If X is a hyperbolic surface $X \cong \mathbb{H}^2/\Gamma$ & $\Gamma \leq \mathrm{Isom}(\mathbb{H}^2)$

acts freely & properly disc.

If X is closed, orientable surface $\Rightarrow$ Γ contains no parabolic elts

Pf) $\hookrightarrow$ define, $\mathrm{inj}_X(y) = \sup \left\{ \varepsilon > 0 \mid B_X(y, \varepsilon) \text{ is isometric to a ball in } \mathbb{H}^2 \right\}$

inj radius

$\hookrightarrow$ note, $\mathrm{inj}_X : X \longrightarrow (0, \infty)$ is 1-Lipschitz, cts

$\Rightarrow$ it has a minimum (achieved as X closed)

$\hookrightarrow \varepsilon_0 > 0$

$\Rightarrow$ all ε_0 ball on $X \cong$ ball in $\mathbb{H}^2$!

If $\gamma \in \Gamma$ is parabolic, $\inf \{ d(z, \gamma(z)) \} = 0$

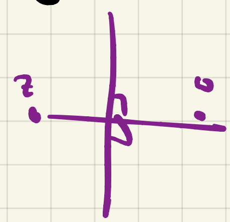
$\Rightarrow \exists$ pt $w \in \mathbb{H}^2$ so $d(w, \gamma(w)) < \varepsilon_0$

$\Rightarrow$ $\overline{w, \gamma(w)}$ projects to a htpically nontrivial curve of len $\leq \varepsilon_0$. But it must be contained in a $\frac{\varepsilon_0}{2}$ ball in X which should be contractible $\square$

Voronoi Diagram, Dirichlet dom

If $z, w \in \mathbb{H}^2$ $\quad H_{z,w} = \{ x \in \mathbb{H}^2 \mid d(z,x) \leq d(w,x) \}$

Consider the perp bisector L to $\overline{zw}$



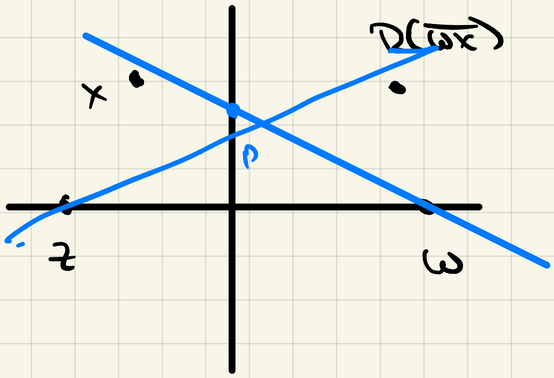
Let $R =$ hyperbolic refl in L

$\hookrightarrow$ will fix L, flip z, w!

$\hookrightarrow$ is an isometry!

If $x \in L$, $d(x, w) = d(x, z)$

Spose $x \in \mathbb{H}^2 \setminus L$ & lies in same comp as z



$$\ell(R(\overline{wp}) \cup \overline{px}) = d(w,x)$$

$\uparrow$ not geodesic

$$d(z,x) < d(w,z)$$

$\Rightarrow$ $H_{z,w}$ is a geodesic half plane bdd by L cont z!

Dirichlet Domain: $X = \mathbb{H}^2/\Gamma \quad z \in \mathbb{H}^2$

$$D_z(\Gamma) = \{x \in \mathbb{H}^2 \mid d(x,z) \leq d(x,\gamma(z)) \ \forall \gamma \in \Gamma\}$$
$$= \bigcap_{\gamma \in \Gamma} \mathbb{H}_{z,\gamma(z)}$$

$\hookrightarrow$ convex (& int of conv is conv)

Claim:

① $D_z(\Gamma)$ is a polygon (sides are locally finite $\hookrightarrow$ i.e $\partial H_{z,\gamma(z)}$)

② it is a fundamental polygon (i.e $\bigcup_{\gamma \in \Gamma} \gamma(D_z(\Gamma)) = \mathbb{H}^2$ and $\gamma(\operatorname{int}(D_z(\Gamma))) \cap \operatorname{int}(D_z(\Gamma)) = \emptyset \ \forall \gamma \neq id$)

③ If F is a face of $D_z(\Gamma)$ then $\exists! \gamma \in \Gamma$ not id so, $\gamma(F)$ is a face of $D_z(\Gamma)$

1) If $w \in \mathbb{H}^2$, $\{\gamma \in \Gamma \mid \partial H_{z,\gamma(z)} \cap B(w,1)\}$

$\Rightarrow d(z,\gamma(z)) \leq 2\,d(z,w) + 2$

$\Rightarrow$ only finitely many as Γ as prop, disc!

If $w \in \mathbb{H}^2$, then $\exists\ \gamma_w \in \Gamma$ s.t., $d(w, \gamma_w(z)) \leq d(w, \gamma(z))$ $\forall \gamma \in \Gamma$

$\Rightarrow w \in \gamma_w(D_z(\Gamma))$

closest orbit pt since no accum by prop disc.

If $\gamma(x) \in \operatorname{int} D_z(\Gamma)$

$\Rightarrow d(\gamma(x), z) < d(\gamma(x), \gamma(z))$

$\Rightarrow d(x, \gamma^{-1}(z)) < d(x, z)$ $\quad \gamma^{-1}$

$\Rightarrow x \notin D_z(\Gamma)$

Fact If L_1, L_2 are non int geodesics in $\mathbb{H}^2$ & don't share endpt.

$\Rightarrow \exists!$ geodesic perp to $\underline{L_1, L_2}$ → common perp

goes from $0 \to \pi$, ctsly $\therefore$ must be $\pi/2$ at some pt

→ can also consider Gauss-Bonnet for uniqueness

non uniqueness is violated

Spse, X is a closed surface, $X = \mathbb{H}^2/\Gamma$

If $\gamma \in \Gamma \setminus \{id\}$ it has an axis & axis(γ) proj to closed geodesic on X.

Cl. 1 If C is htpically non triv simple closed curve on $X \Rightarrow C$ is htpic to unique simple cl. geodesic

Pf] lift $c:\mathbb{R}/\mathbb{Z} \longrightarrow X$ to $\tilde{c}:\mathbb{R} \longrightarrow \mathbb{H}^2$

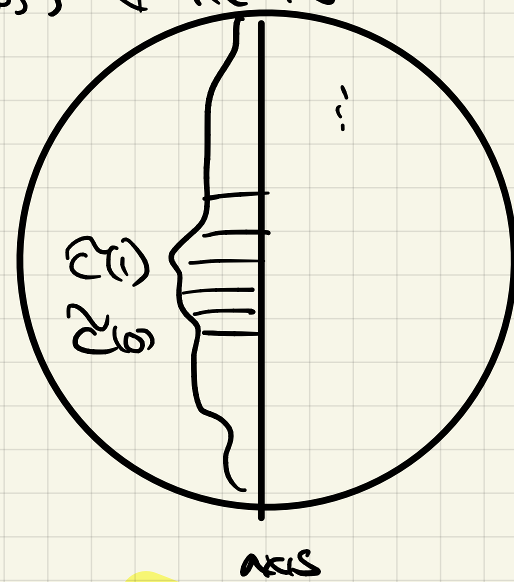
$\exists\, \alpha \in \Gamma$ so $\tilde{c}(n) = \alpha^n(\tilde{c}(0))\ \forall\, n \in \mathbb{N}$

Consider, straightline htpy

$\tilde{H} : [0,1] \times [0,1]$

join $\tilde{c}([0,1])$ to $z\,\alpha(z)$

$z \in \text{axis}(\alpha)$



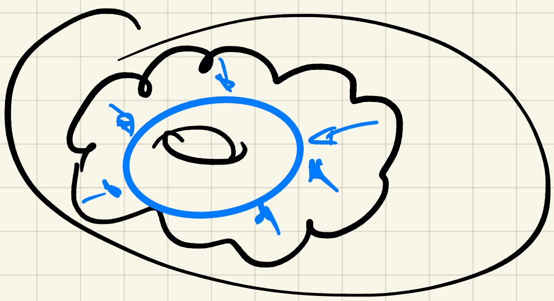
extends equiv to a htpy

b/w $\tilde{c}$ & axis(α)!

↪ preserved axis!

hyperbolic isom!

This proj downstairs to reg htpy!



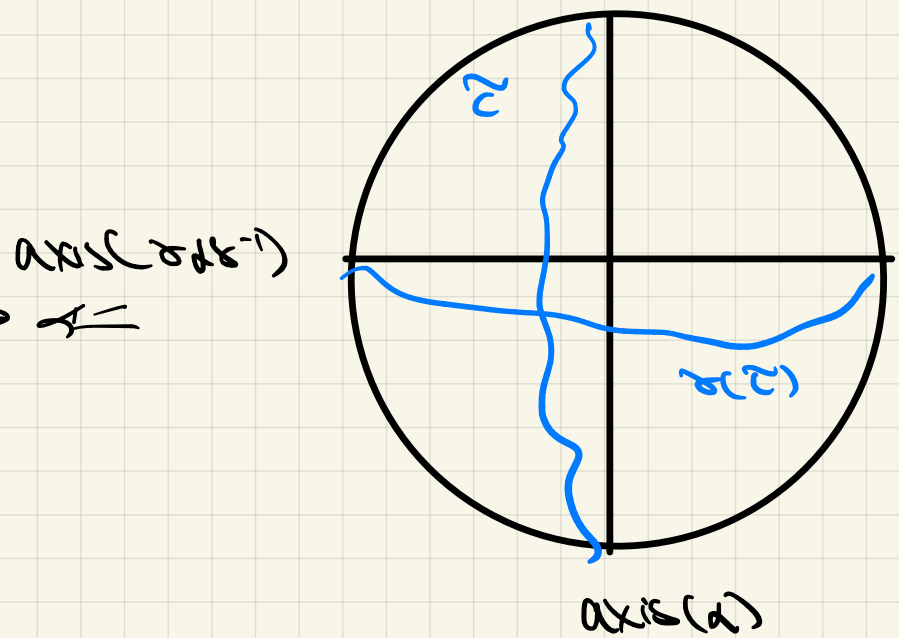
If $\pi(\text{axis}(\alpha))$ not simple

$\Rightarrow$ it has 2 preimages upstairs in $\mathbb{H}^2$ which intersect

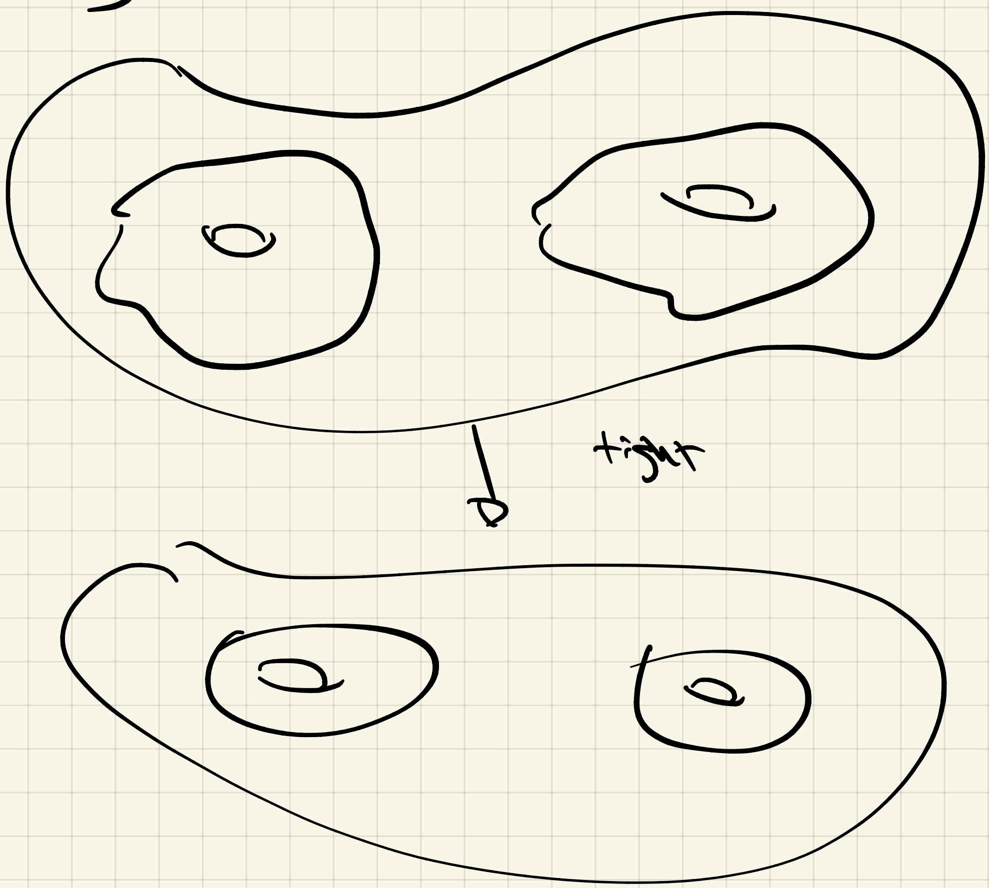
$\Rightarrow$ c has 2 preim which int

$\Rightarrow$ c not simple $\Rightarrow \Leftarrow$

Uniqueness easy?..

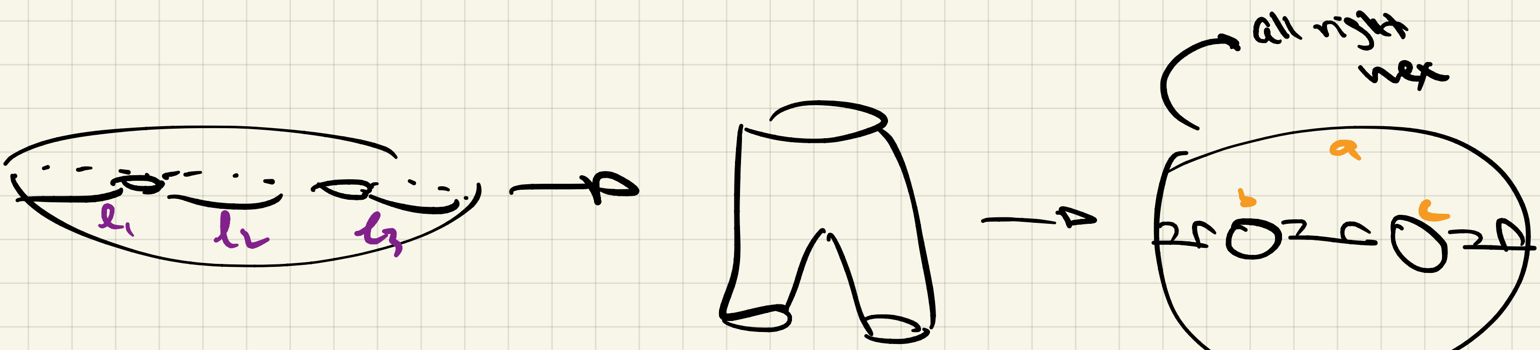


So say,



So can chop! sp into geodesics!

So



pants or

$S^2 \setminus \{0, 1, \infty\}$