**Lemma)** if $c$ is a htpically non-triv curve on a closed hyperbolic surface is htpc to a unique closed geodesic in $c^*$, if $c$ is simple $\Rightarrow c^*$ is simple

**Pf)**

$$\begin{tikzcd} \mathbb{R} \arrow[r, "\tilde{c}"] \arrow[d] & \mathbb{H}^2 \arrow[d] \\ c : \mathbb{R}/\mathbb{Z} \arrow[r] & X = \mathbb{H}^2/\Gamma \end{tikzcd}$$

$c$ lifts to $\tilde{c} : \mathbb{R} \to \mathbb{H}^2$

$\exists\, \alpha \in \Gamma \setminus \{id\}$ s.t.

$\alpha^n(\tilde{c}(t)) = \tilde{c}(t+n) \quad \forall n \in \mathbb{Z},\ t \in \mathbb{R}$

$\langle \alpha \rangle = c_*(\pi_1(\mathbb{R}/\mathbb{Z}))$

Since $X$ is cl, $\alpha$ is hyperbolic, so it has an axis

We can param axis of $\alpha$ by

$\tilde{a} : \mathbb{R} \to \mathbb{H}^2$ so $\alpha^n(\tilde{a}(t)) = \tilde{a}(t+n)$

This descends to a cl. geodesic $a : \mathbb{R}/\mathbb{Z} \to X$

$\tilde{H} : \mathbb{R} \times [0,1] \to \mathbb{H}^2$ a straight line htpy $\tilde{c} \to \tilde{a}$

$\tilde{H}(t \times [0,1]) = \overline{\tilde{a}(t)\,\tilde{c}(t)}$ ↳ geodesic

↳ param proportional to arc len

By constr, $\alpha^n(\tilde{H}(t,s)) = \tilde{H}(t+n, s) \quad \forall s, n, t$

So, $\tilde{H}$ desc. to htpy $H$ from $a$ to $c$.

$\sup_{t \in [0,1]} d(\tilde{a}(t), \tilde{c}(t)) = R$

by equiv, $\tilde{c}(t)$ is in $N_R(\text{axis}(\alpha))$ $\forall t$. ($N_R$ → Nbhd of set)

So, $\lim_{t \to \infty} c(t) = \alpha^+$ (attracting fixed pt $\alpha$)

$\lim_{t \to -\infty} c(t) = \alpha^-$ → repelling

Say we don't have uniqueness,

$\Rightarrow \exists$ geodesics, $c^*, d^*$ on $X$ which are homotopic $J : \mathbb{R}/\mathbb{Z} \times [0,1] \longrightarrow X$

lifts to htpy b/w geodesics $\tilde{c}^*, \tilde{d}^*$

$$D = \sup_{t \in [0,1]} \ell(J(t \times [0,1]))$$

Then, $\tilde{c}^*$ & $\tilde{d}^*$ lie within $D$ of each other

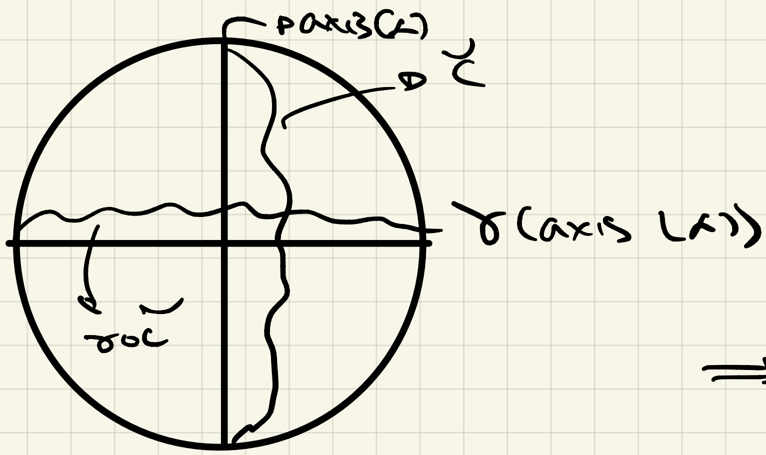
$\Longrightarrow c^* = d^*$ (geodesics det by endpt).

Prop) $c$ is simple

Then note, $\tilde{c}$ & axis($\alpha$) have same endpts

if $c^*$ not simple $\Rightarrow \exists \gamma \in \Gamma \setminus \{id\}$

so $\gamma$(axis($\alpha$)) intersects axis($\alpha$) transversally




$\gamma \circ \tilde{c}$ has same endpts as $\gamma$(axis($\alpha$))

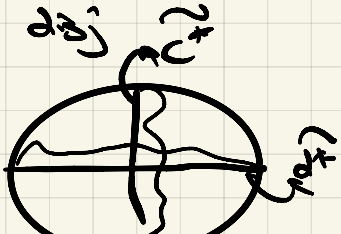
so $\gamma \circ \tilde{c}$ must int $\tilde{c}$ (Jordan curve)

$\Longrightarrow$ $\boxed{c \text{ not simple}}$.

Lemma) if $c, d$ non-htpc, htpcally non triv closed curves on a closed hyperbolic surface

$\Rightarrow c^*, d^*$ are disj

Pf) If $c^*, d^*$ int




$\exists$ lifts $\tilde{c}^*, \tilde{d}^* : \mathbb{R} \to \mathbb{H}^2$ whose images are int geodesic

The htpy b/w $c, c^*$ lifts to htpy b/w $\tilde{c}^*$ & some lift $\tilde{c}$ of $c$

(same w/ $\partial, \partial^*, \tilde{\partial}, \tilde{\partial}^*$)

So, $\tilde{c}, \tilde{c}^*$ same endpts so do $\tilde{\partial}, \tilde{\partial}^*$

$\Rightarrow \tilde{c}, \tilde{\partial}$ intersect $\Rightarrow c, \partial$ intersect!

(Jordan curve w/ $\tilde{c}^*, \tilde{c}$.)

## Pants Decomp.

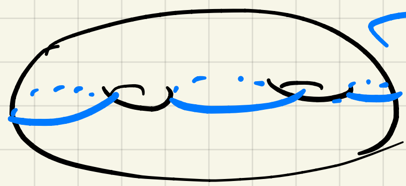
→ non-htpc.

A pants decomposition, is a collection $\{c_1, \ldots, c_n\}$ non-parallel, simple, closed curves s.t every component of $X \setminus \{c_1, \ldots, c_n\}$ is a 3-holed sphere

→ aka pair of pants!

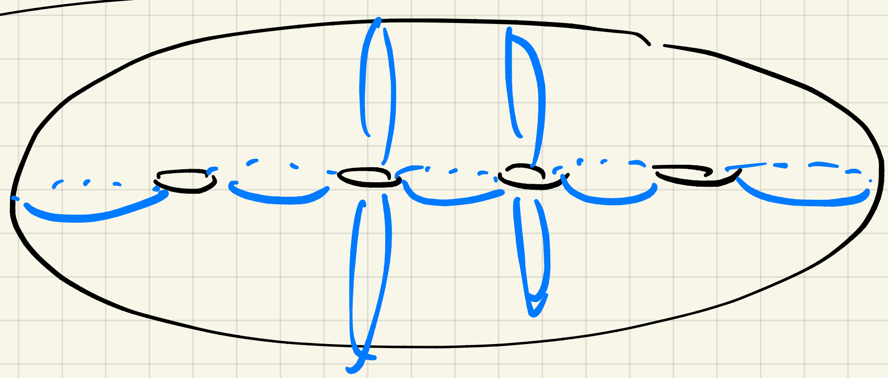
→ not unique.

e.g. 1



→ generalizes to genus g!

like so,



$g+1 + 2(g-2) = 3g-3$ curves!

Each pair of pants has euler char $-1$ & its bdry has euler char $0$ gets glued to each other

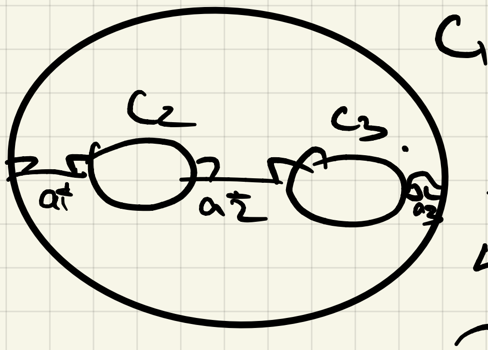
→ genus g surface

$\Rightarrow$ You must have $-\chi(S_g) = 2g-2$ pants

So you have $3g-3$ curves (each pant has 3 curves & each curves lies on ∂ of 2)

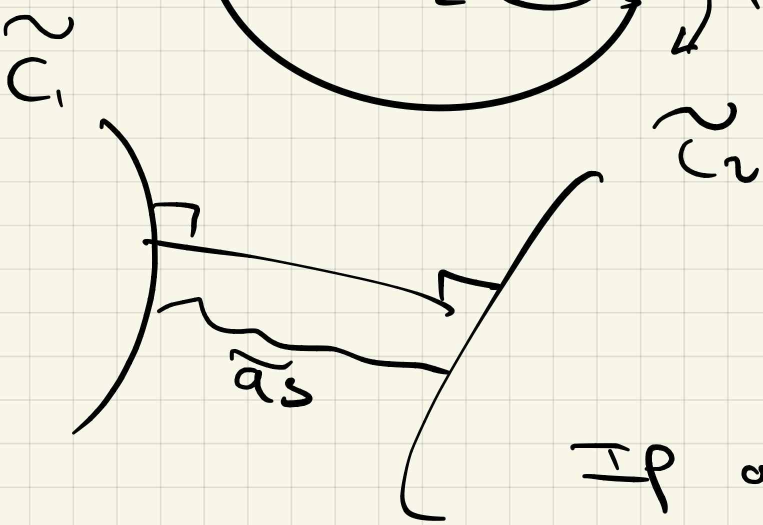
So, each closed hyperbolic surface of genus $g$ admits a geodesic pants decomp w/ $3g-3$ curves, $2g-2$ pants!.

eg)




$a_i$ simple arc joining $c_i$ to $c_{i+1}$ mod 3

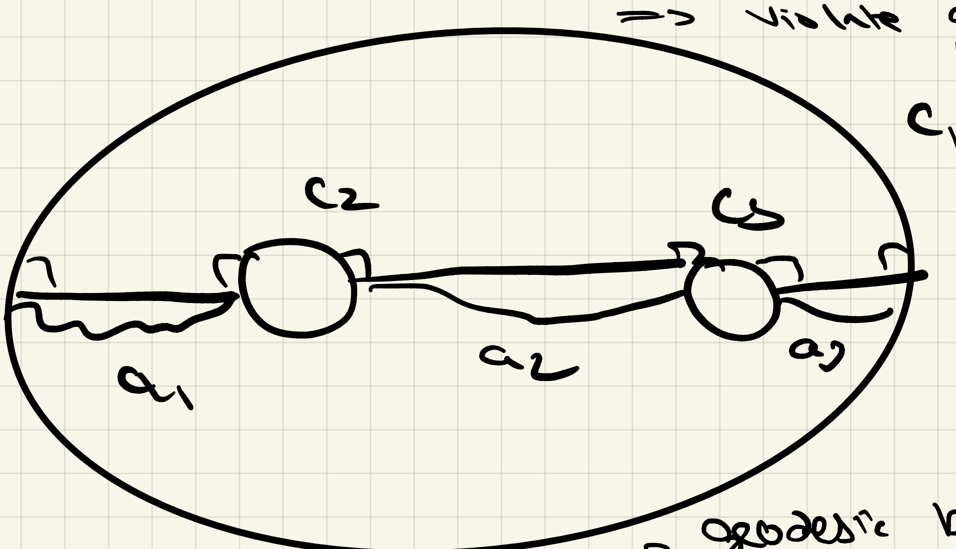
lift to univ cover.




$\exists!$ common perp htpic to $a_i$ $\forall i$

If $a_i^*$, $a_j^*$ int

$\Rightarrow$ violate gauss bonnet.




geodesic bdry (pulled tight)

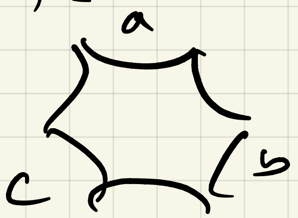
So any geodesic hyperbolic pair of pants is made from 2 hyperbolic hexagons w/ all right angle

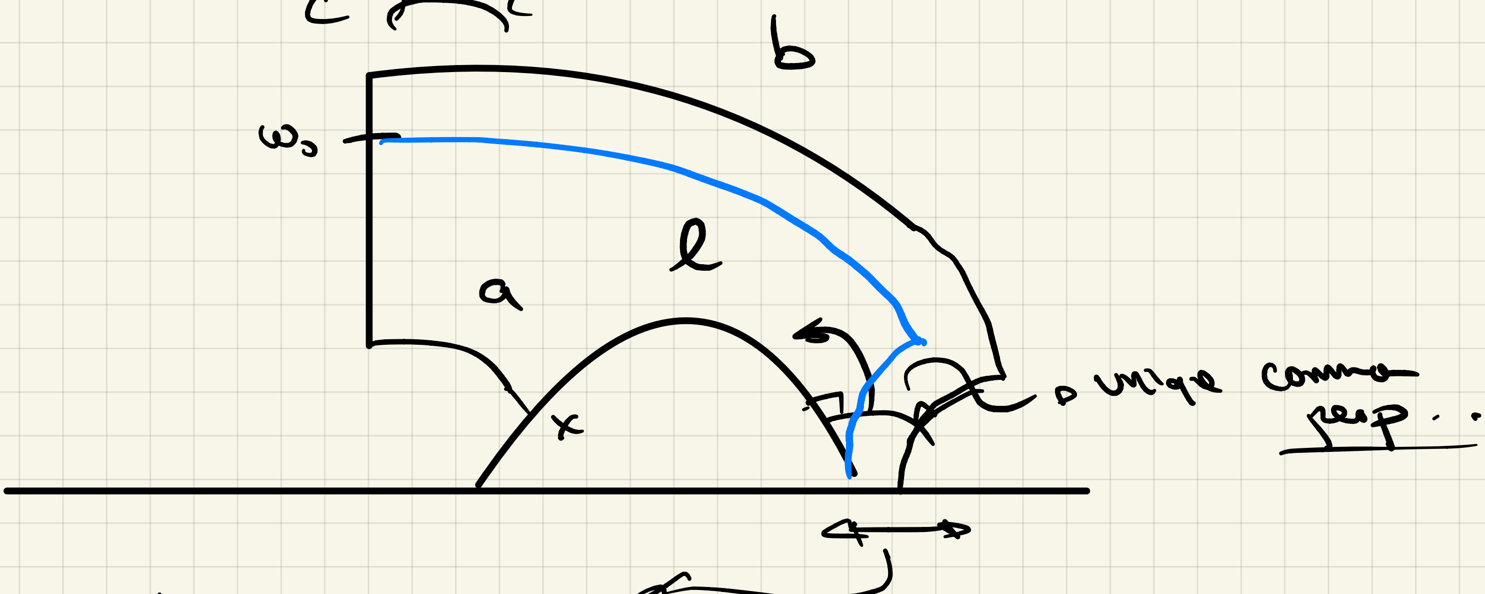
Every all right hexagon has area $4\pi - 6(\pi/2) = \pi$

So each geodesic pants has area $2\pi$.

So every hyperbolic surface $S_g$ genus $g$ has area
$2\pi(2g-2) = 2\pi\,\underline{|\chi(S_g)|}$.

Claim) given $a, b, c > 0$ $\exists!$ (up to cong) all right hex with alternating side lengths $a, b, c$




by moving $w_0$ gives tangency
moving to right, $\ell$ hit c! and unique!