EE7330: Network Information Theory	**2021**

Lecture Notes 7: Fano's Inequality

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Disclaimer: *These notes have not been subjected to the usual scrutiny reserved for formal publications. Please email the course instructor in case of any errors.*

7.1 Fano's inequality

If M **&** $\hat{M}$ **are jointly distributed i.e.** M**,**$\hat{M} \in \mathbb{M}$ **and** $P_e = P_r[M \neq \hat{M}]$ then,

- $H(M|\hat{M}) \leqslant H_2(P_e) + P_e \log_2 |\mathbb{M}|$

7.2 Proof

Let $E = \begin{cases} 1 & \text{if } \hat{M} \neq M \\ 0 & \text{if } \hat{M} = M \end{cases}$

$H(E) = H_2(P_e)$

If $M = \hat{M} \implies H(M|\hat{M}) = 0$

If $M, \hat{M}$ are independent $\implies H(M|\hat{M}) \leqslant H(M)$

$$H(M, E|\hat{M}) = H(M|\hat{M}) + H(E|M, \hat{M}) \tag{7.1}$$

$$H(M, E|\hat{M}) = H(E|\hat{M}) + H(M|E, \hat{M}) \tag{7.2}$$

$$H(M|\hat{M}) = H(E|\hat{M}) + H(M|E, \hat{M}) - H(E|M, \hat{M}) \tag{7.3}$$

Since E is a function of M, $\hat{M}$

$$H(M|\hat{M}) = H(E|\hat{M}) + H(M|E, \hat{M}) - 0 \tag{7.4}$$

Conditioning reduces entropy

$$H(M|\hat{M}) \leqslant H(E) + H(M|E, \hat{M}) \tag{7.5}$$

$$H(M|\hat{M}) \leqslant H_2(P_e) + H(M|E = 0, \hat{M})P_E(0) + H(M|E = 1, \hat{M})P_E(1) \tag{7.6}$$

$$H(M|\hat{M}) \leqslant H_2(P_e) + H(M|\hat{M}, E = 1)P_e \tag{7.7}$$

$$H(M|\hat{M}) \leqslant H_2(P_e) + P_e H(M) \tag{7.8}$$

$$H(M|\hat{M}) \leqslant H_2(P_e) + P_e \log_2 |\mathbb{M}| \tag{7.9}$$

7.3 Converse of channel coding theorem

Let $M^{k_n} \in \{0, 1\}^{k_n}$ be uniformly distributed

DMC

$$M^{k_n} \longrightarrow \boxed{ENC_n} \longrightarrow \boxed{P(Y|X)} \longrightarrow \boxed{DEC_n} \longrightarrow \hat{M}^{k_n}$$

$$X_i \in \mathbb{X} \qquad Y_i \in \mathbb{Y}$$

$C = \max_{P_x} I(X;Y)$

Consider any sequence ENC_n, DEC_n such that

$\lim_{n\to\infty} inf \dfrac{k_n}{n} \geqslant c + \epsilon$

Then,

$\lim_{n\to\infty} inf Pr[\hat{M}^{k_n} \neq M^{k_n}]$