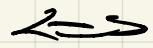


Spec $\phi: \mathbb{R} \times Y \rightarrow Y$ which pres a measure μ .

ϕ is ergodic wrt $\mu \Leftrightarrow$ when $A \subset Y$ & is flow in
($\phi_t(A) = A$ for, then $\mu(A) = 0$
or $\mu(Y - A) = 0$)

(Y compact Haus, μ Radon)



If $f \in L^2(Y)$ & f is flow inv
 $\Rightarrow f$ is constant almost everywhere (a.e.).

Thm) If X is closed hyp. surface, the
geodesic flow is erg wrt Leb meas

Thm) Spec of pres fin meas μ

If $f: Y \rightarrow \mathbb{R}$ is cts. ergodic average.

$$M_f(a) = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t f(\phi_s(a)) ds$$

$$= \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t f(\phi_{-s}(a)) ds$$

1) flow inv set Γ of full meas
P.t., $M_f(a)$ def $\chi_{\Gamma}(a)$ flow inv, $\forall t \in \mathbb{R}$

If, M_f is const a.e. whenever f is cts
 $\Rightarrow \phi$ is ergodic!

Sketch of Proof Say $f: T^*X \rightarrow \mathbb{R}$ st. stable mfd thru (x, \bar{v}) .

Suppose, A, M_f given by thm.

If $(x, \bar{v}) \in A \times L^+(x, v) \subseteq L^+(x, v) \cap A$

$$\lim_{\delta \rightarrow 0} d(\phi_s(x, v) \circ \phi_s(y, \bar{v}), \phi_s(y, \bar{v})) = 0$$

f cts T^*X compact

$$\Rightarrow K = \max |f(z)| \quad z \in T^*X$$

f unif (b)

Given $\epsilon > 0$, $\exists \delta > 0 \quad \forall d(z, w) < \delta$

$$|f(z) - f(w)| < \epsilon$$

But $\exists T$ st. if $\delta > t$ $d(\phi_s(x, v), \phi_s(y, \bar{v})) < \delta$

$$\begin{aligned} & \int_0^t f(\phi_s(x, v)) ds - \int_0^t f(\phi_s(y, \bar{v})) ds \\ & \quad \text{+ int } 0 \rightarrow T \\ & \leq 2TK + (t-T)\epsilon \end{aligned}$$

$$|M_f(x, y) - M_f(y, \bar{v})| \leq \lim_{t \rightarrow \infty} \frac{2KT + (t-T)\epsilon}{t} = \epsilon$$

$$\text{C and, S } M_f(x, v) - M_f(y, \bar{v})$$

if $(x, v) \in A, (y, \bar{v}) \in L^+(x, v) \cap A$

Similarly if $(x, v) \in A \quad (y, \bar{v}) \in L^-(x, v) \quad \frac{\text{Pinst}}{\text{out}}$

$$\Rightarrow \lim_{s \rightarrow \infty} d(\phi_s(x, v), \phi_s(y, \bar{v})) = 0$$

use backward flow $M_f(x, \bar{v}) = M_f(y, \bar{v})$

Locally, L^+, L^-, L^0 give coordinates for $T'X$
 (In $T'W^2$ lines in L^+ are just (\cdot, x, t)
 ————— L^0 are just (ω, x, \cdot)
 ————— L^- are just $(\omega, \cdot, f(\omega))$)

Lebesgue Measure, is in the sense never cover
 as a product measure
 $\lambda^+ \oplus \lambda^- \oplus \lambda^0$
 ↪ some detection of measure 0!

Find $B^+ \times B^- \times B^0 \subseteq A$ & B^+ has full meas π^+
 $B =$
 $B^+ \quad B^- \quad B^0 \quad \rightarrow \lambda^+ \oplus \lambda^- \oplus \lambda^0$

But $\mu_f \approx \text{const}$ on B . & B has full meas.
Def A flow ϕ pres measure μ is mixing
 if, whenever A, B are measurable $\forall \epsilon > 0$,
 then $\mu(\phi_t(A) \cap B) \rightarrow \mu(A) \mu(B)$

Bequiv, $f, g \in L^2(Y)$

$$\Rightarrow \int_Y f(\phi_t(y)) g(y) d\mu(y)$$

$$\rightarrow \int_Y f d\mu \int_Y g d\mu$$

Thm) If X closedhyp surf,
the geodesic flow on $T^1 X$ is mixing,

Given a periodic traj P , you can count a
prob measure supported on P
(are len ab P)
 $\ell(P)$

$P_T = \{ \text{periodic traj on } T^1 X \text{ ab len } \leq T \}$

$$\frac{1}{\#(P)} \sum_{P \in P_T} \mu_P \rightarrow \mu \quad (\text{Leb measure}) \quad \boxed{\text{Thm}}$$

Count) $\#(P_T) \sim \frac{e^{1/T}}{1/T}$, i.e. $\frac{\#(P_T)}{1/T} \rightarrow 1$

(Mirzakhani) X is a (closed) hyperbolic surface of genus g

$$\#\text{Simple closed geodesic} \sim C(X) T^{6g-6}$$

of length $\leq T$

\hookrightarrow func dep on Teich
 $\Rightarrow \hookrightarrow$ surf

Thm) The horocycle flow is mixing & erg
wrt leb measure.

(For starters) Horocycle flow is uniquely erg
i.e. The leb mnu is the only flow inv.
measure. (False for geodesic if it has periodic traj)

(P periodic traj \rightarrow up flow inv.)