We say Γ acts ergodically on $\partial\mathbb{H}^2$ wrt leb measure
if whenever $A \subseteq \partial\mathbb{H}^2$ & A is Γ-inv
then $\mu(A) = 0$ or $\mu(X-A) = 0$

If $\Omega(\Gamma)$ is non-empty, if we can pick a small nbhd U of $x \in \Omega(\Gamma)$ so $\Gamma(\bar{U}) \neq \Omega(\Gamma) \Rightarrow \Gamma$ non act ergo on $\partial\mathbb{H}^2$
↳ gamma-inv meas that not full or 0 μ

Fact! If $X = \mathbb{H}^2/\Gamma$ is a closed surface, then Γ acts ergodically on $\partial\mathbb{H}^2 - \Lambda(\Gamma)$

if have dom of disc not ergodic
↳ this is almost converse!

if $f: S^1 \to \mathbb{R}$ is bdd measurable.
↳ measurable ext of f (sorta like f almost everywhere sequentially or geodesics)

Can define, $Pf : \mathbb{H}^2 \to \mathbb{R}$

$$x \longmapsto \int_{S^1} f(z) \left(\frac{1-|x|^2}{|x-z|^2} \right) d\sigma \qquad d\sigma = \frac{\text{leb meas}}{2\pi}$$

in poincaré disc model

Then, Pf is harmonic $\underbrace{\mathrm{div}(\mathrm{grad}}_{\text{laplacian.}}\, Pf) = 0$
↳ volume pres flow.

This is conf natural,
if f is Γ-inv, so $f(z) = f(\gamma(z))$
$z \in S^1$ $\quad \gamma \in PSL(2,\mathbb{R})$

$\Rightarrow Pf$ is γ inv on $\mathbb{H}^2$.
i.e. $Pf(x) = Pf(\gamma(x))$ for $\boxed{\begin{matrix} x \in \mathbb{H}^2 \\ \gamma \in \Gamma \end{matrix}}$.

Suppose, Γ is not ergodic
$\Rightarrow \exists A \subseteq S^1$ so $\sigma(A) \neq 0$ & $\sigma(S^1 - A) \neq 0$
↳ Γ inv

$h = P\chi_A$ (characteristic) $\longrightarrow$ Γ inv. and $h(x)$ = portion of geodesic rays emanating from x landing at A.

So h descends to a harmonic map
$\hat{h} : X \to (0,1)$

$\nabla \hat{h}$ gives a vol pres flow.

let Φ_1 be a time 1 flow

Pick a regular value, a, $\hat{h}$ → nonzero grad.

$\Rightarrow \Phi_1(\hat{h}^{-1}([a,1])) \subsetneq \hat{h}^{-1}([a,1])$

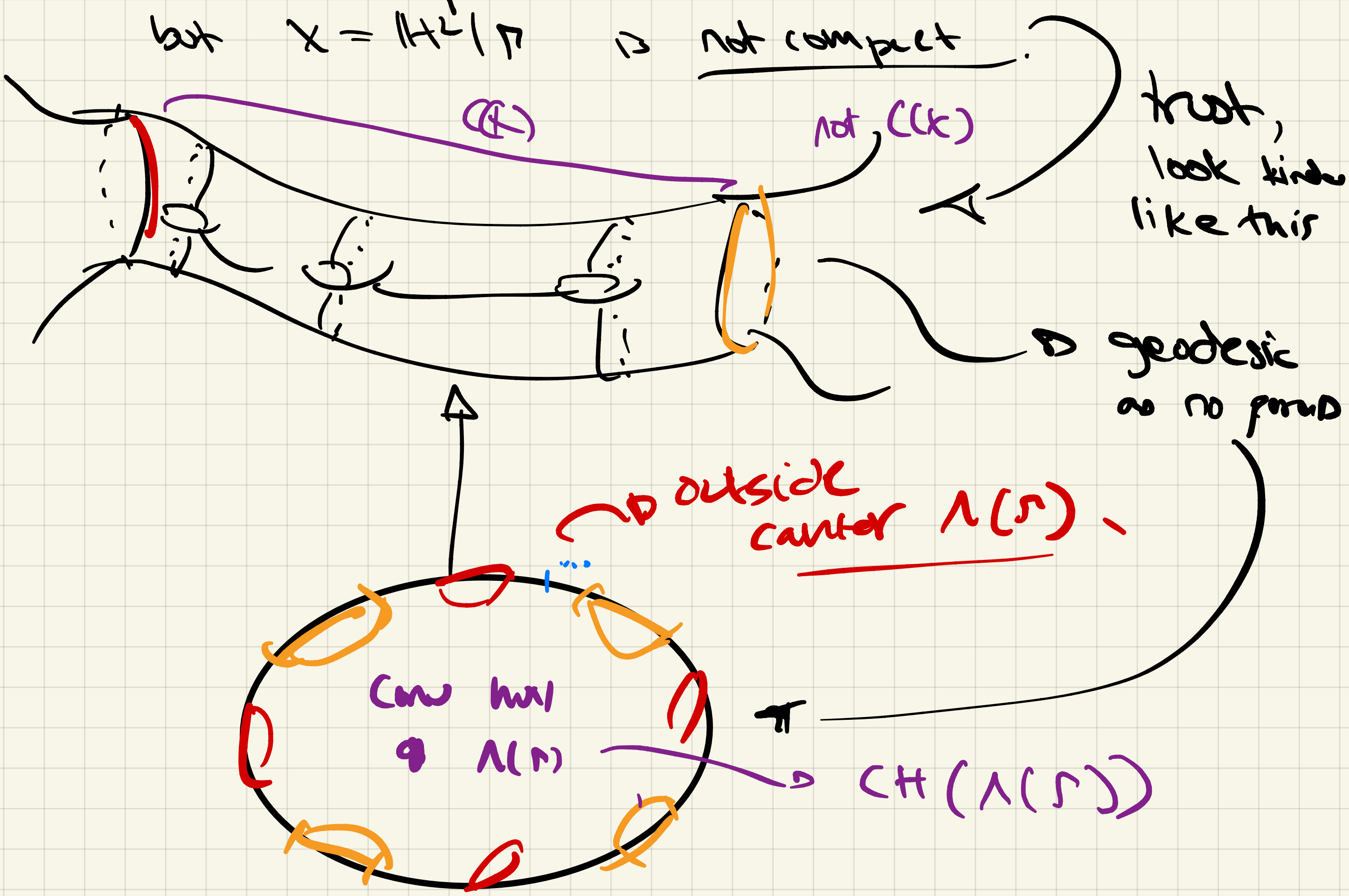
but volume $\hat{h}^{-1}([a,1])$ should be vol $\Phi_1(\hat{h}^{-1}([a,1]))$

OOPS.....

Yau (closed manifold Riem no non const harm maps)

Now spse, Γ is finitely generated & has no parabolic elements but $X = \mathbb{H}^2/\Gamma$ is not compact.

trust, look kinda like this



Suppose, $\sigma(\Lambda(\Gamma)) > 0$ & $\sigma(\Omega(\Gamma)) > 0$

then, $h = P\,\chi_{\Lambda(\Gamma)}$ → char

If $x \in \partial\, CH(\Lambda(\Gamma))$

$h(x) \le \frac{1}{2}$

→ convex core of X

$CH(\Lambda(\Gamma))/_{\Gamma} = C(X)$ (basically push the hull down")

If $\sigma(\Lambda(\Gamma)) > 0$ $\exists$ point of density $z \in \Lambda(\Gamma)$

So, $\sup_{CH(\Lambda(\Gamma))} h = 1$

looking downstairs in $C(X)$.

$h|_{\partial C(X)} \le \frac{1}{2}$ and max $h|_{C(X)} = 1$

so achieves max on interior so bad (should be on ∂)

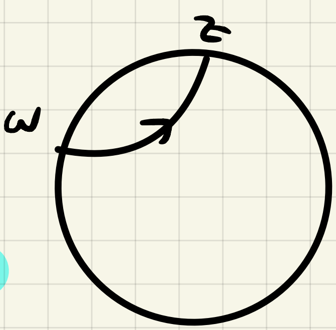
$T^1\mathbb{H}^2 = \{(x, \vec{v}) \mid x \in \mathbb{H}^2 \quad v \in T^1_x \mathbb{H}^2\}$

if $(x, \vec{v})$ $\exists$ unique unit speed geodesic
$c : \mathbb{R} \to \mathbb{H}^2$

so $(x, \vec{v}) = (c(0), c'(0))$

→ Flow!!

$\phi_s(c(0), c'(0)) = (c(s), c'(s))$



If $(w,z) \in (\partial \mathbb{H}^2)^{(2)} \rightsquigarrow$ dist pts

$c_{w,z} : \mathbb{R} \to \mathbb{H}^2$ geodesic starting w ending at z

$$T^1\mathbb{H}^2 \xrightarrow{\text{fibres}} (\partial\mathbb{H}^2)^{(2)}$$

fibres is all points $(c_{wz}(t), c_{wz}'(t))$

$T^1\mathbb{H}^2 = (\partial\mathbb{H}^2)^{(2)} \times \mathbb{R}$ wanna write

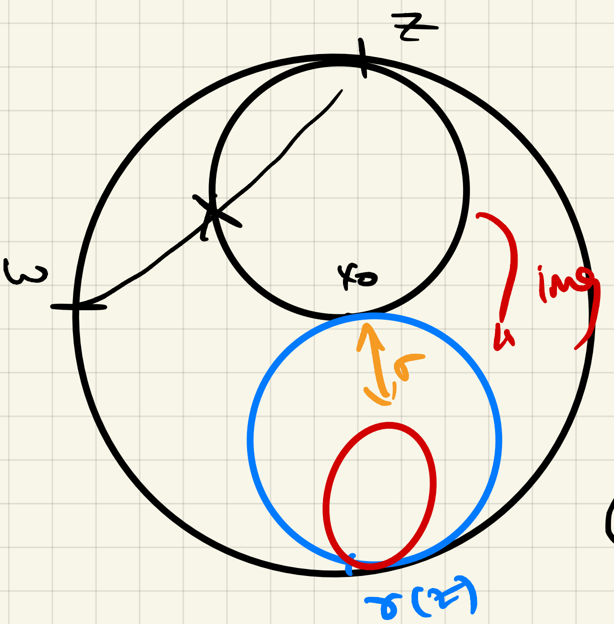
(w,z,t) . ex circle tang

H_z = horo cycle based at passing thru origin

let $c_{wz}(0)$ be int

$c_{wz}(0) = H_z \cap \overline{wz}$

$(c_{wz}(t), c_{wz}'(t)) \sim (w,z,t)$



$\exists\ \sigma : PSL(2,\mathbb{R}) \times \partial\mathbb{H}^2 \to \mathbb{R}$

so $\gamma(w,z,t) = (\gamma(w), \gamma(z), t + \sigma(\gamma, z))$

signed dist b/w $\gamma(H_z)$ & $H_{\gamma(z)}$