COL351 Assignment 1

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1 Minimum Spanning Tree

1.1

1.2

2 Huffman Encoding

2.1

2.2

Since these are 16 bit characters then there are a total of 2^{16} characters that are possible. We will denote 2^{16} by n. Let the frequencies of them be $f_1, f_2, ..., f_n$ and they are in increasing order. It is given that $f_n < 2f_1$. Let's denote the symbol with $a_1, a_2, ..., a_n$

Now let's say we consider any numbers from them. Let them be f_i and f_j.

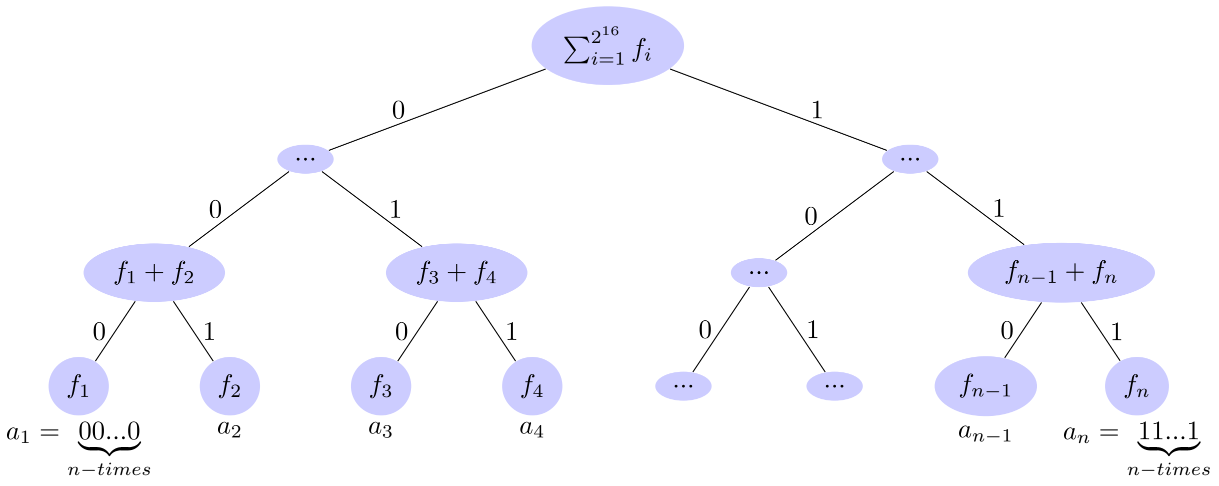
Claim 1: $f_i + f_j > f_n$ (and hence greater than every other frequency).
Proof of claim 1:
$f_i >= f_1$
$f_j >= f_1$
Thus $f_i + f_j >= 2f_1$
Also it is given that $f_n < 2f_1$, thus this directly proves that $f_i + f_j > f_n$.

Now in Huffman encoding we choose the 2 vertices with minimum frequency (say f_1 and f_2) and combine them. Then place a node with value $f_1 + f_2$ and then recursively solve the problem further. The symbols that will be chosen in the next iteration will be f_3 and f_4, since $f_4 <= f_5 <= f_n < f_1 + f_2$. And hence we will join f_3 and f_4 from the set and replace with a node of value $f3 + f4$. This will go on and ultimately we will end with these frequencies in the set: $(f_1 + f_2), (f_3 + f_4), (f_5 + f_6), ..., (f_{n-1} + f_n)$, thus all of the initial $a_i's$ will be combined.

Claim 2: let f be a set of numbers of size n, here n is a power of 2. Let the numbers be $f_1, f_2, f_3, f_4, ..., f_{n-1}, f_n$. If we make another set ff from it such that $ff_i = f_{2i-1} + f_{2i}$, then it is of half the size and also follows the property that maximum element is less than twice the minimum element.
Proof of Claim 2: The minimum element of ff set is $f_1 + f_2$ and the maximum element is $f_{n-1} + f_n$. Now we know that
$f_n < 2f_1$
$f_{n-1} < 2f_1$
and since $f_1 <= f_2$ we can also write that $f_{n-1} < 2f_2$.

Adding both the inequalities we get $f_n + f_{n-1} < 2(f_1 + f_2)$.
Thus for the set ff formed in the above mentioned way, the maximum element is less than twice of the smallest element.

Thus from Claim 2 it is evident that the same pattern will form here and the after combining 2 of them pairwise we will end up nodes of frequency $ff_1 + ff_2, ff_3 + ff_4, ..., ff_{\frac{n}{2}-1} + ff_{\frac{n}{2}}$. Since every successive level is formed after all the nodes from the previous level are exhausted it will take the shape of a perfectly balanced binary tree and every a_i will be at the same level.
Finally the bit encoding of a_i will be the 16 bit representation of $(i - 1)$



Huffman encoding graph for 16-bit characters, $n = 2^{16}$

3 Graduation Party of Alice

3.1

The following problem can be represented as a graph. With the n people as the nodes of the graph and there is an edge between 2 nodes if the two people know each other. Once the graph is ready, we can make an adjacency list for the same in $O(m)$ time. $n->$ no. of people that are invited to the party and $m->$ no. of pairs who know each other (hence the number of edges in the graph will be m). We can maintain an array which will store the degree of each vertex and this can be done in $O(n + m)$ time using the adjacency list we have created above. Degree here will denote the number of people a person knows.

Claim 1: Any node in the graph which has a degree less than 5 cannot be invited to the party.
Proof of Claim 1: Let v_0 be the vertex with degree less than 5. Now since this node has a degree less than 5, it means that he knows less than 5 people out of all the people who can be possibly invited to the party. Their is no such way by which he can know more people and hence the only option that is left with us is to remove him from the list of possible people who can be invited the party.

Now we will keep removing the nodes of the graph which have a degree of less than 5. Notice that as we remove a vertex the degree of it's neighboring vertices will also change and we will have to update their degrees. Removing a node can cause reduction in degree of other nodes. If degree of those vertices fall below 5 then we can't invite them to the party either. We will have to remove them as well. Now this process will continue until every vertex in the graph has degree of at-least 5.

Let the current number of nodes in the graph be n'. Now consider a node whose degree is more than $n' - 6$. Then that person doesn't know less than 5 people and we must do something in this case.

Claim 2: Let V be the current set of nodes and n' be the size of V that is $n' = |V|$. Any vertex whose degree is more than $n' - 6$ cannot be the part of our final optimal solution.
Proof of Claim 2: We will show that their is no final optimal solution in which it doesn't know less than 5 people of all the invited people. Consider the current state of node set V. We know that the final set which will be invited to the party will be a subset of the current V. Let the node with degree more than $n' - 6$ be v_0. Now if any vertex is removed from V (which is not v_0) then there are 2 possible cases. Either it is a neighbour of v_0, that is it knows v_0, or it is not a neighbour of v_0, it doesn't know v_0. If it doesn't know v_0 then removing it from V only reduces the no. of people v_0 doesn't know. If we remove any node which is the neighbor of v_0, then removing them doesn't change the number of people v_0 doesn't know. Hence the only possibility is we have to remove v_0.

Thus any vertex with degree more than $n' - 6$ cannot be the part of our final solution and it must be removed from V. Removing that person might decrease the degree of other vertices as well. From Claim 1 and Claim 2 we know that all the vertices with degree less than 5 must be removed and all the vertices with degree more than $n' - 6$ should also be removed and this process must continue until all the nodes left in the final V has a degree atleast 5 and atmost $n' - 6$.

Once such a V is achieved then we can show that all of these people in V can be invited to the party. Consider any node from the set V, let it be v_1. Now v_1 has a degree of more than 4 and thus it has atleast 5 neighbors and thus knows atleast 5 people from will come to the party (because inviting all of the people present in graph). Also it's degree is atmost $n' - 6$ which means there are atleast 5 vertices that are not connected to v_1. Hence there are atleast 5 people whom v_1 doesn't know.
Hence all of them can be invited to the party.

The problem can be solves in $O(m * log(m))$ time using priority queue, m is the number of edges. The Pseudo code for it is written below. We can initially insert all the nodes in a Binary min heap. Whenever any node is found to have a degree of less than 5 or more than n'-6, then it is removed and marked removed. The process continues until the Heap is empty.

Algorithm:

```
n <- number of people
m <- number of pairs who know each other

edge[m] <- contains the pair of people who know each other

adj[n] <- adjacency list

// Constructing the adjacency list using the edge list

for i <-0 to m-1 do:
    adj[edge[i].First].push(edge[i].Second)
    adj[edge[i].Second].push(edge[i].First)

pq <- Binary Min Heap to store the pairs {degree of vertex, vertex_id}

for i<-0 to n-1 do:
    degree[i] <- adj[i].length()
```

```
18     pq.push(make_pair(degree[i], i))
19
20 n' <- n // this is to store the current size of vertex set of graph
21
22 removed[n] <- to check if some person has been removed from party
23 // removed[i] =1 means the person has been removed from the party
24 // intialized with every value as 0
25
26 while(!pq.empty()) do:
27     d, i = pq.getMinimum()
28     pq.pop()
29
30     if (removed[i]=1) then:
31         continue
32     else if (degree[i]>=5 and degree[i]<= n'-6) then:
33         continue
34     else:
35         removed[i] <- 1
36         n' <- n'-1
37         for j in adj[i] do:
38             if (removed[j]=1) then:
39                 continue
40             else:
41                 degree[j] <- degree[j]-1
42                 // Now push this vertex with updated priority
43                 pq.push(make_pair{degree[j], j})
44
45 invites <- empty list
46
47 // We will invite all of the people who have not been marked removed
48 for i<-0 to n-1 do:
49     if (removed[i] =0) then:
50         invites.push(i)
51
52 // "invites" contains list of all people who will be invited to party
```

Time complexity Analysis:

1. Making adjacency list takes $O(m)$, since we doing only m iterations.
2. Pushing degrees of vertices into Binary Min Heap takes $O(n * log(n))$ time.
3. Every removed node's adjacency list is traversed (atmost 1 time) and the vertex with updated degree is inserted into Binary min heap. This means the maximum possible size of heap could be m. Now since we are iterating in the while loop until the Heap is empty, we do a maximum of m iterations and in each we use the getMinimum function whose complexity is $O(1)$.
4. Traversing the adjacency list takes O(m) time and in each of that we put the updated degree. Adding element in Binary min heap takes O(log(m)) time. Hence the total time taken by the Algorithm from line 37 to 43 over all the iterations of while loop is $O(m * log(m))$

Hence overall time-complexity is $\boldsymbol{O(m * log(m))}$

3.2

Consider all of them are standing in a sorted order of their ages. Let us call them $a_1, a_2, a_3, a_4, ..., a_{n0}$. Thus we know that $age[a_1] <= age[a_2] <= age[a_3]... <= age[a_{n0}]$. Now consider the table on which a_1 is sitting and let this table be T.

Claim 1: If a_k is sitting on T (the table on which a_1 is sitting) in optimal arrangement and there exists a person (say a_i) with age less than a_k not sitting on T (say he is sitting on table T_1), then there also exists an optimal solution in which a_k is sitting in table T_1 and a_i is sitting on table T.
Proof of claim 1: Let's say the person with age less than that of a_k is a_i, a_k is a person sitting on the table T and a_i is not sitting on the table a_k. Let's say a_i was on the table T_1. We can exchange the position of a_i and a_k in this case. Because: a_i can be placed on table of a_1 in place of a_k since $age[a_i] - age[a_1] <= age[a_k] - age[a_1]$. And we can also place a_k in place of a_i since the least age of a member on $T1$ is greater than or equal to $age[a_1]$ and thus the least upper bound possible is $age[a_1] + 9$ and we know that since a_k was sitting on T so $age[a_k] <= age[a_1] + 9$. Thus we can exchange a_i and a_k. Keeping a_k on the other table can provide more flexibility in age constraint on the other table.

Now using the above we can place $S = a_1, a_2, a_3..., a_m$ people on the first table, here $m <= 10$ and $age[a_m] - age[a_1] < 10$. Also $age[a_1] <= age[a_2] <= age[a3]... <= age[am]$. If $m < 10$ then either we don't have enough number of people or $age[m + 1] - age[a1] >= 10$.
Let J be the set of all the people and S be the set that we have placed. Now define $J^* = J \backslash S$.

Claim 2: $opt(J) = opt(J*) + 1$, $opt(J)$ is the minimum number seats required to arrange the set of people J on tables under the required constraints.
Proof of Claim 2:
We will show that both the below inequalities are true

$opt(J) <= opt(J^*) + 1$ ...(i)
$opt(J) - 1 >= opt(J^*)$ ...(ii)

Proof of (i):
After placing S on 1 table we are left with $J \backslash S$ people and $opt(J^*)$ will place all of them on some table. And hence there exists one arrangement in which the arrangement can be done in $opt(J^*) + 1$ no. of tables. Thus $opt(J) <= opt(J^*) + 1$
Proof of (ii):
Let A be the optimal arrangement of J in which all the people in set S are placed on a single table (say table T). Only they can be seated on that table. Now none of the people from $J \backslash S$ can be seated on table T since either T is full or their age is more than $age[a_1] + 9$. Thus the $A \backslash T$ can be used to place all of the members of $J \backslash S$ ie J^*. Thus $J \backslash S$ members can be placed using $opt(J) - 1$ tables. Hence one solution of size $opt(J) - 1$ exists for J^*. This hows that $opt(J^*) <= opt(J) - 1$.

Hence from both (i) and (ii) we have, $opt(J) = opt(J^*) + 1$.

Now for calculating we can maintain a $freq$ array. $freq[i]$ denotes the number of people with age $i+10$. Now we can start placing them on tables in increasing order as we have discussed above, by recursively dividing the problem into a smaller sub-problem.

Algorithm:

```
n0 <- total number of people invited to party
age[n0] <- age of all the n0 people invited to the party

freq[90] <- to store the count of people with some age
// intialize each value with zero
// freq[i] = no. of peopel with age i+10, the array is 0 indexed

// First we complwte the frequency array
for i<- 0 to n0-1 do:
    freq[age[i]-10] <- freq[age[i]-10] +1

// Now we will start placing them on tables

total <- 0  // this maintains the count of tables we will need

i<- 0

while i<90 do:
    if (freq[i]=0) then:
        i <- i+1
        continue
    seats <- 10  // seats on a table
    j <- i
    while(seats != 0 and j<90 and j-i<10) do:
        if (freq[j]> seats) then:
            freq[j]<- frea[j]- seats
            seats <- 0
        else:
            seats <- seats - freq[j]
            freq[j] <- 0
            j <- j+1
    total <- total+1
    i<- j

// "total" is the number of seats required to place them
```

Time complexity Analysis:

1. Making the $freq$ array needs $O(n_0)$ time.
2. j takes values till 90 only over all the iterations but it might be possible it is stuck at same position for some iterations
3. Since the $freq[j]$ decreases by 10 in one iteration, hence the maximum possible number of times these loops can run is $n_0/10$.

Thus the overall Time Complexity is $O(n_0)$