The Relevance of Mathematics in Theoretical Physics

With examples of S-Duality, Quantum Mechanics, Gauge theory, Langlands duality and all that

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The Useless Knowledge

What is the definition of pure mathematics?

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What is the definition of pure mathematics? According to **Bertnard Russell**, it is:

Pure Mathematics is the class of all propositions of the form "p implies q," where p and q are propositions containing one or more variables, the same in the two propositions, and neither p nor q contains any constants except logical constants. And logical constants are all notions definable in terms of the following: Implication, the relation of a term to a class of which it is a member, the notion of such that, the notion of relation, and such further notions as may be involved in the general notion of propositions of the above form. In addition to these, mathematics uses a notion which is not a constituent of the propositions which it considers, namely the notion of truth.

Anyways, this talk is not about the philosophy of either mathematics or physics. I will mention few parallels from past centuries (especially the last year) between theoretical physics and mathematics. Essentially how some of thoughts of both mathematicians and physicists were basically the ‘same’.

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In recent times (mainly since the early 20th century), theoretical physics advancements have gone through a serious play of mathematics. There are a lot many examples of such plays but here we will cover only a few of them - **Quantum Mechanics as Matrix Mechanics, S-Duality, Yang-Mills Gauge Theory and Knot Theory.**

Quantum Mechanics

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Basically, in quantum mechanics, roughly we are concerned about states ψ which represents a certain system. Knowing what exactly ψ of a system is not as trivial as in classical mechanics. However, we can know what is the probabilistic distribution of such ψ (governed by Schrodinger equation) is over some space $\mathbb{R}$. Moreover, operators give us relevant values of numbers that we are interested in, such as energy (by acting with Hamiltonian), position (by acting position operator $\hat{x}$), momentum (by acting momentum operator $\hat{p}$) and so on. The notion of quantization is based when the commutation relation is established $[x, p] = i\hbar$.

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In 1920s, Quantum Mechanics, was studied to explain the atom and related physics but it was later revised with a more concrete use of mathematics by W. Heisenberg known as 'Matrix Mechanics'. In nutshell, this matrix mechanics was use of linear algebra to systematize the quantum mechanics.This can be thought as understanding ψ as vectors and operators as matrix functions. Then the eigenvalues to these operators would be physical relevant quantities (like energy). These states ψ belong to a vector space V and when we introduce the condition of positive-definite inner product (i.e. $\langle a|b\rangle \geq 0, a, b \in V$), then

$$V = \mathcal{H} \tag{1}$$

where $\mathcal{H}$ is called Hilbert space.

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$$H = H^\dagger \tag{3}$$

where $H^\dagger$ is the complex adjoint of H. This is necessary to yield 'real' eigenvalues. The eigenvectors forms a Hilbert space $\mathcal{H}$ of square-integrable functions as previously described.

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Simply speaking, on a region, let us say $\mathbb{R}^2$, one can associate bounded and self-adjoint operators and this will form an algebra $\mathcal{A}$. This is famously known as von Neumann algebra (a type of C^* algebra) if

$$\mathcal{A} = \mathcal{A}'' \tag{4}$$

where $\mathcal{A}''$ is the double-commutant of $\mathcal{A}$.

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Slight Remark: The classical theory that gives a quantum theory can be rigorized in terms of Symplectic phase space and a 2-form ω where instead of Dirac's bracket and commutation relations, we have Poisson's bracket and Hilbert space is replaced by Phase space. There we have the Hamilton's equation

$$\dot{x} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial x} \tag{5}$$

So in the simple case where Hamiltonian is

$$H = \frac{p^2}{2m} + V(x) \tag{6}$$

and we know

$$F = -\frac{\partial V}{\partial x} = m\ddot{x} = ma \tag{7}$$

So Symplectic Phase space is a way to understand the Hamiltonian formulation of classical mechanics. Of course, there are other rigorization of classical mechanics available. Check Arnold's book on classical mechanics for these mathematical motivations.

Yang-Mills Theory and Vector Bundles

A gauge theory is a theory which is invariant under certain 'gauge' transformations (or locally defined transformations). Basically, it represents degrees of freedom of a theory that are redundant and can be fixed. It was introduced by Weyl's paper. An example is changing the phase of a wave

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What is a fiber bundle?

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It is defined as a map $\pi\colon E \to M$ where M is the base space and there are some *fibers* involved in such projections.

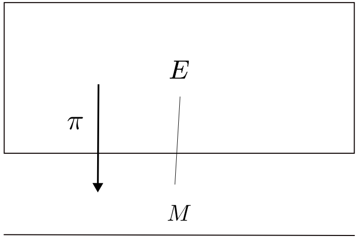
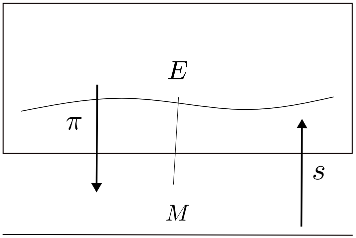


Figure: A fiber bundle $\pi\colon E \to M$.

A section of the fiber bundle is a continous map

$$s\colon M \to E \tag{9}$$

such that for $x \in E$, we have $\pi(s(x)) = x$.



Now, a principal G-bundle is defined for a fiber bundle $\pi\colon P \to X$ where $P \times G \to P$. Basically, for $x \in P$ we have $xg \in P$ where $g \in G$. (Where G is a group.)

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Let us recap some features of a gauge theory. In gauge theory, there is a gauge transformation between variables like

$$g \colon A \to A' \tag{10}$$

where A, A' are gauge fields and g takes value in the gauge group. And there is a field strength F

$$F = dA \wedge A \wedge A \tag{11}$$

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Now, in a principal G-bundle, a connection is defined to be maps which take the value in the structure group G from $P \times G \to P$.

We have seen both fiber bundles and gauge theory. Both has groups features and connections and gauge fields look the same. Both theory has features which are identical.

Yang-Mills theory is described as the G-bundle $E \to M$, where M is a Riemannian manifold. We have the curvature $F = dA + A \wedge A$. The wave equation is

$$D \star F = 0 \tag{12}$$

where $\star$ is the Hodge product and $D = d + A$. In abelian case, such as U(1), D reduces to d and

$$dF = 0 \tag{13}$$

which is the Bianchi identity. This described the propogation of the Maxwell waves. Now, for the non-abelian case we have

$$D \star F = 0 \tag{14}$$

and the waves are nonlinear.

Similarly, GR is described in vacuum by

$$R_{\mu\nu} = 0 \tag{15}$$

which describes the hyperbolic waves. Means that they are evolution to the Cauchy initial data.

In GR, we have collapse of spacetime. And the lightrays cut off. The situation in the quantum gravity is hard because of this only.

For gauge group $G = U(1)$, we observe classical solutions of Maxwell's equations all the time – light waves. For nonabelian G, even though there are beautiful nonlinear classical wave equations, we do not observe these nonlinear classical waves in practice. That is actually because of a phenomenon known as the mass gap.

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TABLE I. Translation of terminology.

Gauge field terminology	Bundle terminology
gauge (or global gauge)	principal coordinate bundle
gauge type	principal fiber bundle
gauge potential b^k_μ	connection on a principal fiber bundle
S_{ba} (see Sec. V)	transition function
phase factor Φ_{QP}	parallel displacement
field strength $f^k_{\mu\nu}$	curvature
source [a] J^K_μ	?
electromagnetism	connection on a $U_1(1)$ bundle
isotopic spin gauge field	connection on a SU_2 bundle
Dirac's monopole quantization	classification of $U_1(1)$ bundle according to first Chern class
electromagnetism without monopole	connection on a trivial $U_1(1)$ bundle
electromagnetism with monopole	connection on a nontrivial $U_1(1)$ bundle

So basically, a gauge theory is equivalent to the fiber bundle mechanism.

So basically, a gauge theory is equivalent to the fiber bundle mechanism. Here is the conversation which Yang had with Chern in 1975

> *"When our conversation turned to fiber bundles, I told him that I had finally learned from Jim Simons the beauty of fiber-bundle theory and the profound Chern-Weil theorem. I said I found it amazing that gauge fields are exactly connections on fiber bundles, which the mathematicians developed without reference to the physical world. I added 'this is both thrilling and puzzling, since you mathematicians dreamed up these concepts out of nowhere.' He immediately protested, 'No, no. These concepts were not dreamed up. They were natural and real."*

S-Duality

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All five kinds of String Theories - Type I, Type IIA, Type IIB, SO(32), $E_8 \times E_8$ are essentially the same in different limits of M-theory.

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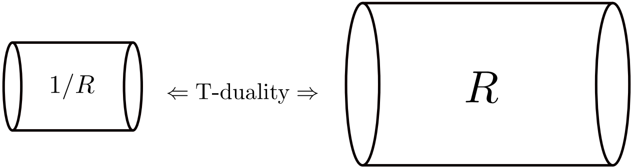

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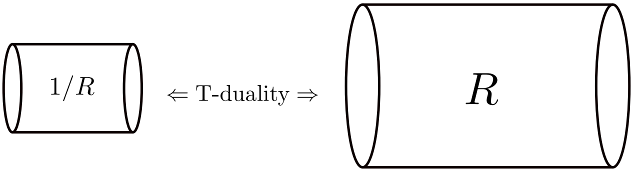

Just like this, we also have an equivalence between coupling constants α and $1/\alpha$which is the S-duality. Which is close to the statement of Dirac's quantization condition $e \Leftrightarrow 1/g$.

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This can be found in works of Sen, Witten-Kapustin, Goddard-Nuyts-Olive and Monotonen-Olive.

A word on Langlands Program

Langlands program is a wide program throughout mathematics that relates the area of representation theory and automorphic forms to Galois theory. One of the many Langlands duality is called **Geometric Langlands Duality**, which is useful in theoretical physics, especially in string theory and conformal field theory.

The statement bowls down to the fact that a (quantum) field theory defined using a group G is equivalent to its Langlands dual G^L. Electric-magnetic duality is a similar statement (given by GNO paper) that G and G^L describe the electric and magnetic charges.

The Connection

We can argue that electric-magnetic duality is just another (lighter) reformulation of geometric Langlands duality. In fact, geometric Langlands duality provides a rigorous definition of **mirror symmetry**, which, in turn, defines these dualities in a way that can be understood.

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This also provides a way for theoretical physics to learn from algebraic geometry, representation theory, number theory, and category theory and to cultivate a 'common' tongue between mathematicians and physicists.

A Bridge between Mathematics and Physics

What we have discussed so far in this talk is how some of the physics were basically mathematics and vice-versa. This depicts that most of us who work in isolated zones actually do the right work and somehow physicists do the work of mathematicians and sometimes the role reverses.

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Right now, the hottest part of the mathematical physics is Supergravity and Langlands Program. However, this is my view. But one thing can be realized that discussions between mathematicians and physicists must be held in order to learn what is happening in the other part of the field as it is evident that sometimes it is just one word written in two different languages.

"The most powerful method of advance [is] to perfect and generalize the mathematical formalism that forms the existing basis of theoretical physics." - **P. Dirac**

Thank you for listening.