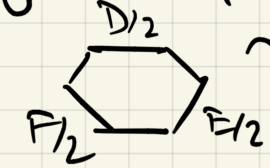
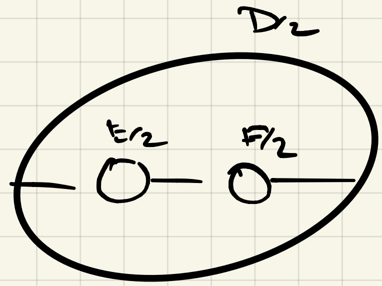
Fact | Every geodesic pair of pants is constr from 2 congruent all-right hexagons.

↳ Recall drawing → union of 2 all right hex showing 3 sides ⇒ 3 noncons lengths.

Fact | Given $D, E, F > 0$ $\exists !$ (up to isom) geodesic pair of pants w ∂ of length D, E, F

↳ 

↝ and double it with other lengths

↪ 

Recall) S is closed oriented surface of genus $g \geq 2$

$$\mathcal{D}(S) = \{(X, h) \mid X \text{ is hyp surface},\ h : S \to X \text{ or pres homeo}\}/\sim$$

→ $h \in \mathrm{homeo}^+(S)$

$$(X_1, h_1) \sim (X_2, h_2) \iff \exists\, g : X_1 \to X_2 \text{ homeo s.t. } g \circ h_1 \simeq h_2$$

Fact | $\exists$ injection $\mathcal{D}(S) \to \mathrm{Hom}(\pi_1(S), PSL(2,\mathbb{R}))/PSL(2,\mathbb{R})$ (act by conj)

$$(X, h) \longmapsto h_* : \pi_1(S) \to \pi_1(X)$$

$$\pi_1(X) = \Gamma \leq PSL(2,\mathbb{R})$$

only def up to conj ←

$$X = \mathbb{H}^2/\Gamma$$

If $\alpha \in PSL(2,\mathbb{R})$ $\quad \alpha : \mathbb{H}^2 \to \mathbb{H}^2$

$\hat{\alpha} : \mathbb{H}^2/\Gamma \to \mathbb{H}^2/_{\alpha\Gamma\alpha^{-1}}$ desc. to isometry

$$\alpha(\gamma(x)) = (\alpha\gamma\alpha^{-1})(\alpha(x))$$

↳ so Γ only def to conjugation.

Top on right

$\pi_1(S)$ gen by $g_1, \ldots, g_n$

Can embed $\mathrm{Hom}(\pi_1(S), PSL(2,\mathbb{R})) \subseteq PSL(2,\mathbb{R})^n$

↳ Real alg variety as rel are poly

↳ so quotient by some polynomial invar

↳ so end up with mfd.

How many dims $\mathcal{T}(S_2)$?

$\mathrm{Hom}(\pi_1(S), PSL(2,\mathbb{R}))$ → PSL dim.

$$\mathrm{dim} \to \underbrace{4}_{4 \text{ gens}}\underbrace{(3)}_{3 \text{ choice}} - 3 = 9$$

Subtract by 3 (quotients by $PSL(2,\mathbb{R})$) to get 6

suggest $\dim \mathcal{T}(S_2) = 6$!

Define a distance on $\mathcal{T}(S)$

$\frac{1}{k} d(x,y) \le d(b(x), b(y)) \le k\, d(x,y)$

$$d([x_1,h_1],[x_2,h_2])$$

$$= \log \inf \{ k \mid \exists\, b : x_1 \to x_2,\ k \text{ bilipschitz},\ b \circ h_1 \simeq h_2 \}$$

↳ $k \ge 1$

now, g is k_1-bilipshitz, h is k_2 bilipshitz

$g \circ h = k_1 + k_2$ bilipshitz

Arzela Ascoli ⟹ lim of k_n-bilip is

$(\lim k_n)$ - bilip

⟹ inf is acvieved.

→ this d gives same top.

Now

$i : \left(\mathcal{T}(S), \text{bilipschitz metric} \right) \longrightarrow \left(\mathcal{T}(S), \text{alg top} \right)$

is continuous shows equivalent topologies.

Say $d([X_n, h_n], [X, h]) \to 0$

$\Rightarrow \exists\ k_n$ - bilip map $b_n : X_n \to X$ s.t. $b_n \circ h_n \simeq h_n$ & $k_n \to 1$.

Now, normalize b_n lifts to $\tilde{b}_n : \mathbb{H}^2 \to \mathbb{H}^2$ is k_n-bilip so $\tilde{b}_n(i) = i$

$\tilde{b}_n$ conjugates action of $\pi_1(X_n)$ to $\pi_1(X)$.

$\tilde{b}_n (h_n)_*(g) \tilde{b}_n^{-1} = h_*(g) \quad \forall g \in \pi_1(S)$

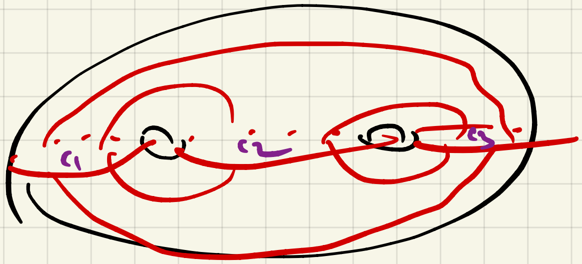
Arzela - Ascoli $\Rightarrow \tilde{b}_n \xrightarrow[\text{conv}]{} \beta \in PSL(2,\mathbb{R})$ (1 bilip $\Rightarrow$ isom).

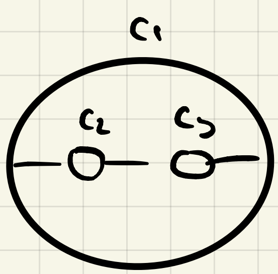
$\Rightarrow (h_n)_*(g) \xrightarrow[\text{conv}]{} \beta^{-1}(h_*(g))\beta \quad \forall g \in \pi_1(S)$

$\Rightarrow [(h_n)_*] \xrightarrow[\text{conv}]{} [h]$

Choose a pants decomp,

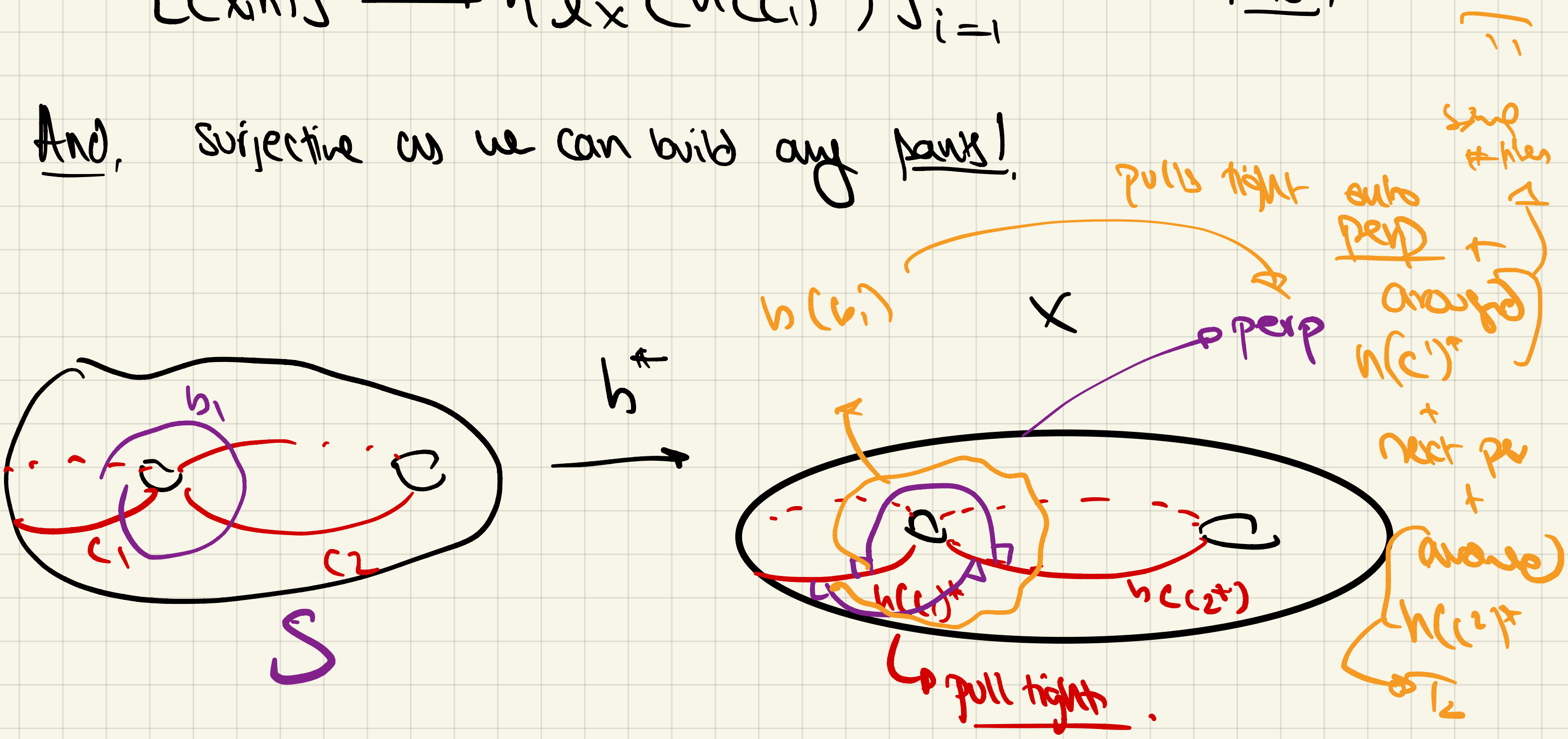
$\{c_1, \ldots, c_{3g-3}\}$ all curves are non-separating $\rightarrow$ seams $\{b_1, \ldots, b_{3g-3}\}$





$$L : \mathcal{D}(S) \longrightarrow \mathbb{R}^{3g-3}_{>0}$$
$$[(X,h)] \longrightarrow \{ \ell_X(h(c_i)^*) \}_{i=1}^{3g-3} \rightsquigarrow \boxed{\text{cts}}$$

__And,__ surjective as we can build any __pants!__



We measure t_1 → $\dfrac{\text{signed length of tight detour along } h(c_1)^*}{\text{leng } h(c_1)^*}$

t_2 — similar for $h(c_2)^*$