On the Ramanujan Conjecture and Langlands Program

And connections to theoretical physics

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Ramanujan's Legacy Week
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In this presentation, we will talk about the Ramanujan conjecture and how it relates to the most ambitious program in mathematics, namely **Langlands Program**. And then how this idea connects to a certain area of theoretical physics, which is *Electric-Magnetic* Duality. Good references include arXiv:hep-th/0604151, hep-th/0512172 and Sarnak, Peter. "Notes on the generalized Ramanujan conjectures." Harmonic analysis, the trace formula, and Shimura varieties 4 (2005): 659-685.

Ramanujan Conjecture

Ramanujan conjecture is a statement about the Fourier coefficients of a cuspidal modular form, which are Ramanujan's τ functions.

$$\Delta(q) = q \prod_{m \geq 1} (1 - q^m)^{24} \tag{1}$$

$$= \sum_{n \geq 1} \tau(n) q^n \tag{2}$$

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$$|\tau(p)| \leq 2p^{1/2} \tag{3}$$

where p is a prime number. This was solved by P. Deligne (1974) in his solutions to Weil conjectures. Mordell in 1917 proved that τ functions are multiplicative, i.e,

$$\tau(mn) = \tau(m)\tau(n) \tag{4}$$

when the normalization $\tau(1) = 1$ is satisfied.

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Nevertheless, we will see how it rather relates to the Langlands program, which, in turn, in its geometric Langlands case, represents the electric-magnetic duality in a mathematical way. This was shown by Goddard, Nuyts, and Olive in 1977 and later by Kapustin and Witten in 2006.

A word on Langlands Program

Langlands program is a wide program throughout mathematics that relates the area of representation theory and automorphic forms to Galois theory. One of the many Langlands duality is called **Geometric Langlands Duality**, which is useful in theoretical physics, especially in string theory and conformal field theory.

The statement bowls down to the fact that a (quantum) field theory defined using a group G is equivalent to its Langlands dual G^L. Electric-magnetic duality is a similar statement (given by GNO paper) that G and G^L describe the electric and magnetic charges.

What is Duality?

A *duality* is a correspondence shared by two different theories. In mathematics, the Galois theory is a form of duality. We can take an easy example of T-duality found in string theory.

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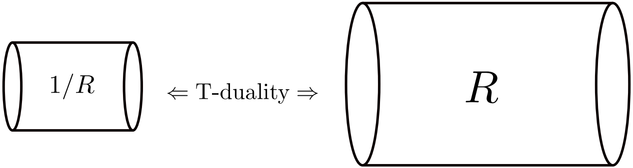

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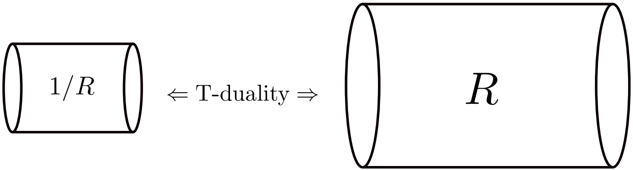

Just like this, we also have an equivalence between coupling constants α and $1/\alpha$. Which is close to the statement of Dirac's quantization condition $e \Leftrightarrow 1/g$.

Generalized Ramanujan Conjecture

Coming back to Ramanujan's conjecture that it can be generalized to any algebraic group, not only modular group $SL(2, \mathbb{Z})$. It was first shown by Satake and it is called **Ramanujan-Petersson** conjecture after Peterson showed that any automorphic forms (of weight greater than 3) can be written in form that we saw for the classical Ramanujan conjecture. For example, the case for GL(2) was done by Satake, later followed by many algebraic geometrists.

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For any automorphic forms (or better Hecke operators), we can study the Langlands program. And Ramanujan was the first person to study this sort of relation until it was generalized.

The Connection

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This also provides a way for theoretical physics to learn from algebraic geometry, representation theory, number theory, and category theory and to cultivate a 'common' tongue between mathematicians and physicists.

Last but not the least...

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We have seen how Ramanujan was the first person to recognize a duality between harmonic analysis and modular forms. Years later, Langlands conjectured his duality, and in 2006, we recognized its geometric importance in theoretical physics. So, Ramanujan's (passive) contributions to physics are not limited to just **black holes**.

Thank you for listening.