SHORT NOTES ON THE COHOMOLOGY OF (QUASI) COHERENT SHEAVES

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ABSTRACT. In this note, we will look at the cohomology of (quasi) coherent sheaves. We also discuss Céch cohomology and discuss Serre's vanishing theorem.

"I just love sheaves. They have algebra this way (and he sliced his hand up and down) and topology this way (and he sliced his hand left to right)" - Donald C. Spencer

This note is a ready waste basket if you are looking for (much) details.
It would be a good idea for us to define quasi-coherent and coherent sheaves (see [1]). We define the sheaf over X as $\mathcal{O}_X$, then the sheaf of $\mathcal{O}_X$-modules make a quasi-coherent sheaf $\mathcal{F}$ which can be restricted on an open sub-scheme $U_i \subset X$ as

$$\mathcal{F}_M \equiv \tilde{M} \tag{1}$$

where $\tilde{M}$ is sheaf defined over the module of the ring R. Locally, quasi-coherent sheaves are the sheaves of the modules over the ring. A morphism between a quasi-coherent sheaf on a scheme X to another is given by the $\mathcal{O}_X$ morphisms.

We can define a coherent sheaf similarly except we need a further condition that $R-$module M should be finitely generated. For the Noetherian scheme, a finitely generated quasi-coherent sheaf will automatically be a coherent sheaf. But for non-Noetherian schemes, it is not guaranteed. That is why, one should be careful defining coherent sheaf as quasi-coherent sheaf which is finitely generated, which is not always true.

The quasi-coherent sheaves on a scheme X form an abelian category and this can be proved if it can be shown that it forms a subcategory of an abelian category.[1] Let us call the category of quasi-coherent sheaves as $\mathcal{Q}_{Coh(X)}$ on a scheme X. It can be

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[1]This is always a better option to do. If one proves that a subgroup of a group exists and if the group is abelian, then the subgroup is also abelian.

shown that is a sub-category of the abelian category $Mod_{\mathcal{O}_X}$

$$\mathcal{Q}_{Coh(X)} \subset Mod_{\mathcal{O}_X}. \tag{2}$$

Proposition. *The category of coherent $R-$modules is a subcategory of an abelian category of $R-$modules.*

Now, let us take a basic look at the Céch Cohomology. We will introduce 'nice' cover $\mathcal{U} = \{U_i\}$ on X. Then, we can introduce p-cochains for the presheaf $\mathcal{F}$ as

$$C^p(\mathcal{U}, \mathcal{F}) = \prod_{\alpha<\beta<\gamma<\cdots<\sigma} \mathcal{F}(U_{\alpha\beta\gamma\cdots\sigma}) \tag{3}$$

where $U_{\alpha\beta\gamma\cdots\sigma} = \mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap \cdots \cap \mathcal{U}_\sigma$ and $\alpha, \beta, \gamma \cdots$ are well ordered. The co-boundary map is defined as $\delta \colon C^p(\mathcal{U}, \mathcal{F}) \to C^{p+1}(\mathcal{U}, \mathcal{F})$ which is a map from p-cochain to $(p+1)$-cochain. A simple exercise is to show that $\delta^2 = 0$. Each co-chain will take values in $\mathcal{F}$.

We can define a sequence

$$0 \longrightarrow \mathcal{F} \xrightarrow{\epsilon} C^0(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta} C^1(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta} C^2(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta} \cdots\cdots \xrightarrow{\delta} C^p(\mathcal{U}, \mathcal{F}) \tag{4}$$

which is exact and where ϵ is defined by the restriction $\mathcal{F}(X) \to \mathcal{F}(U_i)$. Now, one can set $H^p(\mathcal{U}, \mathcal{F}) = H^p(C^p(\mathcal{U}, \mathcal{F}))$. We can now take the inductive limit, as Serre defined [2], and get $\check{H}(X, \mathcal{F})$ as

$$\check{H}(X, \mathcal{F}) = \varinjlim H^p(\mathcal{U}, \mathcal{F}) \tag{5}$$

which is an inductive limit taken over the open covers of X. The covers are ordered by refinement and two covers are equivalent if they refine each other. One can show now that the cohomology group $\check{H}^p$ and H^p are in isomorphism[2] with each other.

A very interesting theorem is provided by Leray which shows an isomorphism between a cohomology with coefficients in a sheaf and the Céch cohomology.

Theorem 1 (Leray)**.** *For a sheaf $\mathcal{F}$ of abelian groups on X and open cover $\mathcal{U}$ on X such that $H^i(U_\alpha \cap U_\beta \cap \cdots \cap U_\sigma, \mathcal{F}) = 0$ for $p > 0$. Then*

$$\check{H}(\mathcal{U}, \mathcal{F}) = H^p(X, \mathcal{F}) \tag{6}$$

and so the natural maps of both sides will be isomorphic.

Proof. See Harstshone [3]. □

Now, we move on to the Serre's vanishing theorem. We will consider *nice* spaces.

[2]Better understood in a more apt language of functors.

Theorem 2 (Serre)**.** *For any affine (separated)*[3] *scheme X and a quasi-coherent sheaf $\mathcal{F}$ on X, we have*

$$H^p(X, \mathcal{F}) = 0 \text{ for } p > 0 \tag{7}$$

For the proof of this theorem, one takes the injective R-modules. Then the (associated) quasi-coherent $\mathcal{O}_X$-module $\tilde{M}$ is *flasque* (or flabby). It is convenient to work with flasque sheaves cause their sections extend and are very helpful in defining sheaf cohomology.

A sheaf $\mathcal{F}$ is called flasque, if for any open subset $U \subset X$, the restriction map $\mathcal{F}(X) \to \mathcal{F}(U)$ is subjective [4]. On an affine scheme X, any invective $\mathcal{O}_X$-module is flasque. It is important to remark that flasque sheaves have trivial higher cohomology and quasicoherent sheaves are not always flasque sheaves.

An important lemma is to prove that an open covering $\mathcal{U}$ on a separated scheme X and a quasi-coherent sheaf $\mathcal{F}$ on X, we have $H^p(X, C^p(\mathcal{U}, \mathcal{F})) = 0$ for $p > 0$. (For proof see [5]) By the property of X which is sepratedness, one can prove Serre's vanishing theorem now. Some details of resolution by flasque sheaves are also required.

A Side Remark. We know that the cohomology with coefficients in a sheaf is isomorphic to Céch cohomology, i.e., $\check{H}(\mathcal{U}, \mathcal{F}) = H^p(X, \mathcal{F})$. In particular, when the presheaf $\mathcal{F} = \mathbb{R}$ is a constant sheaf, then there is a natural isomorphism between the sheaf cohomology and de Rham cohomology

$$H_{\mathrm{dR}}(X) = H(X, \mathbb{R}) \tag{8}$$

where X is now a smooth manifold. One can now always show that the Céch cohomology $\check{H}(\mathcal{U}, \mathbb{R})$ can also be given by de Rham cohomology $H_{\mathrm{dR}(M)}$.

References

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[2] J.-P. Serre, "Faisceaux algébriques cohérents," *Annals of Mathematics* **61** no. 2, (1955) 197–278.

[3] R. Hartshorne, *Algebraic geometry*, vol. 52. Springer Science & Business Media, 2013.

[4] T. Stacks project authors, "The Stacks project." https://stacks.math.columbia.edu, 2025.

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[3] So any finite intersection of the open covers is still affine.