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# On von Neumann Algebra, Holography and Regions

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**Abstract**

We survey the recent results by Leutheusser and Liu on the emergent **type** $III_I$ in large N limit. We discuss the importance of von Neumann algebra in quantum field theory. We also remark on the algebra of de-Sitter spaces, black holes, and other regions of interest.

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# 1 No Entanglement Entropy in QFT

A von Neumann algebra $\mathcal{A}$ is a closed $*-$algebra defined for a region $\mathcal{O}$ in the Hilbert space $\mathcal{H}$. The operators of the algebra $\mathcal{A}$ are bounded and closed under weak operator topology ($\lim_{n\to\infty} \langle\Psi|a_n|\Psi\rangle = \langle\Psi|a|\Psi\rangle$). One can define the causal complement region $\mathcal{O}'$ and the algebra for $\mathcal{O}'$ is $\mathcal{A}'$. Operators in algebra $\mathcal{A}$ and $\mathcal{A}'$ commutes and according to the bi-commutant theorem

$$\mathcal{A} = \mathcal{A}''. \tag{1}$$

An algebra $\mathcal{A}$ is called a *factor* if its center is trivial up to $\mathbb{C}$ multiple. These algebras are divided into five factors, namely **type** $I_n$, **type** $I_\infty$, **type** $II_1$, **type** $II_\infty$, and **type** $III$, studied in [1–4]. Among all these, **type** $III$ is the most evil one and the subject of this section.

We can define these factors on the basis of how they treat trace and renormalization. In other words, if we have a positive sub-algebra[1] $\mathcal{A}^+$ of $\mathcal{A}$ does there exist a renormalization so that our operators become *sane* density matrices. A classification of such kind can be found in [5, 6] and the classification of factors based on the range of the dimensions function[2] constructed from the projection operators can be found in [7, 8]. A **type** $I$ algebra can define renormalized states, and this algebra is a very common example of states in quantum mechanics. While **type** $II_1$ can define traces for all the states, **type** $II_\infty$ has only a few states on which trace can be defined. The algebra of **type** $III$ has no way to define trace at all. Only **type** $I$ algebra has an irreducible representation in a Hilbert space. However, their commutant may or may not be in the same algebra, for example, commutant of **type** $II_1$ is another **type** $II$.

Since **type** $III$ algebras fail to define renormalized state on which we can define trace, there is no notion of entanglement entropy for such algebra. We can now show that the UV divergences in the entanglement entropy for a local relativistic QFT region are precisely because of the fact that **type** $III$ governs such local region [6, 7, 9, 10], see also [11]. There are few subtleties in the local region; for instance, one cannot define a Hilbert space in the local region. Instead of the Hilbert space, we define an open set $\mathcal{W}$ in the spacetime and algebra $\mathcal{A}_\mathcal{W}$ on it. The $\mathcal{A}_\mathcal{W}$ follows some axioms stated in [12] of causality and isotonicity; we do not embark on these properties in this paper.

In Fig. 1, we see the Rindler wedge view in the Minkowski space of a sphere with the slice through the bifurcating in the figure can be said as a splitting of the sphere. We can see that there is a 'full' entanglement between regions $\mathcal{O}$ and $\mathcal{O}'$ [6]. This amounts to UV divergences in the region in the short distances as given by Unruh [13]. There is also an interpretation of this using Tomita-Takesaki theory [14], where the modular Hamiltonian can be thought of as boost operators. Because of the UV divergences, we can not regularize the trace. The divergences in entanglement entropy reduce our

[1]A set of positive operators in an algebra is defined as a set of operators with spectrum $\mathsf{spec} \subset \mathcal{R}^+ \cup \{0\}$.

[2]The dimension function is unique up to a scalar multiple.

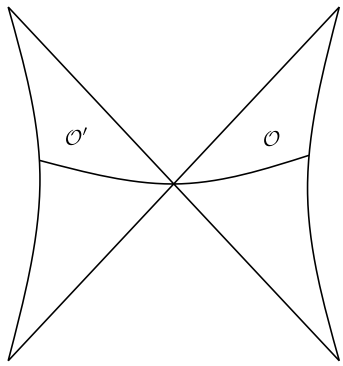


Figure 1: Rindler wedge of a local region in the spacetime where $\mathcal{O}$ can be thought of upper-half of the sphere and $\mathcal{O}'$ of lower-half of the sphere. There are infinite entanglement of modes between regions $\mathcal{O}$ and $\mathcal{O}'$

algebra, let us say $\mathcal{A}_{\mathcal{O}}$, to be of **type** $II_\infty$ and **type** $III$. But since $\mathcal{A}_{\mathcal{O}}$ does not have a trace, the algebra for $\mathcal{A}_{\mathcal{O}}$ is of **type** $III$. This provides us an abstract reasoning for the quantum field theory divergences. Quantum statistical infinite systems also have **type** $III$ algebra [15]; however, one encounters IR divergences in it. One can also provide the conformal mapping between UV divergences of quantum field and IR divergences of quantum statistical system.

A more refined study of the classification of *hyperfinite* factors of **type** $III$ was done by Connes [16] based on the *modular spectrum* of the algebra. It would be interesting to observe how different classifications of **type** $III$ appear in other areas of physics.

## 2 type $III_I$ in large $N$

Recently, Leutheusser and Liu have given an explanation [17, 18], based on algebraic modular flow, for the emergent **type** $III_I$ (a hyperfinite factor of **type** $III$) in the holographic continuum limit. Subsequently, the same authors proposed subregion-subregion duality following the same scheme [19]. While on boundary CFT algebras are of **type** $I$, we see that at large $N$ the algebras are **type** $III_1$ which sheds new lights on holography. Witten showed that in $1/N$ corrections the algebra changes from **type** $III_I$ to **type** $II_\infty$ [20]. We would discuss Leutheusser and Liu results in this section which may serve motivation for many upcoming works, especially works in holographic entanglement entropy and quantum error correction. We now provide a few comments on this proposed emergent time.

We have a no-go condition for **type** $III_I$ solution which is that there exists no projector that can probe $F$ from $\tilde{R}$ because there is no influence detectable, see Fig. 2. The reason is that any projector is an infinite projection for **type** $III_I$, and there is no way to create finite projections.

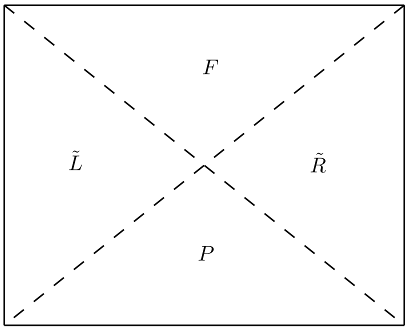


Figure 2: An AdS spacetime. The algebra on the boundary $R$ should be equal to algebra in the bulk region $\tilde{R}$ under duality. Same for region $\tilde{L}$. In large $N$ limit, the algebra in bulk (hence on boundary too) to **type** $III_I$ from simple **type** $I$ [18]. Following the $1/N$ perturbative corrections, the algebra changes to **type** $II_\infty$ [20]. $F$ and $P$ are the future interiors and past interior of black holes.

As stated before, the algebra on the boundary is **type** $I$ but only for finite $N$. In large $N$, there is a sharp horizon and an emergent algebra **type** $III_I$ from which all of the spacetime follows. However, **type** $III_I$ consists of an infinite sharp horizon, so we would regularize it [21, 22]. This can also be used to show that there is a causality between the $\tilde{R}$ and $\tilde{L}$ even from behind the horizons [17].

Let us denote algebra $\mathcal{A}_\mathcal{L}$ for boundary $L$ and $\mathcal{A}_\mathcal{R}$ for boundary $R$. $\mathcal{A}_\mathcal{L}$ and $\mathcal{A}_\mathcal{R}$ are commutant in this case. It should be noted that these algebras can be viewed as representative of algebras in TFD action with a Hilbert space constructed from GNS construction [7]. In the bulk, we can construct Fock states on the Hartle-Hawking vacuum around the black hole, and we can represent it by $\mathcal{H}_{BH}^{Fock}$, by following notation [17]. And of the GNS constructed Hilbert space by $\mathcal{H}_{\rho\beta}^{GNS}$ for a state $\rho_\beta = \frac{1}{Z_\beta} e^{-\beta H_R}$ and $\beta$ is the inverse temperature. Because of the gauge/gravity duality, we have certain dualities between objects

$$\mathcal{H}_{\rho\beta}^{GNS} = \mathcal{H}_{BH}^{Fock} \tag{2}$$

$$\mathcal{A}_\mathcal{L} = \tilde{\mathcal{A}_\mathcal{L}}, \ \mathcal{A}_\mathcal{R} = \tilde{\mathcal{A}_\mathcal{R}} \tag{3}$$

and the duality between the vacuum representation of $\rho$ and Hartle-Hawking vacuum state. In Eq. (3), $\tilde{\mathcal{A}_\mathcal{L}}$ and $\tilde{\mathcal{A}_\mathcal{R}}$ are algebras for bulk regions $\tilde{L}$ and $\tilde{R}$ respectively. Because of the large $N$ limit, the algebra on $\mathcal{A}_\mathcal{L}$ and $\mathcal{A}_\mathcal{R}$ are of **type** $III_I$ and following duality, the algebra on $\tilde{\mathcal{A}_\mathcal{L}}$ and $\tilde{\mathcal{A}_\mathcal{R}}$ are of **type** $III_I$ as well.

The 'smeared' bulk states[3], let us denote it by $\Psi_{\tilde{L},s}$ and $\Psi_{\tilde{R},s}$, are actually in causal connection with each other. This one can see after having a sharp horizon, which constructs the whole causal structure from the black hole. So if one acts on $\Psi_{\tilde{R},s}$ to send it to $F$, then it lies in the light-cone of $\Psi_{\tilde{R},s}$ in $F$, see Fig. 2. To answer what happens

[3]Smearing is done for better purposes, see [6], one of which is to make states stay in the Hilbert space.

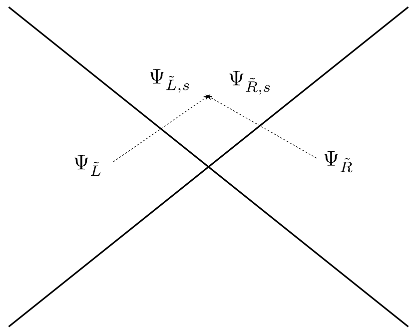


Figure 3: Smeared operators meet behind the horizon. Or as put in [17], if the right state is projected behind the horizon it falls in the light cone of the left state.

between $\Psi_{\tilde{L},s}$ and $\Psi_{\tilde{R},s}$ at singularity is a tough question. One can say that the causal connection between $\tilde{R}$ and $\tilde{L}$ should break down at such a singular point. However, there remains an open question of stability of algebras on the singularity.

To see how the duality works using modular flow techniques, one takes a subalgebra $\mathcal{B}_R$ of $\mathcal{A}_\mathcal{R}$, which is meant to say a boundary subregion. We denote modular conjugation and modular operator by $\mathcal{J}_R$ and $\Delta_R$; see [23] for a discussion. We assume that state $\Omega$ for $\mathcal{A}_\mathcal{R}$ is cyclic and separating, then it is cyclic and separating for $\mathcal{B}_R \subset \mathcal{A}_\mathcal{R}$. The polar decomposition of a Tomita-Takesaki operator is given by

$$S_R = \mathcal{J}_R \Delta_R^{1/2}, \tag{4}$$

and $S_R^2 = \mathbb{1}$. To have a half-sided modular inclusion [24–26] we introduce unitary operations by $U(s)$[4]. In particular, it should leave $\Omega$ invariant. The half-sided inclusion is given by [17]

$$U^\dagger(s)\mathcal{A}_\mathcal{R}U(s) \subseteq \mathcal{A}_\mathcal{R}, \quad \forall s \leq 0 \tag{5}$$

If $U(s)$ acts with $s = -1$, we get $\mathcal{B}_R$

$$\mathcal{B}_R = U^\dagger(-1)\mathcal{A}_\mathcal{R}U(1) \tag{6}$$

and taking $U(s) = e^{iGs}$ and $\mathcal{B}_R(t) = \Delta_R^{-it}\mathcal{B}_R\Delta_R^{it}$ we can construct a function of $s, t$. $\Delta_R^{it}$ generates the one-parameter group of automorphism of algebras[5]. One must note that $\mathcal{A}_\mathcal{L} = \mathcal{J}_R\mathcal{A}_\mathcal{R}\mathcal{J}_R$ and since this is true

$$U^\dagger(s)\mathcal{A}_\mathcal{L}U(s) \subseteq \mathcal{A}_\mathcal{L}, \quad \forall s > 0. \tag{7}$$

This is a very important object, often called 'half-sided modular inclusion', which only works if the algebras in game are of **type** $III_I$ [25]. We omit the discussions of its extension here.

[4] $\mathcal{J}_R U(s)\mathcal{J}_R = U(-s)$

[5] It is also interpreted as a boost operator, see [11].

Now that we have modular flow, which can also be seen as a time translation on boundary[6], we can do the bulk re-construction for this emergent algebra by taking $\mathcal{B}_R$ as a boundary for the causal wedge in the bulk [27]. We do not discuss it here for brevity, but the Poincare transformations in bulk is precisely given by these causal wedges [17,18] and emergent algebras that we just discussed.

## 3 Gravity for Entropy

We have seen that the entropy (and other information objects) is defined as to what factors of von Neumann algebra we are considering. This means that there lies a rigorous connection between entropy and algebraic QFT. In fact, there exists a lot of rigorous proofs of conjecture given by modular flows and a wide array of interests have been born for it.

One interesting observation recently was that presence of gravity should better describe entanglement entropy rather than in absence of it [28, 29]. It is evident that the observable outside the black hole horizons is described by **type** $II_\infty$ [20] and for de-Sitter space by **type** $II_1$ [28]. However, for a local region, as we have discussed, the algebra does not define entropy because of the presence of **type** $III_I$. For **type** $II$, we can define entropy, at least a physical renormalized entropy.

If we consider an empty de-Sitter space (Bunch-Davies states), we see it has maximal entropy and **type** $II_1$ is exactly the algebra that can define the maximal entropy [30,31]. So for de-Sitter, we have algebra described by **type** $II_1$ and the state of empty de-Sitter has the maximum entropy (which is 0) as argued in [28]. As for the black hole, it was argued that since **type** $II_\infty$ has no upper bound of entropy, it should better describe the black hole [20]. Because of the bulk-boundary duality, algebras on the copies can not be **type** $I$ but **type** $II_\infty$.

An important point to notice is that entropy is seldom in **type** $II$, so we need to renormalize it to make sense of it. This renormalization involves a subtraction, up to an additive constant. This makes them to be independent of the state (see [29]). Algebras of **type** $II$ has not been studied much in physical literature, however, we believe that even much of the QFT is **type** $III$, there are many systems described by **type** $II$ (see [28, 32]).

[6]Because of the emergence of **type** $III_I$ in the problem, this time has been dubbed 'emergent time'.

## 4 Brief Remarks on Tomita-Takesaki Theory and Timelike Tube Theorem

We have seen that many physical answers can be provided by the theory of algebras, or at least it can give a new view to much of physics. One thing we did not mention greatly is called 'Tomita-Takesaki theory'. It is also an important concept in algebraic QFT. This has been used in many places, especially to compute modular flow. In Eq. (4), $S_R$ is antilinear operator called Tomita-Takesaki operator, $S_R \mathrm{a} \left|\psi\right\rangle = \mathrm{a}^{\dagger} \left|\psi\right\rangle, \left|\psi\right\rangle \in R$. In particular $\Delta_R$ is called modular operator

$$\Delta_R = S_R^{\dagger} S_R. \tag{8}$$

Using these tools, we have a different way of writing relative entropy, as given by Araki [33]. $S_{rel}(\Psi|\Phi)$ is the relative entropy between $\Psi$ and $\Phi$ in some region. It is given by

$$S_{rel}(\Psi|\Phi) = -\left\langle \Psi|\Delta_{\Psi|\Phi}|\Psi\right\rangle \tag{9}$$

where $\Delta_{\Phi|\Psi}$ is the relative modular operator. $S_{rel}(\Psi|\Phi)$ is a non-negative entropy [33], which is also true when doing general quantum field theory, for example, in [34] proving Bekenstein bound. We also have monotonicity of this modular relative entropy when we have a region nested in another, see [23] for other properties of this modular relative entropy.

Lastly, we comment on Borcher's theorem [35, 36], which is the 'timelike tube theorem'. It has also been reconsidered recently [38, 39]. It is essentially for regions without 'gravity .' For an open set in QFT $\mathcal{O}$, we write the domain of dependence as $D(\mathcal{O})$. One can say then that the algebra of the region $\mathcal{O}$ determines the algebra for $D(\mathcal{O})$

$$\mathcal{A}(\mathcal{O}) = \mathcal{A}(D(\mathcal{O})). \tag{10}$$

In real analytic spacetime, one can be imaginative to construct not just a spacelike envelope but a timelike envelope of $\mathcal{O}$ as $\mathcal{E}(\mathcal{O})$. A timelike envelope can be thought of as a set of points that can be reached from a timelike geodesic $\gamma \subset \mathcal{O}$ by deforming it, see [29]. Timelike tube theorem suggests that

$$\mathcal{A}(\mathcal{O}) = \mathcal{A}(\mathcal{E}(\mathcal{O})). \tag{11}$$

Timelike tube theorem was studied recently for free-field theories in curved spacetime [37] and for non-free field theories in curved spacetime [38, 39]. The mathematical properties of the timelike tube theorem is interesting in its own right. We do not have much rigorous physical application of the timelike tube theorem, but one can be ambitious about this as well. A more detailed analysis of this will appear somewhere else.

We believe a lot of physical interpretation (or the mathematical interpretation of the same events) can be carried out through this von Neumann algebra and its counterparts. For instance, determining the nature of entanglement wedge reconstruction (see [19]), Islands, quantum extremal surfaces[7] [40], JT gravity [32], Hawking radiation in curved spacetime (see [11, 14, 41]), more explanations of algebraic quantum field theory and other works in quantum error correction.

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