Recall

Lemma 1 Given $A, B > 0$ $\exists$ $C > 0$ st if T is cmpt hyp surf w/ geodesic ∂ & $\text{area}(T) = A$ & $\text{length}(\text{comp of } \partial T) < B$

$\Rightarrow$ $\exists$ γ^* simple closed geodesic (not boundary parallel) so, $\underline{\ell(\gamma^*) \le C}$.

Pf Double of T along ∂T, $DT = \mathbb{H}^2/\Gamma$

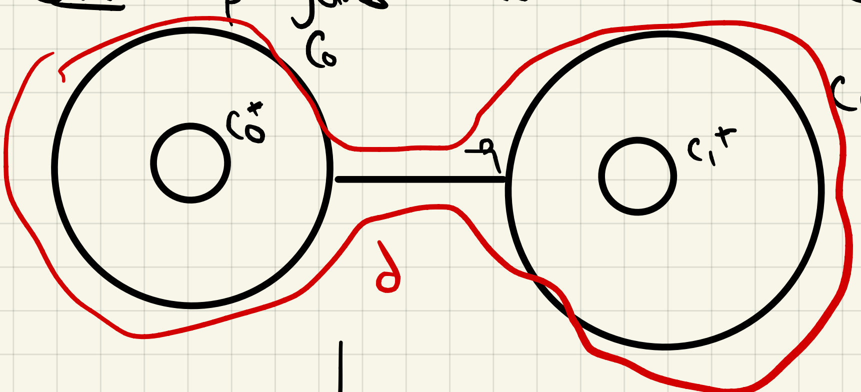
$\mathcal{N}(\partial T)$, if C is int comp of $\partial \mathcal{N}(\partial T)$

$2\sinh^{-1}(1) \le \ell(C) \le L(B)$

Can assume, if $z \in T - \mathcal{N}(\partial T)$, $\underline{inj_{DT}(z) > 2\sinh^{-1}(1)}$.

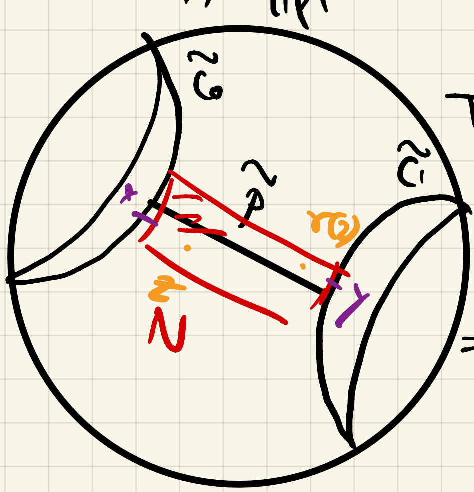
Let p be shortest geodesic arc joining $\partial\mathcal{N}(\partial T)$ to itself

Case 1 p joins $C_0 \to C_1$ & $C_0 \ne C_1$



$\ell(\gamma^*)$
$\le \ell(C_0) + \ell(C_1) + 2\ell(p)$
$\le \underline{2L(B) + 2\ell(p)}$.

lift



Let $N = \{x \in \mathbb{H}^2 \mid \exists z \in \text{int}(\tilde{T})$ st $\overline{zx} \perp \tilde{p}$ and $d(z,x) \le \frac{1}{4}\}$

$\pi : \mathbb{H}^2 \to \mathbb{H}^2/\Gamma = DT$

Cl N embeds in T (under obv. quotient)

$\Rightarrow A \ge \text{area}(\pi(N)) = \text{area}(N) \ge \frac{1}{2}\ell(p)$

$\Rightarrow \underline{\ell(p) \le 2(A)}$.

If not, $\exists z \in N$, $\gamma \in \Gamma \setminus \{id\}$, $\gamma(z) \in N$

$d(z, \tilde{C}_0 \cup \tilde{C}_1) \le \frac{\ell(p)}{2} + \frac{1}{4}$ $\quad d(\gamma(z), \tilde{C}_0 \cup \tilde{C}_1) \le \frac{\ell(p)}{2} + \frac{1}{4}$

Case $x \in \tilde{C}_0 \cup \tilde{C}_0$ st $d(z,x)$ & $y \in \tilde{C}_0 \cup \tilde{C}_1$

Suppose, $d(z,x) < \frac{\ell(p)}{2} - \frac{1}{4}$

But, $\overline{xz} \cup \overline{z\gamma^{-1}(y)} =: q$ $\quad \gamma^{-1}(\overline{\gamma(z)\, y})$

$$\ell(q) < \frac{\ell(p)}{2} - \frac{1}{4} + \frac{\ell(p)}{2} + \frac{1}{4} = \ell(p).$$

q joins $\tilde{C}_x$ to $\gamma^{-1}(\tilde{C}_y)$

$x \in \tilde{C}_x$ $\quad y \in \tilde{C}_y$ $\quad q$ proj to path $\partial \mathcal{H}(\partial T)$ to itself of length $< \ell(p)$

either $\tilde{C}_0$ or $\tilde{C}_1$ whichever has x.

oops!

Similarly, get contra if $d(\gamma(z), y) < \frac{\ell(p)}{2} - \frac{1}{4}$

So,

$$\frac{\ell(p)}{2} - \frac{1}{4} \le \begin{matrix} d(x,z) \\ d(y,\gamma(z)) \end{matrix} \le \frac{\ell(p)}{2} + \frac{1}{4}$$

$\Rightarrow$ m midpt of p

$$d(m,z) \le \frac{1}{2}, \quad d(m,\gamma(z)) \le \frac{1}{2}$$

$\Rightarrow d(z, \gamma(z)) \le 1$

$\Rightarrow \mathrm{inj}_{DT}(\pi(z)) \le \frac{1}{2} < \sinh^{-1}(1)$ OOPS.

Case 2 similar.

Given $\varepsilon > 0$ let $\mathcal{M}_\varepsilon(S) = \{ x \in \mathcal{M}(S) \mid \text{inj}_x(z) \geq \varepsilon \ \forall z \in X \}$

$\hookrightarrow$ moduli sp

$= \{ x \in \mathcal{M}(S) \mid \text{sys}(x) \geq 2\varepsilon \}$

$\hookrightarrow$ length of shortest geodesic on S.

Mumford | $\mathcal{M}_\varepsilon(X)$ is compact!

Sk of Pf | $\mathcal{M}_\varepsilon(X)$ is closed, since

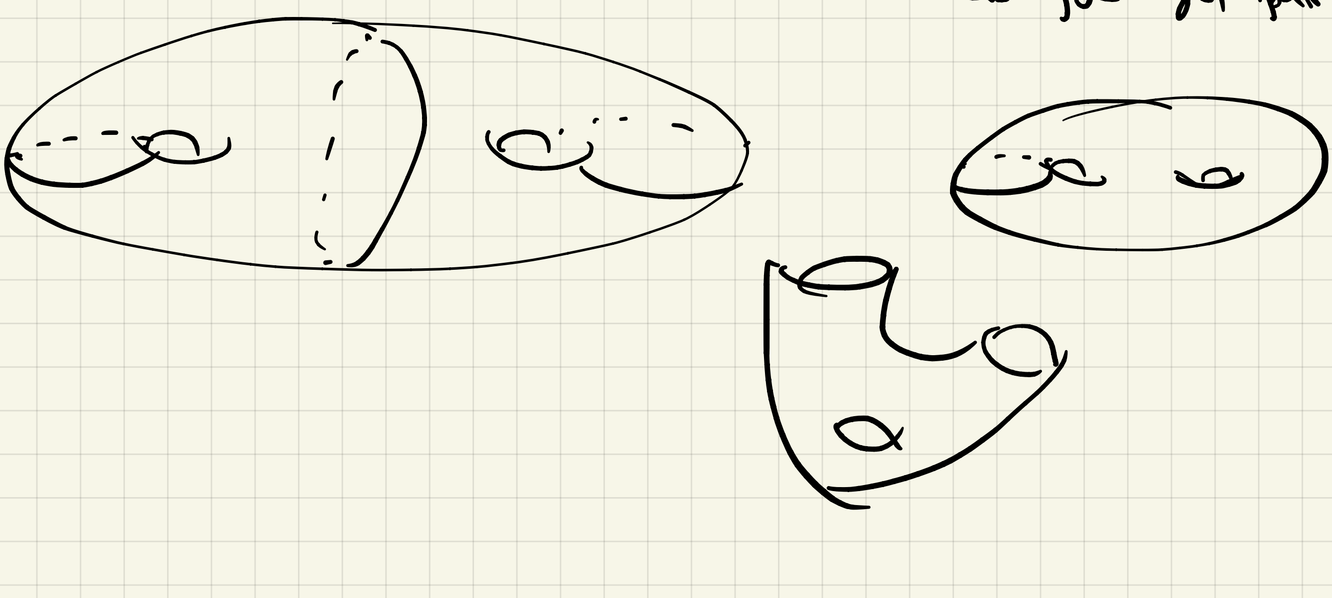
$\text{sys} : \mathcal{M}(X) \longrightarrow \mathbb{R}$ is cts.

Obs | Up to homeo, $\exists$ only fin many $\{P_1, \ldots, P_n\}$ pants decomps of S.

$(g=2)$ Spse P contains a separation curve c then $S - c$ is a union of 2 1-holed tori,

All simple closed geodesics on 1-holed torus are equal up to homeo since you can cut along c'

$\hookrightarrow$ you get pair of pants



given $X \in \mathcal{M}_\varepsilon(x)$

Then $\exists$ geodesic pant decomp P of X (upper bd)

with all curves of length in $[2\varepsilon, M_g]$

and $\exists\, h: S \to X$ s.t. $h(p_i) = P$ for some i.

alter h by Dehn twists about p_i

so that all twists coeff in $[0,1)$

$$(X,h) \in [2\varepsilon, M_g]^{3g-3} \times [0,1)^{3g-3}$$

in F-N coord. assoc to p_i

two forgetful

$$F_i : \mathcal{T}(S) \longrightarrow \mathcal{M}(S)$$

(Teichmuller)

$$\mathcal{T}(S) = \mathbb{R}_{>0}^{3g-3} \times \mathbb{R}^{3g-3}$$

$$(X,h) \in K_i = F_i\left([2\varepsilon, M_g]^{3g-3} \times [0,1]^{3g-3}\right)$$

So, $M_\varepsilon(X) \subseteq K_1 \cup \cdots\cdots \cup K_n$ → finite union of cmpct.

$\therefore$ $M_\varepsilon(x)$ is closed subset cmpct

$\Rightarrow$ $M_\varepsilon(X)$ cmpct