Spse $\Phi : \mathbb{R} \times Y \to Y$ which pres a measure μ.

Φ is ergodic wrt μ $\Longrightarrow$ when $A \subseteq Y$ & is flow inv
($\Phi_t(A) = A$ $\forall t$, then $\mu(A) = 0$ or $\mu(Y - A) = 0$)

(Y compact Haus, μ Radon)

$\Longleftrightarrow$

If $f \in L^2(Y)$ & f is flow inv
$\Rightarrow f$ is constant almost everywhere (a.e).

Thm If X is closed hyp. surface, the geodesic flow is erg wrt Leb measure

Thm Spse Φ pres fin measure μ
If $f : Y \to \mathbb{R}$ is <u>cts</u>. $\leadsto$ ergodic average.

$$M_f(a) = \lim_{t \to \infty} \frac{1}{t} \int_0^t f(\Phi_s(a))\, ds$$

$$= \lim_{t \to \infty} \frac{1}{t} \int_0^t f(\Phi_{-s}(a))\, ds$$

$\exists$ flow inv set A of full meas
s.t. $M_f(a)$ def $\forall a \in A$ & <u>flow inv</u> ($M_f(\Phi_t(a)) = M_f(a)$ <u>$\forall t \in \mathbb{R}$</u>).

If, M_f is const <u>a.e</u>. whenever f is cts
$\Rightarrow$ Φ is <u>ergodic</u>!

Sketch of pf. Say $f : T^1 X \to \mathbb{R}$ cts.

Stable mfd thru $(x, \vec{v})$.

Suppose, A, M_f given by thm.

If $(x, \vec{v}) \in A$ & $(y, \vec{w}) \in L^+(x, v) \cap A$

$$\lim_{s \to \infty} d(\phi_s(x, v), \phi_s(y, \vec{w})) = 0 \quad \text{def.}$$

f cts $T^1 X$ compact

$\Rightarrow K = \max |f(z)| \quad z \in T^1 X$

f unif cts

Given $\varepsilon > 0$, $\exists\ \delta > 0$ so $d(z, w) < \delta$

$|f(z) - f(w)| < \varepsilon$

But $\exists\ T$ s.t. if $s > t$ $\underline{d(\phi_s(x, v), \phi_s(y, \vec{w})) < \delta}$

$t > T$

$$\int_0^t f(\phi_s(x, v))\, ds - \int_0^t f(\phi_s(y, w))\, ds$$

first $0 \to T$

$$< \underline{2TK}, + (t - T)\varepsilon$$

$$|M_f(x, y) - M_f(y, u)| \le \lim_{t \to \infty} \frac{2KT + (t - T)\varepsilon}{t} = \varepsilon$$

ε arb, so $M_f(x, v) - M_f(y, u)$

if $(x, v) \in A$, $(y, u) \in L^+(x, v) \cap A$

Similarly if $(x, v) \in A$ $(y, u) \in L^-(x, y)$ — $\dfrac{\text{unst. mfd}}{(x, v)}$

$$\Rightarrow \lim_{s \to -\infty} d(\phi_s(x, v), \phi_s(y, \vec{w})) = 0$$

use backward flow $M_f(x, \vec{v}) = M_f(y, \vec{w})$

Locally, L^+, L^-, L^0 give coordinates for T^1X

(In $T^1\mathbb{H}^2$ lines in L^+ are just $(\bullet, x, t)$

——— L^0 are just $(\omega, x, \bullet)$

——— L^- are just $(\omega, \bullet, f(\omega, t))$

Lebesgue Measure, is in the same measure class as a product measure,

$$\lambda^+ \oplus \lambda^- \oplus \lambda^0$$

$\to$ some detection of measure 0!

Find $B^+ \times B^- \times B^0 \subseteq A$ & B^+ has full meas λ^+

B B^- ——— λ^-

B^0 ——— λ^0

But M_f is const on B. & B has full meas.

Def A flow Φ pres probability measure μ is mixing if, whenever A, B are measurable $\subseteq Y$, then $\mu(\Phi_t(A) \cap B) \to \Phi(A)\Phi(B)$

Equiv, $f, g \in L^2(Y)$

$$\Rightarrow \int_Y f(\Phi_t(y))\, g(y)\, d\mu(y)$$

$$\longrightarrow \int_Y f\, d\mu \int_Y g\, d\mu$$

Thm | If X closed hyp surf,
the geodesic flow on T^1X is mixing,

Given a periodic traj P, you can construct a prob measure supported on P
$$\left(\frac{\text{arc len on } P}{\ell(P)} \right)$$

$$\mathcal{P}_T = \{ \text{periodic traj on } T^1X \text{ of len} \leq T \}$$
$\hookrightarrow$ finite

$$\frac{1}{\#(\mathcal{P}_T)} \sum_{P \in \mathcal{P}_T} \mu_P \longrightarrow \mu \quad (\text{Leb meas})$$ Thm

Count | $\#(\mathcal{P}_T) \sim \dfrac{e^{1T}}{1T}$, i.e $\dfrac{\#(\mathcal{P}_T)}{\frac{e^T}{T}} \longrightarrow 1$

(Mirzakhani) X is a closed hyperbolic surface of genus g

$\#\{$simple closed geodesic of length $\leq T\} \sim C(X)\, T^{6g-6}$
$\hookrightarrow$ func dep on Teich sp of surf

Thm | The horocycle flow is mixing & erg wrt leb measure.

(Furstenberg) Horocycle flow is uniquely erg, i.e the leb meas is the only flow inv. measure. (false for geodesic b/c have periodic traj)

(P periodic traj $\longrightarrow$ M_p flow inv.)