$\mathbb{H}^n = \{\bar{x} \in \mathbb{R}^n \mid x_n > 0\}$ $\qquad ds^2 = \dfrac{dx_1^2 + \dots + dx_n^2}{x_n^2}$

x_n axis is a geodesic

$\vec{x} \to \lambda \vec{x}$ $\;\lambda > 0$, $\qquad \vec{x} \to \bar{x} + (a_1, \dots, a_{n-1}, 0)$,

reflect in $x_i - x_n$ plane $(i<n)$ are isoms!

Inversion in unit sphere is isom

$\{$inversion in hemi & pl $\perp \partial\mathbb{H}^n\} \subseteq \mathrm{Isom}\,\mathbb{H}^n$

Can show geodesics in $\mathbb{H}^n$ = $\{$segments of circ & line $\perp \partial\mathbb{H}^n\}$

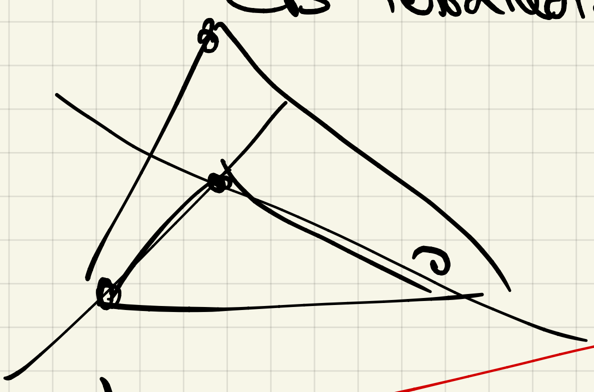
When $n=3$,

$\mathrm{Isom}^+(\mathbb{H}^3) = \{$grp of mobius transf of $\mathbb{CP}^1 = \mathbb{C}\cup\{\infty\}\}$
$\qquad \| $
$\mathrm{PSL}(2,\mathbb{R})$

$\mathbb{B}^n = \{\bar{x} \in \mathbb{H}, \|x\|<1\}$ $\qquad ds^2 = \dfrac{4x_1^2 + \dots dx_n^2}{(1-\|x\|^2)^2}$

Ideal tetrahedron

↳ tetrahedron w/ 4 ideal endpts.



$\to \cosh^{-1}(\sqrt[3]{8})$!

Lemma $\exists A>0$ s.t. if T is ideal tetra & $z \in T \Rightarrow d(z, T^{(1)}) \le A$.

Pf

defn $\mathcal{B} : \mathbb{H}^2 \times \mathbb{H}^2 \times \mathbb{H}^2 \longrightarrow \mathbb{R}$ Busemann Function

$$(z, x, y) \longmapsto d(z,x) - d(z,y)$$

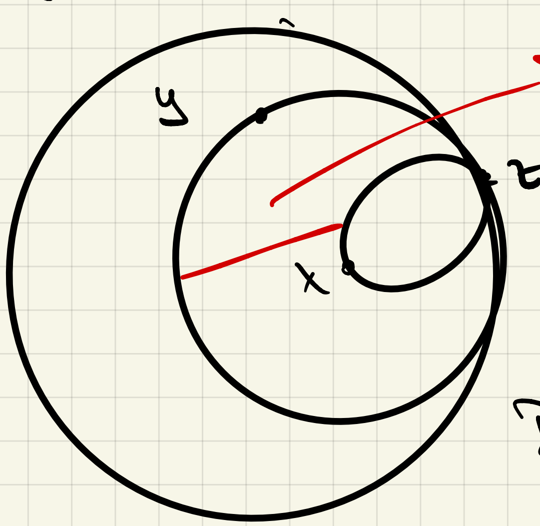
C^{∞} & 1-lipshitz (triangle(ineq))

want to ext $\mathcal{B} : \overline{\mathbb{H}^2} \times \mathbb{H}^2 \times \mathbb{H}^2 \longrightarrow \mathbb{R}$

if $z \in \partial\mathbb{H}^2$ $\mathcal{B}(z,x,y)$ = signed distance b/w $H(x,z)$ (hor cycle based at z thru x) & $H(y,z)$

$\mathcal{B}(z,y,z)$ is +ive if $H(z,y)$ inside $H(x,z)$ else -ive.

So



$\mathcal{B}(z,x,y)$ is this dist but +ive

$\|$

$-\mathcal{B}(z,y,x)$

By constr, $PSL(2,\mathbb{R})$-equivariant

$$\mathcal{B}(\gamma z, \gamma x, \gamma y) = \mathcal{B}(z,x,y)$$

$\forall (x,y,z)$

$\forall \gamma \in PSL(2,\mathbb{R})$

If $\gamma \in PSL(2,\mathbb{R})$ & $z \in \partial\mathbb{H}^2$ define, busemann cocycle $\sigma(\gamma, z)$ to be signed dist, b/w horocycle based at γz going thru x_0 & horocycle at $\gamma(z)$, $\gamma(x_0)$ $[H(\gamma z, x_0), H(\gamma z, \gamma x_0)]$

This says $\sigma(\gamma, z) = B(\gamma(z), x_0, \gamma(x_0))$

$- B(z, \gamma^{-1}x_0, x_0)$

by PSL 2R equiv.

For $z \in \partial \mathbb{H}^2$ define $B_z : \mathbb{H}^2 \times \mathbb{H}^2 \to \mathbb{R}$

This is clearly cts x^{-1} in $\mathbb{H}^2$ $\quad (x, y) \mapsto B(z, x, y)$.

Lemma B is cts (on $\overline{\mathbb{H}^2} \times \mathbb{H}^2 \times \mathbb{H}^2$)

Pf If $z \in \mathbb{H}^2$, B is cts clearly by earlier rmk

If $x_n \to x$ in $\mathbb{H}^2$ & $y_n \to y$ in $\mathbb{H}^2$

& $z_n \to z$ in $\partial \mathbb{H}^2$
in $\partial \mathbb{H}^2$

The cts is clear too (just need to show seq in $\mathbb{H}^2$ w/ lim in $\partial \mathbb{H}^2$)

⋮