CHAPTER 5

Homological Algebra

"For my own sake, I have made a systematic (as yet unfinished) review of my ideas of homological algebra. I find it very agreeable to stick all sorts of things, which are not much fun when taken individually, together under the heading of derived functors."
- Alexander Grothendieck (in his letter to Serre)

5.1. Homological algebra was the most important development by Grothendieck, for example in his Tohuku paper. Our goal in this chapter is to discuss homological algebra in the context of EGA seminars, however, that should not limit us. Of course, some of the material here might be very familiar to you. We will discuss the abstract idea behind chain complexes and them being valued in any category. We will introduce homotopy and chain maps in this chapter. Our focus will be to introduce the homotopy category of chain complexes, do some basic discussions about derived categories as well and projective resolutions and dimensions. Derived functors are discussed as well. One may skip discussions marked with **(*)** as they are not relevant to my original intention behind PtoS.

1. Short Exact Sequences and Chain Complexes

5.2. Most of the focus on abelian categories are given because they admit short exact sequences

$$0 \xrightarrow{f_1} A \xrightarrow{f_2} B \xrightarrow{f_3} C \to 0 \tag{9}$$

such that at each point, it is 'exact' which means that $ker(f_{n+1}) = im(f_n)$ and also notice that $f_{n+1} \circ f_n = 0$.

5.3. With the above definition, we have following examples

(1) A sequence $0 \to V \xrightarrow{f} W$ is exact if f is injective.

This is a part of Prelude of Schemes project available at https://aayushayh.github.io/PtoS.html.
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(2) A sequence $V \xrightarrow{f} W \to 0$ is exact if f is surjective.
(3) A sequence $0 \to V \xrightarrow{f} W \to 0$ is exact if f is an isomorphism.
(4) A sequence $0 \to V \xrightarrow{f} W \xrightarrow{g} U \to 0$ is exact if f is injective, g is surjective and $ker\ g = im\ f$.
(5) Given $V \xrightarrow{f} W$, the following is always exact

$$0 \to ker\ f \xrightarrow{i} V \xrightarrow{f} W \xrightarrow{q} cok\ f \to 0 \tag{10}$$

then $ker\ q = im\ f$ and $ker\ f = im\ i$.

Definition 5.4. A chain complex is defined as a sequence

$$\cdots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots \tag{11}$$

where the $f_n \circ f_{n+1} = 0$. The maps d_n are called differentials.

For example, for $R - Mod$, it is a family of $R - Mod$, $\{C_n\}_{n\in\mathbb{Z}}$ where $C_n \in R - Mod$.

5.5. We can define the kernel of d_n to be the module of n-cycles in $R - Mod$, denoted by $Z_n(C_\bullet)$. Again, we can define image of d_n to be the module of n-boundaries in $R - Mod$, denoted by $B_n(C_\bullet)$. It is clear to see that

$$0 \subseteq Z_n \subset B_n \subset C_n \tag{12}$$

Alternatively[1], $ker(f_n) \subseteq img(f_{n+1})$.d

5.6. For a chain complex to be exact, we just mention that $Z_n = B_n$ for some chain complex $C_\bullet$.

Definition 5.7. When a chain complex fails to be exact, which means that Z_n/B_n is non-trivial quotient group. This quotient group is called 'Homology' $H_n(C_\bullet)$.

This essentially measures by how much the chain complex fails to be 'exact' at n.

5.8. Similarly, one can define cochain complexes and cohomology groups for them. These are dual descriptions.

Example 5.9 (*). A very algebraically clean example of (co)-chain complex and (co)-homology in physics is electromagnetism.[2] We take a complex of differential forms Ω^p on a manifold M

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \Omega^3(M) \to \cdots \tag{13}$$

where $\Omega^0(M)$ is space of functions on M and $\Omega^1(M)$ is a space of 1-forms and so on. The map $d : \Omega^p \to \Omega^{p+1}$ is an exterior derivative map with $d^2 = 0$. The failure of this sequence to be exact is measured by the de Rham co-homology. An excellent resource is [3].

[1]For cochain complexes, with maps $d_n : C_n \to C_{n+1}$, we have $ker(f_{n+1}) \subseteq img(f_n)$

[2]In electromagnetism, the 2-form F curvature solves $dF = 0$.

We will de Rham (co)homology later.

EXAMPLE 5.10. Now we will look at a very interesting example of **s.e.s.** . Given V, W in $Vect_k$, the sequence

$$0 \to V \xrightarrow{i_1} V \oplus W \xrightarrow{\pi_2} W \to 0 \tag{14}$$

where $i_1 : V \to V \oplus W$ is an inclusion map and $\pi_2 : V \oplus W \to W$ is a projection map, the above sequence (14) is an exact sequence since $ker(\pi_2) = im(i_1)$. This sequence will serve as a canonical example of split sequences.

DEFINITION 5.11 (Split Sequence). A short exact sequence

$$0 \to V \xrightarrow{f} W \xrightarrow{g} U \to 0 \tag{15}$$

is said to be split if $\exists\ g' : U \to W$ such that $g \circ g' = 1_U$

$$\begin{tikzcd} 0 \arrow[r] & V \arrow[r] & W \arrow[r, shift left, "g"] & U \arrow[l, dashed, shift left, "g'"] \arrow[r] & 0 \end{tikzcd}$$

For example, the below sequence splits

$$\begin{tikzcd} 0 \arrow[r] & V \arrow[r, "i_1"] & V \oplus W \arrow[r, shift left, "\pi_2"] & W \arrow[l, dashed, shift left, "i_2"] \arrow[r] & 0 \end{tikzcd}$$

We have the following proposition.

5.12. A short exact sequence

$$0 \to V \xrightarrow{f} W \xrightarrow{g} U \to 0 \tag{16}$$

is said to be split if $\exists\ f' : W \to V$ such that $f' \circ f = 1_V$.

So both are equal definitions which means that for our example, the existence of i_2 implies the existence of π_1. These are statements of left split and right split in the Splitting Lemma. In general, the existence of g' will imply that f' exists. To see if this is true, we can look at the following diagram

$$\begin{tikzcd} 0 \arrow[r] & V \arrow[r, "f"] \arrow[d] & W \arrow[r, "g"] \arrow[d, shift right, "\left(\begin{smallmatrix} f' \\ g \end{smallmatrix}\right)"'] & U \arrow[r] & 0 \\ 0 \arrow[r] & V \arrow[r, "i_1"] & V \oplus U \arrow[r, "\pi_2"] \arrow[u, shift right, "(f\ \ g')"'] & U \arrow[r] & 0 \end{tikzcd}$$

where $(f\ \ g')$ is an isomorphism and the diagram must commute.

We suggest that the **s.e.s.**

$$0 \to V \xrightarrow{i_1} V \oplus W \xrightarrow{\pi_2} W \to 0 \tag{17}$$

is the canonical example of a split sequence. Any **s.e.s.** which is isomorphic to above sequence (17) is also a split sequence.

2. Chain Complexes and Chain Maps

DEFINITION 5.13. We define a category of chain complexes $Ch(\mathcal{A})$ with objects as chain complexes, where the morphism are *chain complex maps*. Given two chain complexes $C_\bullet, D_\bullet$, we have a chain map u

$$\begin{tikzcd} \cdots \arrow[r] & C_{n+1} \arrow[r] \arrow[d, dashed, "u_{n+1}"] & C_n \arrow[r] \arrow[d, dashed, "u_n"] & C_{n-1} \arrow[r] \arrow[d, dashed, "u_{n-1}"] & \cdots \\ \cdots \arrow[r] & D_{n+1} \arrow[r] & D_n \arrow[r] & D_{n-1} \arrow[r] & \cdots \end{tikzcd}$$

where each square commutes. It is an important fact that a chain map $u : C_\bullet \to D_\bullet$ induces a map $H_n(C_\bullet) \to H_n(D_\bullet)$ since u sends boundaries $B_n(C_\bullet)$ to boundaries $B_n(D_\bullet)$ and cycles $Z_n(C_\bullet)$ to cycles $Z_n(D_\bullet)$.

DEFINITION 5.14. We define *splitting maps* s_n for some chain complex C_n as $s_n : C_n \to D_{n+1}$ such that it is called a split chain complex such that $d_n^D = d_n^D \circ s_{n-1} \circ d_n^C$. If given two chain complexes C_n and D_n with a chain map $u_n : C_n \to D_n$, we choose maps $s_n : C_n \to D_{n+1}$

$$\begin{tikzcd} \cdots \arrow[r] & C_{n+1} \arrow[r, "d_{n+1}^C"] \arrow[d, "u_{n+1}"'] & C_n \arrow[r, "d_n^C"] \arrow[d, "u_n"] \arrow[dl, "s_n"'] & C_{n-1} \arrow[r, "d_{n-1}^C"] \arrow[d, "u_{n-1}"] & \cdots \\ \cdots \arrow[r] & D_{n+1} \arrow[r, "d_{n+1}^D"] & D_n \arrow[r, "d_n^D"] & D_{n-1} \arrow[r, "d_{n-1}^D"] & \cdots \end{tikzcd}$$

where the commutativity is given by $d_{n+1}^D \circ u_{n+1} = u_n \circ d_{n+1}^C$ and the chain map $u_n = d_{n+1} \circ s_n + s_{n-1} \circ d_n^C$.

5.15. When there exist splitting maps $s_n : C_n \to D_{n+1}$ and a chain map $u_n : C_n \to D_n$ where $u_n = d_{n+1} \circ s_n + s_{n-1} \circ d_n^C$, then u is called a null homotopic chain map. When given two chain maps $u_n, v_n : C_n \to D_n$, we call them *chain homotopic* if their difference is

$$u_n - v_n = d_{n+1} \circ s_n + s_{n-1} \circ d_n^C$$

(18)

and the maps $\{s_n\}$ are called chain homotopy from u to v.

5.16. Furthermore, a chain map $u_n : C_n \to D_n$ is a homotopy equivalence if there exists a map $v_n : D_n \to C_n$ such that uv is chain homotopic to the identity on D and vu is chain homotopic to the identity on C.

3. Snake's Lemma

5.17. Snake's Lemma is a powerful tool to create six term exact sequences from two short exact sequences with a zero object. The proof of the lemma is usually a fun diagram chase. A succinct proof is also available in the beginning of the movie *It's my turn, 1980.* We will focus on the heuristics of the lemma.

5.18. Given two row short exact sequences in an abelian category $\mathcal{A}$, when we have a commutative diagram

$$\begin{tikzcd} & A \arrow[r, "a"] \arrow[d, "f"'] & B \arrow[r, "b"] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "h"'] & 0 \\ 0 \arrow[r] & A' \arrow[r, "a'"] & B' \arrow[r, "b'"] & C' & \end{tikzcd}$$

it gives us a six-term exact sequence

$$ker(f) \longrightarrow ker(g) \longrightarrow ker(h) \xrightarrow{\partial} coker(f) \longrightarrow coker(g) \longrightarrow coker(h)$$

where $\partial : ker(h) \to coker(f)$ is a connecting homomorphism. If the morphism a is a monomorphism, then $ker(f) \to ker(g)$ is a monomorphism and if the morphism b' is an epimorphism, then so is $coker(g) \to coker(h)$.

We obtain the exact sequence by expanding the commutative diagram which gives us a sequence in shape of a 'slithering snake' [A visual diagram of the Snake!]

$$\begin{tikzcd} & & ker(f) \arrow[r, dashed, no head] \arrow[d] & \cancel{ker(g)} \arrow[r, dashed, no head] \arrow[d] & ker(h) \arrow[rr, dashed] \arrow[d] & & \bullet \arrow[d] \\ \bullet \arrow[ddd, dashed] & & A \arrow[r, "a"] \arrow[d, "f"'] & B \arrow[r, "b"] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "h"'] & 0 & \bullet \arrow[llllll, dashed] \\ & 0 \arrow[r] & A' \arrow[r, "a'"] \arrow[d] & B' \arrow[r, "b'"] \arrow[d] & C' \arrow[d] & & \\ \bullet \arrow[rr, dashed, no head] & & \cancel{coker(f)} \arrow[r, dashed, no head] & \cancel{coker(g)} \arrow[r, dashed] & coker(h) & & \end{tikzcd}$$

The connecting homomorphisms ∂ are important in constructing long exact sequences in homological algebra. The proof of the lemma is given by 'diagram chasing'. The proof requires two step: 1) constructing the connecting homomorphism and 2) proving exactness at each point.

A detailed proof will be added in a later draft of these notes.

4. Abelian Categories and Homotopy Category of Complexes

Definition 5.19 (Pre-Additive Category). A category $\mathcal{C}$ is pre-additive if each morphism set $Mor_{\mathcal{C}}(a,b)$ has the structure of an abelian group such that the composition

$$Mor(a,b) \times Mor(b,c) \to Mor(a,c) \tag{19}$$

is bilinear.

Definition 5.20. An object which is both a final object and an initial object in a pre-additive category $\mathcal{C}$ is called a zero object and denoted by 0.

Definition 5.21 (Additive Category). We call a pre-additive category $\mathcal{C}$ *additive* category if it admits finite bi-products[3].

Definition 5.22 (Abelian Category). We call a category $\mathcal{C}$ abelian if it satisfies the following properties

(1) It is an additive category.
(2) Kernels and their dual co-kernels exist.
(3) Every injective morphism is a kernel of its own co-kernel.
(4) Every surjective morphism is a co-kernel of its own kernel.

5.23. Essentially, an abelian category is an abstraction of the (basic) properties of category of abelian groups. It is believed to be introduced by Buchsbaum [**4**] in 1955 as exact categories and later standardized by Grothendieck in his Tohoku paper [**5**] using axiomatic approach. However, the term 'abelian category' was termed by Freyd [**6**].

Example 5.24. A good example of abelian category is category of representation of a group *Rep*(G) or the module category of an artinian algebra Λ over field k.

5.25. The most interesting reason for studying abelian categories is that they admit short exact sequences that we have discussed above. In later parts of this project, we will be interested to read more than exact sequences like *triangles* in triangulated categories.

5.26. We will digress to recall what is a quasi-isomorphism. When we want to just know the homological information of the complexes and their equivalence.

[3]The finite products and fintie co-products coincide here

Definition 5.27. A chain map of complexes $u : C_\bullet \to D_\bullet$ in abelian category $\mathcal{A}$ is called a quasi-isomorphism if the induced homology morphism $u_* = H^n(C_\bullet) \to H^n(D_\bullet)$ is an isomorphism.

From homology point of view, two complexes $C_\bullet$ and $D_\bullet$ become indistinguishable.

Example 5.28. For example, there exists a quasi-isomorphism among any two projective resolutions (or injective resolutions) of same object.

Theorem 5.29. If a map $u : C_\bullet \to D_\bullet$ is a homotopy equivalence, then it is a quasi-isomorphism.

5.30. Note that a quasi-isomorphism is not always a homotopy equivalence.

Definition 5.31. Let $\mathcal{A}$ be an additive category, we define a homotopy category of chain complexes $K(\mathcal{A})$ as follows

- Objects: The objects of $K(\mathcal{A})$ are chain complexes.
- Morphisms: For any two chain complexes $C_\bullet, D_\bullet \in \mathcal{A}$, we define the set of morphisms as the set of homotopy classes of chain maps from $C_\bullet$ to $D_\bullet$
$$\mathrm{Hom}_{K(\mathcal{A})}(C_\bullet, D_\bullet) = \mathrm{Hom}_{Ch(\mathcal{A})}(C_\bullet, D_\bullet)/\sim \tag{20}$$
where any two chain maps u, v are chain homotopic, then $u \sim v$.

5.32. Essentially, we can define a functor $Ch(\mathcal{A}) \to K(\mathcal{A})$ which is identity on the objects and quotient projection on the morphisms. Moreover, $K(\mathcal{A})$ is an additive category as well as *a triangulated category*, however, it is not an abelian category in general.

5.33. The rescue needed to do homological algebras for $K(\mathcal{A})$, since it fails to be abelian category, is provided by triangulated structure on it which contain distinguished triangles from which we get a long exact sequence of homology groups. While $K(\mathcal{A})$ is a 'nice' category to work with homotopical settings. It fails to identify the quasi-isomorphisms and that motivates the construction of derived categories.

5.34. The issue with $K(\mathcal{A})$ being that it does not recognize homotopy equivalence which are not quasi-isomorphism requires an abstract settings of 'derived categories'. The goal is then to construct a category where all quasi-isomorphisms become isomorphisms.

5.35. We define derived category $D(\mathcal{A})$ for an abelian category $\mathcal{A}$ as the localization of the homotopy category $K(\mathcal{A})$ with respect to the class of all quasi-isomorphisms. A morphism between complexes $C_\bullet$ and $D_\bullet$ is no more only class of chain maps. Instead using universality, we have following roof diagram
$$C_\bullet \xleftarrow{f} F_\bullet \xrightarrow{g} D_\bullet \tag{21}$$
where f is a quasi-isomorphism and g is a morphism in $K(\mathcal{A})$. In a sense, it is more natural way to construct homotopy categories.

Derived categories are natural playground for derived functors, higher algebra, representation theory, and mathematical physics.

5. Mapping Cones and Triangulated Category

Triangulated categories were introduced by Jean-Louis Verdier (Grothendieck) in 1963 [7].

DEFINITION 5.36. Let $\mathcal{C}$ be an additive category, then define Σ to be automorphism of $\mathcal{C}$. The automorphism $\Sigma : \mathcal{C} \to \mathcal{C}$ is called the translation functor of $\mathcal{C}$.

5.37. Define a sextuple (X, Y, Z, u, v, w) given by the objects $(X, Y, Z) \in \mathcal{C}$ and morphisms $u : X \to Y$, $v : Y \to Z$ and $w : Z \to \Sigma X$. Basically, we can write the sextuple as

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} Z \to \Sigma X \tag{22}$$

5.38. We can find the morphisms between two sextuples

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, dashed, "g"] & Z \arrow[r, "w"] \arrow[d, dashed, "h"] & \Sigma X \arrow[d, "\Sigma f"] \\ X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & \Sigma X' \end{tikzcd}$$

such that each square commutes.

Now, consider a set T of all sextuples in $\mathcal{C}$, then set T is a triangulation of $\mathcal{C}$ if they satisfy Verdier's axioms. Any element of T is called a **triangle**.

5.39. These Verdier's axioms are as follows.

TR1 For every object X, the following is a distinguished triangle

$$X \to X \to 0 \to \Sigma X \tag{23}$$

(Also, consider any morphism $u : X \to Y$, we can get a triangle $X \xrightarrow{u} Y \to cone(u) \to \Sigma X$)

Any triangle isomorphic to a distinguished triangle is a distinguished triangle.p

TR2 The following is a distinguished triangle precisely

$$X \xrightarrow{u} Y \to Z \to \Sigma X \tag{24}$$

if

$$Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X \xrightarrow{-\Sigma u} \Sigma Y \tag{25}$$

is a distinguished triangle.

TR3 The following diagram of morphism between two triangles commute

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, dashed, "g"] & Z \arrow[r, "w"] \arrow[d, dashed, "h"] & \Sigma X \arrow[d, "\Sigma f"] \\ X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & \Sigma X' \end{tikzcd}$$

TR4 Also called *octahedral axiom*. Consider three triangles and the axioms states that their mapping cones will form a distinguished triangle. [We can ignore this here but they are crucial in determining the non-functoriality of mapping cones[4] in triangulated categories.]

Definition 5.40. An additive category $\mathcal{C}$ with translation functor Σ which admits triangulation is called a **triangulated** category $(\mathcal{C}, \Sigma, T)$.

5.41. Example of our interest: Derived category $D(A)$ of an abelian category A is a triangulated category and so is $D^b(A)$.

We study triangulated categories because it generalizes short exact sequences to distinguished triangles

$$0 \to X \to Y \to Z \to 0 \tag{26}$$

in Abelian category A is generalized as a triangle in $D^b(mod - A)$

$$X \to Y \to Z \to X[1] \tag{27}$$

6. Projective Resolutions and Dimension

5.42. In this section, we wish to discuss projective resolutions (and dually, injective resolutions) and various kinds of definitions of dimensions in homological algebra.

I believe the motivation is slightly to associate to every object (modules, representations and so on) a certain complex which helps us to get a larger picture about the category. This should become apparent in sometime.

Definition 5.43. In an abelian category, an object P is called projective, if given any surjection $f : A \to B$, there exists a map $g : P \to B$ such that one has a universal lifting property

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists h"'] \arrow[d, "g"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

where $g = fh$.

[4] https://aayushayh.blogspot.com/2025/09/the-quill-24-failure-of-functorial.html

5.44. We are mostly dealing with category of (right) modules over an algebra. And projective module is an important object for us. We also find that a projective P helps a sequence of $Hom(P,-)$ to form a short exact sequence, in other words $Hom(P,-)$ is an exact functor iff P is a projective.

PROPOSITION 5.45. A module in $A - mod$ P is projective if it is a direct summand of a free module.

PROOF. A free module is a projective module. Let say F is a free module $F \cong Q \oplus P$. We have the following maps

$$i : P \to F \tag{28}$$

and if P is a projective, then by universal property

$$\pi : F \to P \tag{29}$$

which means $\pi \circ i = id_P$ and

$$0 \to Q \to F \to P \to 0 \tag{30}$$

and hence P is a direct summand of free module F.

PROPOSITION 5.46. If P is a projective, then $Ext^1(P, M) = 0$ for every $A - mod$ M.

PROOF. to be written

REMARK 5.47. Projective modules are closed under direct sums.

5.48. A chain complex with each object a projective is called a chain complex of projectives. Note that it does not have to be a projective object in the category of chain complexes.

5.49. We call an abelian category $\mathcal{A}$ with enough projectives if for every object A in $\mathcal{A}$, there exists a surjective map $P \to A$. A dual definition exist for an abelian category with enough injectives.

5.50. As alluded earlier, for a projective P in $\mathcal{A}$ $Hom(P,-)$ is an exact functor which means that for every **s.e.s.** in $\mathcal{A}$

$$0 \to A \to B \to C \to 0 \tag{31}$$

the functor $Hom(P,-)$ gives a short exact sequence

$$0 \to \mathrm{Hom}(P, A) \to \mathrm{Hom}(P, B) \to \mathrm{Hom}(P, C) \to 0 \tag{32}$$

and dually, $Hom(-, I)$ is an exact functor for an injective object I in $\mathcal{A}$.

5.51. A object P is projective in **Ch** (category of chain complexes) if and only if it is a split exact complex. It is not hard to realize this proposition. Let us look at only if direction. Say, $P_\bullet$ is a projective complex, then each P_n is a projective object.

To realize the split exactness of the complex, we say that a short exact sequence

$$0 \to P_\bullet \xrightarrow{i} Cone(id_{P_\bullet}) \xrightarrow{p} P_\bullet[-1] \to 0 \tag{33}$$

where $Cone(id_{P_\bullet})$ is the mapping cone and since $P_\bullet$ is a projective in **Ch**, every epimorphism $p : Cone(id_{P_\bullet}) \to P_\bullet[-1]$ admits a lift of $id_{P_\bullet}[-1]$ such that there is a chain map

$$q : P_\bullet[-1] \to Cone(id_{P_\bullet}) \tag{34}$$

and $p \circ q = id_{P_\bullet}[-1]$. One can also show split exactness as well. (incomplete, to be added in next update, see Exercise 2.2.1 in [8]).

Definition 5.52 (Projective Resolution). For every object[5] M in $\mathcal{A}$, we write a projective resolution as an exact sequence

$$\cdots \to P_2 \to P_1 \to P_0 \xrightarrow{\epsilon_0} M \to 0 \tag{35}$$

where $\epsilon_0 : P_0 \to M$ is augmentation map. One can think of it as a surjective cover of M which is an epimorphism. Alternatively, we can think of a chain complex $P_\bullet$ which is concentrated in nonnegative degree such that $P_i = 0$ for $i < 0$ along with an augmentation map $\epsilon_0 : P_0 \to M$ such that Eqn. (35) is exact.

Definition 5.53 (Injectives). We have delayed the definition of an injective object but we can always rely on the 'dualness' of projective-injective to mention every definition. One defines an injective object I in $\mathcal{A}$ as following. Given an injection $f : A \to B$, there exist a map $\alpha : A \to I$ such that following commutes

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[dl, dashed, "\exists h"] \\ I & \end{tikzcd}$$

5.54 (Baer's Criterion). For a category of right A-modules $A - mod$, the following statements are equivalent for an object M in $A - mod$

- M is an injective object of category in the sense of Def. 5.53.
- For any right ideal I in A, the linear map $I \to M$ can be extended to $A \to M$.

5.55. As mentioned in 5.50, $Hom(P, -)$ is an exact functor. Dually, $Hom(-, I)$ is an exact functor for the short exact sequence

$$0 \to A \to B \to C \to 0 \tag{36}$$

where $Hom(-, I)$ is a contravariant functor.

[5] Technically, one would need a category with enough projectives and fortunately, an abelian category admits enough projectives which means that there is a projective cover for each object in the category.

Definition 5.56 (Injective Resolution). We will find projective resolutions introduced above in 5.52 very important at many places. Similarly, we can define an injective resolution as

$$M \to I^1 \to I^2 \to I^3 \to \cdots \tag{37}$$

where I_i are injectives.

It is also interesting to see that $P_\bullet \to M$ and $M \to I_\bullet$ are chain map and cochain map respectively and describe the same complex where M is concentrated in degree 0.

5.57. We would like to borrow the story from Weibel [8] to show that a category with enough projectives admit projective resolution for each object M. We briefly alluded to it in footnote 5.

$$\begin{tikzcd}
0 \arrow[dr] & & 0 & & 0 \arrow[dr] & & 0 & & & \\
& M_3 \arrow[ur] \arrow[dr] & & & & M_1 \arrow[ur] \arrow[dr] & & & & \\
\cdots \arrow[ur] \arrow[rr] & & P_3 \arrow[rr, "f_2"'] \arrow[dr] & & P_2 \arrow[rr, "f_2"'] \arrow[ur] & & P_1 \arrow[rr, "f_1"'] \arrow[dr] & & P_0 \arrow[r, "\epsilon_0"'] & M \\
& & & M_2 \arrow[ur] \arrow[dr] & & & & M_0 \arrow[ur] \arrow[dr] & & \\
& & 0 \arrow[ur] & & 0 & & 0 \arrow[ur] & & 0 &
\end{tikzcd}$$

The proof is done by using slicing method and induction. We have a surjection $\epsilon : P_0 \to M$, we can define $M_0 = ker(\epsilon_0)$ and by induction $M_n = ker(\epsilon_n)$ and say $i_n : M_n \to P_n$. Then our map $f_n : P_n \to P_{n-1}$ becomes

$$f_n = i_{n-1} \cdot \epsilon_n \tag{38}$$

which is a chain complex.

5.58. Since these projective resolutions are chain complexes with projective terms, we can define a chain map between two projective resolutions which will be unique up to chain homotopy, see Sec. 2. Let us take objects M, N, then the two projective resolutions are given by

$$\cdots \to P_2 \to P_1 \to P_0 \xrightarrow{\epsilon_0} M \to 0 \tag{39}$$

$$\cdots \to Q_2 \to Q_1 \to Q_0 \xrightarrow{\eta_0} N \to 0 \tag{40}$$

and say $f : M \to N$, then following is a chain map between $P_\bullet$ and $Q_\bullet$ (chain complex of projectives)

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r] \arrow[d, "u_2"] & P_1 \arrow[r] \arrow[d, "u_1"] & P_0 \arrow[r, "\epsilon_0"] \arrow[d, "u_0"] & M \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r] & Q_2 \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r, "\eta_0"] & N \arrow[r] & 0
\end{tikzcd}$$

where u_n chain map lifts the morphism $f \colon M \to N$.

Two projective resolutions of same object M can be chain homotopy equivalent. Say $P_\bullet \to M$ and $Q_\bullet \to M$ are two projective resolutions of object M. Then they are chain homotopy equivalent if there exists

$$u_n : P_\bullet \to Q_\bullet, \quad v_n : Q_\bullet \to P_\bullet \tag{41}$$

such that $u_n \circ v_n \cong id_{Q_\bullet}$ and $v_n \circ u_n \cong id_{P_\bullet}$.

One says that projective resolutions which are equivalent up to homotopy make up a category which sits inside the homotopy category of chain complexes $K(\mathcal{A})$. This means that choice of projective resolutions do not matter (up to chain homotopy).

Lemma 5.59 (Horseshoe Lemma). Let $\mathcal{A}$ be an abelian category with enough projectives and suppose we have been given an extension (we will come to extensions very soon in detail, for now, we can think of it just as a short exact sequence)

$$0 \to M' \to M \to M'' \to 0 \tag{42}$$

and suppose, we have a commutative diagram

$$\begin{tikzcd}
 & & & & 0 \arrow[d] & \\
\cdots \arrow[r] & P_2' \arrow[r] & P_1' \arrow[r] & P_0' \arrow[r, "\epsilon_o^{'}"] & M' \arrow[r] \arrow[d] & 0 \\
 & & & & M \arrow[d] & \\
\cdots \arrow[r] & P_2'' \arrow[r] & P_1'' \arrow[r] & P_0'' \arrow[r, "\epsilon_o^{''}"] & M'' \arrow[r] \arrow[d] & 0 \\
 & & & & 0 &
\end{tikzcd}$$

where the rows are projective resolutions of M' and M'' and both rows and the column are exact. We can now find the projective resolution of M from be

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & P_2' \arrow[r] \arrow[d] & P_1' \arrow[r] \arrow[d] & P_0' \arrow[r, "\epsilon_o'"] \arrow[d] & M' \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & P_2 \arrow[r] \arrow[d] & P_1 \arrow[r] \arrow[d] & P_0 \arrow[r, "\epsilon_0"] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & P_2'' \arrow[r] \arrow[d] & P_1'' \arrow[r] \arrow[d] & P_0'' \arrow[r, "\epsilon_o''"] \arrow[d] & M'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}$$

where we set $P_n = P_n'' \oplus P_n'$ and the exactness of the our initial extension lifts to the extension of $0 \to P_n' \to P_n \to P_n'' \to 0$.

The proof can be achieved by induction, one may see 2.2.8 in [8]. A dual lemma exists for injective resolutions for abelian category with enough injectives.

Definition 5.60 (Projective Dimension). For a module M, we have a projective resolution

$$\cdots \to P_3 \to P_2 \to P_1 \to P_0 \xrightarrow{\epsilon_0} M \to 0 \tag{43}$$

The projective dimension $\mathbf{p.dim(M)}$ of the module M is the minimum integer n such that we have following

$$0 \to P_n \to \cdots P_2 \to P_1 \to P_0 \xrightarrow{\epsilon_0} M \to 0 \tag{44}$$

Definition 5.61 (Injective Dimension). For a module M, injective dimension $\mathbf{i.dim(M)}$ is the minimum integer n such that the resolution is

$$0 \to M \to I_0 \to I_1 \to I_2 \to \cdots \to I_n \to 0 \tag{45}$$

5.62. If such minimal integer n does not exist, we write that the projective dimension (or injective dimension if that is the concern) is infinite. The number $pd(M)$ is a *homological invariant* for module M.

There is also a resolution by flat modules and the definition of flat dimension $fd(M)$ is similar.

Definition 5.63 (Global Dimension). For any algebra A (or ring), global dimension $gl.dim(A)$ is a homological invariant which is given by

$$gl.dim(A) = sup\{p.dim(M) : M \in A - mod\} \tag{46}$$

and equivalently,

$$gl.dim(A) = sup\{i.dim(M) : M \in A - mod\} \tag{47}$$

5.64 (*). For hereditary algebra, the global dimension is at most one[6].

5.65 (*). In modern representation theory, $gl.dim$ is an important invariant. Having a finite global dimension versus having infinite global dimension are interesting in their own rights. For example, one can show that for finite global dimension for finite dimensional, $k-$algebra A (Happel's Theorem [**9**])

$$D^b(A - mod) \simeq \hat{A} - \underline{mod} \tag{48}$$

where the right side is the stable module category of a repetitive algebra $\hat{A}$. However, this is not true for infinite global dimension of A.

7. Derived Functors and Ext

5.66. We will start the discussion on the derived functors in this section. We will first describe Grothendieck's δ-functor [**5**]. Throughout, $\mathcal{A}$ and $\mathcal{B}$ are two abelian categories unless stated otherwise.

Definition 5.67 (δ-functor). Given a functor $\mathcal{F}$ between $\mathcal{A}$ and $\mathcal{B}$

$$\mathcal{F} : \mathcal{A} \to \mathcal{B} \tag{49}$$

and a short exact sequence in $\mathcal{A}$

$$0 \to A \to B \to C \to 0 \tag{50}$$

we define a (homological[7]) δ-functor as a collection of additive functors $\mathcal{F}_n$ for $n \geq 0$ with a connecting morphism

$$\delta_n : \mathcal{F}_n(C) \to \mathcal{F}_{n-1}(A) \tag{51}$$

such that we get a long exact sequence from Eqn. (50)

$$\cdots \to \mathcal{F}_{n+1}(C) \xrightarrow{\delta} \mathcal{F}_n(A) \to \mathcal{F}_n(B) \to \mathcal{F}_n(C) \xrightarrow{\delta} \mathcal{F}_{n-1}(C) \to \cdots \tag{52}$$

5.68. For any two short exact sequences

$$0 \to A \to B \to C \to 0 \tag{53}$$

$$0 \to A' \to B' \to C' \to 0 \tag{54}$$

[6] Basically, at most two projectives in the resolution $P_\bullet \to M$

[7] Of course, a cohomological definition of the functor exists as well.

the following diagram commute

$$\begin{tikzcd} \mathcal{F}_n(C') \arrow[r, "\delta_n"] \arrow[d] & \mathcal{F}_{n-1}(A') \arrow[d] \\ \mathcal{F}_n(C) \arrow[r, "\delta_n"'] & \mathcal{F}_{n-1}(A) \end{tikzcd}$$

5.69. We can define for two delta functors $\mathcal{F}$ and $\mathcal{G}$, a morphism is given by natural transformations $\eta_n : \mathcal{F}_n \to \mathcal{G}_n$ that commute with δ.

Definition 5.70 (Universal δ-functor). We call a δ-functor $\mathcal{F}$ universal if for any given δ-functor $\mathcal{G}$ the natural transformation $\eta_0 : \mathcal{G}_0 \to \mathcal{F}_0$ extends to an unique family of morphisms $\{\eta_n : \mathcal{G}_n \to \mathcal{F}_n\}$.

5.71. Given a short exact sequence

$$0 \to A \to B \to C \to 0 \tag{55}$$

and a $Hom(-, X)$ functor, which is a contravariant functor (a functor for which domain is objects in $\mathcal{C}^{op}$) and is a left-exact functor

$$0 \to \mathrm{Hom}(C, X) \to \mathrm{Hom}(B, X) \to \mathrm{Hom}(A, X) \tag{56}$$

but we can not guarantee the last zero unless X is injective. Conversely, the same argument applies for $Hom(X, -)$ which is a right exact functor and becomes exact if and only if X is projective.

5.72. Before we discuss finding a long exact sequence for any short exact sequence

$$0 \to A \to B \to C \to 0 \tag{57}$$

for any functor $F : \mathcal{A} \to \mathcal{B}$, we should briefly allude to our discussion on Snake's lemma before here. Snake's lemma is a way to get a long exact sequence from short exact sequence. One should emphasize on the connecting morphisms δ.

Example 5.73 (*). You may read Note. 5.100 before reading this part. Let us take an example of sheaves

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0 \tag{58}$$

and the long exact sequences of sheaf (co)homology on a topological space X is given by

$$\cdots \to H^n(X, \mathcal{F}) \to H^n(X, \mathcal{G}) \to H^n(X, \mathcal{H}) \to H^{n+1}(X, \mathcal{G}) \to \cdots \tag{59}$$

A good example in physics would be consider a short exact sequence of sheaves

$$0 \to \mathbb{Z} \to C^\infty(R) \to C^\infty(U(1)) \to 0 \tag{60}$$

and since we have $H^n(X, C^\infty(R)) = 0$ for $n > 0$, we have the isomorphism

$$H^1(X, C^\infty(U(1))) \cong H^2(X, \mathbb{Z}) \tag{61}$$

and

$$H^2(X, C^\infty(U(1))) \cong H^3(X, \mathbb{Z}) \tag{62}$$

write physics relevance

Definition 5.74 (Left Derived Functor). Let $\mathcal{F}$ be a right-exact functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$, then we know that any short exact sequence in $\mathcal{A}$ (which is an abelian category with enough projectives, see Footnote. 5 and Note. 5.57) becomes

$$\mathcal{F}(A) \to \mathcal{F}(B) \to \mathcal{F}(C) \to 0 \tag{63}$$

which is exact in $\mathcal{B}$ since $\mathcal{F}$ is right-exact. We can define left derived functor $L_i(\mathcal{F})$ for $\mathcal{F}$ for $i \geq 0$. Take an object M in $\mathcal{A}$ and write the projective resolution of M

$$\cdots \to P_2 \to P_1 \to P_0 \xrightarrow{\epsilon_0} M \to 0 \tag{64}$$

and define

$$L_i\mathcal{F}(M) = H_i(\mathcal{F}(P_\bullet)) \tag{65}$$

and each $L_i\mathcal{F}$ is an additive functor from $\mathcal{A}$ to $\mathcal{B}$.

5.75. We have $L_0\mathcal{F}(M) \cong \mathcal{F}(M)$ which is true since

$$\cdots \to \mathcal{F}(P_1) \to \mathcal{F}(P_0) \to \mathcal{F}(M) \to 0 \tag{66}$$

is exact at $\mathcal{F}(M)$ and $\mathcal{F}(P_0)$ and zeroth-homology $H_0(\mathcal{F}(P_\bullet)) \cong \mathcal{F}(M)$.

Remark 5.76. For a left derived functor $L_i(\mathcal{F})(M)$, whenever M is a projective, we have $L_i(\mathcal{F})(M) = 0$ for all $i \neq 0$. This is very easy to see, refer to Eqns. (64), (65).

In general, any object A for which $L_i(\mathcal{F})$ for all $i \neq 0$ vanishes is called $\mathcal{F}$-acyclic.

5.77. For $i \geq 0$, $L_i\mathcal{F}$ is an additive functor from $\mathcal{A}$ to $\mathcal{B}$ which is well defined up to canonical isomorphism (independent of chosen projective resolution of M). You can take any two projective resolutions of M, say $P_\bullet \to M$ and $Q_\bullet \to M$ and we have discussed that there exists a comparison theorem of these chain complexes, see Note. 5.58, and two (left) derived functors for two projective resolutions of M will be isomorphic.

Proposition 5.78. The family of functors of $L_i\mathcal{F}$ with connecting morphism δ_i

$$\delta_i : L_i\mathcal{F} \to L_{i-1}\mathcal{F} \tag{67}$$

form a homological δ-functor.

Proof. This is a two-step approach. We start with a short exact sequence in an abelian category $\mathcal{A}$

$$0 \to A \to B \to C \to 0 \tag{68}$$

and from Horseshoe Lemma (see Lemma. 5.59) we have

$$0 \to P_\bullet \to Q_\bullet \to R_\bullet \to 0 \tag{69}$$

where $P_\bullet, Q_\bullet, R_\bullet$ give projective resolutions to A, B, C respectively. Let $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ which is right exact on Eqn. (69)

$$\mathcal{F}(P_\bullet) \to \mathcal{F}(Q_\bullet) \to \mathcal{F}(R_\bullet) \to 0 \tag{70}$$

and exactness holds at each degree (from Lemma. 5.59). Using Snake's lemma (see Def. 5.67), we get a long exact sequence by applying $H_i(\mathcal{F})$

$$\cdots \to H_{i+1}\mathcal{F}(R_\bullet) \xrightarrow{\delta_{i+1}} H_i\mathcal{F}(P_\bullet) \to H_i\mathcal{F}(Q_\bullet) \to H_i\mathcal{F}(R_\bullet) \xrightarrow{\delta_i} H_{i-1}\mathcal{F}(P_\bullet) \to \cdots \tag{71}$$

which concludes the long exact sequence

$$\cdots \to L_{i+1}\mathcal{F}(C) \xrightarrow{\delta_{i+1}} L_i\mathcal{F}(A) \to L_i\mathcal{F}(B) \to L_i\mathcal{F}(C) \xrightarrow{\delta_i} L_{i-1}\mathcal{F}(A) \to \cdots \tag{72}$$

Definition 5.79 ((Co)Effaceable functor). Let there be an additive functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ between two abelian categories, then for every object $A \in \mathcal{A}$ we have a monomorphism $u : A \to M$ such that $\mathcal{F}(u) = 0$, then $\mathcal{F}$ is called effaceable functor[8].

Dually, we have an additive functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ and there exists a epimorphism $u : N \to A$ such that $\mathcal{F}(u) = 0$, then $\mathcal{F}$ is called coeffaceable functor.

5.80 (Tohuku [**5**]). Let $\mathcal{A}, \mathcal{B}$ be abelian categories. Say $\mathcal{F} = (\mathcal{F}_n)_{n\geq 0}$ be a homological δ-functor between $\mathcal{A}, \mathcal{B}$ which is coeffaceable in all positive degrees but $\mathcal{F}_0$ is not coeffaceable, then $\mathcal{F}$ is a universal δ-functor (see Def. 5.70).

A dual definition exists for universal cohomological δ-functors via effaceability.

5.81. Now, we can switch our discussion to right derived functors. The definitions are similar but right derived functors are important. For example, $\mathrm{Ext}^1_{\mathcal{A}}(X, -)$ is the right derived functor for $\mathrm{Hom}_{\mathcal{A}}(X, -)$ functor which is a left exact functor.

Definition 5.82 (Right Derived Functor). Let $\mathcal{F}$ be a left-exact functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$, then we know that any short exact sequence in $\mathcal{A}$ (an abelian category with enough injectives) becomes

$$0 \to \mathcal{F}(A) \to \mathcal{F}(B) \to \mathcal{F}(C) \tag{73}$$

which is exact in $\mathcal{B}$ at $\mathcal{F}(A)$ and $\mathcal{F}(B)$ since $\mathcal{F}$ is left-exact. Now, we can define right derived functor $R^i(\mathcal{F})$ for $\mathcal{F}$ for $i \geq 0$. Take an object M in $\mathcal{A}$ and write the injective resolution $I_\bullet \to M$ of M

$$0 \to M \to I^0 \to I^1 \to \cdots \tag{74}$$

and define

$$R^i\mathcal{F}(M) = H^i(\mathcal{F}(I_\bullet)) \tag{75}$$

and each $R^i\mathcal{F}$ is an additive functor from $\mathcal{A}$ to $\mathcal{B}$.

[8] This just means that $\mathcal{F}(U) : \mathcal{F}(A) \to \mathcal{F}(M)$ is a zero morphism in $\mathcal{B}$.

5.83. Similar to Note. 5.75, since Eqn. (74) is exact, we have $R^0(\mathcal{F}(M)) \cong \mathcal{F}(M)$.

We also note again that right derived functors $R^i(\mathcal{F}(M))$ is independent of the choice of injective resolution of M. Moreover, $R^i(\mathcal{F}(M))$ with a connecting morphism δ_i forms a cohomological δ-functor in the same spirit of 5.78.

Also note that for any injective object M, the right derived functor $R^i\mathcal{F}(M) = 0$ for all $i \neq 0$. An object A is $\mathcal{F}$-acyclic if $R^i\mathcal{F}(A) = 0$ for all $i \neq 0$.

5.84. If one is given a left-exact functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ with enough injectives in $\mathcal{A}$, we can define a right derived functor $R^i\mathcal{F}(M)$ for M in $\mathcal{A}$. But one, by passing to opposite categories, can also see that $\mathcal{F}^{op}$ is a right exact functor $\mathcal{F}^{op} : \mathcal{A}^{op} \to \mathcal{B}^{op}$ where $\mathcal{A}^{op}$ has enough projectives and hence, we can define a left derived functor $L_i(\mathcal{F}^{op}(M))$. This gives us

$$R^i\mathcal{F}(M) = L_i(\mathcal{F}^{op})^{op}(M) \tag{76}$$

and hence, in essence, $R^i\mathcal{F}$ and $L_i(\mathcal{F}^{op})$ encode the same (derived) information. In result, one can work with either of these functors.

5.85. In Grothendieck sense, a cohomological δ-functor being effeceable functor implies that it is a universal δ-functor. See previous discussion 5.79 and 5.80.

We know that $R^i\mathcal{F}(M)$ forms a cohomological δ-functor, so $R^i\mathcal{F}(M)$ is a universal δ-functor if it is a effaceable functor. Similarly, $L_i\mathcal{F}(M)$ is a universal δ-functor if it is a coeffaceable functor.

5.86. We will turn our discussion to the Ext functor and extension group.

In an abelian category $\mathcal{A}$, a short exact sequence

$$0 \to A \to B \to C \to 0 \tag{77}$$

then B is called an extension of C by A. The equivalence class of the short exact sequence forms an abelian group with Baer sum which is called extension group $\mathrm{Ext}^1(C, A)$.

Definition 5.87 (Ext Functor). Let $\mathcal{A}$ be an abelian category with enough injectives and M in $\mathcal{A}$, then $\mathrm{Hom}_{\mathcal{A}}(M, -)$ is a left exact functor. We define its right derived functor to be the Ext functor[9]

$$\mathrm{Ext}^i_{\mathcal{A}}(M, N) = (R^i\ \mathrm{Hom}_{\mathcal{A}}(M, -))(N). \tag{78}$$

where we first take the injective cochain resolution, for say N, $N \to I_\bullet$ and compute the cohomology of resulting cochain complex from $\mathrm{Hom}_{\mathcal{A}}(M, I_\bullet)$.

[9]If one cares about more *notational* comfort, we evaluate $\mathrm{Hom}_{\mathcal{A}}(M, -)(N)$ by taking the injective resolution of N and then taking cohomology of the resulting cochain complex, this gives us $\mathrm{Ext}^i_{\mathcal{A}}(M, N)$.

THEOREM 5.88 (Cartan-Eilenberg [**10**]). For all M, N in $\mathcal{A}$ with enough projectives (and enough injectives), we have[10]

$$\operatorname{Ext}^i_{\mathcal{A}}(M, N) = R^i\operatorname{Hom}_{\mathcal{A}}(M, -) \cong R^i\operatorname{Hom}_{\mathcal{A}}(-, N) \tag{79}$$

and so the choice of projective/injective resolution in Hom does not matter[11]. (One can take an injective resolution $N \to I^\bullet$ or a projective resolution $P_\bullet \to M$.)

PROOF. See Weibel [**8**].

5.89. I will reprint two exercises from [**8**] (Exercise 2.5.1 and 2.5.2) and [**11**, Lemma 015B] since they are very helpful to realize many things about derived functors and in particular, Ext functor, in context of projectives and injectives.

The following are equivalent for $\operatorname{Hom}_{\mathcal{A}}(M, -)$:

(1) M is a projective object in $\mathcal{A}$.
(2) $\operatorname{Hom}_{\mathcal{A}}(M, -)$ is an exact functor, as alluded in Note. 5.50.
(3) $\operatorname{Ext}^1_{\mathcal{A}}(M, A)$ vanishes for all A. (Note that for a projective M, the projective resolution length is 0 and hence the cohomology vanishes at $i = 1$. Def. 5.87 maybe helpful if one intends to show using resolutions.)
(4) $\operatorname{Ext}^i_{\mathcal{A}}(M, A)$ vanishes for all $i > 0$ and all A hence M is $Hom_{\mathcal{A}}(-, A)$-acylic for all A in the sense of 5.83. (Note that $\operatorname{Ext}^i_{\mathcal{A}}$ measures the failure of exactness and $\operatorname{Hom}_{\mathcal{A}}(M, -)$ is an exact functor for M projective, so the higher cohomology vanishes.)

The following are equivalent for $\operatorname{Hom}_{\mathcal{A}}(-, M)$:

(1) M is an injective object in $\mathcal{A}$.
(2) $\operatorname{Hom}_{\mathcal{A}}(-, M)$ is an exact functor, as alluded in Note. 5.55.
(3) $\operatorname{Ext}^1_{\mathcal{A}}(A, M)$ vanishes for all A. (Note that for an injective M, the injective resolution length is 0 and hence the cohomology vanishes at $i = 1$.)
(4) $\operatorname{Ext}^i_{\mathcal{A}}(A, M)$ vanishes for all $i > 0$ and all A hence M is $Hom_{\mathcal{A}}(A, -)$acyclic for all A. (Note that $\operatorname{Ext}^i_{\mathcal{A}}$ measures the failure of exactness and $\operatorname{Hom}_{\mathcal{A}}(-, M)$ is an exact functor for M injective, so the higher cohomology vanishes.)

8. Short Notes on K_0 and Sheaf Cohomology

5.90. We will briefly discuss the idea of K group, the Grothendieck group K_0 which will be needed in PtoS at times. However, a complete exposition will be provided somewhere else

[10]For $R^i\operatorname{Hom}_{\mathcal{A}}(M, -)$, we will find the the injective resolution of N and for $R^i\operatorname{Hom}_{\mathcal{A}}(-, N)$, we will find the projective resolution of M.

[11]Note that a functor $\operatorname{Hom}_{\mathcal{A}}(-, N) : \mathcal{A}^{op} \to \mathsf{Ab}$ is contravariant and left-exact on $\mathcal{A}$. Also see Note. 5.84.

in the project. Some good resources for K-theory include [**11–13**]. The idea of K-theory is also due to Grothendieck. We will mostly talk about the K group in the context of abelian category here.

DEFINITION 5.91. Let $\mathcal{A}$ be an abelian category (skeletally small[12] so that the set of isomorphism classes of objects is a set), we define its Grothendieck group $K_0(\mathcal{A})$ to be the abelian group presented by a generator $[A]$ for each (isomorphism class of) object for $A \in Ob(\mathcal{A})$ such that for each short exact sequence $0 \to A \to B \to C \to 0$ we impose a relation

$$[B] = [A] + [C] \tag{80}$$

The short exact sequence $0 \to 0 \to 0 \to 0 \to 0$ leads to $[0] = 0$ in $K_0(\mathcal{A})$.

5.92. Given two abelian categories and an exact functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$, we have a group homomorphism between Grothendieck groups by

$$K_0(\mathcal{F}) : K_0(\mathcal{A}) \to K_0(\mathcal{B}) \tag{81}$$

where each $[A]$ in $\mathcal{A}$ is set to $K_0(\mathcal{F})([A]) = [\mathcal{F}(A)]$.

If there is an equivalence between two abelian categories $\mathcal{A}$ and $\mathcal{B}$, their Grothendieck groups are naturally isomorphic.

5.93 (Universal Property). We define an additive function $f : Ob(\mathcal{A}) \to \Gamma$ which sends each objects (and is invariant between isomorphic objects) in $\mathcal{A}$ to an abelian group Γ with the relation $f(B) = f(A) + f(C)$ for short exact sequence $0 \to A \to B \to C \to 0$. Essentially, the building of Grothendieck group $K_0(\mathcal{A})$ is such an example. There is a universal property that for each additive function f from $\mathcal{A}$ to Γ, there exist a unique factor

$$\begin{tikzcd}
\mathrm{Ob}(\mathcal{A})/\cong \arrow[r, "f"] \arrow[d] & \Gamma \\
K_0(\mathcal{A}) \arrow[ur, dashed, "\exists! f'"'] &
\end{tikzcd}$$

and a unique group homomorphism $K_0(\mathcal{A}) \to \Gamma$.

DEFINITION 5.94 (Serre subcategory). Let $\mathcal{A}$ be an abelian category. A full subcategory $\mathcal{C}$ of $\mathcal{A}$ is called Serre subcategory such that the exact sequence in $\mathcal{A}$

$$A \to B \to C \tag{82}$$

with $A, C \in \mathcal{C}$, then $B \in \mathcal{C}$.

[12]If the class of objects in $\mathcal{A}$ forms a set, then it is called small category. Grothendieck groups can not be defined for categories which are not skeletally small.

Note that $\mathcal{C}$ is an abelian category and $\mathcal{C} \hookrightarrow \mathcal{A}$ is a fully faithful exact functor. There also exists a notion of thick Serre subcategory or weak Serre subcategory which we will discuss in due course.

5.95 ([**11**], Lemma 02MX (1), Localization Theorem). Let $\mathcal{C}$ be a Serre subcategory of an abelian category $\mathcal{A}$ and we set $\mathcal{B} = \mathcal{A}/\mathcal{C}$, then the exact functors $\mathcal{C} \to \mathcal{A}$ and $\mathcal{A} \to \mathcal{B}$ induce an exact sequence between their Grothendieck groups

$$K_0(\mathcal{C}) \to K_0(\mathcal{A}) \to K_0(\mathcal{B}) \to 0. \tag{83}$$

5.96. We looked at the group co(homology) in the context of representation theory (or modules). Now, we will discuss the extension of the theory to sheaf cohomology which was an important contribution by Jean Leray (see [**14**]) and later worked upon by Serre in FAC [**15**], and Grothendieck among some others. Even though, the discussions about sheaves will be done in Chapter 7, we will give a basic definition of sheaf here and carry out the discussion about sheaf cohomology.

Definition 5.97 (Sheaf). Let X be a topological space, a sheaf $\mathcal{F}$ is defined for each open set $U \subseteq X$ as a set $\mathcal{F}(U)$ with a restriction map $\rho^U_V : \mathcal{F}(U) \to \mathcal{F}(V)$ for each $V \subseteq U$, the set[13] $\Gamma(U, \mathcal{F})$ is a set of section of $\mathcal{F}$, satisfying the following axioms

(1) For an open cover $\{U_i\}_{i\in I}$ on X, for two sections $s, t \in \mathcal{F}(U)$, if $s|_{U_i} = t|_{U_i}$, then $s = t$. So the sections agree on equal cover. This is termed locality.
(2) Let $s_i \in \{\mathcal{F}(U_i)\}$ be the family of sections on the open cover $\{U_i\}_{i\in I}$ on U. If the sections agree on the overlap $U_i \cap U_j$ such that $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$, then one can glue the sections to get a section $s_i = s|_{U_i}$.

Without the two axioms, one instead get a 'presheaf' instead of sheaf from the definition above.

5.98 (*). The second axiom of gluibility is generally seen in gauge theory when one wishes to glue local gauge potentials on overlapping patches to get a global connection which gives the curvature.

(*Disclaimer: This section is under construction!*)

5.99. The concept of sheaf was brought to introduce a new system of coefficients. For example, the sheaf $\mathcal{F} = \mathbb{R}$ is the constant sheaf which is the system of real constants.

5.100. To define sheaf (co)homology, we need the notion of (co)chains. Another exposition of cochains can be found in [**17**].

[13] Both notations are synonymous, $\Gamma(U, \mathcal{F}) = \mathcal{F}(U)$. The former notation is used throughout EGA [**16**].

Let $\{U_i\}$ be a 'good' open cover on X and $\mathcal{F}$ be a sheaf on X, we define cochains $C^n(\mathcal{F})$ of n-degree as

$$\begin{aligned}
C^0(\mathcal{F}) &= \bigsqcup_i \mathcal{F}(U_i) \\
C^1(\mathcal{F}) &= \bigsqcup_{i\neq j} \mathcal{F}(U_i \cap U_j) \\
C^2(\mathcal{F}) &= \bigsqcup_{i\neq j\neq k} \mathcal{F}(U_i \cap U_j \cap U_k) \\
&\vdots \\
C^n(\mathcal{F}) &= \bigsqcup_{i_0\neq i_2\neq i_3\neq\cdots\neq i_n} \mathcal{F}(U_{i_0} \cap U_{i_2} \cap U_{i_3} \cap \cdots \cap U_{i_n})
\end{aligned}$$

and we define a coboundary operator $\delta : C^n(\mathcal{F}) \to C^{n+1}(\mathcal{F})$ by

$$(\delta\sigma)_{i_0,\ldots,i_{n+1}} = \sum_{j=0}^{n+1}(-1)^j \, \sigma_{i_0,\ldots,\widehat{i_j},\ldots,i_{n+1}}\big|_{U_{i_0\cdots i_{n+1}}}. \tag{84}$$

where $\delta^2 = 0$. Now, similar to defining our cohomology for sequences above, we can define a Céch complex of cochains. We define sheaf cohomology as δ-closed cochains modulo δ-exact cochains. However, a more strict definition takes a direct limit over open covers to obtain sheaf cohomology. We denote these cohomology groups as $H^n(X, \mathcal{F})$.

5.101. Given a short exact sequence of sheaves on X

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0 \tag{85}$$

there is long exact sequences of sheaf cohomology on a topological space X is given by

$$\cdots \to H^n(X, \mathcal{F}) \to H^n(X, \mathcal{G}) \to H^n(X, \mathcal{H}) \to H^{n+1}(X, \mathcal{G}) \to \cdots \tag{86}$$

9. Spectral Sequences

'You Could Have Invented Spectral Sequences.' – Timothy Y. Chow [18]

5.102. Spectral sequences have an interesting history (which has been documented here) and one of the most important developments in algebraic topology and algebraic geometry. For an algebraic approach, the references include [8,19,20]. The composition of right derived functors is also achieved using spectral sequences, called Grothendieck's spectral sequence.

5.103. We will discuss here about computing (co)homology spectral sequence and later the Serre spectral sequence and Grothendieck spectral sequence. We have seen earlier in PtoS, that sheaf cohomology forms a long exact sequence for any topological space X. One can also ask about the filtration of a topological space X and a short exact sequence of sheaves.

Instead of asking about cohomology of chain complexes of single grading, we consider a bi-grading. Grading allows to break the computation of homology into smaller pieces and then they can be summed to gain the original homology. However, that is not the case every time. For now, one can think of a lattice with horizontal and vertical differentials. It is not always easy to compute a sequence of (co)homology and sometimes it is very complicated which resulted in the discovery of spectral sequences. In fact, a nice motivation for spectral sequence is of computing the (co)homology of a total chain complex of a first quadrant double complex $C_{p,q}$. Let us say $E^r_{p,q}$ is a bi-graded object where p is horizontal grading and q is vertical grading, and the total degree[14] of $E^r_{p,q}$ is $n = p + q$.

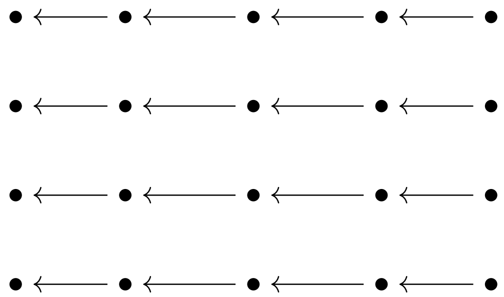
We begin by computing first $E^0_{p,q}$ and then motivate for $r > 0$ as a machine for computing (co)homology. For $E^0_{p,q}$, one ignores the horizontal differentials and apply a vertical differential downside, which means p is fixed, as shown below.

Now, we will write $E^1_{p,q}$ as computing the (vertical) homology of $H_q(E^0_{p,q})$. Intuitively, one may start thinking that each $E^r_{p,q}$ as a page in an infinitely-long book and each page is computing the homology of previous page. However, we are only considering first quadrant of each page. Similarly, $E^2_{p,q}$ would compute the horizontal homology $H_p(E^1_{p,q})$ and so on.

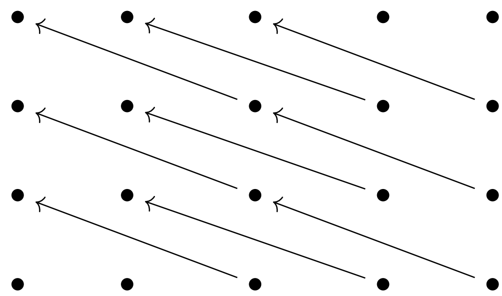
$$\begin{tikzcd}
\bullet \arrow[d] & \bullet \arrow[d] & \bullet \arrow[d] & \bullet \arrow[d] & \bullet \arrow[d] \\
\bullet \arrow[d] & \bullet \arrow[d] & \bullet \arrow[d] & \bullet \arrow[d] & \bullet \arrow[d] \\
\bullet \arrow[d] & \bullet \arrow[d] & \bullet \arrow[d] & \bullet \arrow[d] & \bullet \arrow[d] \\
\bullet & \bullet & \bullet & \bullet & \bullet
\end{tikzcd}$$

$E^0_{p,q}$

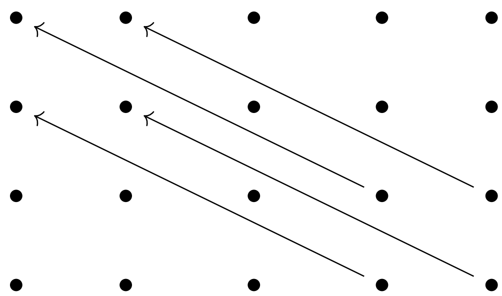
[14]Sometimes, the degree q is called complimentary degree and p is called filtration degree.



$E^1_{p,q}$



$E^2_{p,q}$



$E^3_{p,q}$

In the above r-th pages $E^r_{p,q}$, the (homological) differentials are maps

$$d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1} \tag{87}$$

satisfying $d^r d^r = 0$. This is where it forms chain complexes on each page. Also note that the differentials decrease the total degree ($n = p + q$) by one degree.

Of course, it depends on the context to know more about what exactly are these differentials.

Definition 5.104 (Homology Spectral Sequences). Let $\mathcal{A}$ be an abelian category. A homology spectral sequence consists of

- A family of objects $\{E^r_{p,q}\}$ in $\mathcal{A}$ for all $p, q \in \mathbb{Z}$ and $r \geq a$ where a is some initial page.
- Differential maps $d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}$ with $d^r d^r = 0$.
- Isomorphism between $E^{r+1}_{p,q}$ and the homology of E^r at $E^r_{p,q}$ which means

$$E^{r+1}_{p,q} \cong ker(d^r_{p,q})/im(d^r_{p+r,q-r+1}) \tag{88}$$

5.105. In the case of spectral sequences of first quadrant, then there exists a R such that for $r > R$, we have $E^r_{p,q} \cong E^R_{p,q} \cong E^\infty_{p,q}$. We call such a page $E^\infty_{p,q}$, the stabilized page E^∞.

Definition 5.106. Dually, one can define a cohomology spectral sequence for objects in $\mathcal{A}$ as a family of objects $E_r^{p,q}$ (under notation convenience) with differential maps

$$d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}, \quad d_r d_r = 0 \tag{89}$$

and again, we have the isomorphism between the cohomology of E_r and E_{r+1}.

5.107. For our homology spectral sequences, we can construct a category with morphisms $f : E \to E'$ is a family of maps $f^r_{p,q} : E^r_{pq} \to E'^r_{pq}$ such that each square commutes

$$\begin{tikzcd}
E^r_{p,q} \arrow[r, "d^r_{p,q}"] \arrow[d, "f^r_{p,q}"'] & E^r_{p-r,q+r-1} \arrow[d, "f^r_{p-r,q+r-1}"] \\
E'^r_{p,q} \arrow[r, "d'^r_{p,q}"'] & E'^r_{p-r,q+r-1}
\end{tikzcd}$$

5.108. We discussed about a grading of complexes in 5.103. However, it is not always possible to have a grading on our complexes. For that reason, we work with filtered complexes. A complex $C_\bullet$ is called a *filtered complex* if for each homological degree n, there exists a sequence of submodules F_pC_n:

$$0 = F_0C_n \subseteq F_1C_n \subseteq F_2C_n \subseteq \cdots \subseteq F_kC_n = C_n \tag{90}$$

such that the differential $\partial : C_n \to C_{n-1}$ respects the filtration:

$$\partial(F_pC_n) \subseteq F_pC_{n-1} \tag{91}$$

It is generally not expected to find the grading in complexes in a natural category because that is not-canonical and requires an additional structure. However, a filtration only requires an increasing family of sub-complexes of a complex and respecting the boundary (differential).

A filtration on a complex induces a spectral sequence. In the case where a complex comes from a double complex, this filtration is canonical.

5.109. A spectral sequence E^r is said to converge to some graded object H_*, which is denoted by

$$E^r_{p,q} \Rightarrow H_* \tag{92}$$

if there exists a filtration F on the family of objects H_n in $\mathcal{A}$ such that the infinity page $E^\infty_{p,q}$ has an isomorphism

$$E^\infty_{p,q} \cong \frac{F_pH_{p+q}}{F_{p-1}H_{p+q}} \tag{93}$$

Definition 5.110. A spectral sequence $E^r_{p,q}$ is said to collapse for some Nth term if $d^r = 0$ for $r \geq N$.

This means that $E^N_{p,q}$ is the stabilized page $E^\infty_{*,*}$.

5.111 (*). An interesting use of spectral sequences in the context of BRST and anomalies in gauge theory has been studied in [**21**]. They show that the algebraic structure of anomalies arises naturally from the Weil-BRST algebra.

Definition 5.112 (Exact Couple). In any abelian category $\mathcal{A}$, an exact couple consists of a pair of bigraded objects A and E along with bigraded maps $i : A \to A$, $j : A \to E$, and $k : E \to A$ such that we have the following diagram which is exact

$$\begin{tikzcd} A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

where exactness means

$$im\ k = ker\ i \tag{94}$$
$$im\ i = ker\ j \tag{95}$$
$$im\ j = ker\ k \tag{96}$$

5.113. One can also have a map (ψ, ϕ) between two exact couples $(A_0, E_0, i_0, j_0, k_0)$ and $(A_1, E_1, i_1, j_1, k_1)$ with

$$\psi : A_0 \to A_1 \tag{97}$$
$$\phi : E_0 \to E_1 \tag{98}$$

satisfying

$$\psi \circ i_0 = i_1 \circ \psi \tag{99}$$
$$\phi \circ j_0 = j_1 \circ \psi \tag{100}$$
$$\psi \circ k_0 = k_1 \circ \phi \tag{101}$$
$$\tag{102}$$

5.114. Given an exact couple, we can have a map $d = jk : E \to E$, such that $d^2 = j(kj)k = 0$.

We use this differential to create another exact couple which is called the *derived couple*.

DEFINITION 5.115 (Derived Couple). If given an exact couple (A, E, i, j, k) there is a derived couple (A', E', i', j', k') which is again exact defined by

$$\begin{tikzcd} A' \arrow[rr, "i'"] & & A' \arrow[dl, "j'"] \\ & E' \arrow[ul, "k'"] & \end{tikzcd}$$

with

$$A' = i(A) \tag{103}$$
$$E' = ker(jk)/im(jk) = ker\ d/im\ d \tag{104}$$
$$i' = i|_{i(A)} \tag{105}$$
$$\forall\ i(a) \in A', j'(i(a)) = [j(a)] \in E' \tag{106}$$
$$\forall\ [e] \in E', k'([e]) = k(e) \tag{107}$$

A longer discussion and example(s) of this, as well as the proof that derived couple is exact, can be found in [**20**].

It is interesting to note that one can iterate this process to write[15] n^{th} derived couple $(A^n, E^n, i^n, j^n, k^n)$ starting with an exact couple (A, E, i, j, k).

5.116 (Bockstein Spectral Sequences). An important example of spectral sequence comes from the spectral sequence. The standard way to obtain this spectral sequence is through the long exact sequence of homology. Given a short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\times p} \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \tag{108}$$

and a chain complex of, say, free abelian groups, tensoring with the short exact sequence, we get

$$0 \to C \xrightarrow{p} C \to C \otimes \mathbb{Z}/p\mathbb{Z}. \tag{109}$$

Taking homology produces a long exact sequence in which two of every three terms are same, this is exactly an exact couple

$$\begin{tikzcd} H_*(C) \arrow[rr, "H(\times p)"] & & H_*(C) \arrow[dl] \\ & H_*(C \otimes \mathbb{Z}/p\mathbb{Z}) \arrow[ul] & \end{tikzcd}$$

and the spectral sequence associated with this couple is called **Bockstein spectral sequence**.

[15]This is a notation from [**20**].

One can obviously expect more general short exact sequences giving Bockstein spectral sequences.

5.117 (*). Bockstein spectral sequences offer a deep insight into the study of torsion structure in homology groups as well. Its application to H-spaces were carried by Browder [**22**], alse see Chapter. 10 in [**20**].

10. More on Derived Categories

To be added in ,further updates: Koszul Complex, Hoschild (Co)homology, Motives, Mapping Cones

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