Lec 2

In order for $z \longmapsto \frac{az+b}{cz+d}$ to preserve $\mathbb{H}^2$

$a, b, c, d \in \mathbb{R}$, $ad - bc = 1$ → really need this

Poincaré Disc Model

It is the unit disc, $ds = \frac{2|dz|}{1-|z|^2}$ → Cayley Tr

this is obtained from $\mathbb{H}^2$ model via $z \longmapsto \frac{zi+1}{z+i}$

↳ pullback from

Geodesics in disc are img of geodesics in $\mathbb{H}^2$

Fact geodesics in disc model are lines & circles $\perp$ to $\partial D^2 = S^1$

$\therefore$ $\xi \in S^1$, $0 < r < 1$ $d_{hyp}(0, r\xi) = \int_0^r \frac{2}{1-t^2}\, dt$

$$= \log\left(\frac{1+r}{1-r}\right)\Big|_0^r = \log\left(\frac{1+r}{1-r}\right)$$

So, the circle of hyp R about 0 has euc dist r where $\log\left(\frac{1+r}{1-r}\right) = R$ $\quad \tanh(R/2) = \frac{e^R - 1}{e^R + 1}$

It has euc len $2\pi \tanh(R/2) \cdot \left(\frac{2}{1 - \tanh^2(R/2)}\right)$ ↳ scalar factor

$$= 2\pi \sinh(R/2)$$

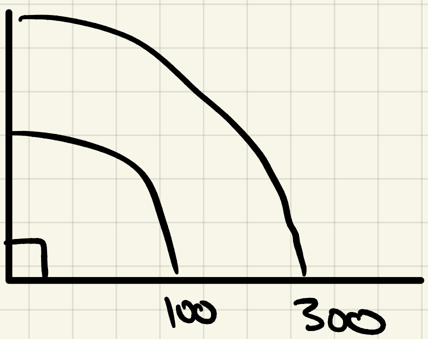
$$\sim \pi e^R$$

As all circles of hyp rad R are congruent, they all have length as abv.

Area of hyperbolic disc radius R,

$$\int_0^R 2\pi\sinh(r)\,dr = 2\pi\cosh R - 2\pi \sim \pi e^R .$$

Baseball



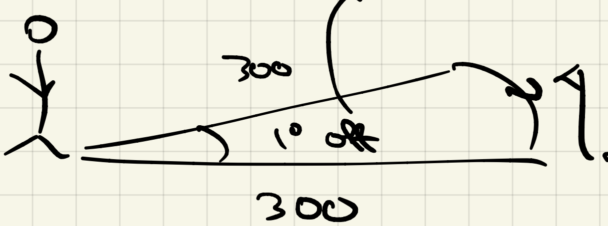
in E^2: Area $= \frac{\pi}{4}\left(300^2 - 100^2\right) \approx 60{,}000$

in H^2: $= \frac{\pi}{2}\left(\cosh 300 - \cosh 100\right) > 10^{100}$.

Golf

how far are we?

1°s from hole E^2



$\frac{2\pi(300)}{360} \approx 5.24$ ft from hole

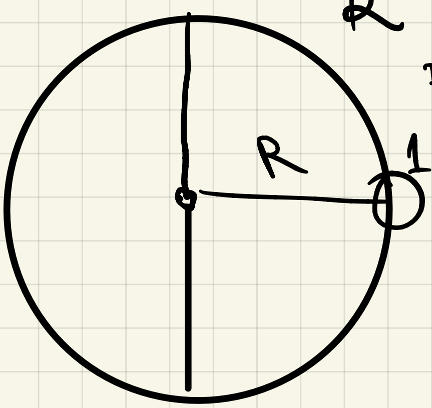
In H^2

same estimate, $\frac{2\pi\sinh(300)}{360} > 10^{100}$ ft.

Actual dist is > 590 feet!

Beach Ball

$R \sim$ field of vision



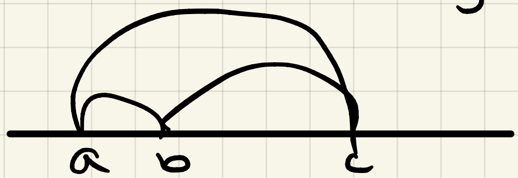
E^2 | ball takes $\frac{1}{\pi R}$ of field of vision.

So if you see smthn if it takes up less than 0.01% of FOV then can see beach ball for 300 ft.

In H^2 | it takes up $\frac{1}{\pi\sinh(R)} \sim \frac{R}{\pi e^R}$

So you can see it less than 7 feet

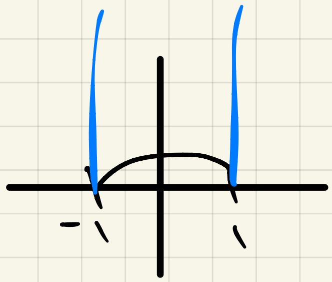
Def A ideal triangle has all 3 endpts at ∞ or $\partial \mathbb{H}^2$
$\hookrightarrow$ 3 vertices joined by geodesics.



Fact: all ideal Δ cong (taken to each other by isom)
$\hookrightarrow$ $\boxed{\text{area } \pi}$

Pf $PSL(2,\mathbb{R})$ acts trans on triples of pts on $\partial \mathbb{H}^2$.

std ideal Δ



$$\text{Area} = \int_{-1}^{1} \int_{\sqrt{1-x^2}}^{\infty} \frac{1}{y^2}\, dy\, dx = \int_{-1}^{1} -\frac{1}{y}\Big|_{\sqrt{1-x^2}}^{\infty} dx$$

$$= \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\, dx \qquad x = \cos\theta$$

$$= \int_{\pi}^{0} -\, d\theta = \pi$$

Def A 2/3 ideal Δ has 2 vert at ∞ & 1 int angle.

Fact All 2/3 ideal Δ with int angle α are congruent & have area $\pi - \alpha$.

Pf Can move 1 ideal endpt to ∞ & other to 1, & scale so side not end at ∞ lies in unit circle

$\longrightarrow$

$$\text{Area}, = \int_{\cos(\pi-\alpha)}^{1} \int_{\sqrt{1-x^2}}^{\infty} \frac{1}{y^2}\, dy\, dx$$

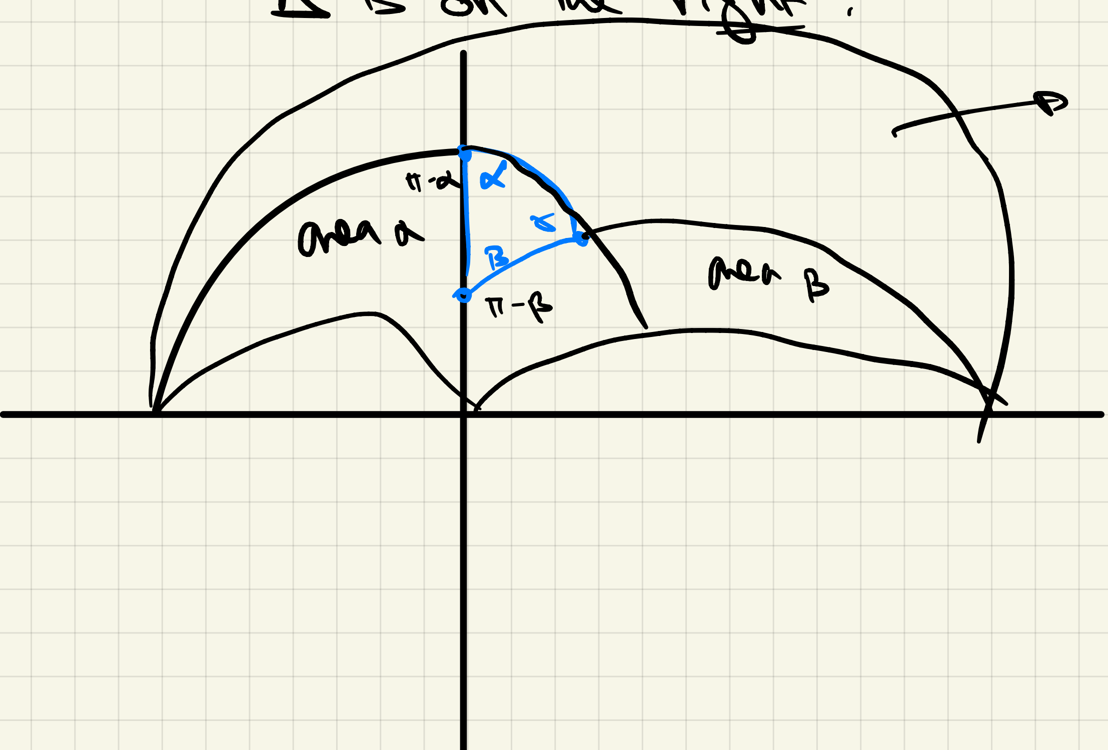
$$\vdots$$

$$= \underline{\pi - \alpha}.$$

Fact! All Δ in $\mathbb{H}^2$ with internal $\angle$ α, β, γ are congruent & have are $\pi - (\alpha + \beta + \gamma)$

In part! $\alpha + \beta + \gamma < \pi$ $\rightsquigarrow$ negative curvature prop
zero curve $(\mathbb{E}^2) = \pi$
pos " $> \pi$.

Move edge ending at vertex with $\angle\alpha$ to y-axis so its highest pt is at i, reflect if necessary so Δ is on the right.

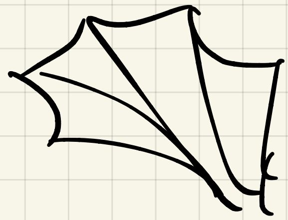


area γ

So, area $\pi - \alpha - \beta - \gamma$

Cor If P is hyperbolic n-gon w/ internal angles $\alpha_1, .. \alpha_n$

$\Rightarrow$ area$(P) = (n-2)\pi - (\sum \alpha_i)$

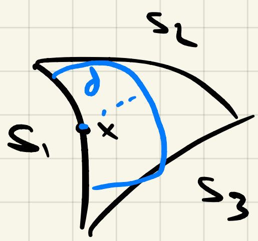


$\rightsquigarrow$ hyperbolic hexagon

Disc. Gauss Bonnet.

A space X is Gromov hyperbolic if $\exists\, \delta > 0$ s.t,
if Δ is a triangle in X w/ sides s_1, s_2, s_3 geodesics
& $x \in s_1$, then $d(x, s_2 \cup s_3) \leq \delta$

Fact H^2 is $\cosh^{-1}(2)$ - Gromov hyperbolic. $\delta = \cosh^{-1}(2)$



$d = d(x, s_2 \cup s_3)$ $\rightsquigarrow$ $\frac{1}{2}$ hole -

$\pi \cosh(d) - \pi \leq \pi$

$\Rightarrow \cosh d \leq 2$

$\Rightarrow d \leq \cosh^{-1}(2)$

Move to origin to see.

Fact α, β, γ w/ $\alpha + \beta + \gamma < \pi$

$\Rightarrow \exists$ triangle w/ angles α, β, γ.