

Method-INewton-Raphson Method

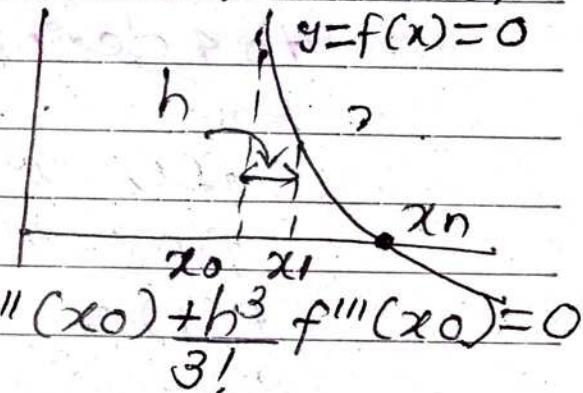
Let  $x_0$  be an initial approximation to the root of the equation  $y = f(x) = 0$ . If  $x_1 = x_0 + h$  be the exact root then  $f(x) = 0$  where  $h$  be difference between  $x_0$  &  $x_1$ , which is very small.

By using Taylor's series

$$f(x_1) = 0$$

$$f(x_0+h) = 0$$

$$f(x_0) + hf'(x_0) + \frac{h^2}{2!} f''(x_0) + \frac{h^3}{3!} f'''(x_0) = 0$$



Since  $h$  is very small, neglecting  $h^2$  & higher power of  $h$ , we get

$$f(x_0) + hf'(x_0) = 0$$

$$\therefore h = -\frac{f(x_0)}{f'(x_0)}$$

$$\therefore x_1 = x_0 + h$$

$$= x_0 - \frac{f(x_0)}{f'(x_0)}$$

Similarly starting with  $x_1$ , better approximation  $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

This process is continued to get the root of desired accuracy.

In general,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

This is called N-R formula.

# Find root of eq<sup>n</sup>  $x^3 - 6x + 4 = 0$  correct to 4 decimal places.

Here

$$f(x) = x^3 - 6x + 4$$

$$f(0) = 4 \text{ (+ve)}$$

$$f(1) = -1 \text{ (-ve)}$$

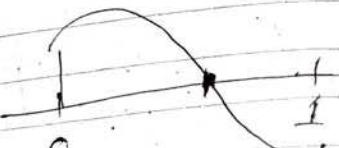
Since  $f(0)$  &  $f(1)$  have opposite sign, the root lies between 0 & 1. Let initial approximation  $x_0 = 0.5$ .

$$\text{Now } f(x) = x^3 - 6x + 4$$

$$f'(x) = 3x^2 - 6$$

We know N-R formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$



The successive approximations are

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.5 - \frac{0.5^3 - 6 \cdot 0.5 + 4}{3(0.5)^2 - 6} = 0.71429$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.71429 - \frac{0.71429^3 - 6 \cdot 0.71429 + 4}{3(0.71429)^2 - 6} = 0.73190$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.73190 - \frac{0.73190^3 - 6 \cdot 0.73190 + 4}{3(0.73190)^2 - 6} = 0.73205$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 0.73205 - \frac{0.73205^3 - 6 \cdot 0.73205 + 4}{3(0.73205)^2 - 6} = 0.73205$$

Since  $x_3 = x_4$

The required root = 0.73205 Ans

2.) Using N-R method to 3 decimal places

$$3x = \cos x + 1$$

Soln:

$$\text{Here } f(x) = 3x - \cos x - 1$$

$$f(0) = -2 \text{ (-ve)}$$

$$f(1) = 1.4597 \text{ (+ve)}$$

Then,

## Exercises

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Let  $x_0 = 0.4$

$$f(x) = 3x - \cos x - 1$$

$$f'(x) = 3 + \sin x$$

By formula

$$\frac{dx}{dt} = x$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

The successive approximations are

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.6127$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.6071$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.6071$$

Since  $x_2 = x_3$

So Root = 0.6071 //

14) Compute a root of each

9)  $x^2 - 5x + 6 = 0$      $x_0 = 5$      $\frac{d}{dx} x \log_{10} x$   
 Soln:

Let  $f(x) = x^2 - 5x + 6 = 0$      $x \cdot \frac{d \log_{10} x + \log_{10} x}{dx}$

$$f(0) = 6 (+ve)$$

$$f(1) = 2 (+ve)$$

$$f(2) = 0$$

$$f(3) = 0$$

$$f(4) = 2$$

$$f(5) =$$

$$0.4343 + \log_{10} x$$

shift calc 1

Q.No.3 Compute a real root.

$$x \log x = 1.2 \text{ correct to 4 decimal places.}$$

Here

$$f(x) = x \log x - 1.2$$

$$f(1) = -1.20000 (-ve)$$

$$f(2) = -0.58794 (-ve)$$

$$f(3) = 0.23136 (+ve)$$

Since  $f(2) & f(3)$  have opposite sign. A root lies between 2 & 3

Let initial approximation  $x_0 = 2.5$

Now,  $f(x) = x \log_{10} x - 1.2$

$$f'(x) = \frac{d}{dx} (x \log_{10} x)$$

$$= x \frac{d}{dx} (\log_{10} x \log x) + \log_{10} x$$

$$= \log_{10} x + 0.43429$$

we have N-R formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

the successive approximations are:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 8.5 - \frac{2.5/\log 8.5 - 1.2}{\log_{10} 8.5 + 0.43429}$$

$$= 8.74651$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 8.74651 - \frac{8.74651/\log 2.94651}{\log_{10} 8.74651 + 0.43429}$$

$$= 8.74065$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 8.74065 - \frac{8.74065/\log 2.74065}{\log_{10} 8.74065 + 0.43429}$$

$$= 8.74065$$

4) Find a real root of square root of 18, correct to 3 decimal places.

$$\text{Let } x = \sqrt{18}$$

$$x^2 = 18$$

$$\therefore f(x) = x^2 - 18$$

$$f(4) = -2.00000$$

$$f(5) = 7(+ve)$$

Since  $f(4)$  &  $f(5)$  have opp. sign. A root lies bet'n 4 & 5

Initial approximation  $x_0 = 4.5$

$$\text{Here } f'(x) = x^2 - 18$$

$$f'(x) = 2x$$

We know,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \approx 4.2500$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 4.2426$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \approx 4.2426$$

$$\text{So root} \approx 4.2426$$

## Method-II

### Bisection method (Bracketing method)

Let  $y=f(x)$  be a continuous function between  $a$  and  $b$  with  $f(a) \neq f(b)$  have opposite sign then a root lies between  $a \& b$

Then first approximation

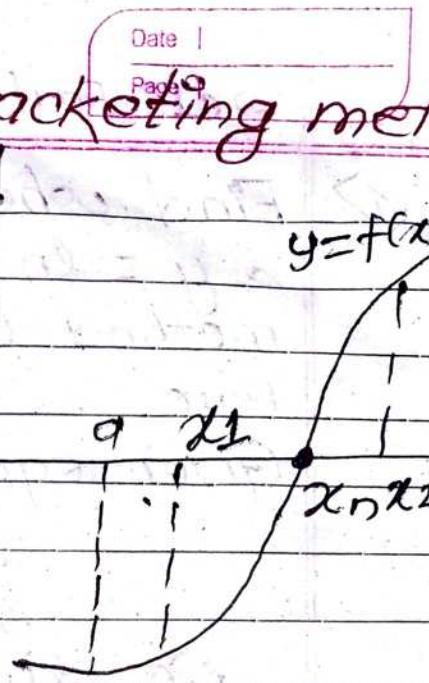
$$x_1 = \frac{a+b}{2}$$

If  $f(x_1) = 0$  then  $x_1$  is the exact root otherwise the root lies between  $a \& x_1$  or  $x_1 \& b$  according to the sign of  $f(x_1)$

In fig  $f(a)$  is negative and  $f(b)$  is +ve then  $x_1 = \frac{a+b}{2}$ . Again  $f(x_1) = -ve$  so the root lies between  $x_1 \& b$ . The second approximation is  $x_2 = \frac{x_1+b}{2}$ .  $f(x_2) +ve$ .

The root lies bet<sup>n</sup>  $x_1 \& x_2$ . The third approximation  $x_3 = \frac{x_1+x_2}{2}$ . This procedure is continued to get root of desired accuracy.

In general  $x_n = \frac{a+b}{2}$  is Bisection formula



setting:  $x = \frac{a+b}{2}$  :  $\ln x - x + 3$

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1) Find where the graph of  $y = x - 3$  &  $y = \ln x$  intersect using bisection method. Get correct to 1 decimal places.

Here

Given equations are

$$y = x - 3$$

$$y = \ln x$$

$$\text{So } \ln x = x - 3$$

$$\text{or } \ln x - x + 3 = 0$$

$$\therefore f(x) = \ln x - x + 3$$

$$\therefore f(4) = 0.38629 (+ve)$$

$$f(5) = -0.39056 (-ve)$$

Since  $f(4)$  and  $f(5)$  have opp. sign.

A root lies between 4 & 5

Let  $a=4, b=5$

We know, the bisection formula

$$x_n = \frac{a+b}{2}$$

Tabulating for successive approximations, we get,

Iteration	$a$ (+ve)	$b$ (-ve)	$x = \frac{a+b}{2}$	$f(x) = \ln x - x + 3$
1.	4	5	4.50000	+ve
2.	4.5	5	4.75000	-ve
3)	4.5	4.75	4.625	-ve

4)	4.5	4.625	4.5625	-ve
5)	4.5	4.5625	4.53125	-ve
6)	4.5	4.53125	4.51563	-ve
7)	4.5	4.51563	4.50782	-ve
8)	4.5	4.50782	4.50391	+ve
9)	4.50391	4.50782	4.50587	-ve
10	4.50391	4.50587	4.50489	+ve
11	4.50489	4.50587	4.50538	-ve
12	4.50489	4.50538	4.50514	+ve
13	4.50514	4.50538	4.50526	-ve
14.	4.50514	4.50526	4.50520	+ve
15	4.50520	4.50526	4.50523	END

Hence  $x_1 = x_{15}$

The required root = 4.5052 Ans

2) Find a root of equation  $xe^x = 3$  by using bracketing method correct to 3 decimal places ( $x_1 = 1, x_2 = 1.5$ )

Solns

$$f(x) = xe^x - 3$$

$$f(1) = -0.2817 \text{ (-ve)}$$

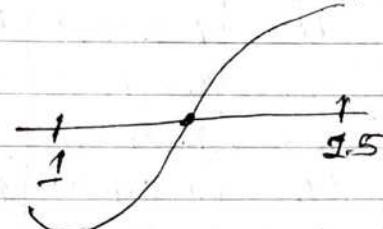
$$f(1.5) = 3.7225 \text{ (+ve)}$$

Since  $f(1)$  &  $f(1.5)$  have opposite sign.

A root lies between 1 & 1.5

Let  $a = x_1 = 1$

$b = x_2 = 1.5$



We know  
the bisection formula:

$$x_n = \frac{a+b}{2}$$

Tabulating for successive approximations, we get.

Iteration	a (ve)	b (ve)	$x = \frac{a+b}{2}$	$f(x) = xe^{-x} - 3$
1	1	1.5	1.2500	+ve
2	1	1.2500	1.1250	+ve
3	1	1.125	1.0625	+ve
4	1	1.0625	1.0313	-ve
5	1.0313	1.0625	1.0469	-ve
6	1.0469	1.0625	1.0547	+ve
7	1.0469	1.0547	1.0508	+ve
8	1.0469	1.0508	1.0489	-ve
9	1.0489	1.0508	1.0499	-ve
10	1.0499	1.0508	1.0504	+ve
11	1.0499	1.0504	1.0502	+ve
12	1.0499	1.0502	1.0501	+ve
13	1.0499	1.0501	1.0500	+ve
14	1.0499	1.0500	1.0499	-ve
15	1.0499	1.0500	1.0499	END

$$\text{root} = 1.0499$$

3) Find negative root

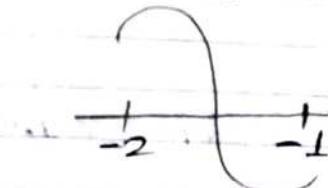
$x^2 - 4x - 10$  correct to 2 decimal

$$f(x) = x^2 - 4x - 10$$

$$f(0) = -10 (-\text{ve})$$

$$f(-1) = -5 (-\text{ve})$$

$$f(-2) = 2 (+\text{ve})$$



Since  $f(-1)$  and  $f(-2)$  have opposite sign so root lies between -2 and -1

$$\text{so } a = -2$$

$$b = -1$$

Then By bisection formula

$$x_n = \frac{a+b}{2}$$

Now Tabulating the successive approximation

Iteration	a (ve)	b (ve)	$x = \frac{a+b}{2}$	$f(x) = x^2 - 4x - 10$
1)	-2	-1	-1.500	-ve
2)	-2	-1.5	-1.750	+ve
3)	-1.750	-1.5	-1.625	-ve
4)	-1.750	-1.625	-1.688	-ve
5)	-1.750	-1.688	-1.718	-ve
6)	-1.750	-1.718	-1.735	-ve
7)	-1.750	-1.735	-1.743	+ve
8)	-1.743	-1.735	-1.739	-ve
9)	-1.743	-1.739	-1.741	-ve

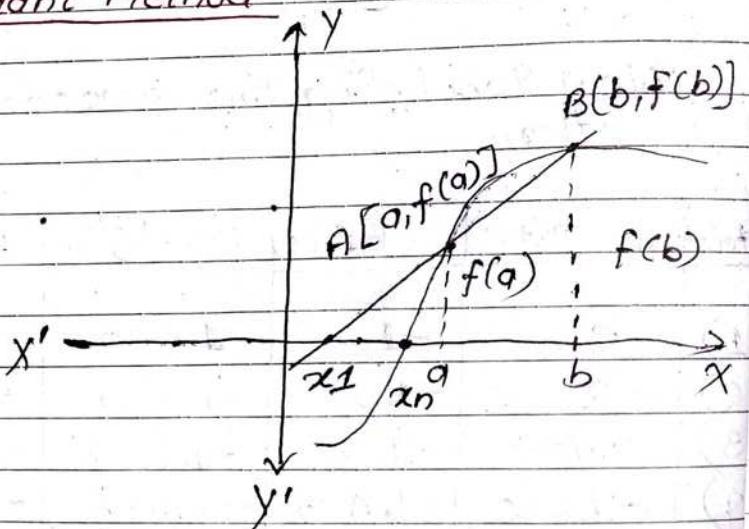
10) 1.743	1.741	-1.742	+ve
11) -1.742	-1.741	-1.742	-ve

Ans = 1.742

Non

## Solution of System of Linear Equations

Method III Secant Method:



This method is an important improvement over the method of false position. It does not require the condition  $f(a) \neq f(b)$  must have the

opposite sign.

Let  $a$  &  $b$  are two initial limits of the interval. The equation of secant (chord) joining the points  $A[a, f(a)]$ ,  $B[b, f(b)]$  is given by.

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1)$$

$$\text{or } y - f(a) = \frac{f(b) - f(a)}{b - a} (x - a)$$

But this line cuts the  $x$ -axis at  $(x, 0)$

$$\therefore 0 - f(a) = \frac{f(b) - f(a)}{b - a} (x_1 - a)$$

$$\text{or, } x_1 - a = -\frac{f(a)(b-a)}{f(b)-f(a)}$$

$$\text{or, } x_1 = -\frac{f(a)(b-a)}{f(b)-f(a)} + a$$

$$\text{or, } x_1 = -\frac{bf(a) + af(a) + af(b) - af(a)}{f(b)-f(a)}$$

$$\therefore x_1 = \frac{af(b) - bf(a)}{f(b)-f(a)}$$

which is our first approximation.

Continuing this procedure we get desire solution.

$$\text{In general } x_n = \frac{af(b) - bf(a)}{f(b)-f(a)}$$

setting  $x =$

No sign

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- 1) Using secant method, Find a real root of  $xe^x - \cos x$  correct to 4 decimal places.

Soln:

Here,

$$f(x) = xe^x - \cos x$$

$$f(0) = -1 \text{ (-ve)}$$

$$f(1) = 0.17798 \text{ (+ve)}$$

$$\text{let } a = 0, b = 1$$

$$f(a) = -1, f(b) = 0.17798 \text{ (+ve)}$$

We know, secant formula

$$x_n = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

The tabulating for successive approximation.

A	B	C	D	$af(b) - bf(a)$
Iteration	a	f(a)	b	$f(b)$
1.	0	-1	1	0.17798
2.	0.31467	-0.51987	0	-1
3.	0.49372	-0.07167	0.31467	-0.51987
4.	0.49372	-0.07167	0.65538	0.46938
5.	0.51513	-0.00796	0.49372	-0.07167
6.	0.51780	0.00014	0.51513	-0.00796
7.	0.51775	-0.00001	0.51780	0.00014
8.			0.51775	-0.00001
9.				(END)

Required root = 0.5177

Setting  $x = \frac{AD - CB}{D - B} - f(x)$

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2. Determine root of  $e^x = x^3 + \cos 25x$ . Using secant method correct to 4 decimal places

Soln:

$$f(x) = e^x - x^3 - \cos 25x$$

$$f(1) = 0.72708$$

$$f(2) = -1.57591$$

Let

$$a = 1, \Rightarrow f(a) = 0.72708$$

$$b = 2 \Rightarrow f(b) = -1.57591$$

We know secant formula,

$$x_n = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

Tabulating for successive approximation

I <sup>n</sup>	a	b	c	d	$x = \frac{af(b) - bf(a)}{f(b) - f(a)}$	$e^x - x^3 - \cos 25x$
1	1	0.72708	2	-1.57591	1.31571	1.85598
2	1.31571	1.35598	1	0.72708	0.63500	2.61704
3	0.63500	2.61704	1.31571	1.35598	2.04766	-1.43698
4	2.04766	-1.43698	0.63500	2.61704	1.54693	0.43338
5	1.54693	0.43338	2.04766	-1.43698	1.66295	1.41919
6	1.66295	1.41919	1.54693	0.43338	1.49593	0.16085
7	1.49593	0.16085	1.66295	1.41919	1.47458	0.49142
8	1.47458	0.49142	1.49593	0.16085	1.50632	0.09311
9	1.50632	0.09311	1.47458	0.49142	1.51374	0.08550
10	1.51374	0.08550	1.50632	0.09311	1.59711	1.47622
11	1.59711	1.47622	1.51374	0.08550	1.50861	0.08711

12	1.50861	0.08711	1.59711	1.47622	1.50306	0.10728
13	1.50306	0.10723	1.50861	0.08711	1.53264	0.21454
14	1.53264	0.21454	1.50306	0.10723	1.47350	0.51396
15	1.47350	0.51396	1.53264	0.21454	1.57501	1.02895
16	1.57501	1.02899	1.47350	0.51396	1.37220	2.82855
17	1.37220	2.32855	1.57501	1.02899	1.73559	0.38590
18	1.73559	-0.38590	1.37220	2.32855	1.68399	0.91834
19	1.68399	0.91834				

## Method-IV

False Position Method

This method is oldest method for computing the real roots of an equation  $f(x)=0$ . For this method we find two numbers  $a$  &  $b$  such that  $f(a)$  &  $f(b)$  are of different signs. Hence a root lies between  $a$  &  $b$ . The curve  $y=f(x)=0$  must cross the  $x$ -axis betw  $a$  &  $b$ .

The equation of chord joining the points  $A[a, f(a)]$  &  $B[b, f(b)]$  is given by

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1)$$

$$\text{or } y - f(a) = \frac{f(b) - f(a)}{b - a} (x - a)$$

But this line cuts  $x$ -axis at  $(x_{10})$

$$\therefore 0 - f(a) = \frac{f(b) - f(a)}{b - a} (x_1 - a)$$

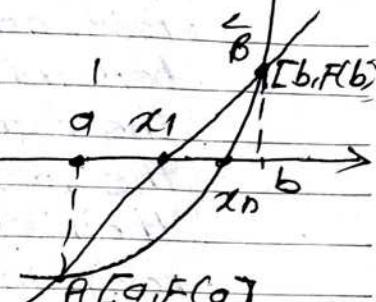
$$\text{or, } (x_1 - a) = \frac{-f(a)(b-a)}{f(b)-f(a)}$$

$$\text{or } x_1 = \frac{-f(a)(b-a)+a}{f(b)-f(a)}$$

$$= \frac{-f(a)b + af(a) + af(b) - af(a)}{f(b)-f(a)}$$

$$= \frac{af(b) - bf(a)}{f(b)-f(a)}$$

$$y = f(x) = \text{chord}$$



which is first approximation  
 If  $f(x_1) = 0$  then  $x_1$  is the exact root. otherwise the root lies betw  $a$  &  $x_1$  or  $x_1$  &  $b$  according the sign of  $f(x_1)$ . Replacing the value by  $x_1$  &  $f(x_1)$  as their sign, we get other appx. This procedure is continual to get root of desired accuracy.

$$x_n = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

Q) Find a real roots  $\log x - \cos x = 0$  correct to 4 decimal places.

$$f(x) = \log x - \cos x$$

$f(0)$  Not defined

$$f(1) = -0.54030 \text{ (-ve)}$$

$$f(2) = 0.71718 \text{ (+ve)}$$

Since  $f(1)$  &  $f(2)$  have opp signs. The root lies betw 1 & 2

$$\text{let } a = 1, f(a) = -0.54030$$

$$b = 2, f(b) = 0.71718$$

We know,

False position formula

$$x_n = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

Tabulating for successive approximation we get

It <sup>n</sup>	a <u>A</u>	f(a) <u>B</u>	b <u>C</u>	f(b) <u>D</u>	$x = \frac{af(b) - bf(a)}{f(b) - f(a)}$	
					-ve	+ve
1)	-0.54030	2				
1)	1	-0.54030	<u>1.42967</u>	0.71718	1.42967	0.014586 (+ve)
2)	1	-0.54030	<u>0.01458</u>	0.01458	1.41838	-0.000036 (-ve)
3)	1.41838	-0.00003	1.42967	0.01458	1.41840	-0.0000096 (-ve)
4)	1.41840	-0.000004	1.42967	0.01458	1.41840	END

Req<sup>n</sup> root = 1.4184 (Ans)

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1) Find a real root of  $e^{\cos x} - \sin x - 1 = 0$  correct to 4 decimal places using false position method.

so/n:

$$f(x) = e^{\cos x} - \sin x - 1$$

$$f(0) = 1.71828 \quad (+ve)$$

$$f(1) = -0.12495 \quad (-ve)$$

Now,

$$a=0, f(a)=1.71828$$

$$b=1 \quad f(b)=-0.12495$$

Since  $f(1)$  and  $f(2)$  have opp. sign

So root lies betw 1 and 2

Now,

From False position.

$$x_n = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

Tabulating successive approximation

	$a$	$b$	$f(a)$	$f(b)$	$x_n = \frac{af(b) - bf(a)}{f(b) - f(a)}$	
1	0	1	1.71828	-0.12495	0.93221	0.01201 (+ve)
2	0.93221	1	0.01201	-0.12495	0.93815	-0.00017 (-ve) 0.000005
3)	0.93221	0.93815	0.01201	-0.00017	0.93807	0 (+ve)
4)	0.93807	0.93815	0.000005	-0.00017	0.93807	ENO

Ans = 0.93807

## Method V

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### Fixed Point Iteration Method

Let  $f(x)=0$  be the given equation where roots are to be determined. In iteration method we write the given eqn in the form of  $x = \phi(x)$ .

Let  $x=x_0$  be the initial approximation of the required root, then the first approximation is given by  $x_1 = \phi(x_0)$

Second, third & other approximations are

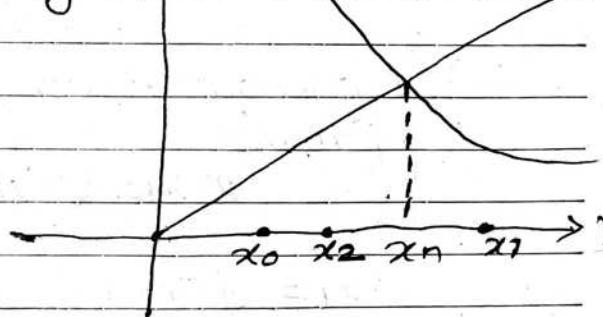
$$x_2 = \phi(x_1)$$

$$x_3 = \phi(x_2) \text{ & so on}$$

In General

$$[x_{n+1} = \phi(x_n)]$$

It is called fixed point iteration formula  
This process is continued to get the root of desired accuracy.



1) By using fixed point iteration  
Find a positive root of  $x^3+x^2-1=0$   
correct to 4 decimal places  
Soln:

$$f(x) = x^3 + x^2 - 1$$

$$f(0) = -1 \text{ (-ve)}$$

$$f(1) = 1 \text{ (+ve)}$$

Since  $f(0)$  &  $f(1)$  have opp signs  
A root lies betw 0 & 1.

Let initial approximation  $x_0=0.5$

Now, Rearranging given eq<sup>n</sup>

we get

$$x^3 + x^2 - 1 = 0$$

$$\text{or } x^2(x+1) = 1$$

$$\text{or, } x = \frac{1}{\sqrt{x+1}} \phi(x)$$

$$\text{where } \phi(x) = \frac{1}{\sqrt{x+1}}$$

We know fixed point iteration formula

$$x_{n+1} = \phi(x_n)$$

$$x_1 = \phi(x_0) = \frac{1}{\sqrt{0.5+1}} = 0.8650$$

$$x_2 = \phi(x_1) = \frac{1}{\sqrt{0.8650+1}} = 0.74196$$

$$x_3 = \phi(x_2) = \frac{1}{\sqrt{0.74196+1}} = 0.75767$$

$$x_4 = \phi(x_3) = \frac{1}{\sqrt{0.75767+1}} = 0.75428$$

$$x_5 = \phi(x_4) = \frac{1}{\sqrt{0.75428+1}} = 0.75501$$

$$x_6 = \phi(x_5) = \frac{1}{\sqrt{0.75501+1}} = 0.75485$$

$$x_7 = \phi(x_6) = \frac{1}{\sqrt{0.75485+1}} = 0.75485$$

Hence  $x_6 = x_7$

So req<sup>n</sup> root = 0.7548

$$\text{Error} = |0.75488 - 0.75485| \\ = 0.00003.$$

2) Using fixed point iteration method. Find square root of 5. correct to 4 decimal places

$$\text{Here } x = \sqrt{5}$$

$$\text{or } x^2 = 5$$

$$\text{or } x^2 - 5 = 0$$

$$\therefore f(x) = x^2 - 5$$

$$f(0) = -5$$

$$f(2) = -1 \text{ (-ve)}$$

$$f(3) = 4 \text{ (+ve)}$$

Since  $f(2)$  &  $f(3)$  have opp sign  
A root lies b/w 2 & 3

Let initial approximation

$$x_0 = 2.5$$

Now Rearranging the given eq we get

Now,

$$x^2 - 5 = 0$$

$$\text{or } x^2 = 5$$

$$\therefore x = \frac{5}{x} = \phi(x)$$

$$\text{where } \phi(x) = \frac{5}{x}$$

By formula  $x_{n+1} = \phi(x_n)$

The successive approximations are

$$x_1 = \phi(x_0) = \frac{5}{2.5} = 2$$

$$x_2 = \phi(x_1) = \frac{5}{2} = 2.5$$

$$x_3 = \phi(x_2) = \frac{5}{2.5} = 2$$

$$x_4 = \phi(x_3) = \frac{5}{2} = 2.5$$

This process does not converge to the solution. This is called oscillatory divergence.

Again, rearranging the given eq we get

$$x^2 = 5$$

$$\text{or } x = \frac{5}{x}$$

$$\text{or } x + x = \frac{5}{x} + x$$

$$\text{or } 2x = \frac{5+x^2}{x}$$

$$\therefore x = \frac{5+x^2}{2x} = \phi(x)$$

Now, the successive approximations

$$x_1 = \phi(x_0) = \frac{5+2.5^2}{2(2.5)} = 2.25000$$

$$x_2 = \phi(x_1) = \frac{5+2.25000^2}{2(2.25000)} = 2.23611$$

$$x_3 = \phi(x_2) = \frac{5+2.23611}{2(2.23611)} = 2.23607$$

$$x_4 = \phi(x_3) = \frac{5+2.23607}{2(2.23607)} = 2.23607$$

3)  $2x - \log_{10} x = 7$  4 decimal

$$f(x) = 2x - \log_{10} x - 7$$

$$f(1) = -5(-ve)$$

$$f(3) = -1.47712(-ve)$$

$$f(4) = 0.39739(+ve)$$

## Chapter-Two

Here  $f(3)$  and  $f(4)$  have opp sign. So root lies betw 3 and 4

$$\text{Now, } x_0 = 3.2$$

Rearranging the given eq<sup>n</sup> is

$$2x - \log_{10} x = 7 \\ \therefore x = \frac{7 + \log_{10} x}{2}$$

Then the successive approx

$$x_{n+1} = \phi(x_n)$$

$$x_1 = \phi(x_0) = \frac{7 + \log(x_0)}{2} = 8.75257$$

$$x_2 = \phi(x_1) = \frac{7 + \log(x_1)}{2} = 3.78716$$

$$x_3 = \phi(x_2) = \frac{7 + \log(x_2)}{2} = 3.78916$$

$$x_4 = \phi(x_3) = \frac{7 + \log(x_3)}{2} = 3.78927$$

$$x_5 = \phi(x_4) = \frac{7 + \log(x_4)}{2} = 3.78928$$

Now,

$$\text{Required root} = 3.7892$$

## Solution of System of linear Equations.

### Method I > Gauss Elimination Method.

In this method, the unknowns are eliminated successively and the system is reduced to an upper triangular matrix form from which the unknowns are found by back substitution

Let us consider a system of linear equations

$$a_1 x + b_1 y + c_1 z = d_1$$

$$a_2 x + b_2 y + c_2 z = d_2$$

$$a_3 x + b_3 y + c_3 z = d_3$$

The augmented matrix form is

$$\left[ \begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array} \right]$$

With the help of  $a_1$  make  $a_2$  &  $a_3$  to zero  
we get

$$\left[ \begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ 0 & b_2' & c_2' & d_2' \\ 0 & b_3' & c_3' & d_3' \end{array} \right]$$

with the help of  $b_2'$  make  $b_3'$  to zero,

$$\left[ \begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ 0 & b_2' & c_2' & d_2' \\ 0 & 0 & c_3''' & d_3''' \end{array} \right]$$

which is an upper triangular matrix  
Using back substitution method

$$\text{From } R_3, c_3''z = d_3'' \Rightarrow z$$

$$\text{From } R_2, b_2'y + c_2'z = d_2' \Rightarrow y$$

$$\text{From } R_1, a_1x + b_1y + c_2z = d_1 \Rightarrow x$$

Hence we get values of  $x, y \text{ & } z$ .

1) Using Gauss Elimination method,

$$2x+y+z=4$$

$$4x+2y+3z=4$$

$$x-y+z=0$$

Here augmented matrix form is

$$\left[ \begin{array}{ccc|c} 2 & 1 & 1 & 4 \\ 4 & 2 & 3 & 4 \\ 1 & -1 & 1 & 0 \end{array} \right]$$

$$R_2 \rightarrow R_2 - 2R_1, R_3 \rightarrow 2R_3 - R_1$$

$$\left[ \begin{array}{ccc|c} 2 & 1 & 1 & 4 \\ 0 & 0 & 1 & -4 \\ 0 & -3 & 1 & -4 \end{array} \right]$$

$R_2 \leftrightarrow R_3$ , we get

$$\left[ \begin{array}{ccc|c} 2 & 1 & 1 & 4 \\ 0 & -3 & 1 & -4 \\ 0 & 0 & 1 & -4 \end{array} \right]$$

then it is an upper triangular matrix

Using back substitution method

$$\text{From } R_3, 1z = -4 \Rightarrow z = -4$$

$$\text{From } R_2, -3y + z = -4$$

$$\text{or } -3y - 4 = -4$$

$$\therefore y = 0$$

$$\text{From } R_1, 2x + y + z = 4$$

$$2x = 4$$

$$\text{or } 2x = 8$$

$$\therefore x = 4$$

$$\text{Hence } x = 4$$

$$y = 0$$

$$z = -4$$

Solve the system of linear equations by using Gauss Elimination method.

Solving:

$$6x_1 - 2x_2 + 2x_3 + 4x_4 = 16$$

$$12x_1 - 8x_2 + 6x_3 + 10x_4 = 26$$

$$3x_1 - 13x_2 + 9x_3 + 3x_4 = -19$$

$$-6x_1 + 4x_2 + x_3 - 18x_4 = -34.$$

Here augmented matrix form

$$\left[ \begin{array}{cccc|c} 8 & -2 & 2 & 4 & : 16 \\ 12 & -8 & 6 & 10 & : 26 \\ 3 & -13 & 9 & 3 & : -19 \\ -6 & 4 & 1 & -18 & : -34 \end{array} \right]$$

$$R_1 \rightarrow R_1 - 2R_2$$

$$R_2 \rightarrow R_2 \div 2, \text{ we get.}$$

$$\left[ \begin{array}{cccc|c} 3 & -1 & 1 & 2 & : 8 \\ 6 & -4 & 3 & 5 & : 13 \\ 3 & -13 & 9 & 3 & : -19 \\ -6 & 4 & 1 & -18 & : -34 \end{array} \right]$$

$$R_2 \rightarrow R_2 - 2R_1$$

$$R_3 \rightarrow R_3 - R_1$$

$$R_4 \rightarrow R_4 + 2R_1$$

$$\left[ \begin{array}{cccc|c} 3 & -1 & 1 & 2 & : 8 \\ 0 & -2 & 1 & 1 & : -3 \\ 0 & -12 & 8 & 1 & : -27 \\ 0 & 2 & 3 & -14 & : -18 \end{array} \right]$$

$$R_2 \rightarrow R_2 - 6R_1$$

$$R_4 \rightarrow R_4 + R_2$$

$$\left[ \begin{array}{cccc|c} 3 & -1 & 1 & 2 & : 8 \\ 0 & -2 & 1 & 1 & : -3 \\ 0 & 0 & 2 & -5 & : -9 \\ 0 & 0 & 4 & -13 & : -21 \end{array} \right]$$

$$\left[ \begin{array}{cccc|c} 3 & -1 & 1 & 2 & : 8 \\ 0 & -2 & 1 & 1 & : -3 \\ 0 & 0 & 2 & -5 & : -9 \\ 0 & 0 & 0 & -3 & : -3 \end{array} \right]$$

which is an upper triangular matrix

Using back substitution we get

$$\text{From } R_4 : -3x_4 = -3 \Rightarrow x_4 = 1$$

$$\text{From } R_3 : 2x_3 - 5x_4 = -9 \Rightarrow x_3 = -2$$

$$\text{From } R_2 : -2x_2 + x_3 + x_4 = -3 \Rightarrow x_2 = 1$$

$$\text{From } R_1 : 3x_1 - x_2 + x_3 + 2x_4 = 8 \Rightarrow x_1 = 3$$

$$\text{Hence } x_1 = 3$$

$$x_2 = 1$$

$$x_3 = -2 \quad \text{answer}$$

$$x_4 = 1$$

## Gauss Elimination Method with partial pivoting:

In the first step of Gauss Elimination method, the numerically largest coefficient of  $x$  is chosen from all the equations and brought as the first pivot by interchanging 1<sup>st</sup> equation with the equation having largest coefficient of  $x$ .

In second step, the numerically largest coefficient of  $y$  is chosen from the remaining equations and brought as the second pivot by interchanging the second equation with the equation having largest coefficient of  $y$ . This process is continued till we arrive at equ<sup>n</sup> with single variable. This modified procedure is called partial pivoting.

1)

# Solve the following by using Gauss Elimination with partial pivoting.

$$2x + 2y + z = 6$$

$$4x + 3y + 3z = 4$$

$$x - y + z = 0$$

Solving

Here the augmented matrix form is

$$\left[ \begin{array}{ccc|c} 2 & 2 & 1 & :6 \\ 4 & 2 & 3 & :4 \\ 1 & -1 & 1 & :0 \end{array} \right]$$

$R_1 \leftrightarrow R_2$ , we get

$$\left[ \begin{array}{ccc|c} 4 & 2 & 3 & :9 \\ 2 & 2 & 1 & :6 \\ 1 & -1 & 1 & :0 \end{array} \right]$$

$$R_2 \rightarrow 2R_2 - R_1$$

$$R_3 \rightarrow 4R_3 - R_1$$

$$\left[ \begin{array}{ccc|c} 4 & 2 & 3 & :9 \\ 0 & 2 & 1 & :6 \\ 0 & -6 & 1 & :-4 \end{array} \right]$$

$R_2 \leftrightarrow R_3$  we get

$$\left[ \begin{array}{ccc|c} 4 & 2 & 3 & :9 \\ 0 & -6 & 1 & :-4 \\ 0 & 2 & -1 & :8 \end{array} \right]$$

$$R_3 \rightarrow 3R_3 + R_2$$

$$\left[ \begin{array}{ccc|c} 4 & 2 & 3 & 4 \\ 0 & -6 & 1 & -4 \\ 0 & 0 & -2 & 20 \end{array} \right]$$

which is an upper triangular matrix.

Using back substitution, we get

$$\text{From } R_3, -2z = 20 \Rightarrow z = -10$$

$$\text{From } R_2 -6y + z = -4$$

$$\text{or, } -6y - 10 = -4$$

$$\text{or, } -6y = 6$$

$$\therefore y = -1$$

$$\text{From } R_1 4x + 2y + 3z = 4$$

$$\text{or, } 4x - 2 - 30 = 4$$

$$\text{or, } 4x = 36$$

$$\therefore x = 9$$

Hence

$$\left. \begin{array}{l} x = 9 \\ y = -1 \\ z = -10 \end{array} \right\}$$

- Q) Solve the equations by using Gauss Elimination method with partial pivoting

$$x + y + z + w = 2$$

$$x + y + 3z - 2w = -6$$

$$2x + 3y - z + 2w = 7$$

$$x + 2y + z - w = -2$$

The augmented matrix form

$$\left[ \begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 3 & -2 & -6 \\ 2 & 3 & -1 & 2 & 7 \\ 1 & 2 & 1 & -1 & -2 \end{array} \right]$$

Now,  $R_1 \leftrightarrow R_3$

$$\left[ \begin{array}{cccc|c} 2 & 3 & -1 & 2 & 7 \\ 1 & 1 & 3 & -2 & -6 \\ 1 & 1 & 1 & 1 & 2 \\ 1 & 2 & 1 & -1 & -2 \end{array} \right]$$

$$R_2 \rightarrow 2R_2 - R_1$$

$$R_3 \rightarrow 2R_3 - R_1$$

$$R_4 \rightarrow 2R_4 - R_1$$

$$\left[ \begin{array}{cccc|c} 2 & 3 & -1 & 2 & 7 \\ 0 & -1 & 7 & -6 & -19 \\ 0 & -1 & 3 & 0 & -3 \\ 0 & 1 & 3 & -4 & -11 \end{array} \right]$$

$$R_3 \leftrightarrow R_2$$

$$\left[ \begin{array}{cccc|c} 2 & 3 & -1 & 2 & 7 \\ 0 & 1 & 3 & -4 & -11 \\ 0 & -1 & 3 & 0 & -3 \\ 0 & -1 & 7 & -6 & -19 \end{array} \right]$$

$$R_3 \rightarrow R_3 + R_2$$

$$R_4 \rightarrow R_3 + R_2$$

$$\left[ \begin{array}{cccc|c} 2 & 3 & -1 & 2 & 7 \\ 0 & 1 & 3 & -4 & -11 \\ 0 & 0 & 6 & -4 & -14 \\ 0 & 0 & 10 & -10 & -30 \end{array} \right]$$

$$R_3 \leftrightarrow R_4$$

$$\left[ \begin{array}{cccc|c} 2 & 3 & -1 & 2 & 7 \\ 0 & 1 & 3 & -4 & -11 \\ 0 & 0 & 10 & -10 & -30 \\ 0 & 0 & 6 & -4 & -14 \end{array} \right]$$

$$R_4 \rightarrow 5R_4 - 3R_3$$

$$\left[ \begin{array}{cccc|c} 2 & 3 & -1 & 2 & 7 \\ 0 & 1 & 3 & -4 & -11 \\ 0 & 0 & 10 & -10 & -30 \\ 0 & 0 & 0 & 10 & 20 \end{array} \right]$$

~~$$\text{From } R_4 \quad 10z = 2$$~~

~~$$\text{From } R_3 \quad 10x - 10z = -30$$~~

~~$$10x = -10$$~~

~~$$x = -1$$~~

$$\text{From } R_2 \quad 1y + 3z - 4w = -11$$

$$\text{or } y - 3 - 4 \times 2 = -11$$

$$y = 0$$

$$\text{From } R_1 \text{ So } 2x + 3y - 1z + 2 = 7$$

$$2x = 7 - 3 \times 0 + 1 - 2$$

$$x = 1$$

$$\text{Hence } x = 1$$

$$y = 0$$

$$z = -1$$

$$w = 2$$

Method II>

### Gauss Jordan Method

This method is a modified form of Gauss-Elimination method. In this method the coefficient matrix is reduced to unit matrix and get the unknowns directly without using back substitution.

Let us consider a system,

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

The augmented matrix form is

$$\begin{bmatrix} a_1 & b_1 & c_1 : d_1 \\ a_2 & b_2 & c_2 : d_2 \\ a_3 & b_3 & c_3 : d_3 \end{bmatrix}$$

-  $R_1 \rightarrow R_1$ , we get

$$\begin{bmatrix} 1 & \frac{b_1}{a_1} & \frac{c_1}{a_1} : d_1' \\ a_2 & b_2 & c_2 : d_2 \\ a_3 & b_3 & c_3 : d_3 \end{bmatrix}$$

With help of 1 making  $a_2$  &  $a_3$  to zero, we get

$$\begin{bmatrix} 1 & b_1' & c_1' : d_1' \\ 0 & b_2' & c_2' : d_2' \\ 0 & b_3' & c_3' : d_3' \end{bmatrix}$$

$R_2 \rightarrow R_2 \div b_2'$  we get

$$\begin{bmatrix} 1 & b_1' & c_1' : d_1' \\ 0 & 1 & c_2' : d_2'' \\ 0 & b_3' & c_3' : d_3' \end{bmatrix}$$

With help of 1 make  $b_1'$  &  $b_3'$  to zero, we get

$$\begin{bmatrix} 1 & 0 & c_1' : d_1'' \\ 0 & 1 & c_2' : d_2'' \\ 0 & 0 & c_3' : d_3' \end{bmatrix}$$

$R_3 \rightarrow R_3 \div c_3''$  we get

$$\begin{bmatrix} 1 & 0 & c_1' : d_1'' \\ 0 & 1 & c_2' : d_2'' \\ 0 & 0 & 1 : d_3'' \end{bmatrix}$$

Making  $c_1''$  &  $c_2''$  to zero with help of 1 we get

$$\begin{bmatrix} 1 & 0 & 0 : d_1''' \\ 0 & 1 & 0 : d_2''' \\ 0 & 0 & 1 : d_3''' \end{bmatrix}$$

$$x = d_1'''$$

$$y = d_2'''$$

$$z = d_3'''$$

## Solution of System of Linear Equations

Solve:

$$2x_1 + 4x_2 - 6x_3 = -8$$

$$x_1 + 3x_2 + x_3 = 10$$

$$2x_1 - 4x_2 - 2x_3 = -12$$

Here augmented matrix form is

$$\begin{bmatrix} 2 & 4 & -6 : -8 \\ 1 & 3 & 1 : 10 \\ 2 & -4 & -2 : -12 \end{bmatrix}$$

$$-1 + \frac{1}{2} \times 9$$

$R_1 \rightarrow R_1 \div 2$ , we get

$$\left[ \begin{array}{cccc|c} 1 & 2 & -3 & : & -4 \\ 1 & 3 & 1 & : & 10 \\ 2 & -4 & -2 & : & -12 \end{array} \right]$$

$R_2 \rightarrow R_2 - R_1$

$R_3 \rightarrow R_3 - 2R_1$

$$\left[ \begin{array}{cccc|c} 1 & 2 & -3 & : & -4 \\ 0 & 1 & 4 & : & 14 \\ 0 & -8 & 4 & : & -4 \end{array} \right]$$

$R_1 \rightarrow R_1 + 2R_2$

$R_3 \rightarrow R_3 + 8R_2$

$$\left[ \begin{array}{cccc|c} 1 & 0 & -11 & : & -32 \\ 0 & 1 & 4 & : & 14 \\ 0 & 0 & 36 & : & 108 \end{array} \right]$$

$R_3 \rightarrow R_3 \div 36$

$$\left[ \begin{array}{cccc|c} 1 & 0 & -11 & : & -32 \\ 0 & 1 & 4 & : & 14 \\ 0 & 0 & 1 & : & 3 \end{array} \right]$$

$R_1 \rightarrow R_1 + 11R_3$

$R_2 \rightarrow R_2 - 4R_3$

$$\left[ \begin{array}{cccc|c} 1 & 0 & 0 & : & 1 \\ 0 & 1 & 0 & : & 2 \\ 0 & 0 & 1 & : & 3 \end{array} \right]$$

Hence  $x_1 = 1$      $x_3 = 3$   
 $x_2 = 2$

2)  $2x_1 + 2x_2 - x_3 + x_4 = 4$

$$4x_1 + 3x_2 - x_3 + 2x_4 = 6$$

$$8x_1 + 5x_2 - 3x_3 + 4x_4 = 12$$

$$x_1 + 3x_2 - 2x_3 + 2x_4 = 6$$

The augmented matrix form

$$\left[ \begin{array}{cccc|c} 2 & 2 & -1 & 1 & : & 4 \\ 4 & 3 & -1 & 2 & : & 6 \\ 8 & 5 & -3 & 4 & : & 12 \\ 1 & 3 & -2 & 2 & : & 6 \end{array} \right]$$

$R_1 \rightarrow R_1 \div 2$

$$\left[ \begin{array}{cccc|c} 1 & 1 & -\frac{1}{2} & \frac{1}{2} & : & 2 \\ 4 & 3 & -1 & 2 & : & 6 \\ 8 & 5 & -3 & 4 & : & 12 \\ 3 & 3 & -2 & 2 & : & 6 \end{array} \right]$$

$R_2 \rightarrow R_2 - 4R_1$ ,  $R_3 \rightarrow R_3 - 8R_1$ ,  $R_4 \rightarrow R_4 - R_1$

$$\left[ \begin{array}{cccc|c} 1 & 1 & -\frac{1}{2} & \frac{1}{2} & : & 2 \\ 0 & -1 & 1 & 0 & : & -2 \\ 0 & -3 & 1 & 0 & : & -4 \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & : & 0 \end{array} \right]$$

$R_2 \rightarrow R_2 \div -1$

$$\left[ \begin{array}{cccc|c} 1 & 1 & -\frac{1}{2} & \frac{1}{2} & : & 2 \\ 0 & 1 & -1 & 0 & : & 2 \\ 0 & -3 & 1 & 0 & : & -4 \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & : & 0 \end{array} \right]$$

$R_1 \rightarrow R_1 - R_2$

$R_3 \rightarrow R_3 + 3R_2$

$$\left[ \begin{array}{cccc} 1 & 0 & \frac{1}{2} & \frac{1}{2} : ① \\ 0 & 1 & -1 & 0 : 2 \\ 0 & 0 & -2 & 0 : 2 \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} : 0 \end{array} \right]$$

$$R_3 \rightarrow R_3 \div -2$$

$$\left[ \begin{array}{cccc} 1 & 0 & \frac{1}{2} & \frac{1}{2} : ① \\ 0 & 1 & -1 & 0 : 2 \\ 0 & 0 & ① & 0 : -1 \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} : 0 \end{array} \right]$$

$$R_1 \rightarrow R_1 - \frac{1}{2} R_3$$

$$R_2 \rightarrow R_2 + R_3$$

$$R_4 \rightarrow R_4 + \frac{1}{2} R_3$$

$$\left[ \begin{array}{cccc} 1 & 0 & 0 & \frac{1}{2} : ① + \frac{1}{2} \\ 0 & 1 & 0 & 0 : 1 \\ 0 & 0 & 1 & 0 : -\frac{1}{2} - 1 \\ 0 & 0 & 0 & \frac{1}{2} : -\frac{1}{2} \end{array} \right]$$

$$R_4 \rightarrow R_4 * 2$$

$$\left[ \begin{array}{cccc} 1 & 0 & 0 & \frac{1}{2} : ① \\ 0 & 1 & 0 & 0 : 1 \\ 0 & 0 & 1 & 0 : -\frac{1}{2} - 1 \\ 0 & 0 & 0 & ① : -2 \end{array} \right] \# \frac{1}{2}$$

$$A - \frac{1}{2}x - 1$$

$$0 - \frac{1}{2}x \quad 0 + \frac{1}{2}$$

$$R_1 \rightarrow R_1 - \frac{1}{2} R_4$$

$$\left[ \begin{array}{cccc} 1 & 0 & 0 & 0 : -\frac{1}{2} 1 \\ 0 & 1 & 0 & 0 : 1 \\ 0 & 0 & 1 & 0 : -1 \\ 0 & 0 & 0 & 1 : -1 \end{array} \right]$$

$$x_1 = 1$$

$$x_2 = 1$$

$$x_3 = -1$$

$$x_4 = -1$$

Inverse of a matrix:a) By using Gauss Elimination Method

We know that  $x$  will be the inverse of  $A$  if  $AX = I$  where  $I$  is the unit matrix of the same order as  $A$ .

Let  $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$  be a given matrix

$X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$  be inverse of  $A$  such that

$$AX = I$$

The augmented matrix form is

$$\left[ \begin{array}{ccc|ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{array} \right]$$

Using Gauss Elimination procedure,  
it reduces to

$$\left[ \begin{array}{ccc|ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ 0 & a_{22}' & a_{23}' & b & c & 0 \\ 0 & 0 & a_{33}'' & d & e & f \end{array} \right]$$

This can be divide into 3 systems:

$$\text{System I } \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}' & a_{23}' \\ 0 & 0 & a_{33}'' \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Using back substitution we get  $\begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix}$

System II

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}' & a_{23}' \\ 0 & 0 & a_{33}'' \end{bmatrix} \begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Using back substitution we get  $\begin{bmatrix} x_{12} \\ x_{22} \\ x_{32} \end{bmatrix}$

System III

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}' & a_{23}' \\ 0 & 0 & a_{33}'' \end{bmatrix} \begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

1) Find inverse of

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$$

let.  $x = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$

be inverse of A such that

$$AX = I$$

The augmented matrix form

$$\left[ \begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & 3 & 0 & 1 & 0 \\ 1 & 4 & 9 & 0 & 0 & 1 \end{array} \right]$$

$$R_2 \rightarrow 2R_2 - 3R_1$$

$$R_3 \rightarrow 2R_3 - R_1$$

$$\left[ \begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -3 & 2 & 0 \\ 0 & 7 & 17 & -1 & 0 & 2 \end{array} \right]$$

$$R_3 \rightarrow R_3 - 7R_2$$

$$\left[ \begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -3 & 2 & 0 \\ 0 & 0 & -4 & 20 & -14 & 2 \end{array} \right]$$

which is upper triangular matrix form

It can write in 3 systems

System

$$\left[ \begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 0 & 1 & 3 & -3 \\ 0 & 0 & -4 & 20 \end{array} \right] \left[ \begin{array}{c} x_{11} \\ x_{21} \\ x_{31} \end{array} \right] = \left[ \begin{array}{c} 1 \\ -3 \\ 20 \end{array} \right]$$

From back substitution

$$\text{From } R_3 \quad -4x_{31} = 20 \Rightarrow x_{31} = -5$$

$$\text{From } R_2 \quad x_{21} + 3x_{31} = -3 \Rightarrow x_{21} = 12$$

$$\text{From } R_1 \quad 2x_{11} + x_{21} + x_{31} = 1 \Rightarrow x_{11} = -3$$

$$\therefore \left[ \begin{array}{c} x_{11} \\ x_{21} \\ x_{31} \end{array} \right] = \left[ \begin{array}{c} -3 \\ 12 \\ -5 \end{array} \right]$$

System II

$$\left[ \begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & -4 & -14 \end{array} \right] \left[ \begin{array}{c} x_{12} \\ x_{22} \\ x_{32} \end{array} \right] = \left[ \begin{array}{c} 0 \\ 2 \\ -14 \end{array} \right]$$

or, From back substitution

$$\text{From } R_3 \quad -4x_{32} = -14$$

$$\therefore x_{32} = \frac{7}{2}$$

$$\text{From } R_2 \quad 2x_{22} + 3x_{32} = 2$$

$$\text{or, } x_{22} = \frac{-17}{2}$$

$$\text{From } R_1 \quad 2x_{12} + x_{22} + x_{32} = 0$$

$$x_{12} = \frac{15}{2}$$

From system III

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_{13} \\ x_{23} \\ x_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

From R<sub>3</sub>:

$$-4x_{33} = 2$$

$$\therefore x_{33} = -\frac{1}{2}$$

From R<sub>2</sub>:

$$x_{23} + 3x_{33} = 0$$

$$\text{or, } x_{23} = -3x_{33}$$

$$= -\frac{3}{2}$$

From R<sub>1</sub>:

$$2x_{13} + x_{23} + x_{33} = 0$$

$$\therefore x_{13} = \frac{1}{2} - \frac{3}{2}$$

$$\therefore x_{13} = -\frac{1}{2}$$

Hence

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} -3 & \frac{5}{2} & -\frac{1}{2} \\ 12 & -\frac{17}{2} & \frac{3}{2} \\ -5 & \frac{7}{2} & -\frac{1}{2} \end{bmatrix}$$

Q) Using Gauss Elimination method.  
Find inverse of

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$

$$X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$$

$$\text{Now, } AX = I$$

The augmented matrix form

$$\left[ \begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 1 & 3 & -3 & 0 & 1 & 0 \\ -2 & -4 & -4 & 0 & 0 & 1 \end{array} \right]$$

$$R_2 \rightarrow R_2 - R_1$$

$$R_3 \rightarrow R_3 + 2R_1$$

$$\left[ \begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & -6 & -1 & 1 & 0 \\ 0 & -2 & 2 & 2 & 0 & 1 \end{array} \right]$$

$$R_3 \rightarrow R_3 + R_2$$

$$\left[ \begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & -6 & -1 & 1 & 0 \\ 0 & 0 & -4 & -1 & 1 & 1 \end{array} \right]$$

which is upper triangular matrix form.

No. o.

System I:

$$\left[ \begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 0 & 2 & -6 & -4 \\ 0 & 0 & -4 & 1 \end{array} \right] \xrightarrow{\text{R1} \times 1/9, \text{R2} \times 1/2, \text{R3} \times 1/(-4)} \left[ \begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & 1 & -1/4 \end{array} \right] \quad \left[ \begin{array}{c} x_{11} \\ x_{21} \\ x_{31} \end{array} \right] = \left[ \begin{array}{c} 1 \\ -1 \\ -1/4 \end{array} \right]$$

From R<sub>3</sub>,

$$-4x_{31} = 1$$

$$\therefore x_{31} = -\frac{1}{4}$$

From R<sub>2</sub>

$$2x_{21} - 6x_{31} = -1$$

$$\text{or } 2x_{21} = -1 + 6 \times \left(-\frac{1}{4}\right)$$

$$\therefore x_{21} = -\frac{5}{4}$$

From R<sub>1</sub>

$$x_{11} + x_{21} + 3x_{31} = 1$$

$$\therefore x_{11} = 1 + \frac{5}{4} - 3 \times \left(-\frac{1}{4}\right)$$

$$= 3$$

System II

$$\left[ \begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 2 & -6 & 1 \\ 0 & 0 & -4 & 1 \end{array} \right] \quad \left[ \begin{array}{c} x_{12} \\ x_{22} \\ x_{32} \end{array} \right] = \left[ \begin{array}{c} 0 \\ 1 \\ 1 \end{array} \right]$$

From R<sub>3</sub>

$$-4x_{32} = 1$$

$$\therefore x_{32} = -\frac{1}{4}$$

From R<sub>2</sub>

$$2x_{22} - 6x_{32} = 1$$

$$\therefore x_{22} = -\frac{1}{4}$$

From R<sub>1</sub>

$$x_{12} + x_{22} + 3x_{32} = 0$$

$$\therefore x_{12} = -\frac{8}{4} = 1$$

System III

$$\left[ \begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 2 & -6 & 0 \\ 0 & 0 & -4 & 1 \end{array} \right] \quad \left[ \begin{array}{c} x_{13} \\ x_{23} \\ x_{33} \end{array} \right] = \left[ \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right]$$

From R<sub>3</sub>

$$-4x_{33} = 1$$

$$\therefore x_{33} = -\frac{1}{4}$$

From R<sub>2</sub>

$$2x_{23} - 6x_{33} = 0$$

$$\therefore x_{23} = \frac{6x_{33}}{2} = \frac{-3}{4}$$

From R<sub>1</sub>

$$x_{13} + x_{23} + 3x_{33} = 0$$

$$\therefore x_{13} = \frac{3}{2}$$

Hence

$$A^{-1} = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} 3 & 1 & \frac{3}{2} \\ -\frac{5}{3} & -\frac{1}{4} & -\frac{3}{4} \\ -\frac{1}{3} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

### Solution of System of linear Equations.

Inverse of a matrix

### b) By Gauss-Jordan Method

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Let  $X$  be the inverse of  $A$  such that

$$AX = I$$

The augmented matrix form is:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} : 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} : 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} : 0 & 0 & 1 \end{bmatrix}$$

By using Gauss-Jordan procedure it is reduced to

$$\begin{bmatrix} 1 & 0 & 0 : a & b & c \\ 0 & 1 & 0 : d & e & f \\ 0 & 0 & 1 : g & h & i \end{bmatrix}$$

then  $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

▷ Find inverse of:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 1 \\ 1 & 0 & \frac{1}{2} \end{bmatrix}$$

The augmented matrix form is

$$\begin{bmatrix} 1 & -1 & 2 : 1 & 0 & 0 \\ 3 & 0 & 1 : 0 & 1 & 0 \\ 1 & 0 & 2 : 0 & 0 & 1 \end{bmatrix}$$

$$R_2 \rightarrow R_2 - 3R_1$$

$$R_3 \rightarrow R_3 - R_1$$

$$\begin{bmatrix} 1 & -1 & 2 : 1 & 0 & 0 \\ 0 & 3 & -5 : -3 & 1 & 0 \\ 0 & 1 & 0 : -1 & 0 & 1 \end{bmatrix}$$

$$R_2 \leftrightarrow R_3$$

$$\left[ \begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 3 & -8 & -3 & 1 & 0 \end{array} \right]$$

$$R_1 \rightarrow R_1 + R_2$$

$$R_3 \rightarrow R_3 - 3R_2$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -5 & 0 & 1 & -3 \end{array} \right]$$

$$R_3 \rightarrow \frac{R_3}{-5}$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -\frac{1}{5} & \frac{3}{5} \end{array} \right]$$

$$R_1 \rightarrow R_1 - 2R_3$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & \frac{2}{5} & -\frac{1}{5} \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -\frac{1}{5} & \frac{3}{5} \end{array} \right]$$

$$A = \left[ \begin{array}{ccc} 0 & \frac{2}{5} & -\frac{1}{5} \\ -1 & 0 & 1 \\ 0 & -\frac{1}{5} & \frac{3}{5} \end{array} \right]$$

### Method III > Factorisation Method:

This method is based on the fact that every square matrix A can be expressed as the product of lower triangular matrix and an upper triangular matrix.

Let us consider a system of linear eqn

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

In matrix form It can be written as  
 $AX = B$

where

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$\text{let } LU = A$$

where

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$$

$$\& U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Now eq<sup>n</sup> ① becomes

$$LUX = B \rightarrow \text{iii}$$

$$\text{let } UX = Z \rightarrow \text{iv}$$

$$\text{then } LZ = B \rightarrow \text{v}$$

Solving eq<sup>n</sup> ⑤ and then ④ we get the value of  $X$ .

On the basis of diagonal elements there are two methods.

The factorisation with  $L$  having unit diagonals is called

9) Factorisation Method      Doolittle LU

$$LU = A$$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

6) The factorisation with  $U$  having unit diagonals is called Cholesky LU factorisation method

$$LU = A$$

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & l_{12} & l_{13} \\ 0 & 1 & l_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

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Solve the system by using Doolittle LU factorisation method.

$$3x_1 + 2x_2 + x_3 = 10$$

$$2x_1 + 3x_2 + 2x_3 = 14$$

$$x_1 + 2x_2 + 3x_3 = 14$$

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$$

$$\begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ 0 & x_{22} & x_{23} \\ 0 & 0 & x_{33} \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Soln:

Here the given system is written in  
 $AX = B$

where

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, B = \begin{bmatrix} 10 \\ 14 \\ 14 \end{bmatrix}$$

Now

$$LU = A$$

$$\text{or } \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} U_{11} & U_{12} & U_{13} \\ l_{21}U_{11} & l_{21}U_{12} + l_{22}U_{22} & l_{21}U_{13} + l_{22}U_{23} \\ l_{31}U_{11} & l_{31}U_{12} + l_{32}U_{22} & l_{31}U_{13} + l_{32}U_{23} + l_{33}U_{33} \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

$$U_{11} = 3, U_{12} = 2, U_{13} = 1$$

Now,

$$l_{21}U_{11} = 2$$

$$l_{21} = \frac{2}{3}$$

$$l_{21}U_{12} + U_{22} = 3$$

$$\text{or } \frac{2}{3} * 2 + U_{22} = 3$$

$$\text{or, } U_{22} = \frac{5}{3}$$

$$l_{21}U_{13} + U_{23} = 2$$

$$\frac{2}{3} * 1 + U_{23} = 2$$

$$\therefore U_{23} = \frac{4}{3}$$

$$l_{31}U_{11} = 1$$

$$\therefore l_{31} = \frac{1}{3}$$

$$l_{31}U_{12} + l_{32}U_{22} = 2$$

$$\frac{1}{3} * 2 + \frac{5}{3} U_{32} = 2$$

$$\text{or, } -\frac{5}{3} U_{32} = \frac{4}{3}$$

$$\therefore U_{32} = \frac{4}{5}$$

$$l_{31}U_{13} + l_{32}U_{23} + U_{33} = 3$$

$$\text{or, } \frac{1}{3} * 1 + \frac{4}{5} * \frac{4}{3} U_{32} + U_{33} = 3$$

$$\text{or, } \frac{1}{3} + \frac{4}{3} * \frac{4}{5} + U_{33} = 3$$

$$\therefore U_{33} = \frac{8}{5} //$$

Now,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{2}{3} & 1 & 0 \\ \frac{1}{3} & \frac{4}{5} & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \\ 0 & \frac{5}{3} & \frac{4}{3} \\ 0 & 0 & \frac{8}{5} \end{bmatrix}$$

From eq<sup>n</sup> (1)

$$AX = B$$

$$LUX = B \quad \text{--- (ii)}$$

$$\text{let } UX = Z \quad \text{--- (iii)}$$

$$\text{From (ii) } LZ = B$$

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{2}{3} & 1 & 0 \\ \frac{1}{3} & \frac{4}{5} & 1 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \\ 14 \end{bmatrix}$$

Using forward substitution,

$$\text{From } R_1, Z_1 = 10$$

$$\text{From } R_2$$

$$\frac{2}{3} Z_1 + Z_2 = 14 \Rightarrow Z_2 = \frac{22}{3}$$

$$\text{From } R_3$$

$$\frac{1}{3} Z_1 + \frac{4}{5} Z_2 + Z_3 = 14 \Rightarrow Z_3 = \frac{24}{5}$$

$$\therefore \begin{bmatrix} Z_1 \\ Z_2 \\ Z_3 \end{bmatrix} = \begin{bmatrix} 10 \\ \frac{22}{3} \\ \frac{24}{5} \end{bmatrix}$$

From eq iii  
 $UX = Z$

$$\begin{bmatrix} 3 & 2 & 1 \\ 0 & \frac{5}{3} & \frac{4}{3} \\ 0 & 0 & \frac{8}{5} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \\ 14 \end{bmatrix}$$

Using back substitution

From R<sub>3</sub>  $\frac{8}{5}X_3 = 14$

$$\therefore X_3 = 3$$

From R<sub>2</sub> :  $\frac{5}{3}X_2 + \frac{4}{3}X_3 = \frac{22}{3}$   
 $\Rightarrow X_2 = 2$

From R<sub>1</sub> :  $3X_1 + 2X_2 + X_3 = 10$

$$\therefore X_1 = 1$$

Hence

$$\left. \begin{array}{l} X_1 = 1 \\ X_2 = 2 \\ X_3 = 3 \end{array} \right\}$$

2) Using LU factorisation method, solve the system

$$2x + 3y + z = 9$$

$$x + 2y + 3z = 6$$

$$3x + y + 2z = 8$$

Here the system can be written in matrix form as

$$AX = B \rightarrow ①$$

where  $A = \begin{bmatrix} 2 & 3 & 1 \\ \frac{1}{3} & 2 & 3 \\ \frac{1}{3} & 1 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$

let  $LU = A$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} U_{11} & U_{12} & U_{13} \\ l_{21}U_{11} & U_{22} + U_{21}U_{12} & U_{23} + U_{21}U_{13} \\ l_{31}U_{11} & l_{32}U_{12} + l_{31}U_{11} & U_{33} + l_{31}U_{13} + l_{32}U_{23} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

Equating the corresponding elements

$$U_{11} = 2, U_{12} = 3, U_{13} = 1$$

$$l_{21}U_{11} = 1 \Rightarrow l_{21} = \frac{1}{2}$$

$$l_{21}U_{12} + U_{22} = 2$$

$$\text{or, } U_{22} = 2 - \frac{1}{2} * 3$$

$$= 2 - \frac{3}{2}$$

$$= \frac{1}{2}$$

$$l_{21} * U_{13} + U_{23} = 3$$

$$\text{or, } U_{23} = 3 - \frac{1}{2} * 1$$

$$= \frac{5}{2}$$

$$l_{31}U_{11} + l_{32}U_{12} + U_{23} + U_{33} = 2$$

$$\text{or, } -U_{33} = 2 - 1 *$$

$$U_{31}U_{11} = 3$$

$$\therefore U_{31} = \frac{3}{2}$$

$$l_{31}U_{12} + l_{32}U_{22} = 1$$

$$\frac{3}{2} * 3 + \frac{1}{2} l_{32} = 1$$

$$\therefore l_{32} = -7$$

$$l_{31}U_{13} + l_{32}U_{23} + U_{33} = 2$$

$$\therefore U_{33} = 18$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & -7 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 18 \end{bmatrix}$$

From eqn ①

$$AX = B$$

$$\text{or } LUX = B$$

$$\text{Let } UX = Z$$

$$LZ = B$$

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & -7 & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

$$\text{From } R_1, z_1 = 9$$

$$R_2, \frac{1}{2}z_1 + z_2 = 6 \Rightarrow z_2 = \frac{3}{2}$$

$$R_3 \div \frac{3}{2}z_1 - 7z_2 + z_3 = 8 \Rightarrow z_3 = 5$$

$$\therefore \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 9 \\ \frac{3}{2} \\ 5 \end{bmatrix}$$

From eq<sup>n</sup> (iii)

$$UX = Z$$
$$\begin{bmatrix} 2 & 3 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 18 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} \frac{9}{2} \\ \frac{5}{2} \\ 5 \end{bmatrix}$$

Using back substitution  
From R<sub>3</sub>  $18Z = 5 \Rightarrow Z = \frac{5}{18}$

$$\text{From } R_2 \quad \frac{1}{2}Y + \frac{5}{2}Z = \frac{5}{2}$$
$$\therefore Y = \frac{29}{18}$$

$$\text{From } 2X + 3Y + Z = 5$$
$$\therefore X = 5 - \frac{5}{18} - 3 \times \frac{29}{18}$$
$$= -\frac{35}{18}$$

Using cholesky (Crout's method)

$$2x_1 + 2x_2 + x_3 = 6 \quad (5, 1-6)$$

$$4x_1 + 2x_2 + 3x_3 = 4$$

$$x_1 + x_2 + x_3 = 0$$

$$AX = B$$

The augmented matrix as:

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 4 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 6 \\ 4 \\ 0 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Now let  $LU = A$

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 \\ 4 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\text{or } \begin{bmatrix} l_{11} & l_{11} \times u_{12} & l_{11}u_{13} \\ l_{21} & l_{21} \times u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31} \times u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 \\ 4 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

Equating the value

$$l_{11} = 2 \quad l_{11}u_{12} = 2 \quad l_{11}u_{13} = 1$$
$$\therefore u_{12} = 1 \quad \therefore u_{13} = \frac{1}{2}$$

$$l_{21} = 4 \quad l_{21}u_{12} + l_{23} = 2 \quad l_{31} = 1$$
$$4 \times 1 + l_{23} = 2$$
$$\therefore l_{23} = -2$$

$$l_{21}u_{13} + l_{22}u_{23} = 3$$
$$4 \times \frac{1}{2} + (-2) \times 1 = 3$$
$$u_{23} = \frac{-1}{2}$$

$$\begin{aligned} l_{31} \cdot u_{12} + l_{32} &= 1 \\ 1 \times 1 + l_{32} &= 1 \\ \therefore l_{32} &= 0 \end{aligned}$$

$$\begin{aligned} l_{31} u_{13} + l_{32} u_{23} + l_{33} &= 0 \\ \text{or, } 1 \cdot 1 + 0 \times \left(\frac{-1}{2}\right) + l_{33} &= 1 \\ \therefore l_{33} &= 1 - \frac{1}{2} = \frac{1}{2} \end{aligned}$$

Now,

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

Here  $Ax = B$

$$LUX = B$$

$$\text{let } UX = Z$$

$$\text{so } LZ = B$$

$$\text{Now, } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 0 \end{bmatrix}$$

$$\text{or, } \begin{bmatrix} 2z_1 \\ 4z_1 - 2z_2 \\ z_1 + \frac{1}{2}z_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 0 \end{bmatrix}$$

Using forward substitution

$$\text{From } R_1 \quad 2z_1 = 6 \Rightarrow z_1 = 3$$

$$\text{From } R_2 \quad 4z_1 - 2z_2 = 4$$

$$\text{or, } 4 \times 3 - 2z_2 = 4$$

$$\text{or, } z_2 = 4$$

$$\text{From } R_3 \quad z_1 + \frac{1}{2}z_3 = 0$$

$$\text{or, } 3 + \frac{1}{2}z_3 = 0$$

$$\text{or, } \frac{1}{2}z_3 = -3$$

$$\therefore z_3 = -6$$

Again  $UX = Z$

$$\text{or, } \begin{bmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -6 \end{bmatrix}$$

$$\text{or, } \begin{bmatrix} x_1 + x_2 + \frac{1}{2}x_3 = 3 \\ x_2 - \frac{1}{2}x_3 = 4 \\ x_3 = -6 \end{bmatrix}$$

From back substitution,

$$\text{From } R_3 \quad x_3 = -6$$

$$\text{From } R_2 \quad x_2 - \frac{1}{2}x_3 = 4$$

$$\text{Hence } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -6 \end{bmatrix}$$

$$\text{or, } x_2 = 4 + 3 = 1$$

$$\text{From } R_1 \quad x_1 + 1 - 3 = 3 \Rightarrow x_1 = 5$$

## Method IV

Iterative Method:

a) Gauss-Seidel Method

b) Gauss-Jacobi's Method:

a) Gauss Seidel Method:

↳ This method is a modified form of Gauss-Jacobi's method.

Let us consider a system of linear equations

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

If the system is diagonally dominant, it can be written as,

$$\text{From eq } (i) x = \frac{1}{a_1} [d_1 - b_1y - c_1z]$$

$$\text{From eq } (ii) y = \frac{1}{b_2} [d_2 - a_2x - c_2z]$$

$$\text{From eq } (iii) z = \frac{1}{c_3} [d_3 - a_3x - b_3y]$$

Let initial approximation be  $x=x_0$ ,  $y=y_0$ ,  $z=z_0$

For 1st approximation

$$\text{put } y=y_0 \text{ & } z=z_0 \text{ in eq } (i) \text{ we get,}$$

$$x_1 = \frac{1}{a_1} [d_1 - b_1y_0 - c_1z_0]$$

$$\text{Put } x=x_1 \text{ & } z=z_0 \text{ in eq } (ii) \text{ we get}$$

$$y_1 = \frac{1}{b_2} [d_2 - a_2x_1 - c_2z_0]$$

$$\text{Put } x=x_1 \text{ & } y=y_1 \text{ in eq } (iii) \text{ we get}$$

$$z_1 = \frac{1}{c_3} [d_3 - a_3x_1 - b_3y_1]$$

As soon as a new approximation for an unknown is found, it is immediately used in next step.

This process is repeated till values of  $x, y, z$  are obtained to desired accuracy.

1) Apply Gauss-Seidel Method. solve the systems:

$$20x + y + 2z = 17$$

$$3x + 20y - z = 18$$

$$2x - 8y + 20z = 25$$

Here the given system is diagonally dominant. It can be written as

$$x = \frac{1}{20} [17 - y + 2z]$$

$$y = \frac{1}{20} [-18 - 3x + z]$$

setting

$$A = \frac{17-B+2C}{20} : B: \frac{-18-2A+C}{20} : C = \frac{25-2A+3B}{20}$$

$$z = \frac{1}{20} [25 - 2x + 3y]$$

Let initial approximation  $x_0 = y_0 = z_0 = 0$   
 Tabulating for successive approximations, we get,

$$\text{It}^n \quad x = \frac{1}{20}(17-y+2z) \quad y = \frac{1}{20}(-18-3x+z) \quad z = \frac{1}{20}(25-2x+3y)$$

Initial	0	0	0
1	0.85	-1.03	1.01
2)	1.00	-1.00	1.00
3)	1.00	-1.00	1.00

From table,

$$x = 1$$

$$y = -1$$

$$z = 1$$

Q) Using Gauss seidal iteration method solve the system.

$$10x_1 - 2x_2 - x_3 - x_4 = 3$$

$$-2x_1 + 10x_2 - x_3 - x_4 = 15$$

$$-x_1 - x_2 + 10x_3 - 2x_4 = 27$$

$$-x_1 - x_2 - 2x_3 + 10x_4 = -9$$

Here the given system is diagonally dominant. It can be written as,

$$x_1 = \frac{3 + 2x_2 + x_3 + x_4}{10}$$

$$x_2 = \frac{15 + 2x_1 + x_3 + x_4}{10}$$

$$x_3 = \frac{27 + x_1 + x_2 + 2x_4}{10}$$

$$x_4 = \frac{(-9 + x_1 + x_2 + 2x_3)}{10}$$

Let initial approximation  $x_1 = x_2 = x_3 = x_4 = 0$   
 Tabulating successive approximation

It <sup>n</sup>	$x_1$	$x_2$	$x_3$	$x_4$
Initial	0	0	0	0
1)	0.30	1.56	2.89	-0.14
2)	0.89	1.95	2.96	-0.02
3)	0.98	1.99	2.99	0.00
4)	1.00	2.00	3.00	0.00
5)	1.00	2.00	3.00	0.00

From table

$$x_1 = 1$$

$$x_2 = 2$$

$$x_3 = 3$$

$$x_4 = 0$$

} Ans

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$$(3) \quad 10x_1 + 6x_2 - 5x_3 = 27$$

$$3x_1 + 8x_2 + 10x_3 = 27$$

$$4x_1 + 10x_2 + 3x_3 = 27$$

Rearranging the given eq<sup>n</sup> to  
make diagonally dominant we  
get

$$10x_1 + 6x_2 - 5x_3 = 27$$

$$4x_1 + 10x_2 + 3x_3 = 27$$

$$8x_1 + 8x_2 + 10x_3 = 27$$

Then

$$x_1 = \frac{27 - 6x_2 + 5x_3}{10}$$

$$x_2 = \frac{27 - 4x_1 - 3x_3}{10}$$

$$x_3 = \frac{27 - 3x_1 - 8x_2}{10}$$

Tabulating successive approximation

It <sup>n</sup>	$x_1$	$x_2$	$x_3$
Initial	0	0	0
1)	2.70	1.62	0.59
2)	2.03	1.71	0.72
3)	2.03		

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setting:

L-1, 1

$$\frac{17-B+2C}{20} :$$

1) Solve the following set of equations by Gauss-Jacobi method.

$$20x + y - 2z = 17$$

$$3x + 20y - z = -18$$

$$2x - 3y + 20z = 25$$

Here the system is diagonally dominant  
it can be written as

$$x = \frac{1}{20} (17 - y + 2z)$$

$$y = \frac{1}{20} (-18 - 3x + z)$$

$$z = \frac{1}{20} (25 - 2x + 3y)$$

Let initial approximation  $x_0 = y_0 = z_0 = 0$   
Tabulating for successive approximation.

It <sup>n</sup>	x	y	z
Initial	0	0	0
1	0.85	-0.90	1.25
2	1.02	-0.97	1.03
3	1.00	-1.00	1.00
4	1.00	-1.00	1.00

From table

$$\left. \begin{array}{l} x=1 \\ y=-1 \\ z=1 \end{array} \right\} \text{ANS}$$

$$2) \quad 2x_1 + x_2 + x_3 = 5$$

$$3x_1 + 5x_2 + 2x_3 = 15$$

$$2x_1 + x_2 + 4x_3 = 8$$

Here the system is diagonally dominant

It can be written as

$$x_1 = \frac{5 - x_2 - x_3}{2}$$

$$x_2 = \frac{15 - 3x_1 - 2x_3}{5}$$

$$x_3 = \frac{8 - x_2 - 2x_1}{4}$$

Let initial approximation

$$x_0 = y_0 = z_0 = 0$$

Tabulating successive approximations

It <sup>n</sup>	$x_1$	$x_2$	$x_3$
Initial	0	0	0
1	2.50	3.00	2.00
2	0.00	0.70	0.00
3	2.15	3.00	1.83
4	0.09	0.98	0.18
5	1.92	2.87	1.71
6	0.21	1.16	0.32
7)	1.76	2.75	1.62

8)	0.32	1.30	0.43
9	1.64	2.64	1.52
10	0.42	1.41	0.52
11	1.54	2.54	1.44
12	0.51	1.5	0.60
13	1.45	2.45	1.37
14	0.59	1.58	0.66
15	1.38	2.38	1.31
16	0.66	1.65	0.72
17	1.32	2.32	1.26
18	0.71	1.70	0.76
19	1.27	2.27	1.22
20	0.76	1.75	0.80
21	1.23	2.22	1.18
22	0.80	1.79	0.83
23	1.19	2.19	1.15
24	0.83	1.83	0.86
25	1.16	2.16	1.13
26	0.86	1.85	0.88
27	1.14	2.13	1.11
28	0.88	1.87	0.90
29	1.12	2.11	1.09
30	0.9	1.89	0.91
31	1.10	2.10	1.08
32	0.91	1.91	0.93
33	1.08	2.08	1.07
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# Eigen value & Eigen vector

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## Problems by power Method

If  $A$  be any square matrix of order  $n \times n$  with elements  $a_{ij}$  we can find a column matrix  $X$  and a constant  $\lambda$  such that  $AX = \lambda X$  where  $\lambda$  is called the eigen value &  $X$  is called the corresponding eigen vector.

Let  $A$  be the given matrix &  $X^{(0)}$  be the initial approximation of eigen vector. We evaluate  $AX^{(0)}$  which is written as  $\lambda^{(1)}X^{(1)}$ . This gives the first approximation. Similarly we evaluate  $AX^{(1)} = \lambda^{(2)}X^{(2)}$  which gives the second approximation. We repeat this procedure to get the repetition of  $X$  as well as  $\lambda$ .

Note:

1) For initial approximation, take

$$X^{(0)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\text{or } \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

2) Evaluate  $AX^{(0)}$  and take numerically largest value as common & divide the whole element of the vector with largest value.

1) Find the largest eigen value and corresponding eigen vector of matrix

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Solving

Here Given matrix is  $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

Let  $X^{(0)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$  be the initial approximation

for eigen vector. Then successive approximations are:

$$AX^{(0)} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \lambda^{(1)}X^{(1)}$$

$$AX^{(1)} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0.5 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 2.5 \\ 0 \end{bmatrix} = 0.5 \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix} = \lambda^{(2)}X^{(2)}$$

$$AX^{(2)} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0.8 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.6 \\ 2.8 \\ 0 \end{bmatrix}$$

$$= 2.8 \begin{bmatrix} 0.93 \\ 1 \\ 0 \end{bmatrix}$$

$$= \lambda^{(3)} X^{(3)}$$

$$AX^{(3)} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0.93 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.93 \\ 2.86 \\ 0 \end{bmatrix}$$

$$= 2.93 \begin{bmatrix} 1 \\ 0.98 \\ 0 \end{bmatrix}$$

$$= \lambda^{(4)} X^{(4)}$$

$$AX^{(4)} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0.98 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.96 \\ 2.98 \\ 0 \end{bmatrix}$$

$$= 2.98 \begin{bmatrix} 0.99 \\ 1 \\ 0 \end{bmatrix}$$

$$= \lambda^{(5)} X^{(5)}$$

$$AX^{(5)} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0.99 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.99 \\ 2.98 \\ 0 \end{bmatrix}$$

$$= 2.99 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \lambda^6 X^{(6)}$$

$$AX^{(6)} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \lambda^{(7)} X^{(7)}$$

$$AX^{(7)} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \lambda^{(8)} X^{(8)}$$

Hence, largest eigenvalue  $\lambda=3$   
corresponding eigen  
vector  $X = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Q)  $A = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Let  $x^{(0)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$  be initial approxi-  
mation for eigen vector. Then  
successive approximation

$$AX^{(0)} = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \lambda^{(1)} X^{(1)}$$

$$AX^{(1)} = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 0 \end{bmatrix}$$

$$= 7 \begin{bmatrix} 1 \\ 0.43 \\ 0 \end{bmatrix} = \lambda^{(2)} x^{(2)}$$

$$AX^{(2)} = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.43 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3.58 \\ 4.86 \\ 0 \end{bmatrix} = 3.58 \begin{bmatrix} 1 \\ 0.52 \\ 0 \end{bmatrix} = \lambda^{(3)} x^{(3)}$$

$$AX^{(3)} = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.52 \\ 0 \end{bmatrix} = \begin{bmatrix} 4.12 \\ 2.04 \\ 0 \end{bmatrix} = 4.12 \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix}$$

$$= \lambda^{(4)} x^{(4)}$$

$$AX^4 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = \lambda^{(5)} x^{(5)}$$

~~$$AX^5 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = \lambda^{(6)} x^{(6)}$$~~

Hence eigen value = 4  
corresponding eigen vector =  $\begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix}$

3) Determine the largest eigen value and the corresponding eigenvector of the matrix using power method.

$$A = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix}$$

Let  $x^{(0)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$  be initial approximation for eigen vector. Then successive approximation

$$AX^{(0)} = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 15 \\ -10 \\ -20 \end{bmatrix} = 20 \begin{bmatrix} 0.75 \\ -0.5 \\ -1 \end{bmatrix} = \lambda^{(1)} x^{(1)}$$

$$AX^{(1)} = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix} \begin{bmatrix} 0.75 \\ -0.5 \\ -1 \end{bmatrix} = \begin{bmatrix} 16.25 \\ -7.50 \\ -15 \end{bmatrix} = 16.25 \begin{bmatrix} 1 \\ -2.17 \\ -0.92 \end{bmatrix} = \lambda^{(2)} x^{(2)}$$

$$AX^{(2)} = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -2.17 \\ -0.92 \end{bmatrix} = \begin{bmatrix} 26.49 \\ -30.52 \\ -23.16 \end{bmatrix} = 30.52 \begin{bmatrix} 0.87 \\ -1 \\ -0.76 \end{bmatrix} = \lambda^{(3)} x^{(3)}$$

$$AX^{(3)} = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix} \begin{bmatrix} 0.87 \\ -1 \\ -0.76 \end{bmatrix} = \begin{bmatrix} 19.33 \\ -16.14 \\ -16.89 \end{bmatrix} = 19.33 \begin{bmatrix} 1 \\ -0.83 \\ -0.87 \end{bmatrix}$$

$$AX^{(4)} = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.83 \\ -0.87 \end{bmatrix} = \begin{bmatrix} 20.93 \\ -19.79 \\ -21.58 \end{bmatrix}$$

$$= 21.58 \begin{bmatrix} 20.93 \\ -0.68 \\ -1 \end{bmatrix}$$

$= \lambda^5 X^{(5)}$

$$AX^{(5)} = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix} \begin{bmatrix} 0.97 \\ -0.68 \\ -1 \end{bmatrix} = \begin{bmatrix} 20.27 \\ -11.86 \\ -20.12 \end{bmatrix}$$

$$= 20.27 \begin{bmatrix} 1 \\ -0.59 \\ -0.99 \end{bmatrix}$$

$= \lambda^6 X^{(6)}$

$$AX^{(6)} = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.59 \\ -0.99 \end{bmatrix} = \begin{bmatrix} 20.33 \\ -11.14 \\ -20.38 \end{bmatrix}$$

$$= 20.38 \begin{bmatrix} 0.99 \\ -0.55 \\ -1 \end{bmatrix}$$

$= \lambda^7 X^{(7)}$

$$AX^{(7)} = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix} \begin{bmatrix} 0.99 \\ -0.55 \\ -1 \end{bmatrix} = \begin{bmatrix} 20.05 \\ -10.50 \\ -20.00 \end{bmatrix}$$

$$= 20.05 \begin{bmatrix} 1 \\ -0.51 \\ -1 \end{bmatrix}$$

$$AX^{(8)} = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.51 \\ -1 \end{bmatrix} = \begin{bmatrix} 20.04 \\ -10.12 \\ -20.04 \end{bmatrix} = 20.04 \begin{bmatrix} 1 \\ -0.5 \\ -1 \end{bmatrix}$$

$$AX^{(9)} = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.5 \\ -1 \end{bmatrix} = \begin{bmatrix} 20.00 \\ -10.00 \\ -20.00 \end{bmatrix} = 20 \begin{bmatrix} 1 \\ -0.5 \\ -1 \end{bmatrix}$$

$$AX^{10} = \begin{bmatrix} 15 & -4 & -3 \\ -10 & 12 & -6 \\ -20 & 4 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.5 \\ -1 \end{bmatrix} = \begin{bmatrix} 20.00 \\ -10.00 \\ -20.00 \end{bmatrix} = 20 \begin{bmatrix} 1 \\ -0.5 \\ -1 \end{bmatrix}$$

Hence largest eigen value = 20  
corresponding eigen vector =  $\begin{bmatrix} 1 \\ -0.5 \\ -1 \end{bmatrix}$

# CHAPTER - Three

## Interpolation and Approximation

Computing the value for a tabulated function at a point is called interpolation.

Suppose we are given the following values of  $y = f(x)$  for a set of values of  $x$

$x$	$x_0$	$x_1$	$x_2$	$\dots$	$x_n$
$y=f(x)$	$f(x_0)$	$f(x_1)$	$f(x_2)$		$f(x_n)$

Then the process of finding the value of corresponding to any value of  $x$  is called interpolation.

### Interpolation with unequal intervals

Let  $f(x_0), f(x_1), f(x_2), f(x_3), \dots$  be the values corresponding to  $x = x_0, x_1, x_2, x_3, \dots$  where  $x_0, x_1, x_2, \dots$  are not equispaced. Then we can find value of  $f(x)$  by following methods:

- 1) Lagrange's interpolation method.
- 2) Divided difference interpolation method.

### Method - I

#### Lagrange's Interpolation method

Let us consider a tabulated function:

$x$	$x_0$	$x_1$	$x_2$	$x_3$
$y=f(x)$	$f(x_0)$	$f(x_1)$	$f(x_2)$	$f(x_3)$

Then

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \times f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} \times f(x_1) + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} \times f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} \times f(x_3)$$

This formula is called lagrange's interpolation formula.

Similarly

$x$	$x_0$	$x_1$	$x_2$	$x_3$	$x_4$
$f(x)$	$f(x_0)$	$f(x_1)$	$f(x_2)$	$f(x_3)$	$f(x_4)$

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)} \times f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)} \times f(x_1) + \frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)} \times f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)} \times f(x_3)$$

$$\frac{+(x-x_0)(x-x_1)(x-x_2)(x-x_3)f(x_4)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)}$$

1) Using Lagrange interpolation  
Find  $f(10)$  from following table

$x$	5	6	9	11
$y$	12	13	14	16

Soln:

Here  $x_0=5$        $f(x_0)=12$   
 $x_1=6$        $f(x_1)=13$   
 $x_2=9$        $f(x_2)=14$   
 $x_3=11$        $f(x_3)=16$   
 $f(x)=?$

Using Lagrange's interpolation formula

$$\begin{aligned}
 f(10) &= \frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)} \times 12 \\
 &\quad + \frac{(10-5)(10-6)(10-11)}{(6-5)(6-9)(6-11)} \times 13 \\
 &\quad + \frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)} \times 14 \\
 &\quad + \frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)} \times 16 \\
 &= 14.67
 \end{aligned}$$

$x$	5	7	11	13	17
$f(x)$	150	392	1452	2366	5202

Using Lagrang's formula evaluate  $f(g)$ 

Soln:

$$\begin{aligned}
 x &= 9 & f(x_0) &= 15 \\
 x_0 &= 5 & f(x_1) &= 392 \\
 x_1 &= 7 & f(x_2) &= 1452 \\
 x_2 &= 11 & f(x_3) &= 2366 \\
 x_3 &= 13 & f(x_4) &= 5202 \\
 x_4 &= 17
 \end{aligned}$$

Now,

$$\begin{aligned}
 f(g) &= \frac{(g-7)(g-11)(g-13)(g-17)}{(5-7)(5-11)(5-13)(5-17)} \times 150 \\
 &\quad + \frac{(g-5)(g-11)(g-13)(g-17)}{(7-5)(7-11)(7-13)(7-17)} \times 392 \\
 &\quad + \frac{(g-5)(g-7)(g-13)(g-17)}{(11-5)(11-7)(11-13)(11-17)} \times 1452 \\
 &\quad + \frac{(g-5)(g-7)(g-11)(g-17)}{(13-5)(13-7)(13-11)(13-17)} \times 2366 \\
 &\quad + \frac{(g-5)(g-7)(g-11)(g-13)}{(17-5)(17-7)(17-11)(17-13)} \times 5202 \\
 &= 810
 \end{aligned}$$

Hence  $f(g) = 810 //$

Find eq<sup>n</sup> f(x)

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3)	x	0	1	3	4
	y	-12	0	6	12

Soln:

$$x_0 = 0 \quad f(x_0) = -12$$

$$x_1 = 1 \quad f(x_1) = 0$$

$$x_2 = 3 \quad f(x_2) = 6$$

$$x_3 = 4 \quad f(x_3) = 12$$

Now,

$$f(x) = \frac{(x-1)(x-3)(x-4)}{(0-1)(0-3)(0-4)} * (-12) +$$

$$+ \frac{(x-0)(x-3)(x-4)}{(1-0)(1-3)(1-4)} * 0$$

$$+ \frac{(x-1)(x-0)(x-4)}{(3-1)(3-0)(3-4)} * 6$$

$$+ \frac{(x-0)(x-1)(x-3)}{(4-0)(4-1)(4-3)} * 12$$

$$= - \frac{12(x^2-x+3)(x-4)}{-1*-3*-4} +$$

$$6(x^2-x)(x-4) + \frac{12(x^2-x)(x-3)}{2*3*-9} * 1$$

$$= - \frac{12(x^3-x^2+3x-4x^2+4x-12)}{-12}$$

$$+ \frac{6(x^3-4x^2+x^2+4x)+12(x^3-3x^2+2x^3)}{6*-9} * 12$$

$$= (x^3-5x^2+7x-12) + x^3-4x^2+3x$$

$$+ \frac{(x^3-5x^2+4x)}{-9}$$

$$= - \frac{9(6x^3-9x^2+10x-12)}{-9} + x^3-5x^2+4x$$

$$= - \frac{18x^3+81x^2-90x+108}{-9} + x^3-5x^2+4x$$

$$= - \frac{17x^3+76x^2-86x+108}{-9}$$

$$= \frac{17x^3+76x^2+86x-108}{9} //$$

- 4) The function  $y = f(x)$  is given at the points  $(7, 3), (8, 1), (9, 1) & (10, 9)$ . Find the value of  $y$  for  $x=9.5$  using Lagrange interpolation formula.

Given table is:

x	7	8	9	10
f(x)	3	1	1	9

Now,

$$f(9.5) = \frac{(9.5-8)(9.5-9)(9.5-10)}{(7-8)(7-9)(7-10)} * 3$$

$$+ \frac{(9.5-7)(9.5-9)(9.5-10)}{(8-7)(8-9)(8-10)} * 1$$

$$+ \frac{(9.5-7)(9.5-8)(9.5-10)}{(9-7)(9-8)(9-10)} * 1$$

$$+ \frac{(9.5-7)(9.5-8)}{(10-7)(10-8)(10-10)} * 9$$

Find Eq<sup>n</sup> f(x)

3)	x	0	1	3	4
	y	-12	0	6	12

Soln:

$$x_0 = 0 \quad f(x_0) = -12$$

$$x_1 = 1 \quad f(x_1) = 0$$

$$x_2 = 3 \quad f(x_2) = 6$$

$$x_3 = 4 \quad f(x_3) = 12$$

Now,

$$f(x) = \frac{(x-1)(x-3)(x-4)}{(0-1)(0-3)(0-4)} * (-12) +$$

$$+ \frac{(x-0)(x-3)(x-4)}{(1-0)(1-3)(1-4)} * 0$$

$$+ \frac{(x-1)(x-0)(x-4)}{(3-1)(3-0)(3-4)} * 6$$

$$+ \frac{(x-0)(x-1)(x-3)}{(4-0)(4-1)(4-3)} * 12$$

$$= - \frac{12(x^2-x+3)(x-4)}{-1*3*-4} +$$

$$6 \frac{(x^2-x)(x-4)}{2*3*-9} + \frac{12(x^2-x)(x-3)}{4*3*1}$$

$$= - \frac{12(x^3-x^2+3x-4x^2+4x-12)}{-12}$$

$$+ \frac{6(x^3-4x^2+x^2+4x)}{6*-9} + \frac{12(x^3-3x^2x^2+3x)}{12}$$

$$= (x^3-5x^2+7x-12) + x^3-4x^2+3x$$

$$+ (x^3-5x^2+4x)$$

-9

$$= -9(2x^3-9x^2+10x-12) + x^3-5x^2+4x$$

-9

$$= - \frac{18x^3+81x^2-90x+108+x^3-5x^2+4x}{-9}$$

$$= - \frac{17x^3+76x^2-86x+108}{-9}$$

$$= \frac{17x^3-76x^2+86x-108}{9}$$

- 4) The function  $y = f(x)$  is given at the points  $(7, 3), (8, 1), (9, 1) & (10, 9)$ . Find the value of  $y$  for  $x=9.5$  using Lagrange interpolation formula.

Here

Given table is:

x	7	8	9	10
f(x)	3	1	1	9

Now,

$$f(9.5) = \frac{(9.5-8)(9.5-9)(9.5-10)}{(7-8)(7-9)(7-10)} * 3$$

$$+ \frac{(9.5-7)(9.5-9)(9.5-10)}{(8-7)(8-9)(8-10)} * 1$$

$$+ \frac{(9.5-7)(9.5-8)(9.5-10)}{(9-7)(9-8)(9-10)} * 1$$

$$+ \frac{(9.5-7)(9.5-8)(9.5-9)}{(10-7)(10-8)(10-9)} * 9$$

$$= \frac{-3}{16} - \frac{5}{16} + \frac{15}{16} + \frac{45}{16}$$

$$= \frac{58}{16}$$

$$= 3.63$$

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4(b)

Determine the largest eigen value and corresponding eigen vector of matrix  $A = \begin{bmatrix} 3 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}$

soln:

$$\text{let } X^{(0)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$AX^{(0)} = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix}$$

$$= 6 \begin{bmatrix} 0.17 \\ 0.67 \\ 1 \end{bmatrix} = A'X^{(1)}$$

$$AX^{(1)} = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.17 \\ 0.67 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.16 \\ 0.36 \\ 8.03 \end{bmatrix} = 8.03 \begin{bmatrix} 0.02 \\ 0.29 \\ 1 \end{bmatrix} = A^{(2)}X^{(2)}$$

$$AX^{(2)} = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.02 \\ 0.29 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.15 \\ 0.24 \\ 5.99 \end{bmatrix} = 5.99 \begin{bmatrix} 0.19 \\ 0.04 \\ 1 \end{bmatrix}$$

$$= A^{(3)}X^{(3)}$$

$$AX^{(3)} = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.19 \\ 0.04 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.07 \\ -0.08 \\ 6.26 \end{bmatrix} = 6.26 \begin{bmatrix} 0.33 \\ -0.01 \\ 1 \end{bmatrix}$$

$$= A^{(4)}X^{(4)}$$

$$AX^{(4)} = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.33 \\ -0.01 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.36 \\ 0.28 \\ 6.95 \end{bmatrix} = 6.95 \begin{bmatrix} 0.04 \\ 1 \\ 1 \end{bmatrix}$$

$$= A^{(5)}X^{(5)}$$

$$AX^{(5)} = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.34 \\ -0.04 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.22 \\ 0.52 \\ 7.16 \end{bmatrix} = 7.16 \begin{bmatrix} 0.07 \\ 1 \\ 1 \end{bmatrix}$$

$$= A^{(6)}X^{(6)}$$

$$AX^{(6)} = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.31 \\ 0.07 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.1 \\ 0.52 \\ 7.07 \end{bmatrix} = 7.07 \begin{bmatrix} 0.07 \\ 1 \\ 1 \end{bmatrix}$$

## Method II

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$$AX^{(7)} = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.3 \\ 0.07 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.09 \\ 0.48 \\ 7.01 \end{bmatrix}$$

$$= 7.01 \begin{bmatrix} 0.3 \\ 0.07 \\ 1 \end{bmatrix}$$

$$AX^{(8)} = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0.3 \\ 0.07 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.09 \\ 0.48 \\ 7.01 \end{bmatrix}$$

$$= 7.01 \begin{bmatrix} 0.3 \\ 0.07 \\ 1 \end{bmatrix}$$

Largest eigen value = 7.01  
corresponding eigen vector-

$$= \begin{bmatrix} 0.3 \\ 0.07 \\ 1 \end{bmatrix} //$$

## Divided Difference Method :-

Divided difference table.

$x$	$y = f(x)$	$\Delta^1 f(x)$
$x_0$	$f(x_0)$	$\frac{f(x_1) - f(x_0)}{x_1 - x_0} = f(x_0, x_1)$
$x_1$	$f(x_1)$	$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f(x_1, x_2)$
$x_2$	$f(x_2)$	$\frac{f(x_3) - f(x_2)}{x_3 - x_2} = f(x_2, x_3)$
$x_3$	$f(x_3)$	$\emptyset$
$\Delta^2 f(x)$	$\Delta^3 f(x)$	
$\frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}$	$\frac{f(x_1, x_2, x_3) - f(x_0, x_1, x_2)}{x_3 - x_0}$	
$= f(x_0, x_1, x_2)$	$= f(x_0, x_1, x_2, x_3)$	
$\frac{f(x_2, x_3) - f(x_1, x_2)}{x_3 - x_1}$		
$= f(x_1, x_2, x_3)$		

$$f(x) = f(x_0) + (x-x_0)f(x_0, x_1) + (x-x_0)(x-x_1)f(x_0, x_1, x_2) + (x-x_0)(x-x_1)(x-x_2)f(x_0, x_1, x_2, x_3).$$

Construct a divided difference table of

$x$	0	1	2	3	5	6
$f(x)$	1	19	55	241	415	

Divided difference table

$x$	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	1	$19-1=18$	$36-36=0$	$19-0=19$
2	19	$55-19=36$	$36-0=36$	$=2$
3	55	$241-55=186$	$93-36=57$	$27-19=8$
5	241	$415-241=174$	$174-93=81$	$6-2=4$
6	415			

$$\Delta^4 f(x)$$

$$\frac{2-2}{6-0} = 0$$

## Interpolation & Approximation

M-

2)	$x$	5	7	11	13	17
	$y$	150	392	1452	2366	5202

Eg Evaluate  $f(9)$

Soln:

The divided difference table

5.	$f(x)=y$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
6.	150	$392-150=242$	$1452-392=1060$	$2366-1452=914$	$5202-2366=2836$
7.	392	$=121$	$=74$	$=32$	$=1$
11.	1452	$11-7=4$	$11-7=4$	$11-7=4$	$11-7=4$
13.	2366	$=265$	$=265$	$=265$	$=265$
17.	5202	$=1$	$=1$	$=1$	$=1$

Using Newton's Divided difference.

$$\begin{aligned}
 f(9) &= 150 + (9-5)*121 + (9-5)*(9-7)*24 \\
 &\quad + (9-5)(9-7)(9-11)*1 \\
 &= 810
 \end{aligned}$$

3) The following table give percentage of criminals for different age groups. Using suitable interpolation formula, Find percentage of criminals under age 35

Under age	25	30	40	50
% of criminals	52	67.3	84.1	94.4

The divided difference table

$x$	$f(x)$	$\Delta^1 f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
25	52	$\frac{67.3 - 52}{30 - 25} = 3.06$	$\frac{1.68 - 3.06}{40 - 25} = -0.092$	
30	67.3	$\frac{84.1 - 67.3}{40 - 30} = 1.68$	$\frac{-0.092 + 0.092}{50 - 25} = 0.00238$	
40	84.1	$\frac{94.4 - 84.1}{50 - 40} = 1.03$	$\frac{1.03 - 1.68}{50 - 30} = -0.0325$	
50	94.4			

$$\begin{aligned}
 f(35) &= 52 + (35-25) \times 3.06 + (35-25)(35-30) \\
 &\quad \times -0.092 + (35-25)(35-30)(35-40) \\
 &\quad \times 0.00238 \\
 &= 77.375
 \end{aligned}$$

Find polynomial eqn.

$x$	0	1	2	5
$f(x)$	2	3	12	147

The divided difference table is:

$x$	$f(x)$	$\Delta^1 f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\dots$
0	2	$\frac{3-2}{1-0} = 1$	$\frac{9+1}{2-0} = 4$		
1	3	$\frac{12-3}{2-1} = 9$	$\frac{45-9}{5-1} = 9$	$\frac{9-9}{5-0} = 1$	
2	12	$\frac{147-12}{5-2} = 45$			
5	147				

$$f(x) = 2 + (x-0) \times 1 + (x-0)(x-1) \times 4 + (x-0)(x-1)(x-2) \times 1$$

$$\begin{aligned}
 &= 2 + x + x(x-1) \times 4 + x(x-1)(x-2) \\
 &= 2 + x + 4x^2 - 4x + x(x^2 - 3x + 2) \\
 &= 2 + x + 4x^2 - 4x + x^3 - 3x^2 + 2x \\
 &= 2 + 3x + x^2 + x^3 - 4x \\
 &= 2 - x + x^2 + x^3 = x^3 + x^2 - x + 2
 \end{aligned}$$

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x	0	1	2	4	5	6
f(x)	1	14	15	5	5	19

The divided difference table

x	f(x)	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
0	1	$\frac{14-1}{1-0} = 13$	$\frac{1-13}{2-0} = -6$	$\frac{-6+6}{4-0} = 0$	$\frac{1-1}{5-0} = 0$
1	14	$\frac{15-14}{2-1} = 1$	$\frac{-5-1}{4-1} = -2$	$\frac{2+2}{5-1} = 0$	
2	15	$\frac{5-15}{4-2} = -5$		$\frac{105}{5-1} = 105$	
4	5	$\frac{6-5}{5-4} = 1$	$\frac{1+5}{5-2} = 2$	$\frac{1-1}{6-1} = 0$	
5	6	$\frac{19-6}{6-5} = 13$	$\frac{13-1}{6-4} = 6$	$\frac{6-2}{6-2} = 1$	
6	19				

$$\begin{aligned}
 f(3) &= 1 + (3-0) 13 + (3-0)(3-1) \times -6 \\
 &\quad + (3-0)(3-1)(3-2) \times 1 + (3-0) \\
 &\quad (3-1)(3-2)(3-4) \times 0 \\
 &= 10.
 \end{aligned}$$

### Method III Interpolation With Equal Intervals

#### Newton Forward Interpolation Method

Formula:

$$f(x) = y_0 + p \Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0 + \dots$$

$$\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!} \Delta^4 y_0 + \dots$$

$$\text{where } p = \frac{x-x_0}{h}$$

$h$  = size of intervals.

This formula is used if  $x$  is near to the beginning value of given tabulated function.  $\Delta y_0$ ,  $\Delta^2 y_0$ ,  $\Delta^3 y_0$  & others are found in difference table.

### Method IV Newton's Backward Interpolation method

$$\text{Formula : } f(x) = y_0 + p \nabla y_0 + \frac{p(p+1)}{2!} \nabla^2 y_0 +$$

$$\frac{p(p+1)(p+2)}{3!} \nabla^3 y_0 + \dots$$

$$\text{where } p = \frac{x-x_0}{h}$$

$h$  = class size

$\nabla y_0$ ,  $\nabla^2 y_0$ ,  $\nabla^3 y_0$  & others are found in difference table.

This formula is used if  $x$  is near to the ending value of tabulated function.

Q No. 1 From following table, estimate the number of students who obtained marks 45.

Marks (x)	40	50	60	70	80
No. of stdf(x)	31	73	124	159	190

Using suitable interpolation  
Solving

Here  $x = 45$  is near to the beginning value of tabulated functn, Newton's Forward interpolation formula is suitable formula.

The difference table is given below

X	$y = f(x)$	$\Delta y$	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	31	42	9	-25	37
50	73	51	-16	120	
60	124	35	-4		
70	159	31			
80	190				

From table :

$$x = 45$$

$$x_0 = 40$$

$$h = 10$$

We know Newton's Forward interpolation formula.

$$f(x) = y_0 + p \Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!} \Delta^4 y_0$$

$$= 31 + 0.5 \times 42 + \frac{0.5(0.5-1)}{2!} \times 9 +$$

$$\frac{0.5(0.5-1)(0.5-2)}{3!} \times (-25) +$$

$$\frac{0.5(0.5-1)(0.5-2)(0.5-3)}{4!} \times 37$$

$$= 47.8672$$

Hence, number of students who obtained marks 45 = 47.8672  
 $= 48$

- 2) The following table gives the pop<sup>n</sup> of the town during last six census. Estimate using Newton's interpolation formula, the increase in pop<sup>n</sup> during period 1946 & 1948.

year	1911	1921	1931	1941	1951	1961
pop <sup>n</sup> (000)	12	13	20	27	39	52

Here,  $x = 1946$  &  $1948$  are near to the ending value of the tabulated funct<sup>n</sup>. Newton's backward interpolation formula is suitable method

$x$	$y = f(x)$	$\nabla y$	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
1911	12	1	6	-6	4	
1921	13	7	0	5	11	
1931	20	7		5	-9	-20
1941	27	5	-4			
1951	39	1				
1961	52	13	$\nabla y_0$	$\nabla^3 y_0$	$\nabla^4 y_0$	$\nabla^5 y_0$
$x_0$	$y_0$					

For 1946

$$h = 10$$

$$x = 1946$$

$$x_0 = 1961$$

$$p = \frac{x - x_0}{h} = \frac{1946 - 1961}{10} = -1.5$$

We know Newton's interpolation formula  
 $f(x) = y_0 + p \nabla y_0 + \frac{p(p+1)}{2!} \nabla^2 y_0 + \frac{p(p+1)(p+2)}{3!} \nabla^3 y_0 + \frac{p(p+1)(p+2)(p+3)}{4!} \nabla^4 y_0$

$$+ \frac{p(p+1)(p+2)(p+3)(p+4)}{5!} \nabla^5 y_0$$

$$\begin{array}{r} -1.5+4 \\ -2.5 \end{array}$$

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$$= 52 + (-1.5) \times 13 - 1.5 \frac{(-1.5+1) \times 1}{2}$$

$$- 1.5 \frac{(-1.5+1)(-1.5+2)(-4)}{3!} +$$

$$1.5 \frac{(-1.5+1)(-1.5+2)(-1.5+3)(-9)}{4!} \times (-9)$$

$$- 1.5 \frac{(-1.5+1)(-1.5+2)(-1.5+3)(-1.5+4)}{5!} \times (-20)$$

$$= 32.180$$

1) For 1948.

$$x = 1948$$

$$x_0 = 1961$$

$$\therefore p = \frac{x-x_0}{h} = -1.3$$

Then

$f(x) =$

$$f(1948) = 52 + (-1.3 \times 13) - 1.5 \frac{(-1.3+1) \times 1}{2}$$

$$- 1.3 \frac{(-1.3+1)(-1.3+2) \times (-4)}{3!}$$

$$- 1.3 \frac{(-1.3+1)(-1.3+2)(-1.3+3)}{4!} \times (-1.3+3)$$

$$\times (-9) - 1.3 (-1.3+1)(-1.3+2) (-1.3+3)(-1.3+4) \times (-20)$$

51

$$= 34.7301$$

Now,

$$\text{Increase in pop}^n = 34.7301 - 32.180 \\ = 2.5501$$

3) From Following data, estimate the no. of persons earning hourly wages between 60 & 70 rupees.

wage Rs	20-40	40-60	60-80	80-100	100-120
pop^n (000)	250	120	100	70	50

Here the difference table is as below

x	y	$\Delta y$	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
Below 40	250				
" 60	370	120			
" 80	470	100	-20		
" 100	540	70	-30	10	
" 120	590	50	-20		

Below 70  
From table

$$x_0 = 40 \quad \therefore p = \frac{x - x_0}{h} = \frac{70 - 40}{20} = 1.5$$

$$h = 20$$

$$x = 70$$

From Newton's Forwarded interpolation formula

$$f(x) = y_0 + p \Delta y_0 + p(p-1) \frac{\Delta^2 y_0}{2!} +$$

$$p(p-1)(p-2) \frac{\Delta^3 y_0}{3!} + p(p-1)(p-2)$$

$$\frac{(p-3)}{4!} \cdot \Delta^4 y_0$$

$$\therefore f(70) = 423.594.$$

$$f(60) = 370$$

then

No. of person's earning below 60  
= 370

No. of person's betw 60 & 70

$$= 423.594 - 370$$

$$= 53.594 \text{ thousand}$$

$$= 53594$$

4) The following table gives the displacement of an object at various of time. Find the velocity & acceleration of object at  $t=1.6$  using suitable interpolation

x	t(sec)
y	

(4)

# Estimate the value of  $\sin \theta$  at  $\theta = 25^\circ$   
using Newton's forward interpolation

$\theta$	10	20	30	40	50
$y = \sin \theta$	0.1736	0.3420	0.5000	0.6428	0.7660

the forward difference table is

$\theta$	$y$	$\Delta y$	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
10	0.1736	0.1684	-0.0104		
20	0.3420	0.1580	-0.0098	0.0004	
30	0.5000	0.1428	-0.0152	-0.0044	
40	0.6428	0.1232	-0.0196		
50	0.7660				

$$\underline{f(28)} =$$

$$x = 25^\circ \quad h = 10$$

$$x_0 = 10$$

$$p = \frac{x - x_0}{h} = \frac{25 - 10}{10} = 1.5$$

Then

$$f(x) = y_0 + p \Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!} \Delta^4 y_0$$

$$= 0.1736 + \frac{1.5(1.5-1)}{2!} \Delta^2 y_0 + \frac{1.5(1.5-1)(1.5-2)}{3!} \Delta^3 y_0 + \frac{1.5(1.5-1)(1.5-2)(1.5-3)}{4!} \Delta^4 y_0$$

- 4) Fitting of a straight line  
 5) Fitting of a parabola  
 6) Fitting of a exponential curve

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## Fitting of a Curve:

### a) Fitting of a straight line:

The equation of straight line is

$$y = a + bx \quad \text{--- (1)}$$

Fitting of a curve means finding the value of parameters  $a$  &  $b$  with help of normal equations

The normal equations are

$$na + b \sum x = \sum y \quad \text{--- (i)}$$

$$a \sum x + b \sum x^2 = \sum xy \quad \text{--- (ii)}$$

Tabulating for  $\sum x$ ,  $\sum y$ ,  $\sum x^2$ ,  $\sum xy$

then solving (i) & (ii) we get value of  $a$  &  $b$ . Substituting value of  $a$  &  $b$  in eqn (1) we get required eqn of straight line

1) Fit a straight line by using least square method to following data

$x$	1	2	3	4	5
$y$	14	27	40	55	68

Estimate value of  $y$  when  $x=6$

Sols:

Let eqn of straight line to be fit is

$$y = a + bx \quad \text{--- (1)}$$

Normal eqn are:

$$na + b \sum x = \sum y$$

$$a \sum x + b \sum x^2 = \sum xy$$

Tabulating for  $\sum x$ ,  $\sum y$ ,  $\sum x^2$  &  $\sum xy$ , we get

$x$	$y$	$x^2$	$\sum xy$
1	14	1	14
2	27	4	54
3	40	9	120
4	55	16	220
5	68	25	340
$\sum x$	$\sum y$	$\sum x^2$	$\sum xy$
15	204	55	748

$n=5$

Normal eqn are

$$na + b \sum x = \sum y$$

$$a \sum x + b \sum x^2 = \sum xy$$

Now

$$5a + 15b = 204$$

$$15a + 55b = 748$$

We get  $a=0$

$$b=13.6$$

From eqn (1)  $y = 13.6x$

when  $x=6$ ,  $y = 13.6 * 6 = 81.6$

Hence req<sup>n</sup> eq<sup>n</sup> of str line is

$$y = 13.6x$$

$$\text{at } x = 6, y = 81.6$$

PV 2014(Fall)

The following table gives height  $x$  cm & weight  $y$  kg of five persons.

$x$	175	165	160	155	145
$y$	68	58	55	52	48

Assuming linear relationship b/w obtain regression line. Obtain  $x$  when  $y = 40$ .

Let linear relationship b/w  $x$  &  $y$  is

$$\text{eq}^n \text{ of straight line } y = a + bx$$

Normal eq<sup>n</sup> are

$$na + b \sum x = \sum y \quad \text{--- (i)}$$

$$a \sum x + b \sum x^2 = \sum xy \quad \text{--- (ii)}$$

$x$	$y$	$\sum x^2$	$\sum xy$
175	68	30625	11900
165	58	27225	9570
160	55	25600	8800
155	52	24025	8060
145	48	21025	6960
$\sum x = 800$		$\sum y$	$\sum x^2$
		$= 281$	$= 128500$
			$= 45290$

then

$$na + b \sum x = \sum y$$

$$a \sum x + b \sum x^2 = \sum xy$$

then

$$5a + b \times 800 = 281$$

$$800a + b \times 128500 = 45290$$

We get

$$a = -49.4$$

$$b = 0.66$$

$$\text{then } y = a + bx$$

$$40 = a + bx$$

$$\text{or } 40 = -49.4 + 0.66x$$

$$\therefore x = 135.45 //$$

The required eq<sup>n</sup> is  $y = -49.4 + 0.66x$   
 $x = 135.45$ .

3) If  $P$  is pull required to lift a load  $w$  by means of pulley. Find a linear law of form  $P = mw + c$  using following data

$x_0$	50	70	100	120
$P$	12	15	21	25

where  $P$  &  $w$  are taken in kg

Here

Given eqn to be fit be

$$P = c + mw$$

Normal eqns are

$$nc + m \sum w = \sum P$$

$$c \sum w + m \sum w^2 = \sum Pw$$

$w$	$P$	$w^2$	$Pw$
50	12	2500	600
70	15	4900	1050
100	21	10000	2100
120	25	14400	3000
$\sum w = 310$	$\sum P = 73$	$\sum w^2 = 31800$	$\sum Pw = 6750$

Then

$$4c + m \cdot 310 = 73$$

$$340c + 31800m = 6750$$

We get  $c = 2.276$

$$m = 0.188$$

$$\text{So } P = 0.188w + 2.276 //$$

4) Find missing value from following data  $y = a + bx$

$x$	3	6	7	9	10
$y$	168	-	120	72	73

Here.

Given data is

$x$	3	7	9	10
$y$	168	120	72	73

We have to find value of  $y$  when  $x=6$

Given eqn of st. line be

$$y = a + bx$$

Normal eqns are

$$na + b \sum ax = \sum y$$

$$a \sum x + b \sum x^2 = \sum xy$$

$x$	$y$	$x^2$	$xy$
3	168	9	504
7	120	49	840
9	72	81	648
10	73	100	730
$\sum x = 29$	$\sum y = 433$	$\sum x^2 = 239$	$\sum xy = 2722$

then

$$4a + 29b = 433$$

$$29a + 239b = 2722$$

We get  $a = 213.46$

$$b = -14.513$$

Now,

$$y = 213.46 - 14.513x$$

$$\text{let } x = 6$$

$$y = 213.46 - 14.513 \times 6 \\ = 126.382 //$$

## b) Fitting of a parabola

The equation of a parabola of degree two is  $y = a + bx + cx^2 \rightarrow ①$

The normal equations are

$$na + b\sum x + c\sum x^2 = \sum y \rightarrow ②$$

$$a\sum x + b\sum x^2 + c\sum x^3 = \sum xy \rightarrow ③$$

$$a\sum x^2 + b\sum x^3 + c\sum x^4 = \sum x^2 y \rightarrow ④$$

Tabulating for  $\sum x, \sum y, \sum x^2, \sum x^3$

$\sum x^4, \sum xy$  &  $\sum x^2 y$  and

solving normal eqns we get a, b & c

## 1) Fit a second degree parabola

x	0	1	2	3	4
y	1	1.8	2.8	2.5	6.3

Find difference between actual value & calculated the value of y when x=2

Solving

Let eqn of parabola to be fit is:

$$y = a + bx + cx^2 \rightarrow ①$$

Normal equations are:

$$na + b\sum x + c\sum x^2 = \sum y \rightarrow ②$$

$$a\sum x + b\sum x^2 + c\sum x^3 = \sum xy \rightarrow ③$$

$$a\sum x^2 + b\sum x^3 + c\sum x^4 = \sum x^2 y \rightarrow ④$$

Tabulating for  $\sum x, \sum y, \sum xy, \sum x^2, \sum x^3, \sum x^4 \notin \sum x^2 y$

x	y	xy	$x^2$	$x^3$	$x^4$	$x^2 y$
0	1	0	0	0	0	0
1	1.8	1.8	1	1	1	1.8
2	2.8	5.6	4	8	16	5.2
3	2.5	7.5	9	27	81	22.5
4	6.3	25.2	16	64	256	100.8
$\Sigma x$	$\Sigma y$	$\Sigma xy$	$\Sigma x^2$	$\Sigma x^3$	$\Sigma x^4$	$\Sigma x^2 y$
10	12.9	37.1	30	-100	354	130.3

$$\begin{aligned} 5a + 10b + 30c &= 12.9 \\ 10a + 30b + 100c &= 37.1 \\ 30a + 100b + 354c &= 130.3 \end{aligned}$$

we get

$$a = 1.42$$

$$b = -1.07$$

$$c = 0.55$$

then  $y = a + bx + cx^2$

$$y = 1.42 - 1.07x + 0.55x^2$$

Also when  $x=2$

$$y = 1.48$$

$$\text{Then difference} = 1.38 - 1.48 \\ = -0.18$$

$$\text{Hence reqd eqn } \therefore y = 1.42 - 1.07x + 0.55x^2$$

$$\text{Difference} = -0.18$$

2) Fit the eqn of parabola

$x$	1	2	3	4
$y$	6	11	18	27

Let eqn of parabola to be fit is:

$$y = a + bx + cx^2$$

Normal equations are:

$$na + b \sum x + c \sum x^2 = \sum y$$

$$a \sum x + b \sum x^2 + c \sum x^3 = \sum xy$$

$$a \sum x^2 + b \sum x^3 + c \sum x^4 = \sum x^2 y$$

Tabulating for  $\sum x, \sum x^2, \sum x^3, \sum x^4, \sum xy, \sum x^2 y$ :

$x$	$y$	$x^2$	$x^3$	$x^4$	$xy$	$x^2 y$
1	6	1	1	1	6	6
2	11	4	8	16	22	84
3	18	9	27	81	54	182
4	27	16	64	256	108	432
$\sum x = 10$	$\sum y = 62$	$\sum x^2 = 30$	$\sum x^3 = 100$	$\sum x^4 = 354$	$\sum xy = 190$	$\sum x^2 y = 644$

Now,

$$4a + 10b + 30c = 62$$

$$10a + 30b + 100c = 190$$

$$30a + 100b + 354c = 644$$

we get

$$a = 3$$

$$b = 2$$

$$c = 1$$

$$\text{then } y = a + bx + cx^2$$

~~$= 3 + 2x + x^2$~~

~~$= x^2 + 2x + 3$~~

~~8th~~

### c) Fitting of a exponential curve:

$$i) y = ax^b$$

Taking log on both sides we get

$$\log y = \log(a x^b)$$

$$\log y = b \log a + b \log x$$

$$\text{or } y = A + b x \quad \textcircled{1}$$

where  $\log y = Y$

$$\log a = A$$

$$\log x = X$$

$$b = b$$

Eqn  $\textcircled{1}$  is eqn of str line

Normal equations are

$$nA + b \sum x = \sum Y$$

$$A \sum x + b \sum x^2 = \sum XY$$

tabulating for  $\sum x, \sum y, \sum x^2, \sum xy$

then solving normal eqn's we get  
a & b.

$$ii) y = a e^{bx}$$

Taking log on both sides

$$\log y = \log(a e^{bx})$$

$$\log y = \log a + \log e^{bx}$$

$$\log y = \log a + bx \log e$$

$$y = A + Bx \quad \textcircled{1}$$

Normal eqn is

$$nA + B \sum x = \sum Y$$

$$A \sum x + B \sum x^2 = \sum XY$$

$$iii) xy^a = b$$

Taking log on both sides

$$\log(xy^a) = \log b$$

$$\text{or } \log x + a \log y = \log b$$

$$\text{or, } a \log y = \log b - \log x$$

$$\log y = \frac{\log b - \log x}{a} = \frac{\log b}{a} - \frac{\log x}{a}$$

i) Fit a curve of form  $y = ae^{bx}$  to the following data

x	0	2	4
y	5.012	10	31.62

Here the curve to be fit

$$y = ae^{bx}$$

Taking log on both sides

$$\log y = \log a + \log e^{bx}$$

$$\text{or } \log y = \log a + b \log e^x$$

$$\text{or } \log y = \log a + bx \log e$$

$$\text{or } \log y = \log a + b \log e \cdot x$$

$$\therefore y = A + Bx \quad \textcircled{1}$$

The eqn  $\textcircled{1}$  is eqn of straight lines

The normal eqns are

$$nA + B \sum x = \sum Y \quad \textcircled{1}$$

$$A \sum x + B \sum x^2 = \sum XY \quad \textcircled{2}$$

x	y	$y = 1000$	$xy$	$x^2$
0	5.02	0.7	0.7	0
2	10	1	2	4
4	31.62	1.5	6	16
$\Sigma x = 6$	$\Sigma y = 3.2$	$\Sigma xy = 8$	$\Sigma x^2 = 20$	

From Normal equations:

$$3A + 6B = 3.2$$

$$6A + 20B = 8$$

We get  $A = 0.6667$ ,  $B = 0.2$

Now,

$$\log a = A \Rightarrow a = \text{Antilog } A \\ = 4.6419$$

$$b/\log e = B \Rightarrow b = \frac{B}{\log e} = 0.4605$$

Then

$$y = a e^{bx} \\ = 4.6419 e^{0.4605x}$$

2) Fit a power curve  $y = ax^b$  to the following data

x	61	26	7	2.6
y	350	400	500	600

Here  $y = ax^b$

Taking log on both sides

$$\log y = \log a + b \log x \\ \text{or, } \log y = A + bx \rightarrow ①$$

where

$$Y = \log y$$

$$A = \log a$$

$$b = b$$

$$X = \log x$$

Then eqn (i) is straight line  
 $nA + b \sum x = \sum Y$

$$A \sum x + b \sum x^2 = \sum XY$$

Finding  $\sum x$ ,  $\sum x^2$ ,  $\sum y$ ,  $\sum xy$

x	y	x	y	xy	$x^2$
61	350	1.7853	2.5441	4.5420	3.1873
26	400	1.4150	2.6021	3.6820	2.0022
7	500	0.8451	2.6990	2.2809	0.7142
2.6	600	0.4150	2.7782	1.1530	0.1722
$\Sigma x =$	$\Sigma y$	$\Sigma xy$	$\Sigma x^2 =$		
14.4604	10.6237	11.6579	6.0759		

Normal eq<sup>n</sup>

$$4A + 4.4604B = 10.6234$$

$$4.4604A + 6.0759B = 11.6579$$

Solving eq<sup>n</sup>'s we get

$$A = 2.8463$$

$$B = -0.1708$$

so,

$$\log a = A = 2.8463$$

$$\therefore a = 701.94$$

$$\text{So, } y = ax^b \\ = 701.94x^{-0.1708}$$

- 3) The pressure and volume of gas are related by the eq<sup>n</sup>  $PV^\gamma = C$   
 where  $\gamma$  &  $C$  being constant  
 Fit this eq<sup>n</sup> to the following data

$x \text{ kg/cm}^2$	$P$	0.5	1.0	1.5	2	2.5	3.0
$y \text{ litres}$	$V$	1.62	1.00	0.75	0.62	0.52	0.46

Here

$$PV^\gamma = C$$

Taking log on both sides

$$\log(PV^\gamma) = \log C$$

$$\text{or, } \log P + \gamma \log V = \log C$$

$$\text{or, } \gamma \log V = \log C - \log P$$

$$\text{or, } \log V = \frac{1}{\gamma} \log C - \frac{1}{\gamma} \log P$$

$$y = A + BX \quad \rightarrow \textcircled{1}$$

where

$$y = \log V$$

$$A = \frac{1}{\gamma} \log C$$

$$B = -\frac{1}{\gamma}$$

$$X = \log P$$

Eq<sup>n</sup> ① is eq<sup>n</sup> of straight line.

The normal equations are

$$nA + B \sum X = \sum Y \quad \rightarrow \textcircled{1}$$

$$A \sum X + B \sum X^2 = \sum XY \quad \rightarrow \textcircled{2}$$

Tabulating for  $\sum X$ ,  $\sum Y$ ,  $\sum XY$ ,  $\sum X^2$ , we get

$X(P)$	$Y(V)$	$X = \log P$	$Y = \log V$	$\sum XY$	$\sum X^2$
0.5	1.62	-0.3010	0.2095	-0.0631	0.0906
1	1	0	0	0	0
1.5	0.75	0.1761	-0.1249	-0.0220	0.0310
2	0.62	0.3010	-0.2076	-0.0625	0.0906
2.5	0.52	0.3979	-0.2840	-0.1130	0.1583
3	0.46	0.4771	-0.3372	-0.1609	0.2276
$\sum X = 10.5$		$\sum Y = 5.21$		$\sum XY = -0.4215$	
		$\sum X^2 = 1.0511$		$\sum X^2 = 0.5981$	

From normal eq<sup>n</sup>

$$6A + 1.0511B = -0.4215$$

$$1.0511A + 0.5981B = -0.4215$$

$$\text{we get } A \approx -0.0008$$

$$B = -0.7033$$

a = -1.3238

Date | b = -0.8453  
Page |

Now,

$$\frac{-1}{r} = B \Rightarrow r = -\frac{1}{B} = -\frac{1}{0.7033} = 1.4219$$

Then

$$\frac{1}{r} \log c = A \Rightarrow \log c = Ax + r$$

$$\log c = (-0.0008x + 1.4219)$$

$$\therefore c = 0.9974$$

So,

$$PV^r = c$$

$$PV^{1.4219} = 0.9974 //$$

4) Fit curve  $y = e^{ax+bx}$

x	-5	-4	-3	-1	0
y	12.96	6.94	4.63	2.11	0.09

Hence  $y = e^{ax+bx}$

$$\log y = (a+bx) \log e$$

$$\text{or } \log y = a \log e + b \log x$$

$$y = A + Bx \quad \text{--- (i)}$$

$$y = \log y$$

$$A = a \log e$$

$$B = b \log e$$

$$x = x$$

Eqn (i) is eqn of straight line

Normal equations are:

$$nA + B \sum x = \sum y$$

$$A \sum x + B \sum x^2 = \sum xy$$

x	y	$y = \log y$	$x^2$	$xy$
-5	12.96	1.1126	25	-5.5630
-4	6.94	0.8414	16	-3.3656
-3	4.63	0.6656	9	-1.9968
-1	2.11	0.3243	1	-0.3243
0	0.09	-1.0458	0	0
$\sum x = -13$		$\sum y = 1.8981$	$\sum x^2 = 54$	$\sum xy = -11.2497$

$$5A + 13B = 1.8981 \quad \text{--- (i)}$$

$$-13A + 51B = -11.2497 \quad \text{--- (ii)}$$

We get  $A = -0.5749$

$$B = -0.3671$$

Then,

$$a = \frac{A}{\log e} = -\frac{0.5749}{\log e} = -1.3238$$

$$b = \frac{B}{\log e} = -\frac{0.3671}{\log e} = -0.8453$$

Hence  $-1.3238 - 0.8453x$

$$y = e$$

//

# Chapter-Four

## Numerical Differentiation and Integration



### Numerical Differentiation

↪ Numerical differentiation is the process of calculating derivatives of a given tabulated function at any point. If  $(x_i, y_i)$  are the given set of values then the process of computing values of  $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots$  etc is called numerical differentiation.

If the values of derivatives are required at a point near to the beginning beginning value we use Newton's forward difference formula for derivatives & if near to the ending values we use backward formula.

#### a) Newton's forward difference formula for derivatives.

We know Newton's forward difference interpolation formula

$$y = y_0 + p \frac{\Delta y_0}{1!} + p(p-1) \frac{\Delta^2 y_0}{2!} + p(p-1)(p-2) \frac{\Delta^3 y_0}{3!}$$

$$+ \frac{\Delta^3 y_0}{4!} + p(p-1)(p-2)(p-3) \frac{\Delta^4 y_0}{4!}$$

$$\text{or } y = y_0 + p \frac{\Delta y_0}{1!} + p^2 \frac{\Delta^2 y_0}{2!} + p^3 - 3p^2 + 2p \frac{\Delta^3 y_0}{3!}$$

$$+ p^4 - 6p^3 + 11p^2 - 6p \frac{\Delta^4 y_0}{4!} + \dots$$

Diff both side wrt p

$$\frac{dy}{dp} = \frac{\Delta y_0}{1!} + \frac{2p-1}{2!} \Delta^2 y_0 + \frac{3p^2-6p+2}{3!} \Delta^3 y_0 +$$

$$+ \frac{4p^3-18p^2+22p-6}{4!} \Delta^4 y_0 + \dots$$

$$\text{where } p = \frac{x-10}{h}$$

$$\therefore \frac{dp}{dx} = \frac{1}{h}$$

$$\text{Now, } \frac{dy}{dx} = \frac{dy}{dp} \cdot \frac{dp}{dx}$$

$$= \frac{1}{h} \left[ \Delta y_0 + \frac{2p-1}{2!} \Delta^2 y_0 + \frac{3p^2-6p+2}{3!} \right]$$

$$+ \frac{4p^3-18p^2+22p-6}{4!} \Delta^4 y_0 + \dots$$

Similarly

$$\frac{d^2y}{dx^2} = \frac{1}{h^2} \left[ \frac{2}{2!} \Delta^2 y_0 + \frac{6p-6}{3!} \Delta^3 y_0 + \frac{12p^2-36p+22}{4!} \Delta^4 y_0 + \dots \right]$$

$$\frac{d^3y}{dx^3} = \frac{1}{h^3} \left[ \frac{6}{3!} \Delta^3 y_0 + \frac{24p-36}{4!} \Delta^4 y_0 \right]$$

1) From the data compute  $\frac{dy}{dx}$

$$\frac{d^2y}{dx^2} \text{ at } x=1$$

x	1	2	3	4	5	6
y	1	8	27	64	125	216

Here  $x=1$  is the beginning value of given function, we use Newton forward difference formula for derivatives. The table is

x	y	$\Delta y$	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	1	7	12		
2	8	19	18	6	0
3	27	37	24	6	
4	64	61	30	6	0
5	125	81			
6	216				

From table

$$x_0 = 1$$

$$x = 1 \therefore p = \frac{x-x_0}{h} = 0$$

$$h = 1$$

We know Forward difference interpolation formula.

$$y = y_0 + p \Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0$$

$$+ \frac{p(p-1)(p-2)(p-3)}{4!} \Delta^4 y_0$$

$$or y = y_0 + p \Delta y_0 + \frac{p^2-p}{2!} \Delta^2 y_0 + \frac{p^3-3p^2+2p}{3!} \Delta^3 y_0$$

$$+ \frac{p^4-6p^3+11p^2-6p}{4!} \Delta^4 y_0$$

Now, diff both sides wrt x we get,

$$\frac{dy}{dx} = \frac{1}{h} \left[ \Delta y_0 + \frac{2p-1}{2!} \Delta^2 y_0 + \frac{3p^2-6p+2}{3!} \Delta^3 y_0 \right]$$

$$+ \frac{4p^3-18p^2+22p-6}{4!} \Delta^4 y_0 \right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2} \left[ \frac{2}{2!} \Delta^2 y_0 + \frac{6p-6}{3!} \Delta^3 y_0 + \frac{12p^2-36p+22}{4!} \Delta^4 y_0 \right]$$

$$\Delta^4 y_0 \right]$$

$$\text{At } x=1, p=0$$

$$\frac{dy}{dx} = \frac{1}{1} \left[ 7 - \frac{1}{2!} \times 12 + \frac{2 \times 6}{3!} - \frac{6 \times 0}{4!} \right]$$

$$= [7 - 6 + 2] \\ = 3$$

$$\frac{d^2y}{dx^2} = \frac{2}{2!} \times 12 + \frac{6}{3!} \times 6 + \frac{22}{4!} > 0$$

$$= 12 - \frac{36}{6}$$

$$= 12 - 6$$

$$= 6 >$$

Hence  $\frac{dy}{dx} = 3$ ,  $\frac{d^2y}{dx^2} = 6$ .

- Q) The pop<sup>n</sup> of a certain town is given below.

Year(x)	1931	1941	1951	1961	1971
pop <sup>n</sup> (y)	40.6	60.8	79.9	103.6	132.7

Here 1961 is near to ending value  
So we use Newton backward difference formula.

x	y	$\Delta y$	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1931	40.6				
1941	60.8	20.20	-1.1	8.57	
1951	79.9	19.1	4.6	-4.9	
1961	103.6	23.70	5.4	0.8	
1971	132.7	29.1			
	$y_0$	$\Delta y_0$	$\Delta^2 y_0$	$\Delta^3 y_0$	

From table

$$x = 1961$$

$$x_0 = 1931$$

$$h = 10$$

$$\therefore p = \frac{x-x_0}{h} = \frac{1961-1931}{10} = \frac{-10}{10} = -1$$

Then

$$y = y_0 + p \Delta y_0 + \frac{p(p+1)}{2!} \Delta^2 y_0 + \frac{p(p+1)(p+2)}{3!} \Delta^3 y_0$$

$$+ \frac{p(p+1)(p+2)(p+3)}{4!} \Delta^4 y_0$$

$$y = y_0 + p \Delta y_0 + \frac{p^2 + p}{2!} \Delta^2 y_0 + \frac{p^3 + 3p^2 + 2p}{3!} \Delta^3 y_0$$

$$+ \frac{p(p+6p^2 + 11p + 6)}{4!} \Delta^4 y_0$$

$$y = y_0 + p \Delta y_0 + \frac{p^2 + p}{2!} \Delta^2 y_0 + \frac{p^3 + 3p^2 + 2p + 3}{3!} \Delta^3 y_0$$

$$+ \frac{p^4 + 6p^3 + 11p^2 - 6p}{4!} \Delta^4 y_0$$

$$\frac{dy}{dp} = p + \frac{2p+1}{2!} \Delta^2 y_0 + \frac{3p^2 + 8p + 2}{3!} \Delta^3 y_0$$

$$+ \frac{4p^3 + 18p^2 + 22p + 6}{4!} \Delta^4 y_0$$

then

$$= \frac{1}{h} \left[ p y_0 + \frac{2p+1}{2!} \nabla y_0 + \frac{3p^2+6p+2}{3!} \nabla^2 y_0 + \frac{4p^3+18p^2+22p+6}{4!} \nabla^4 y_0 \right]$$

$$= \frac{1}{10} \left[ 29.1 + \frac{-2+1}{2!} * 5.4 + \frac{3-6+2}{3!} * 0.8 - \frac{4+18+22+6}{4!} * (-4.9) \right]$$

$$= 2.667$$

3) PU 10 times

A rod is rotating a plane. The following table gives the angle θ radians through which the rod has turned for various values of the time t seconds.

$t$	0	0.2	0.4	0.6	0.8	1.0
$\theta$	0	0.12	0.49	1.12	2.02	3.20

calculate angular velocity and angular acceleration at  $t = 0.6$  s.

⇒ Here  $t = 0.6$  is near to the ending value of given table. Newton's backward difference formula is used.

### The difference table

$x=t$	$y = \theta$	$\nabla y$	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
0.	0	0.12			
0.2	0.12	0.37	0.25	0.01	
0.4	0.49		0.26	0	
0.6	1.12	0.63	0.27	0.01	0
0.8	2.02	0.9	0.28		
1.0	3.20	1.18	$\nabla^2 y_0$	$\nabla^3 y_0$	$\nabla^4 y_0$
	$\nabla y_0$				

From table

$$x_0 = 1$$

$$x = 0.6$$

$$h = 0.2$$

$$\therefore p = \frac{x - x_0}{h} = -2.$$

Then we know,

$$y = y_0 + p \nabla y_0 + \frac{p(p+1)}{2!} \nabla^2 y_0 + \frac{p(p+1)(p+2)}{3!} \nabla^3 y_0 + \frac{p(p+1)(p+2)(p+3)}{4!} \nabla^4 y_0$$

$$\nabla^3 y_0 + \frac{p(p+1)(p+2)(p+3)}{4!} \nabla^4 y_0$$

$$\text{or } y = y_0 + p \nabla y_0 + \frac{p^2 + p}{2!} \nabla^2 y_0 + \frac{p(p+1)}{3!} \nabla^3 y_0 + \frac{p^4 + 6p^3 + 11p^2 + 6p}{4!}$$

$$\frac{p^3 + 3p^2 + 2p}{3!} \nabla^3 y_0 + \frac{p^4 + 6p^3 + 11p^2 + 6p}{4!} \nabla^4 y_0$$

$$\nabla^4 y_0$$

$$\frac{dy}{dx} = \frac{1}{h} \left[ \nabla y_0 + \frac{2p+1}{2!} \nabla^2 y_0 + \frac{3p^2+6p+2}{3!} \nabla^3 y_0 + \dots \right]$$

$$\nabla^3 y_0 + \frac{4p^3+18p^2+22p+6}{4!} \nabla^4 y_0$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2} \left[ \frac{2}{2!} \nabla^2 y_0 + \frac{6p+6}{3!} \nabla^3 y_0 + \dots \right]$$

$$\frac{12p+36p+22}{4!} \nabla^4 y_0$$

Then :

$$v = \frac{dy}{dx} = \frac{1}{0.2} \left[ 1.18 + \frac{-4+1}{2!} * 0.28 + \frac{12-12+2}{3!} * 0.01 + \frac{4*8+18*4-22*2+6*10}{4!} \right]$$

$$= 3.8167 //$$

$$a = \frac{d^2y}{dx^2} = \frac{1}{0.2^2} \left[ 1 * 0.28 + \frac{6 * (-2) + 6}{3!} * 0.01 + 0 \right]$$

$$= 6.75 //$$

$$v = 3.8167 \text{ rad/s}$$

$$a = 6.75 \text{ rad/sec}^2$$

4) Find  $\frac{dy}{dx}$ ,  $\frac{dy^2}{dx^2}$  &  $\frac{dy^3}{dx^3}$  at  $x=1.6$

$x$	1	1.1	1.2	1.3	1.4	1.5	1.6
$y$	7.989	8.403	8.781	9.129	9.451	9.75	10.03

Solns:

$x$	$y$	$\nabla y$	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
1	7.989	0.4140	-0.0360	0.0060	-0.002	
1.1	8.403	0.3780	-0.0300	0.004	-0.001	
1.2	8.781	0.3480	-0.0260	0.003	0.002	
1.3	9.129	0.3220	-0.0230	0.005		
1.4	9.451	0.2990	-0.0180			
1.5	9.75	0.2810				
1.6	10.03	$\nabla y_0$	$\nabla^2 y_0$	$\nabla^3 y_0$	$\nabla^4 y_0$	

From table

$$x = 1.6$$

$$x_0 = 1.6$$

$$h = 0.1$$

$$\therefore p = \frac{x-x_0}{h} = \frac{1.6-1.6}{0.1} = 0$$

$$y = y_0 + p \frac{\nabla y_0}{2!} + \frac{p(p+1)}{3!} \nabla^2 y_0 + \frac{p(p+1)}{2!} \nabla^3 y_0 + \frac{p(p+1)(p+2)(p+3)}{4!} \nabla^4 y_0$$

$$\frac{dy}{dp} = y_0 + p \frac{\nabla y_0}{2!} + \frac{p^2 + p}{2!} \nabla^2 y_0 + \frac{p^3 + 3p^2 + 2p}{3!} \nabla^3 y_0 + \frac{p^4 + 6p^3 + 11p^2 + 6p}{4!} \nabla^4 y_0$$

$$\nabla^4 y_0$$

Now

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{h} \left[ \frac{\nabla y_0 + 2p + 1}{2!} \frac{\nabla^2 y_0 + 3p^2 + 6p + 2}{3!} \right. \\ &\quad \left. \nabla^3 y_0 + \frac{4p^3 + 18p^2 + 22p + 6}{4!} \nabla^4 y_0 \right] \\ &= \frac{1}{0.1} \left[ 0.281 + \frac{1}{2!} (-0.0180) + \frac{2}{3!} (0.005) \right. \\ &\quad \left. + \frac{6}{4!} \times 0.02 \right] \\ &= 2.7867 \end{aligned}$$

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{1}{h^2} \left[ \frac{2}{2!} \nabla^2 y_0 + \frac{6p + 6}{3!} \nabla^3 y_0 + \right. \\ &\quad \left. \frac{12p^2 + 36p + 22}{4!} \nabla^4 y_0 \right] \\ &= \frac{1}{0.1^2} \left[ \frac{2}{2!} (-0.0180) + \frac{6}{3!} \times 0.005 \right. \\ &\quad \left. + \frac{22}{4!} \times 0.02 \right] \end{aligned}$$

$$= \frac{1}{0.01} [-0.018 + 0.005 + 0.0018]$$

$$\begin{aligned} &= \frac{1}{0.01} \times -0.012 \\ &= -1.2 \end{aligned}$$

$$\frac{d^3y}{dx^3} = \frac{1}{h^3} \left[ \frac{6}{3!} \nabla^3 y_0 + \frac{24p + 36}{4!} \nabla^4 y_0 \right]$$

$$\begin{aligned} &= \frac{1}{0.1^3} [0.005 + 0] \\ &= 5 \end{aligned}$$

## Numerical Differentiation

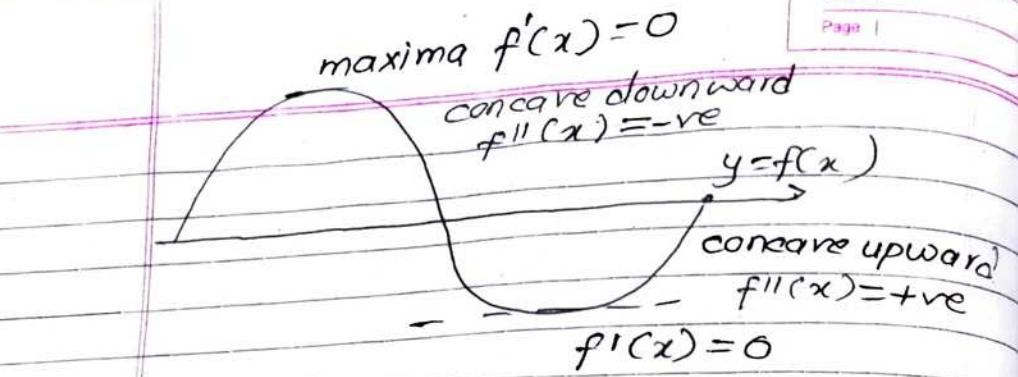
Maxima & minima of tabulated functions

Note

For maxima & minima :  $\frac{dy}{dx} = 0$

i) For maxima  $\frac{d^2y}{dx^2} = -ve$  ~~minima~~

For minima  $\frac{d^2y}{dx^2} = +ve$  ~~maxima~~



We know Newton's forward interpolation  
Forward Formula:

$$y = y_0 + p \frac{\Delta y_0}{1!} + p(p-1) \frac{\Delta^2 y_0}{2!} + p(p-1)(p-2) \frac{\Delta^3 y_0}{3!}$$

$$\text{or } y = y_0 + p \Delta y_0 + \frac{p^2 - p}{2!} \Delta^2 y_0 + \frac{p^3 - 3p^2 + 2p}{3!} \Delta^3 y_0 + \dots$$

Diff both sides wrt p we get

$$\frac{dy}{dp} = \Delta y_0 + 2p - 1 \frac{\Delta^2 y_0}{2!} + 3p^2 - 6p + 2 \frac{\Delta^3 y_0}{3!}$$

$$\frac{d^2y}{dp^2} = \frac{2}{2!} \Delta^2 y_0 + \frac{6p - 6}{3!} \Delta^3 y_0 + \dots$$

For maxima & minima

$$\frac{dy}{dp} = 0$$

$$\text{or } \Delta y_0 + 2p - 1 \frac{\Delta^2 y_0}{2!} + 3p^2 - 6p + 2 \frac{\Delta^3 y_0}{3!} = 0$$

Taking the value of  $\Delta y_0$ ,  $\Delta^2 y_0$  &  $\Delta^3 y_0$  from difference table only up to third difference of solving we get value of p

If  $\frac{d^2y}{dp^2} = +ve$  y has minima

If  $\frac{d^2y}{dp^2} = -ve$  y has maxima

1) From the table below, for what value of x, y is maximum? Also this value of y

x	3	4	5	6	7	8
y	0.205	0.24	0.259	0.262	0.25	0.224

Here different table

x	y	$\Delta y$	$\Delta^2 y$	$\Delta^3 y$
3	0.205	$\Delta y_0$	$\Delta^2 y_0$	$\Delta^3 y_0$
4	0.240	0.035	-0.016	0
5	0.259	0.019	-0.016	0.001
6	0.262	0.003	-0.015	0.001
7	0.25	-0.012	-0.014	
8	0.224	-0.026		

We know Newton's forwarded interpolation formula

$$y = y_0 + p \Delta y_0 + \frac{p^2 - p}{2!} \Delta^2 y_0 + \frac{p^3 - 3p^2 + 2p}{3!} \Delta^3 y_0$$

diff both sides wrt p we get

$$\frac{dy}{dp} = \Delta y_0 + \frac{2p-1}{2!} \Delta^2 y_0 + \frac{3p^2 - 6p + 2}{3!} \Delta^3 y_0$$

$$\frac{d^2y}{dp^2} = \frac{2}{2!} \Delta^2 y_0 + \frac{6p-6}{3!} \Delta^3 y_0$$

For maxima & minima

$$\frac{dy}{dp} = 0$$

$$\text{or } \Delta y_0 + \frac{2p-1}{2!} \Delta^2 y_0 + \frac{3p^2 - 6p + 2}{3!} \Delta^3 y_0 = 0$$

$$\text{or } 0.035 + \frac{2p-1}{2} (-0.015) + \frac{3p^2 - 6p + 2}{6} \times 0 = 0$$

$$\text{or, } 0.035 + (2p-1)(-0.008) = 0$$

$$\text{or } (2p-1) = \frac{-0.035}{-0.008}$$

$$\therefore p = 0.688$$

When  $p = 0.688$

$$\frac{d^2y}{dp^2} = \frac{2}{2!} \Delta^2 y_0 + \frac{6p-6}{3!} \Delta^3 y_0$$

$$= -0.016 \text{ (neg)}$$

This shows that  $y$  has maxima at  $p = 0.688$   
Now,

$$P = \frac{x - x_0}{h}$$

$$\text{or } 0.688 = \frac{x - 3}{1}$$

$$\text{or, } x = 5.688$$

Then

$$y = y_0 + p \Delta y_0 + \frac{p^2 - p}{2!} \Delta^2 y_0 + \frac{p^3 - 3p^2 + 2p}{3!} \Delta^3 y_0$$

$$= 0.205 + 0.688 \times 0.035 + \frac{0.688^2 - 2.688}{2!}$$

$$(-0.016) + 0$$

$$= 0.2628$$

Hence maximum value of  $y = 0.263$  at  
 $x = 5.688$

Q7 From following table, find  $x$  correct to 4 decimal places for which  $y$  is maximum & find this value of  $y$

$x$	1.2	1.3	1.4	1.5	1.6
$y$	0.932	0.9636	0.985	0.9975	0.9996

$x$	$y$	$\Delta y$	$\Delta^2 y$	$\Delta^3 y$
1.2	0.9320	0.0316	-0.0097	
1.3	0.9636	0.0219	-0.0099	-0.0002
1.4	0.9855	0.0120	-0.0099	0
1.5	0.9975	0.0021	-0.0099	
1.6	0.9996			

Using Newton's forward interpolation formula

$$y = y_0 + p \Delta y_0 + \frac{p^2 - p}{2!} \Delta^2 y_0 + \frac{p^3 - 3p^2 + 2p}{3!} \Delta^3 y_0$$

Dif wrt  $p$

$$\frac{dy}{dp} = \Delta y_0 + \frac{2p-1}{2!} \Delta^2 y_0 + \frac{3p^2 - 6p + 2}{3!} \Delta^3 y_0$$

$$\frac{d^2y}{dp^2} = \frac{2}{2!} \Delta^2 y_0 + \frac{6p-6}{3!} \Delta^3 y_0$$

For maxima & minima

$$\frac{dy}{dp} = 0$$

$$\text{or, } \Delta y_0 + \frac{2p-1}{2!} \Delta^2 y_0 + \frac{3p^2 - 6p + 2}{3!} \Delta^3 y_0 = 0$$

$$\text{or, } 0.0316 + \frac{2p-1}{2} * -0.0097 + \frac{3p^2 - 6p + 2}{6}$$

$$* - 0.0002 = 0$$

$$\text{or, } 0.0316 - 0.0049(2p-1) - \frac{3p^2 - 6p + 2}{6} *$$

$$0.0002 = 0$$

$$\text{or, } 0.0316 - 0.0049 * 2p + 0.0049 -$$

$$- \frac{0.0006 p^2 + 0.0012 p + 0.0004}{6} = 0$$

$$\text{or, } 0.1896 - 0.0588 p + 0.0049$$

$$- 0.0006 p^2 + 0.0012 p + 0.0004 = 0$$

$$\text{or, } 0.0006 p^2 + 0.0576 - 0.1949 = 0$$

$$\therefore p = 3.2721$$

$$p = -99.2721$$

→

Now,

$$0.0316 - 0.0049 * 2p + 0.0049 = 0$$

$$\text{or } 0.0365 - 0.0098 p = 0$$

$$\therefore p = 3.7245$$

Now,

$$\frac{d^2y}{dp^2} = \frac{2}{2!} \Delta^2 y_0 + \frac{6p-6}{3!} \Delta^3 y_0$$

$$= 2 * (-0.0097) + \frac{2 * 3.7245 - 6}{6}$$

$$* (-0.0002)$$

$$= -0.0194 - \text{ve.}$$

so there is maxima

$$p = \frac{x-x_0}{h} \therefore x = 1.5725$$

$$\text{or, } 3.7245 = \frac{x - 1.2}{0.1}$$

$$y = y_0 + p^4 y_0 + \frac{p^2}{2!} p^2 y_0 + \frac{p^3 - 3p^2 + 2p}{3!}$$

 $\Delta^3 y_0$ 

$$= 0.9320 + 3.7245 \times 0.0316 + \\ \frac{3.7245^2 - 3.7245 \times (-0.0097)}{2!} \\ = 1.0005 //$$

Hence maximum value of  $y = 1.0005$   
at  $x = 1.5725$

Simpson's  $\frac{3}{8}$  rule $b$ 

$$\int_a^b f(x) dx = \frac{3h}{8} [y_0 + y_b + 3(y_1 + y_2 + y_4 + y_5) + 3(y_3)]$$

# Trapezoidal and Simpson's  
 $\frac{1}{3}$  and  $\frac{3}{8}$ )

## 1) Trapezoidal rule

$$\int_a^b f(x) dx = \frac{h}{2} [y_0 + y_b + 2(y_1 + y_2 + y_3 + y_4 + y_5)]$$

2) Simpson's  $\frac{1}{3}$  rule

$$\int_a^b f(x) dx = \frac{h}{3} [y_0 + y_b + 2(y_2 + 4y_4 + 2y_6 + 4y_8 + 2y_{10}) + 4(y_3 + 2y_5 + 4y_7 + 2y_9 + 4y_{11})]$$

# Numerical Integration

## Method IV) Romberg's Integration method

In Romberg's method we use trapezoidal rule to find

i)  $I_1$  with  $n=2$

ii)  $I_2$  with  $n=4$

iii)  $I_3$  with  $n=8$

then after

$$I_4 = I_2 + \frac{I_2 - I_1}{3}$$

$$I_5 = I_3 + \frac{I_3 - I_2}{3}$$

$$I_6 = I_4 + \frac{I_4 - I_3}{3}$$

This process is continued to get two equal successive values of  $I$

### Evaluate

$$\int_0^1 \frac{1}{1+x^2} dx \text{ by using Romberg's formula.}$$

Here comparing  $\int_0^1 \frac{1}{1+x^2} dx$  with  $\int_a^b f(x) dx$

We get

$$f(x) = \frac{1}{1+x^2}$$

$$a = 0$$

$$b = 1$$

Using Trapezoidal formula, we have  
to find  $I_1, I_2$  &  $I_3$

i) when  $n=2$

$$h = \frac{b-a}{n} = \frac{1}{2}$$

$x$	0	0.5	1
$f(x)$	1	0.8	0.5
	$y_0$	$y_1$	$y_2$

$$I_1 = \frac{h}{2} [y_0 + y_2 + 2y_1]$$

$$= \frac{1}{2} [1 + 0.5 + 2 \times 0.8]$$

$$= 0.7750$$

ii) when  $n=4$

$$h = \frac{b-a}{n} = \frac{1}{4}$$

$x$	0	0.25	0.5	0.75	1
$f(x)$	1	0.9411	0.8	0.64	0.5

$$I_2 = \frac{h}{2} [y_0 + y_4 + 2(y_1 + y_2 + y_3)]$$

$$= \frac{1}{8} [1 + 0.5 + 2(0.9411 + 0.8 + 0.64)]$$

$$= 0.7828$$

iii) When  $n=8$

$$h = \frac{b-a}{n} =$$

$x$	0	0.125	0.25	0.375	0.5	0.625	0.75	0.875	1
$f(x)$	1	0.9846	0.9411	0.8761	0.8	0.7191	0.64	0.5689	0.5
	$y_0$	$y_1$	$y_2$	$y_3$	$y_4$	$y_5$	$y_6$	$y_7$	$y_8$

$$I_3 = \frac{h}{2} [y_0 + y_8 + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)]$$

$$= 0.7847$$

Now,

Using Romberg's formula

$$I_4 = I_2 + \frac{I_2 - I_1}{3} = 0.7828 + \frac{0.7828 - 0.7750}{3}$$

$$= 0.7854$$

$$I_5 = I_3 + \frac{I_3 - I_2}{3} = 0.7847 + \frac{0.7847 - 0.7828}{3}$$

$$= 0.7855$$

Here  $I_4 = I_5$  upto decimal places

$$\int_{-1}^1 \frac{1}{1+x^2} dx = 0.785$$

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Apply Romberg's method to evaluate

$$\int_0^{\pi} \frac{\cos x}{\sqrt{1+\sin x}} dx$$

so in 8

Here  $a = 0$   
 $b = \pi$

Using Trapezoidal formula we find  
 $I_1, I_2, I_3$

i) When  $n=2$

$$h = \frac{b-a}{n} = \frac{\pi}{2}$$

Tabulating:

$x$	0	0.7854	1.5708
$f(x)$	1	0.5412	0

then

$$\begin{aligned} I_1 &= \frac{h}{2} [y_0 + y_2 + 2y_1] \\ &= \frac{\pi}{8} [1 + 0 + 2 \times 0.5412] \\ &= 0.8178 \end{aligned}$$

ii) When  $n=4$

$$h = \frac{b-a}{n} = \frac{\pi}{4}$$

tabulating

$x$	0	0.3927	0.7854	1.1781	1.5708
$f(x)$	1	0.7854	0.5412	0.2759	0 0
$y_0$	$y_1$	$y_2$	$y_3$	$y_4$	$y_5$

$$\begin{aligned} I_2 &= \frac{h}{2} [(y_0 + y_4) + 2(y_1 + y_2 + y_3)] \\ &= 0.8258 \end{aligned}$$

when  $n=8$

$$h = \frac{b-a}{n} = \frac{\pi}{16}$$

$x$	0	0.1963	0.3927	0.5890	0.7854	0.9817	1.1781	1.3744	1.5708
$f(x)$	1	0.8972	0.7854	0.6667	0.5412	0.4165	0.2759	0.1388	0

$$\begin{aligned} I_3 &= \frac{h}{2} [y_0 + y_8 + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)] \\ &= 0.8278 \end{aligned}$$

then  $I_4 = I_2 + \frac{I_2 - I_1}{3}$

$$\begin{aligned} &= 0.8258 + \frac{0.8258 - 0.8178}{3} \\ &= 0.8285 \end{aligned}$$

$$I_5 = I_3 + \frac{I_3 - I_2}{3}$$

$$= 0.8278 + \frac{0.8278 - 0.8258}{3}$$

$$= 0.8285$$

$$\text{Here } I_4 = I_5$$

$$\text{so } \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{1+\sin x}} = 0.8285$$

## Method V > Gauss-Legendre point method

Gauss Legendre integration is based on the concept that accuracy of numerical integration can be improved by choosing the sampling points than on the basis of equal intervals.

$$\text{Formula: } \int_{-1}^1 f(x) dx = \sum_{i=1}^n w_i f(x_i)$$

### Two point formula

$$\int_{-1}^1 f(x) dx = \sum_{i=1}^2 w_i f(x_i)$$

$$= w_1 f(x_1) + w_2 f(x_2)$$

$$\text{where } w_1 = 1 \quad x_1 = \frac{-1}{\sqrt{3}}$$

$$w_2 = 1 \quad x_2 = \frac{1}{\sqrt{3}}$$

### Three point formula

$$\int_{-1}^1 f(x) dx = \sum_{i=1}^3 w_i f(x_i)$$

$$= w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3)$$

where

$$w_1 = \frac{5}{9} \quad x_1 = -\sqrt{\frac{3}{5}}$$

$$w_2 = \frac{8}{9} \quad x_2 = 0$$

$$w_3 = \frac{5}{9} \quad x_3 = \sqrt{\frac{3}{5}}$$

i) Evaluate  $\int_{-1}^1 e^x dx$  by using Gauss

Legendre 2-point, 3-point formula.  
Solving

Here  $a = -1$

$$b = 1$$

$$f(x) = e^x$$

We know Gauss Legendre

ii) 2 point formula

$$\int_{-1}^1 f(x) dx = w_1 f(x_1) + w_2 f(x_2)$$

$$= 1e^{-\frac{1}{\sqrt{3}} + i\theta} \frac{1}{\sqrt{3}}$$

$$= 0.3427$$

ii) 3-point formula

$$\int_{-1}^1 f(x) dx = w_1 f(x_1) + f(x_2) w_2 + w_3 f(x_3)$$

$$= \frac{5}{9} e^{-\frac{\sqrt{3}}{3}} + \frac{8}{9} e^0 + \frac{5}{9} e^{\frac{\sqrt{3}}{3}}$$

$$= 0.3503$$

Hence

$$\int_{-1}^1 e^x dx = 0.8427 \text{ by 2-point formula}$$

$$= 0.3503 \text{ by 3-point formula}$$

Changing limits of integration



$$\text{Let } \int_a^b f(x) dx = c \int_{-1}^1 g(z) dz$$

Assume this transformation between  $x$  & new variable  $z$  satisfies the relation

$$x = Az + B$$

when  $x=a, z=-1$

when  $x=b, z=1$

then  $a = -A + B$

$$b = A + B$$

solving we get

$$A = \frac{b-a}{2}, B = \frac{a+b}{2}$$

From eq? (i)

$$x = \frac{b-a}{2}z + \frac{a+b}{2}$$

$$\text{or } \frac{dx}{dz} = \frac{b-a}{2}$$

$$\text{or } dx = \frac{b-a}{2} dz \Rightarrow c = \frac{b-a}{2}$$

Now,

$$\int_a^b f(x) dx = \frac{b-a}{2} \int_{-1}^1 g(z) dz$$

$$= \frac{b-a}{2} \sum_{i=1}^n w_i g(z_i)$$

2) Evaluate the integral  $\int_0^1 \frac{1}{1+x} dx$

Using Gauss Legendre 3-point formula  
so that

Here  $I = \int \frac{1}{1+x} dx$

$$a=0, b=1, f(x) = \frac{1}{1+x}$$

changing limits of integration we get  
 $x = \frac{b-a}{2}z + \frac{a+b}{2}$

$$x = \frac{z}{2} + \frac{1}{2}$$

$$\therefore f(x) = \frac{1}{1+x}$$

$$g(z) = \frac{1}{1+(\frac{z+1}{2})}$$

$$= \frac{2}{z+3}$$

We know, Gauss Legendre 3-point formula,

$$\int_a^b f(x) dx = \frac{b-a}{2} \int_{-1}^1 g(z) dz$$

$$= \frac{b-a}{2} \sum_{i=1}^3 w_i g(z_i)$$

$$= \frac{b-a}{2} [w_1 g(z_1) + w_2 g(z_2) +$$

$$w_3(g(z_3))]$$

$$= \frac{b-a}{2} \left[ w_1 \left( \frac{2}{z_1+3} \right) + w_2 \left( \frac{2}{z_2+3} \right) + w_3 \left( \frac{2}{z_3+3} \right) \right]$$

$$= \frac{1-0}{2} \left[ \frac{5}{9} \left( -\frac{2}{\sqrt{\frac{2}{3}}+3} \right) + 8 \left( \frac{2}{0+3} \right) + \frac{5}{9} \left( \frac{2}{\sqrt{\frac{2}{3}}+3} \right) \right]$$

$$= \frac{1}{2} 0.6931$$

3) Using Gauss Legendre point formula evaluate

$$I = \int_3^5 \frac{1}{x^6+5} dx$$

$$\text{Here } a=3, b=5, f(x) = \frac{1}{x^6+5}$$

changing limits of integration

$$x = \frac{(b-a)}{2}z + \frac{a+b}{2}$$

$$= \left( \frac{5-3}{2} \right) z + \frac{8}{2}$$

$$\therefore x = z+4$$

$$\therefore f(x) = \frac{1}{x^6+5}$$

$$g(z) = \frac{1}{(z+4)^6+5}$$

We know Gauss Legendre 3-point formula,

$$\int_a^b f(x) dx = \frac{b-a}{2} \int_{-1}^1 g(z) dz$$

$$= \frac{5-3}{2} \int_{-1}^1 (w_i g(z_i))$$

$$= [w_1 g(z_1) + w_2 g(z_2)]$$

$$= \omega_1 \left[ \frac{1}{(z+4)^6 + 5} \right] + \omega_2 \left[ \frac{1}{(\frac{1}{2}z+4)^6 + 5} \right]$$

$$= \frac{5}{9} \left[ \frac{1}{(-\sqrt{\frac{3}{5}}+4)^6 + 5} \right] + \frac{8}{9} \left[ \frac{1}{(0+4)^6 + 5} \right]$$

$$= 0.0007$$

Then For 3 point formula

$$\int_a^b f(x) dx = \frac{b-a}{2} \left[ \omega_1 g(z_1) + \omega_2 g(z_2) + \omega_3 g(z_3) \right]$$

$$= \frac{5-3}{2} \left[ \frac{5}{9} \left[ \frac{1}{(z+4)^6 + 5} \right] + \frac{8}{9} \left[ \frac{1}{(z_2+4)^6 + 5} \right] + \frac{5}{9} \left[ \frac{1}{(z_3+4)^6 + 5} \right] \right]$$

$$= \frac{5}{9} \left[ \frac{1}{(-\sqrt{\frac{3}{5}}+4)^6 + 5} \right] + \frac{8}{9} \left[ \frac{1}{(0+4)^6 + 5} \right] + \frac{5}{9} \left[ \frac{1}{(\sqrt{\frac{3}{5}}+4)^6 + 5} \right]$$

$$= 0.0061$$

## Solution of ordinary Differential Eq's

$\downarrow$   
Solution of ordinary differential equations means finding the value of  $y$  in the equation  $\frac{dy}{dx} = f(x, y)$  with initial condit.

$y(x_0) = y_0$  & step size  $h$ . In this method we apply step by step procedure where

$$x_1 = x_0 + h$$

$$x_2 = x_0 + 2h$$

$$x_3 = x_0 + 3h \text{ & so on}$$

### Methods

i) Taylor's series method

ii) Picard's method

iii) Euler's method

iv) Runge-kutta method.

v) Heun's method

vi) Shooting method

### Taylor's series method

Let  $\frac{dy}{dx} = y' = f(x, y)$  be a differential eq'

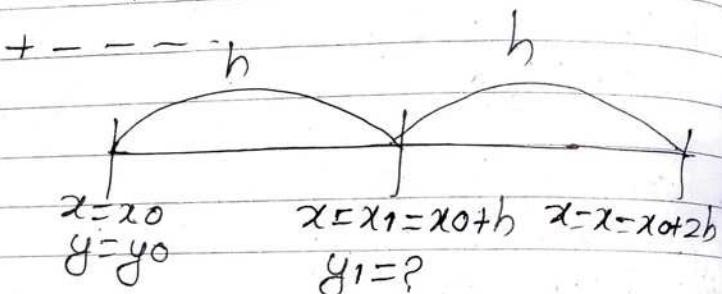
with initial condition  $y(x_0) = y_0 \Rightarrow x = x_0$ ,  $y = y_0$  & step size  $h$ . Then

Taylor's expansion at point  $x_0$  is given by  $f(x) = f(x_0) + \frac{x-x_0}{1!} f'(x_0)$

$$+ \frac{(x-x_0)^2}{2!} f''(x_0) + \frac{(x-x_0)^3}{3!} f'''(x_0) +$$

This can be written as

$$y = y_0 + \frac{(x-x_0)y_0'}{1!} + \frac{(x-x_0)^2 y_0''}{2!} + \frac{(x-x_0)^3 y_0'''}{3!}$$



$$\text{But } x = x_1 = x_0 + h \Rightarrow x - x_0 = h$$

$$y_1 = y_0 + \frac{h y_0'}{1!} + \frac{h^2 y_0''}{2!} + \frac{h^3 y_0'''}{3!} + \frac{h^4 y_0^{IV}}{4!}$$

$$y_{n+1} = y_n + \frac{h y_n'}{1!} + \frac{h^2 y_n''}{2!} + \frac{h^3 y_n'''}{3!} + \frac{h^4 y_n^{IV}}{4!}$$

We have to evaluate up to 4th derivative

Using Taylor's series evaluate  $y(0.3) \frac{dy}{dx} = x+y$  with initial condition.

$$y(0) = 1, \text{ Take } h = 0.1$$

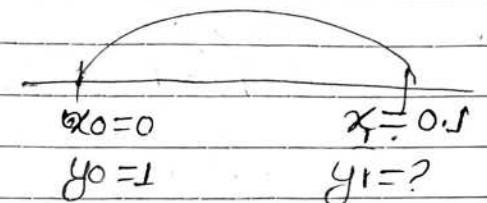
Solving:

$$y' = x+y$$

$$y(0) = 1 \Rightarrow x_0 = 0$$

$$y_0 = 1$$

$$h = 0.1$$



$$x_1 = x_0 + h = 0 + 0.1 = 0.1$$

$$y_1 = ?$$

Now,

$$y_1 = y_0 + \frac{h y_0'}{1!} + \frac{h^2 y_0''}{2!} + \frac{h^3 y_0'''}{3!} + \frac{h^4 y_0^{IV}}{4!} \rightarrow ①$$

where

$$y_0' = x + y \Rightarrow y_0' = x_0 + y_0 = 0 + 1 = 1$$

$$y_0'' = 1 + y' \Rightarrow y_0'' = 1 + y_0' = 1 + 1 = 2$$

$$y_0''' = 0 + y'' \Rightarrow y_0''' = y_0'' = 2$$

$$y^{IV} = y''' \Rightarrow y_0^{IV} = y_0''' = 2$$

From eqn ①

$$y_1 = y_0 + \frac{h y_0'}{1!} + \frac{h^2 y_0''}{2!} + \frac{h^3 y_0'''}{3!} + \frac{h^4 y_0^{IV}}{4!}$$

$$= 1 + 0.1 \times 1 + \frac{0.1^2 \times 2}{2!} + \frac{0.1^3 \times 2}{3!} + \frac{0.1^4 \times 2}{4!}$$

$$= 1.1103$$

Hence,  $y(0.1) = 1.1103$

++ Apply the Taylor's series method to find the value of  $y(1.1)$  &  $y(1.2)$   
 $\frac{dy}{dx} = xy^{\frac{1}{3}}, y(1) = 1, h = 0.1$

$$\begin{array}{ccccc} & & 0.1 & & \\ \text{Here} & & & & \\ y' = xy^{\frac{1}{3}} & & & & \\ y(1) = 1 & x_0 = 1 & y_0 = 1 & & \\ x_0 = 1 & x_1 = 1.1 & & & \\ y_0 = 1 & & & & \\ h = 0.1 & & & & \\ y_1 = ? & & & & \end{array}$$



$$\begin{array}{cccc} x_0 = 1 & & & \\ y_0 = 1 & & & \\ x_1 = 1.1 & & & \\ x_2 = 1.2 & & & \end{array}$$

Now,

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{(4)}$$

where

$$y' = xy^{\frac{1}{3}} \Rightarrow y_0' = x_0 y_0^{\frac{1}{3}} = 1 \cdot 1^{\frac{1}{3}} = 1$$

$$y'' = y^{\frac{1}{3}} \Rightarrow y_0''$$

$$y'' = y^{\frac{1}{3}} + \frac{x}{3y^{\frac{2}{3}}} y' \Rightarrow y_0'' = y_0^{\frac{1}{3}} + x_0 y_0'$$

where

$$y' = xy^{\frac{1}{3}} \Rightarrow y_0' = x_0 y_0^{\frac{1}{3}} = 1 \cdot 1^{\frac{1}{3}} = 1$$

$$y'' = y^{\frac{1}{3}} + \frac{x}{3y^{\frac{2}{3}}} y' \Rightarrow y_0'' = y_0^{\frac{1}{3}} + \frac{x_0 y_0'}{3y_0^{\frac{2}{3}}}$$

$$= 1 + \frac{1}{3} = \frac{4}{3}$$

$$\begin{aligned} y'' &= y^{\frac{1}{3}} + \frac{x}{3y^{\frac{2}{3}}} y' = y^{\frac{1}{3}} + \frac{x}{3y^{\frac{2}{3}}} * xy^{\frac{1}{3}} \\ &= y^{\frac{1}{3}} + \frac{x^2 y^{\frac{1}{3}}}{3} \end{aligned}$$

$$y_0'' = 1 + \frac{1.1}{3} = \frac{4}{3}$$

$$y''' = \frac{1}{3} y^{-\frac{2}{3}} + \frac{1}{3} (y^{-\frac{1}{3}} * 2x + x^2 (-\frac{1}{3}) y^{-\frac{4}{3}} y')$$

$$= \frac{1}{3} y^{-\frac{2}{3}} + \frac{2x}{3y^{\frac{1}{3}}} - \frac{x^2 y^{-\frac{4}{3}}}{3} \cdot xy^{\frac{1}{3}}$$

$$= \frac{1}{3} y^{-\frac{2}{3}} + \frac{2}{3} xy^{-\frac{1}{3}} - \frac{x^3 y^{-1}}{3}$$

$$y_0''' = \frac{1}{3} y_0^{-\frac{2}{3}} + \frac{2}{3} x_0 y_0^{-\frac{1}{3}} - \frac{x_0^3 y_0^{-1}}{3}$$

$$= \frac{1}{3} + \frac{2}{3} - \frac{1}{3}$$

$$= \frac{2}{3}$$

$$y^{(4)} = -\frac{2}{9} y_0^{-\frac{5}{3}} + \frac{2}{3} [y_0^{-\frac{1}{3}} - \frac{1}{3} x_0 y^{-\frac{4}{3}}] - \frac{1}{3} [3x^2 y^{-1} - x^3 y^{-2}]$$

$$y_0^{(4)} = -\frac{2}{9} y_0^{-\frac{5}{3}} + \frac{2}{3} [y_0^{-\frac{1}{3}} - \frac{1}{3} x_0 y^{-\frac{4}{3}}] y_0^{\frac{1}{3}}$$

$$- \frac{1}{3} [3x^2 y_0^{-1} - x_0^3 y_0^{-2}] y_0^{\frac{1}{3}}$$

$$= -\frac{2}{9} + 2 \left[ 1 - \frac{1}{3} - \frac{1}{3} [3 - 1 - 1] \right]$$

$$= -\frac{2}{9} + 2 \left[ 1 - \frac{1}{3} - 1 + \frac{2}{3} \right] = \frac{4}{9}$$

Then

$$y_1 = 1 + 0.1 \times 1 + \frac{0.1^2 \times 4}{2!} + \frac{0.1^3 \times 2}{3!} +$$

$$\frac{0.1^4 \times 4}{4!} + \dots$$

$$= 1.1068$$

Also

$$y_0' = 1.1 \times 1.1068^{\frac{1}{3}} = 1.1378$$

$$y_0'' = 1.1^{\frac{1}{3}} +$$

Picard's method of successive approximations

Suppose the given differential equation is

$$\frac{dy}{dx} = f(x, y) \rightarrow 1$$

Let us solve the given eq<sup>n</sup> with the given initial condition  $y=y_0$  when  $x=x_0$ . The given differential eq<sup>n</sup> can be written as

$$dy = f(x, y) dx \rightarrow 2$$

Integrating eq<sup>n</sup> ② from  $x_0$  to  $x$  wrt  $x$ , we get,

$$\int_{y_0}^y dy = \int_{x_0}^x f(x, y) dx$$

$$\text{or } [y]_{y_0}^y = \int_{x_0}^x f(x, y) dx$$

$$\text{or } y - y_0 = \int_{x_0}^x f(x, y) dx$$

$$\text{or, } y = y_0 + \int_{x_0}^x f(x, y) dx \rightarrow 3$$

As a first approximation, we replace  $y$  by  $y_0$  in  $f(x, y)$  we get

$$y_1 = y_0 + \int_{x_0}^x f(x, y_0) dx$$

Similarly replacing  $y$  by  $y_1$  in  $f(x, y)$  we get second approximation

$$y_2 = y_0 + \int_{x_0}^x f(x, y_1) dx \text{ & so on}$$

$$y_2 = y_0 + \int_{x_0}^x f(x, y_1) dx$$

$$y_{n+1} = y_0 + \int_{x_0}^x f(x, y_n) dx$$

We continue the process we get  
two successive approximation are  
equal.

Q.No 1) Using picard's method, find  $y$  when  
 $x=0.2$  shows that  $y(0)=1$ ,  $\frac{dy}{dx} = x-y$   
Solving:

Here, Given that

$$\frac{dy}{dx} = f(x, y) = x - y$$

$$y(0) = 1 \Rightarrow x_0 = 0, y_0 = 1$$

$$x = 0.2, y = ?$$

We know picard's formula,

$$y_{n+1} = y_0 + \int_{x_0}^x f(x, y_n) dx$$

The successive approximations are

$$y_1 = y_0 + \int_{x_0}^x f(x, y_0) dx$$

$$= 1 + \int_0^{0.2} (x - y_0) dx$$

$$= 0.82$$

$$= 0.82$$

$$2) y_2 = y_0 + \int_{x_0}^x f(x, y_1) dx$$

$$= y_0 + \int_{x_0}^x (x - y_1) dx$$

$$= 1 + \int_0^{0.2} (x - 0.82) dx$$

$$= 0.8560$$

$$3) y_3 = y_0 + \int_{x_0}^x f(x, y_2) dx$$

$$= y_0 + \int_{x_0}^x (x - y_2) dx$$

$$= 1 + \int_0^{0.2} (x - 0.8560) dx$$

$$= 0.8488$$

$$4) y_4 = y_0 + \int_{x_0}^x f(x, y_3) dx$$

$$= 1 + \int_0^{0.2} (x - y_3) dx$$

$$= 0.8502$$

$$5) y_5 = 0.8500$$

$$y(0.2) = 0.85 \text{ Ans}$$

Q) Use picard's method to approximate the value of  $y$  when  $x=0.1, x=0.2, x=0.4$

Soln:

Given that  $y=1$ , when  $x=0$  &  $\frac{dy}{dx} = 1+x+xy$  correct to three decimal places (upto 3<sup>rd</sup> approximation)

Soln:

Here Given that  $\frac{dy}{dx} = f(x,y) = 1+xy$

$$y(0) = 1 \Rightarrow x_0 = 0$$

$$y_0 = 1$$

when  $x=0.1, y=?$

when  $x=0.2, y=?$

when  $x=0.4, y=?$

i) For  $x=0.1$

We know picard's formula

$$y_{n+1} = y_0 + \int_{x_0}^x f(x, y_n) dx$$

The successive approximations are

$$y_1 = y_0 + \int_{x_0}^x f(x, y_0) dx$$

$$= y_0 + \int_{0}^{0.1} (1+xy_0) dx$$

$$= 1 + \int_{0}^{0.1} (1+x \cdot 1) dx$$

$$= 1.1050$$

0.1

$$y_2 = 1 + \int_0^{0.1} (1+x \cdot 1.1050) dx = 1.1055$$

$$y_3 = 1 + \int_0^{0.1} (1+x \cdot 1.1055) dx = 1.1055$$

$$\text{So } y(0.1) = 1.1055$$

ii) for  $x=0.2$

$$y_1 = y_0 + \int_{x_0}^x (1+xy_0) dx$$

$$= 1 + \int_0^{0.2} (1+x \cdot 1) dx$$

$$= 1.2200$$

$$y_2 = 1 + \int_{x_0}^{0.2} (1+x \cdot 1.22) dx$$

$$= 1.2244$$

$$y_3 = 1.2245$$

$$\text{So } y(0.2) = 1.2245$$

ii) For  $x=0.4$

$$y_1 = y_0 + \int_{x_0=0}^{x=0.4} f(x, y_0) dx$$

$$= 1 + \int_0^{0.4} (1+xy_0) dx$$

$$= 1 + \int_0^{0.4} (1+x \cdot 1) dx$$

$$= 1.48$$

$$y_2 = 1 + \int_0^{0.4} (1+x \cdot 1.48) dx$$

$$= 1.5189$$

$$y_3 = 1 + \int_0^{0.4} (1+x \cdot 1.5189) dx$$

$$= 1.5215$$

## Solution of ordinary Differential Equation

### Method III > Euler's method

Consider the first order differential eq<sup>n</sup>

$$\frac{dy}{dx} = f(x, y) \rightarrow ①$$

Let us solve this differential equation under the condition  $y(x_0) = y_0$ . The sol<sup>n</sup> of ① gives  $y$  as a function of  $x$  which may be written as  $y = f(x) \rightarrow ②$

The graph of ② is a curve

in  $xy$  plane & since a

smooth curve is practically

straight for a short  $x'$

distance from any point

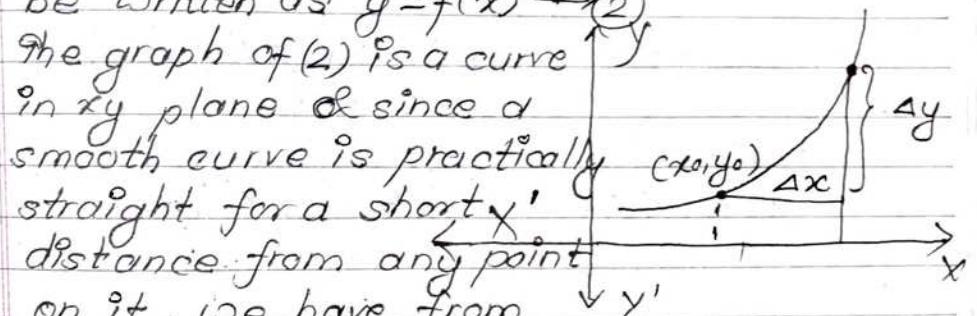
on it. we have from figure.

$$\tan \theta = \frac{\Delta y}{\Delta x}$$

$$\text{or, } \Delta y = \Delta x \tan \theta$$

$$\text{or, } \Delta y = \Delta x \left( \frac{dy}{dx} \right)$$

$$\therefore \text{slope at } (x_0, y_0) = \left( \frac{dy}{dx} \right)_0$$



Now,  $y_1 = y_0 + 4y$   
 or  $y_1 = y_0 + \left(\frac{dy}{dx}\right) \Delta x$   
 or  $y_1 = y_0 + f(x_0, y_0) h$   
 $= y_0 + hy_0'$

where  $h = \Delta x$

Similarly,

$$y_2 = y_1 + h \left( \frac{dy}{dx} \right)$$

$$= y_1 + hy_1'$$

$$y_3 = y_2 + hy_2' \text{ & so on}$$

In general,

$$y_{n+1} = y_n + hy_n'$$

This is called Euler's formula.

1) Solve the equation  $\frac{dy}{dx} = 1-y$  with the

initial condition  $y(0) = 0$  using Euler's method & tabulate the solution at  $x = 0.1, 0.2, 0.3$

so 1%

$$\text{Here } \frac{dy}{dx} = f(x, y) = 1-y$$

$$y(0) = 0 \Rightarrow x_0 = 0$$

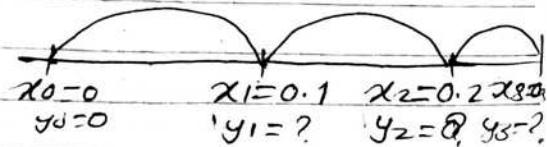
$$y_0 = 0$$

let  $h = 0.1$

$$\therefore x_1 = x_0 + h = 0 + 0.1 = 0.1, y_1 = ?$$

$$x_2 = x_0 + 2h = 0 + 0.1 \times 2 = 0.2, y_2 = ?$$

$$x_3 = x_0 + 3h = 0 + 3 \times 0.1 = 0.3, y_3 = ?$$



We know the Euler's formula,

$$y_{n+1} = y_n + hy_n'$$

$$\text{Now, } y_1 = y_0 + hy_0'$$

$$= y_0 + h f(x_0, y_0)$$

$$= y_0 + h[1 - y_0]$$

$$= 0 + 0.1[1 - 0]$$

$$= 0.1$$

$$y_2 = y_1 + hy_1'$$

$$= 0.1 + h[1 - y_1]$$

$$= 0.1 + 0.1[1 - 0.1]$$

$$= 0.19$$

$$y_3 = y_2 + hy_2'$$

$$= 0.19 + h[1 - 0.19]$$

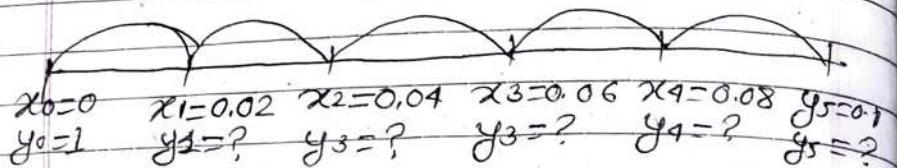
$$= 0.19 + 0.1[1 - 0.19]$$

$$= 0.271$$

Q2) Solve by Euler's method:

$$\frac{dy}{dx} = \frac{y-x}{y+x}, y(0) = 0.1, h = 0.02, \text{ for } x=0.1$$

$$x=0.1$$



$$\text{We know } y_{n+1} = y_n + hy_n'$$

$$y(0) = 0.1 \Rightarrow x_0 = 0, y_0 = 0.1$$

$$\textcircled{1} \quad y_1 = y_0 + hy_0'$$

$$= y_0 + h f(x_0, y_0)$$

$$= y_0 + h \left[ \frac{y_0 - x_0}{y_0 + x_0} \right]$$

$$= 1 + 0.02 \left[ \frac{0.1 - 0}{0.1 + 0} \right]$$

$$= 1.02$$

$$\text{ii) } y_2 = y_1 + hy_1'$$

$$= 1.02 + 0.02 \left[ \frac{y_1 - x_1}{y_1 + x_1} \right]$$

$$= 1.02 + 0.02 \left[ \frac{1.02 - 0.02}{1.02 + 0.02} \right]$$

$$= 1.0392$$

$$\text{iii) } y_3 = y_2 + hy_2'$$

$$= 1.0392 + 0.02 \left[ \frac{1.0392 - 0.04}{1.0392 + 0.04} \right]$$

$$= 1.0577$$

$$\text{iv) } y_4 = y_3 + hy_3'$$

$$= 1.0577 + 0.02 \left[ \frac{1.0577 - 0.08}{1.0577 + 0.08} \right]$$

$$= 1.0758$$

$$\text{v) } y_5 = y_4 + hy_4'$$

$$= 1.0758 + 0.02 \left[ \frac{1.0758 - 0.08}{1.0758 + 0.08} \right]$$

$$= 1.0785 \quad 1.0928$$

$$\text{vi) } y_6 = y_5 + hy_5'$$

$$= 1.0928 + 0.02 \left[ \frac{1.0928 - 0.1}{1.0928 + 0.1} \right]$$

$$= 1.1094$$

3)  $\frac{dy}{dx} = x+y$ ,  $y(0)=1$ ,  $h=0.1$  for  $x=1$

so/n:

$$\frac{dy}{dx} = x+y$$

$$y(0)=1 \Rightarrow y_0=1, x_0=0$$

$$h=0.1$$

$$x_0=0 \quad x_1=0.1 \quad x_2=0.2 \quad x_3=0.3 \quad x_4=0.4 \quad x_5=0.5 \quad x_6=0.6 \quad x_7=0.7 \quad x_8=0.8$$

$$y_0=1 \quad y_1=? \quad y_2=? \quad y_3=? \quad y_4=? \quad y_5=? \quad y_6=? \quad y_7=? \quad y_8=?$$

We know

$$y_{n+1} = y_n + hy_n'$$

Then

$$y_1 = y_0 + hy_0' \\ = 1 + 0.1 [x_0 + y_0]$$

$$= 1 + 0.1 [0 + 1] \\ = 1.1$$

$$y_2 = y_1 + hy_1' \\ = 1.1 + 0.1 [0.1 + 1.1] \\ = 1.22$$

$$x_9=0.9 \\ y_9=? \\ x_{10}=1 \\ y_{10}=?$$

$$y_3 = y_2 + hy_2' \\ = 1.22 + 0.1 [0.2 + 1.22] \\ = 1.362$$

$$y_4 = y_3 + hy_3' \\ = 1.362 + 0.1 [0.3 + 1.362] \\ = 1.5282$$

$$y_5 = y_4 + hy_4' \\ = 1.5282 + 0.1 [0.4 + 1.5282] \\ = 1.7210$$

$$y_6 = 1.7210 + hy_5' \\ = 1.7210 + 0.1 [0.5 + 1.7210] \\ = 1.9431$$

$$y_7 = y_6 + hy_6' \\ = 1.9431 + 0.1 [0.6 + 1.9431] \\ = 2.1974$$

$$y_8 = y_7 + hy_7' \\ = 2.1974 + 0.1 [2.1974 + 0.7] \\ = 2.4871$$

$$y_9 = y_8 + hy_8' \\ = 2.4871 + 0.1 [0.8 + 2.4871] \\ = 2.8158$$

$$\begin{aligned} y_0 &= y_0 + hy_0' \\ &= 2.8158 + 0.1 [0.9 + 2.8158] \\ &= 3.1874 \end{aligned}$$

$$\begin{aligned} y_{10} &= y_0 + hy_0' \\ &= 3.1874 + 0.1 [1 + 3.1874] \\ &= 3.6061 \end{aligned}$$

## Method IV Runge-Kutta Method

This method was devised by Runge, about the year 1934 and extended by Kutta a few years later therefore we call this method as Runge-Kutta method.

### I) Fourth order Runge-Kutta method

This method is commonly used and is often referred to as Runge-Kutta method only or classical Fourth order R-K method.

Let  $\frac{dy}{dx} = f(x, y)$  be a given differential

Equation to be solved under the initial condition  $y(x_0) = y_0$ . If  $h$  be the length of interval between equidistant values. Then

$$y_1 = y_0 + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4)$$

where

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2})$$

$$k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{2k_2}{2})$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

Similarly

$$y_2 = y_1 + \frac{1}{6} (k_2 + 2k_3 + 2k_4 + k_1)$$

where

$$k_1 = hf(x_1, y_1)$$

$$k_2 = hf(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2})$$

$$k_3 = hf(x_1 + \frac{h}{2}, y_1 + \frac{2k_2}{2})$$

$$k_4 = hf(x_1 + h, y_1 + k_3)$$

& so on.

1) Apply Runge-Kutta 4<sup>th</sup> order method to find an approximate value of  $y$  when  $x=0.2$ . Given that  $\frac{dy}{dx} = xy$ ,  $y(0)=1$ ,  $h=0.2$

Soln:

$$\frac{dy}{dx} = f(x, y) = xy$$

$$y(0)=1 \Rightarrow x_0=0, y_0=1$$

$$h=0.2$$

$$\therefore x_1 = x_0 + h = 0 + 0.2 = 0.2$$

$$y_1 = ?$$

We know R-K 4<sup>th</sup> order formula

$$y_1 = y_0 + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4) \rightarrow ①$$

where

$$k_1 = hf(x_0, y_0)$$

$$= 0.2 f(0, 1)$$

$$= 0.2 [0+1]$$

$$= 0.2$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= 0.2 f\left(0 + \frac{0.2}{2}, 1 + \frac{0.2}{2}\right)$$

$$= 0.2 f(0.1, 1.1)$$

$$= 0.2 [0.1 + 1.1]$$

$$= 0.24$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= 0.2 f\left[0 + \frac{0.2}{2}, 1 + \frac{0.24}{2}\right]$$

$$= 0.2 f(0.1, 1.12)$$

$$= 0.2 [0.1 + 1.12]$$

$$= 0.244$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$= 0.2 f(0 + 0.2, 1 + 0.244)$$

$$= 0.2 f(0.2, 1.244)$$

$$= 0.2 [0.2 + 1.244]$$

$$= 0.2888$$

Now,

$$y_1 = y_0 + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1 + \frac{1}{6} (0.2 + 2 \times 0.24 + 2 \times 0.244 + 0.2888)$$

$$y(0.2) = 1.24288$$

2) Solve  $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$  with  $y(0) = 1$  at  $x=0.4$

when  $x=0.2$ . Given  $h=0.2$

Soln:

$$\frac{dy}{dx} = f(x, y) = \frac{y^2 - x^2}{y^2 + x^2}$$

$$\text{Here } y(0) = 1 \Rightarrow x_0 = 0 \\ y_0 = 1$$

$$h = 0.2$$

$$\begin{array}{lll} x_0 = 0 & x_1 = 0.2 & x_2 = 0.4 \\ y_0 = 1 & y_1 = ? & y_2 = ? \end{array}$$

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$$\therefore x_1 = x_0 + h = 0 + 0.2 = 0.2$$

$$x_2 = x_0 + 2h = 0 + 2 \times 0.2 = 0.4$$

We know

$$y_1 = y_0 + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4) \rightarrow ①$$

where

$$k_1 = hf(x_0, y_0)$$

$$= 0.2 f(0, 1)$$

$$= 0.2 \times \left( \frac{1^2 - 0^2}{1^2 + 0^2} \right)$$

$$= 0.2$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= 0.2 f\left(0 + \frac{0.2}{2}, 1 + \frac{0.2}{2}\right)$$

$$= 0.2 f(0.1, 1.1)$$

$$= 0.2 \left( \frac{1.1^2 - 0.1^2}{1.1^2 + 0.1^2} \right)$$

$$= 0.1967$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= 0.2 f\left(0 + \frac{0.2}{2}, 1 + \frac{0.1967}{2}\right)$$

$$= 0.2 f(0.1, 1.0984)$$

$$= 0.2 \left( \frac{1.0984^2 - 0.1^2}{1.0984^2 + 0.1^2} \right)$$

$$= 0.1967$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$= 0.2 f(0 + 0.2, 1 + 0.1967)$$

$$= 0.2 f(0.2, 1.1967)$$

$$= 0.2 \left( \frac{1.1967^2 - 0.2^2}{1.1967^2 + 0.2^2} \right)$$

$$= 0.1891$$

Then

$$y_1 = y_0 + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1 + \frac{1}{6} (0.2 + 2 \times 0.1967 + 2 \times 0.1967 + 0.1891)$$

$$= 1.1960$$

Again,

$$y_2 = y_1 + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4)$$

where

$$k_1 = hf(x_1, y_1)$$

$$= 0.2 f(0.2, 1.1960)$$

$$= 0.2 \left( \frac{1.1960^2 - 0.2^2}{1.1960^2 + 0.2^2} \right)$$

$$= 0.1891$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$

$$= 0.2 f\left(0.2 + 0.1 + 1.1960 + \frac{0.1891}{2}\right)$$

$$= 0.2 f(0.3, 1.2906)$$

$$= 0.2 \left( \frac{1.2906^2 - 0.3^2}{1.2906^2 + 0.3^2} \right)$$

$$= 0.1795$$

$$K_2 = h f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$

$$= 0.2 f(0.2 + 0.1, 1.1960 + \frac{0.1795}{2})$$

$$= 0.2 f(0.3, 1.2858)$$

$$= 0.2 \left( \frac{1.2858^2 - 0.3^2}{1.2858^2 + 0.3^2} \right)$$

$$= 0.1793$$

$$K_4 = h f\left(x_1 + h, y_1 + k_3\right)$$

$$= 0.2 f(0.2 + 0.2, 1.1960 + 0.1793)$$

$$= 0.2 f(0.4, 1.3753)$$

$$= 0.1688$$

Then

$$y_2 = y_1 + \frac{1}{6} (K_1 + 2K_2 + 2K_3 + K_4)$$

$$= 1.1960 + \frac{1}{6} (0.1891 + 2 \times 0.1795 + 2 \times 0.1793 + 0.1688)$$

$$= 1.3753$$

$$\therefore y_2(0.4) = 1.3753$$

II Solution of simultaneous differential eq'

$$\text{Let } \frac{dy}{dx} = f(x, y, z)$$

$$\frac{dz}{dx} = \phi(x, y, z)$$

be simultaneous differential eq's with initial conditions  $y(x_0) = y_0$  &  $z(x_0) = z_0$  & interval size  $h$ . Then

$$y_1 = y_0 + \frac{1}{6} (K_1 + 2K_2 + 2K_3 + K_4)$$

$$z_1 = z_0 + \frac{1}{6} (l_1 + 2l_2 + 2l_3 + l_4)$$

where

$$K_1 = h f(x_0, y_0, z_0)$$

$$K_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$K_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$K_4 = h f(x_0 + h, y_0 + k_3, z_0 + l_3)$$

Similarly

$$l_1 = h \phi(x_0, y_0, z_0)$$

$$l_2 = h \phi\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$l_3 = h \phi\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$l_4 = h \phi(x_0 + h, y_0 + k_3, z_0 + l_3)$$

Replacing  $x_0, y_0, z_0$  by  $x, y, z$  we get  
 $y_2$  &  $z_2$  & so on,

3) Solve  $\frac{dy}{dx} = yz + x$ ,  $\frac{dz}{dx} = xz + y$  with

initial conditions,  $y(0) = 1$  &  $z(0) = -1$   
 $h = 0.1$  for  $x = 0.1$

Solving

Given that,

$$\frac{dy}{dx} = f(x, y, z) = yz + x$$

$$\frac{dz}{dx} = g(x, y, z) = xz + y$$

Initial conditions  $y(0) = 1 \Rightarrow x = 0$

$$\begin{aligned} y_0 &= 1 \\ z(0) &= -1, \quad x_0 = 0 \\ z_0 &= -1 \end{aligned}$$

$$h = 0.1$$

$$\therefore x_1 = x_0 + h = 0 + 0.1 = 0.1$$



$$x_0 = 0$$

$$y_0 = 1$$

$$z_0 = -1$$

$$x_1 = 0.1$$

$$y_1 = ?$$

$$z_1 = ?$$

We know R-K formula,

$$y_1 = y_0 + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4)$$

$$z_1 = z_0 + \frac{1}{6} (l_1 + 2l_2 + 2l_3 + l_4)$$

where

$$\begin{aligned} k_1 &= hf(x_0, y_0, z_0) \\ &= 0.1 f(0, 1, -1) \\ &= 0.1 [1 \times 1 + 0] \\ &= -0.1 \end{aligned}$$

$$\begin{aligned} l_1 &= h\phi(0, 1, -1) \\ &= 0.1 [0 \times -1 + 1] \\ &= 0.1 \end{aligned}$$

$$\begin{aligned} k_2 &= hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right) \\ &= 0.1 f\left(0 + \frac{0.1}{2}, 1 - \frac{0.1}{2}, -1 + \frac{0.1}{2}\right) \\ &= 0.1 f(0.05, 0.95, -0.95) \end{aligned}$$

$$\begin{aligned} l_2 &= h\phi(0.05, 0.95, -0.95) \\ &= 0.1 (0.05 \times 0.95 + 0.95) \\ &= 0.0903 \end{aligned}$$

$$\begin{aligned} &= 0.1 (-0.95 \times 0.95 + 0.05) \\ &= -0.0853 \end{aligned}$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$l_3 = h\phi(0.05, 0.9574, -0.9549)$$

$$\begin{aligned} &= 0.1 f(0.05, 0.9574, -0.9549) \\ &= 0.1 [0.05 \times (-0.9549) + 0.9574] \\ &= -0.0864 \end{aligned}$$

$$l_3 = h\phi(0.05, 0.9574, -0.9549)$$

$$\begin{aligned} &= 0.1 f(-0.9549 \times 0.9574 + 0.05) \\ &= 0.0910 \end{aligned}$$

$$k_4 = hf\left(x_0 + h, y_0 + \frac{k_3}{2}, z_0 + \frac{l_3}{2}\right)$$

$$l_4 = h\phi(0.1, 0.9136, -0.9090)$$

$$\begin{aligned} &= 0.1 f(0 + 0.1, 1 - 0.0864, -1 + 0.091) \\ &= 0.1 [0.1 \times -0.9090 + 0.9136] \\ &= 0.0823 \end{aligned}$$

$$= -0.0730$$

$$y_1 = y_0 + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 1 + \frac{1}{6} (-0.1 + 2 * -0.0853 - 2 * 0.0864 - 0.0730)$$

$$= 0.9169$$

$$z_1 = z_0 + \frac{1}{6} (l_1 + 2l_2 + 2l_3 + l_4)$$

$$= -1 + \frac{1}{6} (0.1 + 2 * 0.0903 + 2 * 0.0910 + 0.0823)$$

$$= -0.9092$$

$$\text{Hence } y(0.1) = 0.9139$$

$$z(0.1) = -0.9092$$

Using R-K method, solve the system  
 $\frac{dx}{dt} = x+z, \frac{dy}{dt} = x-y, y(0) = 0, z(0) = 1$

$t = 0.1$ , for  $x = 0.2$

so,  $t = 0.1$

$$\frac{dy}{dt} = f(x, y, z) = x+z$$

$$\frac{dz}{dt} = g(x, y, z) = x-y$$

$$y(0) = 0 \Rightarrow x_0 = 0, y_0 = 0$$

$$z(0) = 1 \Rightarrow z_0 = 0, z_0 = 1$$

$$h = 0.1$$

$$\begin{aligned} x_0 &= 0 & x_1 &= 0.1 & x_2 &= 0.2 \\ y_0 &= 0 & y_1 &=? & y_2 &=? \\ z_0 &= 1 & z_1 &=? & z_2 &=? \end{aligned}$$

$$\therefore x_1 = x_0 + h = 0 + 0.1 = 0.1$$

$$x_2 = x_0 + 2h = 0 + 0.2 = 0.2$$

$$y_2 = ?$$

$$z_2 = ?$$

R-K method

$$y_1 = y_0 + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4) \rightarrow ①$$

$$z_1 = z_0 + \frac{1}{6} (l_1 + 2l_2 + 2l_3 + l_4) \rightarrow ②$$

where

$$k_1 = hf(x_0, y_0, z_0)$$

$$= 0.1f(0, 0, 1)$$

$$= 0.1(0+1)$$

$$= 0.1$$

$$l_1 = hf(x_0, y_0, z_0)$$

$$= 0.1f(0, 0, 1)$$

$$= 0.1(0-0)$$

$$= 0$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right) l_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$= 0.1f(0.05, 0.05, 1)$$

$$= 0.1(0.05+1)$$

$$= 0$$

$$= 0.1(0.05+1)$$

$$= 0.1050$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right) l_3 = hf(0.05, 0.0525)$$

$$= 0.1f(0.05 + 0.0525, 1)$$

$$= 0.1(0.05+1)$$

$$= 0.1050$$

$$= 0.1(0.05-0.0525)$$

$$= -0.0003$$

$$K_4 = h f(x_0 + h, y_0 + k_3, z_0 + l_3) \quad l_4 = h \phi(0.1, 0.1050, 0.9997)$$

$$= h f(0.1, 0.1050, 0.9997) = 0.1 (0.1050 - 0.100)$$

$$= 0.1 (0.1 + 0.9997) = 0.0005$$

$$= 0.1100$$

Then

$$y_1 = y_0 + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 0 + \frac{1}{6} (0.1 + 2 \times 0.1050 + 2 \times 0.1050 + 0.11)$$

$$= 0.1050$$

$$z_1 = z_0 + \frac{1}{6} (l_1 + 2l_2 + 2l_3 + l_4)$$

$$= 1 + \frac{1}{6} (0 + 2 \times 0 + 2 \times (-0.008) + 0.0005)$$

$$= 0.9998$$

Again

$$y_2 = y_1 + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 0.1050 + \frac{1}{6} (0.1 + 2 \times 0.1050 + 2 \times 0.1050 + 0.11)$$

$$= 0.2100$$

$$z_2 = z_1 + \frac{1}{6} (l_1 + 2l_2 + 2l_3 + l_4)$$

$$= 0.9998 + \frac{1}{6} (0 + 2 \times 0 - 0.0003 \times 2 + 0.0005)$$

$$= 0.9996$$

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$$k_1 = h f(x_1, y_1, z_1)$$

$$= 0.1 f(0.1, 0.1050, 0.9998)$$

$$= 0.1 (0.1 + 0.9998)$$

$$= 0.11$$

$$l_1 = h \phi(0.1, 0.1050, 0.9998)$$

$$= 0.1 \phi(0.1 - 0.1050)$$

$$= -0.0005$$

$$k_2 = h f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}, z_1 + \frac{l_1}{2}\right)$$

$$= 0.1 f(0.15, 0.16, 0.9996)$$

$$= 0.1 (0.15 + 0.9996)$$

$$= 0.1150$$

$$l_2 = h \phi(0.15, 0.16, 0.9996)$$

$$= 0.1 (0.15 - 0.16)$$

$$= -0.001$$

$$k_3 = h f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}, z_1 + \frac{l_2}{2}\right)$$

$$= 0.1 f(0.15, 0.1625, 0.9993)$$

$$= 0.1 (0.15 + 0.9993)$$

$$= 0.1149$$

$$l_3 = h \phi\left(x_1 + \frac{h}{2}, y_1 + \frac{k_3}{2}, z_1 + \frac{l_2}{2}\right)$$

$$= 0.1 \phi(0.15, 0.1625, 0.9993)$$

$$= 0.1 (0.15 - 0.1625)$$

$$= -0.0013$$

$$\begin{aligned}
 k_4 &= h f(x_1 + h, y_1 + k_3, z_1 + l_3) \\
 &= 0.1 f(0.2, 0.2199, 0.9985) \\
 &= 0.1 (0.2 + 0.9985) \\
 &= 0.1199
 \end{aligned}$$

$$\begin{aligned}
 l_4 &= h \phi(0.2, 0.2199, 0.9985) \\
 &= 0.1 (0.2 - 0.2199) \\
 &= -0.0020
 \end{aligned}$$

so,

$$\begin{aligned}
 y_2 &= y_1 + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4) \\
 &= 0.1050 + \frac{1}{6} [0.11 + 2 \times 0.1150 + 2 \times 0.1149 + 0.1199] \\
 &= 0.2200
 \end{aligned}$$

$$\begin{aligned}
 z_2 &= z_1 + \frac{1}{6} (l_1 + 2l_2 + 2l_3 + l_4) \\
 &= 0.9998 + \frac{1}{6} (-0.005 - 2 \times 0.001 \\
 &\quad - 2 \times 0.0013 - 0.002) \\
 &= 0.9986 //
 \end{aligned}$$

Model III >

Solution of Second order differential eqn  
by Runge Kutta method.

let us consider the second order differen-  
-tial equations

$$\frac{d^2y}{dx^2} = \phi(x, y, \frac{dy}{dx}) \rightarrow 1$$

If we put  $\frac{dy}{dx} = z \rightarrow 2$  then

$$\frac{d^2y}{dx^2} = \frac{dz}{dx} \rightarrow 3$$

From eqn 1  $\frac{dz}{dx} = \phi(x, y, z)$

Now, we have two simultaneous differen-  
-tial eqn

$$\frac{dy}{dx} = f(x, y, z)$$

$$\& \frac{dz}{dx} = \phi(x, y, z)$$

with initial conditions  $y(x_0) = y_0$  &  $z(x_0) = z_0$   
solving these eqn's we get,

$$\begin{aligned}
 y_1 &= y_0 + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4) \\
 z_1 &= z_0 + \frac{1}{6} (l_1 + 2l_2 + 2l_3 + l_4)
 \end{aligned}$$

where

$$k_1 = hf(x_0, y_0, z_0)$$

$$l_1 = h\phi(x_0, y_0, z_0)$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$l_2 = h\phi\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$l_3 = h\phi(" " ")$$

$$k_4 = hf\left(x_0 + h, y_0 + k_3, z_0 + l_3\right)$$

$$l_4 = h\phi(" " ")$$

Replacing  $x_0, y_0, z_0$  by  $x_1, y_1, z_1$   
we get  $y_2, z_2$  & so on.

5) Using R-K method of 4th order,  
solve  $\frac{d^2y}{dx^2} = x(\frac{dy}{dx})^2 - y^2$  for  $x=0.2$

(with initial cond'n  $y(0)=1, y'(0)=0$   
take  $h=0.2$ )

Solns:

Here, Given that

$$\frac{d^2y}{dx^2} = x(\frac{dy}{dx})^2 - y^2$$

Let  $\frac{dy}{dx} = z$  then

$$\frac{d^2y}{dx^2} = \frac{-dz}{dx}$$

$$f \frac{dy}{dx} = z$$

Now, we have following simultaneous eq<sup>n</sup>

$$\frac{dy}{dx} = f(x, y, z) = z$$

$$\frac{dz}{dx} = \phi(x, y, z) = xz^2 - y^2$$

$$\text{with } y(0) = 1 \Rightarrow x_0 = 0, y_0 = 1$$

$$y'(0) = 0 \Rightarrow z(0) = 0 \Rightarrow x_0 = 0 \\ z_0 = 0 \\ h = 0.2$$

$$\therefore x_1 = x_0 + h = 0 + 0.2 = 0.2$$

$$y_1 = ?$$

$$z_1 = ?$$

$$\begin{array}{ll} x_0 = 0 & x_1 = 0.2 \\ y_0 = 1 & y_1 = ? \\ z_0 = 0 & z_1 = ? \end{array}$$

We know R-K formula,

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \quad ①$$

$$z_1 = z_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4) \quad ②$$

where

$$\begin{aligned} k_1 &= hf(x_0, y_0, z_0) & l_1 &= h\phi(x_0, y_0, z_0) \\ &= hf(0, 1, 0) & &= 0.2\phi(0, 1, 0) \\ &= 0.2(0.0^3 + 1^2) & &= 0.2(0.0^2 \cdot 1^2) \\ &= +0.2 \times 0 & &= -0.2 \\ &= 0 & & \end{aligned}$$

$$k_2 = h \times f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$\begin{aligned} &= 0.2 f(0.1, 1, -0.1) \\ &= 0.2 (-0.1) \\ &= -0.02 \end{aligned}$$

$$l_2 = h \phi\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$\begin{aligned} &= 0.2 \phi(0.1, 1, -0.1) \\ &= 0.2 (0.1 \times (0.1)^2 - 1) \\ &= -0.1998 \end{aligned}$$

$$k_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$\begin{aligned} &= 0.2 f(0.1, -0.09, -0.0999) \\ &= 0.2 (-0.0999) \\ &= -0.02 \end{aligned}$$

$$\begin{aligned} l_3 &= h \phi(0.1, -0.09, -0.0999) \\ &= 0.2 (0.1 (-0.0999)^2 + 0.09^2) \\ &= 0.0002 - 0.1958 \end{aligned}$$

$$\begin{aligned} k_4 &= h f\left(x_0 + h, y_0 + k_3, z_0 + l_3\right) \\ &= 0.2 f(0.2, 0.9800, -0.1958) \\ &= -0.0392 \end{aligned}$$

$$\begin{aligned} l_4 &= h \phi(0.2, 0.98, -0.1958) \\ &= 0.2 (0.2 (0.1958)^2 - 0.98^2) \\ &= -0.1905 \end{aligned}$$

$$y_1 = y_0 + \frac{1}{6} (k_1 + 2k_2 + 2k_3 + k_4)$$

$$\begin{aligned} &= 1 + \frac{1}{6} (0 + 2 \times 0.2 - 2 \times 0.2 + 0.0392) \\ &= 0.9801 \end{aligned}$$

$$z_1 = z_0 + \frac{1}{6} (l_1 + 2l_2 + 2l_3 + l_4)$$

$$\begin{aligned} &= 0 + \frac{1}{6} (-0.2 - 2 \times 0.1998 - 2 \times 0.1958 \\ &\quad - 0.1905) \\ &= -0.1970 \end{aligned}$$

b) Solve

$$y'' = xy' - y, y(0) = 3, y'(0) = 0 \text{ to } y(0.1)$$

Here

$$\frac{d^2y}{dx^2} = x \frac{dy}{dx} - y$$

$$\text{Let } \frac{dy}{dx} = z$$

$$\frac{d^2y}{dx^2} = \frac{dz}{dx}$$

From eqn ①

$$\frac{dz}{dx} = xz - y$$

Now we have two simultaneous

eqns

$$\frac{dy}{dx} = f(x, y, z) = z,$$

$x_0 = 0$   
 $y_0 = 3$   
 $z_0 = 0$   
 $x_1 = 0.1$   
 $h = 0.1$

$$\frac{dz}{dx} = \phi(x_1, y_1, z) = xyz$$

with initial cond'n

$$y(0) = 3 \Rightarrow x_0 = 0$$

$$y_0 = 3$$

$$y'(0) = 0 \Rightarrow x_0 = 0$$

$$z_0 = 0$$

Taking  $h = 0.1$

$$x_1 = x_0 + h = 0 + 0.1 = 0.1$$

$$y_1 = ?$$

$$z_1 = ?$$

We know - R-K formula

$$y_1 = y_0 + \frac{1}{6} (K_1 + 2K_2 + 2K_3 + K_4)$$

$$z_1 = z_0 + \frac{1}{6} (l_1 + 2l_2 + 2l_3 + l_4)$$

Now,

$$K_1 = hf(x_0, y_0, z_0)$$

$$= 0.1 f(0, 3, 0)$$

$$= 0.1 \times 0$$

$$= 0$$

$$l_1 = h \phi(x_0, y_0, z_0)$$

$$= 0.1 \phi(0, 3, 0)$$

$$= 0.1 (0, 0, -3)$$

$$= -0.3$$

$$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$= 0.1 f(0.05, 3, -0.15)$$

$$= 0.1 (-0.15) = -0.015$$

$$l_2 = h \phi(0.05, 3, -0.15)$$

$$= 0.1 [0.05 \times (-0.15 - 3)]$$

$$= -0.3008$$

$$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$= 0.1 f(0.05, 2.99250, -0.1504)$$

$$= 0.1 (-0.1504)$$

$$= -0.015$$

$$l_3 = h \phi(0.05, 2.9250 - 0.1504)$$

$$= 0.1 (-0.05 \times 0.1504 - 2.9250)$$

$$= -0.2008 = -0.3$$

$$K_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3)$$

$$= 0.1 f(0.1, 2.9850, -0.2000)$$

$$= 0.1 (-0.2000)$$

$$= -0.2000$$

$$l_4 = -0.3015$$

$$= -0.03$$

Then  $y_1 = y_0 + \frac{1}{6} [y_1 + 2K_2 + 2K_3 + K_4]$

$$= 3 + \frac{1}{6} [0 + 2 \times (-0.015) - 2 \times 0.015 - 0.03]$$

$$= 2.9850$$

$$z_1 = z_0 + \frac{1}{6} [l_1 + 2l_2 + 2l_3 + l_4]$$

$$= 0 + \frac{1}{6} [-0.3 - 2 \times 0.3008 - 2 \times 0.3 - 0.3015]$$

$$= -0.3005$$

$$7) \frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 4y = 5$$

$y(0) = 0, y'(0) = 0, h = 0.025$   
for  $0 \leq x \leq 0.5$

Solving

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 4y = 5$$

$$\text{Let } \frac{dy}{dx} = z$$

$$\text{so, } \frac{d^2y}{dx^2} = \left[ 5 + 4y - 2z \right] \frac{1}{20}$$

$$\frac{dz}{dx} = \left[ 5 + 4y - 2z \right] \frac{1}{20}$$

Now,

$$\frac{dy}{dx} = f(x, y, z) = z$$

$$\frac{dz}{dx} = \phi(x, y, z) = \frac{1}{20} [5 + 4y - 2z]$$

Here  $y(0) = 0 \Rightarrow x_0 = 0$

$$y_0 = 0$$

$$y'(0) = 0 \Rightarrow x_0 = 0$$

$$z_0 = 0$$

then,

$$\begin{array}{lll} x_0 = 0 & x_1 = 0.25 & x_2 = 0.5 \\ y_0 = 0 & y_1 = ? & y_2 = ? \\ z_0 = 0 & z_1 = ? & z_2 = ? \end{array}$$

Now, From R-K formula

$$y_1 = y_0 + \frac{1}{6} [k_1 + 2k_2 + 2k_3 + k_4]$$

$$x_1 = x_0 + \frac{1}{6} [l_1 + 2l_2 + 2l_3 + l_4]$$

then

$$\begin{aligned} k_1 &= hf(x_0, y_0, z_0) & l_1 &= \phi(x_0, y_0, z_0) \\ &= 0.25f(0, 0, 0) & &= 0.25\phi(0, 0, 0) \\ &= 0.25 \times 0 & &= 0 \\ &= 0 \end{aligned}$$

$$\begin{aligned} k_2 &= hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right) & l_2 &= h\phi\left(0.125, 0, 0\right) \\ &= 0.25f\left(0.125, 0, 0\right) & &= 0.25 \left[ \frac{5}{20} \right] \\ &= 0.25 \times 0.0625 & &= 0.0625 \end{aligned}$$

$$\begin{aligned} k_3 &= hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right) \\ &= 0.25 \left( 0.125, 0, 0.0313 \right) \\ &= 0.25 (0.0313) \\ &= 0.0078 \end{aligned}$$

$$\begin{aligned} l_3 &= h\phi\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right) \\ &= 0.25 \phi(0.125, 0, 0.0313) \\ &= 0.25 \left[ \frac{1}{20} (5 + 0 - 2 \times 0.0313) \right] \\ &= 0.0617 \end{aligned}$$

$$\begin{aligned}
 K_4 &= hf(x_0 + h, y_0 + k_3, z_0 + l_3) \\
 &= 0.25f(0.25, 0.0078, 0.0617) \\
 &= 0.25 \times 0.0617 \\
 &= 0.0154
 \end{aligned}$$

$$\begin{aligned}
 l_4 &= h\phi(0.25, 0.0078, 0.0617) \\
 &= 0.25 \left[ \frac{1}{20} (5 + 4 \times 0.0078 - 2 \times 0.0617) \right] \\
 &= 0.0613
 \end{aligned}$$

Then

$$\begin{aligned}
 y_1 &= y_0 + \frac{1}{6} [K_1 + 2K_2 + 2K_3 + K_4] \\
 &= 0 + \frac{1}{6} [0 + 2 \times 0.0117 + 2 \times 0.0078 + 0.0154] \\
 &= 0.0052
 \end{aligned}$$

$$\begin{aligned}
 z_1 &= z_0 + \frac{1}{6} [l_1 + 2l_2 + 2l_3 + l_4] \\
 &= 0 + \frac{1}{6} [0 + 2 \times 0.0625 + 2 \times 0.0617 + 0.0613] \\
 &= 0.0516
 \end{aligned}$$

$$\begin{aligned}
 \text{again } k_1 &= hf(x_1, y_1, z_1) \\
 &= 0.25f(0.25, 0.0052, 0.0516) \\
 &= 0.25 \times 0.0516 \\
 &= 0.0129
 \end{aligned}$$

$$\begin{aligned}
 K_1 &= hf(x_1, y_1, z_1) \\
 &= 0.25f(0.25, 0.0052, 0.0516) \\
 &= 0.25 \left[ \frac{1}{20} (5 + 4 \times 0.0052 - 2 \times 0.0516) \right] \\
 &= 0.0615
 \end{aligned}$$

$$\begin{aligned}
 K_2 &= hf(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}, z_1 + \frac{l_1}{2}) \\
 &= 0.25f(0.375, 0.0117, 0.0824) \\
 &= 0.25 \times 0.0824 \\
 &= 0.0206 \\
 l_2 &= h\phi(0.375, 0.0117, 0.0824) \\
 &= 0.25 \left[ \frac{1}{20} (5 + 4 \times 0.0117 - 2 \times 0.0824) \right] \\
 &= 0.0610
 \end{aligned}$$

$$\begin{aligned}
 K_3 &= hf(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}, z_1 + \frac{l_2}{2}) \\
 &= 0.25(0.375, 0.0155, 0.0821) \\
 &= 0.25(0.0821) \\
 &= 0.0205
 \end{aligned}$$

$$\begin{aligned}
 l_3 &= h\phi(0.375, 0.0155, 0.0821) \\
 &= 0.25 \left[ \frac{1}{20} (5 + 4 \times 0.0155 - 2 \times 0.0821) \right] \\
 &= 0.0612
 \end{aligned}$$

$$\begin{aligned}
 K_4 &= hf(x_1 + h, y_1 + k_3, z_1 + l_3) \\
 &= 0.25(0.5, 0.0257, 0.1128) \\
 &= 0.25 \times 0.1128 = 0.0282
 \end{aligned}$$

$$k_1 = h \phi(0.5, 0.0257, 0.1128) \\ = 0.25 \left[ \frac{d\phi}{dx} (5 + 4 \times 0.0257 - 2 \times 0.1128) \right] \\ = 0.0610$$

then,

$$y_2 = y_1 + \frac{1}{6} [k_1 + 2k_2 + 2k_3 + k_4] \\ = 0.0052 + \frac{1}{6} [0.0129 + 2 \times 0.0206 + 2 \times 0.0205 + 0.0282] \\ = 0.0258$$

$$z_3 = z_1 + \frac{1}{6} (l_1 + 2l_2 + 2l_3 + l_4) \\ = 0.0516 + \frac{1}{6} (0.0615 + 2 \times 0.0610 + 2 \times 0.0612 + 0.0610) \\ = 0.1128$$

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Method V) Heun's Method :-  
(R-K 2nd order method)  
or modified Euler's Method.

Formula:

If  $\frac{dy}{dx} = f(x, y)$  with initial condition

$$y(x_0) = y_0 \text{ & step size } h. \text{ Then} \\ y_1 = y_0 + \frac{1}{2} (k_1 + k_2)$$

$$\text{where } k_1 = hf(x_0, y_0)$$

$$k_2 = hf(x_0 + h, y_0 + k_1)$$

Replacing  $x_0, y_0$  by  $x, y$  we get  
 $y_2$  & so on.

Q) Using Heun's method, find an appropriate value of  $y$  for  $x = 0.3$ . Given  $\frac{dy}{dx} = x+y$ ,  $y(0) = 1$ ,  $h = 0.1$

Soln:

$$\text{Given } \frac{dy}{dx} = f(x, y) = x+y$$

$$y(0) = 1 \Rightarrow x_0 = 0 \\ y_0 = 1$$

$$h = 0.1$$

$$x_1 = x_0 + h = 0 + 0.1 = 0.1$$

$$x_2 = x_0 + 2h = 0 + 2 \times 0.1 = 0.2$$

$$x_3 = x_0 + 3h = 0 + 3 \times 0.1 = 0.3$$

We know Heun's formula

$$y_1 = y_0 + \frac{1}{2} (k_1 + k_2)$$

For  $y_1$  where  $k_1 = hf(x_0, y_0)$   
 $= 0.1 f(0, 1)$   
 $= 0.1 [0+1]$   
 $= 0.1$

$$\begin{aligned}k_2 &= hf(x_0+h, y_0+k_1) \\&= 0.1 f(0.1, 1.1) \\&= 0.1 (0.1+1.1) \\&= 0.12\end{aligned}$$

$$\begin{aligned}y_1 &= y_0 + \frac{1}{2} (k_1 + k_2) \\&= 1 + \frac{1}{2} (0.1 + 0.12) \\&= 1.11\end{aligned}$$

then For  $y_2$

$$y_2 = y_1 + \frac{1}{2} (k_1 + k_2)$$

where

$$\begin{aligned}k_1 &= hf(x_1, y_1) \\&= 0.1 f(0.1, 1.11) \\&= 0.1 (0.1+1.11) \\&= 0.1210\end{aligned}$$

$$\begin{aligned}k_2 &= hf(x_1+h, y_1+k_1) \\&= 0.1 f(0.2, 1.11+0.121) \\&= 0.1 f(0.2, 1.232) \\&= 0.1 (0.2+1.232) \\&= 0.1431\end{aligned}$$

$$\begin{aligned}y_2 &= y_1 + \frac{1}{2} (k_1 + k_2) \\&= 1.2421\end{aligned}$$

For  $y_3$

$$\begin{aligned}y_3 &= y_2 + \frac{1}{2} (k_1 + k_2) \\&\text{where}\end{aligned}$$

$$\begin{aligned}k_1 &= hf(x_2, y_2) \\&= 0.1 f(0.2, 1.2421) \\&= 0.1 (0.2+1.2421) \\&= 0.1442\end{aligned}$$

$$\begin{aligned}k_2 &= hf(x_2+h, y_2+k_1) \\&= 0.1 f(0.3, 1.3863) \\&= 0.1686\end{aligned}$$

Then

$$\begin{aligned}y_3 &= y_2 + \frac{1}{2} (k_1 + k_2) \\&= 1.2421 + \frac{1}{2} (0.1442 + 0.1686) \\&= 1.3985\end{aligned}$$

Hence  $\boxed{y(0.3) = 1.3985}$

Heun's method for second order differential

solve for  $y(0.2)$

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 6x, y(0) = 0, y'(0) = 1$$

$$h = 0.2$$

so eqn 8

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 6x$$

$$\text{Put } \frac{dy}{dx} = z$$

$$\frac{d^2y}{dx^2} = \frac{dz}{dx}$$

From eqn 7

$$\frac{dz}{dx} + 2z - 3y = 6x$$

$$\text{or } \frac{dz}{dx} = 6x + 3y - 2z$$

Now we have simultaneous diff eqn

$$\frac{dy}{dx} = f(x, y, z) = z$$

$$\frac{dz}{dx} = \phi(x, y, z) = 6x + 3y - 2z$$

with initial conditions

$$y(0) = 0 \Rightarrow x_0 = 0$$

$$y_0 = 0$$

$$y'(0) = 1 \Rightarrow z_0 = 1, x_0 = 0$$

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$$\begin{aligned} x_0 &= 0 \\ y_0 &= 0 \\ z_0 &= 1 \end{aligned}$$

$$\begin{aligned} x_1 &= 0.2 \\ y_1 &=? \\ z_1 &=? \end{aligned}$$

$$\therefore x_1 = x_0 + h = 0 + 0.2, y_1 = ? \\ z_1 = ?$$

we know Heun's formula

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$$

$$z_1 = z_0 + \frac{1}{2}(l_1 + l_2)$$

where

$$\begin{aligned} k_1 &= hf(x_0, y_0, z_0) & l_1 &= h\phi(x_0, y_0, z_0) \\ &= 0.2f(0, 0, 1) & &= 0.2\phi(0, 0, 1) \\ &= 0.2(1) & &= 0.2(6(0) + 3(0) - 2(1)) \\ &= 0.2 & &= -0.4 \end{aligned}$$

$$\begin{aligned} k_2 &= hf(x_0 + h, y_0 + k_1, z_0 + l_1) & l_2 &= h\phi(x_0 + h, y_0 + k_1, z_0 + l_1) \\ &= 0.2f(0.2, 0.2, 0.6) & &= 0.2(6(0.2) + 3(0.2) - 2(0.6)) \\ &= 0.2 \times 0.6 & &= 0.12 \\ &= 0.12 & & \end{aligned}$$

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$$

$$= 0 + \frac{1}{2}(0.2 + 0.12)$$

$$= 0.16$$

$$z_1 = z_0 + \frac{1}{2}(l_1 + l_2)$$

$$= 0 + \frac{1}{2}(-0.4 + 0.12)$$

$$= -0.14$$

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solve  $\frac{dy}{dx} = 1+xz$ ,  $\frac{dz}{dx} = -xy$  for  
 $y(0.6), z(0.6)$ , Given  $y=0, z=1$   
at  $x=0$  by using Heun's method  
Take  $h=0.3$ .

Solving

Here Given

$$\frac{dy}{dx} = 1+xz$$

$$\frac{dz}{dx} = \phi(x, y, z) = -xy$$

$$y(0)=0 \text{ & } z(0)=1 \Rightarrow x_0=0$$

$$\text{Now, } x_1 = x_0 + h = 0 + 0.3 \quad y_0 = 0 \\ = 0.3 \quad z_0 = 1$$

$$x_2 = x_0 + 2h = 0 + 2 \times 0.3 = 0.6$$

$$y_2 = ?$$

$$z_2 = ?$$

Now,

$x_0 = 0$	$x_1 = 0.3$	$x_2 = 0.6$
$y_0 = 0$	$y_1 = ?$	$y_2 = ?$
$z_0 = 1$	$z_1 = ?$	$z_2 = ?$

For  $y_1$ ,

$$\begin{aligned} k_1 &= hf(x_0, y_0, z_0) & k_2 &= h\phi(x_0, y_0, z_0) \\ &= 0.3(0, 0, 1) & &= 0.3\phi(0, 0, 1) \\ &= 0.3(1+0 \times 1) & &= 0.3(-0 \times 0) \\ &= 0.3 & &= 0 \end{aligned}$$

$$\begin{aligned} k_2 &= hf(x_0+h, y_0+k_1, z_0+l_1) & l_2 &= h\phi(0.3, 0.3, 1) \\ &= 0.3f(0.3, 0.3, 1) & &= 0.3(-0.3 \times 0.3) \\ &= 0.3(1 + 0.3 \times 1) & &= -0.027 \\ &= 0.39 \end{aligned}$$

Then

$$\begin{aligned} y_1 &= y_0 + \frac{1}{2}(k_1 + k_2) \\ &= 0 + \frac{1}{2}(0.3 + 0.39) \end{aligned}$$

$$= 0.3450$$

$$z_1 = z_0 + \frac{1}{2}(l_1 + l_2)$$

$$\begin{aligned} &= 1 + \frac{1}{2}(0 - 0.027) \\ &= 0.9865 \end{aligned}$$

Again for  $y_2$ 

$$\begin{aligned} k_1 &= hf(x_1, y_1, z_1) \\ &= 0.3f(0.3, 0.345, 0.9865) \\ &= 0.3(1 + 0.3 \times 0.9865) \\ &= 0.3888 \end{aligned}$$

$$\begin{aligned} l_1 &= h\phi(0.3, 0.345, 0.9865) \\ &= 0.3(-0.3 \times 0.345) \\ &= -0.0311 \end{aligned}$$

$$\begin{aligned} k_2 &= hf(x_1+h, y_1+k_1, z_1+l_1) \\ &= 0.3f(0.6, 0.7338, 0.9554) \\ &= 0.3(1 + 0.6 \times 0.9554) \\ &= 0.4720 \\ l_2 &= 0.3(-0.6 \times 0.7338) \\ &= -0.1321 \end{aligned}$$

$$y_2 = y_1 + \frac{1}{2} (k_1 + k_2)$$

$$= 0.3450 + \frac{1}{2} (0.3888 + 0.4720) \\ = 0.7754$$

$$z_2 = z_1 + \frac{1}{2} (l_1 + l_2)$$

$$= 0.9865 + \frac{1}{2} (-0.0311 - 0.1321) \\ = 0.87 \\ = 0.9049$$

Hence  $y(0.6) = 0.7754$

$$z(0.6) = 0.9049 //$$

Method VI

## Shooting Method

Consider the eqn

$$\frac{d^2y}{dx^2} = \phi(x, y, \frac{dy}{dx}) \text{ or } y'' = \phi(x, y, y')$$

$$y(a) = A \text{ & } y(b) = B$$

$$\text{Let } \frac{dy}{dx} = z$$

$$\frac{d^2y}{dx^2} = \frac{dz}{dx}$$

$$\therefore \frac{dz}{dx} = \phi(x, y, z)$$

In order to solve this set as an initial value problem, we need two conditions at  $x=a$ . We have one condition for  $f(a)=A$  and therefore require another condition for  $z$  at  $x=a$ .

Let us assume that  $z(a) = M_1$ , where  $M_1$  is the guess.  $M_1$  represents the slope  $y'$  at  $x=a$ . Thus  $y' = z$ ,  $y(a) = A$

$$z = \phi(x, y, z) \quad z(a) = M_1 \quad (1)$$

Eqn (1) can be solved for  $y$  &  $z$  using Heun's method using steps of  $h$  until  $x=b$  below.

If  $B_1 = B$  then we have obtained the required solution. If  $B_1 \neq B$  then we obtain the solution with another guess, say  $z(a) = M_2$ .

Let the new estimate of  $y$  at  $x=b$  be  $B_2$ .

If  $B_2 \neq B$  then process may be continued till we obtain the correct estimate of  $y$ .

However the procedure can be accelerated by using an improved guess for  $z(a)$

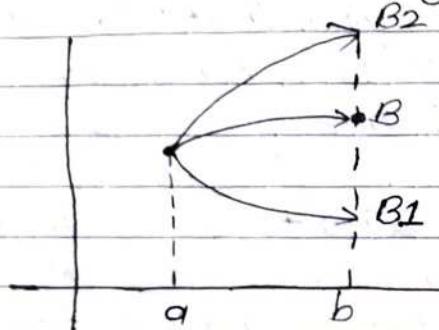
after estimates of  $B_1$  &  $B_2$ . Let us consider that  $z(a) = M_3$  leads to the value  $y(b) = B$ . If we assume that the values of  $M$  &  $B$  are linearly related then

$$\frac{M_3 - M_1}{B - B_2} = \frac{M_2 - M_1}{B_2 - B_1}$$

$$\text{or, } M_3 = M_2 + \frac{B - B_2}{B_2 - B_1} * (M_2 - M_1)$$

$$= M_2 - \frac{B_2 - B}{B_2 - B_1} * (M_2 - M_1)$$

Now with  $z(a) = M_3$ , we can again obtain solution of  $y$ :



1) Using shooting method, solve the eq "  $\frac{d^2y}{dx^2} = 6x$ ,  $y(1)=2$ ,  $y(2)=9$  in

interval  $(1,2)$

Soln:

$$\text{Here } \frac{d^2y}{dx^2} = 6x \rightarrow ①$$

Let  $\frac{dy}{dx} = z$  then

$$\frac{d^2y}{dx^2} = \frac{dz}{dx}$$

$$\text{From eq } ① \quad \frac{dz}{dx} = 6x$$

$$\therefore \frac{dz}{dx} = \phi(x, y, z) = 6x$$

$$\frac{dy}{dx} = f(x, y, z) = z$$

Initial cond'n

$$y(a) = A \text{ or } y(1) = 2 \Rightarrow a = 1, A = 2$$

$$y(b) = B \text{ or } y(2) = 9 \Rightarrow b = 2, B = 9$$

$$y(1) = 2 \Rightarrow x_0 = 1$$

$$y_0 = 2$$

$$\text{Let } x_0 = 2 \text{ (M)}$$

$$\text{Then } x_0 = 1$$

$$y_0 = 2$$

$$x_0 = 2$$

$$\text{let, } h = 1 \quad x_1 = x_0 + h = 1 + 1 = 2, y_1 = ? \quad z = ?$$

Using Heun's method we get

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$$

$$z_1 = z_0 + \frac{1}{2}(l_1 + l_2)$$

$$\begin{aligned} \text{where } k_1 &= hf(x_0, y_0, z_0) \\ &= 1 \cdot f(1, 2, 2) \\ &= 2 \end{aligned}$$

$$\begin{aligned} l_1 &= h \phi(x_0, y_0, z_0) \\ &= h \phi(1, 2, 2) \\ &= 6 \times 1 \times 1 \\ &= 6 \end{aligned}$$

$$\begin{aligned}
 k_2 &= h f(x_0 + h, y_0 + k_1, z_0 + l_1) \\
 &= 1 f(2, 8, 8) \\
 &= 8 \\
 l_2 &= h \phi(2, 4, 8) \\
 &= 12
 \end{aligned}$$

Then

$$\begin{aligned}
 y_1 &= y_0 + \frac{1}{2} (k_1 + k_2) & z_1 &= z_0 + \frac{1}{2} (l_1 + l_2) \\
 &= 2 + \frac{1}{2} (8 + 12) & &= 2 + \frac{1}{2} (4 + 10) \\
 &= 11 & &= 9 = 13
 \end{aligned}$$

This gives  $B_1 = 7$  which is less than  $g$ .  
Now assume  $z(1) = 4 = M_2$

$$x_0 = 1$$

$$y_0 = 2$$

$$z_0 = 4$$

$$y_1 = y_0 + \frac{1}{2} (k_1 + k_2) = 2 + \frac{1}{2} (6)$$

$$z_1 = z_0 + \frac{1}{2} (l_1 + l_2)$$

where

$$\begin{aligned}
 k_1 &= h f(x_0, y_0, z_0) & l_1 &= h \phi(x_0, y_0, z_0) \\
 &= 1 f(1, 2, 4) & &= 1 (6 \times 1) \\
 &= 4 & &= 6
 \end{aligned}$$

$$\begin{aligned}
 k_2 &= h f(x_0 + h, y_0 + k_1, z_0 + l_1) \\
 &= 1 f(2, 6, 10) \\
 &= 10
 \end{aligned}$$

$$\begin{aligned}
 l_2 &= h \phi(x_0 + h, y_0 + k_1, z_0 + l_1) \\
 &= 1 (6 \times 2) \\
 &= 12
 \end{aligned}$$

$$\begin{aligned}
 \text{So, } y_1 &= y_0 + \frac{1}{2} (k_1 + k_2) & z_1 &= z_0 + \frac{1}{2} (l_1 + l_2) \\
 &= 2 + \frac{1}{2} (4 + 10) & &= 4 + \frac{1}{2} (6 + 12) \\
 &= 9 = 13 & &= 13
 \end{aligned}$$

$$\text{so } y(1) = 2, y(2) = 9, z(1) = 4$$

Hence  $y(1) = 2, z(1) = 4$  Ans

Solve

$$\frac{d^2y}{dx^2} = 6x, y(1) = 2, y(2) = 9, h = 0.5$$

Solving

$$\frac{d^2y}{dx^2} = 6x$$

$$\text{Here Let } \frac{dy}{dx} = z$$

$$\text{so, } \frac{d^2y}{dx^2} = \frac{dz}{dx}$$

$$\text{Then } \frac{dz}{dx} = 6x$$

so simultaneous eqn is

$$\frac{dy}{dx} = f(x, y, z) = z$$

$$\frac{dz}{dx} = \phi(x, y, z) = 6x$$

Initial cond'

$$y(1)=2 \Rightarrow x_0=1, y_0=2, h=0.5$$

$$\therefore x_1 = x_0 + h = 1 + 0.5 = 1.5$$

$$x_2 = x_0 + 2h = 1 + 2 \times 0.5 = 2$$

$$y_2 = ?$$

$$z_2 = ?$$

$$\therefore y(2) = 9 \Rightarrow y = 9 \text{ when } x=2$$

$b=2, B=9.$

$$\text{let } z(1)=2=M$$

$$\begin{array}{ccc} & & \\ & & \\ x_0=1 & & x_1=1.5 \\ & & \\ y_0=2 & & \\ & & \\ z_0=2 & & \end{array}$$

Using Heun's formula

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$$

$$z_1 = z_0 + \frac{1}{2}(l_1 + l_2)$$

where

$$k_1 = hf(x_0, y_0, z_0)$$

$$= 0.5 f(1, 2, 2)$$

$$= 1$$

$$l_1 = h\phi(1, 2, 2)$$

$$= 0.5 (6 \times 1)$$

$$= 3$$

$$k_2 = hf(x_0+h, y_0+k_1, z_0+l_1)$$

$$= 0.5 f(1.5, 3, 5)$$

$$= 0.5 \times 5$$

$$= 2.5$$

$$l_2 = h\phi(1.5, 3, 5)$$

$$= 0.5 \times 6 \times 1.5$$

$$= 4.5$$

$$\text{Then } y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$$

$$= 2 + \frac{1}{2}(1+2.5)$$

$$= 3.75$$

$$z_1 = z_0 + \frac{1}{2}(l_1 + l_2)$$

$$= 2 + \frac{1}{2}(3+4.5)$$

$$= 5.75$$

II step

$$\text{where } k_1 = hf(x_1, y_1, z_1)$$

$$= 0.5 f(1.5, 3.75, 5.75)$$

$$= 0.5 \times 5.75$$

$$= 2.8750$$

$$l_1 = h\phi(1.5, 3.75, 5.75)$$

$$= 0.5 \times 6 \times 1.5$$

$$= 4.5$$

$$k_2 = hf(x_1+h, y_1+k_1, z_1+l_1)$$

$$= 0.5 f(2, 6, 6250, 10.25)$$

$$= 0.5 (10.25)$$

$$= 5.125$$

$$\ell_2 = h \phi(2, 6.625, 10.25) \\ = 0.5(6*2) \\ = 6$$

So,

$$y_2 = y_1 + \frac{1}{2}(k_1 + k_2) \\ = 3.75 + \frac{1}{2}(8.875 + 5.125) \\ = 7.75 = B_1$$

This given  $B_1 = 7.75$  which is less than  $B = 9$ . Let us assume another guess  $z(1) = 4 = M_2$

$$x_0 = 1 \quad x_1 = 1.5 \quad x_2 = 2 \\ y_0 = 2 \quad y_2 = 9 \\ z_0 = 4$$

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$$

$$z_1 = z_0 + \frac{1}{2}(\ell_1 + \ell_2)$$

where

$$k_1 = h f(x_0, y_0, z_0) \quad \ell_1 = 0.5 \phi(1, 2, 4) \\ = 0.5 f(1, 2, 4) \quad = 0.5 * 6 * 1 \\ = 0.5 * 4 \\ = 2$$

$$k_2 = h f(x_0 + h, y_0 + k_1, z_0 + \ell_1) \\ = 0.5 f(1.5, 4, 7) \\ = 0.5 * 7$$

$$= 3.5$$

$$\ell_2 = h \phi(1.5, 4, 7) \\ = 0.5 * 6 * 1.5 \\ = 4.5$$

Then:

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2) \\ = 2 + \frac{1}{2}(2 + 3.5)$$

$$= 4.75$$

$$z_1 = z_0 + \frac{1}{2}(\ell_1 + \ell_2) \\ = 4 + \frac{1}{2}(3 + 4.5)$$

$$= 7.75$$

$$\text{Again } y_2 = y_1 + \frac{1}{2}(k_1 + k_2) \\ z_2 = z_1 + \frac{1}{2}(k_1 + k_2)$$

where

$$k_1 = h f(x_1, y_1, z_1) \\ = 0.5 f(1.5, 4.75, 7.75) \\ = 0.5 * 7.75 \\ = 3.8750$$

$$\ell_1 = h \phi(1.5, 4.75, 7.75) \\ = 0.5 * 6 * 1.5 \\ = 4.5$$

$$k_2 = hf(2, 8.6250, 12.25) \\ = 0.5 \times 6 \times 12.25 - 0.5 \times 12.25 \\ = 36.75 - 6.1250$$

Then

$$g_2 = g_1 + \frac{1}{2} (k_1 + k_2) \\ = 4.75 + \frac{1}{2} (3.875 + 6.125) \\ = 9.625 = B_2 \neq B$$

Then

$$\frac{M_3 - M_2}{B - B_2} = \frac{M_2 - M_1}{B_2 - B_1}$$

$$\text{or, } \frac{M_3 - 4}{9 - 9.75} = \frac{4 - 2}{9.75 - 7.75}$$

$$\text{or, } \frac{M_3 - 4}{-0.75} = \frac{-2}{2}$$

$$\text{or, } M_3 - 4 = -0.75$$

$$\therefore M_3 = 3.25 //$$

## Chapter - Six

### Solution of partial differential eqn:



The second order equation involving two independent variable in general form as

$$\frac{\partial^2 f}{\partial x^2} + b \frac{\partial^2 f}{\partial x \partial y} + c \frac{\partial^2 f}{\partial y^2} = f(x, y, f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) \quad (1)$$

where coefficients,  $a, b, c$  may be constants or function of  $x, y$ . Depending on the value of these coefficients, this equations may be classified as,

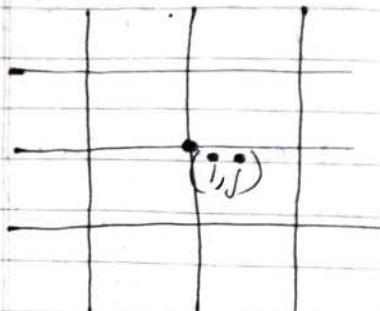
- i) Elliptic if  $b^2 - 4ac < 0$
- ii) parabolic if  $b^2 - 4ac = 0$
- iii) Hyperbolic if  $b^2 - 4ac > 0$

### Elliptic Equation:-

A differential equation is the elliptic in a region  $R$  if  $b^2 - 4ac < 0$  at all points of the region. The boundary condition of this type of equation specify the function  $u$  at every point of the closed boundary of the region  $R$  with in which the solution  $u(x, y)$  is to be determined

$R$   
closed Region

Boundary conditions are given at every or some point of the boundary.



### Method I) Laplace Equations

If  $a=1, b=0, c=1$  &  $F(x, y, f, f_x, f_y) = 0$   
in eq<sup>n</sup>  $a \frac{\partial^2 f}{\partial x^2} + b \frac{\partial^2 f}{\partial y^2} + c \frac{\partial^2 f}{\partial z^2} = F(x, y, f) \frac{\partial f}{\partial x}$

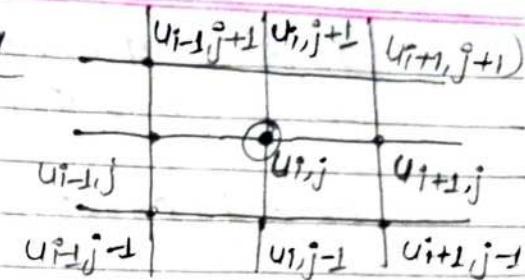
$$\frac{\partial f}{\partial y}) \rightarrow ①$$

$$\text{Then } \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

$$\text{or. } \nabla^2 f = 0$$

or.  $f_{xx} + f_{yy} = 0$  is called laplace equation.

### Laplace formula



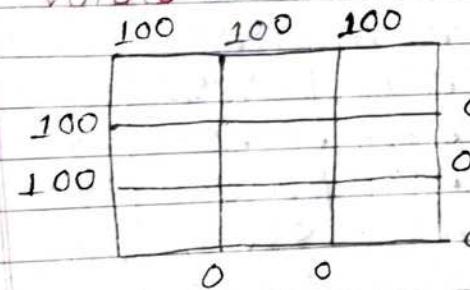
Standard 5-point formula

$$u_{i,j} = \frac{1}{4} [u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1}]$$

Diagonal 5-point formula

$$u_{i,j} = \frac{1}{4} [u_{i+1,j+1} + u_{i-1,j+1} + u_{i-1,j-1} + u_{i+1,j-1}]$$

1) Solve the elliptic eq<sup>n</sup>  $u_{xx} + u_{yy} = 0$  for following square mesh with boundary values as



Here using standard 5-point formula  
 $U_1 = \frac{1}{4}(U_2 + U_0 + U_1 + U_3) = \frac{1}{4}(U_2 + U_5 + 2U_0)$

$$U_2 = \frac{1}{4}(0 + 100 + U_1 + U_4) = \frac{1}{4}(U_1 + U_4 + 100)$$

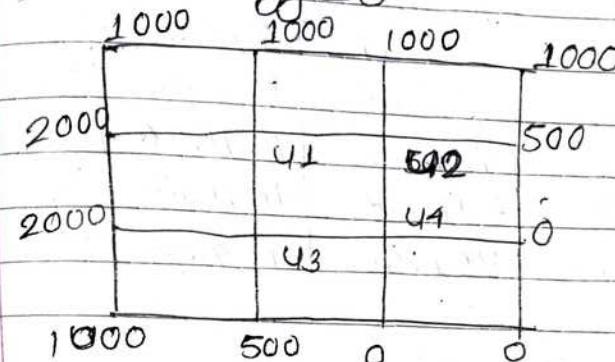
$$U_3 = \frac{1}{4}(U_4 + U_1 + 100 + 0) = \frac{1}{4}(U_1 + U_4 + 100)$$

$$U_4 = \frac{1}{4}(0 + U_2 + U_3 + 0) = \frac{1}{4}(U_2 + U_3)$$

Solving eqns by Gauss seidel method, we get

<i>i<sup>th</sup></i> Initial	$U_1$	$U_2$	$U_3$	$U_4$
0	0	0	0	0
50	37.5	37.5	18.75	
68.75	46.875	46.8750	23.4375	
73.4375	49.2188	49.2188	24.6094	
74.6094	49.8047	49.8047	24.9023	
74.9023	49.9512	49.9512	24.9756	Int <sup>n</sup>
74.9756	49.9878	49.9878	24.9939	Int <sup>n</sup>
74.9939	49.9969	49.9969	24.9985	1
74.9985	49.9992	49.9992	24.9996	2
74.9996	49.9998	49.9998	24.9999	3
74.9999	50.000	50.000	25.000	4
75	50	50	25	5
75	50	50	25	6

2)  $U_{xx} + U_{yy} = 0$



Here using standard 5-point formula

$$U_1 = \frac{1}{4}(1000 + U_3 + 2000 + U_2) - \frac{1}{4}(3000 + U_5 + U_4)$$

$$U_2 = \frac{1}{4}(1000 + U_1 + U_4 + 500) - \frac{1}{4}(U_1 + U_4 + 1000)$$

$$U_3 = \frac{1}{4}(U_1 + 500 + 2000 + U_4) - \frac{1}{4}(2500 + U_1 + U_4)$$

$$U_4 = \frac{1}{4}(U_2 + U_3 + 0 + 0) - \frac{1}{4}(U_2 + U_3)$$

then

	$U_1$	$U_2$	$U_3$	$U_4$
1	0	0	0	0
2	750	562.5	812.5	343.75
3	1093.75	734.375	984.375	429.6875
4	1179.6875	777.3438	1027.3438	451.17
5	1201.17	777.34	451.17	1201.17
6	1206.54	788.09	1038.09	456.54
	1206.54	790.77	1040.77	457.89

7) 1207.89	791.44	1041.44	458.22
8) 1208.22	791.61	1041.61	458.31
9) 1208.31	791.65	1041.65	458.33
10) 1208.33	791.66	1041.66	458.33
11) 1208.38	791.67	1041.67	458.33
12) 1208.39	791.67	1041.67	458.33

Hence  $u_1 = 1208.33$

$$u_2 = 791.67$$

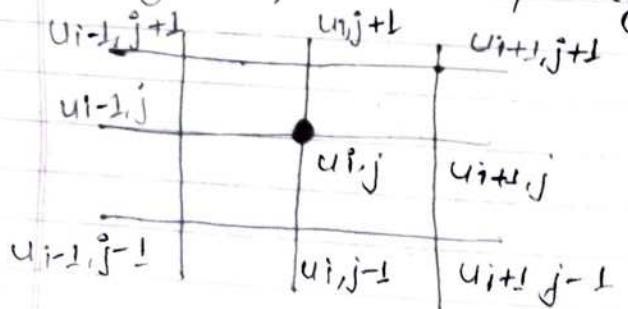
$$u_3 = 1041.67$$

$$u_4 = 458.33$$

## Method II > Poisson Equations

The partial differential equation of the form  $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$  is

called poisson equation where  $f(x, y)$  is a given function of  $x$  &  $y$



## Formula

$$u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1} - 4u_{i,j} = h^2 f(ih, jh)$$

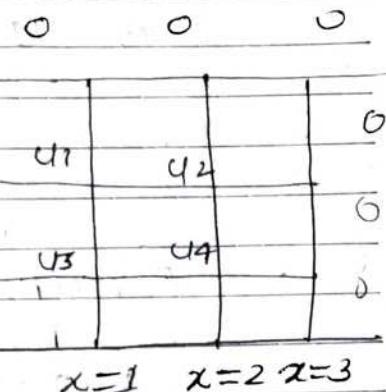
1) Solve the equation  $\nabla^2 u = -10(x^2 + y^2 + 10)$  over the square with sides  $x=0=y$  &  $x=3=y$  with  $u=0$  on the boundary and mesh length = 1 unit

Here

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -10(x^2 + y^2 + 10)$$

It is poission equation.

We know, poission standard formula -



$$u_{i+1,j} + u_{i,j+1} + u_{i-1,j} + u_{i,j-1} - 4u_{i,j} = h^2 f(ih, jh)$$

For  $u_1$ ,  $x=1$

$$j=y=2$$

$$u_2 + u_3 - 4u_1 = 1^2 f(1, 2)$$

$$\text{or, } u_2 + u_3 - 4u_1 = -10(1+4+10)$$

$$\text{or, } u_2 + u_3 - 4u_1 = -150$$

$$u_1 = \frac{u_2 + u_3 + 150}{4} \quad (1)$$

For  $U_2$  :  $i=2, j=2$

$$0+0+U_1+U_4-4U_2 = 1^2 f(2,2)$$

$$\text{or, } U_1+U_4-4U_2 = -10(4+4+10)$$

$$\text{or, } U_1+U_4-4U_2 = -180$$

$$\therefore U_2 = \frac{U_1+U_4+180}{4} \quad \text{(ii)}$$

For  $U_3$  :  $i=1, j=1$

$$\text{or, } U_1+U_4+0+0-4U_3 = 1^2 f(1,1)$$

$$U_1+U_4-4U_3 = -10(12)$$

$$\therefore U_3 = \frac{U_1+U_4+120}{4} \quad \text{(iii)}$$

For  $U_4$  :  $i=2, j=1$

$$0+0+U_2+U_3-4U_4 = 1^2 f(2,1)$$

$$\text{or, } U_2+U_3-4U_4 = -10(4+1+10)$$

$$\text{or, } U_2+U_3-4U_4 = -150$$

$$\therefore U_4 = \frac{U_2+U_3+150}{4} \quad \text{(iv)}$$

Using Gauss seidel method

$I + n$	$U_1$	$U_2$	$U_3$	$U_4$
Int <sup>n</sup>	0	0	0	0
1	37.50	54.38	39.38	60.94
2	60.94	75.47	60.47	71.48
3.	71.48	80.74	65.74	79.12
4.	74.12	82.06	67.06	74.78
5.	74.78	82.39	67.39	74.95
6.	74.95	82.47	67.47	74.99
7	74.99	82.49	67.49	75.00
8	75	82.5	67.5	75.00
9	75	82.5	67.5	75.00

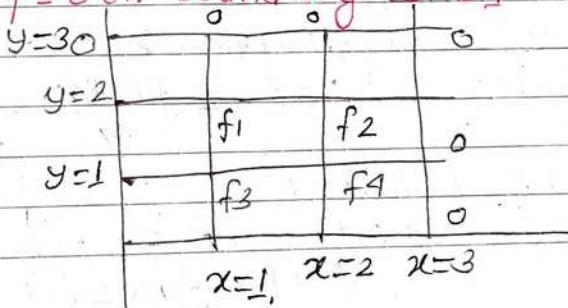
$$\text{Hence } U_1 = 75$$

$$U_2 = 82.5$$

$$U_3 = 67.5$$

$$U_4 = 75$$

2) Solve the poisson eq<sup>n</sup>  $\nabla^2 f = 2x^2y^2$  over square domain  $0 \leq x \leq 3, 0 \leq y \leq 3$  with  $f=0$  on boundary &  $h=1$



Here

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 2x^2y^2$$

This is poisson eqn

By using poisson formula

$$U_{i+1,j} + U_{i,j+1} + U_{i-1,j} + U_{i,j-1} - 4U_{ij} = 1/4(f_{i,j})$$

For  $f_1 \quad i=1, j=2$

$$\text{Then } 0 + 0 + f_2 + f_3 - 4f_1 = 1/4 f(1,2)$$

$$\text{or } f_2 + f_3 - 4f_1 = 2$$

$$\text{or, } f_2 + f_3 - 4f_1 = 8$$

$$\therefore f_1 = \frac{f_2 + f_3 - 8}{4} \rightarrow \textcircled{1}$$

Again For  $f_2, i=2, j=2$

$$0 + f_1 + f_3 - 4f_2 = 1/4 f(2,2)$$

$$\text{or, } f_1 + f_3 - 4f_2 = 2 \times 4 \times 4$$

$$\text{or, } f_2 = \frac{f_1 + f_3 - 32}{4} \rightarrow \textcircled{2}$$

For  $f_3 \quad i=1, j=1$

$$0 + f_1 + f_2 - 4f_3 = 1/4 f(1,1)$$

$$\text{or, } f_1 + f_2 - 4f_3 = 2$$

$$\therefore f_3 = \frac{f_1 + f_2 - 2}{4} \rightarrow \textcircled{3}$$

For  $f_4: \quad i=2, j=1$

$$0 + f_2 + f_3 - 4f_4 = 1/4 f(2,1)$$

$$f_2 + f_3 - 4f_4 = 2 \times 4$$

$$\therefore f_4 = \frac{f_1 + f_3 - 8}{4}$$

Using Gauss seidel method.

JH^n	U <sub>1</sub>	U <sub>2</sub>	U <sub>3</sub>	U <sub>4</sub>
Initial	0	0	0	0
1	-2	-8.5	-1	-2.75
2	-4.38	-9.78	-2.28	-3.66
3	-5.02	-10.17	-2.67	-3.92
4	-5.21	-10.28	-2.78	-4.00
5	-5.27	-10.32	-2.82	-4.02
6	-5.28	-10.33	-2.83	-4.03
7	-5.29	-10.33	-2.83	-4.03
8	-5.29	-10.33	-2.83	-4.03

$$\text{Hence } U_1 = -5.29$$

$$U_2 = -10.33$$

$$U_3 = -2.83$$

$$U_4 = -4.03 //$$

## Solution of System of Linear Equations.

### Method VI Relaxation method

$$20x + y - 2z = 17$$

$$3x + 20y - z = 18$$

$$2x - 3y + 20z = 25$$

Solving

Here residuals are given by

$$R_x = 17 - 20x - y + 2z$$

$$R_y = 18 - 3x - 20y + z$$

$$R_z = 25 - 2x + 3y - 20z$$

The operation table is

	$\delta R_x$	$\delta R_y$	$\delta R_z$
$\delta x = 1$	-20	-1	2
$\delta y$	-3	-20	1
$\delta z$	-2	3	-20

The relation table is

	$R_x$	$R_y$	$R_z$
$x = y = z = 0$	17	18	25
$\delta z = 1$	19	19	5
$\delta x = 1$	-1	16	3
$\delta y = 1$	-2	-4	6
$\delta z = 0.3$	-1.4	-3.7	0
$\delta y = -0.185$	-1.215	0	-0.555
$\delta x = -0.061$	0.005	0.183	-0.433

$$\begin{aligned} \sum \delta x &= 1 - 0.061 = 0.939 \\ \sum \delta y &= 1 - 0.185 = 0.823 \\ \sum \delta z &= 1 + 0.3 = 1.3 \end{aligned} \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \text{Ans}$$