Dynamics of a 2-DOF Planar Robot Arm

August 12, 2025

1 Project Overview

We derive the equations of motion for a 2-degree-of-freedom (2-DOF) planar robot arm using Lagrangian mechanics. The goal is to:

- Model the arm as two rigid links with point masses at the distal ends (neglecting moment of inertia, MOI)
- Compute total kinetic (KE) and potential energy (PE)
- Derive the Lagrangian $\mathcal{L} = T - V$
- Obtain the nonlinear ODEs governing the arm's dynamics

2 Robot Arm Description

- **Joints**: Revolute joints at shoulder (θ_1) and elbow (θ_2)
- **Links**: Massless rods of lengths L_1, L_2 with point masses m_1, m_2 at endpoints
- **Assumptions**:
  - Motion is constrained to the xy-plane
  - MOI is negligible (simplified model)
  - Gravity g acts downward

3 Why MOI is Neglected

We ignore moment of inertia because:

- The masses are modeled as *point masses* at the ends of massless links
- Rotational KE about the center of mass is zero (no mass distribution along links)
- This simplification is valid when:
  - Link masses are concentrated at endpoints
  - The arm moves slowly (minimal rotational effects)

4 Kinetic Energy (KE) Calculation

4.1 Link 1 (Shoulder to Elbow)

Position of m_1:

$$\mathbf{r}_1 = \begin{bmatrix} L_1 \cos\theta_1 \\ L_1 \sin\theta_1 \end{bmatrix}$$

Velocity:

$$\dot{\mathbf{r}}_1 = \begin{bmatrix} -L_1\dot{\theta}_1 \sin\theta_1 \\ L_1\dot{\theta}_1 \cos\theta_1 \end{bmatrix}$$

KE:

$$T_1 = \frac{1}{2} m_1 \|\dot{\mathbf{r}}_1\|^2 = \frac{1}{2} m_1 L_1^2 \dot{\theta}_1^2$$

4.2 Link 2 (Elbow to End-Effector)

Position of m_2:

$$\mathbf{r}_2 = \begin{bmatrix} L_1 \cos\theta_1 + L_2 \cos(\theta_1 + \theta_2) \\ L_1 \sin\theta_1 + L_2 \sin(\theta_1 + \theta_2) \end{bmatrix}$$

Velocity:

$$\dot{\mathbf{r}}_2 = \begin{bmatrix} -L_1\dot{\theta}_1 \sin\theta_1 - L_2(\dot{\theta}_1 + \dot{\theta}_2)\sin(\theta_1 + \theta_2) \\ L_1\dot{\theta}_1 \cos\theta_1 + L_2(\dot{\theta}_1 + \dot{\theta}_2)\cos(\theta_1 + \theta_2) \end{bmatrix}$$

KE:

$$T_2 = \frac{1}{2}m_2\|\dot{\mathbf{r}}_2\|^2 = \frac{1}{2}m_2\left[L_1^2\dot{\theta}_1^2 + L_2^2(\dot{\theta}_1 + \dot{\theta}_2)^2 + 2L_1L_2\dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2)\cos\theta_2\right]$$

4.3 Total KE

$$T = T_1 + T_2 = \frac{1}{2}(m_1 + m_2)L_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2L_2^2(\dot{\theta}_1 + \dot{\theta}_2)^2 + m_2L_1L_2\dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2)\cos\theta_2$$

5 Potential Energy (PE) Calculation

5.1 Link 1

$$V_1 = m_1gy_1 = m_1gL_1\sin\theta_1$$

5.2 Link 2

$$V_2 = m_2gy_2 = m_2g\left(L_1\sin\theta_1 + L_2\sin(\theta_1 + \theta_2)\right)$$

5.3 Total PE

$$V = V_1 + V_2 = (m_1 + m_2)gL_1\sin\theta_1 + m_2gL_2\sin(\theta_1 + \theta_2)$$

6 Lagrangian

$$\mathcal{L} = T - V = \frac{1}{2}(m_1 + m_2)L_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2L_2^2(\dot{\theta}_1 + \dot{\theta}_2)^2 + m_2L_1L_2\dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2)\cos\theta_2 - (m_1 + m_2)gL_1\sin\theta_1 - m_2gL_2\sin(\theta_1 + \theta_2)$$

7 Equations of Motion

Apply the Euler-Lagrange equations for each generalized coordinate θ_i:

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}_i}\right) - \frac{\partial\mathcal{L}}{\partial\theta_i} = \tau_i \quad \text{(where } \tau_i \text{ is the applied torque)}$$

7.1 For θ_1

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}_1}\right) = (m_1 + m_2)L_1^2\ddot{\theta}_1 + m_2L_2^2(\ddot{\theta}_1 + \ddot{\theta}_2) + m_2L_1L_2(2\ddot{\theta}_1 + \ddot{\theta}_2)\cos\theta_2 - m_2L_1L_2(2\dot{\theta}_1 + \dot{\theta}_2)\dot{\theta}_2\sin\theta_2$$

$$\frac{\partial\mathcal{L}}{\partial\theta_1} = -(m_1 + m_2)gL_1\cos\theta_1 - m_2gL_2\cos(\theta_1 + \theta_2)$$

7.2 For θ_2

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}_2}\right) = m_2L_2^2(\ddot{\theta}_1 + \ddot{\theta}_2) + m_2L_1L_2\ddot{\theta}_1\cos\theta_2 - m_2L_1L_2\dot{\theta}_1\dot{\theta}_2\sin\theta_2$$

$$\frac{\partial\mathcal{L}}{\partial\theta_2} = -m_2L_1L_2\dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2)\sin\theta_2 - m_2gL_2\cos(\theta_1 + \theta_2)$$

7.3 Final Nonlinear ODEs

The complete dynamics are described by:

$$\begin{aligned}
\tau_1 = &\left[(m_1 + m_2)L_1^2 + m_2 L_2^2 + 2m_2 L_1 L_2 \cos\theta_2\right]\ddot{\theta}_1 \\
&+ \left[m_2 L_2^2 + m_2 L_1 L_2 \cos\theta_2\right]\ddot{\theta}_2 \\
&- m_2 L_1 L_2 (2\dot{\theta}_1 + \dot{\theta}_2)\dot{\theta}_2 \sin\theta_2 \\
&+ (m_1 + m_2) g L_1 \cos\theta_1 + m_2 g L_2 \cos(\theta_1 + \theta_2)
\end{aligned}$$

$$\begin{aligned}
\tau_2 = &\left[m_2 L_2^2 + m_2 L_1 L_2 \cos\theta_2\right]\ddot{\theta}_1 \\
&+ m_2 L_2^2 \ddot{\theta}_2 \\
&+ m_2 L_1 L_2 \dot{\theta}_1^2 \sin\theta_2 \\
&+ m_2 g L_2 \cos(\theta_1 + \theta_2)
\end{aligned}$$

8 Constraints in the System

8.1 Holonomic Constraints

The 2-DOF arm has two **holonomic constraints**:

- **Rigid Link Constraint 1**: $x_1^2 + y_1^2 = L_1^2$ (fixes length of first link)
- **Rigid Link Constraint 2**: $(x_2 - x_1)^2 + (y_2 - y_1)^2 = L_2^2$ (fixes length of second link)

These are holonomic because:

- They depend only on coordinates (x_i, y_i) *not* velocities
- They reduce the system from 4 variables (x_1, y_1, x_2, y_2) to 2 DOF (θ_1, θ_2)

8.2 Why Only Holonomic?

The system has **no non-holonomic constraints** because:

- There are no velocity-dependent restrictions (e.g., no ”no-slip” conditions)
- All constraints can be integrated to position-level equations

9 Torque as Non-Conservative Force

The RHS of Euler-Lagrange equations contains τ_i because:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_i}\right) - \frac{\partial \mathcal{L}}{\partial \theta_i} = \tau_i$$

- **Gravity** is conservative → already included in $\partial\mathcal{L}/\partial\theta_i$ via V
- **Motor Torques** (τ_1, τ_2) are non-conservative because:
  - They are externally applied forces not derivable from a potential
  - Their work depends on the path taken by the system

10 Kinetic Energy (KE) Calculation

[... Previous KE derivation sections ...]

11 Potential Energy (PE) Calculation

[... Previous PE derivation sections ...]

12 Lagrangian

[... Previous Lagrangian section ...]

13 Equations of Motion

[... Previous equations of motion ...]

$$\tau_1 = \text{Inertia terms} + \text{Coriolis/centripetal} + \text{Gravity}$$
$$\tau_2 = \text{Inertia terms} + \text{Coriolis/centripetal} + \text{Gravity}$$

13.1 Physical Interpretation

- τ_i appears on RHS because it's an *external non-conservative force*
- Gravity terms appear on LHS because they're *conservative forces* (embedded in $\mathcal{L}$)
- Coriolis terms arise from $\mathbf{C}(\theta, \dot{\theta})\dot{\theta}$ due to velocity coupling

14 Feedback Linearization

Feedback linearization is a nonlinear control technique that algebraically transforms a nonlinear system into a linear one through state feedback. For robotic systems, this allows us to leverage linear control tools for trajectory tracking.

15 Dynamic Equations

The dynamics of a 2-DOF planar robot arm are given by:

$$\mathbf{M}(\theta)\ddot{\theta} + \mathbf{C}(\theta, \dot{\theta})\dot{\theta} + \mathbf{G}(\theta) = \tau \tag{1}$$

where:

- $\theta = [\theta_1, \theta_2]^T$: joint angles
- $\mathbf{M}(\theta)$: inertia matrix (symmetric, positive-definite)
- $\mathbf{C}(\theta, \dot{\theta})$: Coriolis/centripetal matrix
- $\mathbf{G}(\theta)$: gravity vector
- τ: control torque

15.1 Why This Specific τ?

We select the control input τ as:

$$\tau = \mathbf{M}(\theta)\mathbf{u} + \mathbf{C}(\theta, \dot{\theta})\dot{\theta} + \mathbf{G}(\theta) \tag{2}$$

Substituting into (1) yields the *linearized* system:

$$\ddot{\theta} = \mathbf{u} \tag{3}$$

This transformation:

- Cancels all nonlinear terms ($\mathbf{C}\dot{\theta} + \mathbf{G}$)
- Reshapes the inertia to identity
- Decouples the joint dynamics

16 Control Design for the Linearized System

With the system now linear ($\ddot{\theta} = \mathbf{u}$), we can apply linear control techniques.

17 PD Control Law

For trajectory tracking, we define $\mathbf{u}$ as:

$$\mathbf{u} = \ddot{\theta}_d + \mathbf{K}_v(\dot{\theta}_d - \dot{\theta}) + \mathbf{K}_p(\theta_d - \theta) \tag{4}$$

where:

- $\theta_d(t)$: desired trajectory
- $\mathbf{K}_p$, $\mathbf{K}_v$: positive definite gain matrices

17.1 Closed-Loop Dynamics

Substituting (4) into (3) gives error dynamics ($e = \theta_d - \theta$):

$$\ddot{e} + \mathbf{K}_v\dot{e} + \mathbf{K}_p e = 0 \tag{5}$$

Properly chosen gains ensure exponential stability.

18 What is a Control Law?

A control law is a mathematical expression that determines the actuator inputs (τ) based on:

- Current state (θ, $\dot{\theta}$)
- Desired behavior (e.g., trajectory tracking)
- System dynamics

19 Why Not Just $\mathbf{u} = -\mathbf{K}\mathbf{x}$?

The intuitive answer is in this case, the theta or the angle of the robot joints is changing with time. Hence the theta dot and theta double dot terms. If we only use u=-kx, we are essentially disregarding these velocity and acceleration terms. The state feedback law $\mathbf{u} = -\mathbf{K}\mathbf{x}$ is insufficient because:

- It lacks the feedforward term $\ddot{\theta}_d$ needed for trajectory tracking
- It cannot compensate for initial errors in acceleration
- For our robot, we need:

  $$\mathbf{u} = \ddot{\theta}_d - \mathbf{K}_v\dot{e} - \mathbf{K}_p e$$

  to achieve $\ddot{e} + \mathbf{K}_v\dot{e} + \mathbf{K}_p e = 0$

20 Example: Trajectory Tracking

20.1 System Parameters

$$L_1 = 1.0\,\mathrm{m}, \quad L_2 = 0.8\,\mathrm{m}$$
$$m_1 = 2.0\,\mathrm{kg}, \quad m_2 = 1.5\,\mathrm{kg}$$
$$\theta(0) = [0,0]^T\ \mathrm{rad}, \quad \dot{\theta}(0) = [0,0]^T\ \mathrm{rad/s}$$

20.2 Desired Trajectory

$$\theta_d(t) = \begin{bmatrix} \sin(t) \\ 0.5\cos(t) \end{bmatrix} \tag{6}$$

20.3 Control Computation

At $t = 1\,\text{s}$ with $\mathbf{K}_p = 100\mathbf{I}$, $\mathbf{K}_v = 20\mathbf{I}$:

$$\begin{aligned}
\theta(1) &= [0.5, 0.2]^T \\
\dot{\theta}(1) &= [0.8, -0.3]^T \\
\theta_d(1) &= [0.841, 0.270]^T \\
\dot{\theta}_d(1) &= [0.540, -0.420]^T \\
\ddot{\theta}_d(1) &= [-0.841, -0.270]^T \\
e &= [0.341, 0.070]^T \\
\dot{e} &= [-0.260, -0.120]^T \\
\mathbf{u} &= \begin{bmatrix} -0.841 \\ -0.270 \end{bmatrix} + 20 \begin{bmatrix} -0.260 \\ -0.120 \end{bmatrix} + 100 \begin{bmatrix} 0.341 \\ 0.070 \end{bmatrix} \\
&= \begin{bmatrix} 25.7 \\ 8.3 \end{bmatrix} \text{rad/s}^2
\end{aligned}$$

21 Complete Control Law

The final control torque is:

$$\tau = \mathbf{M}(\theta) \begin{bmatrix} 25.7 \\ 8.3 \end{bmatrix} + \mathbf{C}(\theta, \dot{\theta})\dot{\theta} + \mathbf{G}(\theta) \tag{7}$$

22 Complete Torque Calculation

Using the computed $\mathbf{u}$ from (7), we now calculate the final control torque τ.

22.1 Compute Inertia Matrix $\mathbf{M}(\theta)$

At $t = 1\,s$ with $\theta(1) = [0.5, 0.2]^T$ rad:

$$\begin{aligned}
\mathbf{M}(\theta) &= \begin{bmatrix} (m_1 + m_2)L_1^2 + m_2 L_2^2 + 2m_2 L_1 L_2 c_2 & m_2 L_2^2 + m_2 L_1 L_2 c_2 \\ m_2 L_2^2 + m_2 L_1 L_2 c_2 & m_2 L_2^2 \end{bmatrix} \\
&= \begin{bmatrix} (3.5)(1) + 1.5(0.64) + 2(1.5)(1)(0.8)\cos(0.2) & 1.5(0.64) + 1.5(1)(0.8)\cos(0.2) \\ \text{Symmetric} & 1.5(0.64) \end{bmatrix} \\
&= \begin{bmatrix} 3.5 + 0.96 + 2.35 & 0.96 + 1.18 \\ 0.96 + 1.18 & 0.96 \end{bmatrix} \\
&= \begin{bmatrix} 6.81 & 2.14 \\ 2.14 & 0.96 \end{bmatrix} \text{kg} \cdot \text{m}^2
\end{aligned}$$

22.2 Compute Coriolis Matrix $\mathbf{C}(\theta, \dot{\theta})$

With $\dot{\theta} = [0.8, -0.3]^T$ rad/s:

$$\begin{aligned}
\mathbf{C}(\theta, \dot{\theta}) &= \begin{bmatrix} -2m_2 L_1 L_2 \dot{\theta}_2 s_2 & -m_2 L_1 L_2 \dot{\theta}_2 s_2 \\ m_2 L_1 L_2 \dot{\theta}_1 s_2 & 0 \end{bmatrix} \\
&= \begin{bmatrix} -2(1.5)(1)(0.8)(-0.3)\sin(0.2) & -1.5(1)(0.8)(-0.3)\sin(0.2) \\ 1.5(1)(0.8)(0.8)\sin(0.2) & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0.144\sin(0.2) & 0.072\sin(0.2) \\ 0.96\sin(0.2) & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0.0285 & 0.0142 \\ 0.1904 & 0 \end{bmatrix} \text{N} \cdot \text{m} \cdot \text{s}
\end{aligned}$$

22.3 Compute Gravity Vector $\mathbf{G}(\theta)$

$$\begin{aligned}
\mathbf{G}(\theta) &= \begin{bmatrix} (m_1+m_2)gL_1c_1 + m_2gL_2c_{12} \\ m_2gL_2c_{12} \end{bmatrix} \\
&= \begin{bmatrix} 3.5(9.81)(1)\cos(0.5) + 1.5(9.81)(0.8)\cos(0.7) \\ 1.5(9.81)(0.8)\cos(0.7) \end{bmatrix} \\
&= \begin{bmatrix} 34.34(0.8776) + 11.77(0.7648) \\ 11.77(0.7648) \end{bmatrix} \\
&= \begin{bmatrix} 30.14 + 9.00 \\ 9.00 \end{bmatrix} \\
&= \begin{bmatrix} 39.14 \\ 9.00 \end{bmatrix} \mathrm{N\cdot m}
\end{aligned}$$

22.4 Compute Final Torque τ

Using $\mathbf{u} = [25.7, 8.3]^T$ from (7):

$$\begin{aligned}
\tau &= \mathbf{M}(\theta)\mathbf{u} + \mathbf{C}(\theta,\dot{\theta})\dot{\theta} + \mathbf{G}(\theta) \\
&= \begin{bmatrix} 6.81 & 2.14 \\ 2.14 & 0.96 \end{bmatrix} \begin{bmatrix} 25.7 \\ 8.3 \end{bmatrix} + \begin{bmatrix} 0.0285 & 0.0142 \\ 0.1904 & 0 \end{bmatrix} \begin{bmatrix} 0.8 \\ -0.3 \end{bmatrix} + \begin{bmatrix} 39.14 \\ 9.00 \end{bmatrix} \\
&= \begin{bmatrix} 6.81(25.7) + 2.14(8.3) \\ 2.14(25.7) + 0.96(8.3) \end{bmatrix} + \begin{bmatrix} 0.0285(0.8) + 0.0142(-0.3) \\ 0.1904(0.8) + 0(-0.3) \end{bmatrix} + \begin{bmatrix} 39.14 \\ 9.00 \end{bmatrix} \\
&= \begin{bmatrix} 175.0 + 17.8 \\ 55.0 + 8.0 \end{bmatrix} + \begin{bmatrix} 0.0228 - 0.0043 \\ 0.1523 + 0 \end{bmatrix} + \begin{bmatrix} 39.14 \\ 9.00 \end{bmatrix} \\
&= \begin{bmatrix} 192.8 \\ 63.0 \end{bmatrix} + \begin{bmatrix} 0.0185 \\ 0.1523 \end{bmatrix} + \begin{bmatrix} 39.14 \\ 9.00 \end{bmatrix} \\
&= \begin{bmatrix} 231.96 \\ 72.15 \end{bmatrix} \mathrm{N\cdot m}
\end{aligned}$$

23 Interpretation

The computed torque values:

- $\tau_1 = 231.96\,\mathrm{N\cdot m}$
- $\tau_2 = 72.15\,\mathrm{N\cdot m}$

are the actual motor torques required at $t = 1\,s$ to:

- Cancel all nonlinear effects (Coriolis, gravity)
- Compensate for tracking errors
- Achieve the desired acceleration $\ddot{\theta}_d$