Γ discr. t.f, non-elt $\subset PSL(2,\mathbb{R})$

$X = \mathbb{H}^2/\Gamma$ $\quad T^1X = T^1\mathbb{H}^2/\Gamma$ $\quad T^1\mathbb{H}^2 = (\partial\mathbb{H}^2)^{(2)} \times \mathbb{R}$

$T^1X^{(n.w)} = \Lambda(\Gamma)^{(2)} \times \mathbb{R}/\Gamma$

$\hookrightarrow$ non wandering

$\exists$ Patterson-Sullivan measure μ on $\Lambda(\Gamma)$

$\gamma_* \mu(z) = (\gamma^{-1})'(z)^{\delta} \mu(z)$

$\tilde{m}(w,z,s) = e^{-2\delta(\eta_{x_0}(w,z)} d\mu(w)\, d\mu(z)\, dt(s)$

$\hookrightarrow$ Leb meas.

$\tilde{m}$ supported on $\Lambda(\Gamma)^{(2)} \times \mathbb{R}$, Γ-invariant.

$\hookrightarrow$ desc. to meas supp on $T^1X^{(n.w)}$

Thm 1 Γ convex co-compact, geodesic flow $T^1X^{(n.w)}$ is ergodic w.r.t m.

Cor 1 Γ conv co-compact

$\Downarrow$

(1) Γ acts erg on pairs of dist. fixed pts wrt $\mu \otimes \mu$

(2) Γ acts erg on $\Lambda(\Gamma)$ wrt μ

Pf 1 (1)

If $A \subseteq \Lambda(\Gamma)^{(2)}$ is Γ-inv and not zero or full measure,

$A \times \mathbb{R} \subseteq T^1\mathbb{H}^2$, $A \times \mathbb{R}/\Gamma$ flow inv and has neither zero nor full measure μ. oops.

Cor Γ conv. cocomp $\Rightarrow$ P.S measure unique!

Pf Suppose μ_1, μ_2 P.S measures

look at $\nu = \frac{1}{2}(\mu_1 + \mu_2)$ also P.S measure.

$\frac{\partial \nu}{\partial \mu_1} = f$ is measurable Γ inv fnct

(Radon Nykodim)

$\Rightarrow f$ const a.e.

Prob measure

$\Rightarrow f = 1$, $\Rightarrow \nu = \mu_1 = \mu_2$

Thm Γ convex co-compact $\Rightarrow$ geodesic flow is mixing on $T^1 X^{(\text{nw})}$ wrt m

If $A, B \subseteq T^1 X^{(\text{nw})}$ measble

$$m(A \cap \phi_t(B)) \xrightarrow[t \to \infty]{} m(A)\, m(B)$$

(to ind var vals)

Thm $P_T = \{$closed geodesics on X of length $\leq T\}$

$$\#(P_T) \sim \frac{e^{\delta T}}{\delta T}$$

Thm (Sullivan) Γ conv cocomp

$$\dim_H(\Lambda(\Gamma)) = \delta(\Gamma)$$

Thm (Bishop Jones) In gen

$$\dim_H(\Lambda^{\text{con}}(\Gamma)) = \delta(\Gamma)$$

Hausdorff Dim

Say $A \subseteq Y$ a metric sad.

If $d \geq 0$ & $\delta > 0$

$$H_d^\delta(A) = \inf \left\{ \sum_{i=1}^{\infty} \varepsilon_i^d \;\middle|\; A \subseteq \bigcup B(z_i, \varepsilon_i),\ z_i \in A,\ \varepsilon_i \in (0, \delta) \right\}$$

$H_d(A) = \lim_{\delta \to 0} H_d^\delta(A)$ well def by monotonicity.

$$\dim_H(A) = \inf\{d \mid H_d(A) < +\infty\} = \inf\{d \mid H_d(A) = 0\}.$$

① $\dim_H(\Lambda(\Gamma)) \leq \delta(\Gamma)$

$R_1 = \operatorname{diam}(C(x)) \Rightarrow \Lambda(\Gamma) = \Lambda_R^{con}(\Gamma)$

$R = R_1 + 1$.

Lemma $\forall\, r > 0\ \exists\, a > b > 0$ & if $y \in \mathbb{H}^2$

$$b e^{-d(0,y)} \leq S_r(0,y) \leq a e^{-d(0,y)}$$

If $\delta > 0$, $\Gamma_\delta = \{\gamma \in \Gamma \mid e^{-d(0,\gamma(0))} < d\}$

& Γ / Γ_δ is finite as Γ is discr.

$$\Lambda(\Gamma) \subseteq \bigcup_{\gamma \in \Gamma_\delta} S_R(0, \gamma(0)) \subseteq B(z_\gamma, a e^{-d(0,\gamma(0))})$$

$z_\gamma = \overrightarrow{0\,\gamma(0)} \cap \partial \mathbb{H}^2$

$$H^{\delta}_{s}(\Lambda(\Gamma)) \leq \sum_{\gamma \in \Gamma_r} \left(a e^{-d(0,\gamma(0))} \right)^{s}$$

$$\leq a^{s} \sum_{\gamma \in \Gamma} e^{-s\, d(0,\gamma(0))} \leq a^{s} Q_{\Gamma}(s, x_0) < +\infty$$

$$\Rightarrow H_s(\ast) \leq a^{s} Q_{\Gamma}(s, x_0) < +\infty$$

$$\Rightarrow s \geq \dim_H(\Gamma) \quad \text{as } s \geq \delta(\Gamma),$$

so $\boxed{\delta(\Gamma) \geq \dim_H(\Gamma)}$.

Other direction "Sullivans Shadow lemma"

"Frostmans lemma abt Haus dim"