Numerical integration

The process of finding integration of any discrete function is called Numerical integration. The various Numerical integration methods are :

(i) Newton cote's quadrature formula
(ii) Gauss legendre formula / Gaussian integration
(iii) Romberg's integration.

1) Newton cotes quadrature formula :-

let $I = \int_a^b y\,dx$, where $y = f(x)$ be the integrand which is to be evaluated numerically from the given set of tabulated values.

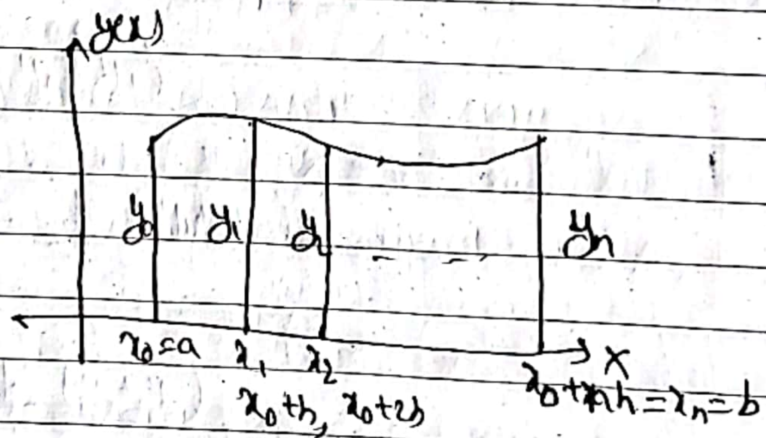

fig : process of numerical integration.

let the values of arguments $x_1, x_2, x_3, --- x_n$ be equispaced with $x_0 = a$, $x_1 = x_0+h$, $x_2 = x_0+2h$ --- $x_n = x_0+nh$, such that

$$h = \frac{b-a}{n}$$

then,
$$I = \int_a^b f(x)\,dx = \int_{x_0}^{x_0+nh} f(x)\,dx \quad \text{--- (1)}$$

putting $x = x_0 + rh \Rightarrow \frac{dx}{dr} = h$

$\Rightarrow dx = h \cdot dr$

then,

for lower limit
$x_0 = x_0 + rh$
$\Rightarrow r = 0$

and for upper limit
$x_0 + nh = x_0 + rh$
$r = n$

$$\int_{x_0}^{x_0+nh} f(x)\,dx = h \int_0^n f(x_0 + rh)\,dr \quad \text{--- (11)}$$

Applying, Newton forward interpolation formula,

$$I = h \cdot \int_0^n \left(y_0 + r\,\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \frac{r(r-1)(r-2)(r-3)}{4!}\Delta^4 y_0 + \cdots \right) dr$$

Now, intergarating by parts, we get

$$I = \int_a^b y\,d(x) = n \cdot h \left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \frac{n}{4!}\left(\frac{n^3}{5} - \frac{3n^2}{2} + \frac{11n}{3} - 3\right) + \cdots \right]$$

This is ~~ca~~ called Newton cote's quadrature formula.

equ^n ③ becomes so complex to compute. Integration for simplicity, we compute ③ by taking $n = 1, 2, 3 \dots$ to give various useful formulas as follows:

① for $n=1$ (Trapezoidal rule)

Taking $n=1$ in equ^n ③ the curve $f(x)$ will pass through the points (x_0, y_0) & (x_1, y_1) so that the difference greater than Δy_0 becomes zero then equ^n ③ can be written as,

$$\int_{x_0}^{x_0+h} f(x)\, d = h\left[y_0 + \tfrac{1}{2}\Delta y_0\right]$$

$$= \frac{h}{2}(2y_0 + y_1 - y_0)$$

$$= \frac{h}{2}(y_0 + y_1)$$

and,

$$\int_{x_0+h}^{x_0+2h} f(x)\, dx = \frac{h}{2}(y_1 + y_2)$$

also,

$$\int_{x_0+2h}^{x_0+3h} f(x)\, dx = \frac{h}{2}(y_2 + y_3)$$

Similarly proceeding upto n^{th} term

$$\int_{x_0+(n-1)h}^{x_0+nh} f(x)\, dx = \frac{h}{2}(y_{n-1} + y_n)$$

$$\therefore \int_{x_0}^{x_0+nh} f(x)\, dx = \left[\int_{x_0}^{x_0+h} + \int_{x_0+h}^{x_0+2h} + \dots + \int_{x_0+(n-1)h}^{x_0+nh}\right] f(x)\, dx$$

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+y_3+\dots+y_{n-1})\right] \quad \text{--- (1)}$$

which is known as trapezoidal rule.

(2) for $n=2$ (Simpson's $\frac{1}{3}^{rd}$ rule)

Taking $n=2$ in (3), the curve $f(x)$ will pass through the points (x_0,y_0), (x_1,y_1), (x_2,y_2), so that the difference greater than $\Delta^2 y_0$ becomes zero. Then equ^n (3) can be written as,

$$\int_{x_0}^{x_0+2h} f(x)\,dx = 2h\left[y_0+\Delta y_0+\frac{1}{6}\Delta^2 y_0\right]$$

$$= \frac{h}{3}(6y_0+6y_1-6y_0+y_2-2y_1+y_0)$$

$$= \frac{h}{3}(y_0+4y_1+y_2)$$

and,

$$\int_{x_0+2h}^{x_0+4h} f(x)\,dx = \frac{h}{3}(y_2+4y_3+y_4)$$

also,

$$\int_{x_0+4h}^{x_0+6h} f(x)\,dx = \frac{h}{3}(y_4+4y_5+y_6)$$

upto n^{th} term,

$$\int_{x_0+(n-2)h}^{x_0+nh} f(x)\,dx = \frac{h}{3}(y_{n-2}+4y_{n-1}+y_n)$$

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \left[\int_{x_0}^{x_0+2h} + \int_{x_0+2h}^{x_0+4h} + \cdots + \int_{x_0+(n-2)h}^{x_0+nh}\right] f(x)\,dx$$

$$\therefore I = \int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{3}\left[(y_0+y_n) + 4(y_1+y_3+y_5+\cdots+y_{n-1}) + 2(y_2+y_4+y_6+\cdots+y_{n-2})\right] \quad \text{--- (5)}$$

which is called Simpson's $1/3^{rd}$ rule.

(2) for $n=3$ (simpson's $3/8^{th}$ rule)

Taking $n=3$ in equ^n (3), the curve $f(x)$ will pass through the points (x_i, y_i) for $i=0,1,2,3$. So that the difference between greater than $\Delta^3 y_0$ becomes zero. then equ^n (3) can be written as,

$$\int_{x_0}^{x_0+3h} f(x)\,dx = 3h\left(y_0 + \tfrac{3}{2}\Delta y_0 + \tfrac{3}{4}\Delta^2 y_0 + \tfrac{1}{8}\Delta^3 y_0\right)$$

$$= \frac{3h}{8}\left[8y_0 + 12y_1 - 12y_0 + 6(y_0 - 2y_1 + y_2) + (y_3 - 3y_2 + 3y_1 - y_0)\right]$$

$$= \frac{3h}{8}\left(y_0 + 3y_1 + 3y_2 + y_3\right)$$

and,

$$\int_{x_0+3h}^{x_0+6h} f(x)\,dx = \frac{3h}{8}\left(y_3 + 3y_4 + 3y_5 + y_6\right)$$

$$\int_{x_0+6h}^{x_0+9h} f(x)\,dx = \frac{3h}{8}\left(y_6 + 3y_7 + 3y_8 + y_9\right)$$

Similarly,

$$\int_{x_0+(n-3)h}^{x_0+nh} f(x)\,dx = \frac{3h}{8}\left(y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n\right)$$

$$\therefore \int_{x_0}^{x_0+nh} f(x)\,dx = \left[\int_{x_0}^{x_0+3h} + \int_{x_0+3h}^{x_0+6h} + \cdots + \int_{x_0+(n-3)h}^{x_0+nh}\right] f(x)\,dx$$

$$\Rightarrow \frac{3h}{8}\left[(y_0+y_n) + 3(y_1+y_2+y_4+y_5+\cdots+y_{n-1}) + 2(y_3+y_6+y_9+\cdots+y_{n-3})\right] \quad \text{--- (6)}$$

Which is called simpson's $\frac{3}{8}$th rule.

<u>Note</u>
for trapezoidal rule
$n \geq 2$

for simpson's $\frac{1}{3}$rd rule
$n \% 2 == 0$

for $\frac{3}{8}$th rule
$n \% 3 == 0$

Q. Integration $I = \int_0^6 \frac{1}{1+x^2} dx$

using,

(i) Trapezoidal rule.

(ii) simpsones $1/3^{rd}$ rule

(iii) simpsons $3/8^{th}$ rule

Soln. Given, $I = \int_a^b f(x)\,dx = \int_0^6 \frac{1}{1+x^2} dx$

where

$$a = 0,\ b = 6,\ f(x) = \frac{1}{1+x^2}$$

$$n = 6$$

$$h = \frac{b-a}{n} = \frac{6}{6} = 1$$

x	0	1	2	3	4	5	6
y	1	0.5	0.2	~~0.58~~ 0.1	0.0588	0.0384	0.027

By appyying trapezoidal rule

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}\left[(y_0+y_6) + 2(y_1+y_2+y_3+y_4+y_5)\right]$$

$$= \frac{1}{2}\left[(1+0.027) + 2(0.5+0.2+0.1+0.0588+0.0$$

$$= 1.4107$$

$$I = \int_0^{\pi/2} \sin x \, dx$$

Again, Simpson's $\frac{1}{3}$ rd rule,

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{2h}{3}\left[(y_0+y_6) + 4(y_1+y_3+y_5) + 2(y_2+y_4)\right]$$

$$= \frac{1}{3}\left[(1+0.027) + 4(0.5+0.1+0.0384) + 2(0.2+0.05\ldots\right]$$

$$= 1.366$$

Again, $\frac{3}{8}$ th rule

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{3h}{8}\left[(y_0+y_6) + 3(y_1+y_2+y_4+y_5) + 2(y_3)\right]$$

$$= \frac{3}{8}\left[(1+0.027) + 3(0.5+0.2+0.0588+0.0384) + 2(0.1)\right]$$

$$= \frac{3}{8}\left[1.027 + 3.4284 + 0.2\right]$$

$$= 1.356$$

2) Gauss Integration method :

This method gives better accuracy than Newton Cote's quadrature formula with same number of n but different spacing. Gauss Integration formula is given by,

$$I = \int_{-1}^{1} f(x)\, dx = \sum_{i=1}^{n} w_i f(x_i) \quad \text{---- ①}$$

Where, w_i & x_i is the weights and abscissal values respectively. The various values of w_i & x_i is given below for different value of n:

n	w_i	x_i		n	w_i	$f(x_i)$
2	1	$-1/\sqrt{3}$		4	0.3478	-0.861136
	1	$1/\sqrt{3}$			0.6521	-0.3399
3	5/9	$-\sqrt{3/5}$			0.6521	0.3399
	8/9	0			0.3478	0.8611
	5/9	$\sqrt{3/5}$				

↳ The Gauss integration method has a limitation that it can only be applied for the range [-1, 1], Hence to integrate between [a, b] it should be firstly converted into [-1, 1] by using transformation

$$\boxed{x = \frac{b-a}{2} \times u + \frac{b+a}{2}}$$

Q. find the integration of $I = \int_{-1}^{1} \frac{1}{1+x^2} dx$ using Gauss integration formula for $n=2$ & $n=3$

Sol^n: Given,

$$I = \int_{-1}^{1} f(x)\, dx = \int_{-1}^{1} \frac{1}{1+x^2} dx$$

Applying Gauss integration formula, we get

$$I = \int_{-1}^{1} f(x)\, dx = \sum_{i=1}^{n} w_i f(x_i)$$

for $n=2$,

$$I = w_1 f(x_1) + w_2 f(x_2)$$

where $w_1 = 1$, $x_1 = -1/\sqrt{3}$

$w_2 = 1$, $x_2 = 1/\sqrt{3}$

$$I = 1 \times f(-1/\sqrt{3}) + 1 \times f(1/\sqrt{3})$$

$$= 1 \times \frac{1}{1+(-1/\sqrt{3})^2} + 1 \times \frac{1}{1+(1/\sqrt{3})^2}$$

$$= 1.5$$

for $n=3$,

$w_1 = 5/9$, $x_1 = -\sqrt{3/5}$

$w_2 = 8/9$, $x_2 = 0$

$w_4 = 5/9$, $x_3 = \sqrt{3/5}$

$$I = 5/9 \times f(-\sqrt{3/5}) + 9/8 \times f(0) + 5/9\, f(\sqrt{3/5})$$

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$= \frac{5}{9} * \frac{1}{1+\sqrt{3/5}} + \frac{8}{9} \times \frac{1}{1+0} + \frac{5}{9} * \frac{1}{1+\sqrt{3/5}}$

$= 1.58$

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$I = \int_0^1 \frac{1}{1+x}\, dx$ $n=2$ & $n=3$

Given, $I = \int_a^b f(x)\, dx$

Applying Gauss formula, changing limit $[a,b]$ to $[-1,1]$

I

$x = \frac{b-a}{2}u + \frac{b+a}{2} = \frac{u+1}{2}$

$\frac{dx}{du} = \frac{1}{2}, \Rightarrow dx = \frac{1}{2} du$

$\therefore \int_0^1 \frac{1}{1+x}\, dx = \int_{-1}^{1} \frac{1}{1+\frac{u+1}{2}} \cdot \frac{1}{2}\, du$

$= \int_{-1}^{1} \frac{1}{u+3}\, du$

Now, for $n=2$,

$I = w_1 f(x_1) + w_2 f(x_2)$

$= 1 \times f(-1/\sqrt{3}) + 1 \times f(1/\sqrt{3})$

$=$

Applying Gauss integration formula,

$$I = \int_{-1}^{1} f(u)\, du = \sum_{i=1}^{n} w_i f(u_i)$$

for $n=2$,

$$I = w_1 f(u_1) + w_2 f(u_2)$$
$$= 1 \times f(-1/\sqrt{3}) + 1 \times f(1/\sqrt{3})$$
$$= \left(1 \times \frac{1}{-1/\sqrt{3}+3} + \frac{1}{1/\sqrt{3}+3}\right)$$
$$= 0.4127 + 0.2795$$
$$= 0.6923$$

for $n=3$

$$I = w_1 f(u_1) + w_2 f(u_2) + w_3 f(u_3)$$
$$= 5/9 \times f(-\sqrt{3/5}) + 8/9 f(0) + 5/9 \times f(\sqrt{3/5})$$

$$I = \frac{5}{9} \times \frac{1}{-\sqrt{3/5}+3} + \frac{8}{9} \times \frac{1}{3} + \frac{5}{9} \times \frac{1}{\sqrt{3/5}+3}$$

$= 0.6931$

Romberg's Integration:

Romberg's integration is the modification of Trapezoidal rule to get better approximation.

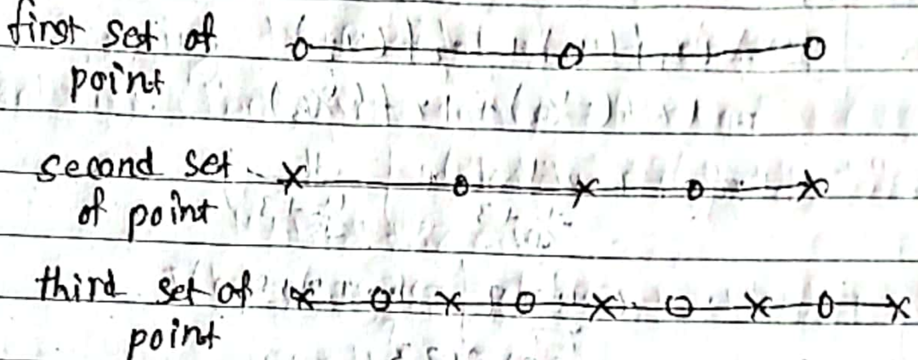

fig: Sampling for Romberg's Integration.

Where o → new sampling point
x → old sampling point

- firstly we determine the integral values by taking interval h using Trapezoidal rule and determing the integral value by dividing interval each time by ½ using Trapezoidal rule to obtain the various improved values as $I(h)$, $I(h/2)$, $I(h/4)$, $I(h/8)$ etc.
- To evaluate improved values of integral I systematically we take $h_1 = h$, $h_2 = h/2$ & the improvement formula is given as,

$$\boxed{I(h_1, h_2) = \frac{4I(h_2) - I(h_1)}{3}}$$

The schema for the improved value is given as,

$$
\begin{array}{l}
I(h) \\
I(h/2) \\
I(h/4) \\
I(h/8)
\end{array}
\quad
\begin{array}{l}
I(h, h/2) \\
I(h/2, h/4) \\
I(h/4, h/8)
\end{array}
\quad
\begin{array}{l}
I(h, h/2, h/4) \\
I(h/2, h/4, h/8)
\end{array}
\quad
I(h, h/2, h/4, h/8)
$$

this process is repeated until we get the value correct upto desired accuracy.

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Integrate the given integral using Romberg's integration.

$$\int_1^2 \frac{1}{1+x^3}\,dx$$

Given, $I = \int_a^b f(x)\,dx = \int_1^2 \frac{1}{1+x^3}\,dx$

where,

$a = 1,\ b = 2,\ f(x) = \frac{1}{1+x^3}$

taking $n = 2$, $h = \frac{b-a}{n} = \frac{1}{2} = 0.5$

Then

x	1	1.5	2
y	0.5	0.2285	0.1111
	y_0	y_1	y_2

Applying Trapezoidal rule

$$I = \frac{h}{2}\left[(y_0 + y_2) + 2 \times y_1\right]$$

$$= \frac{0.5}{2}\left[(0.5 + 0.1111) + 2 \times 0.2285\right]$$

$$= 0.2670$$

Again taking $h = h/2 = \frac{0.5}{2} = 0.25$ then

x	1	1.25	1.5	1.75	2
y	0.5	0.338	0.2285	0.1572	0.1111

Applying trapezoidal rule, we get

$$I = \frac{h}{2}\left[(y_0+y_4)+2(y_1+y_2+y_3)\right]$$

$$I(h/2) = \frac{0.25}{2}\left[(0.338+0.1111)+2(0.3386+0.2285+0.1572)\right]$$

$$= 0.25774$$

Similarly, taking $h = \frac{h}{4} = \frac{0.25}{2} = 0.125$

x	1	1.125	1.25	1.375	1.5	1.625	1.75	0.875	2
y	0.5	0.4125	0.3386	0.2778	0.2285	0.1889	0.1572	0.1317	0.1111

trapezoidal rule,

$$I = \frac{h}{2}\left[(y_0+y_8)+2(y_1+y_2+y_3+y_4+y_5+y_6+y_7)\right]$$

$$I(h/4) = 0.25509$$

Now, Applying Romberg's Integration formula, we get

$$I(h, h/2) = \frac{4I(h/2) - I(h)}{3}$$

$I(h) = 0.2670$ ⟩ 0.2842 ⟩ 0.25433

$I(h/2) = 0.2574$ ⟩ 0.2543

$I(h/4) = 0.25508$

$I = \int_1^2 \frac{1}{1+x^3}dx = 0.2843$

Direct method:-

Gauss Elimination method :-

In this method, the unknown variables are determined in two step of calculation.

1) Forward Elimination :-
In this step the values of unknowns are eliminated from successive equation to reduced the system of equ^n into upper triangular matrix.

2) Backward substitution :-
From the upper triangular matrix the values of unknowns found and substituted to successive upper equation to find unknowns is called backward substitution

- The equ^n with which other equ^n coefficient is made zero is called pivot equ^n & the coeff. w.r.t. which other coeff. is eliminated is called pivot point.

Partial pivoting :-
If largest coeff. of x chosen from all equ^n is bought as first pivot by changing first equ^n with largest coeff. equ^n. In second step numerically largest coeff. of y is chosen from remaining equ^n leaving the first equ^n & brought as 2^nd pivot equ^n having largest coeff. of y this process continued until we arrive to single equ^n. this procedure is called partial pivoting.

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Q.) Solve the given system of equⁿ by using Gauss elimination method.

$$2x+2y+z=6$$
$$4x+2y+3z=4$$
$$x-y+z=0$$

Applying the given system of equⁿ we get

$$4x+2y+3z=4$$
$$2x+2y+z=6$$
$$x-y+z=0$$

Applying $R_2 \leftarrow 2R_2 - R_1$, $R_3 \leftarrow 4R_3 - R_1$

$$4x+2y+3z=4$$
$$0+2y-z=8$$
$$0-6y+z=-4$$

Again, $R_2 \leftrightarrow R_3$

$$4x_1+2y+3x=4$$
$$0-6y+z=-4$$
$$0+2y-z=8$$

Applying $R_3 \leftarrow 3R_3 + R_2$

$$4x_0+2y+3z=4 \quad \text{(i)}$$
$$0-6y+z=-4 \quad \text{(ii)}$$
$$0-2z=20 \quad \text{(iii)}$$

Now by using backward substitution, from (iii)

$$z = -10$$

$$-6y + -10 = -4$$

$$y = +1$$

$$x + 2 + 30 = 4$$

$$x = 9$$

(2) Gauss - Jordan method :-
It is the improvement over the gauss-elimination method such that the elimination process is not only applied below to pivot point but also above the pivot point to reduce the system of equ to diagonal matrix. Hence the values of unknowns are determined readily from corresponding reduced system of equ. Thus this method doesnot need backward substitution at the cost of additional computation.

Q.) Solve the given system of equ by Gauss-Jordan method.

$$5x - 2y + z = 4$$

$$7x + y - 5z = 8$$

$$3x + 7y + 4z = 10$$

Solve given system of equ is

$$7x + y - 5z = 8$$

$$3x + 7y + 4z = 10$$

$$5x - 2y + z = 4$$

Applying $R_2 \leftarrow 7R_2 - 3R_1$, $R_3 \leftarrow 7R_3 - 5R_1$

$$7x + 4y - 5z = 8$$
$$0 + 46y + 43z = 46$$
$$0 - 19y + 32z = -12$$

Applying $R_1 \leftarrow 46R_1 - R_2$, $R_3 \leftarrow 46R_3 + 19R_2$

$$322x + 0 - 273z = 322$$
$$0 + 46y + 43z = 46$$
$$0 + 0 + 2289z = 322$$

Applying $R_1 \leftarrow R_1 + 273R_3$, $R_2 \leftarrow R_2 - 43R_3$

$$322x + 0 + 0 = \frac{39284}{109}$$
$$0 + 46y + 0 = \frac{1308}{327}$$
$$0 + 0 + z = \frac{322}{2289}$$

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8.) Find inverse of matrix by using Gauss-Jordan.

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$

Augmented matrix is given by

$$A = \left[\begin{array}{ccc:ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 1 & 3 & -3 & 0 & 1 & 0 \\ -2 & -4 & -4 & 0 & 0 & 1 \end{array}\right]$$

Applying $R_2 \leftarrow R_2 - R_1$, $R_3 \rightarrow R_3 - 2R_1$

$$A = \left[\begin{array}{ccc:ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & -6 & -1 & 1 & 0 \\ 0 & -2 & 2 & 2 & 0 & 1 \end{array}\right]$$

Applying $R_2 \leftarrow R_2/2$

$$A = \left[\begin{array}{ccc:ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -1/2 & 1/2 & 0 \\ 0 & -2 & 2 & 2 & 0 & 1 \end{array}\right]$$

Applying $R_1 \leftarrow R_1 - R_2$, $R_3 \leftarrow R_3 + 2R_2$

$$A = \left[\begin{array}{ccc:ccc} 1 & 0 & 6 & 3/2 & -1/2 & 0 \\ 0 & 1 & -3 & -1/2 & 1/2 & 0 \\ 0 & 0 & -4 & 1 & 1 & 1 \end{array}\right]$$

Applying $R_3 \leftarrow R_3/4$

$$A\left[\begin{array}{ccc:ccc} 1 & 0 & 6 & 3/2 & -1/2 & 0 \\ 0 & 1 & -3 & -1/2 & 1/2 & 0 \\ 0 & 0 & 1 & -1/4 & -1/4 & -1/4 \end{array}\right]$$

Applying $R_1 \leftarrow R_1 - 6R_3$, $R_2 \leftarrow R_2 + 3R_3$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 3 & 1 & 3/2 \\ 0 & 1 & 0 & -5/4 & -1/4 & -3/4 \\ 0 & 0 & 1 & -1/4 & -1/4 & -1/4 \end{array}\right]$$

$$\therefore A^{-1} = \begin{bmatrix} 3 & 1 & 3/2 \\ -5/4 & -1/4 & -3/4 \\ -1/4 & -1/4 & -1/4 \end{bmatrix}$$

Find the inverse of matrix

Q.) ~~Solve the given system of eqn~~ by Gauss elimination method

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$

Soln:

Augmented matrix is given by,

$$\left[\begin{array}{ccc:ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 1 & 3 & -3 & 0 & 1 & 0 \\ -2 & -4 & -4 & 0 & 0 & 1 \end{array}\right]$$

Applying $R_2 \leftarrow R_2 - R_1$, $R_3 \leftarrow R_3 + 2R_1$

$$\left[\begin{array}{ccc:ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & -6 & -1 & 1 & 0 \\ 0 & -2 & 2 & 2 & 0 & 1 \end{array}\right]$$

Applying $R_3 \rightarrow R_3 + R_2$

$$\left[\begin{array}{ccc:ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & -6 & -1 & 1 & 0 \\ 0 & 0 & -4 & 1 & 1 & 1 \end{array}\right]$$

Writting the Augmented matrix as:

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & -6 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad -(1)$$

from equ^n (1)

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & -6 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

~~$a_{11} = 3, \quad a_{22} = -5/4, \quad a$~~

$a_{31} = -1/4$

$a_{21} = -5/4$

$a_{11} = 3$

Again,

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & -6 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$a_{32} = -1/4$

$a_{22} = -1/4$

$a_{12} = 1$

also,

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & -6 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$a_{33} = -1/4$

$a_{23} = -3/4$

$a_{13} = 3/2$

$$\therefore \quad A^{-1} = \begin{bmatrix} 3 & -1 & -3/2 \\ -5/4 & -1/4 & -3/4 \\ -1/4 & -1/4 & -1/4 \end{bmatrix} \text{ Ans.}$$

Factorization method

This method is based on the fact that every square matrix can be divided into lower and upper triangular matrix such that all principal minors are non-zero i.e,

$$|a_{11}| \neq 0; \quad \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0; \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0 \text{ etc.}$$

Consider a given system of equ^n

$$\left.\begin{aligned} a_{11}x + a_{12}y + a_{13}z = b_1 \\ a_{21}x + a_{22}y + a_{23}z = b_2 \\ a_{31}x + a_{32}y + a_{33}z = b_3 \end{aligned}\right\} \text{ --- (1)}$$

then equ^n (1) can be written as:

$$A * X = B \text{ --- (1')}$$

Where,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}; \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Now,

factorizing matrix a into L & U, i.e,

$$A = L \cdot U \text{ --- (2)}$$

The various ways of factorizing A are:

(1) Do-little method

$$A = L \cdot U$$

Where, $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$ & $U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

2) Crouts Decomposition

$$A = L \cdot U$$

Where,

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \quad \& \quad U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

3) Cholesky decomposition:-

Only be applied to symmetric matrix given by,

$$\boxed{A = L \cdot L^T = U^T \cdot U}$$

where,

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \quad \& \quad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

from equ^n (2) & (3)

$$L \cdot U \cdot X = B \quad \text{---} (4)$$

Writing

$$U \cdot X = V \quad \text{---} (5)$$

equ^n (4) becomes,

$$L \cdot V = B \quad \text{---} (6)$$

Solving ⑥ we get values of V, putting the value of V in ⑤ we get the value of x by using backward substitution method. Thus the system of linear eqn is solved by factorization method.

2015 fall

Q.) Solve the given system of eqn by factorization method.

$$3x+2y+z=10$$
$$2x+3y+2z=14$$
$$x+2y+3z=14$$

Sol: Writing the given system of eqn in the form of
$$AX = B \quad \text{--- ①}$$

we get,
$$A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}; \quad B = \begin{bmatrix} 10 \\ 14 \\ 14 \end{bmatrix}; \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Now, factorizing A into L & U by Do-little decomposition
$$A = L \cdot U \quad \text{---- ②}$$

$$\text{or, } \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

Now, multiplying L & U & equating with element of A we get.

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① // first column × first column, & equating

$u_{11} = a_{11} = 3$, $u_{12} = a_{12} = 2$, $u_{13} = a_{13} = 1$

② // first row × by first column & equating

$l_{21} u_{11} = a_{21}$, $l_{31} \cdot u_{11} = a_{31}$

$l_{21} = 2/3$, $l_{31} = 1/3$

③ // second colu^n × 2nd ~~col~~ row

$$l_{21} \cdot u_{12} + u_{22} = a_{22}$$
$$\Rightarrow u_{22} = a_{22} - l_{21} \cdot u_{12}$$
$$= 3 - 2/3 \times 2$$
$$= 5/3$$

$$l_{21} \cdot u_{13} + u_{23} = a_{23}$$
$$u_{23} = a_{23} - l_{21} \cdot u_{13}$$
$$= 2 - 2/3 \times 1 = \frac{4}{3}$$

④ //

$$l_{31} \cdot u_{12} + l_{32} \cdot u_{22} = a_{32}$$
$$l_{32} = \frac{1}{u_{22}} \left[a_{32} - l_{31} \cdot u_{12} \right]$$
$$= \frac{3}{5} \left[2 - 1/3 \times 2 \right]$$
$$= 4/5$$

⑤

$$l_{31} \cdot u_{13} + l_{32} \cdot u_{23} + u_{33} = a_{33}$$
$$u_{33} = a_{33} - l_{31} \cdot u_{13} - l_{32} \cdot u_{23}$$
$$= 3 - 1/3 \times 1 - 4/5 \times 4/3$$
$$= 8/5$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1/3 & 4/5 & 1 \end{bmatrix} \quad \& \quad U = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 5/3 & 4/3 \\ 0 & 0 & 8/5 \end{bmatrix}$$

from (I) & (II)

$$LU \cdot X = B \quad \text{---} (3)$$

writing

$$U \cdot X = V \quad \text{---} (4)$$ from equ^n (3) become

$$L \cdot V = B \quad \text{---} (5)$$

from (5)

$$\begin{bmatrix} 1 & 0 & 0 \\ 2/3 & 1 & 0 \\ 1/3 & 4/5 & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \\ 14 \end{bmatrix}$$

on solving which we get

$$V_1 = 10$$

&

$$\frac{2}{3} V_1 + V_2 = 14$$

$$V_2 = 14 - \frac{2}{3} \times 10 = \frac{22}{3}$$

also,

$$\frac{1}{3} V_1 + \frac{4}{5} V_2 + V_3 = 14$$

$$V_3 = 14 - \frac{1}{3} \times 10 - \frac{4}{5} \times \frac{22}{3}$$

$$= \frac{14 \times 15 - 50 - 88}{15}$$

$$= \frac{24}{5}$$

$$\therefore \quad V = \begin{bmatrix} 10 \\ 22/3 \\ 24/5 \end{bmatrix}$$

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putting the value of v in (10) we get,

$$\begin{bmatrix} 3 & 2 & 1 \\ 0 & 5/3 & 4/3 \\ 0 & 0 & 8/5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 22/3 \\ 24/5 \end{bmatrix}$$

Solving by backward substitution

$$\frac{8}{5} z = \frac{24}{5} \Rightarrow z = 3$$

& $\frac{5}{3} y + \frac{4}{3} \times 3 = \frac{22}{3}$

$$y = 2$$

also, $3x + 2y + z = 10$

$$\Rightarrow x = \frac{1}{3}[10 - 4 - 3]$$

$$= 1$$

$$\left.\begin{array}{l} x = 1 \\ y = 2 \\ z = 3 \end{array}\right\} *$$

Solve the following system of equ^n by Crout's algorithm.

$$2x - 3y + 10z = 3$$
$$-x + 4y + 2z = 20$$
$$5x + 2y + z = -12$$

Writting the given equ^n in the form of

$$A * X = B \quad --- (I)$$

we get

$$A = \begin{bmatrix} 2 & -3 & 10 \\ -1 & 4 & 2 \\ 5 & 2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 3 \\ 20 \\ -12 \end{bmatrix}$$

Now, factorizing A into L & U by Crout's algorithm.

$$A = L \cdot U \quad --- (II)$$

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 10 \\ -1 & 4 & 2 \\ 5 & 2 & 1 \end{bmatrix}$$

Now, multiplying L & U and equating with element of A.

(1) $l_{11} = 2, \quad l_{21} = -1, \quad l_{31} = 5$

(2) $u_{12} = -3/2 \quad ; \quad u_{13} = 10/2 = 5$

(3) $l_{21} \times u_{12} + l_{22} = a_{22}$

$-1 \times -3/2 + l_{22} = 4$

$l_{22} = 5/2$

$l_{31} \cdot u_{12} + l_{32} = a_{32}$

$l_{32} = a_{32} - l_{31} \cdot u_{12}$

$= 2 - 5 \times (-3/2)$

$= 19/2$

(4) $l_{21} \cdot u_{13} + l_{22} \cdot u_{23} = a_{23}$

$$u_{23} = \frac{1}{l_{22}}\left[a_{22} - l_{21} \cdot u_{13}\right] = \frac{2}{5}\left[2-(-1)\times 5\right] = \frac{14}{5}$$

(5) $l_{31} \cdot u_{13} + l_{32} \cdot u_{23} + l_{33} = a_{33}$

$$l_{33} = a_{33} - l_{31} \cdot u_{13} - l_{32} \cdot u_{23}$$

$$= 1 - 5\times 5 - \frac{19}{2} * \frac{14}{5} \Rightarrow \frac{10 - 250 - 19\times 14}{10}$$

$$= -\frac{253}{5}$$

$$L = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 5/2 & 0 \\ 5 & 19/2 & -253/5 \end{bmatrix} \quad \& \quad U = \begin{bmatrix} 1 & -3/2 & 5 \\ 0 & 1 & 14/5 \\ 0 & 0 & 1 \end{bmatrix}$$

Now, from (1) & (2), $L \cdot U \cdot X = B$ —— (3)

writting $U \cdot X = V$ —— (4) we get

$$L \cdot V = B \text{ —— (5)}$$

$$\Rightarrow \begin{bmatrix} 2 & 0 & 0 \\ -1 & 5/2 & 0 \\ 5 & 19/2 & -253/5 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 20 \\ -12 \end{bmatrix}$$

on solving which we get,

$$2V_1 = 3$$

$$V_1 = 3/2$$

and, $-V_1 + \frac{5}{2} V_2 = 20$

$$V_2 = \frac{2}{5}\left(20 + \frac{3}{2}\right) = \frac{43}{5}$$

also, $5v_1 + \frac{19}{2}v_2 - \frac{253}{5}v_3 = -12$

$$v_3 = \left(-12 + \frac{15}{2} - \frac{43}{5} \times \frac{19}{2}\right) \frac{-5}{253}$$

$$v_3 = 2$$

Hence $v = \begin{bmatrix} 3/2 \\ 43/5 \\ 2 \end{bmatrix}$

puting the value of v in equ^n (1)

$$U \cdot X = V$$

$$\begin{bmatrix} 1 & -3/2 & 5 \\ 0 & 1 & 14/5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3/2 \\ 43/5 \\ 2 \end{bmatrix}$$

$$z = 2,$$

$$y + \frac{28}{5} = \frac{43}{5}$$

$$y = 3$$

$$x - \frac{9}{2} + 10 = \frac{3}{2}$$

$$x = -4$$

$$\left.\begin{array}{l} x = -4 \\ y = 3 \\ z = 2 \end{array}\right\} \#$$

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Q.) factorize the given matrix by Cholesky decomposition :-

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 22 \\ 3 & 22 & 82 \end{bmatrix}$$

Given,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 22 \\ 3 & 22 & 82 \end{bmatrix}$$

$A = L \cdot L^{-1} = U^T \cdot U$

$\therefore \quad A = L \cdot L^T$

$$= \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{22} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 8 & 8 & 22 \\ 3 & 22 & 82 \end{bmatrix}$$

(1) $l_{11}^2 = 1$; $l_{11} = 2$ | $l_{21} \cdot l_{11} = 2$; $l_{21} = 2$ | $l_{31} \cdot l_{11} = a_{31}$; $l_{31} = 3$

(2) $l_{21}^2 + l_{22}^2 = 8$

$l_{22} = \sqrt{8-4}$

$= 2$

(3) $l_{31} \cdot l_{21} + l_{32} \cdot l_{22} = a_{32}$

$l_{32} = \frac{1}{l_{22}} \left[a_{32} - l_{31} l \right.$

$= \frac{1}{2} [22 - 3 \times 2]$

$= 8$

(iii) $l_{31}^2 + l_{32}^2 + l_{33}^2 = a_{33}$

$l_{33} = \sqrt{82+9-64} = 3$

$$\therefore L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 8 & 3 \end{bmatrix} \quad \& \quad L^T = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 8 \\ 0 & 0 & 3 \end{bmatrix}$$

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Iterative methods:

In this method the result is obtained by repeating the same step of calculation ~~unit~~ until we get the results correct upto desired accuracy. The various iterative techniques are:

1. Jacobi-iteration method
2. Gauss-seidal method
3. Relaxation method

Jacobi-iteration method

Consider the given system of eqn,

$$\left.\begin{aligned} a_1x+b_1y+c_1z=d_1 \\ a_2x+b_2y+c_2z=d_2 \\ a_3x+b_3y+c_3z=d_3 \end{aligned}\right\} \quad \text{——— (1)}$$

Assuming that a_1, b_2 and c_3 be the largest coefficients among the all coeff. then eqn (1) can be written as

$$\left.\begin{aligned} x &= \frac{1}{a_1}(d_1-b_1y-c_1z) \\ y &= \frac{1}{b_2}(d_2-c_2z-a_2x) \\ z &= \frac{1}{c_3}(d_3-b_3y-a_3x) \end{aligned}\right\} \quad \text{——— (2)}$$

Now, starting with initial value of x, y & z as x_0, y_0 & z_0 which can be taken as zero and evaluating (2) we get,

$$x_1 = \frac{1}{a_1}(d_1 - b_1 y_0 - c_1 z_0)$$

$$y_1 = \frac{1}{b_2}(d_2 - a_2 x_0 + c_2 z_0) \quad \text{--- (3)}$$

$$z_1 = \frac{1}{a_3}(d_3 - a_3 x_0 - b_3 y_0)$$

Again, putting the value of x_1, y_1 & z_1 from (3) to equ^n (2), we get

$$x_2 = \frac{1}{a_1}(d_1 - b_1 y_1 - c_1 z_1)$$

$$y_2 = \frac{1}{b_2}(d_2 - a_2 x_1 - c_2 z_1) \quad \text{--- (4)}$$

$$z_2 = \frac{1}{a_3}(d_3 - a_3 x_1 - b_3 y_1)$$

This process is repeated until we get the value of x, y, z correct upto desired accuracy

i.e, $|x^{(n)} - x^{(n-1)}| \simeq 0$. Hence, the process is called Jacobi-iteration or Gauss Jacobi method.

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Q.1) Solve the given system of eqn by Jacobi-iteration method

$$20x + y - 2z = 17$$
$$3x + 20y - z = -18$$
$$2x - 3y + 20z = 25$$

Solⁿ: Writing the given system of eqn in terms of x, y, z we get,

$$x = \frac{1}{20}(17 - y + 2z)$$

(1) $$y = \frac{1}{20}(-18 - 2x + z)$$

$$z = \frac{1}{20}(25 - 2x + 3y)$$

Now, starting with $x_0 = y_0 = z_0 = 0$ and solving ① by using Jacobi-iteration method successively,

No. of iteration	$x = \frac{1}{20}(17-y+2z)$	$y = \frac{1}{20}(-18-2x+z)$	$z = \frac{1}{20}(25-2x+3y)$	
0	0	0	0	
1	0.85	-0.9	1.25	
2	1.02	-0.965	1.03	
3	1.00125	-1.0015	1.00325	
4	1.0004	-1.000025	0.999725	
5	0.99997	-1.00007875	0.999956	$x^{(5)}$
6	0.999999	0.99999	0.99999	$x^{(6)}$

At 6th $|x^{(6)} - x^{(5)}| \simeq 0$

2) Gauss - seidal method

It is the improvement over the Jacobi iteration method such that the value of x, y and z are placed immediately to the equ as soon as it is found insted of wait for next iteration like Jacobi method. Hence, the method is faster than Jacobi method.

$$x_1 = \frac{1}{a_1}(d_1 - b_1 y_0 - c_1 z_0)$$

$$y_1 = \frac{1}{b_2}(d_2 - a_2 x_1 - c_2 z_0)$$

$$z_1 = \frac{1}{c_3}(d_3 - a_3 x_1 - b_3 y_1)$$

2016 spring

Solve by using Gauss- seidal method

$$10x_1 + 6x_2 - 5x_3 = 27$$
$$3x_1 + 8x_2 + 10x_3 = 27$$
$$4x_1 + 10x_2 + 3x_3 = 27$$

Writting the given system of eqn in terms of x_1, x_2 & x_3

$$x_1 = \frac{1}{10}(27 - 6x_2 + 5x_3)$$

$$x_2 = \frac{1}{10}(27 - 3x_3 - 4x_1)$$

$$x_3 = \frac{1}{10}(27 - 8x_2 - 3x_1)$$

Now, starting with $x_1 = x_2 = x_3 = 0$ and solving the given system of eqn by appling Gauss-seidal method

No. of iteration	$x_1 = \frac{1}{10}(27 - 6x_2 + 5x_3)$	$x_2 = \frac{1}{10}(27 - 4x_1 - 3x_3)$	$x_3 = \frac{1}{10}(27 - 3x_1 - 8x_2)$
0	0	0	0
1	2.7	1.62	0.594
2	2.025	1.7118	0.784306
3	2.03445	1.6693	0.75422
4	2.0755	1.6435	0.7625
5	2.095	1.63319	0.7649
6	2.1025	1.6295	0.7656
7	2.1051	1.6282	0.7658

iii) Relaxation method

Starting $a_1x + b_1y + c_1z = d_1$
$a_2x + b_2y + c_2z = d_2$ ...... (1)
$a_3x + b_3y + c_3z = d_3$

Then residuals of the given system is written as,

$$R_x = d_1 - a_1x - b_1y - c_1z$$
$$R_y = d_2 - a_2x - b_2y - c_2z$$
$$R_z = d_3 - a_3x - b_3y - c_3z$$

The operation table for residual is written as,

	δR_x	δR_y	δR_z
$\delta x = 1$	$-a_1$	$-a_2$	$-a_3$
$\delta y = 1$	$-b_1$	$-b_2$	$-b_3$
$\delta z = 1$	$-c_1$	$-c_2$	$-c_3$

Starting from $x = y = z = 0$ if x is increased by small value 1 keeping y and z constant the R_x, R_y and R_z will decrease by a_1, a_2 & a_3 respectively other values constant is shown in above operation table.

In other word, to reduce R_x by a particular value p, x should be increased by a factor p/a_1.

Now, reducing largest residuals among R_x, R_y & R_z step by step with the help of operation table to make all the residuals almost zero. Finally the value of unknowns are determined by adding all the successive increments. i.e, $x = \Sigma \delta x$, $y = \Sigma \delta y$ & $z = \Sigma \delta y$

∴ This process is called Relaxation method.

Q) Solve the given system of eqn by Relaxation method.

$$20x + y - 2z = 17$$
$$3x + 20y - z = 18$$
$$2x - 3y + 20z = 25$$

Soln: the residuals for given eqn is,

$$R_x = 17 - 20x - y + 2z$$
$$R_y = 18 - 3x - 20y + z$$
$$R_z = 25 - 2x + 3y - 20z$$

The operation table for residuals are :

	δR_x	δR_y	δR_z
$\delta x = 1$	-20	-3	-2
$\delta y = 1$	-1	-20	3
$\delta z = 1$	2	1	-20

Now, the ~~retaxation~~ relation table is obtained as,

Increments	R_x	R_y	R_z
$x=y=z=0$	17	18	28
$\delta z = 1$	18	19	15
$\delta x = 0.99$	0	16.15	3.1
$\delta y = 0.8075$	-10.8075	0	5.5225
$\delta z = 0.277629$	-0.25275	0.277629	0
$\delta y = 0.013881$	-0.26613	0	0.04164 ~~0.01388~~
$\delta x = -0.0133$	0	0.0399	0.06824

Hence,

$$x = \Sigma \delta x = 0.9362$$
$$y = \Sigma \delta y = 0.8213$$
$$z = \Sigma \delta z = 1.277628$$

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Eigen value & eigen vector :

- Eigen vectors are those vectors (non-zero) which do not changes the direction when any linear transformation is applied. It changes only by a scale factor.
- In other word if A is a linear transformation from a vector space V & x is a vector in V, which is not a zero vector, then V is an eigen vector of A if $A * x$ is a scalar multiple of x.
- Eigen values are the special set of scalars associated with the system of linear eqn which is a scalar quantity that is used to transform eigen vector.

★ Properties of eigen values :-

↳ Eigen values are real & symmetric.

↳ Eigen values of unitary & orthogonal matrix are of unit modulus $|\lambda| = 1$.

↳ Eigen value of A = Eigen value of A^T

↳ Sum of Eigen value A = Traces of A (sum of diagonal of A)

↳ Product of Eigen values = $|A|$.

Power method

↳ It is an iterative method to find larger or smallest eigen value & corresponding eigen vector the largest eigen value & corresponding eigen vector is determined by using formula.

$$\boxed{A * x = \lambda * x} \quad \text{--- (1)}$$

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Where A = Any given matrix
λ = Eigen value
x = Eigen vector

Now, multiplying both side with A^{-1} in ① we get

$$\Rightarrow A * A^{-1} * x = A^{-1} * \lambda * x$$

$$\Rightarrow \boxed{A^{-1} * x = \frac{1}{\lambda} * x} \quad ②$$

Which gives the smallest eigen value & corresponding eigen vector.

Q.) Determine the largest eigen value & corresponding eigen vector of the matrix

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix}$$

Soln: Starting with initial vector $x^{(0)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and

Applying formula of power method, we have

$$A * x = \lambda * x$$

Iteration 1,

$$A * X^{(0)} = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} = 3 \begin{bmatrix} 1/3 \\ 1 \\ -1/3 \end{bmatrix}$$

$$\lambda^{(1)} = 3, \quad X^{(1)} = \begin{bmatrix} 0.3333 \\ 1 \\ -0.3333 \end{bmatrix}$$

Iteration 2,

$$A * X^{(1)} = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix} \begin{bmatrix} 1/3 \\ 1 \\ -1/3 \end{bmatrix} = \begin{bmatrix} 3.666 \\ 1.666 \\ -0.333 \end{bmatrix} = 3.666 \begin{bmatrix} 1 \\ 0.4545 \\ 0.0909 \end{bmatrix}$$

$$\lambda^{(2)} = 3.6666, \quad X^{(2)} = \begin{bmatrix} 1 \\ 0.4545 \\ 0.0909 \end{bmatrix}$$

Iteration 3,

$$A * X^{(2)} = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix} \begin{bmatrix} 1 \\ 0.4545 \\ 0.0909 \end{bmatrix} = \begin{bmatrix} 2.2726 \\ 4.2726 \\ 1.727 \end{bmatrix} = 4.2726 \begin{bmatrix} 0.5319 \\ 1 \\ 0.4042 \end{bmatrix}$$

$$\lambda^{(3)} = 4.2726, \quad X^{(3)} = \begin{bmatrix} 0.5319 \\ 1 \\ 0.4042 \end{bmatrix}$$

Iteration 4,

$$A * X^{(3)} = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix} \begin{bmatrix} 0.5319 \\ 1 \\ 0.4042 \end{bmatrix} = \begin{bmatrix} 3.1277 \\ 5.2125 \\ 7.5101 \end{bmatrix} = 7.5101 \begin{bmatrix} 0.4164 \\ 0.6943 \\ 1 \end{bmatrix}$$

$$\therefore \lambda^{(4)} = 7.5101, \quad X^{(4)} = \begin{bmatrix} 0.4164 \\ 0.6943 \\ 1 \end{bmatrix}$$

Iteration 5,

$$A * X^{(4)} = \begin{bmatrix} 1 & 3 & -1 \\ 3 & 2 & 4 \\ -1 & 4 & 10 \end{bmatrix} \cdot \begin{bmatrix} 0.4164 \\ 0.6943 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.4993 \\ 6.6378 \\ 12.36 \end{bmatrix} = 12.36 \begin{bmatrix} 0.1213 \\ 0.5370 \\ 1 \end{bmatrix}$$

$$\lambda^{(5)} = 12.36 \quad \& \quad X^{(5)} = \begin{bmatrix} 0.1213 \\ 0.5370 \\ 1 \end{bmatrix}$$

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Soln of the ordinary differential equation:

Introduction:

Soln to the ordinary differential equn are of 2 types:

(A) Soln to ~~...~~ initial value problem

(B) Soln to Boundary value problem.

(A) Soln to initial value problem.

The ordinary differential equn $dy/dx = f(x,y)$ defined over single point $x = x_0$ then it is called initial value problem. The techniques of solving initial value problems are:-

(i) Picard's method / method of integration.

(ii) Taylor's series method / method of differentiation.

(iii) Euler's method / R-K 1st order method.

(iv) Modified Euler's method / Heun's method / R-K 2nd order method

(v) R-K 4th order / Range Kutta method.

(B) Boundary value problem:-

If any ordinary differential equn $dy/dx = f(x,y)$ is define over more value problems are:

(1) finite-difference method (2) Shooting method.

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Picard's method

Consider 1st order ordinary differential equ^n $\frac{dy}{dx}=f(x,y)$ with initial condition, $y(x_0)=y_0$ i.e. $y=y_0$ when $x=x_0$ then, equ^n ① can be written as,

$$dy = f(x,y)\,dx$$

Integrating on both side taking limits, we get:

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x,y)\,dx$$

$$\Rightarrow y-y_0 = \int_{x_0}^{x} f(x,y)\,dx$$

$$\Rightarrow y = y_0+\int_{x_0}^{x} f(x,y)\,dx \quad \text{--- ②}$$

putting $y=y_0$ in $f(x,y)$ equ^n ② gives,

$$y_1 = y_0+\int_{x_0}^{x} f(x,y_0)\,dx \quad \text{--- ③}$$

Now, putting $y=y_1$ in $f(x,y)$ equ^n ② becomes,

$$y_2 = y_0+\int_{x_0}^{x} f(x,y_1)\,dx \quad \text{--- ④}$$

Similarly proceeding upto n^{th} term we get,

$$\boxed{y_n = y_0+\int_{x_0}^{x} f(x,y_{n-1})\,dx} \quad \text{--- ⑤}$$

which is generalized formula of picards method. This process is repeated until the value of y for two consicutine iterations are same or correct upto desired accuracy.

Q.) Using picards method determine the value of y at $x = 0.1, 0.2, 0.3$ for the first order ordinary different equ^n : $\frac{dy}{dx} = 1+xy$ with initial Cond^n $y(0) = 1$

Sol^n:- Given, $\frac{dy}{dx} = 1+xy$ ---- ①

integrating both side, we get

$$\int_1^y dy = \int_0^x (1+xy)\,dx$$

$$\Rightarrow y = 1 + \int_0^x (1+xy)\,dx \text{ ---- ②}$$

Now, putting $y = y_0 = 1$ in ② then,

$$y_1 = 1 + \int_0^x (1+x)\,dx$$

$$y_1 = 1 + x + \frac{x^2}{2}$$

Again putting $y = y_1$ in ②, then,

$$y_2 = 1 + \int_0^x \left(1 + x + x^2 + \frac{x^3}{2}\right) dx$$

$y_2 = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8}$

Similarly,

putting $y = y_2$ in ② we get

$$y_3 = 1 + \int_0^1 \left(1 + x + x^2 + \frac{x^3}{2} + \frac{x^4}{3} + \frac{x^5}{8}\right) dx$$

$$y_3 = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{6} + \frac{x^5}{15} + \frac{x^6}{48}$$

At $x = 0.1$, $y_1 = 1.105$ $\quad y_2 = 1.1053$ $\quad y_3 = 1.10534$

$x = 0.2$, $y_1 = 1.22$ $\quad y_2 = \cancel{1.2053}$ 1.2228 $\quad y_3 = 1.22288$

$x = 0.3$, $y_1 = 1.345$ $\quad y_2 = 1.3950$ $\quad y_3 = 1.35518$

$\therefore y(0.1) = 1.10534$

$y(0.2) = 1.22288$

$y(0.3) = 1.35518$

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② Taylor's series method

Consider 1st order ordinary differential eqn $dy/dx = f(x,y)$ — ①

with the initial condition $y(x_0) = y_0$. The it's solution by taylor's series is given by,

$$y(x) = y(0) + \frac{(x-x_0)}{1!} y'(0) + \frac{(x-x_0)^2}{2!} y''(0) + \frac{(x-x_0)^3}{3!} y'''(0)$$

$$+ \frac{(x-x_0)^4}{4!} y^{iv}(0) + \cdots \quad ②$$

Hence the method is also known as method of differentiation.

Q) Using Taylor's series solve $dy/dx = x^2 - y$ with the initial condition $y(0) = 1$, to determine the value of y at $x = 0.1, 0.2, 0.3$.

Soln: Given $x_0 = 0$

$y_0 = 1$

$y' = x^2 - y \quad y'(0) = -1$

diff. y' successively and putting the initial condition we get,

$y'' = 2x - y' \Rightarrow y''(0) = 1$

$y''' = 2 - y'' \Rightarrow y'''(0) = 1$

$y^{iv} = -y''' \Rightarrow y^{iv}(0) = -1$

Now, Applying Taylor's series we get.

$$y(x) = y(0) + (x - x_0)\, y'(0) + \frac{(x - x_0)^2}{2!} y''(0) + \frac{(x - x_0)^3}{3!} y'''(0)$$

$$+ \frac{(x - x^4)}{4!} y^{iv}(0) + \cdots$$

$$y(x) = 1 - x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$$

At $x = 0.1$ $\quad y(0.1) = 0.905$

at $x = 0.2$ $\quad y(0.2) = 0.8212$

at $x = 0.3$ $\quad y(0.3) = 0.0.7491$

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3) Euler's method / R-K 1st order

Euler's method is the simple one-step method and has a limited application because of its low accuracy.

Consider the differential equ^n,

$$\frac{dy}{dx} = f(x,y) \quad \text{--- (1)}$$

Where, $y(x_0) = y_0$

Suppose that we wish to find successively $y_1, y_2, y_3 \dots y_n$ where y_n is the value of y corresponding to $x = x_n$ where $x_n = x_0 + nh$ for $n = 1, 2, 3 \dots$ h being small. Here, we use the property that in a small interval, Thus in the interval x to x_1 at x_1 we approximate the curve by the tangent at the point (x_0, y_0).

Therefore the equ^n of tangent at (x_0, y_0) is,

$$y - y_0 = \left(\frac{dy}{dx}\right)_{(x_0, y_0)} (x - x_0)$$

$$y = y_0 + f(x_0, y_0) \cdot (x - x_0) \quad \text{--- (2)}$$

putting $x = x_1$ & $y = y_1$ equ^n (2) becomes,

$$y_1 = y_0 + f(x_0, y_0)(x_1 - x_0)$$

$$\Rightarrow \boxed{y_1 = y_0 + h * f(x_0, y_0)} \quad \text{--- (3)}$$

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Now starting from point (x_1, y_1) equ^n ② becomes

$$y_2 = y_1 + h \cdot f(x_1, y_1) \quad \text{--- ⑦}$$

Similarly, proceeding from n^{th} term,

$$y_n = y_{n-1} + h \cdot f(x_{n-1}, y_{n-1}) \quad \text{--- ⑧}$$

Which is the generalized formula of Euler's method for solving 1st order ordinary ~~eq~~ differential equ^n.

Q) Given $\frac{dy}{dx} = \frac{y-x}{y+x}$ with $y = 1$ for $x = 0$. Find y approximat for $x = 0.1$ by euler method.

Sol^n: Given, $\frac{dy}{dx} = f(x,y) = \frac{y-x}{y+x}$

$x_0 = 0$

$y_0 = 1$

$x_n = 0.1$

$y_n = ?$

Taking $n = 5$, $h = \frac{x_n - x_0}{5} = \frac{0.1}{5} = 0.02$

Applying Euler's method

$$y_1 = y_0 + h \cdot f(x_0, y_0)$$

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no. of iteration	A x_0	B y_0	$C = \frac{B-A}{B+A}$ $f(x_0, y_0)$	$y = B + 0.02 * C$ $y_1 = y_0 + h * f(x_0, y_0)$
0	0	1		
1	0.02	1.02		
2	0.04	1.02		
3	0.06	1.0392		
4	0.08	1.0577		
5	0.1	1.0755		

(4) R-K 2nd order / modified Euler's method / Heun's method.

Consider the 1st order ordinary differential eqn

Q.) Given $\frac{dy}{dx} = x^2 - y$, $y(0) = 1$. Find $y(0.1)$, $y(0.2)$ using R-K 2nd order method.

Soln: Given, $\frac{dy}{dx} = x^2 - y$

$y_0 = 1$

$x_0 = 0$

$y(0) = 1$

$y(0.1) = ?$

$y(0.2) = ?$

$K_1 = hf(x_0, y_0)$
$= 0.1 f(0, 1)$
$= 0.1 \times (0^2 - 1)$
$= -0.1$

$K_2 = hf(x_0 + h, y_0 + $
$= 0.1 \times f(0.1, 0.9$
$= 0.1 (0.1^2 - 0.9$
$= 0.089$

$x_0 = 0.1$

$y_1 = y_0 + \frac{1}{2}(K_1 + K_2)$

$y(x_0 + h) = 1 + \frac{1}{2}(-0.1 - 0.089)$

$= 0.9055$

Iteration max 2 step
$h \geq 0.25$ ×

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1st Iteration,

$x_0 = 0.1$

$y_0 = 0.9055$

$K_1 = h \cdot f(x_0, y_0)$	$K_2 = h \cdot f(x_0+h, y_0+K_1)$
$= 0.1\, f(0.1, 0.9055)$	$= 0.1\, f(0.2, 0.81595)$
$= 0.1\,(0.1^2 - 0.9055)$	$= 0.1 * (0.2^2 - 0.81595)$
$= -0.08955$	$= 0.077595$

$\therefore\ y_1 = y_0 + \frac{1}{2}(K_1 + K_2)$

$y(x_0+h) = 0.9055 + \frac{1}{2}(-0.08955 - 0.077595)$

$y(0.2) = 0.8219275$

$y(0.1) = 0.9055$

R-K 4^{th} order for $y(0.2)$

$$\frac{dy}{dx} = xy + y^2 \quad \text{with } y(0) = 1$$

$$\frac{dy}{dx} = xy + y^2$$

$x_0 = 0$

$y_0 = 1$

taking $h = 0.1$, and applying 4^{th} order,

1^{st} iteration.

$K_1 = h \cdot f(x_0, y_0)$

$= 0.1\, f(0, 1)$

$= 0.1 \times (0 * 1 + 1^2)$

$= 0.1$

$K_2 = h \cdot f(x_0 + h/2, y_0 + K_1/2)$

$= 0.1\, f(0.05, 1.05)$

$= 0.1 * (0.05 * 1.05 + 1.05$

$= 0.1155$

$K_3 = h f(x_0 + h/2, y_0 + K_2/2)$

$K_4 = h f(x_0 + h/2 + y_0 + K_3)$

$$y_1 = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

Boundary value problem :-

Shooting Method

In this method, Boundary value problem is transformed into initial value problem. This initial value problem is solved by Taylor's series method or either of R-K method. Finally boundary value problem is solved.

Consider the boundary value problem.

$$y''(x) = y(x) \; ; \; y(a) = A \text{ and } y(b) = B \quad ----(1)$$

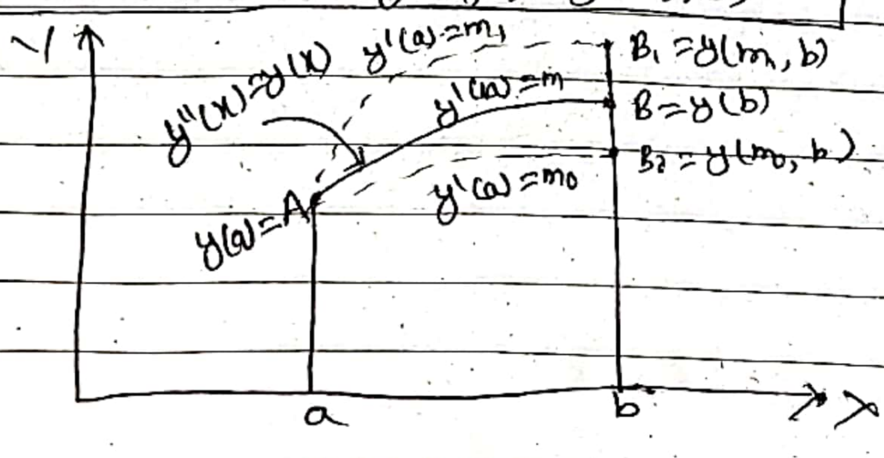
One condition is $y(a) = A$ and let us assume $y'(a) = m$ which represents slope we starts with the initial guess form, then find values of $y(b)$ using any initial value method.

Consider two initial guess be m_0 & m_1 so corresponding values of $y(b)$ are $y(m_0, b)$ and $y(m_1, b)$. Assuming the value of m & $y(b)$ are linearly related, we obtain better approximation, m_2 for m using relation.

$$\frac{m_2 - m_1}{y(b) - y(m_1, b)} = \frac{m_1 - m_0}{y(m_1, b) - y(m_0, b)}$$

Which gives,

$$\boxed{m_2 = m_1 - (m_1 - m_0)\frac{y(m_1, b) - y(b)}{y(m_1, b) - y(m_0, b)}}$$



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We, now solve the initial value problem $y''(x) = y(x)$; $y(a) = A$ & $y'(a) = m_2$ and obtain the solⁿ $y(m_2, b)$
This process is repeated until $y(m_n, b)$ matches with $y(b)$.
Hence the process is called shooting method.

Q.) Use shooting method to solve boundary value problem
$y''(x) = y(x)$, $y(0) = 0$ & $y(1) = 1.17$

Solⁿ: Given,

$$y''(x) = y(x) \quad \text{----} \quad (1)$$

$$y(a) = y(x_0) = y_0$$
$$x_0 = 0$$
$$y_0 = 0$$
$$y(b) = y(x_n) = y_n$$
$$x_n = 1$$
$$y_n = 1.17$$
$$y(b) = 1.17$$

Assuming
$y'(a) = y'(0) = m$ be the slope of eqⁿ then,
$$y''(0) = y(0) \Rightarrow y''(0) = 0$$

Now, differentiating (1) sucessively w.r.t. x and putting the initial values to it, we get.

$$y'''(x) = y'(x) \Rightarrow y'''(0) = y'(0) = m$$
$$y^{iv}(x) = y''(x) \Rightarrow y^{iv}(0) = y''(0) = 0$$
$$y^{v}(x) = y'''(x) \Rightarrow y^{v}(0) = y'''(0) = m \quad \text{and so on.}$$

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Now, the solution by Taylor's series method is,

$$y(x) = y(0) + (x-x_0)\, y'(0) + \frac{(x-x_0)^2}{2!} y''(0) + \frac{(x-x_0)^3}{3!} y'''(0) + \cdots$$

$$y(x) = m\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots\right) \quad \cdots (11)$$

$\therefore$ At $x=1$

$$y(1) = m\left(1 + \frac{1}{3!} + \frac{1}{5!} + \frac{1}{7!} + \cdots\right)$$

Assuming $m_0 = 0.3$ & $m_1 = 1$, be the initial value for Slope m. then,

$$y(m_0, 1) = 0.3525, \quad \& \quad y(m_1, 1) = 1.1751$$

Now, Applying Shooting method, m_2 we get

$$m_2 = m_1 - (m_1 - m_0)\frac{y(m_1, b) - y(b)}{y(m_1, b) - y(m_0, b)}$$

$$= 1 - (1 - 0.3)\frac{1.1751 - 1.17}{1.1751 - 0.3527}$$

$$= 0.99565$$

and, $y(m_2, b) = 1.17008 \approx y(b)$

$$\therefore y'(0) = 0.99565$$

Hence, Generalized sol^n is,

$$y(x) = 0.99565\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots\right)$$

Chapter-6

Solution to the partial differential equ^n

Introduction

The second order partial differential is of the form,

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + B(x,y)\frac{\partial^2 u}{\partial x \cdot \partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} + F\left(x,y, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0 \quad \text{--- (1)}$$

is of

(i) elliptic type if $B^2 - 4AC < 0$

(ii) Parabolic type if $B^2 - 4AC = 0$

(iii) hyperbolic type if $B^2 - 4AC > 0$

Solution to the elliptic equation

Elliptic equ^n are two types :

1) Laplace equ^n :

The equ^n of the form,

$$\left.\begin{aligned} &\nabla^2 u = 0 \\ \text{or } &\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \\ \text{or } &u_{xx} + u_{yy} = 0 \end{aligned}\right\} \text{is called laplace equ}^n.$$

2) Poisson's equ^n :

The equ^n of the form

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$$\nabla^2 u = f(x,y)$$
or $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial^2 y} = f(x,y)$$
or $$u_{xx} + u_{yy} = f(x,y)$$
} is called poisson's equⁿ

Solution to laplace equⁿ

Consider the laplace equⁿ

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad ----- (1)$$

Assume, a rectangular region for which $u(x,y)$ is known at boundary, Divide this region into network of square mesh of side h. Replacing the derivatives in (1) by their difference approximation. We have,

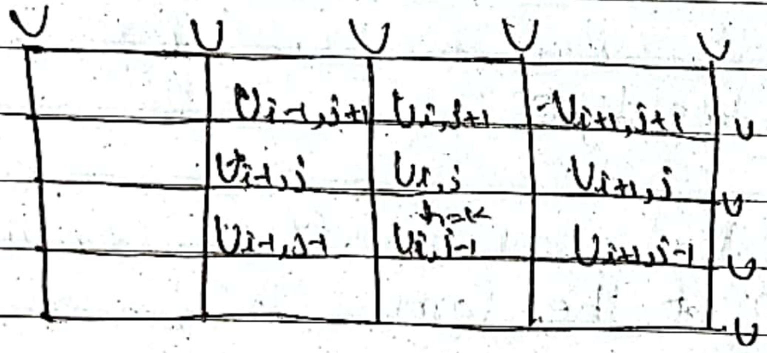


fig: network of square mesh at which laplace equⁿ is applied.

from figure,

$$\frac{\partial u}{\partial x} = \frac{u_{i,j} - u_{i-1,j}}{h} = \frac{u_{i+1,j} - u_{i,j}}{h}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = \frac{\frac{u_{i+1,j} - u_{i,j}}{h} - \frac{u_{i,j} - u_{i-1,j}}{h}}{h}$$

$$\frac{\partial^2 x}{\partial x^2} = \frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{h^2}$$

Also,

$$\frac{\partial^2 U}{\partial y^2} = \frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{k^2}$$

putting these value in eqn ① we get

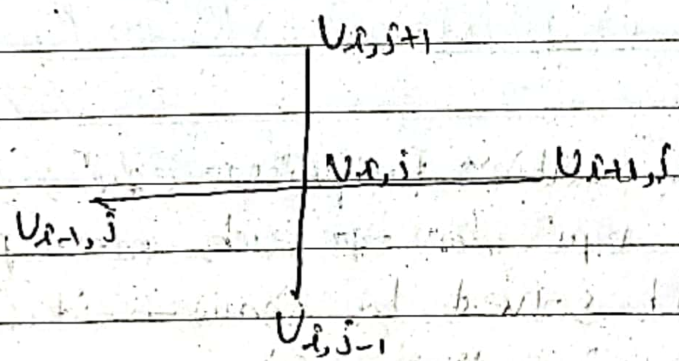
$$\frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{h^2} + \frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{k^2} = 0$$

for square mesh $h = k$

$$\frac{U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1} - 4U_{i,j}}{h^2} = 0$$

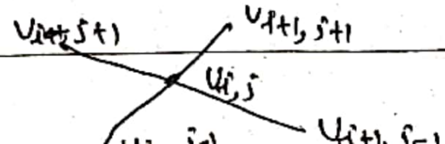
$$\therefore U_{i,j} = \frac{1}{4}\left[U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1}\right] \quad \text{——— ②}$$

Which is known as std. five point formula.



Also, diagonal five point formula is given by

$$U_{i,j} = \frac{1}{4}\left[U_{i+1,j+1} + U_{i-1,j+1} + U_{i+1,j-1} + U_{i-1,j-1}\right] \quad \text{——— ③}$$



Now, equ^n for each mesh point are determined by using either of formula explained above. And the simultaneous equ^n are solved by using gauss seidal method or any of iterative method for solving simultaneous linear equ^n. Hence laplace equ^n is solved.

Solution to the poisson equ^n

Consider the poisson equ^n

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y)$$

assuming this equ^n is applied over the rectangular region with square mesh of side h. The replacing the derivatives by corresponding approximated differences, we get,

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2} = f(ih, jh)$$

$$\therefore \boxed{u_{i+1,j} + u_{i-1,j} + u_{i,j-1} + 4u_{i,j} = f(ih, jh)h^2} \quad \text{--- (2)}$$

which is the solution for poisson equ^n.

The equ^n for ~~eqn~~ each mesh points are derived using (2) and solved by Gauss-seidal or any of iterative method for solving linear simultaneous equ^n.

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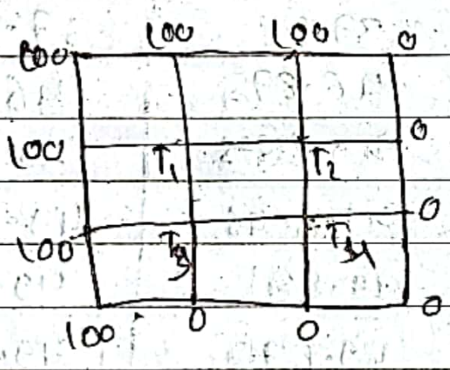
8) The steady state 2 dimensional heat flow in a metal plate of size 30 * 30 cm is defined by $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$

~~To~~ Two adjacent sides are placed at 100°C and other two sides are at 0°C. Find tempr at inner point assuming grid size of 10 * 10.

Sol^n: The laplace



Assuming T_1, T_2, T_3 & T_4 be the inner mesh points,
The solution to laplace is given by,

$$T_{i,j} = \frac{1}{4}\left[T_{i-1,j} + T_{i+1,j} + T_{i,j+1} + T_{i,j-1}\right] \quad \text{——— (I)}$$

At point T_1,

$$T_1 = \frac{200 + T_2 + T_3}{4} \quad \text{——— (II)}$$

At point T_2,

$$T_2 = \frac{T_1 + T_4 + 100}{4} \quad \text{——— (III)}$$

At point T_3,

$$T_3 = \frac{T_1 + T_4 + 100}{4} \quad \text{——— (IV)}$$

At point T_4,

$$T_4 = \frac{T_2 + T_3}{4} \quad \text{——— (v)}$$

Assuming $T_1 = T_2 = T_3 = T_4 = 0$ and solving by using gauss seidal method, we get.

no. of itr	$T_1 = \frac{T_2 + T_3 + 200}{4}$	$T_2 = \frac{T_1 + T_4 + 100}{4}$	$T_3 = \frac{T_1 + T_4 + 100}{4}$	$T_4 = \frac{T_2 + T}{4}$
0	0	0	0	0
1	50	37.5	37.5	18.75
2	68.75	46.875	46.875	23.43
3	73.4375	49.21875	49.21875	24.009
4.	74.609	49.8046	49.8046	24.90
5.	74.9028	49.9511	49.9511	24.97
6.	74.9751	49.877	49.9877	24.99
7.	74.9938	49.9877	49.877	24.98
8.	74.9984	49.999	49.999	24.999

$\therefore$ At 8^{th} iteration $|x^{(8)} - x^{(7)}| \approx 0$

$$\therefore \left.\begin{aligned} T_1 &= 74.9984 \approx 75 \\ T_2 &= 49.999 \approx 50 \\ T_3 &= 49.999 \approx 50 \\ T_4 &= 24.999 \approx 25 \end{aligned}\right\}$$

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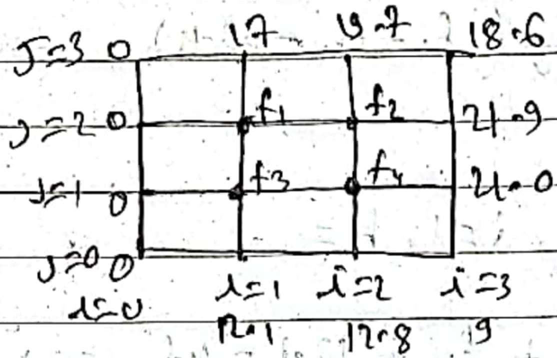
Solve the poisson's equ^n $\nabla^2 f = 4x^2y + 3x^2y^2$, over the square domain of $0 \le x \le 3$ and $0 \le y \le 3$ with f on the boundary is given in fig. below. Take $h = k = 1$.



Sol^n:- The poisson's equ^n applied over the square region is,

$$\nabla^2 f = 4x^2y + 3x^2y^2$$

$$\text{i.e., } f(x,y) = 4x^2y^2 + 3x^2y^2$$

The sol^n of poisson's equ^n is given by,

$$f_{i-1,j} + f_{i+1,j} + f_{i,j+1} + f_{i,j-1} - 4f_{i,j} = h^2 f(ih, jh)$$

Assuming f_1, f_2, f_3 & f_4 being the interval mess points then at point f_1,

$i = 1$, $j = 2$, $h = 1$

$$\Rightarrow 0 + 17 + f_2 + f_3 - 4f_1 = 1^2 f(1,2)$$

$$\Rightarrow f_1 = \frac{1}{4}(f_2 + f_3 - 3) \quad \text{—— (1)}$$

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at point f_2

$i=2, j=2, h=1$

$$f_1 + 19.7 + 21.9 + f_4 - 4f_2 = i^2 f(2,2)$$

$$\Rightarrow f_2 = \frac{1}{4}(f_1 + f_4 - 38.4) \quad \text{(2)}$$

at point f_3

$i=1, j=1, h=1$

$$0 + f_1 + f_4 + 12.1 - 4f_3 = 1 f(1,1)$$

$$f_3 = \frac{1}{4}(f_1 + f_4 + 5.1) \quad \text{(3)}$$

at point f_4

$i=2, j=1, h=1$

$$f_3 + f_2 + 21.0 + 12.8 - 4f_4 = 1 f(2,1)$$

$$f_4 = \frac{1}{4}(f_2 + f_3 + 5.8) \quad \text{(4)}$$

no of itr	$f_1 = \frac{1}{4}(f_2+f_3-3)$	$f_2 = \frac{1}{4}(f_1+f_4-38.4)$	$f_3 = \frac{1}{4}(f_1+f_4+5.1)$	$f_4 = \frac{1}{4}(f_2+f_3+5.8)$
0	0	0	0	0
1	$-3/4$	-9.7875	1.0875	-0.7
2	-2.925	-10.5125	0.3625	-1.0875

[illegible]