CHAPTER 5

Homological Algebra

> "For my own sake, I have made a systematic (as yet unfinished) review of my ideas of homological algebra. I find it very agreeable to stick all sorts of things, which are not much fun when taken individually, together under the heading of derived functors."
> - Alexander Grothendieck (in his letter to Serre)

5.1. Homological algebra was the most important development by Grothendieck, for example in his Tohuku paper. Our goal in this chapter is to discuss homological algebra in the context of EGA seminars, however, that should not limit us. Of course, some of the material here might be very familiar to you. We will discuss the abstract idea behind chain complexes and them being valued in any category. We will introduce homotopy and chain maps in this chapter. Our focus will be to introduce the homotopy category of chain complexes, do some basic discussions about derived categories as well and projective resolutions and dimensions. Derived functors are discussed as well[1]. One may skip discussions marked with **(*)** as they are not relevant to my original intention behind PtoS.

1. Short Exact Sequences and Chain Complexes

5.2. Most of the focus on abelian categories are given because they admit short exact sequences

$$0 \xrightarrow{f_1} A \xrightarrow{f_2} B \xrightarrow{f_3} C \to 0 \tag{3}$$

such that at each point, it is 'exact' which means that $ker(f_{n+1}) = im(f_n)$ and also notice that $f_{n+1} \circ f_n = 0$.

5.3. With the above definition, we have following examples

(1) A sequence $0 \to V \xrightarrow{f} W$ is exact if f is injective.
(2) A sequence $V \xrightarrow{f} W \to 0$ is exact if f is surjective.
(3) A sequence $0 \to V \xrightarrow{f} W \to 0$ is exact if f is an isomorphism.

[1]A part of these notes are taken from my written notes (and a poster) in a fall 2025 course in representation theory of quivers by Prof. Amit Kuber.

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(4) A sequence $0 \to V \xrightarrow{f} W \xrightarrow{g} U \to 0$ is exact if f is injective, g is surjective and $ker\ g = im\ f$.

(5) Given $V \xrightarrow{f} W$, the following is always exact

$$0 \to ker\ f \xrightarrow{i} V \xrightarrow{f} W \xrightarrow{q} cok\ f \to 0 \tag{4}$$

then $ker\ q = im\ f$ and $ker\ f = im\ i$.

Definition 5.4. A chain complex is defined as a sequence

$$\cdots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots \tag{5}$$

where the $f_n \circ f_{n+1} = 0$. The maps d_n are called differentials.

For example, for $R - Mod$, it is a family of $R - Mod$, $\{C_n\}_{n \in \mathbb{Z}}$ where $C_n \in R - Mod$.

5.5. We can define the kernel of d_n to be the module of n-cycles in $R - Mod$, denoted by $Z_n(C_\bullet)$. Again, we can define image of d_n to be the module of n-boundaries in $R - Mod$, denoted by $B_n(C_\bullet)$. It is clear to see that

$$0 \subseteq Z_n \subset B_n \subset C_n \tag{6}$$

Alternatively, $ker(f_{n+1}) \subseteq img(f_n)$.

5.6. For a chain complex to be exact, we just mention that $Z_n = B_n$ for some chain complex $C_\bullet$.

Definition 5.7. When a chain complex fails to be exact, which means that Z_n/B_n is non-trivial quotient group. This guotient group is called 'Homology' $H_n(C_\bullet)$.

This essentially measures by how much the chain complex fails to be 'exact' at n.

5.8. Similarly, one can define cochain complexes and cohomology groups for them. These are dual descriptions.

Example 5.9 (*). A very algebraically clean example of (co)-chain complex and (co)-homology in physics is electromagnetism.[2] We take a complex of differential forms Ω^p on a manifold M

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \Omega^3(M) \to \cdots \tag{7}$$

where $\Omega^0(M)$ is space of functions on M and $\Omega^1(M)$ is a space of 1-forms and so on. The map $d : \Omega^p \to \Omega^{p+1}$ is an exterior derivative map with $d^2 = 0$. The failure of this sequence to be exact is measured by the de Rham co-homology. An excellent resource is [3].

We will possibly de Rham (co)homology possibly later.

[2] In electromagnetism, the 2-form F curvature solves $dF = 0$.

EXAMPLE 5.10. Now we will look at a very interesting example of **s.e.s.** . Given V, W in $Vect_k$, the sequence

$$0 \to V \xrightarrow{i_1} V \oplus W \xrightarrow{\pi_2} W \to 0 \tag{8}$$

where $i_1 : V \to V \oplus W$ is an inclusion map and $\pi_2 : V \oplus W \to W$ is a projection map, the above sequence (8) is an exact sequence since $ker(\pi_2) = im(i_1)$. This sequence will serve as a canonical example of split sequences.

DEFINITION 5.11 (Split Sequence). A short exact sequence

$$0 \to V \xrightarrow{f} W \xrightarrow{g} U \to 0 \tag{9}$$

is said to be split if $\exists\ g' : U \to W$ such that $g \circ g' = 1_U$

$$\begin{tikzcd} 0 \arrow[r] & V \arrow[r] & W \arrow[r, shift left, "g"] & U \arrow[l, shift left, dashed, "g'"] \arrow[r] & 0 \end{tikzcd}$$

For example, the below sequence splits

$$\begin{tikzcd} 0 \arrow[r] & V \arrow[r, "i_1"] & V \oplus W \arrow[r, shift left, "\pi_2"] & W \arrow[l, shift left, dashed, "i_2"] \arrow[r] & 0 \end{tikzcd}$$

We have the following proposition.

5.12. A short exact sequence

$$0 \to V \xrightarrow{f} W \xrightarrow{g} U \to 0 \tag{10}$$

is said to be split if $\exists\ f' : W \to V$ such that $f' \circ f = 1_V$.

So both are equal definitions which means that for our example, the existence of i_2 implies the existence of π_1. These are statements of left split and right split in the Splitting Lemma. In general, the existence of g' will imply that f' exists. To see if this is true, we can look at the following diagram

$$\begin{tikzcd} 0 \arrow[r] & V \arrow[r, "f"] \arrow[d] & W \arrow[r, "g"] \arrow[d, shift right, "\binom{f'}{g}"'] & U \arrow[r] & 0 \\ 0 \arrow[r] & V \arrow[r, "i_1"] & V \oplus U \arrow[r, "\pi_2"] \arrow[u, shift right, "(f\ g')"'] & U \arrow[r] & 0 \end{tikzcd}$$

where $(f\ g')$ is an isomorphism and the diagram must commute.

We suggest that the **s.e.s.**

$$0 \to V \xrightarrow{i_1} V \oplus W \xrightarrow{\pi_2} W \to 0 \tag{11}$$

is the canonical example of a split sequence. Any **s.e.s.** which is isomorphic to above sequence (11) is also a split sequence.

2. Chain Complexes and Chain Maps

DEFINITION 5.13. We define a category of chain complexes $Ch(\mathcal{A})$ with objects as chain complexes, where the morphism are *chain complex maps*. Given two chain complexes $C_\bullet, D_\bullet$, we have a chain map u

$$\begin{tikzcd} \cdots \arrow[r] & C_{n+1} \arrow[r] \arrow[d, dashed, "u_{n+1}"] & C_n \arrow[r] \arrow[d, dashed, "u_n"] & C_{n-1} \arrow[r] \arrow[d, dashed, "u_{n-1}"] & \cdots \\ \cdots \arrow[r] & D_{n+1} \arrow[r] & D_n \arrow[r] & D_{n-1} \arrow[r] & \cdots \end{tikzcd}$$

where each square commutes. It is an important fact that a chain map $u : C_\bullet \to D_\bullet$ induces a map $H_n(C_\bullet) \to H_n(D_\bullet)$ since u sends boundaries $B_n(C_\bullet)$ to boundaries $B_n(D_\bullet)$ and cycles $Z_n(C_\bullet)$ to cycles $Z_n(D_\bullet)$.

DEFINITION 5.14. We define *splitting maps* s_n for some chain complex C_n as $s_n : C_n \to C_{n+1}$ such that it is called a split chain complex if $d_n = d_{n+1} \circ s_n \circ d_n$. If given two chain complexes C_n and D_n with a chain map $u_n : C_n \to D_n$, we choose maps $s_n : C_n \to D_{n+1}$

$$\begin{tikzcd} \cdots \arrow[r] & C_{n+1} \arrow[r, "d^C_{n+1}"] \arrow[d, "u_{n+1}"'] & C_n \arrow[r, "d^C_n"] \arrow[d, "u_n"] \arrow[dl, "s_n"'] & C_{n-1} \arrow[r, "d^C_{n-1}"] \arrow[d, "u_{n-1}"] & \cdots \\ \cdots \arrow[r] & D_{n+1} \arrow[r, "d^D_{n+1}"] & D_n \arrow[r, "d^D_n"] & D_{n-1} \arrow[r, "d^D_{n-1}"] & \cdots \end{tikzcd}$$

where the commutativity is given by $d^D_{n+1} \circ u_{n+1} = u_n \circ d^C_{n+1}$ and the chain map $u_n = d_{n+1} \circ s_n + s_{n-1} \circ d^C_n$.

5.15. When there exist splitting maps $s_n : C_n \to D_{n+1}$ and a chain map $u_n : C_n \to D_n$ where $u_n = d_{n+1} \circ s_n + s_{n-1} \circ d^C_n$, then u is called a null homotopic chain map. When given two chain maps $u_n, v_n : C_n \to D_n$, we call them *null homotopic* if their difference is

$$u_n - v_n = d_{n+1} \circ s_n + s_{n-1} \circ d^C_n$$

(12)

and the maps $\{s_n\}$ are called chain homotopy from u to v.

5.16. Furthermore, a chain map $u_n : C_n \to D_n$ is a homotopy equivalence if there exists a map $v_n : D_n \to C_n$ such that uv is chain homotopic to the identity on D and vu is chain homotopic to the identity on C.

3. Snake's Lemma

5.17. Snake's Lemma is a powerful tool to create six term exact sequences from two short exact sequences with a zero object. The proof of the lemma is usually a fun diagram chase. A succinct proof is also available in the beginning of the movie *It's my turn, 1980.* We will focus on the heuristics of the lemma.

5.18. Given two row short exact sequences in an abelian category $\mathcal{A}$, when we have a commutative diagram

$$\begin{tikzcd}
 & A \arrow[r, "a"] \arrow[d, "f"'] & B \arrow[r, "b"] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "h"'] & 0 \\
0 \arrow[r] & A' \arrow[r, "a'"] & B' \arrow[r, "b'"] & C' &
\end{tikzcd}$$

it gives us a six-term exact sequence

$$ker(f) \longrightarrow ker(g) \longrightarrow ker(h) \xrightarrow{\partial} coker(f) \longrightarrow coker(g) \longrightarrow coker(h)$$

where $\partial : ker(h) \to coker(f)$ is a connecting homomorphism. If the morphism a is a monomorphism, then $ker(f) \to ker(g)$ is a monomorphism and if the morphism b' is an epimorphism, then so is $coker(g) \to coker(h)$.

We obtain the exact sequence by expanding the commutative diagram which gives us a sequence in shape of a 'slithering snake' [A visual diagram of the Snake!]

$$\begin{tikzcd}
 & & ker(f) \arrow[r, dash, dashed] \arrow[d] & \cancel{ker(g)} \arrow[r, dash, dashed] \arrow[d] & ker(h) \arrow[rr, dashed] \arrow[d] & & \bullet \arrow[d] \\
\bullet \arrow[dd, dashed] & & A \arrow[r, "a"] \arrow[d, "f"'] & B \arrow[r, "b"] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "h"'] & 0 & \bullet \arrow[llllll, dashed] \\
 & 0 \arrow[r] & A' \arrow[r, "a'"] \arrow[d] & B' \arrow[r, "b'"] \arrow[d] & C' \arrow[d] & & \\
\bullet \arrow[rr, dash, dashed] & & \cancel{coker(f)} \arrow[r, dash, dashed] & \cancel{coker(g)} \arrow[r, dashed] & coker(h) & &
\end{tikzcd}$$

The connecting homomorphisms ∂ are important in constructing long exact sequences in homological algebra. The proof of the lemma is given by 'diagram chasing'. The proof requires two step: 1) constructing the connecting homomorphism and 2) proving exactness at each point.

A detailed proof will be added in a later draft of these notes.

4. Abelian Categories and Homotopy Category of Complexes

DEFINITION 5.19 (Pre-Additive Category). A category $\mathcal{C}$ is pre-additive if each morphism set $Mor_{\mathcal{C}}(a, b)$ has the structure of an abelian group such that the composition

$$Mor(a, b) \times Mor(b, c) \rightarrow Mor(a, c) \tag{13}$$

is bilinear.

DEFINITION 5.20. An object which is both a final object and an initial object in a pre-additive category $\mathcal{C}$ is called a zero object and denoted by 0.

DEFINITION 5.21 (Additive Category). We call a pre-additive category $\mathcal{C}$ *additive* category if it admits finite bi-products[3].

DEFINITION 5.22 (Abelian Category). We call a category $\mathcal{C}$ abelian if it satisfies the following properties

(1) It is an additive category.
(2) Kernels and their dual co-kernels exist.
(3) Every injective morphism is a kernel of its own co-kernel.
(4) Every surjective morphism is a co-kernel of its own kernel.

5.23. Essentially, an abelian category is an abstraction of the (basic) properties of category of abelian groups. It is believed to be introduced by Buchsbaum [**4**] in 1955 as exact categories and later standardized by Grothendieck in his Tohoku paper [**5**] using axiomatic approach. However, the term 'abelian category' was termed by Freyd [**6**].

EXAMPLE 5.24. A good example of abelian category is category of representation of a group *Rep*(G) or the module category of an artinian algebra Λ over field k.

5.25. The most interesting reason for studying abelian categories is that they admit short exact sequences that we have discussed above. In later parts of this project, we will be interested to read more than exact sequences like *triangles* in triangulated categories.

5.26. We will digress to recall what is a quasi-isomorphism. When we want to just know the homological information of the complexes and their equivalence.

DEFINITION 5.27. A chain map of complexes $u : C_\bullet \rightarrow D_\bullet$ in abelian category $\mathcal{A}$ is called a quasi-isomorphism if the induced homology morphism $u_* = H^n(C_\bullet) \rightarrow H^n(D_\bullet)$ is an isomorphism.

From homology point of view, two complexes $C_\bullet$ and $D_\bullet$ become distinguishable.

EXAMPLE 5.28. For example, there exists a quasi-isomorphism among any two projective resolutions (or injective resolutions) of same object.

[3]The finite products and fintie co-products coincide here

THEOREM 5.29. If a map $u : C_\bullet \to D_\bullet$ is a homotopy equivalence, then it is a quasi-isomorphism.

5.30. Note that a quasi-isomorphism is not always a homotopy equivalence.

DEFINITION 5.31. Let $\mathcal{A}$ be an additive category, we define a homotopy category of chain complexes $K(\mathcal{A})$ as follows

- Objects: The objects of $K(\mathcal{A})$ are chain complexes.
- Morphisms: For any two chain complexes $C_\bullet, D_\bullet \in \mathcal{A}$, we define the set of morphisms as the set of homotopy classes of chain maps from $C_\bullet$ to $D_\bullet$
$$Hom_{K(\mathcal{A})}(C_\bullet, D_\bullet) = Hom_{Ch(\mathcal{A})}(C_\bullet, D_\bullet)/\sim \tag{14}$$
where any two chain maps u, v are chain homotopic, then $u \sim v$.

5.32. Essentially, we can define a functor $Ch(\mathcal{A}) \to K(\mathcal{A})$ which is identity on the objects and quotient projection on the morphisms. Moreover, $K(\mathcal{A})$ is an additive category as well as *a triangulated category*, however, it is not an abelian category in general.

5.33. The rescue needed to do homological algebras for $K(\mathcal{A})$, since it fails to be abelian category, is provided by triangulated structure on it which contain distinguished triangles from which we get a long exact sequence of homology groups. While $K(\mathcal{A})$ is a 'nice' category to work with homotopical settings. It fails to identify the quasi-isomorphisms and that motivates the construction of derived categories.

5.34. The issue with $K(\mathcal{A})$ being that it does not recognize homotopy equivalence which are not quasi-isomorphism requires an abstract settings of 'derived categories'. The goal is then to construct a category where all quasi-isomorphisms become isomorphisms.

5.35. We define derived category $D(\mathcal{A})$ for an abelian category $\mathcal{A}$ as the localization of the homotopy category $K(\mathcal{A})$ with respect to the class of all quasi-isomorphisms. A morphism between complexes $C_\bullet$ and $D_\bullet$ is no more only class of chain maps. Instead using universality, we have following roof diagram
$$C_\bullet \xleftarrow{f} F_\bullet \xrightarrow{g} D_\bullet \tag{15}$$
where f is a quasi-isomorphism and g is a morphism in $K(\mathcal{A})$. In a sense, it is more natural way to construct homotopy categories.

Derived categories are natural playground for derived functors, higher algebra, representation theory, and mathematical physics.

5. Mapping Cones and Triangulated Category

Triangulated categories were introduced by Jean-Louis Verdier (Grothendieck) in 1963 [7].

DEFINITION 5.36. Let $\mathcal{C}$ be an additive category, then define Σ to be automorphism of $\mathcal{C}$. The automorphism $\Sigma : \mathcal{C} \to \mathcal{C}$ is called the translation functor of $\mathcal{C}$.

5.37. Define a sextuple (X, Y, Z, u, v, w) given by the objects $(X, Y, Z) \in \mathcal{C}$ and morphisms $u : X \to Y$, $v : Y \to Z$ and $w : Z \to \Sigma X$. Basically, we can write the sextuple as

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} Z \to \Sigma X \tag{16}$$

5.38. We can find the morphisms between two sextuples

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, dashed, "g"] & Z \arrow[r, "w"] \arrow[d, dashed, "h"] & \Sigma X \arrow[d, "\Sigma f"] \\ X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & \Sigma X' \end{tikzcd}$$

such that each square commutes.

Now, consider a set T of all sextuples in $\mathcal{C}$, then set T is a triangulation of $\mathcal{C}$ if they satisfy Verdier's axioms. Any element of T is called a **triangle**.

5.39. These Verdier's axioms are as follows.

TR1 For every object X, the following is a distinguished triangle

$$X \to X \to 0 \to \Sigma X \tag{17}$$

(Also, consider any morphism $u : X \to Y$, we can get a triangle $X \xrightarrow{u} Y \to cone(u) \to \Sigma X$)

Any triangle isomorphic to a distinguished triangle is a distinguished triangle.p

TR2 The following is a distinguished triangle precisely

$$X \xrightarrow{u} Y \to Z \to \Sigma X \tag{18}$$

if

$$Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X \xrightarrow{-\Sigma u} \Sigma Y \tag{19}$$

is a distinguished triangle.

TR3 The following diagram of morphism between two triangles commute

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, dashed, "g"] & Z \arrow[r, "w"] \arrow[d, dashed, "h"] & \Sigma X \arrow[d, "\Sigma f"] \\ X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & \Sigma X' \end{tikzcd}$$

TR4 Also called *octahedral axiom.* Consider three triangles and the axioms states that their mapping cones will form a distinguished triangle. [We can ignore this for this talk but they are crucial in determining the non-functoriality of mapping cones[4] in triangulated categories.]

DEFINITION 5.40. An additive category $\mathcal{C}$ with translation functor Σ which admits triangulation is called a **triangulated** category $(\mathcal{C}, \Sigma, T)$.

5.41. Example of our interest: Derived category $D(A)$ of an abelian category A is a triangulated category and so is $D^b(A)$.

We study triangulated categories because it generalizes short exact sequences to distinguished triangles

$$0 \to X \to Y \to Z \to 0 \tag{20}$$

in Abelian category A is generalized as a triangle in $D^b(mod - A)$

$$X \to Y \to Z \to X[1] \tag{21}$$

6. Projective Resolutions and Dimension

5.42. In this section, we wish to discuss projective resolutions (and dually, injective resolutions) and various kinds of definitions of dimensions in homological algebra.

I believe the motivation is slightly to associate to every object (modules, representations and so on) a certain complex which helps us to get a larger picture about the category. This should become apparent in sometime.

DEFINITION 5.43. In an abelian category, an object P is called projective, if given any surjection $f : A \to B$, there exists a map $g : P \to B$ such that one has a universal lifting property

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists h"'] \arrow[d, "g"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

where $g = fh$.

5.44. We are mostly dealing with category of (right) modules over an algebra. And projective module is an important object for us. We also find that a projective P helps a sequence of $Hom(P, -)$ to form a short exact sequence, in other words $Hom(P, -)$ is an exact functor iff P is a projective.

PROPOSITION 5.45. A module in $A - mod$ P is projective if it is a direct summand of a free module.

[4] https://aayushayh.blogspot.com/2025/09/the-quill-24-failure-of-functorial.html

PROOF. A free module is a projective module. Let say F is a free module $F \cong Q \oplus P$. We have the following maps

$$i : P \to F \tag{22}$$

and if P is a projective, then by universal property

$$\pi : F \to P \tag{23}$$

which means $\pi \circ i = id_P$ and

$$0 \to Q \to F \to P \to 0 \tag{24}$$

and hence P is a direct summand of free module F.

PROPOSITION 5.46. If P is a projective, then $Ext^1(P, M) = 0$ for every $A - mod$ M.

PROOF. to be written

REMARK 5.47. Projective modules are closed under direct sums.

5.48. A chain complex with each object a projective is called a chain complex of projectives. Note that it does not have to be a projective object in the category of chain complexes.

5.49. We call an abelian category $\mathcal{A}$ with enough projectives if for every object A in $\mathcal{A}$, there exists a surjective map $P \to A$. A dual definition exist for an abelian category with enough injectives.

5.50. As alluded earlier, for a projective P in $\mathcal{A}$ $Hom(P, -)$ is an exact functor which means that for every **s.e.s.** in $\mathcal{A}$

$$0 \to A \to B \to C \to 0 \tag{25}$$

the functor $Hom(P, -)$ gives a short exact sequence

$$0 \to Hom(P, A) \to Hom(P, B) \to Hom(P, C) \to 0 \tag{26}$$

and dually, $Hom(-, I)$ is an exact functor for an injective object I in $\mathcal{A}$.

5.51. A object P is projective in **Ch** (category of chain complexes) if and only if it is a split exact complex. It is not hard to realize this proposition. Let us look at only if direction. Say, $P_\bullet$ is a projective complex, then each P_n is a projective object.

To realize the split exactness of the complex, we say that a short exact sequence

$$0 \to P_\bullet \xrightarrow{i} Cone(id_{P_\bullet}) \xrightarrow{p} P_\bullet[-1] \to 0 \tag{27}$$

where $Cone(id_{P_\bullet})$ is the mapping cone and since $P_\bullet$ is a projective in **Ch**, every epimorphism $p : Cone(id_{P_\bullet}) \to P_\bullet[-1]$ admits a lift of $id_{P_\bullet}[-1]$ such that there is a chain map

$$q : P_\bullet[-1] \to Cone(id_{P_\bullet}) \tag{28}$$

and $p \circ q = id_{P_\bullet}[-1]$. One can also show split exactness as well. (incomplete, to be added in next update, see Exercise 2.2.1 in [8]).

DEFINITION 5.52 (Projective Resolution). For every object[5] M in $\mathcal{A}$, we write a projective resolution as an exact sequence

$$\cdots \to P_2 \to P_1 \to P_0 \xrightarrow{\epsilon_0} M \to 0 \tag{29}$$

where $\epsilon_0 : P_0 \to M$ is augmentation map. One can think of it as a surjective cover of M which is an epimorphism. Alternatively, we can think of a chain complex $P_\bullet$ which is concentrated in nonnegative degree such that $P_i = 0$ for $i < 0$ along with an augmentation map $\epsilon_0 : P_0 \to M$ such that Eqn. (29) is exact.

DEFINITION 5.53 (Injectives). We have delayed the definition of an injective object but we can always rely on the 'dualness' of projective-injective to mention every definition. One defines an injective object I in $\mathcal{A}$ as following. Given an injection $f : A \to B$, there exist a map $\alpha : A \to I$ such that following commutes

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[dl, dashed, "\exists h"] \\ I & \end{tikzcd}$$

5.54 (Baer's Criterion). For a category of right A-modules $A - mod$, the following statements are equivalent for an object M in $A - mod$

- M is an injective object of category in the sense of Def. 5.53.
- For any right ideal I in A, the linear map $I \to M$ can be extended to $A \to M$.

5.55. As mentioned in 5.50, $Hom(P, -)$ is an exact functor. Dually, $Hom(-, I)$ is an exact functor for the short exact sequence

$$0 \to A \to B \to C \to 0 \tag{30}$$

where $Hom(-, I)$ is a contravariant functor.

DEFINITION 5.56 (Injective Resolution). We will find projective resolutions introduced above in 5.52 very important at many places. Similarly, we can define an injective resolution as

$$M \to I^1 \to I^2 \to I^3 \to \cdots \tag{31}$$

where I_i are injectives.

It is also interesting to see that $P_\bullet \to M$ and $M \to I_\bullet$ are chain map and cochain map respectively and describe the same complex where M is concentrated in degree 0.

[5]Technically, one would need a category with enough projectives and fortunately, an abelian category admits enough projectives which means that there is a projective cover for each object in the category.

5.57. We would like to borrow the story from Weibel [8] to show that a category with enough projectives admit projective resolution for each object M. We briefly alluded to it in footnote 5.

$$\begin{tikzcd}
0 \arrow[dr] & & 0 & & 0 \arrow[dr] & & 0 & & & \\
& M_3 \arrow[ur] \arrow[dr] & & & & M_1 \arrow[ur] \arrow[dr] & & & & \\
\cdots \arrow[ur] \arrow[rr] & & P_3 \arrow[rr, "f_2"'] \arrow[dr] & & P_2 \arrow[rr, "f_2"'] \arrow[ur] & & P_1 \arrow[rr, "f_1"'] \arrow[dr] & & P_0 \arrow[r, "\epsilon_0"'] & M \\
& & & M_2 \arrow[ur] \arrow[dr] & & & & M_0 \arrow[ur] \arrow[dr] & & \\
& & 0 \arrow[ur] & & 0 & & 0 \arrow[ur] & & 0 &
\end{tikzcd}$$

The proof is done by using slicing method and induction. We have a surjection $\epsilon : P_0 \to M$, we can define $M_0 = ker(\epsilon_0)$ and by induction $M_n = ker(\epsilon_n)$ and say $i_n : M_n \to P_n$. Then our map $f_n : P_n \to P_{n-1}$ becomes

$$f_n = i_{n-1} \cdot \epsilon_n \tag{32}$$

which is a chain complex.

5.58. Since these projective resolutions are chain complexes with projective terms, we can define a chain map between two projective resolutions which will be unique up to chain homotopy, see Sec. 2. Let us take objects M, N, then the two projective resolutions are given by

$$\cdots \to P_2 \to P_1 \to P_0 \xrightarrow{\epsilon_0} M \to 0 \tag{33}$$

$$\cdots \to Q_2 \to Q_1 \to Q_0 \xrightarrow{\eta_0} N \to 0 \tag{34}$$

and say $f : M \to N$, then following is a chain map between $P_\bullet$ and $Q_\bullet$ (chain complex of projectives)

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r] \arrow[d, "u_2"] & P_1 \arrow[r] \arrow[d, "u_1"] & P_0 \arrow[r, "\epsilon_0"] \arrow[d, "u_0"] & M \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r] & Q_2 \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r, "\eta_0"] & N \arrow[r] & 0
\end{tikzcd}$$

where u_n chain map lifts the morphism $f\colon M \to N$.

Two projective resolutions of same object M can be chain homotopy equivalent. Say $P_\bullet \to M$ and $Q_\bullet \to M$ are two projective resolutions of object M. Then they are chain homotopy equivalent if there exists

$$u_n : P_\bullet \to Q_\bullet, \quad v_n : Q_\bullet \to P_\bullet \tag{35}$$

such that $u_n \circ v_n \cong id_{Q_\bullet}$ and $v_n \circ u_n \cong id_{P_\bullet}$.

One says that projective resolutions which are equivalent up to homotopy make up a category which sits inside the homotopy category of chain complexes $K(\mathcal{A})$. This means that choice of projective resolutions do not matter (up to chain homotopy).

LEMMA 5.59 (Horseshoe Lemma). Let $\mathcal{A}$ be an abelian category with enough projectives and suppose we have been given an extension (we will come to extensions very soon in detail, for now, we can think of it just as a short exact sequence)

$$0 \to M' \to M \to M'' \to 0 \tag{36}$$

and suppose, we have a commutative diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
\cdots \arrow[r] & P_2' \arrow[r] & P_1' \arrow[r] & P_0' \arrow[r, "\epsilon_o'"] & M' \arrow[r] \arrow[d] & 0 \\
 & & & & M \arrow[d] & \\
\cdots \arrow[r] & P_2'' \arrow[r] & P_1'' \arrow[r] & P_0'' \arrow[r, "\epsilon_o''"] & M'' \arrow[r] \arrow[d] & 0 \\
 & & & & 0 &
\end{tikzcd}$$

where the rows are projective resolutions of M' and M'' and both rows and the column are exact. We can now find the projective resolution of M from be

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & P_2' \arrow[r] \arrow[d] & P_1' \arrow[r] \arrow[d] & P_0' \arrow[r, "\epsilon_o'"] \arrow[d] & M' \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & P_2 \arrow[r] \arrow[d] & P_1 \arrow[r] \arrow[d] & P_0 \arrow[r, "\epsilon_0"] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & P_2'' \arrow[r] \arrow[d] & P_1'' \arrow[r] \arrow[d] & P_0'' \arrow[r, "\epsilon_o''"] \arrow[d] & M'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}$$

where we set $P_n = P_n'' \oplus P_n'$ and the exactness of the our initial extension lifts to the extension of $0 \to P_n' \to P_n \to P_n'' \to 0$.

The proof can be achieved by induction, one may see 2.2.8 in [8]. A dual lemma exists for injective resolutions for abelian category with enough injectives.

Definition 5.60 (Projective Dimension). For a module M, we have a projective resolution

$$\cdots \to P_3 \to P_2 \to P_1 \to P_0 \xrightarrow{\epsilon_0} M \to 0 \tag{37}$$

The projective dimension $\mathbf{p.dim(M)}$ of the module M is the minimum integer n such that we have following

$$0 \to P_n \to \cdots P_2 \to P_1 \to P_0 \xrightarrow{\epsilon_0} M \to 0 \tag{38}$$

Definition 5.61 (Injective Dimension). For a module M, injective dimension **i.dim(M)** is the minimum integer n such that the resolution is

$$0 \to M \to I_0 \to I_1 \to I_2 \to \cdots \to I_n \to 0 \tag{39}$$

5.62. If such minimal integer n does not exist, we write that the projective dimension (or injective dimension if that is the concern) is infinite. The number $pd(M)$ is a *homological invariant* for module M.

There is also a resolution by flat modules and the definition of flat dimension $fd(M)$ is similar.

Definition 5.63 (Global Dimension). For any algebra A (or ring), global dimension $gl.dim(A)$ is a homological invariant which is given by

$$gl.dim(A) = sup\{p.dim(M) : M \in A - mod\} \tag{40}$$

and equivalently,

$$gl.dim(A) = sup\{i.dim(M) : M \in A - mod\} \tag{41}$$

5.64 (*). For hereditary algebra, the global dimension is at most one[6].

5.65 (*). In modern representation theory, $gl.dim$ is an important invariant. Having a finite global dimension versus having infinite global dimension are interesting in their own rights. For example, one can show that for finite global dimension for finite dimensional, $k-$algebra A (Happel's Theorem [9])

$$D^b(A - mod) \simeq \underline{Mod}(\hat{A}) \tag{42}$$

where the right side is the stable module category of a repetitive algebra $\hat{A}$. However, this is not true for infinite global dimension of A.

[6]Basically, at most two projectives in the resolution.

7. Derived Functors and Ext

5.66. We will start the discussion on the derived functors in this section. We will first describe Grothendieck's δ-functor [5]. Throughout, $\mathcal{A}$ and $\mathcal{B}$ are two abelian categories unless stated otherwise.

DEFINITION 5.67 (δ-functor). Given a functor $\mathcal{F}$ between $\mathcal{A}$ and $\mathcal{B}$

$$\mathcal{F} : \mathcal{A} \to \mathcal{B} \tag{43}$$

and a short exact sequence in $\mathcal{A}$

$$0 \to A \to B \to C \to 0 \tag{44}$$

we define a (homological[7]) δ-functor as a collection of additive functors $\mathcal{F}_n$ for $n \geq 0$ with a connecting morphism

$$\delta_n : \mathcal{F}_n(C) \to \mathcal{F}_{n-1}(A) \tag{45}$$

such that we get a long exact sequence from Eqn. (44)

$$\cdots \to \mathcal{F}_{n+1}(C) \xrightarrow{\delta} \mathcal{F}_n(A) \to \mathcal{F}_n(B) \to \mathcal{F}_n(C) \xrightarrow{\delta} \mathcal{F}_{n-1}(C) \to \cdots \tag{46}$$

5.68. For any two short exact sequences

$$0 \to A \to B \to C \to 0 \tag{47}$$

$$0 \to A' \to B' \to C' \to 0 \tag{48}$$

the following diagram commute

$$\begin{tikzcd}
\mathcal{F}_n(C') \arrow[r, "\delta_n"] \arrow[d] & \mathcal{F}_{n-1}(A') \arrow[d] \\
\mathcal{F}_n(C) \arrow[r, "\delta_n"'] & \mathcal{F}_{n-1}(A)
\end{tikzcd}$$

5.69. We can define for two delta functors $\mathcal{F}$ and $\mathcal{G}$, a morphism is given by natural transformations $\eta_n : \mathcal{F}_n \to \mathcal{G}_n$ that commute with δ.

DEFINITION 5.70 (Universal δ-functor). We call a δ-functor $\mathcal{F}$ universal if for any given δ-functor $\mathcal{G}$ the natural transformation $\eta_0 : \mathcal{G}_0 \to \mathcal{F}_0$ extends to an unique family of morphisms $\{\eta_n : \mathcal{G}_n \to \mathcal{F}_n\}$.

5.71. Given a short exact sequence

$$0 \to A \to B \to C \to 0 \tag{49}$$

[7]Of course, a cohomological definition of the functor exists as well.

and a $Hom(-,X)$ functor, which is a contravariant functor (a functor for which domain is objects in $\mathcal{C}^{op}$) and is a left-exact functor

$$0 \to Hom(C,X) \to Hom(B,X) \to Hom(A,X) \tag{50}$$

but we can not guarantee the last zero unless X is injective. Conversely, the same argument applies for $Hom(X,-)$ which is a right exact functor and becomes exact if and only if X is projective.

5.72. Before we discuss finding a long exact sequence for any short exact sequence

$$0 \to A \to B \to C \to 0 \tag{51}$$

for any functor $F : \mathcal{A} \to \mathcal{B}$, we should briefly allude to our discussion on Snake's lemma before here. Snake's lemma is a way to get a long exact sequence from short exact sequence. One should emphasize on the connecting morphisms δ.

Example 5.73 (*). Let us take an example of sheaves

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0 \tag{52}$$

and the long exact sequences of sheaf (co)homology on a topological space X is given by

$$\cdots \to H^n(X,\mathcal{F}) \to H^n(X,\mathcal{G}) \to H^n(X,\mathcal{H}) \to H^{n+1}(X,\mathcal{G}) \to \cdots \tag{53}$$

A good example in physics would be consider a short exact sequence of sheaves

$$0 \to \mathbb{Z} \to C^\infty(R) \to C^\infty(U(1)) \to 0 \tag{54}$$

and since we have $H^n(X, C^\infty(R)) = 0$ for $n > 0$, we have the isomorphism

$$H^1(X, C^\infty(U(1))) \cong H^2(X,\mathbb{Z}) \tag{55}$$

and

$$H^2(X, C^\infty(U(1))) \cong H^3(X,\mathbb{Z}) \tag{56}$$

write physics relevance

Definition 5.74 (Left Derived Functor). Let $\mathcal{F}$ be a right-exact functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$, then we know that any short exact sequence in $\mathcal{A}$ (which is an abelian category with enough projectives, see Footnote. 5 and Note. 5.57) becomes

$$\mathcal{F}(A) \to \mathcal{F}(B) \to \mathcal{F}(C) \to 0 \tag{57}$$

which is exact in $\mathcal{B}$ since $\mathcal{F}$ is right-exact. We can define left derived functor $L_i(\mathcal{F})$ for $\mathcal{F}$ for $i \geq 0$. Take an object M in $\mathcal{A}$ and write the projective resolution of M

$$\cdots \to P_2 \to P_1 \to P_0 \xrightarrow{\epsilon_0} M \to 0 \tag{58}$$

and define

$$L_i\mathcal{F}(M) = H_i(\mathcal{F}(P_\bullet)) \tag{59}$$

and each $L_i\mathcal{F}$ is an additive functor from $\mathcal{A}$ to $\mathcal{B}$.

5.75. We have $L_0\mathcal{F}(M) \cong \mathcal{F}(M)$ which is true since

$$\cdots \to \mathcal{F}(P_1) \to \mathcal{F}(P_0) \to \mathcal{F}(M) \to 0 \tag{60}$$

is exact at $\mathcal{F}(M)$ and $\mathcal{F}(P_0)$ and zeroth-homology $H_0(\mathcal{F}(P_\bullet)) \cong \mathcal{F}(M)$.

Remark 5.76. For a left derived functor $L_i(\mathcal{F})(M)$, whenever M is a projective, we have $L_i(\mathcal{F})(M) = 0$ for all $i \neq 0$. This is very easy to see, refer to Eqns. (58), (59).

In general, any object A for which $L_i(\mathcal{F})$ for all $i \neq 0$ vanishes is called $\mathcal{F}$-acyclic.

5.77. For $i \geq 0$, $L_i\mathcal{F}$ is an additive functor from $\mathcal{A}$ to $\mathcal{B}$ which is well defined up to canonical isomorphism (independent of chosen projective resolution of M). You can take any two projective resolutions of M, say $P_\bullet \to M$ and $Q_\bullet \to M$ and we have discussed that there exists a comparison theorem of these chain complexes, see Note. 5.58, and two (left) derived functors for two projective resolutions of M will be isomorphic.

Proposition 5.78. The family of functors of $L_i\mathcal{F}$ with connecting morphism δ_i

$$\delta_i : L_i\mathcal{F} \to L_{i-1}\mathcal{F} \tag{61}$$

form a homological δ-functor.

Proof. This is a two-step approach. We start with a short exact sequence in an abelian category $\mathcal{A}$

$$0 \to A \to B \to C \to 0 \tag{62}$$

and from Horseshoe Lemma (see Lemma. 5.59) we have

$$0 \to P_\bullet \to Q_\bullet \to R_\bullet \to 0 \tag{63}$$

where $P_\bullet, Q_\bullet, R_\bullet$ give projective resolutions to A, B, C respectively. Let $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ which is right exact on Eqn. (63)

$$\mathcal{F}(P_\bullet) \to \mathcal{F}(Q_\bullet) \to \mathcal{F}(R_\bullet) \to 0 \tag{64}$$

and exactness holds at each degree (from Lemma. 5.59). Using Snake's lemma (see Def. 5.67), we get a long exact sequence by applying $H_i(\mathcal{F})$

$$\cdots \to H_{i+1}\mathcal{F}(R_\bullet) \xrightarrow{\delta_{i+1}} H_i\mathcal{F}(P_\bullet) \to H_i\mathcal{F}(Q_\bullet) \to H_i\mathcal{F}(R_\bullet) \xrightarrow{\delta_i} H_{i-1}\mathcal{F}(P_\bullet) \to \cdots \tag{65}$$

which concludes the long exact sequence

$$\cdots \to L_{i+1}\mathcal{F}(C) \xrightarrow{\delta_{i+1}} L_i\mathcal{F}(A) \to L_i\mathcal{F}(B) \to L_i\mathcal{F}(C) \xrightarrow{\delta_i} L_{i-1}\mathcal{F}(A) \to \cdots \tag{66}$$

Definition 5.79 ((Co)Effaceable functor). Let there be an additive functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ between two abelian categories, then for every object $A \in \mathcal{A}$ we have a monomorphism $u : A \to M$ such that $\mathcal{F}(u) = 0$, then $\mathcal{F}$ is called effaceable functor[8].

[8]This just means that $\mathcal{F}(U) : \mathcal{F}(A) \to \mathcal{F}(M)$ is a zero morphism in $\mathcal{B}$.

Dually, we have an additive functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ and there exists a epimorphism $u : N \to A$ such that $\mathcal{F}(u) = 0$, then $\mathcal{F}$ is called coeffaceable functor.

5.80 (Tohuku [5]). Let $\mathcal{A}, \mathcal{B}$ be abelian categories. Say $\mathcal{F} = (\mathcal{F}_n)_{n\geq 0}$ be a homological δ-functor between $\mathcal{A}, \mathcal{B}$ which is coeffaceable in all positive degrees but $\mathcal{F}_0$ is not coeffaceable, then $\mathcal{F}$ is a universal δ-functor (see Def. 5.70).

A dual definition exists for universal cohomological δ-functors via effaceability.

5.81. Now, we can switch our discussion to right derived functors. The definitions are similar but right derived functors are important. For example, $\mathrm{Ext}^1_{\mathcal{A}}(X, -)$ is the right derived functor for $\mathrm{Hom}_{\mathcal{A}}(X, -)$ functor which is a left exact functor.

Definition 5.82 (Right Derived Functor). Let $\mathcal{F}$ be a left-exact functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$, then we know that any short exact sequence in $\mathcal{A}$ (an abelian category with enough injectives) becomes

$$0 \to \mathcal{F}(A) \to \mathcal{F}(B) \to \mathcal{F}(C) \tag{67}$$

which is exact in $\mathcal{B}$ at $\mathcal{F}(A)$ and $\mathcal{F}(B)$ since $\mathcal{F}$ is left-exact. Now, we can define right derived functor $R^i(\mathcal{F})$ for $\mathcal{F}$ for $i \geq 0$. Take an object M in $\mathcal{A}$ and write the injective resolution $I_\bullet \to M$ of M

$$0 \to M \to I^0 \to I^1 \to \cdots \tag{68}$$

and define

$$R^i\mathcal{F}(M) = H^i(\mathcal{F}(I_\bullet)) \tag{69}$$

and each $R^i\mathcal{F}$ is an additive functor from $\mathcal{A}$ to $\mathcal{B}$.

5.83. Similar to Note. 5.75, since Eqn. (68) is exact, we have $R^0(\mathcal{F}(M)) \cong \mathcal{F}(M)$.

We also note again that right derived functors $R^i(\mathcal{F}(M))$ is independent of the choice of injective resolution of M. Moreover, $R^i(\mathcal{F}(M))$ with a connecting morphism δ_i forms a cohomological δ-functor in the same spirit of 5.78.

Also note that for any injective object M, the right derived functor $R^i\mathcal{F}(M) = 0$ for all $i \neq 0$. An object A is $\mathcal{F}$-acyclic if $R^i\mathcal{F}(A) = 0$ for all $i \neq 0$.

5.84. If one is given a left-exact functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ with enough injectives in $\mathcal{A}$, we can define a right derived functor $R^i\mathcal{F}(M)$ for M in $\mathcal{A}$. But one, by passing to opposite categories, can also see that $\mathcal{F}^{op}$ is a right exact functor $\mathcal{F}^{op} : \mathcal{A}^{op} \to \mathcal{B}^{op}$ where $\mathcal{A}^{op}$ has enough projectives and hence, we can define a left derived functor $L_i(\mathcal{F}^{op}(M))$. This gives us

$$R^i\mathcal{F}(M) = L_i(\mathcal{F}^{op})^{op}(M) \tag{70}$$

and hence, in essence, $R^i\mathcal{F}$ and $L_i(\mathcal{F}^{op})$ encode the same (derived) information. In result, one can work with either of these functors.

5.85. In Grothendieck sense, a cohomological δ-functor being effeceable functor implies that it is a universal δ-functor. See previous discussion 5.79 and 5.80.

We know that $R^i\mathcal{F}(M)$ forms a cohomological δ-functor, so $R^i\mathcal{F}(M)$ is a universal δ-functor if it is a effaceable functor. Similarly, $L_i\mathcal{F}(M)$ is a universal δ-functor if it is a coeffaceable functor.

5.86. We will turn our discussion to the Ext functor and extension group.

In an abelian category $\mathcal{A}$, a short exact sequence

$$0 \to A \to B \to C \to 0 \tag{71}$$

then B is called an extension of C by A. The equivalence class of the short exact sequence forms an abelian group with Baer sum which is called extension group $\mathrm{Ext}^1(C, A)$.

Definition 5.87 (Ext Functor). Let $\mathcal{A}$ be an abelian category without enough injectives and M in $\mathcal{A}$, then $\mathrm{Hom}_{\mathcal{A}}(M, -)$ is a left exact functor. We define its right derived functor to be the Ext functor

$$Ext^i_{\mathcal{A}}(M, -) = R^i \; Hom_{\mathcal{A}}(M, -). \tag{72}$$

8. More on Derived Categories

9. Spectral Sequences