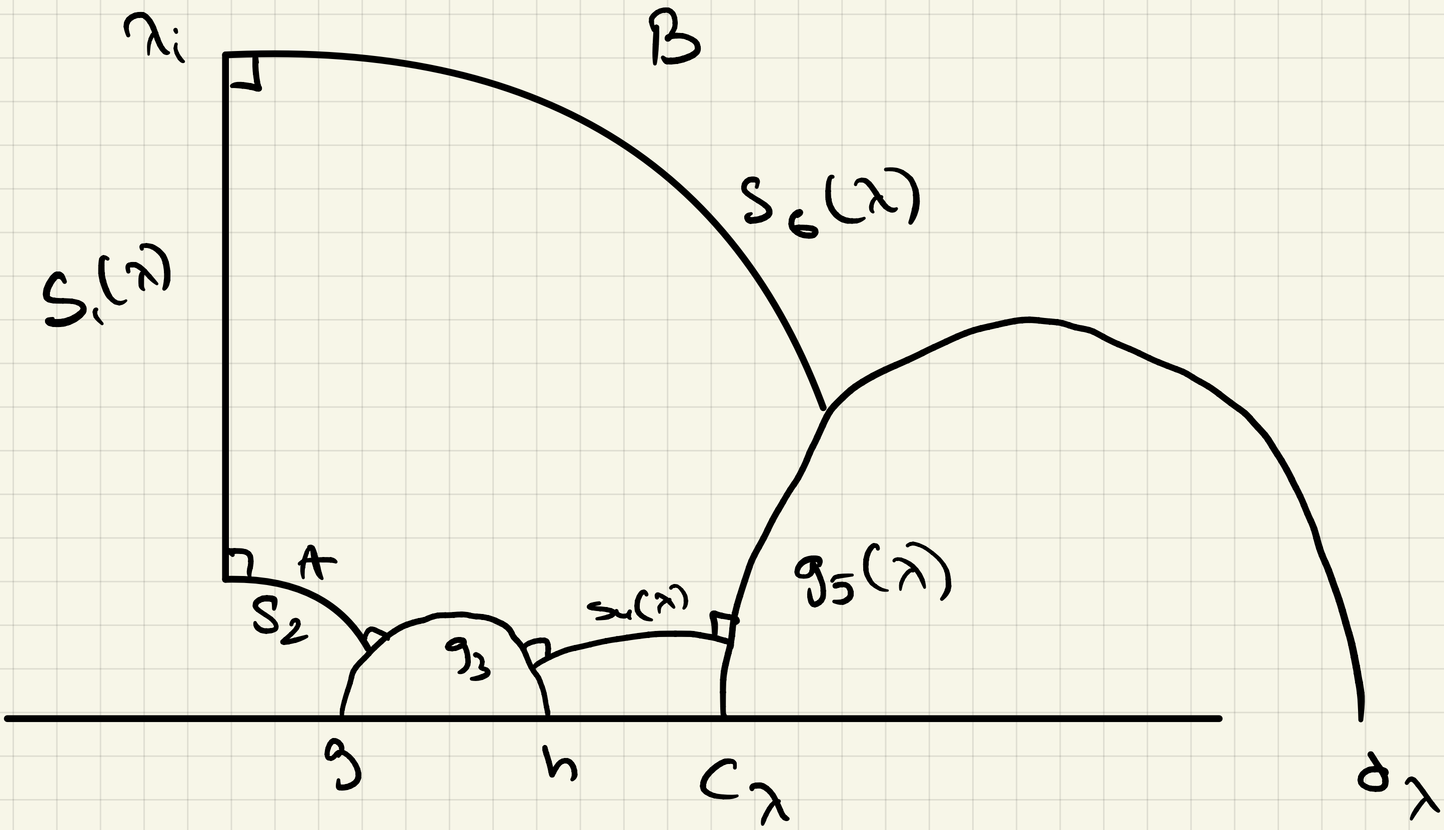
Consider diagram,



$c_\lambda = \lambda c_1 \quad d_\lambda = \lambda d_1$ via isometry $z \mapsto \lambda z$

Choose λ_0 s.t. $\lambda_0 c_1 = h$

If $\lambda > \lambda_0$, $c_\lambda > h$

$\Rightarrow \exists!$ common perp called $s_4(\lambda)$ join g_3 to $g_5(\lambda)$

$C(\lambda) = \ell(s_4(\lambda))$

as $\lambda \to \lambda_0$ $C(\lambda) \to 0$ as $\lambda \to \infty$ $C(\lambda) \to \infty$

↳ intuitive!

↳ think of banana nbd.

Co $\boxed{\text{cts}}$

$\forall c \ \exists$ some λ so $C(\lambda) = c$

if $x < y < z < w$ let $[x, y, z, w] = \dfrac{(z-x)(w-y)}{(y-x)(w-z)}$

$\mathbb{R} \cup \{\infty\}$

cross ratio

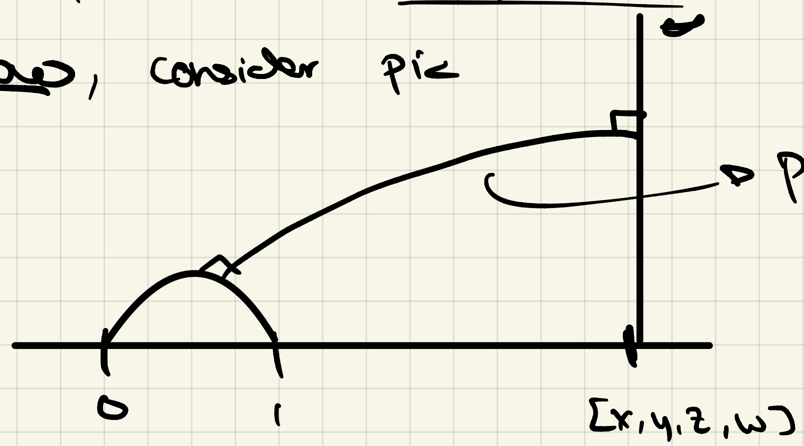
This $[x,y,z,w]$ is inv under $PSL(2,\mathbb{C})$

If x,y,z,w distinct in $\partial\mathbb{H}^2$ then $\exists A \in PSL(2,\mathbb{R})$

$A(x)=0$, $A(y)=1$, $A(w)=\infty$

Then, $A(z) = \underline{[x,y,z,w]}$.

<u>Now</u>, consider pic



$\to$ prse by $PSL(2,\mathbb{R})$.

unique common perp

$\therefore\ d(\overline{xy}, \overline{zw})$ is a func of $[x,y,z,w]$

strictly increasing func of $[x,y,z,w]$ (unique common perp realizes smallest dist)

$C : (\lambda_0, \infty) \longrightarrow \mathbb{R}$

$C = f([g, h, c_1\lambda, d_1\lambda])$ f is strictly increasing

now show * is strictly <u>increasing</u>!

$$* \ \frac{(\lambda c_1 - g)(\lambda d_1 - h)}{(h-g)(\lambda c_1 - \lambda d_1)} = \frac{1}{(h-g)(d-c)} \frac{c_1 d_1 \lambda^2 - \lambda(d_1 g + c_1 h) + gh}{\lambda}$$

derivative

$$\to \frac{1}{(h-g)(d-c)}\left(\frac{\lambda(2c_1 d_1\lambda - d_1 g - c_1 h) - (c_1 d_1 \lambda^2 - \lambda(d_1 g + c_1 h) + gh)}{\lambda^2}\right)$$

$$= \frac{1}{\lambda^2 (h-g)(d-c)}\left(c_1 d_1 \lambda^2 - gh\right)$$

$c_1\lambda > g$, $d_1\lambda > h$ (P <u>increasing</u> $\Rightarrow$ abov positive) upshot → monotonicity of $c(x)$ ⟹ uniqueness

So, get thm

Thm for $A, B, C > 0$

$\exists !$ (up to congr) all-right hexagon with alt sides of len $\underline{A, B, C}$!

⟹ both hex in pair of pants are $\underline{\text{congruent}}$!

Fact given $A, B, C > 0$ $\exists !$ up to congr hyperbolic pair of pants with geodesic bdry curves of length $\underline{A, B, C}$.

$\underline{\text{Btw}}$ $d(\overline{xy}, \overline{zw}) = 2\cosh^{-1}\left(([x,y,z,w])^{-1/2} \right)$

Teichmüller Space

↳ it is the space of all hyperbolic metrics $\underline{\text{in } S}$ ← a closed $\underline{\text{surface}}$

up to $\underline{\text{isotopy}}$ → homotopy through homeomorphism

$\mathcal{T}(S) = \{\text{hyperbolic metrics on } S\} / \mathrm{Diff}_0(S)$

↳ mathcal (T)

↳ diffeo isotopic to identity

Baer Thm If S is closed surface $f, g : S \to S$ homeos

f is isotopic to g ⟺ $\underline{f \text{ is htpic to } g}$

Def Marked hyperbolic surface (X, h)

X hyperbolic surface, $h : S \to X$ is an or pres $\underline{\text{homeo}}$!

$\boxed{\text{fixed}}$

Def) $(X_1, h_1) \sim (X_2, h_2)$ if $\exists j : X_1 \to X_2$

$$\begin{tikzcd} & X_1 \arrow[d, "j"] \\ S \arrow[ur, "h_1"] \arrow[r, "h_2"'] & X_2 \end{tikzcd}$$

$j : X_1 \to X_2$ is an isometry & $j \circ h_1$ is homotopic $\simeq h_2$.

(commute up to htpy).

Fix S closed oriented surface of genus $g \geq 2$.

$$\mathcal{T}(S) = \frac{\{(X,h) \mid X \text{ hyp surface } h : S \to X \text{ or pres homeo}\}}{\sim}$$

(working def)

There is a Moduli sp

$$\mathcal{M}(S) = \frac{\{X \mid X \text{ admits or pres homeo to } S\}}{X \sim Y \text{ if } \exists \text{ or pres isom } X \to Y}$$

$\exists$ forgetful map $\mathcal{T}(S) \to \mathcal{M}(S)$.

$$\begin{tikzcd} \mathcal{T}(S) \overset{?}{\cong} \mathbb{R}^{6g-6} \arrow[d] \\ \mathcal{M}(S) \text{ not a mfd} \end{tikzcd}$$

$\mathrm{Mod}(S) = \{$isotopy class of o.p self homeo of $S\}$

& so

$\mathcal{T}(S)/\mathrm{Mod}(S) \cong \mathcal{M}(S)$

($\to$ orbifold)

Prop disc but not free action.