① Given an abstract simpl. complex $X$

$X$ is a **cone** if $\exists$ a vertex $v$ (cone pt) in $X$ st $X = \mathrm{Star}(v)$ [not nec unique]

i.e. $\forall$ simpl. $\sigma$ in $X$, $\sigma \cup \{v\}$ is a simplex

$$X = \mathrm{Star}_X(v) \overset{\text{WTS}}{=} v * \mathrm{Link}_X(v) = \mathrm{Cone}(\mathrm{Link}_X(v))$$

$\Rightarrow X$ is contractible

② $X$ abs simpl. cmplx

$X$ is a **near cone** assoc to vertex $x$

if, $X = \mathrm{Star}_X(v) \cup \{\sigma\}_{\sigma \in B}$ ⟶ additional simplices glued to $\mathrm{Star}_X(v)$ along whole boundary.

Up to htpy, can contract $\mathrm{Star}_X(v)$

⟶ so $X \simeq \bigvee_{\sigma \in B} \sigma / \partial\sigma$ ⟶ sphere of dim $\sigma$

as $X \simeq X / \mathrm{Star}_X(v)$.

Precisely: $\forall$ simpl. $\sigma$ of $X$ either

① $\sigma \in \mathrm{Star}_X(v)$ or

② $\forall\, w \in \sigma$, $(\sigma \setminus \{w\}) \cup \{v\}$ a simplex in $X$. ⟵ formal bdry condit

⟶ i.e. face $\sigma \setminus \{w\}$ of $\sigma$ is contained in $\mathrm{Star}_X(v)$.

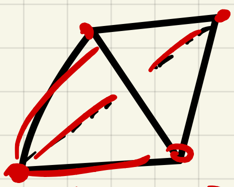
Warm up! $(\Delta^n)^{(k)}$ k skel n-simpl.

vertices, $[n+1]$

simplices $\longleftrightarrow$ all subsets of size $\leq k+1$

Choose $1 = v$

$\subset \Delta^k$ simpl



1 $\quad \mathrm{link}_X(1)$

So, $(\Delta^n)^{(k)} = \mathrm{Star}_X(1) \cup \left\{ \text{subsets size } k+1 \text{ that don't have } 1 \right\}$

not in $\mathrm{Star}_X(1)$ but every proper face is

this satisfies (2)

all the $< k+1$ not with $1$ is in $\mathrm{Star}_X(1)$ as can add $\sigma \cup \{1\}$ & simplex

good near cone decomp

$$\Rightarrow (\Delta^n)^{(k)} \simeq \bigvee_{B} S^k \qquad \#B = \binom{n}{k+1}$$

$$\simeq \bigvee_{\binom{n}{k+1}} S^k$$