ECE 657 Assignment 1 Answer 3.-

To prove: when input vector $x^{(k)}$ is presented at iteration $(k+1)$ then the weight vector change decreases by factor $(1-\eta)$ going from iteration $(k)$ to iteration $(k+1)$.

$$\Delta w^{(k+1)} = (1-\eta)\,\Delta w^{(k)}$$

Given:

1) Normalized Widrow Hoff learning rule/
Normalized least mean square learning rule expressed as:

$$\Delta w^{(k)} = \eta\left(t^{(k)} - w^{(k)} x^{(k)}\right) \frac{x^{(k)}}{\| x^{(k)} \|^2} \quad -①$$

2) weight vector change from iteration $(k)$ to $(k+1)$

$$\Delta w^{(k)} = w^{(k+1)} - w^{(k)}$$

or $$w^{(k+1)} = w^{(k)} + \Delta w^{(k)} \quad -②$$

3) $t^{(k)}$ : expected output for $x^{(k)}$ at iteration $(k)$.

4) $w^{(k)}$ : weight of $x^{(k)}$ at iteration $(k)$.

5) $\eta$ : learning rate $0 < \eta < 1$
It determines the severity of the weight and threshold changes.

6) $\| x^k \|$ : Euclidean norm of input vector $x^{(k)}$ at iteraction $(k)$.

It can also be written as

$$\| x^k \| = \sqrt{x^k \cdot x^k} \quad -③$$

Proof: The Delta Rule for $(k+1)$ iteration is given by

$$\Rightarrow \Delta w^{(k+1)} = \eta\left(t^{(k+1)} - w^{(k+1)} x^{(k+1)}\right) \frac{x^{(k+1)}}{\|x^{(k+1)}\|^2} \quad -⑥$$

As the problem states that the same input vector $x^{(k)}$ is presented for iteration $(k+1)$

$$\Rightarrow x^{(k+1)} = x^{(k)} \quad -④$$

Since, $t^{(k+1)}$ is expected output and $x^{(k+1)} = x^{(k)}$

$$\Rightarrow t^{(k+1)} = t^{(k)} \quad -⑤$$

In equation ⑥, substitute the values of eq ②, ④, ⑤

We get

$$\Rightarrow \Delta w^{(k+1)} = \eta\left(t^{(k)} - \left(w^{(k)} + \Delta w^{(k)}\right) x^{(k)}\right) \frac{x^{(k)}}{\|x^{(k)}\|^2}$$

$$\Rightarrow \Delta w^{(k+1)} = \eta\left(t^{(k)} - w^{(k)} x^{(k)} - \Delta w^{(k)} x^{(k)}\right) \frac{x^{(k)}}{\|x^{(k)}\|^2}$$

$$\Rightarrow \Delta w^{(k+1)} = \boxed{\eta\left(t^{(k)} - w^{(k)} x^{(k)}\right) \frac{x^{(k)}}{\|x^{(k)}\|^2}} - \eta\left(\frac{\Delta w^{(k)} (x^{(k)})^2}{\|x^{(k)}\|^2}\right)$$

This is $\Delta w^{(k)}$ from eq ①

$$\Rightarrow \Delta w^{(k+1)} = \Delta w^{(k)} - \eta\, \Delta w^{(k)} \boxed{\frac{(x^{(k)})^2}{\|x^{(k)}\|^2}} = 1$$

We know that from eq ③ $\|x^{(k)}\|^2 = (x^{(k)})^2$.

$$\Rightarrow \therefore \Delta w^{(k+1)} = \Delta w^{(k)} - \eta\, \Delta w^{(k)}$$

Taking $\Delta w^{(k)}$ common.

$$\Delta w^{(k+1)} = \Delta w^{(k)}(1-\eta)$$

$$\frac{\Delta w^{(k+1)}}{\Delta w^{(k)}} = 1-\eta$$

Hence Proved.