Forever Γ is a discr, tors-free subgrp of $PSL(2,\mathbb{R})$

$X = \mathbb{H}^2/_\Gamma$ is a hyp surf (not necc closed)

Pick $x_0 \in \mathbb{H}^2$, limit set $\Lambda(\Gamma) = \overline{\Gamma(x_0)} \setminus \Gamma(x_0)$

↳ just accum pts

Fact ① $\Lambda(\Gamma) \subseteq \partial\mathbb{H}^2$ (by disc) $\subseteq \mathbb{H}^2 \cup \partial\mathbb{H}^2$

② $\Lambda(\Gamma)$ is closed

③ $\Lambda(\Gamma)$ is Γ-invariant

④ does not depend on choice of x_0

Pf 1) as Γ is discr, accums at only $\partial\mathbb{H}^2$

2) $\Lambda(\Gamma) = \overline{\Gamma(x_0)} \cap \partial\mathbb{H}^2$ ↳ int of closed

3) $\Lambda(\Gamma)$ is Γ-inv if $z \in \Lambda(\Gamma)$ & $\alpha \in \Gamma$

$\exists \{\gamma_n\}$ so $\gamma_n(x_0) \to z$

$\alpha\gamma_m(x_0) \to \alpha(z) \Rightarrow \alpha(z) \in \Lambda(\Gamma)$

4) Say y_0 is some other point in $\mathbb{H}^2$

$r = d(x_0, y_0)$. If $z \in \Lambda_{x_0}(\Gamma)$

$\exists \{\gamma_n\}$ so $\gamma_n(x_0) \to z$

but $d(\gamma_n(x_0), \gamma_n(y_0)) = r$ as isometry

$\Rightarrow \gamma_n(y_0) \in \overline{B}_r(\gamma_n(x_0))$

In the disc model, $\text{diam}_{euc}(B_r(\gamma_n(x_0))) \to 0$ as $\gamma_n(x_0) \to z \in \partial\mathbb{H}^2$

$\Rightarrow \overline{B_r}(\gamma_n(x_0)) \to z$

$\Rightarrow \gamma_n(y_0) \to z$

so $z \in \Lambda_{y_0}(\Gamma)$

Symmetry of arg gets other subset incl

If $X = \mathbb{H}^2/\Gamma$ is a closed hyperbolic surface
$\Rightarrow \Lambda(\Gamma) = \partial\mathbb{H}^2$

Pf let $D = \operatorname{diam}(X)$. If $z \in \partial\mathbb{H}^2$

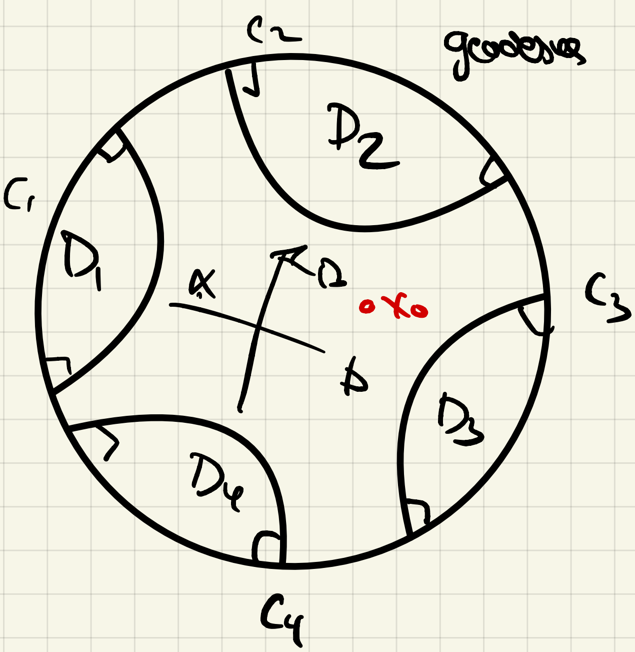
$\Rightarrow \exists$ seq in $\mathbb{H}^2$ so $x_n \to z$

covering sp thry, $\Rightarrow \forall n \; \exists \gamma_n \in \Gamma$ so
$d(\gamma_n(x_0), x_n) \le D$

$\Rightarrow \gamma_n(x_0) \in \overline{B}_D(x_n) \xrightarrow[\text{as before}]{} z$

$\Rightarrow \gamma_n(x_0) \to z$

$\Rightarrow z \in \Lambda(\Gamma)$



$\alpha(C_1) = C_3$, $\alpha(D_1) = \mathbb{H}^2 - D_3$

$\beta(C_2) = C_4$, $\alpha(D_2) = \mathbb{H}^2 - D_4$

$\langle \alpha, \beta \rangle \cong \mathbb{F}_2$ and discr! $\rightsquigarrow$ in $PSL(2,\mathbb{R})$

write a word in α, β but also prop disc.

$\rightsquigarrow \alpha\beta\alpha^{-1}\beta^{-1}\alpha^{-1}\beta$

$\left.\begin{array}{l} \beta(x_0) \in D_4 \\ \alpha^{-1}\beta(x_0) \in D_1 \\ \vdots \end{array}\right\}$ always in $D_1 \dots D_4$ $\rightsquigarrow$ ping pong

$\Rightarrow w(x_0) \notin \mathbb{H}^2 - (D_1 \cup \dots \cup D_4)$

so, $w(x_0)$ non triv $\Rightarrow$ free!

bracke & forte

Also, discrete for similar reasons.

limit set is cantor set.

$\mathbb{H}^2 - (D_1 \cup \dots \cup D_4)$ is fund domain for $\langle \alpha, \beta \rangle$.

Γ is elementary if $\Lambda(\Gamma)$ is finite.

Fact) Γ elementary $\Rightarrow$ $\Gamma \cong \mathbb{Z}$ or 1 & $|\Lambda(\Gamma)| \leq 2$

limset sizes $0, 1, 2$, unctbl

Pf) If $\Gamma \neq 1$, $\gamma \in \Gamma$ acts as perm of $\Lambda(\Gamma)$

$\Rightarrow \exists n > 0$ so γ^n fixes all pts in $\Lambda(\Gamma)$

$\hookrightarrow$ if γ^n parabolic $\Rightarrow |\Lambda(\Gamma)| = 1$ (fixed pt)

γ^n hyp $\Rightarrow |\Lambda(\Gamma)| \underline{\underline{\leq}} 2$.

$\gamma^n(x_0) \to \gamma^+$

$\gamma^{-n}(x_0) \to \gamma^-$ $\Rightarrow \gamma^+, \gamma^- \in \Lambda(\Gamma)$

Fact) fixed pts of nontriv elts in Γ lie in $\Lambda(\Gamma)$

Fact) wts Action of Γ on $\Lambda(\Gamma)$ is minimal.

i.e., if F is any closed, non-empty Γ inv ss $\Lambda(\Gamma)$ $\Rightarrow F = \Lambda(\Gamma)$. topological idea

$\Rightarrow$ set of hyperbolic fixed pts is dense in the limit set $\Lambda(\Gamma)$.

$\{(\gamma^-, \gamma^+) \mid \gamma \in \Gamma \text{ hyp}\}$ is dense in $\Lambda(\Gamma)^2$.

$\hookrightarrow$ fixed pt