CHAPTER 6

(Just Enough) Algebraic Geometry

"whatever doesn't kill you simply makes you stranger"
— Joker

"I am sad to hear that one cannot present Weil's results without juggling with generic points; as a matter of fact, the unbridled abuse of generic points necessarily hides the few situations in which their use is truly essential, such as the proof that every endomorphism of the Jacobian comes from a correspondence. As Chevalley says, one feels frustrated when faced with a proof like that one."
— Grothendieck to Serre in a letter on November 22, 1956

6.1. There are two routes to algebraic geometry. The first one goes through the theory of varieties and the second one, which is relatively modern, goes directly through the theory of schemes. In this book, we will discuss about the latter mostly. However, historically speaking, varieties have been crucial for algebraic geometry too and, of course, play a significant role in many parts of mathematics (and physics!). So in the initial parts of this chapter, we will discuss the classical theory of varieties.

6.2. Scheme has been standard for more than over six decades. Before the language of schemes, there was the definition of algebraic variety in the pens of Weil [**24**] and Zariski (and the abstract variety in the sense of Serre). Affine variety is just a special case of the schemes. The formalization of scheme theory was done by Grothendieck (with the help of works by Serre, Chevalley) in Éléments de Géométrie Algébrique.

In this chapter, we will discuss some basics of varieties that will comfort one who has not seen them so far. We will swiftly change our discussion to schemes. This will be an important chapter in Prelude to Schemes as we will discuss most of EGA work herein.

Textbooks [**25**, **26**] do a nice job to introduce varieties and [**22**] is an excellent resource algebraic geometry in the languages of schemes straightway, and this will be closer to our hearts in this chapter. For the part about varieties, we will follow [**25**].

This is a part of Prelude of Schemes project available at `https://aayushayh.github.io/PtoS.html`.
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6.3. Throughout the chapter, we fix k to be an algebraically closed field unless stated otherwise.

1. A little bit of Variety does not kill you

DEFINITION 6.4 (Affine space over k). An affine n-space over k, denoted as A_k^n, is a set of all n-tuples of elements of k. An element P of A_k^n is called point and the components of $P \in A_k^n$ are called coordinates of point P. So if $P = (a_1, \cdots, a_n)$ with $a_i \in k$, then we call a_i to be coordinates.

In a very loose sense, A_k^n is just an arbitrary geometric way to write k^n (without an origin). But we will postpone this discuss to a later stage.

6.5. Given a polynomial ring[1] over k, $A = k[x_1, \cdots, x_n]$, we can write elements of A as elements $f \in A$ which is a function $f : A_k^n \to k^n$,

$$f(P) = f(a_1, \cdots, a_n), \quad f \in A \text{ and } P \in A_k^n \tag{114}$$

and we can define the zeroes of f as a set $Z(f)$ but say we have a subset $T \subseteq A$, so define the zero set of T as the common zeroes of all the elements of T

$$Z(T) = \{P \in A_k^n | \forall f \in T, f(P) = 0\} \tag{115}$$

6.6. We will now denote affine $n-$space A_k^n by $\mathbf{A}^n$.

DEFINITION 6.7 (Algebraic Sets). A subset V of $\mathbf{A}^n$ is called *algebraic set* if $V = Z(T)$ for some T.

DEFINITION 6.8. An *algebraic plane curve* over k is an algebraic set $C \subseteq \mathbf{A}^2$. We will hopefully discuss affine curves in detail in detail.

6.9. We will define the Zariski topology on $\mathbf{A}^n$. Now, the intersection of two algebraic sets is an algebraic set and so is the union of two algebraic sets. (See [**25**] for a proof.)

The Zariski topology on $\mathbf{A}^n$ is defined by taking the complements of algebraic sets to be open subsets. Since their intersections and unions will be open, it is a topology. And the empty set and whole space being algebraic sets, are both open.

6.10. A non-empty subset W of a topological space X is said to be irreducible if it cannot be written as the union of two proper subsets of X and closed in Y, i.e.,

$$W = W_1 \cup W_2. \tag{116}$$

(A curve is said to be irreducible if its polynomial equation is irreducible.)

[1] A polynomial ring is a unique factorization domain (UFD).

DEFINITION 6.11 (Affine Algebraic Variety). We define an *affine algebraic variety* as an irreducible closed subset of $\mathbf{A}^n$. Moreover, if we take an open subset of an affine variety, then it is called a *quasi-affine variety*.

6.12. Now, the subsets of $\mathbf{A}^n$ can be mapped to ideals in A. Given any subset $V \subseteq \mathbf{A}^n$, we define

$$I(V) = \{f \in A = k[x_1, \cdots, x_n] | f(P) = 0 \ \forall P \in V\} \tag{117}$$

THEOREM 6.13 (Hilbert's Weak Nullstellansatz, from [22]). Let $A = k[x_1, \cdots, x_n]$ be a polynomial ring and k be an algebraically closed field (see 6.15 for why), then the maximal ideals in A are precisely those ideals of the form $(x_1 - a_1, \cdots, x_n - a_n)$, where $a_i \in k$.

6.14. The statement of theorem 6.13 might not be the best way to introduce Hilbert's Nullstellansatz in a progression of varieties. But a statement exists about the correspondence between ideals and varieties.

Note that in Point. 6.5, we defined the zero set of a subset of $A = k[x_1, \cdots, x_n]$. Given an ideal $J \subseteq A$, we can define a zero set, let us call it *zero locus* J and re-notate it by

$$V(J) = \{P \in \mathbf{A}^n | \forall f \in J \ f(P) = 0\} \tag{118}$$

and when $V(J) = \varnothing$ that means that there are no points in $\mathbf{A}^n$ where polynomials in J vanish simultaneously.

6.15. There could be two situations when $V(J) = \emptyset$. Firstly, when $J = A$, then $1 \in J$ and for any point $P \in \mathbf{A}^n$, we have $1(P) \neq 0$. Therefore $V(J) = \varnothing$ and hence no point lies in $V(J)$. One can deduce that the question of if $V(J) =$ becomes non-trivial only if J is a proper ideal in A. Second instance when $V(J) = \varnothing$ happens when k is not algebraically closed, for examples $\mathbb{Q}, \mathbb{R}$. Given $k = \mathbb{R}$ or $k = \mathbb{Q}$ and an ideal $J = (x^2+1) \subseteq \mathbb{R}[x]$ generated by x^2+1, the zero-locus is empty $V(J) = \varnothing$. Hence, one is required to have an algebraically closed field k.

- It is also worth noting that for $J = 0$, we have $V(J) = \mathbf{A}^n$.
- If $I \subset J$, then $V(I) \supset V(J)$ which is known as inclusion-reversing.
- Since the union and intersection of algebraic sets is algebraic set (see 6.9) and any algebraic set is equal to $V(J)$ for some ideal J, we have $V(I) \cup V(J) = V(I \cap J)$ and for any collection of ideals $\{J_\alpha\}$, the intersection $\bigcap_\alpha V(J_\alpha) = V(\sum_\alpha J_\alpha)$.

Now, before moving ahead, some important points.

PROPOSITION 6.16 (See [27]). If A and B are integral domains with every element of B integral over A (in other words, B is integral over A), then A is a field iff B is a field.

LEMMA 6.17 (Noether's Normalization, [2]). Let k be an algebraically closed field and $A = k[x_1, \cdots, x_n]$ a finitely generated k-algebra.[2] There exists some $m \leq n$, for which there exist elements $y_1, y_2, \cdots, y_m$ algebraically independent over k such that A is a finitely generated module over the k-subalgebra $k[y_1, y_2, \cdots, y_m]$.

LEMMA 6.18 (Zariski's Lemma). Let $k \to L$ be an extension of fields such that L is of finite type over k, then L is finite over k

THEOREM 6.19 (Hilbert's Weak Nullstellensatz (again), [28]). For any algebraically closed field k, for any proper ideal $J \subsetneq k[x_1, \cdots, x_n]$ we have $V(J) \neq \varnothing$.

PROOF. For the ideal J is proper in $A = k[x_1, \cdots, x_n]$, it would be contained in a maximal ideal $\mathfrak{m} \subseteq A$ (by Zorn's Lemma) and $V(\mathfrak{m}) \subseteq V(J)$ by inclusion-reversing 6.15. So just show that $V(\mathfrak{m}) \neq \varnothing$. Let us look at the quotient $L = A/\mathfrak{m}$ and since $\mathfrak{m}$ is maximal ideal, L is a field and L is a finitely generated k-algebra. We can take k to be a sub-field of L by a natural homomorphism $k \hookrightarrow L$. Since k is an algebraically closed field, k does not have any non-trivial finite field extensions, by Lemma. 6.18, we have $k \cong L$.

The quotient map (which is a projection) $\pi : A \to L = A/\mathfrak{m}$ is $x_i \mapsto a_i \in k$. The kernel of this quotient map is just $ker(\pi) = \mathfrak{m}$. Let us pick $f \in J \subseteq \mathfrak{m}$, then we have

$$f(a_1, \cdots, a_n) = f(\pi(x_1), \cdots, \pi(x_n)) = \pi(f) = 0 \tag{119}$$

and hence $P = \{a_1, \cdots, a_n\} \in \mathbf{A}^n$ lies in $V(J)$.

REMARK 6.20. The statements in 6.13 and 6.19 are actually the same.

[2] Here, we gave the statement for a polynomial ring as finitely generated k-algebra but Noether's normalization works for any finitely generated k-algebra.

Bibliography

[1] Bourbaki, *Algèbre*. Springer, 2007.
[2] M. F. Atiyah and I. G. Macdonald, *Introduction to commutative algebra*. CRC Press, 2018.
[3] R. Bott and L. W. Tu, *Differential forms in algebraic topology*, vol. 82. Springer Science & Business Media, 2013.
[4] D. A. Buchsbaum, "Exact categories and duality," *Transactions of the American Mathematical Society* **80** no. 1, (1955) 1–34.
[5] A. Grothendieck, "Sur quelques points d'algèbre homologique," *Tohoku Mathematical Journal, Second Series* **9** no. 2, (1957) 119–183. English translation available at https://www.math.mcgill.ca/barr/papers/gk.pdf.
[6] F. Peter, "Abelian Categories: An Introduction to the Theory of Functors,".
[7] J.-L. Verdier, "Catégories dérivées Quelques résultats (état 0)," in *Cohomologie Etale: Séminaire de Géométrie Algébrique du Bois-Marie SGA 4 1/2*, pp. 262–311. Springer, 1977.
[8] C. A. Weibel, *An introduction to homological algebra*. No. 38. Cambridge university press, 1994.
[9] D. Happel, *Triangulated categories in the representation theory of finite dimensional algebras*, vol. 119. Cambridge University Press, 1988.
[10] S. MacLane, "Henri Cartan and Samuel Eilenberg, Homological Algebra,".
[11] T. Stacks project authors, "The Stacks project." https://stacks.math.columbia.edu, 2025.
[12] C. A. Weibel, *The K-book: An Introduction to Algebraic K-theory*, vol. 145. American Mathematical Soc., 2013.
[13] M. Atiyah, *K-theory*. CRC press, 2018.
[14] H. Miller, "Leray in Oflag XVIIA: The origins of sheaf theory, sheaf cohomology, and spectral sequences,".
[15] J.-P. Serre, "Faisceaux algébriques cohérents," *Annals of Mathematics* **61** no. 2, (1955) 197–278.
[16] A. Grothendieck, "Éléments de géométrie algébrique: I. Le langage des schémas," *Publications Mathématiques de l'IHÉS* **4** (1960) 5–228.
[17] A. Verma, "Local aspects of topological quantization and the Wu-Yang Monopoles," *arXiv:2406.18799* (2024) .
[18] T. Y. Chow, "You could have invented spectral sequences," *Notices of the AMS* **53** (2006) 15–19.
[19] A. HATCHER, "Algebraic Topology," *http://www. math. cornell. edu/˜ hatcher/AT/ATpage. html* (2002) .
[20] J. McCleary, *A user's guide to spectral sequences*. No. 58. Cambridge University Press, 2001.
[21] M. Dubois-Violette, "The Weil-BRS algebra of a Lie algebra and the anomalous terms in gauge theory," *Journal of Geometry and Physics* **3** no. 4, (1986) 525–565.
[22] R. Vakil, "The rising sea: Foundations of algebraic geometry,". available at https://math.stanford.edu/~vakil/216blog/FOAGjul2724public.pdf.
[23] W. Browder, "Torsion in H-spaces," *Annals of Mathematics* **74** no. 1, (1961) 24–51.
[24] A. Weil, *Foundations of algebraic geometry*, vol. 29. American Mathematical Soc., 1946.

[25] R. Hartshorne, *Algebraic geometry*, vol. 52. Springer Science & Business Media, 2013.
[26] D. Mumford, *The red book of varieties and schemes: includes the Michigan lectures (1974) on curves and their Jacobians*, vol. 1358. Springer, 2004.
[27] D. Dummit and R. Foote, *Abstract Algebra*. Wiley, 2003.
[28] D. Hilbert, "Über die vollen Invariantensysteme," in *Algebra· Invariantentheorie Geometrie*, pp. 287–344. Springer, 1970.