Rightarrow \gamma(\lambda_1) > \gamma(\lambda_2)$

$\Rightarrow$ all values of γ b/w 0, $\pi - \alpha - \beta$ occur exactly 1 time (monotone)

Notation

$\overline{xy}$ geodesic joining x, y in $\mathbb{H}^2$

if y in $\partial\mathbb{H}^2$, $\overrightarrow{xy}$

if $z \in \partial\mathbb{H}^2$ too, $\overleftrightarrow{yz}$

Classification of Isometries

$A \in PSL(2,\mathbb{R})$ is conjugate to either, 2 jordan normal.

$\pm\begin{bmatrix} \sqrt{\lambda} & 0 \\ 0 & \frac{1}{\sqrt{\lambda}} \end{bmatrix}$ $\lambda > 1$ tr is $\pm(\lambda + \frac{1}{\lambda}) \in \pm(2,\infty)$ hyperbolic isom,

$\pm\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, tr is 2 parabolic isom.

$\pm\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ tr is $\pm 2\cos(\theta) \in (-2,2)$ elliptic isom

Obs! tr enough to classify!

$\begin{bmatrix} \sqrt{\lambda} & 0 \\ 0 & \sqrt{\frac{1}{\lambda}} \end{bmatrix} \rightsquigarrow H_\lambda$ $\quad z \longmapsto \lambda z$ fixes $0, \infty \in \mathbb{H}^2$ & pres y axis.

trans y axis to $\log\lambda$

$d(bi, \lambda bi) = \log\left(\frac{\lambda bi}{bi}\right) = \log\lambda$

If $pr : \mathbb{H}^2 \to \mathbb{H}^2$ is hyperbolic orth proj to y-axis

$\Rightarrow pr(re^{i\theta}) = ri$

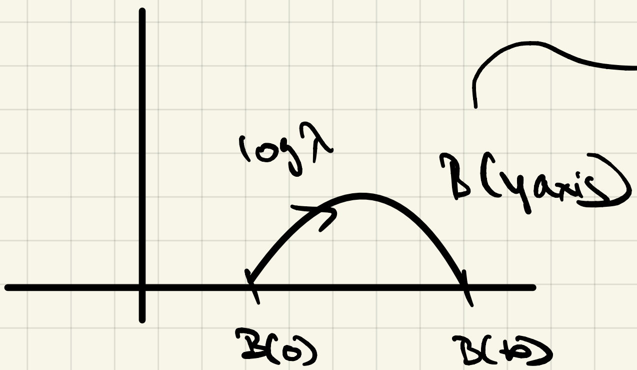
$d(pr(z), pr(w)) \leq d(z,w)$ with eq $\iff$ z, w on $y$axis.

also, $pr \circ H_\lambda = H_\lambda \circ pr$

So, if $z \in \mathbb{H}^2$, $d(z, H_\lambda(z)) \geq \log \lambda$
with eq iff z on y axis.

Spse, $A = B \circ H_\lambda \circ B^{-1}$ w/ $B \in PSL(2,\mathbb{R})$

A fixes $B(0)$ & $B(\infty)$
pres $B(y\text{-axis})$ ⟶ The axis of A



⟶ translates a distance $\log\lambda$ in axis of A

$\therefore$ $d(z, A(z)) \geq \log \lambda$
if $z \in$ axis(A).

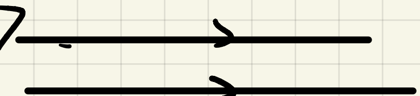
Now $P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $P(z) = z+1$ P fixes $\infty \in \partial\mathbb{H}^2$
but pres no geodesic (must pres 2 pts)

$$d(xi, xi+1) \propto \frac{1}{x}$$

As $x \to \infty$ $d(xi, xi+1) \to 0$

$\inf_{z \in \mathbb{H}^2} d(z, P(z)) = 0$ but inf not achieved as no pt in $\mathbb{H}^2$ is fixed!

Horocircles preserved



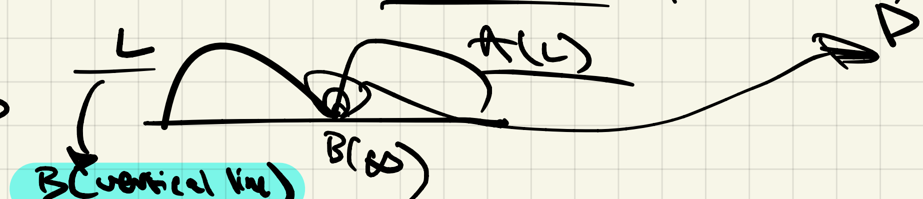
Spse parabolic.

$\Rightarrow A = BPB^{-1}$

A fixes $B(\infty)$ and nothing else.

$\inf_{z \in \mathbb{H}} d(z, A(z)) = 0$ not achieved.

Horocircles ⟶



Now elliptic

$$E_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$ fixes i & only i

$$E_\theta(z) = \frac{\cos\theta\, z + \sin\theta}{-\sin\theta + \cos\theta}$$

$\curvearrowright$ hyperbolic rotation is traversing the radial geodesic.

$$E'_\theta(z) = e^{2i\theta}$$

So it acts by hyperbolic rotation by 2θ.

$A^{\neq I} \in PSL(2,\mathbb{R})$ is hyp $\iff$ A fixes 2 pts in $\partial\mathbb{H}^2$

$\iff \inf_{z\in\mathbb{H}^2} d(z, A(z)) > 0$ & achieved.

$\iff |\operatorname{tr} A| > 2$

A is parab $\iff$ A fixes exactly 1 pt $\partial\mathbb{H}^2$

$\iff \inf_{z\in\mathbb{H}^2} d(z, A(z)) = 0$ not achieved

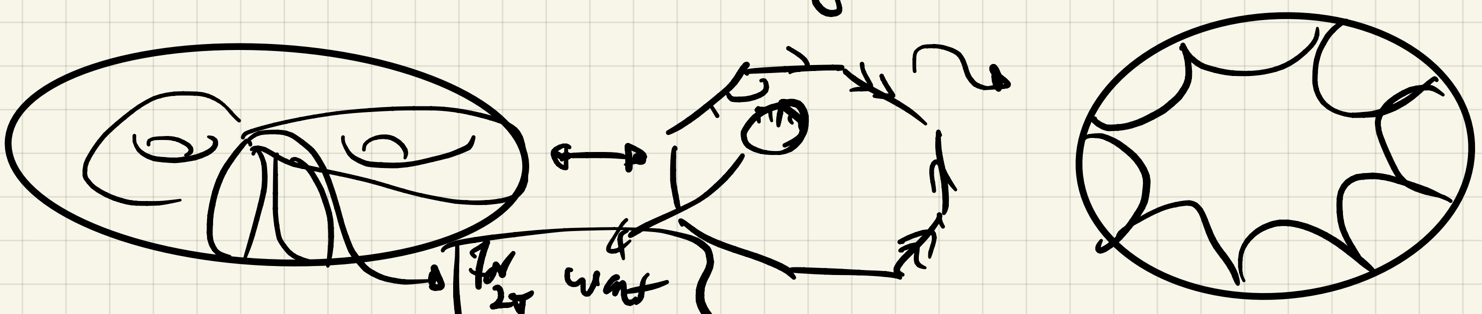
$\iff |\operatorname{tr} A| = 2$

A is elliptic. $\iff$ A fixes 1 pt in $\mathbb{H}^2$

$\iff$ inf $= 0$ is achieved

$\iff |\operatorname{tr} A| \in (0,2)$

Def A (riemannian) metric surface is hyp pt it is complete & locally $\cong$ to $\mathbb{H}^2$.

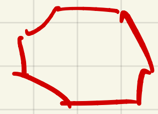


latter ⇒ hyperbolic octagon w all angles

↳ if we collapse to pt get euclidean w/ Σ is 2π

→ <u>achieve</u>, $\pi/4$ some<u>where</u>!

<u>Area</u>, $6\pi - 8(\pi/4) - 4\pi$ | $\boxed{\text{Gauss Bonnet}}$

Every hyperbolic surface genus 2 ⇒ obt from hyp  !!