

CHAPTER 6

(Just Enough) Algebraic Geometry

“whatever doesn’t kill you simply makes you stranger”

— Joker

6.1. There are two routes to algebraic geometry. The first one goes through the theory of varieties and the second one, which is relatively modern, goes directly through the theory of schemes. In this book, we will discuss about the latter mostly. However, historically speaking, varieties have been crucial for algebraic geometry too and, of course, play a significant role in many parts of mathematics (and physics!). So in the initial parts of this chapter, we will discuss the classical theory of varieties.

6.2. Scheme has been standard for more than over six decades. Before the language of schemes, there was the definition of algebraic variety in the pens of Weil [24] and Zariski (and the abstract variety in the sense of Serre). Affine variety is just a special case of the schemes. The formalization of scheme theory was done by Grothendieck (with the help of works by Serre, Chevalley) in *Éléments de Géométrie Algébrique*.

In this chapter, we will discuss some basics of varieties that will comfort one who has not seen them so far. We will swiftly change our discussion to schemes. This will be an important chapter in Prelude to Schemes as we will discuss most of EGA work herein.

Textbooks [25, 26] do a nice job to introduce varieties and [22] is an excellent resource algebraic geometry in the languages of schemes straightway, and this will be closer to our hearts in this chapter. For the part about varieties, we will follow [25].

6.3. Throughout the chapter, we fix k to be an algebraically closed field unless stated otherwise.

1. A little bit of Variety does not kill you

DEFINITION 6.4 (Affine space over k). An affine n -space over k , denoted as A_k^n , is a set of all n -tuples of elements of k . An element P of A_k^n is called point and the components of

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$P \in A_k^n$ are called coordinates of point P . So if $P = (a_1, \dots, a_n)$ with $a_i \in k$, then we call a_i to be coordinates.

In a very loose sense, A_k^n is just an arbitrary geometric way to write k^n (without an origin). But we will postpone this discuss to a later stage.

6.5. Given a polynomial ring¹ over k , $A = k[x_1, \dots, x_n]$, we can write elements of A as elements $f \in A$ which is a function $f : A_k^n \rightarrow k^n$,

$$f(P) = f(a_1, \dots, a_n), \quad f \in A \text{ and } P \in A_k^n \quad (114)$$

and we can define the zeroes of f as a set $Z(f)$ but say we have a subset $T \subseteq A$, so define the zero set of T as the common zeroes of all the elements of T

$$Z(T) = \{P \in A_k^n \mid \forall f \in T, f(P) = 0\} \quad (115)$$

6.6. We will now denote affine n -space A_k^n by \mathbf{A}^n .

DEFINITION 6.7 (Algebraic Sets). A subset V of \mathbf{A}^n is called *algebraic set* if $V = Z(T)$ for some T .

DEFINITION 6.8. An *algebraic plane curve* over k is an algebraic set $C \subseteq \mathbf{A}^2$. We will hopefully discuss affine curves in detail in detail.

6.9. We will define the Zariski topology on \mathbf{A}^n . Now, the intersection of two algebraic sets is an algebraic set and so is the union of two algebraic sets. (See [25] for a proof.)

The Zariski topology on \mathbf{A}^n is defined by taking the complements of algebraic sets to be open subsets. Since their intersections and unions will be open, it is a topology. And the empty set and whole space being algebraic sets, are both open.

6.10. A non-empty subset W of a topological space X is said to be irreducible if it cannot be written as the union of two proper subsets of X and closed in Y , i.e.,

$$W = W_1 \cup W_2. \quad (116)$$

(A curve is said to be irreducible if its polynomial equation is irreducible.)

DEFINITION 6.11 (Affine Algebraic Variety). We define an *affine algebraic variety* as an irreducible closed subset of \mathbf{A}^n . Moreover, if we take an open subset of an affine variety, then it is called a *quasi-affine variety*.

6.12. Now, the subsets of \mathbf{A}^n can be mapped to ideals in A . Given any subset $V \subseteq \mathbf{A}^n$, we define

$$I(V) = \{f \in A = k[x_1, \dots, x_n] \mid f(P) = 0 \forall P \in V\} \quad (117)$$

¹A polynomial ring is a unique factorization domain (UFD).

THEOREM 6.13 (Hilbert's Weak Nullstellensatz, from [22]). Let $A = k[x_1, \dots, x_n]$ be a polynomial ring and k be an algebraically closed field (see 6.15 for why), then the maximal ideals in A are precisely those ideals of the form $(x_1 - a_1, \dots, x_n - a_n)$, where $a_i \in k$.

6.14. The statement of theorem 6.13 might not be the best way to introduce Hilbert's Nullstellensatz in a progression of varieties. But a statement exists about the correspondence between ideals and varieties.

Note that in Point. 6.5, we defined the zero set of a subset of $A = k[x_1, \dots, x_n]$. Given an ideal $J \subseteq A$, we can define a zero set, let us call it *zero locus* J and re-notate it by

$$V(J) = \{P \in \mathbf{A}^n \mid \forall f \in J \ f(P) = 0\} \quad (118)$$

and when $V(J) = \emptyset$ that means that there are no points in \mathbf{A}^n where polynomials in J vanish simultaneously.

6.15. There could be two situations when $V(J) = \emptyset$. Firstly, when $J = A$, then $1 \in J$ and for any point $P \in \mathbf{A}^n$, we have $1(P) \neq 0$. Therefore $V(J) = \emptyset$ and hence no point lies in $V(J)$. One can deduce that the question of if $V(J) = \emptyset$ becomes non-trivial only if J is a proper ideal in A . Second instance when $V(J) = \emptyset$ happens when k is not algebraically closed, for examples \mathbb{Q}, \mathbb{R} . Given $k = \mathbb{R}$ or $k = \mathbb{Q}$ and an ideal $J = (x^2 + 1) \subseteq \mathbb{R}[x]$ generated by $x^2 + 1$, the zero-locus is empty $V(J) = \emptyset$. Hence, one is required to have an algebraically closed field k .

- It is also worth noting that for $J = 0$, we have $V(J) = \mathbf{A}^n$.
- If $I \subset J$, then $V(I) \supset V(J)$ which is known as inclusion-reversing.
- Since the union and intersection of algebraic sets is algebraic set (see 6.9) and any algebraic set is equal to $V(J)$ for some ideal J , we have $V(I) \cup V(J) = V(I \cap J)$ and for any collection of ideals $\{J_\alpha\}$, the intersection $\bigcap_\alpha V(J_\alpha) = V(\sum_\alpha J_\alpha)$.

THEOREM 6.16 (Hilbert's Weak Nullstellensatz (again)). For any algebraically closed field k , for any proper ideal $J \subsetneq k[x_1, \dots, x_n]$ we have $V(J) \neq \emptyset$.

PROOF. To be written

6.17. The statements in 6.13 and 6.16 are actually the same.

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