

Lecture Notes 7: Fano's Inequality

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7.1 Fano's inequality

If M & \hat{M} are jointly distributed i.e. $M, \hat{M} \in \mathbb{M}$ and $P_e = P_r[M \neq \hat{M}]$ then,

- $H(M|\hat{M}) \leq H_2(P_e) + P_e \log_2 |\mathbb{M}|$

7.2 Proof

$$\text{Let } E = \begin{cases} 1 & \text{if } \hat{M} \neq M \\ 0 & \text{if } \hat{M} = M \end{cases}$$

$$H(E) = H_2(P_e)$$

$$\text{If } M = \hat{M} \implies H(M|\hat{M}) = 0$$

$$\text{If } M, \hat{M} \text{ are independent} \implies H(M|\hat{M}) = H(M)$$

$$H(M, E|\hat{M}) = H(M|\hat{M}) + H(E|M, \hat{M}) \tag{7.1}$$

$$H(M, E|\hat{M}) = H(E|\hat{M}) + H(M|E, \hat{M}) \tag{7.2}$$

$$H(M|\hat{M}) = H(E|\hat{M}) + H(M|E, \hat{M}) - H(E|M, \hat{M}) \tag{7.3}$$

Since E is a function of M, \hat{M}

$$H(M|\hat{M}) = H(E|\hat{M}) + H(M|E, \hat{M}) - 0 \tag{7.4}$$

Conditioning reduces entropy

$$H(M|\hat{M}) \leq H(E) + H(M|E, \hat{M}) \tag{7.5}$$

$$H(M|\hat{M}) \leq H_2(P_e) + H(M|E=0, \hat{M})P_E(0) + H(M|E=1, \hat{M})P_E(1) \tag{7.6}$$

$$H(M|\hat{M}) \leq H_2(P_e) + H(M|\hat{M}, E=1)P_e \tag{7.7}$$

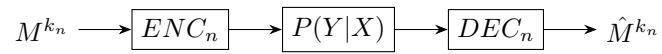
$$H(M|\hat{M}) \leq H_2(P_e) + P_e H(M) \tag{7.8}$$

$$H(M|\hat{M}) \leq H_2(P_e) + P_e \log_2 |\mathbb{M}| \tag{7.9}$$

7.3 Converse of channel coding theorem

Let $M^{k_n} \in \{0, 1\}^{k_n}$ be uniformly distributed

DMC



$$X_i \in \mathbb{X} \qquad Y_i \in \mathbb{Y}$$

$$C = \max_{P_x} I(X; Y)$$

Consider any sequence ENC_n, DEC_n such that

$$\lim_{n \rightarrow \infty} \inf \frac{k_n}{n} \geq c + \epsilon$$

Then,

$$\lim_{n \rightarrow \infty} \inf Pr[\hat{M}^{k_n} \neq M^{k_n}]$$